

A note on the existence of L^2 valued solutions for a hyperbolic system with boundary conditions

Stefano Marchesani

Stefano Olla

February 10, 2019

Abstract

We prove existence of L^2 -weak solutions of a quasilinear wave equation with boundary conditions. This is done using a vanishing viscosity approximation with mixed Dirichlet-Neumann boundary conditions. In this setting we obtain a uniform a priori estimate in L^2 , allowing us to use L^2 Young measures, together with the classical tools of compensated compactness. We then prove that the viscous solutions converge to weak solutions of the quasilinear wave equation strongly in L^p , for any $p \in [1, 2)$, that satisfy, in a weak sense, the boundary conditions in the sense given by Definition 2.1. Furthermore, Clausius inequality is in force for these solutions.

Keywords: hyperbolic conservation laws, quasi-linear wave equation, boundary conditions, weak solutions, vanishing viscosity, compensated compactness

Mathematics Subject Classification numbers: 35L40, 35D40

1 Introduction

The problem of existence of weak solutions for hyperbolic systems of conservation law in a bounded domain has been studied for solutions that are of bounded variation or in L^∞ [6]. In the scalar case some works extend to L^∞ solutions, obtained from viscous approximations [17]. But viscous approximations require extra boundary conditions, that are usually taken of Dirichlet type.

We present here an approach based on viscosity approximations, where the extra boundary conditions are of Neumann type, to reflect the conservative nature of the viscous approximation. We consider here the quasilinear wave equation

$$\begin{cases} r_t - p_x = 0 \\ p_t - \tau(r)_x = 0 \end{cases}, \quad (t, x) \in (0, \infty) \times (0, 1) \quad (1.1)$$

where $\tau(r)$ is a strictly increasing regular function of r such that $0 < c_1 \leq \tau'(r) \leq c_2$, for some constant c_1, c_2 . We shall also later on require other technical assumption for τ . We add to the system the following boundary conditions:

$$p(t, 0) = 0, \quad r(t, 1) = \tau^{-1}(\bar{\tau}(t)) \quad (1.2)$$

and initial data

$$r(0, x) = r_0(x), \quad p(0, x) = p_0(x) \quad (1.3)$$

with $r_0, p_0 \in L^2(0, 1)$ which satisfy suitable compatibility with the boundary conditions.

The boundary tension $\bar{\tau} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is smooth. Moreover, there are a time T_* and a real number τ_1 such that $\bar{\tau}(t) = \tau_1$ for all $t \geq T_*$. This implies that all the derivatives of $\bar{\tau}$ are zero on (T_*, ∞) . In particular, $\bar{\tau}$ is bounded together with all its derivatives.

Throughout this paper, if a vector field u belongs to $L^p(\Omega; \mathbb{R}^2)$, for some $\Omega \subset \mathbb{R}^2$ and $p \geq 1$, we will simply write $u \in L^p(\Omega)$. Similarly, we will write $u \in L^p_{loc}(\Omega)$ in place of $u \in L^p_{loc}(\Omega; \mathbb{R}^2)$. We shall construct weak solutions $u(t, y) = (r(t, y), p(t, y))$ to the quasilinear wave equation which are in $L^2_{loc}(\mathbb{R}_+ \times [0, 1])$ and satisfy the initial and boundary conditions in the following weak sense:

$$\int_0^1 \varphi(0, x) r_0(x) dx + \int_0^\infty \int_0^1 (\varphi_t r - \varphi_x p) dx dt = 0 \quad (1.4)$$

$$\int_0^1 \psi(0, x) p_0(x) dx + \int_0^\infty \int_0^1 (\psi_t p - \psi_x \tau(r)) dx dt + \int_0^\infty \psi(t, 1) \bar{\tau}(t) dt = 0 \quad (1.5)$$

for all functions $\varphi, \psi \in C^1_c(\mathbb{R}_+ \times [0, 1])$ such that $\varphi(t, 1) = \psi(t, 0) = 0$ for all $t \geq 0$.

Define the free energy of the system, associated to a profile $u(y) = (r(y), p(y)) \in L^2(0, 1)$, as

$$\mathcal{F}(u) := \int_0^1 \left(\frac{p^2(y)}{2} + F(r(y)) \right) dx \quad (1.6)$$

where $F(r)$ is a primitive of $\tau(r)$ ($F'(r) = \tau(r)$), such that $\frac{c_1}{2} r^2 \leq F(r) \leq \frac{c_2}{2} r^2$ for any $r \in \mathbb{R}$. This is possible thanks to the bounds we required on τ' .

The solution $\bar{u} \in L^2_{loc}(\mathbb{R}_+ \times [0, 1])$ of (1.5) that we obtain has the following properties:

- $\bar{u} \in L^\infty(\mathbb{R}_+; L^2(0, 1))$ and, for all $t \geq 0$, $\bar{u}(t, \cdot) \in L^2(0, 1)$. Moreover, $\bar{u}(0, x) = u_0(x)$ for a.e. x ;
- \bar{u} satisfies *Clausius inequality*:

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(u_0) \leq W(t), \quad \forall t \geq 0, \quad (1.7)$$

where $u_0 = (r_0, p_0)$ and

$$W(t) := - \int_0^t \bar{\tau}'(s) \int_0^1 \bar{r}(s, x) dx ds + \bar{\tau}(t) \int_0^1 \bar{r}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0(x) dx \quad (1.8)$$

is the work done by the external tension up to time. In this sense we call our solution an *entropy solution*.

Remark. Moreover, if $\bar{r}(t, x)$ is differentiable with respect to time, we may perform an integration by parts and obtain

$$W(t) = \int_0^t \bar{\tau}(s) dL(s), \quad (1.9)$$

where

$$L(s) := \int_0^1 \bar{r}(s, x) dx \quad (1.10)$$

is the total length of the chain at time s . This recovers the usual definition of the work W .

The construction of the solution is obtained from the following viscosity approximation

$$\begin{cases} r_t^\delta - p_x^\delta = \delta r_{xx}^\delta \\ p_t^\delta - \tau(r^\delta)_x = \delta p_{xx}^\delta \end{cases}, \quad (t, x) \in (0, \infty) \times (0, 1) \quad (1.11)$$

with boundary conditions

$$p^\delta(t, 0) = 0, \quad r^\delta(t, 1) = \tau^{-1}(\bar{\tau}(t)), \quad p_x^\delta(t, 1) = 0, \quad r_x^\delta(t, 0) = 0 \quad (1.12)$$

and initial data

$$r^\delta(0, x) = r_0^\delta(x), \quad p^\delta(0, x) = p_0^\delta(x) \quad (1.13)$$

such that r_0^δ and p_0^δ are compatible with the boundary conditions, regular enough and converge to r_0 and p_0 , respectively, as $\delta \rightarrow 0$.

