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ON PRO- p -IWAHORI INVARIANTS OF R -REPRESENTATIONS OF REDUCTIVE p -ADIC GROUPS

N. ABE, G. HENNIART, AND M.-F. VIGNÉRAS

ABSTRACT. Let F be a locally compact field with residue characteristic p , and let \mathbf{G} be a connected reductive F -group. Let \mathcal{U} be a pro- p Iwahori subgroup of $G = \mathbf{G}(F)$. Fix a commutative ring R . If π is a smooth $R[G]$ -representation, the space of invariants $\pi^{\mathcal{U}}$ is a right module over the Hecke algebra \mathcal{H} of \mathcal{U} in G .

Let P be a parabolic subgroup of G with a Levi decomposition $P = MN$ adapted to \mathcal{U} . We complement a previous investigation of Ollivier-Vignéras on the relation between taking \mathcal{U} -invariants and various functor like Ind_P^G and right and left adjoints. More precisely the authors' previous work with Herzig introduced representations $I_G(P, \sigma, Q)$ where σ is a smooth representation of M extending, trivially on N , to a larger parabolic subgroup $P(\sigma)$, and Q is a parabolic subgroup between P and $P(\sigma)$. Here we relate $I_G(P, \sigma, Q)^{\mathcal{U}}$ to an analogously defined \mathcal{H} -module $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$, where $\mathcal{U}_M = \mathcal{U} \cap M$ and $\sigma^{\mathcal{U}_M}$ is seen as a module over the Hecke algebra \mathcal{H}_M of \mathcal{U}_M in M . In the reverse direction, if \mathcal{V} is a right \mathcal{H}_M -module, we relate $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes \text{c-Ind}_{\mathcal{U}}^G \mathbf{1}$ to $I_G(P, \mathcal{V} \otimes_{\mathcal{H}_M} \text{c-Ind}_{\mathcal{U}_M}^M \mathbf{1}, Q)$. As an application we prove that if R is an algebraically closed field of characteristic p , and π is an irreducible admissible representation of G , then the contragredient of π is 0 unless π has finite dimension.

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1. INTRODUCTION

1.1. The present paper is a companion to [AHV] and is similarly inspired by the classification results of [AHHV17]; however it can be read independently. We recall the setting. We have a non-archimedean locally compact field F of residue characteristic p and a connected reductive F -group \mathbf{G} . We fix a commutative ring R and study the smooth R -representations of $G = \mathbf{G}(F)$.

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In [AHHV17] the irreducible admissible R -representations of G are classified in terms of supersingular ones when R is an algebraically closed field of characteristic p . That classification is expressed in terms of representations $I_G(P, \sigma, Q)$, which make sense for any R . In that notation, P is a parabolic subgroup of G with a Levi decomposition $P = MN$ and σ a smooth R -representation of the Levi subgroup M ; there is a maximal parabolic subgroup $P(\sigma)$ of G containing P to which σ inflated to P extends to a representation $e_{P(\sigma)}(\sigma)$, and Q is a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$. Then

$$I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e_{P(\sigma)}(\sigma) \otimes \text{St}_Q^{P(\sigma)}),$$

where Ind stands for parabolic induction and $\text{St}_Q^{P(\sigma)} = \text{Ind}_Q^{P(\sigma)} R / \sum \text{Ind}_{Q'}^{P(\sigma)} R$, the sum being over parabolic subgroups Q' of G with $Q \subsetneq Q' \subset P(\sigma)$. Alternatively, $I_G(P, \sigma, Q)$ is the quotient of $\text{Ind}_Q^G(e_Q(\sigma))$ by $\sum \text{Ind}_{Q'}^G e_{Q'}(\sigma)$ with Q' as above, where $e_Q(\sigma)$ is the restriction of $e_{P(\sigma)}(\sigma)$ to Q , similarly for Q' .

In [AHV] we mainly studied what happens to $I_G(P, \sigma, Q)$ when we apply to it, for a parabolic subgroup P_1 of G , the left adjoint of $\text{Ind}_{P_1}^G$, or its right adjoint. Here we tackle a different question. We fix a pro- p Iwahori subgroup \mathcal{U} of G in good position with respect to P , so that in particular $\mathcal{U}_M = \mathcal{U} \cap M$ is a pro- p Iwahori subgroup of M . One of our main goals is to identify the R -module $I_G(P, \sigma, Q)^\mathcal{U}$ of \mathcal{U} -invariants, as a right module over the Hecke algebra $\mathcal{H} = \mathcal{H}_G$ of \mathcal{U} in G - the convolution algebra on the double coset space $\mathcal{U} \backslash G / \mathcal{U}$ - in terms of the module $\sigma^{\mathcal{U}_M}$ over the Hecke algebra \mathcal{H}_M of \mathcal{U}_M in M . That goal is achieved in section 4, Theorem 4.17.

1.2. The initial work has been done in [OV17, §4] where $(\text{Ind}_P^G \sigma)^\mathcal{U}$ is identified. Precisely, writing M^+ for the monoid of elements $m \in M$ with $m(\mathcal{U} \cap N)m^{-1} \subset \mathcal{U} \cap N$, the subalgebra \mathcal{H}_{M^+} of \mathcal{H}_M with support in M^+ , has a natural algebra embedding θ into \mathcal{H} and [OV17, Proposition 4.4] identifies $(\text{Ind}_P^G \sigma)^\mathcal{U}$ with $\text{Ind}_{\mathcal{H}_{M^+}}^{\mathcal{H}}(\sigma^{\mathcal{U}_M}) = \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+}} \mathcal{H}$. So in a sense, this paper is a sequel to [OV17] although some of our results here are used in [OV17, §5].

As $I_G(P, \sigma, Q)$ is only a subquotient of $\text{Ind}_P^G \sigma$ and taking \mathcal{U} -invariants is only left exact, it is not straightforward to describe $I_G(P, \sigma, Q)^\mathcal{U}$ from the previous result. However, that takes care of the parabolic induction step, so in a first approach we may assume $P(\sigma) = G$ so that $I_G(P, \sigma, Q) = e_G(\sigma) \otimes \text{St}_Q^G$. The crucial case is when moreover σ is e -minimal, that is, not an extension $e_M(\tau)$ of a smooth R -representation τ of a proper Levi subgroup of M . That case is treated first and the general case in section 4 only.

1.3. To explain our results, we need more notation. We choose a maximal F -split torus T in G and a minimal parabolic subgroup $B = ZU$ with Levi component Z the G -centralizer of T . We assume that $P = MN$ contains B and M contains Z , and that \mathcal{U} corresponds to an alcove in the apartment associated to T in the adjoint building of G . It turns out that when σ is e -minimal and $P(\sigma) = G$, the set Δ_M of simple roots of T in $\text{Lie}(M \cap U)$ is orthogonal to its complement in the set Δ of simple roots of T in $\text{Lie}U$. We assume until the end of this section that Δ_M and $\Delta_2 = \Delta \setminus \Delta_M$ are orthogonal. If M_2 is the Levi subgroup - containing Z - corresponding to Δ_2 , both M and M_2 are normal in G , $M \cap M_2 = Z$ and $G = MM_2$. Moreover the normal subgroup M_2' of G generated by N is included in M_2 and $G = MM_2'$.

We say that a right \mathcal{H}_M -module \mathcal{V} is extensible to \mathcal{H} if T_z^M acts trivially on \mathcal{V} for $z \in Z \cap M'_2$ (section 3.3). In this case, we show that there is a natural structure of right \mathcal{H} -module $e_{\mathcal{H}}(\mathcal{V})$ on \mathcal{V} such that $T_g \in \mathcal{H}$ corresponding to $\mathcal{U}g\mathcal{U}$ for $g \in M'_2$ acts as in the trivial character of G (section 3.4). We call $e_{\mathcal{H}}(\mathcal{V})$ the extension of \mathcal{V} to \mathcal{H} though \mathcal{H}_M is not a subalgebra of \mathcal{H} . That notion is already present in [Abe] in the case where R has characteristic p . Here we extend the construction to any R and prove some more properties. In particular we produce an \mathcal{H} -equivariant embedding $e_{\mathcal{H}}(\mathcal{V})$ into $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \mathcal{V}$ (Lemma 3.10). If Q is a parabolic subgroup of G containing P , we go further and put on $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\text{Ind}_Q^G R)^{\mathcal{U}}$ and $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$ structures of \mathcal{H} -modules (Proposition 3.15 and Corollary 3.17) - note that \mathcal{H} is not a group algebra and there is no obvious notion of tensor product of \mathcal{H} -modules.

If σ is an R -representation of M extensible to G , then its extension $e_G(\sigma)$ is simply obtained by letting M'_2 act trivially on the space of σ ; moreover it is clear that $\sigma^{\mathcal{U}_M}$ is extensible to \mathcal{H} , and one shows easily that $e_G(\sigma)^{\mathcal{U}} = e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$ as an \mathcal{H} -module (section 3.5). Moreover, the natural inclusion of $e_G(\sigma)$ into $\text{Ind}_P^G \sigma$ induces on taking pro- p Iwahori invariants an embedding $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \rightarrow (\text{Ind}_P^G \sigma)^{\mathcal{U}}$ which, via the isomorphism of [OV17], yields exactly the above embedding of \mathcal{H} -modules of $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$ into $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma^{\mathcal{U}_M})$.

Then we show the \mathcal{H} -modules $(e_G(\sigma) \otimes_R \text{Ind}_Q^G R)^{\mathcal{U}}$ and $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\text{Ind}_Q^G R)^{\mathcal{U}}$ are equal, and similarly $(e_G(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}}$ and $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$ are equal (Theorem 4.9).

1.4. We turn back to the general case where we do not assume that Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. Nevertheless, given a right \mathcal{H}_M -module \mathcal{V} , there exists a largest Levi subgroup $M(\mathcal{V})$ of G - containing Z - corresponding to $\Delta_M \cup \Delta_1$ where Δ_1 is a subset of $\Delta \setminus \Delta_M$ orthogonal to Δ_M , such that \mathcal{V} extends to a right $\mathcal{H}_{M(\mathcal{V})}$ -module $e_{M(\mathcal{V})}(\mathcal{V})$ with the notation of section 1.3. For any parabolic subgroup Q between P and $P(\mathcal{V}) = M(\mathcal{V})U$ we put (Definition 4.12)

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} (e_{M(\mathcal{V})}(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}).$$

We refer to Theorem 4.17 for the description of the right \mathcal{H} -module $I_G(P, \sigma, Q)^{\mathcal{U}}$ for any smooth R -representation σ of \mathcal{U} . As a special case, it says that when σ is e -minimal then $P(\sigma) \supset P(\sigma^{\mathcal{U}_M})$ and if moreover $P(\sigma) = P(\sigma^{\mathcal{U}_M})$, then $I_G(P, \sigma, Q)^{\mathcal{U}}$ is isomorphic to $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$.

Remark 1.1. In [Abe] are attached similar \mathcal{H} -modules to (P, \mathcal{V}, Q) ; here we write them as $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ because their definition uses, instead of $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}$, a different kind of induction, which we call coinduction. In [Abe] those modules are used to give, when R is an algebraically closed field of characteristic p , a classification of simple \mathcal{H} -modules in terms of supersingular modules - that classification is similar to the classification of irreducible admissible R -representations of G in [AHHV17]. Using the comparison between induced and coinduced modules established in [Vig15b, 4.3] for any R , our Corollary 4.24 expresses $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ as a module $I_{\mathcal{H}}(P_1, \mathcal{V}_1, Q_1)$; consequently we show in section 4.5 that the classification of [Abe] can also be expressed in terms of modules $I_{\mathcal{H}}(P, \mathcal{V}, Q)$.

1.5. In a reverse direction one can associate to a right \mathcal{H} -module \mathcal{V} a smooth R -representation $\mathcal{V} \otimes_{\mathcal{H}} R[\mathcal{U} \setminus G]$ of G (seeing \mathcal{H} as the endomorphism ring of the $R[G]$ -module $R[\mathcal{U} \setminus G]$).

If \mathcal{V} is a right \mathcal{H}_M -module, we construct, again using [OV17], a natural $R[G]$ -map

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G] \rightarrow \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\mathcal{V} \otimes_{\mathcal{H}_M} R[\mathcal{U}_M \backslash M]) \otimes_R \text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})}),$$

with the notation of section 1.4. We show in section 5 that it is an isomorphism under a mild assumption on the \mathbb{Z} -torsion in \mathcal{V} ; in particular it is an isomorphism if $p = 0$ in R .

1.6. In the final section 6, we turn back to the case where R is an algebraically closed field of characteristic p . We prove that the smooth dual of an irreducible admissible R -representation V of G is 0 unless V is finite dimensional - that result is new if F has positive characteristic, a case where the proof of Kohlhaase [Koh] for $\text{char}(F) = 0$ does not apply. Our proof first reduces to the case where V is supercuspidal (by [AHHV17]) then uses again the \mathcal{H} -module $V^{\mathcal{U}}$.

2. NOTATION, USEFUL FACTS, AND PRELIMINARIES

2.1. **The group G and its standard parabolic subgroups $P = MN$.** In all that follows, p is a prime number and F is a local field with finite residue field k of characteristic p . We denote an algebraic group over F by a bold letter, like \mathbf{H} , and use the same ordinary letter for the group of F -points, $H = \mathbf{H}(F)$. We fix a connected reductive F -group \mathbf{G} . We fix a maximal F -split subtorus \mathbf{T} and write \mathbf{Z} for its \mathbf{G} -centralizer; we also fix a minimal parabolic subgroup \mathbf{B} of \mathbf{G} with Levi component \mathbf{Z} , so that $\mathbf{B} = \mathbf{Z}\mathbf{U}$ where \mathbf{U} is the unipotent radical of \mathbf{B} . Let $X^*(\mathbf{T})$ be the group of F -rational characters of \mathbf{T} and let Φ be the subset of roots of \mathbf{T} in the Lie algebra of \mathbf{G} . Then \mathbf{B} determines a subset Φ^+ of positive roots - the roots of \mathbf{T} in the Lie algebra of \mathbf{U} - and a subset of simple roots Δ . The \mathbf{G} -normalizer $\mathbf{N}_{\mathbf{G}}$ of \mathbf{T} acts on $X^*(\mathbf{T})$ and through that action, $\mathbf{N}_{\mathbf{G}}/\mathbf{Z}$ identifies with the Weyl group of the root system Φ . Set $\mathcal{N} := \mathbf{N}_{\mathbf{G}}(F)$ and note that $\mathbf{N}_{\mathbf{G}}/\mathbf{Z} \simeq \mathcal{N}/Z$; we write \mathbb{W} for \mathcal{N}/Z .

A standard parabolic subgroup of \mathbf{G} is a parabolic F -subgroup containing \mathbf{B} . Such a parabolic subgroup \mathbf{P} has a unique Levi subgroup \mathbf{M} containing \mathbf{Z} , so that $\mathbf{P} = \mathbf{M}\mathbf{N}$ where \mathbf{N} is the unipotent radical of \mathbf{P} - we also call \mathbf{M} standard. By a common abuse of language to describe the preceding situation, we simply say “let $P = MN$ be a standard parabolic subgroup of G ”; we sometimes write N_P for N and M_P for M . The parabolic subgroup of G opposite to P will be written \overline{P} and its unipotent radical \overline{N} , so that $\overline{P} = M\overline{N}$, but beware that \overline{P} is not standard! We write \mathbb{W}_M for the Weyl group $(M \cap \mathcal{N})/Z$.

If $\mathbf{P} = \mathbf{M}\mathbf{N}$ is a standard parabolic subgroup of G , then $\mathbf{M} \cap \mathbf{B}$ is a minimal parabolic subgroup of \mathbf{M} . If Φ_M denotes the set of roots of \mathbf{T} in the Lie algebra of \mathbf{M} , with respect to $\mathbf{M} \cap \mathbf{B}$ we have $\Phi_M^+ = \Phi_M \cap \Phi^+$ and $\Delta_M = \Phi_M \cap \Delta$. We also write Δ_P for Δ_M as P and M determine each other, $P = MU$. Thus we obtain a bijection $P \mapsto \Delta_P$ from standard parabolic subgroups of G to subsets of Δ , with B corresponding to Φ and G to Δ . If I is a subset of Δ , we sometimes denote by $P_I = M_I N_I$ the corresponding standard parabolic subgroup of G . If $I = \{\alpha\}$ is a singleton, we write $P_\alpha = M_\alpha N_\alpha$. We note a few useful properties. If P_1 is another standard parabolic subgroup of G , then $P \subset P_1$ if and only if $\Delta_P \subset \Delta_{P_1}$; we have $\Delta_{P \cap P_1} = \Delta_P \cap \Delta_{P_1}$ and the parabolic subgroup corresponding to $\Delta_P \cup \Delta_{P_1}$ is the subgroup $\langle P, P_1 \rangle$ of G generated by P and P_1 . The standard parabolic subgroup of M associated to $\Delta_M \cap \Delta_{M_1}$ is $M \cap P_1 = (M \cap M_1)(M \cap N_1)$ [Car85, Proposition 2.8.9]. It is convenient to write G' for the subgroup of G generated by the unipotent

radicals of the parabolic subgroups; it is also the normal subgroup of G generated by U , and we have $G = ZG'$. For future reference, we now give a useful lemma extracted from [AHHV17].

Lemma 2.1. *The group $Z \cap G'$ is generated by the $Z \cap M'_\alpha$, α running through Δ .*

Proof. Take $I = \emptyset$ in [AHHV17, II.6.Proposition]. \square

Let v_F be the normalized valuation of F . For each $\alpha \in X^*(T)$, the homomorphism $x \mapsto v_F(\alpha(x)) : T \rightarrow \mathbb{Z}$ extends uniquely to a homomorphism $Z \rightarrow \mathbb{Q}$ that we denote in the same way. This defines a homomorphism $Z \xrightarrow{v} X_*(T) \otimes \mathbb{Q}$ such that $\alpha(v(z)) = v_F(\alpha(z))$ for $z \in Z, \alpha \in X^*(T)$.

An interesting situation occurs when $\Delta = I \sqcup J$ is the union of two orthogonal subsets I and J . In that case, $G' = M'_I M'_J$, M'_I and M'_J commute with each other, and their intersection is finite and central in G [AHHV17, II.7 Remark 4].

2.2. $I_G(P, \sigma, Q)$ and minimality. We recall from [AHHV17] the construction of $I_G(P, \sigma, Q)$, our main object of study.

Let σ be an R -representation of M and let P_σ be the standard parabolic subgroup with $\Delta_{P_\sigma} = \Delta_\sigma$ where

$$\Delta_\sigma = \{\alpha \in \Delta \setminus \Delta_P \mid Z \cap M'_\alpha \text{ acts trivially on } \sigma\}.$$

We also let $P(\sigma)$ be the standard parabolic subgroup with

$$\Delta_{P(\sigma)} = \Delta_\sigma \cup \Delta_P.$$

This is the largest parabolic subgroup $P(\sigma)$ containing P to which σ extends, here $N \subset P$ acts on σ trivially. Clearly when $P \subset Q \subset P(\sigma)$, σ extends to Q and the extension is denoted by $e_Q(\sigma)$. The restriction of $e_{P(\sigma)}(\sigma)$ to Q is $e_Q(\sigma)$. If there is no risk of ambiguity, we write

$$e(\sigma) = e_{P(\sigma)}(\sigma).$$

Definition 2.2. An $R[G]$ -triple is a triple (P, σ, Q) made out of a standard parabolic subgroup $P = MN$ of G , a smooth R -representation of M , and a parabolic subgroup Q of G with $P \subset Q \subset P(\sigma)$. To an $R[G]$ -triple (P, σ, Q) is associated a smooth R -representation of G :

$$I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \text{St}_Q^{P(\sigma)}),$$

where $\text{St}_Q^{P(\sigma)}$ is the quotient of $\text{Ind}_Q^{P(\sigma)} \mathbf{1}$, $\mathbf{1}$ denoting the trivial R -representation of Q , by the sum of its subrepresentations $\text{Ind}_{Q'}^{P(\sigma)} \mathbf{1}$, the sum being over the set of parabolic subgroups Q' of G with $Q \subsetneq Q' \subset P(\sigma)$.

