

A note on the existence of L^2 valued solutions for a hyperbolic system with boundary conditions

Stefano Marchesani
Stefano Olla

Abstract

We prove existence of L^2 -weak solutions of a p -system with boundary conditions. This is done using the vanishing viscosity with mixed Dirichlet-Neumann boundary conditions. Under these boundary conditions the free energy decreases and provides a uniform a priori estimate in L^2 , allowing us to use L^2 Young measures, together with the classical tools of compensated compactness. We then obtain that the viscous solutions converge to weak solutions of the p -system strongly in L^p , for any $p \in [1, 2)$, that satisfy the boundary conditions in the sense given by Definition 2.1. Furthermore the free energy decreases along these solutions.

Keywords: hyperbolic conservation laws, boundary conditions, weak solutions, vanishing viscosity, compensated compactness

Mathematics Subject Classification numbers: 35L40, 35D40

1 Introduction

The problem of existence of weak solutions for hyperbolic systems of conservation law in a bounded domain has been studied for solutions that are of bounded variation or in L^∞ [4]. In the scalar case some works extend to L^∞ solutions, obtained from viscous approximations [13]. But viscous approximations require extra boundary conditions, that are usually taken of Dirichlet type.

We present here an approach based of viscosity approximations, where the extra boundary conditions are of Neumann type, to reflect the conservative nature of the viscous approximation. Under these boundary conditions the free energy of the system cannot increase, providing uniform bounds in L^2_{loc} .

We consider here the p -system

$$\begin{cases} r_t - p_x = 0 \\ p_t - \tau(r)_x = 0 \end{cases}, \quad (t, x) \in (0, \infty) \times (0, 1) \quad (1.1)$$

where $\tau(r)$ is a strictly increasing regular function of r with bounded derivative, with boundary conditions

$$p(t, 0) = 0, \quad \tau(r(t, 1)) = \bar{\tau}, \quad (1.2)$$

where $\bar{\tau}$ is a real constant. We shall construct weak solutions $u(t, y) = (r(t, y), p(t, y))$ to the p -system which are in $L^2_{loc}(\mathbb{R}_+ \times [0, 1])$ and satisfy the boundary conditions in the following weak sense:

$$\int_0^\infty \int_0^1 (\varphi_t r - \varphi_x p) dx dt = 0 \quad (1.3)$$

$$\int_0^\infty \int_0^1 (\psi_t p - \psi_x \tau(r)) dx dt + \int_0^\infty \psi(t, 1) \bar{\tau} dt = 0 \quad (1.4)$$

for all functions $\varphi, \psi \in C^2(\mathbb{R}_+ \times [0, 1])$ with the following properties:

- $\varphi(\cdot, x)$ and $\psi(\cdot, x)$ are compactly supported in $(0, \infty)$ for all $x \in [0, 1]$;
- $\varphi(t, 1) = \psi(t, 0) = 0$ for all $t \geq 0$.

Define the free energy of the system, associated to a profile $u(y) = (r(y), p(y)) \in (L^2(0, 1))^2$, as

$$\mathcal{F}(u) := \int_0^1 \left(\frac{p^2(y)}{2} + F(r(y)) - \bar{\tau}r(y) \right) dx \quad (1.5)$$

where $F(r)$ is a primitive of $\tau(r)$ ($F'(r) = \tau(r)$), such that $\mathcal{F}(u) \geq 0$ for any $u \in (L^2(0, 1))^2$. This is possible since $F(r)$ grows more than linearly. The solution $u \in (L^2_{\text{loc}}(\mathbb{R}_+ \times [0, 1]))^2$ of (1.4) that we obtain has the following properties:

- $u \in L^\infty(\mathbb{R}_+, (L^2(0, 1))^2)$,
- Clausius inequality holds:

$$\mathcal{F}(u(t)) \leq \mathcal{F}(u(0)), \quad \forall t \geq 0 \quad (1.6)$$

In this sense we call our solution an *entropy solution*.