Note that in the viscous approximation we have added two Neumann boundary conditions, that reflect the *conservative* nature of the viscous perturbation. Under these conditions we have

$$\int_0^1 |u^\delta|^2 dx + \delta \int_0^t \int_0^1 |u_x^\delta|^2 dx ds \leq C, \quad \forall t \geq 0 \quad (1.14)$$

where C is independent of t and δ . If we integrate on a finite time interval as well, it is clear that $\{u^\delta\}_{\delta>0}$ and $\{\sqrt{\delta}u_x^\delta\}_{\delta>0}$ are uniformly bounded in $L^2_{loc}(\mathbb{R}_+ \times (0,1))$. Then we rely in the existence of a family of bounded Lax entropy-entropy fluxes as in [20], [18] and [19], that allows us to apply the compensated compactness in the L^2 version. The conditions assumed on $\tau(r)$ are in fact those required to apply [20] results. Under a slight different set of conditions, another L^p extension of the compensated compactness argument can be found in [11].

1.1 Physical motivations

The problem arises naturally considering hydrodynamic limit for a non-linear chain of oscillators (of FPU type) in contact with a heat bath at a given temperature [13]. This models an isothermal transformation governed by (1.1).

Consider $N + 1$ particles on the real line and call q_i and p_i the positions and the momenta of the i -th particle, respectively. Particles i and $i - 1$ interact via a nonlinear potential $V(q_i - q_{i-1})$. Then, defining $r_i := q_i - q_{i-1}$ we have a system with Hamiltonian

$$H_N = \frac{p_0^2}{2} + \sum_{i=1}^N \left(\frac{p_i^2}{2} + V(r_i) \right) \quad (1.15)$$

and $u_i := (r_i, p_i)$ evolve, after a time-scaling, accordingly to the Newton's equations

$$du_i = NJ_{i-1,i} dt. \quad (1.16)$$

The microscopic currents are defined by

$$\begin{aligned} J_{i-1,i} &= (J_{i-1,i}^1, J_{i-1,i}^2) := (p_i - p_{i-1}, V'(r_{i+1}) - V'(r_i)), \quad 1 \leq i \leq N-1 \\ J_{N-1,N} &= (J_{N-1,N}^1, J_{N-1,N}^2) := (p_N - p_{N-1}, \bar{\tau}(t) - V'(r_N)) \end{aligned}$$

together with $p_0 \equiv 0$. Thus, the boundary conditions have the following microscopic interpretation: particle number 0 is not moving, while on particle number N is applied a force $\bar{\tau}(t)$. The microscopic nonlinearity τ is fully determined as the expectation in equilibrium of the microscopic force V' . The system is then put in contact with a heat bath at temperature β^{-1} that acts as a microscopic stochastic viscosity, and the evolution equations are given by the following system of stochastic differential equations:

$$\begin{cases} dr_1 = Np_1 dt + N^2 \delta J_{0,1}^2 dt - \sqrt{2\beta^{-1}N^2\delta} d\tilde{w}_1 \\ dr_i = N\nabla p_{i-1} dt + N^2 \delta \nabla J_{i-2,i-1}^2 dt - \sqrt{2\beta^{-1}N^2\delta} \nabla d\tilde{w}_{i-1}, & 2 \leq i \leq N \\ dp_j = N\nabla V'(r_j) dt + N^2 \delta \nabla J_{j-1,j}^1 dt - \sqrt{2\beta^{-1}N^2\delta} \nabla dw_{j-1}, & 1 \leq j \leq N-1 \\ dp_N = N(\bar{\tau}(t) - V'(r_N)) dt - N^2 \delta J_{N-1,N}^1 dt + \sqrt{2\beta^{-1}N^2\delta} dw_{N-1} \end{cases} \quad (1.17)$$

together with $\tilde{w}_N = w_0 = 0$. Here $\beta^{-1} > 0$ is the temperature and $\{w_i\}_{i=1}^{N-1}$, $\{\tilde{w}_i\}_{i=1}^{N-1}$ are families of independent Brownian motions. The parameter δ is chosen depending on N and vanishing as $N \rightarrow \infty$.

Finally, ∇ is a discrete gradient, and acts on sequences $\{a_i\}_{i \in \mathbb{N}}$ as follows:

$$\nabla a_i := a_{i+1} - a_i. \quad (1.18)$$

The hydrodynamic limit consists in proving that, for any continuous function $G(x)$ on $[0, 1]$,

$$\frac{1}{N} \sum_{i=1}^N G\left(\frac{i}{N}\right) \begin{pmatrix} r_i(t) \\ p_i(t) \end{pmatrix} \xrightarrow{N \rightarrow \infty} \int_0^1 G(x) \begin{pmatrix} r(t, x) \\ p(t, x) \end{pmatrix} dx, \quad (1.19)$$

in probability, with $(r(t, x), p(t, x))$ satisfying (1.4), (1.5). Of course a complete proof would require the uniqueness of such L^2 valued solutions that satisfy (1.7): this remains an open problem. The results contained in [12] states that the limit distribution of the empirical distribution defined on the RHS of (1.19), concentrates on the possible solutions of (1.4) and (1.5) that satisfy (1.7).

This stochastic model was already considered by Fritz [10] in the infinite volume without boundary conditions, and in [13], but without the characterisation of the boundary conditions. In the hydrodynamic limit only L^2 bounds are available and we are constrained to consider L^2 valued solutions. Since these solutions do not have definite values on the boundary, boundary conditions have only a dynamical meaning in the sense of an evolution in L^2 given by (1.4), (1.5).

2 Hyperbolic system and definition of weak solutions

For $r, p : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$, consider the hyperbolic system

$$\begin{cases} r_t - p_x = 0 & p(t, 0) = 0 & r(t, 1) = \tau^{-1}(\bar{\tau}(t)) \\ p_t - \tau(r)_x = 0 & p(0, x) = p_0(x) & r(0, x) = r_0(x) \end{cases}, \quad (2.1)$$

The nonlinearity $\tau \in C^3(\mathbb{R})$ is chosen to have the following properties.

(τ -i) $c_1 \leq \tau'(r) \leq c_2$ for some $c_1, c_2 > 0$ and all $r \in \mathbb{R}$;

(τ -ii) $\tau''(r) \neq 0$ for all $r \in \mathbb{R}$;

(τ -iii) $\tau''(r), \tau'''(r) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

We also assume that $\bar{\tau} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is smooth. Moreover, there is a time T_* such that $\bar{\tau}'(t) = 0$ for all $t \geq T_*$. The initial data $r_0, p_0 \in L^2(0, 1)$ are compatible with the boundary conditions.

Remark. Conditions (τ -i) and (τ -ii) ensure that the system is strictly hyperbolic and genuinely nonlinear, respectively. Condition (τ -iii) is used later on to ensure some boundedness properties of the Lax entropies.

Definition 2.1. We say that $(r, p) \in L^2_{loc}(\mathbb{R}_+ \times [0, 1])$ is a weak solution of (2.1) provided

$$\int_0^1 \varphi(0, x) r_0(x) dx + \int_0^\infty \int_0^1 (\varphi_t r - \varphi_x p) dx dt = 0 \quad (2.2)$$

$$\int_0^1 \psi(0, x) p_0(x) dx + \int_0^\infty \int_0^1 (\psi_t p - \psi_x \tau(r)) dx dt + \int_0^\infty \psi(t, 1) \bar{\tau}(t) dt = 0 \quad (2.3)$$

for all functions $\varphi, \psi \in C_c^1(\mathbb{R}_+ \times [0, 1])$ such that $\varphi(t, 1) = \psi(t, 0) = 0$ for all $t \geq 0$.