Note that $I_G(P, \sigma, Q)$ is naturally isomorphic to the quotient of $\text{Ind}_Q^G(e_Q(\sigma))$ by the sum of its subrepresentations $\text{Ind}_{Q'}^G(e_{Q'}(\sigma))$ for $Q \subsetneq Q' \subset P(\sigma)$ by [AHV, Lemma 2.5].

It might happen that σ itself has the form $e_P(\sigma_1)$ for some standard parabolic subgroup $P_1 = M_1 N_1$ contained in P and some R -representation σ_1 of M_1 . In that case, $P(\sigma_1) = P(\sigma)$ and $e(\sigma) = e(\sigma_1)$. We say that σ is e -minimal if $\sigma = e_P(\sigma_1)$ implies $P_1 = P, \sigma_1 = \sigma$.

Lemma 2.3 ([AHV, Lemma 2.9]). *Let $P = MN$ be a standard parabolic subgroup of G and let σ be an R -representation of M . There exists a unique standard parabolic subgroup $P_{\min, \sigma} = M_{\min, \sigma} N_{\min, \sigma}$ of G and a unique e -minimal representation of σ_{\min} of $M_{\min, \sigma}$ with $\sigma = e_P(\sigma_{\min})$. Moreover $P(\sigma) = P(\sigma_{\min})$ and $e(\sigma) = e(\sigma_{\min})$.*

Lemma 2.4. *Let $P = MN$ be a standard parabolic subgroup of G and let σ be an e -minimal R -representation of M . Then Δ_P and $\Delta_{P(\sigma)} \setminus \Delta_P$ are orthogonal.*

That comes from [AHHV17, II.7 Corollary 2]. That corollary of [AHHV17] also shows that when R is a field and σ is supercuspidal, then σ is e -minimal. Lemma 2.4 shows that $\Delta_{P_{\min, \sigma}}$ and $\Delta_{P(\sigma_{\min})} \setminus \Delta_{P_{\min, \sigma}}$ are orthogonal.

Note that when Δ_P and Δ_σ are orthogonal of union $\Delta = \Delta_P \sqcup \Delta_\sigma$, then $G = P(\sigma) = MM'_\sigma$ and $e(\sigma)$ is the R -representation of G simply obtained by extending σ trivially on M'_σ .

Lemma 2.5 ([AHV, Lemma 2.11]). *Let (P, σ, Q) be an $R[G]$ -triple. Then we have that $(P_{\min, \sigma}, \sigma_{\min}, Q)$ is an $R[G]$ -triple and $I_G(P, \sigma, Q) = I_G(P_{\min, \sigma}, \sigma_{\min}, Q)$.*

2.3. Pro- p Iwahori Hecke algebras. We fix a special parahoric subgroup \mathcal{K} of G fixing a special vertex x_0 in the apartment \mathcal{A} associated to T in the Bruhat-Tits building of the adjoint group of G . We let \mathcal{B} be the Iwahori subgroup fixing the alcove \mathcal{C} in \mathcal{A} with vertex x_0 contained in the Weyl chamber (of vertex x_0) associated to B . We let \mathcal{U} be the pro- p radical of \mathcal{B} (the pro- p Iwahori subgroup). The pro- p Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$ is the convolution ring of compactly supported functions $G \rightarrow \mathbb{Z}$ constant on the double classes of G modulo \mathcal{U} . We denote by $T(g)$ the characteristic function of $\mathcal{U}g\mathcal{U}$ for $g \in G$, seen as an element of \mathcal{H} . Let R be a commutative ring. The pro- p Iwahori Hecke R -algebra \mathcal{H}_R is $R \otimes_{\mathbb{Z}} \mathcal{H}$. We will follow the custom to still denote by h the natural image $1 \otimes h$ of $h \in \mathcal{H}$ in \mathcal{H}_R .

For $P = MN$ a standard parabolic subgroup of G , the similar objects for M are indexed by M , we have $\mathcal{K}_M = \mathcal{K} \cap M$, $\mathcal{B}_M = \mathcal{B} \cap M$, $\mathcal{U}_M = \mathcal{U} \cap M$, the pro- p Iwahori Hecke ring $\mathcal{H}_M = \mathcal{H}(M, \mathcal{U}_M)$, $T^M(m)$ the characteristic function of $\mathcal{U}_M m \mathcal{U}_M$ for $m \in M$, seen as an element of \mathcal{H}_M . The pro- p Iwahori subgroup \mathcal{U} of G satisfies the Iwahori decomposition with respect to P :

$$\mathcal{U} = \mathcal{U}_N \mathcal{U}_M \mathcal{U}_{\overline{N}},$$

where $\mathcal{U}_N = \mathcal{U} \cap N$, $\mathcal{U}_{\overline{N}} = \mathcal{U} \cap \overline{N}$. The linear map

$$(2.1) \quad \mathcal{H}_M \xrightarrow{\theta} \mathcal{H}, \quad \theta(T^M(m)) = T(m) \quad (m \in M)$$

does not respect the product. But if we introduce the monoid M^+ of elements $m \in M$ contracting \mathcal{U}_N , meaning $m\mathcal{U}_N m^{-1} \subset \mathcal{U}_N$, and the submodule $\mathcal{H}_{M^+} \subset \mathcal{H}_M$ of functions with support in M^+ , we have [Vig15b, Theorem 1.4]:

\mathcal{H}_{M^+} is a subring of \mathcal{H}_M and \mathcal{H}_M is the localization of \mathcal{H}_{M^+} at an element $\tau^M \in \mathcal{H}_{M^+}$ central and invertible in \mathcal{H}_M , meaning $\mathcal{H}_M = \bigcup_{n \in \mathbb{N}} \mathcal{H}_{M^+} (\tau^M)^{-n}$. The map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ is injective and its restriction $\theta|_{\mathcal{H}_{M^+}}$ to \mathcal{H}_{M^+} respects the product.

These properties are also true when (M^+, τ^M) is replaced by its inverse $(M^-, (\tau^M)^{-1})$ where $M^- = \{m^{-1} \in M \mid m \in M^+\}$.

3. PRO- p IWAHORI INVARIANTS OF $I_G(P, \sigma, Q)$

3.1. Pro- p Iwahori Hecke algebras: Structures. Here we supplement the notation of sections 2.1 and 2.3. The subgroups $Z^0 = Z \cap \mathcal{K} = Z \cap \mathcal{B}$ and $Z^1 = Z \cap \mathcal{U}$

are normal in \mathcal{N} and we put

$$W = \mathcal{N}/Z^0, \quad W(1) = \mathcal{N}/Z^1, \quad \Lambda = Z/Z^0, \quad \Lambda(1) = Z/Z^1, \quad Z_k = Z^0/Z^1.$$

We have $\tilde{\mathcal{N}} = (\mathcal{N} \cap \mathcal{K})Z$ so that we see the finite Weyl group $\mathbb{W} = \mathcal{N}/Z$ as the subgroup $(\mathcal{N} \cap \mathcal{K})/Z^0$ of W ; in this way W is the semidirect product $\Lambda \rtimes \mathbb{W}$. We put $\mathcal{N}_{G'} = \mathcal{N} \cap G'$. The image $W_{G'} = W'$ of $\mathcal{N}_{G'}$ in W is an affine Weyl group generated by the set S^{aff} of affine reflections determined by the walls of the alcove \mathcal{C} . The group W' is normal in W and W is the semidirect product $W' \rtimes \Omega$ where Ω is the image in W of the normalizer $\mathcal{N}_{\mathcal{C}}$ of \mathcal{C} in \mathcal{N} . The length function ℓ on the affine Weyl system (W', S^{aff}) extends to a length function on W such that Ω is the set of elements of length 0. We also view ℓ as a function of $W(1)$ via the quotient map $W(1) \rightarrow W$. We write

(3.1)

$(\hat{w}, \tilde{w}, w) \in \mathcal{N} \times W(1) \times W$ corresponding via the quotient maps $\mathcal{N} \rightarrow W(1) \rightarrow W$.

When $w = s$ in S^{aff} or more generally w in $W_{G'}$, we will most of the time choose \hat{w} in $\mathcal{N} \cap G'$ and \tilde{w} in the image ${}_1W_{G'}$ of $\mathcal{N} \cap G'$ in $W(1)$.

We are now ready to describe the pro- p Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$ [Vig16]. We have $G = \mathcal{U}\mathcal{N}\mathcal{U}$ and for $n, n' \in \mathcal{N}$ we have $\mathcal{U}n\mathcal{U} = \mathcal{U}n'\mathcal{U}$ if and only if $nZ^1 = n'Z^1$. For $n \in \mathcal{N}$ of image $w \in W(1)$ and $g \in \mathcal{U}\mathcal{N}\mathcal{U}$ we denote $T_w = T(n) = T(g)$ in \mathcal{H} . The relations among the basis elements $(T_w)_{w \in W(1)}$ of \mathcal{H} are:

- (1) Braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.
- (2) Quadratic relations: $T_{\tilde{s}}^2 = q_s T_{\tilde{s}^2} + c_{\tilde{s}} T_{\tilde{s}}$

for $\tilde{s} \in W(1)$ lifting $s \in S^{\text{aff}}$, where $q_s = q_G(s) = |\mathcal{U}/\mathcal{U} \cap \hat{s}\mathcal{U}(\hat{s})^{-1}|$ depends only on s , and $c_{\tilde{s}} = \sum_{t \in Z_k} c_{\tilde{s}}(t) T_t$ for integers $c_{\tilde{s}}(t) \in \mathbb{N}$ summing to $q_s - 1$.

We shall need the basis elements $(T_w^*)_{w \in W(1)}$ of \mathcal{H} defined by:

- (1) $T_w^* = T_w$ for $w \in W(1)$ of length $\ell(w) = 0$.
- (2) $T_{\tilde{s}}^* = T_{\tilde{s}} - c_{\tilde{s}}$ for $\tilde{s} \in W(1)$ lifting $s \in S^{\text{aff}}$.
- (3) $T_{ww'}^* = T_w^* T_{w'}^*$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.

We need more notation for the definition of the admissible lifts of S^{aff} in \mathcal{N}_G . Let $s \in S^{\text{aff}}$ fixing a face \mathcal{C}_s of the alcove \mathcal{C} and \mathcal{K}_s the parahoric subgroup of G fixing \mathcal{C}_s . The theory of Bruhat-Tits associates to \mathcal{C}_s a certain root $\alpha_s \in \Phi^+$ [Vig16, §4.2]. We consider the group \mathcal{G}'_s generated by $U_{\alpha_s} \cup U_{-\alpha_s}$ where $U_{\pm\alpha_s}$ the root subgroup of $\pm\alpha_s$ (if $2\alpha_s \in \Phi$, then $U_{2\alpha_s} \subset U_{\alpha_s}$) and the group \mathcal{G}_s generated by $\mathcal{U}_{\alpha_s} \cup \mathcal{U}_{-\alpha_s}$ where $\mathcal{U}_{\pm\alpha_s} = U_{\pm\alpha_s} \cap \mathcal{K}_s$. When $u \in \mathcal{U}_{\alpha_s} - \{1\}$, the intersection $\mathcal{N}_G \cap \mathcal{U}_{-\alpha_s} u \mathcal{U}_{-\alpha_s}$ (equal to $\mathcal{N}_G \cap U_{-\alpha_s} u U_{-\alpha_s}$ [BT72, 6.2.1 (V5)], [Vig16, §3.3 (19)]) possesses a single element $n_s(u)$. The group $Z'_s = Z \cap \mathcal{G}'_s$ is contained in $Z \cap \mathcal{K}_s = Z^0$; its image in Z_k is denoted by $Z'_{k,s}$.

The elements $n_s(u)$ for $u \in \mathcal{U}_{\alpha_s} - \{1\}$ are the admissible lifts of s in \mathcal{N}_G ; their images in $W(1)$ are the admissible lifts of s in $W(1)$. By [Vig16, Theorem 2.2, Proposition 4.4], when $\tilde{s} \in W(1)$ is an admissible lift of s , $c_{\tilde{s}}(t) = 0$ if $t \in Z_k \setminus Z'_{k,s}$, and

$$(3.2) \quad c_{\tilde{s}} \equiv (q_s - 1) |Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} T_t \pmod{p}.$$

The admissible lifts of S in \mathcal{N}_G are contained in $\mathcal{N}_G \cap \mathcal{K}$ because $\mathcal{K}_s \subset \mathcal{K}$ when $s \in S$.

Definition 3.1. An admissible lift of the finite Weyl group \mathbb{W} in \mathcal{N}_G is a map $w \mapsto \hat{w} : \mathbb{W} \rightarrow \mathcal{N}_G \cap \mathcal{K}$ such that \hat{s} is admissible for all $s \in S$ and $\hat{w} = \hat{w}_1 \hat{w}_2$ for $w_1, w_2 \in \mathbb{W}$ such that $w = w_1 w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$.

Any choice of admissible lifts of S in $\mathcal{N}_G \cap \mathcal{K}$ extends uniquely to an admissible lift of \mathbb{W} ([AHHV17, IV.6], [OV17, Proposition 2.7]).

Let $P = MN$ be a standard parabolic subgroup of G . The groups $Z, Z^0 = Z \cap \mathcal{K}_M = Z \cap \mathcal{B}_M, Z^1 = Z \cap \mathcal{U}_M$ are the same for G and M , but $\mathcal{N}_M = \mathcal{N} \cap M$ and $M \cap G'$ are subgroups of \mathcal{N} and G' . The monoid M^+ (section 2.3) contains $(\mathcal{N}_M \cap \mathcal{K})$ and is equal to $M^+ = \mathcal{U}_M \mathcal{N}_{M^+} \mathcal{U}_M$ where $\mathcal{N}_{M^+} = \mathcal{N} \cap M^+$. An element $z \in Z$ belongs to M^+ if and only if $v_F(\alpha(z)) \geq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_M^+$ (see [Vig15b, Lemme 2.2]). Put $W_M = \mathcal{N}_M / Z^0$ and $W_M(1) = \mathcal{N}_M / Z^1$.

Let $\epsilon = +$ or $\epsilon = -$. We denote by $W_{M^\epsilon}, W_{M^\epsilon}(1)$ the images of \mathcal{N}_{M^ϵ} in $W_M, W_M(1)$. We see the groups $W_M, W_M(1), {}_1W_{M'}$ as subgroups of $W, W(1), {}_1W_{G'}$. As θ (section 2.3), the linear injective map

$$(3.3) \quad \mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}, \quad \theta^*(T_w^{M,*}) = T_w^*, \quad (w \in W_M(1)),$$

respects the product on the subring \mathcal{H}_{M^ϵ} . Here $T_w^{M,*} \in \mathcal{H}_M$ is defined in the same way as T_w^* for \mathcal{H}_M . Note that θ and θ^* satisfy the obvious transitivity property with respect to a change of parabolic subgroups.

3.2. Orthogonal case. Let us examine the case where Δ_M and $\Delta \setminus \Delta_M$ are orthogonal, writing $M_2 = M_{\Delta \setminus \Delta_M}$ as in section 1.3.

From $M \cap M_2 = Z$ we get $W_M \cap W_{M_2} = \Lambda, W_M(1) \cap W_{M_2}(1) = \Lambda(1)$, the semisimple building of G is the product of those of M and M_2 . The set S^{aff} is the disjoint union of S_M^{aff} and $S_{M_2}^{\text{aff}}$, the group $W_{G'}$ is the direct product of $W_{M'}$ and $W_{M_2'}$. For $\tilde{s} \in W_M(1)$ lifting $s \in S_M^{\text{aff}}$, the elements $T_{\tilde{s}}^M \in \mathcal{H}_M$ and $T_{\tilde{s}} \in \mathcal{H}$ satisfy the same quadratic relations. A word of caution is necessary for the lengths ℓ_M of W_M and ℓ_{M_2} of W_{M_2} different from the restrictions of the length ℓ of W_M , for example $\ell_M(\lambda) = 0$ for $\lambda \in \Lambda \cap W_{M_2'}$.

Lemma 3.2. *We have $\Lambda = (W_{M^\epsilon} \cap \Lambda)(W_{M_2'} \cap \Lambda)$.*

Proof. We prove the lemma for $\epsilon = -$. The case $\epsilon = +$ is similar. The map $v : Z \rightarrow X_*(T) \otimes \mathbb{Q}$ defined in section 2.1 is trivial on Z^0 and we also write v for the resulting homomorphism on Λ . For $\lambda \in \Lambda$ there exists $\lambda_2 \in W_{M_2'} \cap \Lambda$ such that $\lambda \lambda_2 \in W_{M^-}$, or equivalently $\alpha(v(\lambda \lambda_2)) \leq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_M^+ = \Phi_{M_2}^+$. It suffices to have the inequality for $\alpha \in \Delta_{M_2}$. The matrix $(\alpha(\beta^\vee))_{\alpha, \beta \in \Delta_{M_2}}$ is invertible, hence there exists $n_\beta \in \mathbb{Z}$ such that $\sum_{\beta \in \Delta_{M_2}} n_\beta \alpha(\beta^\vee) \leq -\alpha(v(\lambda))$ for all $\alpha \in \Delta_{M_2}$. As $v(W_{M_2'} \cap \Lambda)$ contains $\bigoplus_{\alpha \in \Delta_{M_2}} \mathbb{Z} \alpha^\vee$ where α^\vee is the coroot of α [Vig16, after formula (71)], there exists $\lambda_2 \in W_{M_2'} \cap \Lambda$ with $v(\lambda_2) = \sum_{\beta \in \Delta_{M_2}} n_\beta \beta^\vee$. \square

The groups $\mathcal{N} \cap M'$ and $\mathcal{N} \cap M_2'$ are normal in \mathcal{N} , and

$$\mathcal{N} = (\mathcal{N} \cap M') \mathcal{N}_c(\mathcal{N} \cap M_2') = Z(\mathcal{N} \cap M')(\mathcal{N} \cap M_2'),$$

and

$$W = W_{M'} \Omega W_{M_2'} = W_M W_{M_2'} = W_{M^+} W_{M_2'} = W_{M^-} W_{M_2'}.$$

The first two equalities are clear, the equality $W_M W_{M_2'} = W_{M^\epsilon} W_{M_2'}$ follows from $W_M = \mathbb{W}_M \Lambda, \mathbb{W}_M \subset W_{M^\epsilon}$ and the lemma. The inverse image in $W(1)$ of these

groups are

(3.4)

$$W(1) = {}_1W_{M'} \Omega(1) {}_1W_{M'_2} = W_M(1) {}_1W_{M'_2} = W_{M^+}(1) {}_1W_{M'_2} = W_{M^-}(1) {}_1W_{M'_2}.$$

We recall the function $q_G(n) = q(n) = |\mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)|$ on \mathcal{N} [Vig16, Proposition 3.38] and we extend to \mathcal{N} the functions q_M on $\mathcal{N} \cap M$ and q_{M_2} on $\mathcal{N} \cap M_2$:

$$(3.5) \quad q_M(n) = |\mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_Mn)|, \quad q_{M_2}(n) = |\mathcal{U}_{M_2}/(\mathcal{U}_{M_2} \cap n^{-1}\mathcal{U}_{M_2}n)|.$$

The functions q, q_M, q_{M_2} descend to functions on $W(1)$ and on W , also denoted by q, q_M, q_{M_2} .