The construction of the solution is obtained from the following viscosity approximation

$$\begin{cases} r_t^\delta - p_x^\delta = \delta_1 \tau(r^\delta)_{xx} \\ p_t^\delta - \tau(r^\delta)_x = \delta_2 p_{xx}^\delta \end{cases}, \quad (t, x) \in (0, \infty) \times (0, 1) \quad (1.7)$$

with boundary conditions

$$p^\delta(t, 0) = 0, \quad \tau(r^\delta(t, 1)) = \bar{\tau}, \quad p_x^\delta(t, 1) = 0, \quad r_x^\delta(t, 0) = 0. \quad (1.8)$$

Notice that we have added two Neumann boundary conditions, that reflect the *conservative* nature of the viscous perturbation. Under these conditions we have that (1.6) is still valid for u^δ and

$$\mathcal{F}(u^\delta(t)) - \mathcal{F}(u(0)) = - \int_0^t \int_0^1 \left(\delta_1 \tau(r^\delta)_x^2 + \delta_2 (p_x^\delta)^2 \right) dx ds \leq 0. \quad (1.9)$$

From this we obtain the necessary bounds to apply a standard compensated compactness in the L^2 version [14].

1.1 Physical motivations

The problem arises naturally considering hydrodynamic limit for a non-linear chain of oscillators (of FPU type) in contact with a heat path at a give temperature. This models an isothermal transformation governed by (1.1).

Consider $N + 1$ particles on the real line and call q_i and p_i the positions and the momenta of the i -th particle, respectively. Particles i and $i - 1$ interacts via a nonlinear potential $V(q_i - q_{i-1})$. Then, defining $r_i := q_i - q_{i-1}$ we have a Hamiltonian system with Hamiltonian

$$H_N = \frac{p_0^2}{2} + \sum_{i=1}^N \left(\frac{p_i^2}{2} + V(r_i) \right) \quad (1.10)$$

and r_i and p_i evolve, after a time-scaling, accordingly to the Newton's equations

$$\begin{cases} dr_i = N(p_i - p_{i-1})dt & , \quad 1 \leq i \leq N \\ dp_i = N(V'(r_{i+1}) - V'(r_i))dt & , \quad 1 \leq i \leq N - 1 \\ dp_N = N(\bar{\tau} - V'(r_N))dt \end{cases} \quad (1.11)$$

together with $p_0 \equiv 0$. Thus, the boundary conditions have the following microscopic interpretation: particle number 0 is not moving, while particle number N is pulled (or pushed) with a constant force $\bar{\tau}$. The microscopic nonlinearity τ is fully determined as the expectation in equilibrium of the

microscopic force V' . The system is then put in contact with a heat bath that acts as a microscopic stochastic viscosity, and the evolution equations are given by the following system of stochastic differential equations:

$$\begin{cases} dr_1 = Np_1 dt + N^2 \delta_1 (V'(r_2) - V'(r_1)) dt - \sqrt{2\beta^{-1}N^2 \delta_1} d\tilde{w}_1 \\ dr_i = N(p_i - p_{i-1})dt + N^2 \delta_1 (V'(r_{i+1}) + V'(r_{i-1}) - 2V'(r_i)) dt + \sqrt{2\beta^{-1}N^2 \delta_1} (d\tilde{w}_{i-1} - d\tilde{w}_i) \\ dr_N = N(p_N - p_{N-1})dt + N^2 \delta_1 (\bar{\tau} - 2V'(r_N) + V'(r_{N-1})) dt + \sqrt{2\beta^{-1}N^2 \delta_1} d\tilde{w}_{N-1} \\ dp_1 = N(V'(r_2) - V'(r_1))dt + N^2 \delta_2 (p_2 - 2p_1) dt - \sqrt{2\beta^{-1}N^2 \delta_2} dw_1 \\ dp_i = N(V'(r_{i+1}) - V'(r_i))dt + N^2 \delta_2 (p_{i+1} + p_{i-1} - 2p_i) dt + \sqrt{2\beta^{-1}N^2 \delta_2} (dw_{i-1} - dw_i) \\ dp_N = N(\bar{\tau} - V'(r_N))dt + N^2 \delta_2 (p_{N-1} - p_N) dt + \sqrt{2\beta^{-1}N^2 \delta_2} dw_{N-1} \end{cases} \quad (1.12)$$

for $2 \leq i \leq N-1$, together with $p_0 \equiv 0$. Here $\beta^{-1} > 0$ is the temperature and $\{w_i\}_{i=1}^\infty, \{\tilde{w}_i\}_{i=1}^\infty$ are independent families of independent Brownian motions. The parameters δ_1, δ_2 are chosen depending on N and vanishing as $N \rightarrow \infty$.