We shall prove the following

Theorem 2.2. System (2.1) admits a weak solution $\bar{u} = (\bar{r}, \bar{p})$ in the sense of Definition 2.1, and such that

- $\bar{u} \in L^\infty(\mathbb{R}_+; L^2(0, 1))$ and, for all $t \geq 0$, $\bar{u}(t, \cdot) \in L^2(0, 1)$. and $\bar{u}(0, x) = u_0(x)$ for a.e. x ;
- \bar{u} satisfies the Clausius inequality:

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(u_0) \leq W(t), \quad \forall t \geq 0 \quad (2.4)$$

with W as in (1.8).

3 Viscous approximation and energy estimates

We consider the following parabolic approximation of the hyperbolic system (2.1)

$$\begin{cases} r_t^\delta - p_x^\delta = \delta r_{xx}^\delta \\ p_t^\delta - \tau(r^\delta)_x = \delta p_{xx}^\delta \end{cases}, \quad (t, x) \in (0, \infty) \times (0, 1) \quad (3.1)$$

for $\delta > 0$, with the boundary conditions:

$$p^\delta(t, 0) = 0, \quad r^\delta(t, 1) = \tau^{-1}(\bar{\tau}(t)), \quad p_x^\delta(t, 1) = 0, \quad r_x^\delta(t, 0) = 0,$$

and initial data:

$$p^\delta(0, x) = p_0^\delta(x), \quad r^\delta(0, x) = r_0^\delta(x).$$

The initial data $r_0^\delta, p_0^\delta \in C^\infty([0, 1])$ are mollifications of r_0 and p_0 compatible with the boundary conditions:

$$p_0^\delta(0) = 0, \quad r_0^\delta(1) = \tau^{-1}(\bar{\tau}(0)), \quad \partial_x p_0^\delta(1) = 0, \quad \partial_x r_0^\delta(0) = 0.$$

and there is C independent of δ such that

$$\|r_0^\delta\|_{L^2} + \|p_0^\delta\|_{L^2} + \|\sqrt{\delta}\partial_x r_0^\delta\|_{L^2} + \|\sqrt{\delta}\partial_x p_0^\delta\|_{L^2} \leq C$$

Finally, $(r_0^\delta, p_0^\delta) \rightarrow (r_0, p_0)$ strongly in $L^2(0, 1)$.

As shown in [2] in a more general setting, this system admits a global classical solution (r^δ, p^δ) , with

$$r^\delta, p^\delta \in C^1(\mathbb{R}_+; C^0([0, 1])) \cap C^0(\mathbb{R}_+; C^2([0, 1])).$$

Remark. (i) We added two extra Neumann conditions, namely $p_x^\delta(t, 1) = r_x^\delta(t, 0) = 0$. These conditions reflect the conservative nature of the viscous perturbation, and are required in order to obtain the correct production of free energy.

(ii) One could introduce a *nonlinear* viscosity term: $\delta\tau(r^\delta)_{xx}$ in place of δr_{xx}^δ . This is a term which comes naturally from a microscopic derivation of system (3.1), as described in the introduction. Nevertheless, this does not drastically change the problem, thus we shall consider only the linear viscosity δr_{xx}^δ .

Theorem 3.1 (Energy estimate). *There there is a constant $C > 0$ independent of t and δ such that*

$$\int_0^1 |u^\delta(t, x)|^2 dx + \delta \int_0^t \int_0^1 |u_x^\delta(s, x)|^2 dx ds \leq C \quad (3.2)$$

for all $t \geq 0$ and $\delta > 0$.

Proof. Let F be a primitive of τ such that $\frac{c_1}{2}r^2 \leq F(r) \leq \frac{c_2}{2}r^2$. By a direct calculation we have

$$\begin{aligned} \int_0^1 \left(\frac{(p^\delta)^2}{2} + F(r^\delta) \right) dx \Big|_{t=0}^{t=T} + \int_0^T \int_0^1 \left(\delta (r_x^\delta)^2 + \delta (p_x^\delta)^2 \right) dx &= \int_0^T \bar{\tau}(t) \int_0^1 r_t^\delta dx dt \\ &= \left(\bar{\tau}(t) \int_0^1 r^\delta dx \right) \Big|_{t=0}^{t=T} - \int_0^T \bar{\tau}'(t) \int_0^1 r^\delta dx dt. \end{aligned} \quad (3.3)$$

Write, for some $\varepsilon > 0$ to be chosen later,

$$\bar{\tau}(T) \int_0^1 r^\delta(T, x) dx \leq |\bar{\tau}(T)| \left(\frac{1}{2\varepsilon} + \frac{\varepsilon}{2} \int_0^1 (r^\delta)^2(T, x) dx \right) \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + \frac{C_{\bar{\tau}}\varepsilon}{2} \int_0^1 (r^\delta)^2(T, x) dx \quad (3.4)$$

where $C_{\bar{\tau}} = \sup_{t \geq 0} (|\bar{\tau}(t)| + |\bar{\tau}'(t)|)$ depends on $\bar{\tau}$ only.

Using $F(r) \geq \frac{c_1}{2}r^2$ we obtain

$$\begin{aligned} \left(\frac{c_1}{2} - \frac{C_{\bar{\tau}}\varepsilon}{2}\right) \int_0^1 (r^\delta)^2(T, x) dx + \frac{1}{2} \int_0^1 (p^\delta)^2(T, x) dx + \int_0^T \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx dt \\ \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 - \int_0^T \bar{\tau}'(t) \int_0^1 r^\delta(t, x) dx dt. \end{aligned} \quad (3.5)$$

Recall that there is $T_\star > 0$ such that $\bar{\tau}'(t) = 0$ for $t \geq T_\star$. Then, for $T < T_\star$, we write

$$\begin{aligned} \left(\frac{c_1}{2} - \frac{C_{\bar{\tau}}\varepsilon}{2}\right) \int_0^1 (r^\delta)^2(T, x) dx + \frac{1}{2} \int_0^1 (p^\delta)^2(T, x) dx + \int_0^T \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx dt \\ \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 + \frac{C_{\bar{\tau}}^2}{2}T + \frac{1}{2} \int_0^T \int_0^1 (r^\delta)^2(t, x) dx \\ \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 + \frac{C_{\bar{\tau}}^2}{2}T + \frac{1}{2} \int_0^T \int_0^1 \left((r^\delta)^2(t, x) + (p^\delta)^2(t, x)\right) dx dt + \frac{1}{2} \int_0^T \int_0^t \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx ds dt \end{aligned} \quad (3.6)$$

where C_0 depends on the initial data only. Choosing $\varepsilon = c_1/(2C_{\bar{\tau}})$ gives

$$\frac{c_1}{4}J(T) \leq C_0 + \frac{C_{\bar{\tau}}^2}{c_1} + \frac{C_{\bar{\tau}}^2}{2}T + \frac{1}{2} \int_0^T J(t) dt, \quad (3.7)$$

where

$$J(t) = \int_0^1 \left((r^\delta)^2(t, x) + (p^\delta)^2(t, x)\right) dx + \int_0^t \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx ds. \quad (3.8)$$

We apply Gronwall's inequality. This, together with $T < T_\star$, gives

$$\begin{aligned} J(T) &\leq \frac{4c_1C_0 + 2C_{\bar{\tau}}^2(2 + c_1T)}{c_1^2} \exp\left(\frac{2T}{c_1}\right), \\ &\leq \frac{4c_1C_0 + 2C_{\bar{\tau}}^2(2 + c_1T_\star)}{c_1^2} \exp\left(\frac{2T_\star}{c_1}\right) := C_0(c_1, \bar{\tau}), \end{aligned} \quad (3.9)$$

for all $T \in [0, T_\star)$, where $C_0(c_1, \bar{\tau})$ is independent of T , δ and δ .