Lemma 3.3. *Let $n \in \mathcal{N}$ of image $w \in W$. We have*

- (1) $q(n) = q_M(n)q_{M_2}(n)$.
- (2) $q_M(n) = q_M(n_M)$ if $n = n_M n_2$, $n_M \in \mathcal{N} \cap M, n_2 \in \mathcal{N} \cap M'_2$ and similarly when M and M_2 are permuted.
- (3) $q(w) = 1 \Leftrightarrow q_M(\lambda w_M) = q_{M_2}(\lambda w_{M_2}) = 1$, if $w = \lambda w_M w_{M_2}, (\lambda, w_M, w_{M_2}) \in \Lambda \times \mathbb{W}_M \times \mathbb{W}_{M_2}$.
- (4) On the coset $(\mathcal{N} \cap M'_2)\mathcal{N}_C n$, q_M is constant equal to $q_M(n_{M'})$ for any element $n_{M'} \in M' \cap (\mathcal{N} \cap M'_2)\mathcal{N}_C n$. A similar result is true when M and M_2 are permuted.

Proof. We put $\mathcal{U}_{M'} = \mathcal{U} \cap M'$ and $\mathcal{U}_{M'_2} = \mathcal{U} \cap M'_2$. The product map

$$(3.6) \quad Z^1 \prod_{\alpha \in \Phi_{M,red}} \mathcal{U}_\alpha \prod_{\alpha \in \Phi_{M_2,red}} \mathcal{U}_\alpha \rightarrow \mathcal{U}$$

with $\mathcal{U}_\alpha = U_\alpha \cap \mathcal{U}$, is a homeomorphism. We have $\mathcal{U}_M = Z^1 \mathcal{Y}_{M'}$, $\mathcal{U}_{M'} = (Z^1 \cap M') \mathcal{Y}_{M'}$ where $\mathcal{Y}_{M'} = \prod_{\alpha \in \Phi_{M,red}} \mathcal{U}_\alpha$ and $\mathcal{N} \cap M'_2$ commutes with $\mathcal{Y}_{M'}$, in particular $\mathcal{N} \cap M'_2$ normalizes $\mathcal{Y}_{M'}$. Similar results are true when M and M_2 are permuted, and $\mathcal{U} = \mathcal{U}_{M'} \mathcal{U}_{M_2} = \mathcal{U}_M \mathcal{U}_{M'_2}$.

Writing $\mathcal{N} = Z(\mathcal{N} \cap M')(\mathcal{N} \cap M'_2)$ (in any order), we see that the product map

$$(3.7) \quad Z^1(\mathcal{Y}_{M'} \cap n^{-1}\mathcal{Y}_{M'}n)(\mathcal{Y}_{M'_2} \cap n^{-1}\mathcal{Y}_{M'_2}n) \rightarrow \mathcal{U} \cap n^{-1}\mathcal{U}n$$

is a homeomorphism. The inclusions induce bijections

$$(3.8) \quad \mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap n^{-1}\mathcal{Y}_{M'}n) \simeq \mathcal{U}_{M'}/(\mathcal{U}_{M'} \cap n^{-1}\mathcal{U}_{M'}n) \simeq \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_Mn),$$

similarly for M_2 , and also a bijection

$$(3.9) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n) \simeq (\mathcal{Y}_{M'_2}/(\mathcal{Y}_{M'_2} \cap n^{-1}\mathcal{Y}_{M'_2}n)) \times (\mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap n^{-1}\mathcal{Y}_{M'}n)).$$

From (3.8) and (3.9), we get

$$(3.10) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n) \simeq (\mathcal{U}_{M'_2}/(\mathcal{U}_{M'_2} \cap n\mathcal{U}_{M'_2}n^{-1})) \times (\mathcal{U}_{M'}/(\mathcal{U}_{M'} \cap n\mathcal{U}_{M'}n^{-1}))$$

which implies the assertion (1) in the lemma.

The assertion (2) follows from (3.7) since $\mathcal{N} \cap M'_2$ normalizes $\mathcal{Y}_{M'}$; with (1), it implies the assertion (3).

A subgroup of \mathcal{N} normalizes \mathcal{U}_M if and only if it normalizes $\mathcal{Y}_{M'}$ by (3.8) if and only if $q_M = 1$ on this group. The group $\mathcal{N} \cap M'_2$ normalizes $\mathcal{Y}_{M'}$. Therefore the group $(\mathcal{N} \cap M'_2)\mathcal{N}_C$ normalizes \mathcal{U}_M . The coset $(\mathcal{N} \cap M'_2)\mathcal{N}_C n$ contains an element $n_{M'} \in M'$. For $x \in (\mathcal{N} \cap M'_2)\mathcal{N}_C$, $(xn_{M'})^{-1} \mathcal{U} xn_{M'} = n_{M'}^{-1} \mathcal{U} n_{M'}$ hence $q_M(xn_{M'}) = q_M(n_{M'})$. \square

3.3. Extension of an \mathcal{H}_M -module to \mathcal{H} . This section is inspired by similar results for the pro- p Iwahori Hecke algebras over an algebraically closed field of characteristic p [Abe, Proposition 4.16]. We keep the setting of section 3.2 and we introduce ideals:

- \mathcal{J}_ℓ (resp., \mathcal{J}_r) the left (resp., right) ideal of \mathcal{H} generated by $T_w^* - 1_{\mathcal{H}}$ for all $w \in {}_1W_{M'_2}$,
- $\mathcal{J}_{M,\ell}$ (resp., $\mathcal{J}_{M,r}$) the left (resp., right) ideal of \mathcal{H}_M generated by $T_\lambda^{M,*} - 1_{\mathcal{H}_M}$ for all λ in ${}_1W_{M'_2} \cap W_M(1) = {}_1W_{M'_2} \cap \Lambda(1)$.

The next proposition shows that the ideals $\mathcal{J}_\ell = \mathcal{J}_r$ are equal and similarly $\mathcal{J}_{M,\ell} = \mathcal{J}_{M,r}$. After the proposition, we will drop the indices ℓ and r .

Proposition 3.4. *The ideals \mathcal{J}_ℓ and \mathcal{J}_r are equal to the submodule \mathcal{J}' of \mathcal{H} generated by $T_w^* - T_{ww_2}^*$ for all $w \in W(1)$ and $w_2 \in {}_1W_{M'_2}$.*

The ideals $\mathcal{J}_{M,\ell}$ and $\mathcal{J}_{M,r}$ are equal to the submodule \mathcal{J}'_M of \mathcal{H}_M generated by $T_w^{M,} - T_{w\lambda_2}^{M,*}$ for all $w \in W_M(1)$ and $\lambda_2 \in \Lambda(1) \cap {}_1W_{M'_2}$.*

Proof.

(1) We prove $\mathcal{J}_\ell = \mathcal{J}'$. Let $w \in W(1), w_2 \in {}_1W_{M'_2}$. We prove by induction on the length of w_2 that $T_w^*(T_{w_2}^* - 1) \in \mathcal{J}'$. This is obvious when $\ell(w_2) = 0$ because $T_w^*T_{w_2}^* = T_{ww_2}^*$. Assume that $\ell(w_2) = 1$ and put $s = w_2$. If $\ell(ws) = \ell(w) + 1$, as before $T_w^*(T_s^* - 1) \in \mathcal{J}'$ because $T_w^*T_s^* = T_{ws}^*$. Otherwise $\ell(ws) = \ell(w) - 1$ and $T_w^* = T_{ws^{-1}}^*T_s^*$ hence

$$T_w^*(T_s^* - 1) = T_{ws^{-1}}^*(T_s^*)^2 - T_w^* = T_{ws^{-1}}^*(q_s T_{s^2}^* - T_s^* c_s) - T_w^* = q_s T_{ws}^* - T_w^*(c_s + 1).$$

Since $c_s + 1 = \sum_{t \in Z'_k} c_s(t) T_t$ with $c_s(t) \in \mathbb{N}$ and $\sum_{t \in Z'_k} c_s(t) = q_s$ [Vig16, Proposition 4.4],

$$q_s T_{ws}^* - T_w^*(c_s + 1) = \sum_{t \in Z'_k} c_s(t) (T_{ws}^* - T_w^* T_t^*) = \sum_{t \in Z'_k} c_s(t) (T_{ws}^* - T_{ws s^{-1} t}^*) \in \mathcal{J}'.$$

Assume now that $\ell(w_2) > 1$. Then, we factorize $w_2 = xy$ with $x, y \in {}_1W_{M'_2}$ of length $\ell(x), \ell(y) < \ell(w_2)$ and $\ell(w_2) = \ell(x) + \ell(y)$. The element $T_w^*(T_{w_2}^* - 1) = T_w^* T_x^*(T_y^* - 1) + T_w^*(T_x^* - 1)$ lies in \mathcal{J}' by induction.

Conversely, we prove $T_{ww_2}^* - T_w^* \in \mathcal{J}_\ell$. We factorize $w = xy$ with $y \in {}_1W_{M'_2}$ and $x \in {}_1W_{M'} \Omega(1)$. Then, we have $\ell(w) = \ell(x) + \ell(y)$ and $\ell(ww_2) = \ell(x) + \ell(yw_2)$. Hence

$$T_{ww_2}^* - T_w^* = T_x^*(T_{yw_2}^* - T_y^*) = T_x^*(T_{yw_2}^* - 1) - T_x^*(T_y^* - 1) \in \mathcal{J}_\ell.$$

This ends the proof of $\mathcal{J}_\ell = \mathcal{J}'$.

By the same argument, the right ideal \mathcal{J}_r of \mathcal{H} is equal to the submodule of \mathcal{H} generated by $T_{w_2 w}^* - T_w^*$ for all $w \in W(1)$ and $w_2 \in {}_1W_{M'_2}$. But this latter submodule is equal to \mathcal{J}' because ${}_1W_{M'_2}$ is normal in $W(1)$. Therefore we proved $\mathcal{J}' = \mathcal{J}_r = \mathcal{J}_\ell$.

(2) Proof of the second assertion. We prove $\mathcal{J}_{M,\ell} = \mathcal{J}'_M$. The proof is easier than in (1) because for $w \in W_M(1)$ and $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$, we have $\ell(w\lambda_2) = \ell(w) + \ell(\lambda_2)$ hence $T_w^{M,*}(T_{\lambda_2}^{M,*} - 1) = T_{w\lambda_2}^{M,*} - T_w^{M,*}$. We have also $\ell(\lambda_2 w) = \ell(\lambda_2) + \ell(w)$ hence $(T_{\lambda_2}^{M,*} - 1)T_w^{M,*} = T_{\lambda_2 w}^{M,*} - T_w^{M,*}$ hence $\mathcal{J}_{M,r}$ is equal to the submodule of \mathcal{H}_M generated by $T_{\lambda_2 w}^{M,*} - T_w^{M,*}$ for all $w \in W_M(1)$ and $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$. This latter submodule is \mathcal{J}'_M , as ${}_1W_{M'_2} \cap \Lambda(1) = {}_1W_{M'_2} \cap W_M(1)$ is normal in $W_M(1)$. Therefore $\mathcal{J}'_M = \mathcal{J}_{M,r} = \mathcal{J}_{M,\ell}$. \square

By Proposition 3.4, a basis of \mathcal{J} is $T_w^* - T_{ww_2}^*$ for w in a system of representatives of $W(1)/{}_1W_{M_2'}$, and $w_2 \in {}_1W_{M_2'} \setminus \{1\}$. Similarly a basis of \mathcal{J}_M is $T_w^{M,*} - T_{w\lambda_2}^{M,*}$ for w in a system of representatives of $W_M(1)/(\Lambda(1) \cap {}_1W_{M_2'})$. and $\lambda_2 \in (\Lambda(1) \cap {}_1W_{M_2'}) \setminus \{1\}$.

Proposition 3.5. *The natural ring inclusion of \mathcal{H}_{M^-} in \mathcal{H}_M and the ring inclusion of \mathcal{H}_{M^-} in \mathcal{H} via θ^* induce ring isomorphisms*

$$\mathcal{H}_M/\mathcal{J}_M \xleftarrow{\sim} \mathcal{H}_{M^-}/(\mathcal{J}_M \cap \mathcal{H}_{M^-}) \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

Proof.

(1) The left map is obviously injective. We prove the surjectivity. Let $w \in W_M(1)$. Let $\lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1)$ such that $w\lambda_2^{-1} \in W_{M^-}(1)$ (see (3.4)). We have $T_{w\lambda_2^{-1}}^{M,*} \in \mathcal{H}_{M^-}$ and $T_w^{M,*} = T_{w\lambda_2^{-1}}^{M,*} T_{\lambda_2}^{M,*} = T_{w\lambda_2^{-1}}^{M,*} + T_{w\lambda_2^{-1}}^{M,*} (T_{\lambda_2}^{M,*} - 1)$. Therefore $T_w^{M,*} \in \mathcal{H}_{M^-} + \mathcal{J}_M$. As w is arbitrary, $\mathcal{H}_M = \mathcal{H}_{M^-} + \mathcal{J}_M$.

(2) The right map is surjective: let $w \in W(1)$ and $w_2 \in {}_1W_{M_2'}$ such that $ww_2^{-1} \in W_{M^-}(1)$ (see (3.4)). Then $T_w^* - T_{ww_2^{-1}}^* \in \mathcal{J}$ with the same arguments as in (1), using Proposition 3.4. Therefore $\mathcal{H} = \theta^*(\mathcal{H}_{M^-}) + \mathcal{J}$.

We prove the injectivity: $\theta^*(\mathcal{H}_{M^-}) \cap \mathcal{J} = \theta^*(\mathcal{H}_{M^-} \cap \mathcal{J}_M)$. Let $\sum_{w \in W_{M^-}(1)} c_w T_w^{M,*}$, with $c_w \in \mathbb{Z}$, be an element of \mathcal{H}_{M^-} . Its image by θ^* is $\sum_{w \in W(1)} c_w T_w^*$ where we have set $c_w = 0$ for $w \in W(1) \setminus W_{M^-}(1)$. We have $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}$ if and only if $\sum_{w_2 \in {}_1W_{M_2'}} c_{ww_2} = 0$ for all $w \in W(1)$. If $c_{ww_2} \neq 0$, then $w_2 \in {}_1W_{M_2'} \cap W_M(1)$, that is, $w_2 \in {}_1W_{M_2'} \cap \Lambda(1)$. The sum $\sum_{w_2 \in {}_1W_{M_2'}} c_{ww_2}$ is equal to $\sum_{\lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1)} c_{w\lambda_2}$. By Proposition 3.4, $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}$ if and only if $\sum_{w \in W_{M^-}(1)} c_w T_w^{M,*} \in \mathcal{J}_M$. \square

We construct a ring isomorphism

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}$$

by using Proposition 3.5. For any $w \in W(1)$, $T_w^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$ where $w_{M^-} \in W_{M^-}(1) \cap {}_1W_{M_2'}$ (see (3.4)), because by Proposition 3.4, $T_w^* + \mathcal{J} = T_{w_{M^-}}^* + \mathcal{J}$ and $T_{w_{M^-}}^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$ by construction of e^* . We check that e^* is induced by θ^* .

Theorem 3.6. *The linear map $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$ induces a ring isomorphism*

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

Proof. Let $w \in W_M(1)$. We have to show that $T_w^* + \mathcal{J} = e^*(T_w^{M,*} + \mathcal{J}_M)$. We saw above that $T_w^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$ with $w = w_{M^-}\lambda_2$ with $\lambda_2 \in {}_1W_{M_2'} \cap W_M(1)$. As $\ell_M(\lambda_2) = 0$, $T_w^{M,*} = T_{w_{M^-}}^{M,*} T_{\lambda_2}^{M,*} \in T_{w_{M^-}}^{M,*} + \mathcal{J}_M$. Therefore $T_{w_{M^-}}^{M,*} + \mathcal{J}_M = T_w^{M,*} + \mathcal{J}_M$. This ends the proof of the theorem. \square

We now wish to compute e^* in terms of the T_w instead of the T_w^* .

Proposition 3.7. *Let $w \in W(1)$. Then, $T_w + \mathcal{J} = e^*(T_{w_M}^M q_{M_2}(w) + \mathcal{J}_M)$ for any $w_M \in W_M(1) \cap {}_1W_{M_2'}$.*

Proof. The element w_M is unique modulo right multiplication by an element $\lambda_2 \in W_M(1) \cap {}_1W_{M'_2}$ of length $\ell_M(\lambda_2) = 0$ and $T_{w_M}^M q_{M_2}(w) + \mathcal{J}_M$ does not depend on the choice of w_M . We choose a decomposition (see (3.4)):

$$w = \tilde{s}_1 \dots \tilde{s}_a u \tilde{s}_{a+1} \dots \tilde{s}_{a+b}, \quad \ell(w) = a + b,$$

for $u \in \Omega(1)$, $\tilde{s}_i \in {}_1W_{M'}$ lifting $s_i \in S_M^{\text{aff}}$ for $1 \leq i \leq a$ and $\tilde{s}_i \in {}_1W_{M'_2}$ lifting $s_i \in S_{M_2}^{\text{aff}}$ for $a+1 \leq i \leq a+b$, and we choose $u_M \in W_M(1)$ such that $u \in u_M {}_1W_{M'_2}$. Then

$$w_M = \tilde{s}_1 \dots \tilde{s}_a u_M \in W_M(1) \cap w {}_1W_{M'_2}$$

and $q_{M_2}(w) = q_{M_2}(\tilde{s}_{a+1} \dots \tilde{s}_{a+b})$ (Lemma 3.3 (4)). First we check the proposition in three simple cases:

Case 1. Let $w = \tilde{s} \in {}_1W_{M'}$ lifting $s \in S_M^{\text{aff}}$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(T_{\tilde{s}}^M + \mathcal{J}_M)$ because $T_{\tilde{s}}^* - e^*(T_{\tilde{s}}^{M,*}) \in \mathcal{J}$, $T_{\tilde{s}}^* = T_{\tilde{s}}^* + c_{\tilde{s}}$, $T_{\tilde{s}}^M = T_{\tilde{s}}^{M,*} + c_{\tilde{s}}$ and $1 = q_{M_2}(\tilde{s})$.

Case 2. Let $w = u \in W(1)$ of length $\ell(u) = 0$ and $u_M \in W_M(1)$ such that $u \in u_M {}_1W_{M'_2}$. We have $\ell_M(u_M) = 0$ and $q_{M_2}(u) = 1$ (Lemma 3.3). We deduce $T_u + \mathcal{J} = e^*(T_{u_M}^M + \mathcal{J}_M)$ because $T_u^* + \mathcal{J} = T_{u_M}^* + \mathcal{J} = e^*(T_{u_M}^{M,*} + \mathcal{J}_M)$, and $T_u = T_{u_M}^*$, $T_{u_M}^M = T_{u_M}^{M,*}$.