The hydrodynamic limit consists in proving that, for any continuous function $G(x)$ on $[0, 1]$,

$$\frac{1}{N} \sum_{i=1}^N G\left(\frac{i}{N}\right) \begin{pmatrix} r_i(t) \\ p_i(t) \end{pmatrix} \xrightarrow{N \rightarrow \infty} \int_0^1 G(x) \begin{pmatrix} r(t, x) \\ p(t, x) \end{pmatrix} dx, \quad (1.13)$$

in probability, with $(r(t, x), p(t, x))$ satisfying (1.3), (1.4). Of course a complete proof would require the uniqueness of such L^2 valued solutions that satisfy Clausius inequality (1.6): this remains an open problem. The results contained in [8] states that the limit distribution of the empirical distribution defined on the RHS of (1.13), concentrates on the possible solutions of (1.3) and (1.4) that satisfy (1.6).

This stochastic model was already considered by Fritz [7] in the infinite volume without boundary conditions, and in [9], but without the characterisation of the boundary conditions. In the hydrodynamic limit only L^2 bounds are available and we are constrained to consider L^2 valued solutions. Since these solutions do not have definite values on the boundary, boundary conditions have only a dynamical meaning in the sense of an evolution in L^2 given by (1.3), (1.4).

2 Hyperbolic system and definition of weak solutions

For $r, p : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$, consider the hyperbolic system

$$\begin{cases} r_t - p_x = 0 \\ p_t - \tau(r)_x = 0 \end{cases}, \quad p(t, 0) = 0, \quad \tau(r(t, 1)) = \bar{\tau}. \quad (2.1)$$

The nonlinearity τ is chosen to have the following properties.

- (τ -i) $c_1 \leq \tau'(r) \leq c_2$ for some $c_1, c_2 > 0$ and all $r \in \mathbb{R}$;
- (τ -ii) $\tau''(r) > 0$ for all $r \in \mathbb{R}$;
- (τ -iii) $\tau''(r)(\tau'(r))^{-5/4}$ and $\tau'''(r)(\tau'(r))^{-7/4}$ are in $L^2(\mathbb{R})$, while $\tau''(r)(\tau'(r))^{-3/4}$ and $\tau'''(r)(\tau'(r))^{-2}$ are in $L^\infty(\mathbb{R})$.

$\bar{\tau}$ is a given real number.

Remark. Condition (τ -i) and (τ -ii) ensure that the system is strictly hyperbolic and genuinely nonlinear, respectively. Condition (τ -iii) is used later on to ensure some boundedness properties of the Lax entropies.

Definition 2.1. We say that $(r, p) \in L^2_{loc}(\mathbb{R}_+ \times [0, 1])$ is a weak solution of the p -system (2.1) provided

$$\int_0^\infty \int_0^1 (\varphi_t r - \varphi_x p) dx dt = 0 \quad (2.2)$$

$$\int_0^\infty \int_0^1 (\psi_t p - \psi_x \tau(r)) dx dt + \int_0^\infty \psi(t, 1) \bar{r} dt = 0 \quad (2.3)$$

for all functions $\varphi, \psi \in C^2(\mathbb{R}_+ \times [0, 1])$ with the following properties:

- $\varphi(\cdot, x)$ and $\psi(\cdot, x)$ are compactly supported in $(0, \infty)$ for all $x \in [0, 1]$;
- $\varphi(t, 1) = \psi(t, 0) = 0$ for all $t \geq 0$.

We shall prove the following

Theorem 2.1. *The p -system (2.1) admits a weak solution $\bar{u} = (\bar{r}, \bar{p})$ in the sense of Definition 2.1. Moreover, \bar{u} is entropic, in the sense that*

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(\bar{u}(0)) \leq 0, \quad \forall t \geq 0 \quad (2.4)$$

We shall not deal with uniqueness. This is the reason why we are not interested in what happens at time $t = 0$.

3 Viscous approximation and energy estimates

We consider the following parabolic approximation of the hyperbolic system (2.1)

$$\begin{cases} r_t^\delta - p_x^\delta = \delta_1 \tau(r^\delta)_{xx} \\ p_t^\delta - \tau(r^\delta)_x = \delta_2 p_{xx}^\delta \end{cases}, \quad (t, x) \in (0, \infty) \times (0, 1) \quad (3.1)$$

with the boundary conditions:

$$p^\delta(t, 0) = 0, \quad \tau(r^\delta(t, 1)) = \bar{\tau}, \quad p_x^\delta(t, 1) = 0, \quad r_x^\delta(t, 0) = 0$$

and δ -independent initial data

$$p^\delta(0, x) = p_0(x), \quad r^\delta(0, x) = r_0(x).$$

Remark. (i) We added two extra Neumann conditions, namely $p_x^\delta(t, 1) = r_x^\delta(t, 0) = 0$. These conditions reflect the conservative nature of the viscous perturbation, and are required in order to obtain the correct production of free energy and get the corresponding Clausius inequality.