On the other hand, if $T \geq T_\star$, we have

$$\begin{aligned} \left(\frac{c_1}{2} - \frac{C_{\bar{\tau}}\varepsilon}{2}\right) \int_0^1 (r^\delta)^2(T, x) dx + \frac{1}{2} \int_0^1 (p^\delta)^2(T, x) dx + \int_0^T \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx dt \\ \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 - \int_0^{T_\star} \bar{\tau}'(t) \int_0^1 r^\delta(t, x) dx dt \end{aligned} \quad (3.10)$$

and the integral at the right-hand side is uniformly bounded in T , δ and δ , since

$$\begin{aligned} - \int_0^{T_\star} \bar{\tau}'(t) \int_0^1 r^\delta(t, x) dx dt &\leq C_{\bar{\tau}} \int_0^{T_\star} \int_0^1 |r^\delta(t, x)| dx dt \\ &\leq C_{\bar{\tau}} T_\star \left(\frac{1}{T_\star} \int_0^{T_\star} \int_0^1 (r^\delta)^2(t, x) dx dt\right)^{1/2} \\ &\leq C_{\bar{\tau}} \sqrt{T_\star} \left(\int_0^{T_\star} J(t) dt\right)^{1/2} \leq C_{\bar{\tau}} T_\star \sqrt{C_0(c_1, \bar{\tau})} \end{aligned} \quad (3.11)$$

□

From (3.3) we also immediately obtain the following

Corollary 3.2 (Viscous Clausius inequality).

$$\mathcal{F}(u^\delta(t)) - \mathcal{F}(u_0^\delta) \leq - \int_0^t \bar{\tau}'(s) \int_0^1 r^\delta(s, x) dx + \bar{\tau}(t) \int_0^1 r^\delta(t, x) dx - \bar{\tau}(0) \int_0^1 r_0^\delta(x) dx. \quad (3.12)$$

4 L^p Young measures and compensated compactness

By Theorem 3.1 and after a time integration over $(0, T)$ for any $T > 0$, we obtain

$$\|u^\delta\|_{L^2((0,T)\times(0,1))} \leq C \quad (4.1)$$

for some C independent of δ . Thus we can extract from $\{u^\delta\}_{\delta>0}$ a subsequence that is weakly convergent in $L^2_{loc}(\mathbb{R}_+ \times (0, 1))$. Namely, up to a subsequence, there exists $\bar{u} = (\bar{r}, \bar{p}) \in L^2_{loc}(\mathbb{R}_+ \times (0, 1))$ such that

$$\lim_{\delta \rightarrow 0} \int_0^\infty \int_0^1 u^\delta \varphi = \int_0^\infty \int_0^1 \bar{u} \varphi, \quad \forall \varphi \in L^2_{loc}(\mathbb{R}_+ \times (0, 1)). \quad (4.2)$$

Our aim is to show that our solution u^δ converges to a weak solution of the hyperbolic system (2.1), in the sense of Section 2. We focus on obtaining (2.3), as the other equation is easier, being linear. Let $\psi \in C^1_c(\mathbb{R}_+ \times [0, 1])$ with $\psi(t, 0) = 0$. Then, we have

$$\begin{aligned} 0 &= \int_0^\infty \int_0^1 \left(\psi p_t^\delta - \psi \tau(r^\delta)_x - \delta \psi p_{xx}^\delta \right) dx dt \\ &= - \int_0^1 \psi(0, x) p_0^\delta(x) dx - \int_0^\infty \int_0^1 \left(\psi_t p^\delta - \psi_x \tau(r^\delta) - \delta \psi_x p_x^\delta \right) dx dt - \int_0^\infty \psi(t, 1) \bar{\tau}(t) dt. \end{aligned} \quad (4.3)$$

where we have used the boundary conditions $\tau(r^\delta(t, 1)) = \bar{\tau}(t)$ and $p_x(t, 1) = 0$, as well as $p^\delta(0, x) = p_0^\delta(x)$ and $\psi(t, 0) = 0$. Since p_0^δ converges to p_0 in $L^2(0, 1)$, we have

$$\lim_{\delta \rightarrow 0} \int_0^1 \psi(0, x) p_0^\delta(x) dx = \int_0^1 \psi(0, x) p_0(x) dx. \quad (4.4)$$

Furthermore, Theorem 3.1 implies $\sqrt{\delta} p_x^\delta \in L^2_{loc}(\mathbb{R}_+ \times [0, 1])$. Thus, since ψ_x is compactly supported in time and $\delta p_x^\delta = \sqrt{\delta} (\sqrt{\delta} p_x^\delta)$, the term

$$\int_0^\infty \int_0^1 \delta \psi_x p_x^\delta dx dt \quad (4.5)$$

vanishes as $\delta \rightarrow 0$.

Moreover, (4.2) implies

$$\lim_{\delta \rightarrow 0} \int_0^\infty \int_0^1 \psi_t p^\delta dx dt = \int_0^\infty \int_0^1 \psi_t \bar{p} dx dt. \quad (4.6)$$

We are left to show that

$$\lim_{\delta \rightarrow 0} \int_0^\infty \int_0^1 \psi_x \tau(r^\delta) dx dt = \int_0^\infty \int_0^1 \psi_x \tau(\bar{r}) dx dt \quad (4.7)$$

Note that the sense in which the boundary is attained is encoded in the last term of (4.3). Therefore, we may focus on the convergence (4.7) only on the spatial interior $(0, 1)$. This is done using a L^p version of the compensated compactness, which is usually performed in L^∞ .

For any $T > 0$ let $Q_T := [0, T] \times [a, b]$ be a compact subset of $\mathbb{R}_+ \times (0, 1)$. From the solution $u^\delta(t, x)$, we define the following Young measure on $Q_T \times \mathbb{R}^2$:

$$\nu_{t,x}^\delta := \delta_{u^\delta(t,x)}, \quad (4.8)$$

which is a Dirac mass centred at u^δ , i.e.

$$\int_{Q_T} J(t, x) f(u^\delta(t, x)) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\nu_{t,x}^\delta(\xi) dx dt$$

for all measurable $J : Q_T \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Since we have L^2 bounds on u^δ , we refer at $\nu_{t,x}^\delta$ as a L^2 -Young measure [3].