Case 3. Let $w = \tilde{s} \in {}_1W_{M'_2}$ lifting $s \in S_{M_2}^{\text{aff}}$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(q_{M_2}(\tilde{s}) + \mathcal{J}_M)$ because $T_{\tilde{s}}^* - 1, c_{\tilde{s}} - (q_s - 1) \in \mathcal{J}$, $T_{\tilde{s}}^* = T_{\tilde{s}}^* + c_{\tilde{s}} \in q_s + \mathcal{J}$ and $q_s = q_{M_2}(\tilde{s})$.

In general, the braid relations $T_w = T_{\tilde{s}_1} \dots T_{\tilde{s}_a} T_u T_{\tilde{s}_{a+1}} \dots T_{\tilde{s}_{a+b}}$ give a similar product decomposition of $T_w + \mathcal{J}$, and the simple cases 1, 2, 3 imply that $T_w + \mathcal{J}$ is equal to

$$\begin{aligned} & e^*(T_{\tilde{s}_1}^M + \mathcal{J}_M) \dots e^*(T_{\tilde{s}_a}^M + \mathcal{J}_M) e^*(T_{u_M}^M + \mathcal{J}_M) e^*(q_{M_2}(\tilde{s}_{a+1}) \\ & \quad + \mathcal{J}_M) \dots e^*(q_{M_2}(\tilde{s}_{a+b}) + \mathcal{J}_M) \\ & = e^*(T_{w_M}^M q_{M_2}(w) + \mathcal{J}_M). \end{aligned}$$

The proposition is proved. \square

Propositions 3.4, 3.5, 3.7, and Theorem 3.6 are valid over any commutative ring R (instead of \mathbb{Z}).

The two-sided ideal of \mathcal{H}_R generated by $T_w^* - 1$ for all $w \in {}_1W_{M'_2}$ is $\mathcal{J}_R = \mathcal{J} \otimes_{\mathbb{Z}} R$, the two-sided ideal of $\mathcal{H}_{M,R}$ generated by $T_\lambda^* - 1$ for all $\lambda \in {}_1W_{M'_2} \cap \Lambda(1)$ is $\mathcal{J}_{M,R} = \mathcal{J}_M \otimes_{\mathbb{Z}} R$, and we get as in Proposition 3.5 isomorphisms

$$\mathcal{H}_{M,R} / \mathcal{J}_{M,R} \xleftarrow{\sim} \mathcal{H}_{M^-,R} / (\mathcal{J}_{M,R} \cap \mathcal{H}_{M^-,R}) \xrightarrow{\sim} \mathcal{H}_R / \mathcal{J}_R,$$

giving an isomorphism $\mathcal{H}_{M,R} / \mathcal{J}_{M,R} \rightarrow \mathcal{H}_R / \mathcal{J}_R$ induced by θ^* . Therefore, we have an isomorphism from the category of right $\mathcal{H}_{M,R}$ -modules where \mathcal{J}_M acts by 0 onto the category of right \mathcal{H}_R -modules where \mathcal{J} acts by 0.

Definition 3.8. A right $\mathcal{H}_{M,R}$ -module \mathcal{V} where \mathcal{J}_M acts by 0 is called extensible to \mathcal{H} . The corresponding \mathcal{H}_R -module where \mathcal{J} acts by 0 is called its extension to \mathcal{H} and denoted by $e_{\mathcal{H}}(\mathcal{V})$ or $e(\mathcal{V})$.

With the element basis T_w^* , \mathcal{V} is extensible to \mathcal{H} if and only if

$$(3.11) \quad v T_{\lambda_2}^{M,*} = v \text{ for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1).$$

The \mathcal{H} -module structure on the R -module $e(\mathcal{V}) = \mathcal{V}$ is determined by

$$(3.12) \quad v T_{w_2}^* = v, \quad v T_w^* = v T_w^{M,*} \text{ for all } v \in \mathcal{V}, w_2 \in {}_1W_{M'_2}, w \in W_M(1).$$

It is also determined by the action of T_w^* for $w \in {}_1W_{M'_2} \cup W_{M^+}(1)$ (or $w \in {}_1W_{M'_2} \cup W_{M^-}(1)$). Conversely, a right \mathcal{H} -module \mathcal{W} over R is extended from an \mathcal{H}_M -module if and only if

$$(3.13) \quad vT_{w_2}^* = v \quad \text{for all } v \in \mathcal{W}, w_2 \in {}_1W_{M'_2}.$$

In terms of the basis elements T_w instead of T_w^* , this says the following.

Corollary 3.9. *A right \mathcal{H}_M -module \mathcal{V} over R is extensible to \mathcal{H} if and only if*

$$(3.14) \quad vT_{\lambda_2}^M = v \quad \text{for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1).$$

Then, the structure of an \mathcal{H} -module on the R -module $e(\mathcal{V}) = \mathcal{V}$ is determined by

$$(3.15) \quad vT_{w_2} = vq_{w_2}, \quad vT_w = vT_w^M q_{M_2}(w) \quad \text{for all } v \in \mathcal{V}, w_2 \in {}_1W_{M'_2}, w \in W_M(1).$$

($W_{M^+}(1)$ or $W_{M^-}(1)$ instead of $W_M(1)$ is enough.) A right \mathcal{H} -module \mathcal{W} over R is extended from an \mathcal{H}_M -module if and only if

$$(3.16) \quad vT_{w_2} = vq_{w_2} \quad \text{for all } v \in \mathcal{W}, w_2 \in {}_1W_{M'_2}.$$

3.4. $\sigma^{\mathcal{U}_M}$ is extensible to \mathcal{H} of extension $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$. Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal, and let σ be a smooth R -representation of M extensible to G . Let $P_2 = M_2N_2$ denote the standard parabolic subgroup of G with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

Recall that $G = MM'_2$, that $M \cap M'_2 = Z \cap M'_2$ acts trivially on σ , $e(\sigma)$ is the representation of G equal to σ on M and trivial on M'_2 . We will describe the \mathcal{H} -module $e(\sigma)^{\mathcal{U}}$ in this section. We first consider $e(\sigma)$ as a subrepresentation of $\text{Ind}_P^G \sigma$. For $v \in \sigma$, let $f_v \in (\text{Ind}_P^G \sigma)^{M'_2}$ be the unique function with value v on M'_2 . Then, the map

$$(3.17) \quad v \mapsto f_v : \sigma \rightarrow \text{Ind}_P^G \sigma$$

is the natural G -equivariant embedding of $e(\sigma)$ in $\text{Ind}_P^G \sigma$. As $\sigma^{\mathcal{U}_M} = e(\sigma)^{\mathcal{U}}$ as R -modules, the image of $e(\sigma)^{\mathcal{U}}$ in $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$ is made out of the f_v for $v \in \sigma^{\mathcal{U}_M}$.

We now recall the explicit description of $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$. For each $d \in \mathbb{W}_{M_2}$, we fix a lift $\hat{d} \in {}_1W_{M'_2}$ and for $v \in \sigma^{\mathcal{U}_M}$ let $f_{P\hat{d}\mathcal{U},v} \in (\text{Ind}_P^G \sigma)^{\mathcal{U}}$ for the function with support contained in $P\hat{d}\mathcal{U}$ and value v on $\hat{d}\mathcal{U}$. As $Z \cap M'_2$ acts trivially on σ , the function $f_{P\hat{d}\mathcal{U},v}$ does not depend on the choice of the lift $\hat{d} \in {}_1W_{M'_2}$ of d . By [OV17, Lemma 4.5], recalling that $w \in \mathbb{W}_{M_2}$ is of minimal length in its coset $w\mathbb{W}_M = \mathbb{W}_M w$ as Δ_M and Δ_{M_2} are orthogonal to each other:

The map $\bigoplus_{d \in \mathbb{W}_{M_2}} \sigma^{\mathcal{U}_M} \rightarrow (\text{Ind}_P^G \sigma)^{\mathcal{U}}$ given on each d -component by $v \mapsto f_{P\hat{d}\mathcal{U},v}$, is an \mathcal{H}_{M^+} -equivariant isomorphism where \mathcal{H}_{M^+} is seen as a subring of \mathcal{H} via θ , and induces an \mathcal{H}_R -module isomorphism

$$(3.18) \quad v \otimes h \mapsto f_{P\mathcal{U},v}h : \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H} \rightarrow (\text{Ind}_P^G \sigma)^{\mathcal{U}}.$$

In particular for $v \in \sigma^{\mathcal{U}_M}$, $v \otimes T(\hat{d})$ does not depend on the choice of the lift $\hat{d} \in {}_1W_{M'_2}$ of d and

$$(3.19) \quad f_{P\hat{d}\mathcal{U},v} = f_{P\mathcal{U},v}T(\hat{d}).$$

As G is the disjoint union of $P\hat{d}\mathcal{U}$ for $d \in \mathbb{W}_{M_2}$, we have $f_v = \sum_{d \in \mathbb{W}_{M_2}} f_{P\hat{d}\mathcal{U}, v}$ and f_v is the image of $v \otimes e_{M_2}$ in (3.18), where

$$(3.20) \quad e_{M_2} = \sum_{d \in \mathbb{W}_{M_2}} T(\hat{d}).$$

Recalling (3.17) we get the following.

Lemma 3.10. *The map $v \mapsto v \otimes e_{M_2} : e(\sigma)^{\mathcal{U}} \rightarrow \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H}$ is an \mathcal{H}_R -equivariant embedding.*

Remark 3.11. The trivial map $v \mapsto v \otimes 1_{\mathcal{H}}$ is not an \mathcal{H}_R -equivariant embedding.

We describe the action of $T(n)$ on $e(\sigma)^{\mathcal{U}}$ for $n \in \mathcal{N}$. By definition for $v \in e(\sigma)^{\mathcal{U}}$,

$$(3.21) \quad vT(n) = \sum_{y \in \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)} yn^{-1}v.$$

Proposition 3.12. *We have $vT(n) = vT^M(n_M)q_{M_2}(n)$ for any $n_M \in \mathcal{N} \cap M$ is such that $n = n_M(\mathcal{N} \cap M'_2)$.*

Proof. The description (3.10) of $\mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)$ gives

$$vT(n) = \sum_{y_1 \in \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_M n)} y_1 \sum_{y_2 \in \mathcal{U}_{M'_2}/(\mathcal{U}_{M'_2} \cap n^{-1}\mathcal{U}_{M'_2} n)} y_2 n^{-1}v.$$

As M'_2 acts trivially on $e(\sigma)$, we obtain

$$vT(n) = q_{M_2}(n) \sum_{y_1 \in \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_M n)} y_1 n_M^{-1}v = q_{M_2}(n) vT^M(n_M).$$

□

Theorem 3.13. *Let σ be a smooth R -representation of M . If $P(\sigma) = G$, then $\sigma^{\mathcal{U}_M}$ is extensible to \mathcal{H} of extension $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$. Conversely, if $\sigma^{\mathcal{U}_M}$ is extensible to \mathcal{H} and generates σ , then $P(\sigma) = G$.*

Proof.

(1) The \mathcal{H}_M -module $\sigma^{\mathcal{U}_M}$ is extensible to \mathcal{H} if and only if $Z \cap M'_2$ acts trivially on $\sigma^{\mathcal{U}_M}$. Indeed, for $v \in \sigma^{\mathcal{U}_M}$, $z_2 \in Z \cap M'_2$,

$$vT^M(z_2) = \sum_{y \in \mathcal{U}_M/(\mathcal{U}_M \cap z_2^{-1}\mathcal{U}_M z_2)} yz_2^{-1}v = \sum_{y \in \mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap z_2^{-1}\mathcal{Y}_{M'} z_2)} yz_2^{-1}v = z_2^{-1}v,$$

by (3.21), then (3.8), then the fact that z_2^{-1} commutes with the elements of \mathcal{Y}_M .

(2) $P(\sigma) = G$ if and only if $Z \cap M'_2$ acts trivially on σ (the group $Z \cap M'_2$ is generated by $Z \cap M'_\alpha$ for $\alpha \in \Delta_{M_2}$ by Lemma 2.1). The R -submodule $\sigma^{Z \cap M'_2}$ of elements fixed by $Z \cap M'_2$ is stable by M , because $M = ZM'$, the elements of M' commute with those of $Z \cap M'_2$ and Z normalizes $Z \cap M'_2$.

(3) Apply (1) and (2) to get the theorem except the equality $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$ when $P(\sigma) = G$ which follows from Propositions 3.12 and 3.7. □

Let $\mathbf{1}_M$ denote the trivial representation of M over R (or $\mathbf{1}$ when there is no ambiguity on M). The right \mathcal{H}_R -module $(\mathbf{1}_G)^{\mathcal{U}} = \mathbf{1}_{\mathcal{H}}$ (or $\mathbf{1}$ if there is no ambiguity) is the trivial right \mathcal{H}_R -module: for $w \in W_M(1)$, $T_w = q_w \text{id}$ and $T_w^* = \text{id}$ on $\mathbf{1}_{\mathcal{H}}$.

Example 3.14. The \mathcal{H} -module $(\text{Ind}_P^G \mathbf{1})^\mathcal{U}$ is the extension of the \mathcal{H}_{M_2} -module $(\text{Ind}_{M_2 \cap B}^{M_2} \mathbf{1})^{\mathcal{U}_{M_2}}$. Indeed, the representation $\text{Ind}_P^G \mathbf{1}$ of G is trivial on N_2 , as $G = MM'_2$ and $N_2 \subset M'$ (as $\Phi = \Phi_M \cup \Phi_{M_2}$). For $g = mm'_2$ with $m \in M, m'_2 \in M'_2$ and $n_2 \in N_2$, we have $Pgn_2 = Pm'_2n_2 = Pn_2m'_2 = Pm'_2 = Pg$. The group $M_2 \cap B = M_2 \cap P$ is the standard minimal parabolic subgroup of M_2 and $(\text{Ind}_P^G \mathbf{1})|_{M_2} = \text{Ind}_{M_2 \cap B}^{M_2} \mathbf{1}$. Apply Theorem 3.13 as follows.

3.5. The \mathcal{H}_R -module $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$. Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal, let \mathcal{V} be a right $\mathcal{H}_{M,R}$ -module which is extensible to \mathcal{H}_R of extension $e(\mathcal{V})$, and let Q be a parabolic subgroup of G containing P . Let $P_2 = M_2N_2$ denote the standard parabolic subgroup of G with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

We define on the R -module $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ a structure of a right \mathcal{H}_R -module as follows.

Proposition 3.15.

- (1) *The diagonal action of T_w^* for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ defines a structure of a right \mathcal{H}_R -module.*
- (2) *The action of the T_w is also diagonal and satisfies:*

$$((v \otimes f)T_w, (v \otimes f)T_w^*) = (vT_{uw_{M'}}, \otimes fT_{uw_{M'_2}}, vT_{uw_{M'}}^* \otimes fT_{uw_{M'_2}}^*),$$

where $w = uw_{M'}w_{M'_2}$ with $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M'_2} \in {}_1W_{M'_2}$.

Proof. If the lemma is true for P it is also true for Q , because the R -module $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ naturally embedded in $e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$ is stable by the action of \mathcal{H} defined in the lemma. So, we suppose $Q = P$.

For each element in ${}_1S^{\text{aff}}$ we fix an admissible lift and denote the set of admissible lifts by ${}_1S^{\text{aff}}$. We also use the obvious notation ${}_1S_M^{\text{aff}}$ and ${}_1S_{M_2}^{\text{aff}}$. Suppose that T_w^* for $w \in W(1)$ acts on $e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$ as in (1). The braid relations obviously hold. The quadratic relations hold because T_s^* with $s \in {}_1S^{\text{aff}}$, acts trivially either on $e(\mathcal{V})$ or on $(\text{Ind}_P^G \mathbf{1})^\mathcal{U}$. Indeed, ${}_1S^{\text{aff}} = {}_1S_M^{\text{aff}} \cup {}_1S_{M_2}^{\text{aff}}$, T_s^* for $s \in {}_1S_M^{\text{aff}}$, acts trivially on $(\text{Ind}_P^G \mathbf{1})^\mathcal{U}$ which is extended from an \mathcal{H}_{M_2} -module (Example 3.14), and T_s^* for $s \in {}_1S_{M_2}^{\text{aff}}$, acts trivially on $e(\mathcal{V})$ which is extended from an \mathcal{H}_M -module. This proves (1).

We describe now the action of T_w instead of T_w^* on the \mathcal{H} -module $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$. Let $w \in W(1)$. We write $w = uw_{M'}w_{M'_2} = uw_{M'_2}w_{M'}$ with $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M'_2} \in {}_1W_{M'_2}$. We have $\ell(w) = \ell(w_{M'}) + \ell(w_{M'_2})$ hence $T_w = T_uT_{w_{M'}}T_{w_{M'_2}}$.

For $w = u$, we have $T_u = T_u^*$ and $(v \otimes f)T_u = (v \otimes f)T_u^* = vT_u^* \otimes fT_u^* = vT_u \otimes fT_u$.

For $w = w_{M'}$, $(v \otimes f)T_w^* = vT_w^* \otimes f$; for $s \in {}_1S_M^{\text{aff}}$, $c_s = \sum_{t \in Z_k \cap {}_1W_{M'}} c_s(t)T_t^*$ in particular, we have $(v \otimes f)T_s = (v \otimes f)(T_s^* + c_s) = v(T_s^* + c_s) \otimes f = vT_s \otimes f$. Hence $(v \otimes f)T_w = vT_w \otimes f$.

For $w = w_{M'_2}$, we have similarly $(v \otimes f)T_w^* = v \otimes fT_w^*$ and $(v \otimes f)T_w = v \otimes fT_w$. \square

Example 3.16. Let \mathcal{X} be a right \mathcal{H}_R -module. Then $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$ where the T_w^* acts diagonally is an \mathcal{H}_R -module isomorphic to \mathcal{X} . But the action of the T_w on $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$ is not diagonal.

It is known [Ly15] that $(\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ and $(\text{St}_Q^G)^\mathcal{U}$ are free R -modules and that $(\text{St}_Q^G)^\mathcal{U}$ is the cokernel of the natural \mathcal{H}_R -map

$$(3.22) \quad \bigoplus_{Q \subsetneq Q'} (\text{Ind}_{Q'}^G \mathbf{1})^\mathcal{U} \rightarrow (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$$

although the invariant functor $(-)^\mathcal{U}$ is only left exact.

Corollary 3.17. *The diagonal action of T_w^* for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}$ defines a structure of a right \mathcal{H}_R -module satisfying Proposition 3.15(2).*

4. HECKE MODULE $I_{\mathcal{H}}(P, \mathcal{V}, Q)$

4.1. Case \mathcal{V} extensible to \mathcal{H} . Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal, let \mathcal{V} be a right $\mathcal{H}_{M,R}$ -module extensible to \mathcal{H}_R of extension $e(\mathcal{V})$, and let Q be a parabolic subgroup of G containing P . As Q and M_Q determine each other: $Q = M_Q U$, we denote also $\mathcal{H}_{M_Q} = \mathcal{H}_Q$ and $\mathcal{H}_{M_Q,R} = \mathcal{H}_{Q,R}$ when $Q \neq P, G$. When $Q = G$ we drop G and we denote $e_{\mathcal{H}}(\mathcal{V}) = e(\mathcal{V})$.