(ii) We have introduced a *nonlinear* viscosity term: $\delta_1 \tau(r^\delta)_{xx}$. This is a term which comes naturally from a microscopic derivation of system (3.1), as described in the introduction. But all the results are still valid for the linear viscosity $\delta_1 r_{xx}^\delta$.

Assume that there exists a $T < \infty$ independent of δ such that we have a strong solution of (3.1) in $Q_T = [0, T] \times [0, 1]$.

We evaluate

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(u^\delta(t)) &= \int_0^1 \left(p^\delta p_t^\delta + \tau(r^\delta) r_t^\delta - \bar{\tau} r_t^\delta \right) dx \\ &= \int_0^1 \left(p^\delta \tau(r^\delta)_x + \delta_2 p^\delta p_{xx}^\delta + \tau(r^\delta) p_x^\delta + \delta_1 \tau(r^\delta) \tau(r^\delta)_{xx} - \bar{\tau} p_x^\delta - \delta_1 \bar{\tau} \tau(r^\delta)_{xx} \right) dx \end{aligned} \quad (3.2)$$

Using the boundary conditions $p^\delta(t, 0) = 0$, $\tau(r^\delta(t, 1)) = \bar{\tau}$, $p_x^\delta(t, 1) = r_x^\delta(t, 0) = 0$ after an integration by parts in space the boundary terms cancel and we obtain, after a time integration

$$\mathcal{F}(u^\delta(t)) - \mathcal{F}(r_0, p_0) = - \int_0^t \int_0^1 (\delta_1 \tau(r^\delta)_x^2 + \delta_2 p_x^\delta{}^2) dx ds \leq 0. \quad (3.3)$$

As a consequence, and thanks to the fact that $F(r)$ grows quadratically as $|r| \rightarrow +\infty$ and τ' is bounded from below by positive constants, we obtain the following

Proposition 3.1 (A priori estimates). *For any $t, \delta_1, \delta_2 \geq 0$, we have*

- (i) $\mathcal{F}(u^\delta(t)) \leq C$;
- (ii) $\int_0^1 ((p^\delta)^2 + (r^\delta)^2) dx \leq C$;
- (iii) $\int_0^t ds \int_0^1 dx (\delta_1 \tau(r^\delta)_x^2 + \delta_2 (p_x^\delta)^2) \leq C$;
- (iv) $\int_0^t ds \int_0^1 dx (\delta_1 (r_x^\delta)^2 + \delta_2 (p_x^\delta)^2) \leq C$

4 L^p Young measures and compensated compactness

By (ii) of Proposition 3.1, we can extract from $\{u^\delta\}_{\delta>0}$ a subsequence that is weakly convergent in $L^2(Q_T)$. Namely, up to a subsequence, there exists $\bar{u} = (\bar{r}, \bar{p}) \in L^2(Q_T)^2$ such that

$$\lim_{\delta \rightarrow 0} \int_{Q_T} u^\delta \varphi = \int_{Q_T} \bar{u} \varphi, \quad \forall \varphi \in L^2(Q_T)^2. \quad (4.1)$$

Our aim is to show that our solution u^δ converges to a weak solution of the hyperbolic system (2.1), in the sense of Section 2. We focus on (2.3), as the other equation is easier. Let $\psi(t, x)$ be C^1 in time, C^2 in space, with $\psi(\cdot, x)$ compactly supported in $(0, \infty)$ and $\psi(t, 0) = 0$. Then, we have

$$\begin{aligned} 0 &= \int_0^\infty \int_0^1 \left(\psi p_t^\delta - \psi \tau(r^\delta)_x - \delta_2 \psi p_{xx}^\delta \right) dx dt \\ &= - \int_0^\infty \int_0^1 \left(\psi_t p^\delta - \psi_x \tau(r^\delta) - \delta_2 \psi_x p_x^\delta \right) dx dt - \int_0^\infty \psi(t, 1) \bar{r} dt. \end{aligned} \quad (4.2)$$

where we have used the boundary conditions $\tau(r^\delta(t, 1)) = \bar{r}(t)$, $\partial_x p(t, 1) = 0$, as well as $\psi(t, 0) = 0$.