We call \mathcal{Y} the set of Young measures on $Q_T \times \mathbb{R}^2$ and we make it a metric space by endowing it with the Prohorov's metric. Then (cf. [3]), if there is a compact set $K \subset \mathcal{Y}$ such that $\nu_{t,x}^\delta \in K$ for all $\delta > 0$, there exists $\bar{\nu}_{t,x} \in \mathcal{Y}$ so that, up to a subsequence,

$$\lim_{\delta \rightarrow 0} \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) d\nu_{t,x}^\delta(\xi) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) d\bar{\nu}_{t,x}(\xi) dx dt \quad (4.9)$$

for all continuous and *bounded* $J : Q_T \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We shall simply write $\nu^\delta \rightarrow \nu$ in place of (4.9). We proceed by finding such compact set K and then by extending the convergence to functions f with subquadratic growth. This is done by adapting the argument of [4].

Proposition 4.1. *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ be continuous and such that*

$$\lim_{|\xi| \rightarrow +\infty} h(\xi) = +\infty. \quad (4.10)$$

Then, for any $C > 0$, the set

$$K_C := \left\{ \nu \in \mathcal{Y} : \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}(\xi) dx dt \leq C \right\} \quad (4.11)$$

is compact.

Proof. K_C is relatively compact if and only if it is tight, namely if for all $\varepsilon > 0$ there exists a compact subset $A_\varepsilon \subset \mathbb{R}^2$ such that

$$\nu(\mathbb{R}^2 \setminus A_\varepsilon) \leq \varepsilon \quad (4.12)$$

for all $\nu \in K_C$. Thus, K_C is compact if and only if it closed and tight.

K_C is closed. In fact, if $\nu^\delta \in K_C$ and $\nu^\delta \rightarrow \bar{\nu}$,

$$\int_{Q_T} \int_{\mathbb{R}^2} h(\xi) \bar{\nu}_{t,x}(\xi) dx dt \leq \liminf_{\delta \rightarrow 0} \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) \nu_{t,x}^\delta(\xi) dx dt \leq C, \quad (4.13)$$

so that $\bar{\nu} \in K_C$. K_C is also tight. Let $\varepsilon, R > 0$ and set

$$H(R) := \inf_{|\xi| > R} h(\xi), \quad (4.14)$$

so that $\lim_{R \rightarrow +\infty} H(R) = +\infty$. Let $\bar{B}_0(R) \subset \mathbb{R}^2$ be the closed ball of center 0 and radius R and define $A_R = Q_T \times \bar{B}_0(R)$. Then, for any $\nu \in K_C$,

$$H(R) \nu(\mathbb{R}^2 \setminus A_R) \leq \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}(\xi) dx dt \leq C \quad (4.15)$$

and therefore

$$\nu(\mathbb{R}^2 \setminus A_R) \leq \varepsilon \quad (4.16)$$

provided R is large enough. \square

Proposition 4.2. *Let h be as in Proposition 4.1 and let $\nu^\delta \in \mathcal{Y}$ be such that*

$$\int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}^\delta(\xi) dx dt \leq C \quad (4.17)$$

for all $\delta > 0$. Then there exists $\bar{\nu} \in \mathcal{Y}$ such that:

(i)

$$\int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\bar{\nu}_{t,x}(\xi) dx dt \leq C; \quad (4.18)$$

(ii) up to a subsequence

$$\lim_{\delta \rightarrow 0} \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\nu_{t,x}^\delta(\xi) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\bar{\nu}_{t,x}(\xi) dx dt \quad (4.19)$$

for any continuous $J : [0, T] \times [0, 1] \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f(\xi)/h(\xi) \rightarrow 0$ as $|\xi| \rightarrow +\infty$.

Proof. By the previous proposition, $\nu^\delta \in K_C$ and K_C is compact: this implies the existence of a Young measure $\bar{\nu}$ such that, up to a subsequence, $\nu^\delta \rightarrow \bar{\nu}$ and such that (i) holds.

In order to prove (ii), let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, non-negative non-increasing function supported in $[0, 2]$ which is identically equal to 1 on $[0, 1]$. For $R > 1$ and $a \in \mathbb{R}$ define $\chi_R(a) := \chi(a/R)$.

Define

$$I_\delta := \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\nu_{t,x}^\delta(\xi) dx dt, \quad I := \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\bar{\nu}_{t,x}(\xi) dx dt. \quad (4.20)$$

so that we need to prove

$$\lim_{\delta \rightarrow 0} |I_\delta - I| = 0. \quad (4.21)$$

We further define

$$\begin{aligned} I_\delta^R &:= \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) d\nu_{t,x}^\delta(\xi) dx dt, \\ I^R &:= \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) d\bar{\nu}_{t,x}(\xi) dx dt, \end{aligned} \quad (4.22)$$

and estimate

$$|I_\delta - I| \leq |I_\delta - I_\delta^R| + |I_\delta^R - I^R| + |I^R - I|. \quad (4.23)$$

The first term on the right hand side gives

$$|I_\delta - I_\delta^R| \leq \int_{Q_T} \int_{\mathbb{R}^2} |J(t, x)| |f(\xi)| \left(1 - \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) \right) d\nu_{t,x}^\delta(\xi) dx dt. \quad (4.24)$$

Since and $f(\xi)/h(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ we have

$$1 - \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) \leq 1_{\frac{h(\xi)}{1 + |f(\xi)|} > R}(\xi) \leq \frac{h(\xi)}{R(1 + |f(\xi)|)}, \quad (4.25)$$

for any $R > 1$. This implies

$$|I_\delta - I_\delta^R| \leq \frac{\|J\|_{L^\infty}}{R} \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}^\delta(\xi) dx dt \leq \frac{C \|J\|_{L^\infty}}{R}. \quad (4.26)$$

Analogously, we have

$$|I - I^R| \leq \frac{C \|J\|_{L^\infty}}{R}, \quad (4.27)$$

which gives

$$|I_\delta - I| \leq \frac{2C \|J\|_{L^\infty}}{R} + |I_\delta^R - I^R|. \quad (4.28)$$

Since f may diverge only at infinity and $f(\xi)/h(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$, then $f(\xi) \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right)$ is continuous and bounded and hence $|I_\delta^R - I^R| \rightarrow 0$. Thus the conclusion follows from (4.28) by taking the limits $\delta \rightarrow 0$ and then $R \rightarrow +\infty$. \square

Remark. This proposition applies to our case, as (4.17) with $h(\xi) = |\xi|^2$ is nothing but our energy estimate and τ is chosen to grow linearly at infinity.

Going back to (4.3), passing to the Young measures and taking the limit $\delta \rightarrow 0$ gives, for functions $\psi \in C^1(\mathbb{R}_+ \times [0, 1])$ such that $\psi(\cdot, x)$ is compactly supported in $[0, \infty)$ and $\psi(t, 0) = 0$,

$$\int_0^1 \psi(0, x) p_0(x) dx + \int_0^\infty \int_0^1 \int_{\mathbb{R}^2} (\psi_t \xi_p - \psi_x \tau(\xi_r)) d\bar{\nu}_{t,x}(\xi) dx dt + \int_0^\infty \psi(t, 1) \bar{\tau}(t) dt = 0, \quad (4.29)$$

where $\xi := (\xi_r, \xi_p)$.

In order to obtain weak solutions to the hyperbolic system (2.1), we need to prove that the limit Young measure $\bar{\nu}$ is a Dirac mass: $\bar{\nu}_{t,x} = \delta_{\bar{u}(t,x)}$, for some $\bar{u} \in L^2(Q_T)$ and a.e. $(t, x) \in Q_T$. This is done using the classical argument by Tartar and Murat.