Lemma 4.1. *\mathcal{V} is extensible to an $\mathcal{H}_{Q,R}$ -module $e_{\mathcal{H}_Q}(\mathcal{V})$.*

Proof. This is straightforward. By Corollary 3.9, \mathcal{V} extensible to \mathcal{H} means that $T^M(z)$ acts trivially on \mathcal{V} for all $z \in \mathcal{N}_{M'_2} \cap Z$. We have $M_Q = M M'_{2,Q}$ with $M'_{2,Q} \subset M_Q \cap M'_2$ and $\mathcal{N}_{M'_{2,Q}} \subset \mathcal{N}_{M'_2}$; hence $T^M(z)$ acts trivially on \mathcal{V} for all $z \in \mathcal{N}_{M'_{2,Q}} \cap Z$ meaning that \mathcal{V} is extensible to \mathcal{H}_Q . \square

Remark 4.2. We cannot say that $e_{\mathcal{H}_Q}(\mathcal{V})$ is extensible to \mathcal{H} of extension $e(\mathcal{V})$ when the set of roots Δ_Q and $\Delta \setminus \Delta_Q$ are not orthogonal (Definition 3.8).

Let Q' be an arbitrary parabolic subgroup of G containing Q . We are going to define an \mathcal{H}_R -embedding $\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H}$ defining an \mathcal{H}_R -homomorphism

$$\bigoplus_{Q \subsetneq Q' \subset G} \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

of cokernel isomorphic to $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}$. In the extreme case $(Q, Q') = (P, G)$, the \mathcal{H}_R -embedding $e(\mathcal{V}) \xrightarrow{\iota(P,G)} \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$ is given in the following lemma where f_G and $f_{PU} \in (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$ denote the characteristic functions of G and PU , $f_G = f_{PU} e_{M_2}$ (see (3.20)).

Lemma 4.3. *There is a natural \mathcal{H}_R -isomorphism*

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{PU} : \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \mathcal{V} \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U},$$

and compatible \mathcal{H}_R -embeddings

$$(4.1) \quad v \mapsto v \otimes f_G : e(\mathcal{V}) \rightarrow e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U},$$

$$(4.2) \quad v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P,G)} \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}).$$

Proof. We show first that the map

$$(4.3) \quad v \mapsto v \otimes f_{PU} : \mathcal{V} \rightarrow e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$$

is \mathcal{H}_{M^+} -equivariant. Let $w \in W_{M^+}(1)$. We write $w = uw_{M'}w_{M'_2}$ as in Proposition 3.15 (2), so that $f_{PU}T_w = f_{PU}T_{uw_{M'_2}}$. We have $f_{PU}T_{uw_{M'_2}} = f_{PU}$ because ${}_1W_{M'} \subset W_{M^+}(1) \cap W_{M^-}(1)$ hence $uw_{M'_2} = ww_{M'}^{-1} \in W_{M^+}(1)$ and in $\mathbf{1}_{\mathcal{H}_M} \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H}$ we have $(1 \otimes \mathbf{1}_{\mathcal{H}})T_{uw_{M'_2}} = 1T_{uw_{M'_2}}^M \otimes \mathbf{1}_{\mathcal{H}}$, and $T_{uw_{M'_2}}^M$ acts trivially in $\mathbf{1}_{\mathcal{H}_M}$ because $\ell_M(uw_{M'_2}) = 0$. We deduce $(v \otimes f_{PU})T_w = vT_w \otimes f_{PU}T_w = vT_w^M \otimes f_{PU}$.

By adjunction (4.3) gives an \mathcal{H}_R -equivariant linear map

$$(4.4) \quad v \otimes \mathbf{1}_{\mathcal{H}} \mapsto v \otimes f_{PU} : \mathcal{V} \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}.$$

We prove that κ_P is an isomorphism. Recalling $\hat{d} \in \mathcal{N} \cap M'_2, \tilde{d} \in {}_1W_{M'_2}$ lift d , one knows that

$$(4.5) \quad \mathcal{V} \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes T_{\tilde{d}}, \quad e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes f_{P\hat{d}\mathcal{U}},$$

where each summand is isomorphic to \mathcal{V} . The left equality follows from section 4.1 and Remark 3.7 in [Vig15b] recalling that $w \in \mathbb{W}_{M_2}$ is of minimal length in its coset $\mathbb{W}_M w = w\mathbb{W}_M$ as Δ_M and Δ_{M_2} are orthogonal; for the second equality see section 3.4 (3.19). We have $\kappa_P(v \otimes T_{\tilde{d}}) = (v \otimes f_{PU})T_{\tilde{d}} = v \otimes f_{PU}T_{\tilde{d}}$ (Proposition 3.15). Hence κ_P is an isomorphism.

We consider the composite map

$$v \mapsto v \otimes \mathbf{1} \mapsto v \otimes f_{PU}e_{M_2} : e(\mathcal{V}) \rightarrow e(\mathcal{V}) \otimes_R \mathbf{1}_{\mathcal{H}} \rightarrow e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}},$$

where the right map is the tensor product $e(\mathcal{V}) \otimes_R -$ of the \mathcal{H}_R -equivariant embedding $\mathbf{1}_{\mathcal{H}} \rightarrow (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ sending $\mathbf{1}_R$ to $f_{PU}e_{M_2}$ (Lemma 3.10); this map is injective because $(\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}/\mathbf{1}$ is a free R -module; it is \mathcal{H}_R -equivariant for the diagonal action of the T_w^* on the tensor products (Example 3.16 for the first map). By compatibility with (4.4), we get the \mathcal{H}_R -equivariant embedding $v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P, G)} \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$. \square

For a general (Q, Q') the \mathcal{H}_R -embedding $\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q, Q')} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$ is given in the next proposition generalizing Lemma 4.3. The element e_{M_2} of \mathcal{H}_R appearing in the definition of $\iota(P, G)$ is replaced in the definition of $\iota(Q, Q')$ by an element $\theta_{Q'}(e_{Q'}^{Q'}) \in \mathcal{H}_R$ that we define first.

Until the end of section 4, we fix an admissible lift $w \mapsto \hat{w} : \mathbb{W} \rightarrow \mathcal{N} \cap \mathcal{K}$ (Definition 3.1) and \tilde{w} denotes the image of \hat{w} in $W(1)$. We denote $\mathbb{W}_{M_Q} = \mathbb{W}_Q$ and by ${}^{\mathbb{W}_Q}\mathbb{W}$ the set of $w \in \mathbb{W}$ of minimal length in their coset $\mathbb{W}_Q w$. The group G is the disjoint union of $Q\hat{d}\mathcal{U}$ for d running through ${}^{\mathbb{W}_Q}\mathbb{W}$ [OV17, Lemma 2.15 (2)]: $G = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}} Q\hat{d}\mathcal{U}$. Since $Q\hat{d}\mathcal{U} \subset Q'\mathcal{U}$ if and only if $\hat{d} \in Q'$, namely $d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}$, we have

$$(4.6) \quad Q'\mathcal{U} = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} Q\hat{d}\mathcal{U}.$$

Set

$$(4.7) \quad e_Q^{Q'} = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} T_{\hat{d}}^{M_{Q'}}.$$

We write $e_P^G = e_Q$. We have $e_P^Q = \sum_{d \in \mathbb{W}_{M_2, Q}} T_{\hat{d}}^{M_Q}$.

Remark 4.4. Note that ${}^{\mathbb{W}^M}\mathbb{W} = \mathbb{W}_{M_2}$ and $e_P = e_{M_2}$, where M_2 is the standard Levi subgroup of G with $\Delta_{M_2} = \Delta \setminus \Delta_M$, as Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. More generally, ${}^{\mathbb{W}^Q}\mathbb{W}_{M_{Q'}} = {}^{\mathbb{W}^{M_2, Q}}\mathbb{W}_{M_2, Q'}$ where $M_{2, Q'} = M_2 \cap M_{Q'}$.

Note that $e_Q^{Q'} \in \mathcal{H}_{M^+} \cap \mathcal{H}_{M^-}$. We consider the linear map

$$\theta_Q^{Q'} : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'} \quad T_w^{M_Q} \mapsto T_w^{M_{Q'}} \quad (w \in W_{M_Q}(1)).$$

We write $\theta_Q^G = \theta_Q$ so that $\theta_Q(T_w^{M_Q}) = T_w$. When $Q = P$ this is the map θ defined earlier. Similarly we denote by $\theta_Q^{Q', *}$ the linear map sending the $T_w^{M_Q, *}$ to $T_w^{M_{Q'}, *}$ and $\theta_Q^{G, *} = \theta_Q^*$. We have

$$(4.8) \quad \theta_{Q'}(e_Q^{Q'}) = \sum_{d \in {}^{\mathbb{W}^Q}\mathbb{W}_{Q'}} T_{\tilde{d}}, \quad \theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q)\theta_{Q'}(e_Q^{Q'}).$$

Proposition 4.5. *There exists an \mathcal{H}_R -isomorphism*

$$(4.9) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{QU} : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H} \xrightarrow{\kappa_Q} e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}},$$

and compatible \mathcal{H}_R -embeddings

$$(4.10) \quad v \otimes f_{Q'U} \mapsto v \otimes f_{Q'U} : e_{\mathcal{H}_{Q'}}(\mathcal{V}) \otimes_R (\text{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \rightarrow e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}},$$

$$(4.11) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}(e_Q^{Q'}) : \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q, Q')} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})).$$

Proof. We have the $\mathcal{H}_{M_Q, R}$ -embedding

$$v \mapsto v \otimes e_P^Q : e_{\mathcal{H}_Q}(\mathcal{V}) \rightarrow \mathcal{V} \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H}_Q = \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})$$

by Lemma 4.3 (4.2) as Δ_M is orthogonal to $\Delta_{M_Q} \setminus \Delta_M$. Applying the parabolic induction which is exact, we get the \mathcal{H} -embedding

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes e_P^Q \otimes 1_{\mathcal{H}} : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})).$$

Note that $T_{\tilde{d}}^{M_Q} \in \mathcal{H}_{M_Q^+}$ for $d \in \mathbb{W}_{M_Q}$. By transitivity of the parabolic induction, it is equal to the \mathcal{H}_R -embedding

$$(4.12) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_Q(e_P^Q) : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}).$$

On the other hand we have the \mathcal{H}_R -embedding

$$(4.13) \quad v \otimes f_{QU} \mapsto v \otimes \theta_Q(e_P^Q) : e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \rightarrow \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$$

given by the restriction to $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ of the \mathcal{H}_R -isomorphism given in Lemma 4.3 (4.1), from $e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ to $\mathcal{V} \otimes_{\mathcal{H}_{M^+, \theta}} \mathcal{H}$ sending $v \otimes f_{PU}$ to $v \otimes 1_{\mathcal{H}}$, noting that $v \otimes f_{QU} = (v \otimes f_{PU})\theta_Q(e_P^Q)$ by Proposition 3.15, $f_{QU} = f_{PU}\theta_Q(e_P^Q)$ and $\theta_Q(e_P^Q)$ acts trivially on $e(\mathcal{V})$ (this is true for $T_{\tilde{d}}$ for $\tilde{d} \in {}_1W_{M_2}$). Comparing the embeddings (4.12) and (4.13), we get the \mathcal{H}_R -isomorphism (4.9).

We can replace Q by Q' in the \mathcal{H}_R -homomorphisms (4.9), (4.12), and (4.13). With (4.12) we see $\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ and $\text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$ as \mathcal{H}_R -submodules of $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$. As seen in (4.8) we have $\theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q)\theta_{Q'}(e_Q^{Q'})$. We deduce the \mathcal{H}_R -embedding (4.11).

By (3.19) for Q and (4.6),

$$f_{Q'}u = \sum_{d \in {}^{\mathbb{W}Q}\mathbb{W}_{Q'}} f_{Qu}T_{\bar{d}} = f_{Qu}\theta_{Q'}(e_{Q'}^{Q'})$$

in $(\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$. We deduce that the \mathcal{H}_R -embedding corresponding to (4.11) via κ_Q and $\kappa_{Q'}$ is the \mathcal{H}_R -embedding (4.10). \square

We recall that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal and that \mathcal{V} is extensible to \mathcal{H} of extension $e(\mathcal{V})$.

Corollary 4.6. *The cokernel of the \mathcal{H}_R -map*

$$\bigoplus_{Q \subseteq Q' \subset G} \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the $\iota(Q, Q')$, is isomorphic to $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$ via κ_Q .

4.2. Invariants in the tensor product. We return to the setting where $P = MN$ is a standard parabolic subgroup of G , σ is a smooth R -representation of M with $P(\sigma) = G$ of extension $e(\sigma)$ to G , and Q a parabolic subgroup of G containing P . We still assume that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal.

The \mathcal{H}_R -modules $e(\sigma^{\mathcal{U}M}) = e(\sigma)^{\mathcal{U}}$ are equal (Theorem 3.13). We compute $I_G(P, \sigma, Q)^{\mathcal{U}} = (e(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}}$.

Theorem 4.7. *The natural linear maps $e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \rightarrow (e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ and $e(\sigma)^{\mathcal{U}} \otimes_R (\text{St}_Q^G)^{\mathcal{U}} \rightarrow (e(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}}$ are isomorphisms.*

Proof. We need some preliminaries. In [GK14, Ly15], are introduced a finite free \mathbb{Z} -module \mathfrak{M} (depending on Δ_Q) and a \mathcal{B} -equivariant embedding $\text{St}_Q^G \mathbb{Z} \xrightarrow{\iota} C_c^\infty(\mathcal{B}, \mathfrak{M})$ (we indicate the coefficient ring in the Steinberg representation) which induces an isomorphism $(\text{St}_Q^G \mathbb{Z})^{\mathcal{B}} \simeq C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$.

Lemma 4.8.

- (1) $(\text{Ind}_Q^G \mathbb{Z})^{\mathcal{B}}$ is a direct factor of $\text{Ind}_Q^G \mathbb{Z}$.
- (2) $(\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}$ is a direct factor of $\text{St}_Q^G \mathbb{Z}$.

Proof.

(1) [AHV, Example 2.2].

(2) As \mathfrak{M} is a free \mathbb{Z} -module, $C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$ is a direct factor of $C_c^\infty(\mathcal{B}, \mathfrak{M})$. Consequently, $\iota((\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}) = C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$ is a direct factor of $\iota(\text{St}_Q^G \mathbb{Z})$. As ι is injective, we get (2). \square

We now prove Theorem 4.7. We may and do assume that σ is e -minimal (because $P(\sigma) = P(\sigma_{\min})$, $e(\sigma) = e(\sigma_{\min})$) so that Δ_M and $\Delta \setminus \Delta_M$ are orthogonal and we use the same notation as in section 3.2 in particular $M_2 = M_{\Delta \setminus \Delta_M}$. Let V be the space of $e(\sigma)$ on which M_2' acts trivially. The restriction of $\text{Ind}_Q^G \mathbb{Z}$ to M_2 is $\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z}$, that of $\text{St}_Q^G \mathbb{Z}$ is $\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z}$.

As in [AHV, Example 2.2], $((\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z}) \otimes V)^{\mathcal{U}_{M_2}'} \simeq (\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}'} \otimes V$. We have

$$(\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}'} = (\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}} = (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}}.$$

The first equality follows from $M_2 = (Q \cap M_2)\mathbb{W}_{M_2}\mathcal{U}_{M_2}$, $\mathcal{U}_{M_2} = Z^1\mathcal{U}_{M_2}'$ and Z^1 normalizes \mathcal{U}_{M_2}' and is normalized by \mathbb{W}_{M_2} . The second equality follows from $\mathcal{U} =$

$\mathcal{U}_M \mathcal{U}_{M_2}$ and $\text{Ind}_Q^G \mathbb{Z}$ is trivial on M' . Therefore $((\text{Ind}_Q^G \mathbb{Z}) \otimes V)^{\mathcal{U}_{M'_2}} \simeq (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V$. Now taking fixed points under \mathcal{U}_M , as $\mathcal{U} = \mathcal{U}_{M'_2} \mathcal{U}_M$,

$$((\text{Ind}_Q^G \mathbb{Z}) \otimes V)^{\mathcal{U}} \simeq ((\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V)^{\mathcal{U}_M} = (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}_M}.$$

The equality uses that the \mathbb{Z} -module $\text{Ind}_Q^G \mathbb{Z}$ is free. We get the first part of the theorem as $(\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}_M} \simeq (\text{Ind}_Q^G R)^{\mathcal{U}} \otimes_R V^{\mathcal{U}_M}$.