The term proportional to δ_2 vanishes as $\delta_2 \rightarrow 0$ thanks to inequality (iii) of Proposition 3.1.

Concerning the linear term, (4.1) implies

$$\lim_{\delta \rightarrow 0} \int_0^\infty \int_0^1 \psi_t p^\delta dx dt = \int_0^\infty \int_0^1 \psi_t \bar{p} dx dt. \quad (4.3)$$

We are left to show that

$$\lim_{\delta \rightarrow 0} \int_0^\infty \int_0^1 \psi_x \tau(r^\delta) dx dt = \int_0^\infty \int_0^1 \psi_x \tau(\bar{r}) dx dt. \quad (4.4)$$

This is done using a L^p version of the compensated compactness, which is usually performed in L^∞ .

From the solution $u^\delta(t, x)$, we define the following Young measures on $Q_T \times \mathbb{R}^2$:

$$\nu_{t,x}^\delta := \delta_{u^\delta(t,x)}, \quad (4.5)$$

which are Dirac masses centred at u^δ . Since we have only L^2 bounds on u^δ , we refer at $\nu_{t,x}^\delta$ as L^2 -Young measures.

We call \mathcal{Y} the set of Young measures on $Q_T \times \mathbb{R}^2$ and we make it a metric space by endowing it with the Prohorov's metric. Then (cf [1]), if there is a compact set $K \subset \mathcal{Y}$ such that $\nu_{t,x}^\delta \in K$ for all $\delta > 0$, there exists $\bar{\nu}_{t,x} \in \mathcal{Y}$ so that, up to a subsequence,

$$\lim_{\delta \rightarrow 0} \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\nu_{t,x}^\delta(\xi) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\bar{\nu}_{t,x}(\xi) dx dt \quad (4.6)$$

for all continuous and bounded $J : Q_T \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We shall simply write $\nu^\delta \rightarrow \nu$ in place of (4.6). We proceed by finding such compact set K and then by extending the convergence to functions f with subquadratic growth. This is done by adapting the argument of [2].

Proposition 4.1. *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ be continuous and such that*

$$\lim_{|\xi| \rightarrow +\infty} h(\xi) = +\infty. \quad (4.7)$$

Then, for any $C > 0$, the set

$$K_C := \left\{ \nu \in \mathcal{Y} : \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}(\xi) dx dt \leq C \right\} \quad (4.8)$$

is compact.

Proof. K_C is relatively compact if and only if it is tight, namely if for all $\varepsilon > 0$ there exists a compact subset $A_\varepsilon \subset \mathbb{R}^2$ such that

$$\nu(\mathbb{R}^2 \setminus A_\varepsilon) \leq \varepsilon \quad (4.9)$$

for all $\nu \in K_C$. Thus, K_C is compact if and only if it is closed and tight.

K_C is closed. In fact, if $\nu^\delta \in K_C$ and $\nu^\delta \rightarrow \bar{\nu}$,

$$\int_{Q_T} \int_{\mathbb{R}^2} h(\xi) \bar{\nu}_{t,x}(\xi) dx dt \leq \liminf_{\delta \rightarrow 0} \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) \nu_{t,x}^\delta(\xi) dx dt \leq C, \quad (4.10)$$

so that $\bar{\nu} \in K_C$. K_C is also tight. Let $\varepsilon, R > 0$ and set

$$H(R) := \inf_{|\xi| > R} h(\xi), \quad (4.11)$$

so that $\lim_{R \rightarrow +\infty} H(R) = +\infty$. Let $\bar{B}_0(R) \subset \mathbb{R}^2$ be the closed ball of center 0 and radius R and define $A_R = Q_T \times \bar{B}_0(R)$. Then, for any $\nu \in K_C$,

$$H(R) \nu(\mathbb{R}^2 \setminus A_R) \leq \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}(\xi) dx dt \leq C \quad (4.12)$$

and therefore

$$\nu(\mathbb{R}^2 \setminus A_R) \leq \varepsilon \quad (4.13)$$

provided R is large enough. \square

Proposition 4.2. *Let h be as in the Proposition 4.1 and let $\nu^\delta \in \mathcal{Y}$ be such that*

$$\int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}^\delta(\xi) \leq C \quad (4.14)$$

for all $\delta > 0$. Then there exists $\bar{\nu} \in \mathcal{Y}$ such that:

(i)

$$\int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\bar{\nu}_{t,x}(\xi) \leq C; \quad (4.15)$$

(ii) up to a subsequence

$$\lim_{\delta \rightarrow 0} \int_0^T \int_0^1 \int_{\mathbb{R}^2} J(t,x) f(\xi) d\nu_{t,x}^\delta(\xi) = \int_0^T \int_0^1 \int_{\mathbb{R}^2} J(t,x) f(\xi) d\bar{\nu}_{t,x}(\xi) \quad (4.16)$$

for any continuous $J : [0, T] \times [0, 1] \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f(\xi)/h(\xi) \rightarrow 0$ as $|\xi| \rightarrow +\infty$.

Proof. By the previous proposition, $\nu^\delta \in K_C$ and K_C is compact: this implies the existence of a Young measure $\bar{\nu}$ such that, up to a subsequence, $\nu^\delta \rightarrow \bar{\nu}$ and such that (i) holds.

In order to prove (ii), let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous, non-negative non-increasing function supported in $[0, 2]$ which is identically equal to 1 on $[0, 1]$. For $R > 1$ and $a \in \mathbb{R}$ define $\chi_R(a) := \chi(a/R)$.

Define

$$I_\delta := \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) d\nu_{t,x}^\delta(\xi) dx dt, \quad I := \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) d\bar{\nu}_{t,x}(\xi) dx dt. \quad (4.17)$$

so that we need to prove

$$\lim_{\delta \rightarrow 0} |I_\delta - I| = 0. \quad (4.18)$$

We further define

$$\begin{aligned} I_\delta^R &:= \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) d\nu_{t,x}^\delta(\xi) dx dt, \\ I^R &:= \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) d\bar{\nu}_{t,x}(\xi) dx dt, \end{aligned} \quad (4.19)$$

and estimate

$$|I_\delta - I| \leq |I_\delta - I_\delta^R| + |I_\delta^R - I^R| + |I^R - I|. \quad (4.20)$$

The first term on the right hand side gives

$$|I_\delta - I_\delta^R| \leq \int_{Q_T} \int_{\mathbb{R}^2} |J(t, x)| |f(\xi)| \left(1 - \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) \right) d\nu_{t,x}^\delta(\xi) dx dt. \quad (4.21)$$

Since and $f(\xi)/h(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ we have

$$1 - \chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right) \leq 1_{\frac{h(\xi)}{1 + |f(\xi)|} > R}(\xi) \leq \frac{h(\xi)}{R(1 + |f(\xi)|)}, \quad (4.22)$$

for any $R > 1$. This implies

$$|I_\delta - I_\delta^R| \leq \frac{\|J\|_{L^\infty}}{R} \int_{Q_T} \int_{\mathbb{R}^2} h(\xi) d\nu_{t,x}^\delta(\xi) dx dt \leq \frac{C \|J\|_{L^\infty}}{R}. \quad (4.23)$$

Analogously, we have

$$|I - I^R| \leq \frac{C \|J\|_{L^\infty}}{R}, \quad (4.24)$$

which gives

$$|I_\delta - I| \leq \frac{2C \|J\|_{L^\infty}}{R} + |I_\delta^R - I^R|. \quad (4.25)$$

Since f may diverge only at infinity and $f(\xi)/h(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$, then $f(\xi)\chi_R \left(\frac{h(\xi)}{1 + |f(\xi)|} \right)$ is continuous and bounded and hence $|I_\delta^R - I^R| \rightarrow 0$. Thus the conclusion follows from (4.25) by taking the limits $\delta \rightarrow 0$ and then $R \rightarrow +\infty$. \square

Remark. This proposition applies to our case, as (4.14) with $h(\xi) = |\xi|^2$ is nothing but our energy estimate and τ is chosen to grow at most at infinity.