Definition 4.3. A Lax entropy-entropy flux pair for system (2.1) is a couple of differentiable functions $(\eta, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\begin{cases} \eta_r + q_p = 0 \\ \tau'(r)\eta_p + q_r = 0 \end{cases}. \quad (4.30)$$

We show that Tartar's equation holds for any two suitable entropy pairs (η, q) and (η', q') to be specified below and almost all t, x :

$$\langle \eta q' - \eta' q, \bar{\nu}_{t,x} \rangle = \langle \eta, \bar{\nu}_{t,x} \rangle \langle q', \bar{\nu}_{t,x} \rangle - \langle \eta', \bar{\nu}_{t,x} \rangle \langle q, \bar{\nu}_{t,x} \rangle, \quad (4.31)$$

where

$$\langle f, \bar{\nu}_{t,x} \rangle := \int_{\mathbb{R}^2} f(\xi) d\bar{\nu}_{t,x}(\xi) \quad (4.32)$$

for any measurable f . We employ the following argument due to Shearer [20].

Accordingly to Lemma 2 in [20], there exists a family of half-plane supported entropy-entropy fluxes (η, q) such that η and q are bounded together with their first and second derivatives. These are explicitly given as follows. We define $z(r) := \int_0^r \sqrt{\tau'(\rho)} d\rho$ and we define the Riemann coordinates $w_1 = p + z$, $w_2 = p - z$. We also pass from the dependent variables η, q to H, Q as follows:

$$\eta = \frac{1}{2}(\tau')^{-1/4} (H + Q) \quad (4.33)$$

$$q = \frac{1}{2}(\tau')^{+1/4} (H - Q) \quad (4.34)$$

so that (4.30) becomes

$$\begin{cases} H_{w_1} = aQ \\ H_{w_2} = -aQ \end{cases}, \quad (4.35)$$

where

$$a(w_1 - w_2) = \frac{\tau'' \left(r \left(\frac{w_1 - w_2}{2} \right) \right)}{8 \left(\tau' \left(r \left(\frac{w_1 - w_2}{2} \right) \right) \right)^{3/2}}. \quad (4.36)$$

Then we fix $\bar{w}_1, \bar{w}_2 \in \mathbb{R}$ and we solve (4.35) with Goursat data given on the lines $w_1 = \bar{w}_1$ and $w_2 = \bar{w}_2$:

$$\begin{aligned} H(\bar{w}_1, w_2) &= g(w_2) \\ Q(w_1, \bar{w}_2) &= 0, \end{aligned} \quad (4.37)$$

where g is continuous and compactly supported. Then one can explicitly solve (4.35) and get

$$\begin{aligned} H(w_1, w_2) &= g(w_2) + \sum_{n=1}^{\infty} (\mathcal{A}^n g)(w_1, w_2) \\ Q(w_1, w_2) &= - \int_{\bar{w}_2}^{w_2} a(w_1 - v) H(w_1, v) dv, \end{aligned} \quad (4.38)$$

where the operator \mathcal{A} acts on functions $f \in L^1_{loc}(\mathbb{R}^2)$ as follows:

$$(\mathcal{A}f)(w_1, w_2) = - \int_{\bar{w}_1}^{w_1} \int_{\bar{w}_2}^{w_2} a(v - w_2)a(v - u)f(v, u)dudv. \quad (4.39)$$

Finally, going back to η and q and using our assumptions on τ it is possible to show ([20], Lemma 2) that η and q are bounded, together with their first and second derivatives.

Now we have a suitable family of entropy-entropy flux pair, we use Tartar-Murat Lemma in order to derive Tartar's equation (4.31). We evaluate (η, q) along the approximate solutions u^δ and compute the entropy production:

$$\eta(u^\delta)_t + q(u^\delta)_x = \delta \left(\eta_r r_x^\delta + \eta_p p_x^\delta \right)_x - \delta \left(\eta_{rr} (r_x^\delta)^2 + \eta_{pp} (p_x^\delta)^2 + 2\eta_{rp} r_x^\delta p_x^\delta \right) \quad (4.40)$$

Since η_r and η_p are bounded and $\sqrt{\delta}r_x^\delta, \sqrt{\delta}p_x^\delta$ are bounded in $L^2(Q_T)$, we have

$$\lim_{\delta \rightarrow 0} \delta \left(\eta_r r_x^\delta + \eta_p p_x^\delta \right)_x = 0 \quad \text{in } H^{-1}(Q_T), \quad (4.41)$$

while

$$\left\| \delta \left(\eta_{rr} (r_x^\delta)^2 + \eta_{pp} (p_x^\delta)^2 + 2\eta_{rp} r_x^\delta p_x^\delta \right) \right\|_{L^1(Q_T)} \leq C \quad (4.42)$$

uniformly with respect to δ . Thus we have obtained an equality of the form

$$\eta(u^\delta)_t + q(u^\delta)_x = \chi^\delta + \psi^\delta,$$

where $\{\chi^\delta\}_{\delta>0}$ lies in a compact set of $H^{-1}(Q_T)$ and $\{\psi^\delta\}_{\delta>0}$ is bounded in $L^1(Q_T)$. Moreover, since η and q are bounded, $\{\eta(u^\delta)_t + q(u^\delta)_x\}_{\delta>0}$ is bounded in $W^{-1,p}(Q_T)$ for some $p > 2$.

Therefore, we can apply Tartar-Murat and the div-curl lemma (cf [8], Theorem 16.2.1 and Lemma 16.2.2) and obtain Tartar's equation (4.31).

The final step is to use Tartar's equation to prove that the support of the limit Young measure $\bar{\nu}_{t,x}$ is a point. This is done in lemmas 4 to 7 of [20] and leads to the following

Proposition 4.4. *There exists a $\bar{u} \in L^2(Q_T)$ such that $\bar{\nu}_{t,x} = \delta_{\bar{u}(t,x)}$ for almost all $(t, x) \in Q_T$. Moreover, $u^\delta \rightarrow \bar{u}$ strongly in $L^p(Q_T)$ for any $p \in [1, 2)$.*

4.1 Regularity of the solutions and Clausius inequality

Proposition 4.5. $\bar{u} \in L^\infty(\mathbb{R}_+; L^2(0, 1))$.

Proof. Since $u^\delta \rightarrow \bar{u}$ in L^p strong for $p < 2$, we can extract a subsequence $\{u^{\delta_k}\}_{k \in \mathbb{N}}$ that converges pointwise to \bar{u} for almost all t and x . In particular, for almost all t , the sequence $u^{\delta_k}(t, x)$ converges for almost all x . Therefore, by Fatou lemma and Theorem 3.1,

$$\int_0^1 |\bar{u}(t, x)|^2 dx \leq \liminf_{k \rightarrow \infty} \int_0^1 |u^{\delta_k}(t, x)|^2 dx \leq C \quad (4.43)$$

for almost all $t \geq 0$. □

The proof of next lemma is standard and therefore omitted.