Tensoring with R the usual exact sequence defining $\text{St}_Q^G \mathbb{Z}$ gives an isomorphism $\text{St}_Q^G \mathbb{Z} \otimes R \simeq \text{St}_Q^G R$ and in [GK14, Ly15], it is proved that the resulting map $\text{St}_Q^G R \xrightarrow{\iota_R} C^\infty(\mathcal{B}, \mathfrak{M} \otimes R)$ is also injective. Their proof in no way uses the ring structure of R , and for any \mathbb{Z} -module V , tensoring with V gives a \mathcal{B} -equivariant embedding $\text{St}_Q^G \mathbb{Z} \otimes V \xrightarrow{\iota_V} C_c^\infty(\mathcal{B}, \mathfrak{M} \otimes V)$. The natural map $(\text{St}_Q^G \mathbb{Z})^{\mathcal{B}} \otimes V \rightarrow \text{St}_Q^G \mathbb{Z} \otimes V$ is also injective by Lemma 4.8 (2). Taking \mathcal{B} -fixed points we get inclusions

$$(4.14) \quad (\text{St}_Q^G \mathbb{Z})^{\mathcal{B}} \otimes V \rightarrow (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{B}} \rightarrow C_c^\infty(\mathcal{B}, \mathfrak{M} \otimes V)^{\mathcal{B}} \simeq \mathfrak{M} \otimes V.$$

The composite map is surjective, so the inclusions are isomorphisms. The image of ι_V consists of functions which are left Z^0 -invariant, and $\mathcal{B} = Z^0 \mathcal{U}'$ where $\mathcal{U}' = G' \cap \mathcal{U}$. It follows that ι yields an isomorphism $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} \simeq C_c^\infty(Z^0 \backslash \mathcal{B}, \mathfrak{M})^{\mathcal{U}'}$ again consisting of the constant functions. So that in particular $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} = (\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}$ and reasoning as previously we get isomorphisms

$$(4.15) \quad (\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} \otimes V \simeq (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{U}'} \simeq \mathfrak{M} \otimes V.$$

The equality $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} = (\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}$ and the isomorphisms remain true when we replace \mathcal{U}' by any group between \mathcal{B} and \mathcal{U}' . We apply these results to $\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z} \otimes V$ to get that the natural map $(\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M'_2}} \otimes V \rightarrow (\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z} \otimes V)^{\mathcal{U}_{M'_2}}$ is an isomorphism and also that $(\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M'_2}} = (\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}}$. We have $\mathcal{U} = \mathcal{U}_M \mathcal{U}_{M_2}$ so $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} = (\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}}$ and the natural map $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V \rightarrow (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{U}_{M'_2}}$ is an isomorphism. The \mathbb{Z} -module $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}}$ is free and the $V^{\mathcal{U}_M} = V^{\mathcal{U}}$, so taking fixed points under \mathcal{U}_M , we get $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}} \simeq (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{U}}$. We have $\text{St}_Q^G \mathbb{Z} \otimes V = \text{St}_Q^G R \otimes_R V$ and $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}} = (\text{St}_Q^G R)^{\mathcal{U}} \otimes_R V^{\mathcal{U}}$. This ends the proof of the theorem. \square

Theorem 4.9. *The \mathcal{H}_R -modules $(e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} = e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ are equal. The \mathcal{H}_R -modules $(e(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}} = e(\sigma)^{\mathcal{U}} \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$ are also equal.*

Proof. We already know that the R -modules are equal (Theorem 4.7). We show that they are equal as \mathcal{H} -modules. The \mathcal{H}_R -modules $e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} = e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ are equal (Theorem 3.13), they are isomorphic to $\text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\sigma^{\mathcal{U}_M}))$ (Proposition 4.5), to $(\text{Ind}_Q^G(e_Q(\sigma)))^{\mathcal{U}}$ [OV17, Proposition 4.4], and to $(e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ [AHV, Lemma 2.5]). We deduce that the \mathcal{H}_R -modules $e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} = (e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ are equal. The same is true when Q is replaced by a parabolic subgroup Q' of G containing Q . The representation $e(\sigma) \otimes_R \text{St}_Q^G$ is the cokernel of the natural $R[G]$ -map

$$\bigoplus_{Q \subsetneq Q'} e(\sigma) \otimes_R \text{Ind}_{Q'}^G \mathbf{1} \xrightarrow{\alpha_Q} e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$$

and the \mathcal{H}_R -module $e(\sigma)^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U}$ is the cokernel of the natural \mathcal{H}_R -map

$$\bigoplus_{Q \subsetneq Q'} e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}_{Q'}^G \mathbf{1})^\mathcal{U} \xrightarrow{\beta_Q} e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$$

obtained by tensoring (3.22) by $e(\sigma)^\mathcal{U}$ over R , because the tensor product is right exact. The maps $\beta_Q = \alpha_Q^\mathcal{U}$ are equal and the R -modules $e(\sigma)^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U} = (e(\sigma) \otimes_R \text{St}_Q^G)^\mathcal{U}$ are equal. This implies that the \mathcal{H}_R -modules $e(\sigma)^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U} = (e(\sigma) \otimes_R \text{St}_Q^G)^\mathcal{U}$ are equal. \square

Remark 4.10. The proof shows that the representations $e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$ and $e(\sigma) \otimes \text{St}_Q^G$ of G are generated by their \mathcal{U} -fixed vectors if the representation σ of M is generated by its \mathcal{U}_M -fixed vectors. Indeed, the R -modules $e(\sigma)^\mathcal{U} = \sigma^{\mathcal{U}_M}, (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}_{M_2}} = (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ are equal. If $\sigma^{\mathcal{U}_M}$ generates σ , then $e(\sigma)$ is generated by $e(\sigma)^\mathcal{U}$. The representation $\text{Ind}_Q^G \mathbf{1}|_{M_2}$ is generated by $(\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ (this follows from the lemma below), we have $G = MM_2$ and M_2 acts trivially on $e(\sigma)$. Therefore the $R[G]$ -module generated by $\sigma^\mathcal{U} \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ is $e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$. As $e(\sigma) \otimes_R \text{St}_Q^G$ is a quotient of $e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$, the $R[G]$ -module generated by $\sigma^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U}$ is $e(\sigma) \otimes_R \text{St}_Q^G$.

Lemma 4.11. *For any standard parabolic subgroup P of G , the representation $\text{Ind}_P^G \mathbf{1}|_{G'}$ is generated by its \mathcal{U} -fixed vectors.*

Proof. Because $G = PG'$ it suffices to prove that if J is an open compact subgroup of \overline{N} the characteristic function 1_{PJ} of PJ is a finite sum of translates of $1_{PU} = 1_{P\mathcal{U}_{\overline{N}}}$ by G' . For $t \in T$ we have $P\mathcal{U}t = Pt^{-1}\mathcal{U}_{\overline{N}}t$ and we can choose $t \in T \cap J'$ such that $t^{-1}\mathcal{U}_{\overline{N}}t \subset J$. \square

4.3. General triples. Let $P = MN$ be a standard parabolic subgroup of G . We now investigate situations where Δ_P and $\Delta \setminus \Delta_P$ are not necessarily orthogonal. Let \mathcal{V} be a right $\mathcal{H}_{M,R}$ -module.

Definition 4.12. Let $P(\mathcal{V}) = M(\mathcal{V})N(\mathcal{V})$ be the standard parabolic subgroup of G with $\Delta_{P(\mathcal{V})} = \Delta_P \cup \Delta_{\mathcal{V}}$ and

$$\Delta_{\mathcal{V}} = \{\alpha \in \Delta \text{ orthogonal to } \Delta_M, T^M(z) \text{ acts trivially on } \mathcal{V} \text{ for all } z \in Z \cap M'_\alpha\}.$$

If Q is a parabolic subgroup of G between P and $P(\mathcal{V})$, the triple (P, \mathcal{V}, Q) called an \mathcal{H}_R -triple, defines a right \mathcal{H}_R -module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ equal to

$$\text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}} (e(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) = (e(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V})+R}, \theta} \mathcal{H}_R,$$

where $e(\mathcal{V})$ is the extension of \mathcal{V} to $\mathcal{H}_{M(\mathcal{V})}$.

This definition is justified by the fact that $M(\mathcal{V})$ is the maximal standard Levi subgroup of G such that the $\mathcal{H}_{M,R}$ -module \mathcal{V} is extensible to $\mathcal{H}_{M(\mathcal{V})}$.

Lemma 4.13. $\Delta_{\mathcal{V}}$ is the maximal subset of $\Delta \setminus \Delta_P$ orthogonal to Δ_P such that $T_\lambda^{M,*}$ acts trivially on \mathcal{V} for all $\lambda \in \Lambda(1) \cap {}_1W_{M'_{\mathcal{V}}}$.

Proof. For $J \subset \Delta$ let M_J denote the standard Levi subgroup of G with $\Delta_{M_J} = J$. The group $Z \cap M'_J$ is generated by the $Z \cap M'_\alpha$ for all $\alpha \in J$ (Lemma 2.1). When J is orthogonal to Δ_M and $\lambda \in \Lambda_{M'_J}(1)$, $\ell_M(\lambda) = 0$ where ℓ_M is the length associated to S_M^{aff} , and the map $\lambda \mapsto T_\lambda^{M,*} = T_\lambda^M : \Lambda_{M'_J}(1) \rightarrow \mathcal{H}_M$ is multiplicative. \square

The following is the natural generalization of Proposition 4.5 and Corollary 4.6. Let Q' be a parabolic subgroup of G with $Q \subset Q' \subset P(\mathcal{V})$. Applying the results of section 4.1 to $M(\mathcal{V})$ and its standard parabolic subgroups $Q \cap M(\mathcal{V}) \subset Q' \cap M(\mathcal{V})$, we have an $\mathcal{H}_{M(\mathcal{V}),R}$ -isomorphism

$$\begin{aligned} & \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V})) \\ &= e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M(\mathcal{V})}^+, \theta} \mathcal{H}_{M(\mathcal{V}),R} \xrightarrow{\kappa_{Q \cap M(\mathcal{V})}} e(\mathcal{V}) \otimes_R (\text{Ind}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1})^{\mathcal{U}_{M(\mathcal{V})}} \\ & \quad v \otimes \mathbf{1}_{\mathcal{H}} \mapsto v \otimes f_{QU \cap M(\mathcal{V})} : \end{aligned}$$

and an $\mathcal{H}_{M(\mathcal{V}),R}$ -embedding

$$\begin{aligned} & \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota_{(Q \cap M(\mathcal{V}), Q' \cap M(\mathcal{V}))}} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V})) \\ & \quad v \otimes \mathbf{1}_{\mathcal{H}_{M(\mathcal{V})}} \mapsto v \otimes \theta_{Q'}^{P(\mathcal{V})}(e_{Q'}^{Q'}). \end{aligned}$$

Applying the parabolic induction $\text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}$ which is exact and transitive, we obtain an \mathcal{H}_R -isomorphism $\kappa_Q = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\kappa_{Q \cap M(\mathcal{V})})$,

$$(4.16) \quad \begin{aligned} & \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\kappa_Q} \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\text{Ind}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1}_{M_Q})^{\mathcal{U}_{M(\mathcal{V})}}) \\ & \quad v \otimes \mathbf{1}_{\mathcal{H}} \mapsto v \otimes f_{QU_{M(\mathcal{V})}} \otimes \mathbf{1}_{\mathcal{H}} \end{aligned}$$

and an \mathcal{H}_R -embedding $\iota(Q, Q') = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\iota(Q, Q')^{M(\mathcal{V})})$

$$(4.17) \quad v \otimes \mathbf{1}_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}(e_{Q'}^{Q'}) : \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q, Q')} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})).$$

Applying Corollary 4.6 we obtain:

Theorem 4.14. *Let (P, \mathcal{V}, Q) be an \mathcal{H}_R -triple. Then, the cokernel of the \mathcal{H}_R -map*

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})),$$

defined by the $\iota(Q, Q')$ is isomorphic to $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ via the \mathcal{H}_R -isomorphism κ_Q .

Let σ be a smooth R -representation of M and let Q be a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$.

Remark 4.15. The \mathcal{H}_R -module $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$ is defined if $\Delta_Q \setminus \Delta_P$ and Δ_P are orthogonal because $Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_M})$ (Theorem 3.13).

We denote here by $P_{\min} = M_{\min} N_{\min}$ the minimal standard parabolic subgroup of G contained in P such that $\sigma = e_P(\sigma|_{M_{\min}})$ (Lemma 2.3, we drop the index σ). The sets of roots $\Delta_{P_{\min}}$ and $\Delta_{P(\sigma|_{M_{\min}})} \setminus \Delta_{P_{\min}}$ are orthogonal (Lemma 2.4). The groups $P(\sigma) = P(\sigma|_{M_{\min}})$, the representations $e(\sigma) = e(\sigma|_{M_{\min}})$ of $M(\sigma)$, the representations $I_G(P, \sigma, Q) = I_G(P_{\min}, \sigma|_{M_{\min}}, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \text{St}_Q^{P(\sigma)})$ of G , and the R -modules $\sigma^{\mathcal{U}_{M_{\min}}} = \sigma^{\mathcal{U}_M}$ are equal. From Theorem 3.13,

$$P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}}), \quad e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) = e(\sigma)^{\mathcal{U}_{M(\sigma)}},$$

and $P(\sigma^{\mathcal{U}_{M_{\min}}}) = P(\sigma)$ if $\sigma^{\mathcal{U}_{M_{\min}}}$ generates the representation $\sigma|_{M_{\min}}$. The \mathcal{H}_R -module

$$I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) = \text{Ind}_{\mathcal{H}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}}^{\mathcal{H}}(e(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_Q^{P(\sigma^{\mathcal{U}_{M_{\min}}})})^{\mathcal{U}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}})$$

is defined because $\Delta_{P_{\min}}$ and $\Delta_{P(\sigma^{\mathcal{U}_{M_{\min}}})} \setminus \Delta_{P_{\min}}$ are orthogonal and $P \subset Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}})$.

Remark 4.16. If $\sigma^{\mathcal{U}_{M_{\min}}}$ generates the representation $\sigma|_{M_{\min}}$ (in particular if R is an algebraically closed field of characteristic p and σ is irreducible), then $P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}})$ hence

$$I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) = \text{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

Applying Theorem 4.9 to $(P_{\min} \cap M(\sigma), \sigma|_{M_{\min}}, Q \cap M(\sigma))$, the $\mathcal{H}_{M(\sigma), R}$ -modules

$$(4.18) \quad e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}} = (e_{M(\sigma)}(\sigma) \otimes_R \text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}$$

are equal. We have the \mathcal{H}_R -isomorphism [OV17, Proposition 4.4]:

$$\begin{aligned} I_G(P, \sigma, Q)^{\mathcal{U}} &= (\text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \text{St}_Q^{P(\sigma)}))^{\mathcal{U}} \\ &\xrightarrow{ov} \text{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}((e(\sigma) \otimes_R \text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}) \\ &= f_{P(\sigma)\mathcal{U}, x} \mapsto x \otimes 1_{\mathcal{H}} \quad (x \in (e(\sigma) \otimes_R \text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}). \end{aligned}$$

We deduce the following.

Theorem 4.17. *Let (P, σ, Q) be an $R[G]$ -triple. Then, we have the \mathcal{H}_R -isomorphism*

$$I_G(P, \sigma, Q)^{\mathcal{U}} \xrightarrow{ov} \text{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

In particular,

$$I_G(P, \sigma, Q)^{\mathcal{U}} \simeq \begin{cases} I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) & \text{if } P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}}), \\ I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q) & \text{if } P = P_{\min}, P(\sigma) = P(\sigma^{\mathcal{U}_M}). \end{cases}$$

4.4. Comparison of the parabolic induction and coinduction. Let $P = MN$ be a standard parabolic subgroup of G , let \mathcal{V} be a right \mathcal{H}_R -module, and let Q be a parabolic subgroup of G with $Q \subset P(\mathcal{V})$. When R is an algebraically closed field of characteristic p , in [Abe], we associated to (P, \mathcal{V}, Q) an \mathcal{H}_R -module using the parabolic coinduction

$$\text{Coind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = \text{Hom}_{\mathcal{H}_{M_Q^-, \theta^*}}(\mathcal{H}, -) : \text{Mod}_R(\mathcal{H}_{M_Q}) \rightarrow \text{Mod}_R(\mathcal{H})$$

instead of the parabolic induction $\text{Ind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = - \otimes_{\mathcal{H}_{M_Q^+, \theta}} \mathcal{H}$. The index θ^* in the parabolic coinduction means that $\mathcal{H}_{M_Q^-}$ embeds in \mathcal{H} by θ_Q^* . Our terminology is different from the one in [Abe] where the parabolic coinduction is called induction. For a parabolic subgroup Q' of G with $Q \subset Q' \subset P(\mathcal{V})$, there is a natural inclusion of \mathcal{H}_R -modules

$$(4.19) \quad \text{Hom}_{\mathcal{H}_{M_{Q'}, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{i(Q, Q')} \text{Hom}_{\mathcal{H}_{M_Q^-, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

because $\theta^*(\mathcal{H}_{M_Q^-}) \subset \theta^*(\mathcal{H}_{M_{Q'}^-})$ as $W_{M_Q^-}(1) \subset W_{M_{Q'}^-}(1)$, and $vT_w^{M_{Q'}, *}$ = $vT_w^{M_Q, *}$ for $w \in W_{M_Q^-}(1)$ and $v \in \mathcal{V}$. (This is [Abe, Proposition 4.19] when R is an algebraically closed field of characteristic p . This follows from our formulation of the extension for any R .)

Definition 4.18. Let $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ denote the cokernel of the map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Hom}_{\mathcal{H}_{M_{Q'}, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Hom}_{\mathcal{H}_{M_Q^-, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the \mathcal{H}_R -embeddings $i(Q, Q')$.

When R is an algebraically closed field of characteristic p , we showed that the \mathcal{H}_R -module $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ is simple when \mathcal{V} is simple and supersingular (Definition 4.25), and that any simple \mathcal{H}_R -module is of this form for an \mathcal{H}_R -triple (P, \mathcal{V}, Q) where \mathcal{V} is simple and supersingular, P, Q and the isomorphism class of \mathcal{V} are unique [Abe]. The aim of this section is to compare the \mathcal{H}_R -modules $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ with the \mathcal{H}_R -modules $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ and to show that the classification is also valid with the \mathcal{H}_R -modules $I_{\mathcal{H}}(P, \mathcal{V}, Q)$.

It is already known that a parabolically coinduced module is a parabolically induced module and vice versa [Abe, Proposition 4.15], [Vig15b, Theorem 1.8]. To make it more precise we need to introduce notation.

We lift the elements w of the finite Weyl group \mathbb{W} to $\hat{w} \in \mathcal{N}_G \cap \mathcal{K}$ as in [AHHV17, IV.6], [OV17, Proposition 2.7]: they satisfy the braid relations $\hat{w}_1 \hat{w}_2 = (w_1 w_2)$ when $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ and when $s \in S$, \hat{s} is admissible, in particular lies in ${}_1 W_{G'}$.

Let $\mathbf{w}, \mathbf{w}_M, \mathbf{w}^M$ denote, respectively, the longest elements in \mathbb{W}, \mathbb{W}_M and $\mathbf{w}\mathbf{w}_M$. We have $\mathbf{w} = \mathbf{w}^{-1} = \mathbf{w}^M \mathbf{w}_M, \mathbf{w}_M = \mathbf{w}_M^{-1}, \hat{\mathbf{w}} = \hat{\mathbf{w}}^M \hat{\mathbf{w}}_M$,

$$\mathbf{w}^M(\Delta_M) = -\mathbf{w}(\Delta_M) \subset \Delta, \quad \mathbf{w}^M(\Phi^+ \setminus \Phi_M^+) = \mathbf{w}(\Phi^+ \setminus \Phi_M^+).$$

Let $\mathbf{w}.M$ be the standard Levi subgroup of G with $\Delta_{\mathbf{w}.M} = \mathbf{w}^M(\Delta_M)$ and $\mathbf{w}.P$ the standard parabolic subgroup of G with Levi $\mathbf{w}.M$. We have

$$\mathbf{w}.M = \hat{\mathbf{w}}^M M (\hat{\mathbf{w}}^M)^{-1} = \hat{\mathbf{w}} M (\hat{\mathbf{w}})^{-1}, \quad \mathbf{w}^{\mathbf{w}.M} = \mathbf{w}_M \mathbf{w} = (\mathbf{w}^M)^{-1}.$$

The conjugation $w \mapsto \mathbf{w}^M w (\mathbf{w}^M)^{-1}$ in W gives a group isomorphism $W_M \rightarrow W_{\mathbf{w}.M}$ sending S_M^{aff} onto $S_{\mathbf{w}.M}^{\text{aff}}$, respecting the finite Weyl subgroups $\mathbf{w}^M \mathbb{W}_M (\mathbf{w}^M)^{-1} = \mathbb{W}_{\mathbf{w}.M} = \mathbf{w} \mathbb{W}_M \mathbf{w}^{-1}$, and exchanging W_{M^+} and $W_{(\mathbf{w}.M)^-} = \mathbf{w} W_{M^+} \mathbf{w}^{-1}$. The conjugation by $\tilde{\mathbf{w}}^M$ restricts to a group isomorphism $W_M(1) \rightarrow W_{\mathbf{w}.M}(1)$ sending $W_{M^+}(1)$ onto $W_{(\mathbf{w}.M)^-}(1)$. The linear isomorphism

$$(4.20) \quad \mathcal{H}_M \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w}.M} \quad T_w^M \mapsto T_{\tilde{\mathbf{w}}^M w (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M} \text{ for } w \in W_M(1),$$

is a ring isomorphism between the pro- p -Iwahori Hecke rings of M and $\mathbf{w}.M$. It sends the positive part \mathcal{H}_{M^+} of \mathcal{H}_M onto the negative part $\mathcal{H}_{(\mathbf{w}.M)^-}$ of $\mathcal{H}_{\mathbf{w}.M}$ [Vig15b, Proposition 2.20]. We have $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}_M \tilde{\mathbf{w}}^{\mathbf{w}.M} = \tilde{\mathbf{w}}^M \tilde{\mathbf{w}}_M, (\tilde{\mathbf{w}}^M)^{-1} = \tilde{\mathbf{w}}^{\mathbf{w}.M} t_M$ where $t_M = \tilde{\mathbf{w}}^2 \tilde{\mathbf{w}}_M^{-2} \in Z_k$.