Going back to (4.2), passing to the Young measures and taking the limit $\delta \rightarrow 0$ gives, for functions $\psi(t, y)$ that are C^1 in time, C^2 in space such that $\psi(\cdot, x)$ is compactly supported in $(0, \infty)$ and $\psi(t, 0) = 0$,

$$\int_0^\infty \int_0^1 \int_{\mathbb{R}^2} (\psi_t \xi_p - \psi_x \tau(\xi_r)) d\bar{\nu}_{t,x}(\xi) dx dt + \int_0^\infty \psi(t, 1) \bar{\tau} dt = 0, \quad (4.26)$$

where $\xi := (\xi_r, \xi_p)$. Thus, we obtain weak solutions to the hyperbolic system (2.1) provided the limit Young measure $\bar{\nu}$ is a Dirac mass: $\bar{\nu}_{t,x} = \delta_{\bar{u}(t,x)}$, for some $\bar{u} \in L^2(Q_T)$ and a.e. $(t, x) \in Q_T$.

This is standard and is done using the classical argument by Tartar and Murat. We find Lax entropy-entropy flux pairs (η, q) relative to the hyperbolic system:

Definition 4.1. A Lax entropy-entropy flux pair for system (2.1) is a couple of differentiable functions $(\eta, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\begin{cases} \eta_r + q_p = 0 \\ \tau'(r)\eta_p + q_r = 0 \end{cases} \quad (4.27)$$

Then, we show that Tartar's equation holds true for any two pairs (η_1, q_1) and (η_2, q_2) :

$$\langle \eta_1 q_2 - \eta_2 q_1, \bar{\nu}_{t,x} \rangle = \langle \eta_1, \bar{\nu}_{t,x} \rangle \langle q_2, \bar{\nu}_{t,x} \rangle - \langle \eta_2, \bar{\nu}_{t,x} \rangle \langle q_1, \bar{\nu}_{t,x} \rangle. \quad (4.28)$$

We follow the argument in Shearer [14], where he studies the same hyperbolic system, but with a linear viscosity in the parabolic approximation. In fact, we may use the same Lax entropy-entropy flux (η, q) he uses, as these objects do not depend on the viscosity.

Accordingly to [14], Lemma 2, we may find a family of entropy-entropy fluxes (η, q) such that η , its first and second derivatives are bounded. This is where we make use of assumption $(\tau$ -iii).

We then evaluate η and q along solutions u^δ of the viscous system (3.1) and calculate the entropy production $\eta(u^\delta)_t + q(u^\delta)_x$. A direct computation gives

$$\eta(u^\delta)_t + q(u^\delta)_x = (\delta_1 \eta_r \tau'(r) r_x + \delta_2 \eta_p p_x)_x - [\delta_1 \eta_{rr} \tau'(r) r_x^2 + \delta_2 \eta_{pp} p_x^2 + (\delta_1 \tau'(r) + \delta_2) \eta_{rp} r_x p_x] \quad (4.29)$$

Since η_r and η_p are bounded, $\sqrt{\delta_2} r_x$ and $\sqrt{\delta_1} p_x$ are in $L^2(Q_T)$, and τ' is bounded

$$(\delta_1 \eta_r \tau'(r) r_x + \delta_2 \eta_p p_x)_x \rightarrow 0 \quad (4.30)$$

in $H^{-1}(Q_T)$ as $\delta \rightarrow 0$, while

$$\delta_1 \eta_{rr} \tau'(r) r_x^2 + \delta_2 \eta_{pp} p_x^2 + (\delta_1 \tau'(r) + \delta_2) \eta_{rp} r_x p_x \quad (4.31)$$

is uniformly bounded in $L^1(Q_T)$ with respect to δ_1 and δ_2 . Thus we can apply the argument Tartar and Murat, and deduce

Proposition 4.3. *There exists a $\bar{u} \in L^2(Q_T)$ such that $\bar{v}_{t,x} = \delta_{\bar{u}(t,x)}$ for almost all $(t,x) \in Q_T$. Moreover, $u^\delta \rightarrow \bar{u}$ strongly in $L^p(Q_T)$ for any $p \in [1, 2)$.*

Remark. As a consequence of the previous theorem, it is easy to check that the solution \bar{u} satisfies Clausius inequality

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(r_0, p_0) \leq 0. \quad (4.32)$$

Thus, such solutions are natural candidates for being the thermodynamic entropy solution of the equation (3.1) and one can conjecture that such limit is unique.

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Stefano Olla

CEREMADE, UMR-CNRS, Université de Paris Dauphine, PSL Research University

Place du Maréchal De Lattre De Tassigny, 75016 Paris, France

`olla@ceremade.dauphine.fr`

Stefano Marchesani

GSSI,

Viale F. Crispi 7, 67100 L'Aquila, Italy

`stefano.marchesani@gssi.it`