Lemma 4.6. *Let $a(t) := \tau^{-1}(\bar{\tau}(t))$. Then, the solutions (r^δ, p^δ) of the viscous system (3.1) can be written as follows:*

$$\begin{aligned} r^\delta(t, x) &= a(t) + \int_0^1 G_r^\delta(x, x', t)(r_0^\delta(x') - a(0))dx' + \\ &+ \int_0^t \int_0^1 G_r^\delta(x, x', t - t')(\partial_{x'} p^\delta(t', x') - a(t'))dx'dt' \end{aligned} \quad (4.44)$$

$$p^\delta(t, x) = \int_0^1 G_p^\delta(x, x', t) p_0^\delta(x') dx' + \int_0^t \int_0^1 G_p^\delta(x, x', t-t') \partial_{x'} \tau(r^\delta(t', x')) dx' dt' \quad (4.45)$$

where the G_r^δ and G_p^δ are Green functions of the heat operator $\partial_t - \delta \partial_{xx}$ with homogeneous boundary conditions:

$$G_r^\delta(1, x', t) = \partial_x G_r^\delta(0, x', t) = 0 \quad (4.46)$$

$$G_p^\delta(0, x', t) = \partial_x G_p^\delta(1, x', t) = 0 \quad (4.47)$$

for all $x' \in [0, 1]$, $t \geq 0$ and $\delta > 0$.

It is possible to show that $G_r^\delta(x, x', t)$ and $G_p^\delta(x, x', t)$ are symmetric under the exchange of x and x' . Moreover we have the following identities

$$\partial_x G_p^\delta(x, x', t) = -\partial_{x'} G_r^\delta(x, x', t), \quad (4.48)$$

$$\partial_x G_r^\delta(x, x', t) = -\partial_{x'} G_p^\delta(x, x', t). \quad (4.49)$$

Finally, the Green's functions G_r^δ and G_p^δ are subprobability density, meaning

$$0 \leq \int_0^1 G_r^\delta(x, x', t) dx' \leq 1 \quad (4.50)$$

$$0 \leq \int_0^1 G_p^\delta(x, x', t) dx' \leq 1 \quad (4.51)$$

for any x, t and δ .

Remark. Although we will not use them, explicit expressions are available for the G_r^δ and G_p^δ , namely

$$G_p^\delta(x, x', t) = \sum_{n \text{ odd}} e^{-t\delta\lambda_n} \sin(\sqrt{\lambda_n}x) \sin(\sqrt{\lambda_n}x') \quad (4.52)$$

$$G_r^\delta(x, x', t) = \sum_{n \text{ odd}} e^{-t\delta\lambda_n} \cos(\sqrt{\lambda_n}x) \cos(\sqrt{\lambda_n}x'), \quad (4.53)$$

with $\lambda_n = \frac{n^2\pi^2}{4}$.

Lemma 4.7. For any $\phi \in C^1([0, 1])$, the application

$$t \mapsto I_\phi(t) := \int_0^1 \phi(x) \bar{u}(t, x) dx \quad (4.54)$$

is Lipschitz continuous. Consequently $\bar{u}(t, \cdot) \in L^2(0, 1)$ for all $t \geq 0$.

Proof. We prove the statement for \bar{p} , as the proof for \bar{r} is similar. Furthermore, we prove the proposition only between 0 and t , as in the general case, say between t_1 and t , it is enough to replace the initial term $p_0^\delta(x)$ with $p^\delta(t_1, x)$. We let

$$I_\phi^\delta(t) := \int_0^1 \phi(x) p^\delta(t, x) dx \quad (4.55)$$

and evaluate

$$\begin{aligned} I_\phi(t) - I_\phi(0) &= \int_0^1 \int_0^1 \phi(x) G_p^\delta(x, x', t) p_0^\delta(x') dx' dx - \int_0^1 \phi(x) p_0^\delta(x) dx + \\ &+ \int_0^t \int_0^1 \int_0^1 \phi(x) G_p^\delta(x, x', t-t') \partial_{x'} \tau(r^\delta(t', x')) dx dx' dt' \end{aligned} \quad (4.56)$$

$$\begin{aligned}
&= \int_0^1 \phi(x) \int_0^1 \left[G_p^\delta(x, x', t) p_0^\delta(x') - p_0^\delta(x) \right] dx' dx + \\
&\quad + \int_0^t \int_0^1 \int_0^1 \phi(x) \partial_x G_r^\delta(x, x', t-t') \tau(r^\delta(t', x')) dx dx' dt' + \int_0^t \int_0^1 \phi(x) G_p^\delta(x, 1, t-t') \bar{\tau}(t') dx dt',
\end{aligned} \tag{4.57}$$

where we have used the symmetry of G_p^δ as well as the property $\partial_{x'} G_p^\delta = -\partial_x G_r^\delta$. The boundary term is estimated as

$$\left| \int_0^t \int_0^1 \phi(x) G_p^\delta(x, 1, t-t') \bar{\tau}(t') dx dt' \right| \leq C_{\bar{\tau}} \int_0^t \left| \int_0^1 \phi(x) G_p^\delta(x, 1, t-t') dx \right| dt \leq t C_{\bar{\tau}} \|\phi\|_{L^2}, \tag{4.58}$$

by Jensen's inequality and since G_p^δ is a subprobability density. \square

We estimate the term involving τ as

$$\left| \int_0^t \int_0^1 \int_0^1 \phi(x) \partial_x G_r^\delta(x, x', t-t') \tau(r^\delta(t', x')) dx dx' dt' \right| = \tag{4.59}$$

$$\begin{aligned}
&= \left| \int_0^t \int_0^1 \int_0^1 \phi'(x) G_r^\delta(x, x', t-t') \tau(r^\delta(t', x')) dx dx' dt' \right| + \\
&\quad + \left| \phi(0) \int_0^t \int_0^1 G_r^\delta(0, x', t-t') \tau(r^\delta(t', x')) dx' dt' \right|
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&\leq t \|\phi'\|_{L^2} \left\| \frac{1}{t} \int_0^t \int_0^1 G_p^\delta(\cdot, x', t-t') \tau(r^\delta(x', t')) dx' dt' \right\|_{L^2} + \\
&\quad + t \|\phi\|_{L^\infty} \left\| \frac{1}{t} \int_0^t \int_0^1 G_r^\delta(0, x', t-t') \tau(r^\delta(t', x')) dx' dt' \right\|
\end{aligned} \tag{4.61}$$

$$\leq t (\|\phi'\|_{L^2} + \|\phi\|_{L^\infty}) \left(\frac{1}{t} \int_0^t \int_0^1 \tau(r^\delta(x', t'))^2 dx' dt' \right)^{1/2} \leq Ct \tag{4.62}$$

where C is independent of t and δ .