Definition 4.19. The twist $\tilde{\mathbf{w}}^M.\mathcal{V}$ of \mathcal{V} by $\tilde{\mathbf{w}}^M$ is the right $\mathcal{H}_{\mathbf{w}.M}$ -module deduced from the right \mathcal{H}_M -module \mathcal{V} by functoriality: as R -modules $\tilde{\mathbf{w}}^M.\mathcal{V} = \mathcal{V}$ and for $v \in \mathcal{V}, w \in W_M(1)$ we have $v T_{\tilde{\mathbf{w}}^M w (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M} = v T_w^M$.

We can define the twist $\tilde{\mathbf{w}}^M.\mathcal{V}$ of \mathcal{V} with the $T_w^{M,*}$ instead of T_w^M .

Lemma 4.20. For $v \in \mathcal{V}, w \in W_M(1)$ we have $v T_{\tilde{\mathbf{w}}^M w (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M,*} = v T_w^{M,*}$ in $\tilde{\mathbf{w}}^M.\mathcal{V}$.

Proof. By the ring isomorphism $\mathcal{H}_M \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w}.M}$, we have $c_{\tilde{\mathbf{w}}^M \tilde{s} (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M} = c_{\tilde{s}}^M$ when $\tilde{s} \in W_M(1)$ lifts $s \in S_M^{\text{aff}}$. So the equality of the lemma is true for $w = \tilde{s}$. Apply the braid relations to get the equality for all $w \in W_M(1)$. \square

We return to the \mathcal{H}_R -module $\text{Hom}_{\mathcal{H}_{M^-, \theta^*}}(\mathcal{H}, V)$ parabolically coinduced from \mathcal{V} . It has a natural direct decomposition indexed by the set $\mathbb{W}^{\mathbb{W}_M}$ of elements d in

the finite Weyl group \mathbb{W} of minimal length in the coset $d\mathbb{W}_M$. Indeed it is known that the linear map

$$f \mapsto (f(T_{\bar{d}}))_{d \in \mathbb{W}^{\mathbb{W}_M}} : \text{Hom}_{\mathcal{H}_{M^-}, \theta^*}(\mathcal{H}, \mathcal{V}) \rightarrow \bigoplus_{d \in \mathbb{W}^{\mathbb{W}_M}} \mathcal{V}$$

is an isomorphism. For $v \in \mathcal{V}$ and $d \in \mathbb{W}^{\mathbb{W}_M}$, there is a unique element

$$f_{\bar{d}, v} \in \text{Hom}_{\mathcal{H}_{M^-}, \theta^*}(\mathcal{H}, \mathcal{V}) \text{ satisfying } f(T_{\bar{d}}) = v \text{ and } f(T_{\bar{d}'}) = 0 \text{ for } d' \in \mathbb{W}^{\mathbb{W}_M} \setminus \{d\}.$$

It is known that the map $v \mapsto f_{\tilde{\mathbf{w}}^M, v} : \tilde{\mathbf{w}}^M \cdot \mathcal{V} \rightarrow \text{Hom}_{\mathcal{H}_{M^-}, \theta^*}(\mathcal{H}, \mathcal{V})$ is $\mathcal{H}_{(\mathbf{w}, M)^+}$ -equivariant: $f_{\tilde{\mathbf{w}}^M, v} T_w^{\mathbf{w}, M} = f_{\tilde{\mathbf{w}}^M, v} T_w$ for all $v \in \mathcal{V}, w \in W_{\mathbf{w}, M^+}(1)$. By adjunction, this $\mathcal{H}_{(\mathbf{w}, M)^+}$ -equivariant map gives an \mathcal{H}_R -homomorphism from an induced module to a coinduced module:

$$(4.21) \quad v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M, v} : \tilde{\mathbf{w}}^M \cdot \mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}, M)^+}, \theta} \mathcal{H} \xrightarrow{\mu_P} \text{Hom}_{\mathcal{H}_{M^-}, \theta^*}(\mathcal{H}, \mathcal{V}).$$

This is an isomorphism [Abe, Proposition 4.15], [Vig15b, Theorem 1.8].

The naive guess that a variant μ_Q of μ_P induces an \mathcal{H}_R -isomorphism between the \mathcal{H}_R -modules $I_{\mathcal{H}}(\mathbf{w}, P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w}, Q)$ and $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ turns out to be true. The proof is the aim of the rest of this section.

The \mathcal{H}_R -module $I_{\mathcal{H}}(\mathbf{w}, P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w}, Q)$ is well defined because the parabolic subgroups of G containing \mathbf{w}, P and contained in $P(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$ are \mathbf{w}, Q for $P \subset Q \subset P(\mathcal{V})$, as follows from Lemma 4.21.

Lemma 4.21. $\Delta_{\tilde{\mathbf{w}}^M \cdot \mathcal{V}} = -\mathbf{w}(\Delta_{\mathcal{V}})$.

Proof. Recall that $\Delta_{\mathcal{V}}$ is the set of simple roots $\alpha \in \Delta \setminus \Delta_M$ orthogonal to Δ_M and $T^M(z)$ acts trivially on \mathcal{V} for all $z \in Z \cap M'_\alpha$, and the corresponding standard parabolic subgroup $P_{\mathcal{V}} = M_{\mathcal{V}} N_{\mathcal{V}}$. The $Z \cap M'_\alpha$ for $\alpha \in \Delta_{\mathcal{V}}$ generate the group $Z \cap M'_{\mathcal{V}}$. A root $\alpha \in \Delta \setminus \Delta_M$ orthogonal to Δ_M is fixed by \mathbf{w}_M so $\mathbf{w}^M(\alpha) = \mathbf{w}(\alpha)$ and

$$\hat{\mathbf{w}}^M M_{\mathcal{V}} (\hat{\mathbf{w}}^M)^{-1} = \hat{\mathbf{w}} M_{\mathcal{V}} (\hat{\mathbf{w}})^{-1}.$$

The proof of Lemma 4.21 is straightforward as $\Delta = -\mathbf{w}(\Delta), \Delta_{\mathbf{w}, M} = -\mathbf{w}(\Delta_M)$. \square

Before going further, we check the commutativity of the extension with the twist. As $Q = M_Q U$ and M_Q determine each other we denote $\mathbf{w}_{M_Q} = \mathbf{w}_Q, \mathbf{w}^{M_Q} = \mathbf{w}^Q$ when $Q \neq P, G$.

Lemma 4.22. $e_{\mathcal{H}_{\mathbf{w}, Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) = \tilde{\mathbf{w}}^Q \cdot e_{\mathcal{H}_Q}(\mathcal{V})$.

Proof. As R -modules $\mathcal{V} = e_{\mathcal{H}_{\mathbf{w}, Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) = \tilde{\mathbf{w}}^Q \cdot e_{\mathcal{H}_Q}(\mathcal{V})$. A direct computation shows that the Hecke element $T_w^{\mathbf{w}, Q, *}$ acts in the \mathcal{H}_R -module $e_{\mathcal{H}_{\mathbf{w}, Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$, by the identity if $w \in \tilde{\mathbf{w}}^Q {}_1 W_{M'_2}(\mathbf{w}^Q)^{-1}$ and by $T_{(\tilde{\mathbf{w}}^Q)^{-1} w \tilde{\mathbf{w}}^Q}^{M, *}$ if $w \in \tilde{\mathbf{w}}^Q {}_1 W_{M'_2}(\mathbf{w}^Q)^{-1}$ where M_2 denotes the standard Levi subgroup with $\Delta_{M_2} = \Delta_Q \setminus \Delta_P$. Whereas in the \mathcal{H}_R -module $\tilde{\mathbf{w}}^Q \cdot e_{\mathcal{H}_Q}(\mathcal{V})$, the Hecke element $T_w^{\mathbf{w}, Q, *}$ acts by the identity if $w \in {}_1 W_{\mathbf{w}, M'_2}$ and by $T_{(\tilde{\mathbf{w}}^M)^{-1} w \tilde{\mathbf{w}}^M}^{M, *}$ if $w \in W_{\mathbf{w}, M}(1)$. So the lemma means that

$${}_1 W_{\mathbf{w}, M'_2} = \tilde{\mathbf{w}}^Q {}_1 W_{M'_2}(\mathbf{w}^Q)^{-1}, \quad (\tilde{\mathbf{w}}^Q)^{-1} w \tilde{\mathbf{w}}^Q = (\tilde{\mathbf{w}}^M)^{-1} w \tilde{\mathbf{w}}^M \text{ if } w \in W_{\mathbf{w}, M}(1).$$

These properties are easily proved using that ${}_1 W_{G'}$ is normal in $W(1)$ and that the sets of roots Δ_P and $\Delta_Q \setminus \Delta_P$ are orthogonal: $\mathbf{w}_Q = \mathbf{w}_{M_2} \mathbf{w}_M$, the elements \mathbf{w}_{M_2} and \mathbf{w}_M normalize W_M and W_{M_2} , the elements of \mathbb{W}_{M_2} commutes with the elements of \mathbb{W}_M . \square

We return to our guess. The variant μ_Q of μ_P is obtained by combining the commutativity of the extension with the twist and the isomorphism (4.21) applied to $(Q, e_{\mathcal{H}_Q}(\mathcal{V}))$ instead of (P, \mathcal{V}) . The \mathcal{H}_R -isomorphism μ_Q is

$$(4.22) \quad v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M, v} : \text{Ind}_{\mathcal{H}_{\mathbf{w}.M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})) \xrightarrow{\mu_Q} \text{Hom}_{\mathcal{H}_{M_Q}^-, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})).$$

Our guess is that μ_Q induces an \mathcal{H}_R -isomorphism from the cokernel of the \mathcal{H}_R -map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Ind}_{\mathcal{H}_{\mathbf{w}.Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_{\mathbf{w}.Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}))$$

defined by the \mathcal{H}_R -embeddings $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$, isomorphic to $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M \mathcal{V}, \mathbf{w}.\bar{Q})$ via $\kappa_{\mathbf{w}.Q}$ (Theorem 4.14), onto the cokernel $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ the \mathcal{H}_R -map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Hom}_{\mathcal{H}_{M_Q}^-, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Hom}_{\mathcal{H}_{M_Q}^-, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the \mathcal{H}_R -embeddings $i(Q, Q')$. This is true if $i(Q, Q')$ corresponds to $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$ via the isomorphisms $\mu_{Q'}$ and μ_Q . This is the content of the next proposition.

Proposition 4.23. *For all $Q \subsetneq Q' \subset P(\mathcal{V})$ we have*

$$i(Q, Q') \circ \mu_{Q'} = \mu_Q \circ \iota(\mathbf{w}.Q, \mathbf{w}.Q').$$

We postpone to section 4.6 the rather long proof of the proposition.

Corollary 4.24. *The \mathcal{H}_R -isomorphism $\mu_Q \circ \kappa_{\mathbf{w}.Q}^{-1}$ induces an \mathcal{H}_R -isomorphism*

$$I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M \mathcal{V}, \mathbf{w}.\bar{Q}) \rightarrow CI_{\mathcal{H}}(P, \mathcal{V}, Q).$$

4.5. Supersingular \mathcal{H}_R -modules, classification of simple \mathcal{H}_R -modules. We recall first the notion of supersingularity based on the action of the center of \mathcal{H} .

The center of \mathcal{H} [Vig14, Theorem 1.3] contains a subalgebra \mathcal{Z}_{T^+} isomorphic to $\mathbb{Z}[T^+/T_1]$ where T^+ is the monoid of dominant elements of T and T_1 is the pro- p -Sylow subgroup of the maximal compact subgroup of T .

Let $t \in T$ of image $\mu_t \in W(1)$ and let $(E_o(w))_{w \in W(1)}$ denote the alcove walk basis of \mathcal{H} associated to a closed Weyl chamber o of \mathbb{W} . The element

$$E_o(C(\mu_t)) = \sum_{\mu'} E_o(\mu')$$

is the sum over the elements in μ' in the conjugacy class $C(\mu_t)$ of μ_t in $W(1)$. It is a central element of \mathcal{H} and does not depend on the choice of o . We write also $z(t) = E_o(C(\mu_t))$.

Definition 4.25. A non-zero right \mathcal{H}_R -module \mathcal{V} is called supersingular when, for any $v \in \mathcal{V}$ and any non-invertible $t \in T^+$, there exists a positive integer $n \in \mathbb{N}$ such that $v(z(t))^n = 0$. If one can choose n independent on (v, t) , then \mathcal{V} is called uniformly supersingular.

Remark 4.26. One can choose n independent on (v, t) when \mathcal{V} is finitely generated as a right \mathcal{H}_R -module. If R is a field and \mathcal{V} is simple we can take $n = 1$.

When G is compact modulo the center, $T^+ = T$, and any non-zero \mathcal{H}_R -module is supersingular.

The induction functor $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} : \text{Mod}(\mathcal{H}_{M,R}) \rightarrow \text{Mod}(\mathcal{H}_R)$ has a left adjoint $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}$ and a right adjoint $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}$ [Vig15b]: for $\mathcal{V} \in \text{Mod}(\mathcal{H}_R)$,

$$(4.23) \quad \mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \tilde{\mathbf{w}}^{\mathbf{w}.M} \circ (\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}), \quad \mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V}).$$

In the left adjoint, \mathcal{V} is seen as a right $\mathcal{H}_{(\mathbf{w}.M)^-}$ -module via the ring homomorphism $\theta_{\mathbf{w}.M}^* : \mathcal{H}_{(\mathbf{w}.M)^-} \rightarrow \mathcal{H}$; in the right adjoint, \mathcal{V} is seen as a right \mathcal{H}_{M^+} -module via the ring homomorphism $\theta_M : \mathcal{H}_{M^+} \rightarrow \mathcal{H}$ (section 2.3).

Proposition 4.27. *Assume that \mathcal{V} is a supersingular right \mathcal{H}_R -module and that p is nilpotent in \mathcal{V} . Then $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$, and if \mathcal{V} is uniformly supersingular $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$.*

Proof. This is a consequence of three known properties:

- (1) \mathcal{H}_M is the localization of \mathcal{H}_{M^+} (resp., \mathcal{H}_{M^-}) at T_{μ}^M for any element $\mu \in \Lambda_T(1)$, central in $W_M(1)$ and strictly N -positive (resp., N -negative), and $T_{\mu}^M = T_{\mu}^{M,*}$. See [Vig15b, Theorem 1.4].
- (2) When o is anti-dominant, $E_o(\mu) = T_{\mu}$ if $\mu \in \Lambda^+(1)$ and $E_o(\mu) = T_{\mu}^*$ if $\mu \in \Lambda^-(1)$.
- (3) Let an integer $n > 0$ and $\mu \in \Lambda(1)$ such that the \mathbb{W} -orbit of $v(\mu) \in X_*(T) \otimes \mathbb{Q}$ (definition in section 2.1) and of μ have the same number of elements. Then

$$(E_o(C(\mu)))^n E_o(\mu) - E_o(\mu)^{n+1} \in p\mathcal{H}.$$

See [Vig15a, Lemma 6.5], where the hypotheses are given in the proof (but not written in the lemma).

Let $\mu \in \Lambda_T^+(1)$ satisfying (1) for M^+ and (3), similarly let $\mathbf{w}.\mu \in \Lambda_T^-(1)$ satisfying (1) for $(\mathbf{w}.M)^-$ and (3). For (R, \mathcal{V}) as in the proposition, let $v \in \mathcal{V}$ and $n > 0$ such that $vE_o(C(\mu))^n = vE_o(C(\mathbf{w}.\mu))^n = 0$. Multiplying by $E_o(\mu)$ or $E_o(\mathbf{w}.\mu)$, and applying (3) and (2) for o anti-dominant we get:

$$vE_o(\mu^{n+1}) = vT_{\mu}^{n+1} \in p\mathcal{V}, \quad vE_o((\mathbf{w}.\mu)^{n+1}) = v(T_{\mathbf{w}.\mu}^*)^{n+1} \in p\mathcal{V}.$$

The proposition follows from: $vT_{\mu}^{n+1}, v(T_{\mathbf{w}.\mu}^*)^{n+1}$ in $p\mathcal{V}$ (as explained in [Abe16, Proposition 5.17] when $p = 0$ in R). From $v(T_{\mathbf{w}.\mu}^*)^{n+1}$ in $p\mathcal{V}$, we get $v \otimes (T_{\mathbf{w}.\mu}^{\mathbf{w}.M,*})^{n+1} = v(T_{\mathbf{w}.\mu}^*)^{n+1} \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$ in $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}$. As $T^{\mathbf{w}.M,*} = T^{\mathbf{w}.M}$ is invertible in $\mathcal{H}_{\mathbf{w}.M}$ we get $v \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$ in $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}$. As v was arbitrary, $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M} \subset p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}$. If p is nilpotent in \mathcal{V} , then $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M} = 0$. Suppose now that there exists $n > 0$ such that $\mathcal{V}(z(t))^n = 0$ for any non-invertible $t \in T^+$; then $\mathcal{V}T_{\mu}^{n+1} \subset p\mathcal{V}$ where $\mu = \mu_t$ and hence $\varphi(h) = \varphi(hT_{\mu}^{M-n-1})T_{\mu}^{n+1}$ in $p\mathcal{V}$ for an arbitrary $\varphi \in \text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V})$ and an arbitrary $h \in \mathcal{H}_M$. We deduce $\text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V}) \subset \text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, p\mathcal{V})$. If p is nilpotent in \mathcal{V} , then $\text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V}) = 0$. \square

Recalling that $\tilde{\mathbf{w}}^M \mathcal{V}$ is obtained by functoriality from \mathcal{V} and the ring isomorphism $\iota(\tilde{\mathbf{w}}^M)$ defined in (4.20), the equivalence between \mathcal{V} supersingular and $\tilde{\mathbf{w}}^M \mathcal{V}$ supersingular follows from Lemma 4.28

Lemma 4.28.

- (1) *Let $t \in T$. Then t is dominant for U_M if and only if $\hat{\mathbf{w}}^M t (\hat{\mathbf{w}}^M)^{-1} \in T$ is dominant for $U_{\mathbf{w}.M}$.*

- (2) The R -algebra isomorphism $\mathcal{H}_{M,R} \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w},M,R}$, $T_w^M \mapsto T_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w},M}$ for $w \in W_M(1)$ sends $z^M(t)$ to $z^{\mathbf{w},M}(\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1})$ for $t \in T$ dominant for U_M .

Proof. The conjugation by $\hat{\mathbf{w}}^M$ stabilizes T , sends U_M to $U_{\mathbf{w},M}$, and sends the \mathbb{W}_M -orbit of $t \in T$ to the $\mathbb{W}_{\mathbf{w},M}$ -orbit of $\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1}$, as $\mathbf{w}^M \mathbb{W}_M(\mathbf{w}^M)^{-1} = \mathbb{W}_{\mathbf{w},M}$. It is known that $\iota(\tilde{\mathbf{w}}^M)$ respects the anti-dominant alcove walk bases [Vig15b, Proposition 2.20]: it sends $E^M(w)$ to $E^{\mathbf{w},M}(\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1})$ for $w \in W_M(1)$. \square

We deduce the following.