In order to estimate the first term of (4.57) we write

$$\int_0^1 \left[G_p^\delta(x, x', t) p_0^\delta(x') - p_0^\delta(x) \right] dx' = \left(e^{t\delta\partial_{xx}} - 1 \right) p_0^\delta(x) \tag{4.63}$$

$$= \int_0^{t\delta} \partial_{xx} e^{s\partial_{xx}} p_0^\delta(x) ds. \tag{4.64}$$

Therefore, we obtain

$$\left| \int_0^1 \int_0^1 \left[G_p^\delta(x, x', t) p_0^\delta(x') - p_0^\delta(x) \right] dx' dx \right| \leq \int_0^{t\delta} \left| \int_0^1 \phi(x) \partial_{xx} e^{s\partial_{xx}} p_0^\delta(x) dx \right| ds \tag{4.65}$$

$$= \int_0^{t\delta} \left| \int_0^1 \phi'(x) \partial_x e^{s\partial_{xx}} p_0^\delta(x) dx \right| ds \tag{4.66}$$

$$\leq \|\phi'\|_{L^2} \int_0^{t\delta} \left\| \partial_x e^{s\partial_{xx}} p_0^\delta \right\|_{L^2} ds. \tag{4.67}$$

Finally, since, by standard results on the heat equation, the application

$$s \mapsto \left\| \partial_x e^{s\partial_{xx}} p_0^\delta \right\|_{L^2} \tag{4.68}$$

is decreasing, we have

$$\int_0^{t\delta} \left\| \partial_x e^{s\partial_{xx}} p_0^\delta \right\|_{L^2} ds \leq t\sqrt{\delta} \|\sqrt{\delta}\partial_x p_0^\delta\|_{L^2} \leq Ct, \quad (4.69)$$

where we have used the assumption that $\{\sqrt{\delta}\partial_x p_0^\delta\}_{\delta>0}$ is bounded in $L^2(0, 1)$. We have also assumed, without loss of generality, $\delta \leq 1$.

Putting everything together, we have obtained

$$\left| I_\phi^\delta(t) - I_\phi^\delta(0) \right| \leq Ct \quad (4.70)$$

for some constant C independent of t and δ . This leads to the conclusion after passing to the limit $\delta \rightarrow 0$.

Proposition 4.8. *The solution \bar{u} satisfies Clausius inequality*

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(u_0) \leq W(t) \quad (4.71)$$

for all $t \geq 0$, where

$$W(t) = - \int_0^t \bar{\tau}'(s) \int_0^1 \bar{r}(s, x) dx + \bar{\tau}(t) \int_0^1 \bar{r}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0(x) dx. \quad (4.72)$$

Proof. By Proposition 4.5, Corollary 3.1, and Lemma 4.7, we have, for all $t \geq 0$,

$$\int_0^1 \left(\frac{\bar{p}^2(t, x)}{2} + F(\bar{r}(t, x)) \right) dx \leq \liminf_{k \rightarrow \infty} \int_0^1 \left(\frac{(p^{\delta_k})^2(t, x)}{2} + F(r^{\delta_k}(t, x)) \right) dx \quad (4.73)$$

$$\leq \lim_{\delta \rightarrow 0} \left(\mathcal{F}(u_0^{\delta_k}) - \int_0^t \bar{\tau}'(s) \int_0^1 r^{\delta_k}(s, x) dx ds + \bar{\tau}(t) \int_0^1 r^{\delta_k}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0^{\delta_k}(x) dx \right) \quad (4.74)$$

$$= \mathcal{F}(u_0) - \int_0^t \bar{\tau}'(s) \int_0^1 \bar{r}(s, x) dx ds + \bar{\tau}(t) \int_0^1 \bar{r}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0(x) dx, \quad (4.75)$$

where we have used the fact that u_0^δ converges to u_0 in L^2 strong in order to conclude that $\mathcal{F}(u_0^\delta) \rightarrow \mathcal{F}(u_0)$. Moreover, all the integrals are well defined, since the application

$$t \mapsto \int_0^1 \bar{r}(t, x) dx$$

is Lipschitz and so in particular continuous. \square

Thanks to the Clausius inequality, the solutions we have constructed are natural candidates for being the thermodynamic entropy solution of the equation (3.1) and one can conjecture that such limit is unique.

Acknowledgments

This work has been partially supported by the grants ANR-15-CE40-0020-01 LSD of the French National Research Agency. We thank Olivier Glass for very helpful discussions.

References

- [1] P Acquistapace and B Terreni. On quasilinear parabolic systems. *Mathematische Annalen*, 282(2):315–335, 1988.
- [2] L Alasio and S Marchesani. Global existence for a class of viscous systems of conservation laws. *Preprint*, <https://arxiv.org/abs/1902.02714v1>, 2019.
- [3] J M Ball. A version of the fundamental theorem for young measures. In *PDEs and continuum models of phase transitions*, pages 207–215. Springer, 1989.
- [4] F Berthelin and J Vovelle. Stochastic isentropic Euler equations. *to appear in Annales de l'ENS*, 2017.
- [5] S Bianchini and A Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Annals of Mathematics*, 161:223–342, 2005.
- [6] G-Q Chen and H Frid. Vanishing viscosity limit for initial-boundary value problems for conservation laws. *Contemporary Mathematics*, 238:35–51, 1999.
- [7] G-Q G Chen and M Perepelitsa. Vanishing viscosity limit of the navier-stokes equations to the euler equations for compressible fluid flow. *Communications on Pure and Applied Mathematics*, 63(11):1469–1504, 2010.
- [8] C M Dafermos. Hyperbolic conservation laws in continuum physics, volume 325 of *grundlehren der mathematischen wissenschaften [fundamental principles of mathematical sciences]*, 2010.
- [9] R Di Perna. Convergence of approximate solutions to conservation laws. *Archiv Rational Mech. Anal.*, 82:27–70, 1983.
- [10] J Fritz. Microscopic theory of isothermal elastodynamics. *Archiv Rational Mech. Anal.*, 201(1):209–249, 2011.
- [11] P Lin. Young measures and an application of compensated compactness to one-dimensional nonlinear elastodynamics. *Transactions of the AMS*, 329(1):377–413, January 1992.
- [12] S Marchesani and S Olla. In preparation.
- [13] S Marchesani and S Olla. Hydrodynamic limit for an anharmonic chain under boundary tension. *Nonlinearity*, 31(11):4979–5035, 2018.
- [14] F Murat. Compacité par compensation. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 5(3):489–507, 1978.
- [15] F Murat. Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 8(1):69–102, 1981.
- [16] S Olla. Microscopic derivation of an isothermal thermodynamic transformation. In *From Particle Systems to Partial Differential Equations*, pages 225–238. Springer, 2014.
- [17] F Otto. Initial-boundary value problem for a scalar conservation law. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 322:729–734, 1996.
- [18] D Serre. La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d'espace. *J. Math. Pures Appl.*, 65(4):423–468, 1986.
- [19] D Serre and J Shearer. Convergence with physical viscosity for nonlinear elasticity. *Preprint, unpublished*, 1994.
- [20] J W Shearer. Global existence and compactness in L^p for the quasi-linear wave equation. *Communications in Partial Differential Equations*, 19(11):1829–1878, 1994.
- [21] L Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear Analysis and Mechanics: Heriot-Watt symposium*, volume 4, pages 136–212, 1979.
- [22] L Tartar. The compensated compactness method applied to systems of conservation laws. In *Systems of Nonlinear Partial Differential Equations*, pages 263–285. Springer, 1983.

Stefano Olla
CEREMADE, UMR-CNRS, Université de Paris Dauphine, PSL Research University
Place du Maréchal De Lattre De Tassigny, 75016 Paris, France
`olla@ceremade.dauphine.fr`

Stefano Marchesani
GSSI,
Viale F. Crispi 7, 67100 L'Aquila, Italy
`stefano.marchesani@gssi.it`