Corollary 4.29. *Let \mathcal{V} be a right $\mathcal{H}_{M,R}$ -module. Then \mathcal{V} is supersingular if and only if the right $\mathcal{H}_{\mathbf{w},M,R}$ -module $\tilde{\mathbf{w}}^M \mathcal{V}$ is supersingular.*

Assume R is an algebraically close field of characteristic p . The supersingular simple $\mathcal{H}_{M,R}$ -modules are classified in [Vig15a]. By Corollaries 4.24 and 4.29, the classification of the simple \mathcal{H}_R -modules in [Abe] remains valid with the \mathcal{H}_R -modules $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ instead of $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$:

Corollary 4.30 (Classification of simple \mathcal{H}_R -modules). *Assume R is an algebraically closed field of characteristic p . Let (P, \mathcal{V}, Q) be an \mathcal{H}_R -triple where \mathcal{V} is simple and supersingular. Then, the \mathcal{H}_R -module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ is simple. A simple \mathcal{H}_R -module is isomorphic to $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ for an \mathcal{H}_R -triple (P, \mathcal{V}, Q) where \mathcal{V} is simple and supersingular, P, Q and the isomorphism class of \mathcal{V} are unique.*

4.6. A commutative diagram. We prove in this section Proposition 4.23. For $Q \subset Q' \subset P(\mathcal{V})$ we show by an explicit computation that

$$\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'} \text{Ind}_{\mathcal{H}_{\mathbf{w},Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w},Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_{\mathbf{w},Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w},Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}))$$

is equal to $\iota(\mathbf{w}, Q, \mathbf{w}, Q')$. The R -module $e_{\mathcal{H}_{\mathbf{w},Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) \otimes 1_{\mathcal{H}}$ generates the \mathcal{H}_R -module $e_{\mathcal{H}_{\mathbf{w},Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) \otimes_{\mathcal{H}_{\mathbf{w},Q'}, R, \theta^+} \mathcal{H}_R = \text{Ind}_{\mathcal{H}_{\mathbf{w},Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w},Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}))$ and by (4.17)

$$(4.24) \quad \iota(\mathbf{w}, Q, \mathbf{w}, Q')(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in {}^{\mathbb{W}_{M,Q}} \mathbb{W}_{M,Q'}} T_{\bar{d}}$$

for $v \in \mathcal{V}$ seen as an element of $e_{\mathcal{H}_{\mathbf{w},Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$ in the LHS and an element of $e_{\mathcal{H}_{\mathbf{w},Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$ in the RHS.

Lemma 4.31. $(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in {}^{\mathbb{W}_{M,Q}} \mathbb{W}_{M,Q'}} qd T_{\tilde{\mathbf{w}}^M Q(\tilde{\mathbf{w}}^M Q' \bar{d})^{-1}}^*$.

Proof. $\mu_{Q'}(v \otimes 1_{\mathcal{H}})$ is the unique homomorphism $f_{\tilde{\mathbf{w}}^M Q', v} \in \text{Hom}_{\mathcal{H}_{M_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ sending $T_{\tilde{\mathbf{w}}^M Q'}$ to v and vanishing on $T_{\bar{d}'}$ for $d' \in \mathbb{W}^{\mathbb{W}_{M_{Q'}}} \setminus \{\mathbf{w}^{Q'}\}$ by (4.22). By (4.19), $i(Q, Q')$ is the natural embedding of $\text{Hom}_{\mathcal{H}_{M_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ in $\text{Hom}_{\mathcal{H}_{M_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$ therefore $i(Q, Q')(f_{\tilde{\mathbf{w}}^M Q', v})$ is the unique homomorphism $\text{Hom}_{\mathcal{H}_{M_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$ sending $T_{\tilde{\mathbf{w}}^M Q'}$ to v and vanishing on $T_{\bar{d}'}$ for $d' \in \mathbb{W}^{\mathbb{W}_{M_{Q'}}} \setminus \{\mathbf{w}^{Q'}\}$. As $\mathbb{W}^{\mathbb{W}_{M_Q}} = \mathbb{W}^{\mathbb{W}_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$, this homomorphism vanishes on $T_{\bar{w}}$ for w not

in $\mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. By [Abe16, Lemma 2.22], the inverse of μ_Q is the \mathcal{H}_R -isomorphism

$$(4.25) \quad \text{Hom}_{\mathcal{H}_{M_Q^-}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\mu_Q^{-1}} \text{Ind}_{\mathcal{H}_{\mathbf{w}, M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}, Q}}(\tilde{\mathbf{w}}^M \mathcal{V}))$$

$$f \mapsto \sum_{d \in \mathbb{W}^{\mathbb{W}_M}} f(T_{\tilde{d}}) \otimes T_{\tilde{\mathbf{w}}^M \tilde{d}^{-1}}^*$$

where $\mathbb{W}^{\mathbb{W}_M}$ is the set of $d \in \mathbb{W}$ with minimal length in the coset $d\mathbb{W}_M$. We deduce the explicit formula

$$(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = \sum_{w \in \mathbb{W}^{\mathbb{W}_{M_Q}}} i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{w}}) \otimes T_{\tilde{\mathbf{w}}^{M_Q} \tilde{w}^{-1}}^*$$

Some terms are zero: the terms for $w \in \mathbb{W}^{\mathbb{W}_{M_Q}}$ not in $\mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. We analyze the other terms for w in $\mathbb{W}^{\mathbb{W}_{M_Q}} \cap \mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$; this set is $\mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. Let $w = \mathbf{w}^{M_{Q'}} d$, $d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$, and $\tilde{w} = \tilde{\mathbf{w}}^{M_{Q'}} \tilde{d}$ with $\tilde{d} \in {}_1W_{G'}$ lifting d . By the braid relations $T_{\tilde{w}} = T_{\tilde{\mathbf{w}}^{M_{Q'}}} T_{\tilde{d}}$. We have $T_{\tilde{d}} = \theta^*(T_{\tilde{d}}^{M_{Q'}})$ by the braid relations because $d \in \mathbb{W}_{M_{Q'}}$, $S_{M_{Q'}} \subset S^{\text{aff}}$ and $\theta^*(c_{\tilde{s}}^{M_{Q'}}) = c_{\tilde{s}}$ for $s \in S_{M_{Q'}}$. As $\mathbb{W}_{M_{Q'}} \subset W_{M_{Q'}^-} \cap W_{M_{Q'}^+}$, we deduce:

$$\begin{aligned} i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{w}}) &= i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{\mathbf{w}}^{M_{Q'}}} T_{\tilde{d}}) \\ &= i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{\mathbf{w}}^{M_{Q'}}}) T_{\tilde{d}}^{M_{Q'}} \\ &= v T_{\tilde{d}}^{M_{Q'}} = q_d v. \end{aligned}$$

Corollary 3.9 gives the last equality. \square

The formula for $(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}})$ given in Lemma 4.31 is different from the formula (4.24) for $\iota(\mathbf{w}, Q, \mathbf{w}, Q')(v \otimes 1_{\mathcal{H}})$. It needs some work to prove that they are equal.

A first reassuring remark is that ${}^{\mathbb{W}_{M_{\mathbf{w}, Q}}} \mathbb{W}_{M_{\mathbf{w}, Q'}} = \{\mathbf{w} d^{-1} \mathbf{w} \mid d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$, so the two summation sets have the same number of elements. But better,

$${}^{\mathbb{W}_{M_{\mathbf{w}, Q}}} \mathbb{W}_{M_{\mathbf{w}, Q'}} = \{\mathbf{w}^Q (\mathbf{w}^{Q'} d)^{-1} \mid d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$$

because $\mathbf{w}_{Q'} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}} \mathbf{w}_Q = \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. To prove the latter equality, we apply the criterion: $w \in \mathbb{W}_{M_{Q'}}$ lies in $\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ if and only if $w(\alpha) > 0$ for all $\alpha \in \Delta_Q$ noticing that $d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ implies $\mathbf{w}_Q(\alpha) \in -\Delta_Q$, $d\mathbf{w}_Q(\alpha) \in -\Phi_{M_{Q'}}$, $\mathbf{w}_{Q'} d\mathbf{w}_Q(\alpha) > 0$. Let $x_d = \mathbf{w}^Q (\mathbf{w}^{Q'} d)^{-1}$. We have $\tilde{\mathbf{w}}^{M_Q} (\tilde{\mathbf{w}}^{M_{Q'}} \tilde{d})^{-1} = \tilde{x}_d$ because the lifts \tilde{w} of the elements $w \in \mathbb{W}$ satisfy the braid relations and $\ell(x_d) = \ell(\mathbf{w}_Q d^{-1} \mathbf{w}_{Q'}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d) = -\ell(\mathbf{w}^{Q'}) + \ell(\mathbf{w}^Q) - \ell(d)$. We have $q_d = q_{\mathbf{w}_{\mathbf{w}, Q} x_d \mathbf{w}_{\mathbf{w}, Q'}}$ because $\mathbf{w} d^{-1} \mathbf{w} = \mathbf{w}_{\mathbf{w}, Q} x_d \mathbf{w}_{\mathbf{w}, Q'}$, and $q_d = q_{d^{-1}} = q_{\mathbf{w} d^{-1} \mathbf{w}}$. So

$$\sum_{d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}} q_d T_{\tilde{\mathbf{w}}^Q (\tilde{\mathbf{w}}^{Q'} \tilde{d})^{-1}}^* = \sum_{x_d \in {}^{\mathbb{W}_{M_{\mathbf{w}, Q}}} \mathbb{W}_{M_{\mathbf{w}, Q'}}} q_{\mathbf{w}_{\mathbf{w}, Q} x_d \mathbf{w}_{\mathbf{w}, Q'}} T_{\tilde{x}_d}^*$$

In the RHS, only $\tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w} \cdot Q, \mathbf{w} \cdot Q'$ appear. The same holds true in the formula (4.24). The map $(P, \mathcal{V}, Q, Q') \mapsto (\mathbf{w} \cdot P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w} \cdot Q, \mathbf{w} \cdot Q')$ is a bijection of the set of triples (P, \mathcal{V}, Q, Q') where $P = MN, Q, Q'$ are standard parabolic subgroups of G , \mathcal{V} a right \mathcal{H}_R -module, $Q \subset Q' \subset P(\mathcal{V})$ by Lemma 4.21. So we can replace $(\mathbf{w} \cdot P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w} \cdot Q, \mathbf{w} \cdot Q')$ by (P, \mathcal{V}, Q, Q') . Our task is reduced to prove in $e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+}, \theta} \mathcal{H}_R$:

$$(4.26) \quad v \otimes \sum_{d \in {}^{\mathbb{W}}M_Q \mathbb{W}_{M_{Q'}}} T_{\tilde{d}} = v \otimes \sum_{d \in {}^{\mathbb{W}}M_Q \mathbb{W}_{M_{Q'}}} q_{\mathbf{w}_Q d \mathbf{w}_{Q'}} T_{\tilde{d}}^*.$$

A second simplification is possible: we can replace $Q \subset Q'$ by the standard parabolic subgroups $Q_2 \subset Q'_2$ of G with $\Delta_{Q_2} = \Delta_Q \setminus \Delta_P$ and $\Delta_{Q'_2} = \Delta_{Q'} \setminus \Delta_P$, because Δ_P and $\Delta_{P(\mathcal{V})} \setminus \Delta_P$ are orthogonal. Indeed, $\mathbb{W}_{M_{Q'}} = \mathbb{W}_M \times \mathbb{W}_{M_{Q'_2}}$ and $\mathbb{W}_{M_Q} = \mathbb{W}_M \times \mathbb{W}_{M_{Q_2}}$ are direct products, the longest elements $\mathbf{w}_{Q'} = \mathbf{w}_M \mathbf{w}_{Q'_2}, \mathbf{w}_Q = \mathbf{w}_M \mathbf{w}_{Q_2}$ are direct products and

$${}^{\mathbb{W}}M_Q \mathbb{W}_{M_{Q'}} = {}^{\mathbb{W}}M_{Q_2} \mathbb{W}_{M_{Q'_2}}, \quad \mathbf{w}_Q d \mathbf{w}_{Q'} = \mathbf{w}_{Q_2} d \mathbf{w}_{Q'_2}.$$

Once this is done, we use the properties of $e_{\mathcal{H}_Q}(\mathcal{V})$: $vh \otimes 1_{\mathcal{H}} = v \otimes \theta_Q(h)$ for $h \in \mathcal{H}_{M_{Q_2}^+}$, and $T_w^{Q,*}$ acts trivially on $e_{\mathcal{H}_Q}(\mathcal{V})$ for $w \in {}_1W_{M_{Q'_2}}' \cup (\Lambda(1) \cap {}_1W_{M_{Q'_2}}')$. Set ${}_1\mathbb{W}_{M_{Q'_2}}' = \{w \in {}_1W_{M_{Q'_2}}' \mid w \text{ is a lift of some element in } \mathbb{W}_{M_{Q'_2}}\}$ and ${}_1\mathbb{W}_{M_{Q_2}}'$ similarly. Then $Z_k \cap {}_1\mathbb{W}_{M_{Q'_2}}' \subset (\Lambda(1) \cap {}_1W_{M_{Q'_2}}') \cap {}_1W_{M_{Q_2}^+}$ and ${}_1\mathbb{W}_{M_{Q_2}}' \subset {}_1W_{M_{Q_2}}' \cap {}_1W_{M_{Q_2}^+}$. This implies that (4.26) where $Q \subset Q'$ has been replaced by $Q_2 \subset Q'_2$ follows from a congruence

$$(4.27) \quad \sum_{d \in {}^{\mathbb{W}}M_{Q_2} \mathbb{W}_{M_{Q'_2}}} T_{\tilde{d}} \equiv \sum_{d \in {}^{\mathbb{W}}M_{Q_2} \mathbb{W}_{M_{Q'_2}}} q_{\mathbf{w}_{Q_2} d \mathbf{w}_{Q'_2}} T_{\tilde{d}}^*$$

in the finite subring $H({}_1\mathbb{W}_{M_{Q_2}'}')$ of \mathcal{H} generated by $\{T_w \mid w \in {}_1\mathbb{W}_{M_{Q_2}'}'\}$ modulo the right ideal \mathcal{J}_2 with generators $\{\theta_Q(T_w^{Q,*}) - 1 \mid w \in (Z_k \cap {}_1\mathbb{W}_{M_{Q'_2}}') \cup {}_1\mathbb{W}_{M_{Q_2}'}'\}$.

Another simplification concerns $T_{\tilde{d}}^*$ modulo \mathcal{J}_2 for $d \in \mathbb{W}_{M_{Q'_2}}$. We recall that for any reduced decomposition $d = s_1 \dots s_n$ with $s_i \in S \cap \mathbb{W}_{M_{Q'_2}}$ we have $T_{\tilde{d}}^* = (T_{\tilde{s}_1} - c_{\tilde{s}_1}) \dots (T_{\tilde{s}_n} - c_{\tilde{s}_n})$ where the \tilde{s}_i are admissible. For \tilde{s} admissible, by (3.2)

$$c_{\tilde{s}} \equiv q_s - 1.$$

Therefore

$$T_{\tilde{d}}^* \equiv (T_{\tilde{s}_1} - q_{s_1} + 1) \dots (T_{\tilde{s}_n} - q_{s_n} + 1).$$

Let $\mathcal{J}' \subset \mathcal{J}_2$ be the ideal of $H({}_1\mathbb{W}_{M_{Q_2}'}')$ generated by $\{T_t - 1 \mid t \in Z_k \cap {}_1W_{M_{Q_2}'}'\}$. Then the ring $H({}_1\mathbb{W}_{M_{Q_2}'}')/\mathcal{J}'$ and its right ideal $\mathcal{J}_2/\mathcal{J}'$ are the specialization of the generic finite ring $H(\mathbb{W}_{M_{Q_2}})^g$ over $\mathbb{Z}[(q_s)_{s \in S_{M_{Q_2}}}]$ where the q_s for $s \in S_{M_{Q_2}} = S \cap \mathbb{W}_{M_{Q_2}}$ are indeterminates, and of its right ideal \mathcal{J}_2^g with the same generators. The similar congruence modulo \mathcal{J}_2^g in $H(\mathbb{W}_{M_{Q_2}})^g$ (the generic congruence) implies the congruence (4.27) by specialization.

We will prove the generic congruence in a more general setting where H is the generic Hecke ring of a finite Coxeter system (\mathbb{W}, S) and parameters $(q_s)_{s \in S}$ such that $q_s = q_{s'}$ when s, s' are conjugate in \mathbb{W} . The Hecke ring H is a $\mathbb{Z}[(q_s)_{s \in S}]$ -free

module of basis $(T_w)_{w \in \mathbb{W}}$ satisfying the braid relations and the quadratic relations $T_s^2 = q_s + (q_s - 1)T_s$ for $s \in S$. The other basis $(T_w^*)_{w \in \mathbb{W}}$ satisfies the braid relations and the quadratic relations $(T_s^*)^2 = q_s - (q_s - 1)T_s^*$ for $s \in S$, and is related to the first basis by $T_s^* = T_s - (q_s - 1)$ for $s \in S$, and more generally $T_w T_{w^{-1}}^* = T_{w^{-1}}^* T_w = q_w$ for $w \in \mathbb{W}$ [Vig16, Proposition 4.13].

Let $J \subset S$ and \mathcal{J} is the right ideal of H with generators $T_w^* - 1$ for all w in the group \mathbb{W}_J generated by J .

Lemma 4.32. *A basis of \mathcal{J} is $(T_{w_1}^* - 1)T_{w_2}^*$ for $w_1 \in \mathbb{W}_J \setminus \{1\}, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$, and adding $T_{w_2}^*$ for $w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$ gives a basis of H . In particular, \mathcal{J} is a direct factor of \mathcal{H} .*

Proof. The elements $(T_{w_1}^* - 1)T_w^*$ for $w_1 \in \mathbb{W}_J, w \in \mathbb{W}$ generate \mathcal{J} . We write $w = u_1 w_2$ with unique elements $u_1 \in \mathbb{W}_J, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$, and $T_w^* = T_{u_1}^* T_{w_2}^*$. Therefore, $(T_{w_1}^* - 1)T_{u_1}^* T_{w_2}^*$. By an induction on the length of u_1 , one proves that $(T_{w_1}^* - 1)T_{u_1}^*$ is a linear combination of $(T_{v_1}^* - 1)$ for $v_1 \in \mathbb{W}_J$ as in the proof of Proposition 3.4. It is clear that the elements $(T_{w_1}^* - 1)T_{w_2}^*$ and $T_{w_2}^*$ for $w_1 \in \mathbb{W}_J \setminus \{1\}, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$ form a basis of H . \square

Let \mathbf{w}_J denote the longest element of \mathbb{W}_J and $\mathbf{w} = \mathbf{w}_S$.

Lemma 4.33. *In the generic Hecke ring H , the congruence modulo \mathcal{J}*

$$\sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J d \mathbf{w}} T_d^*$$

holds true.

Proof.

Step 1. We show

$${}^{\mathbb{W}_J}\mathbb{W} = \mathbf{w}_J {}^{\mathbb{W}_J}\mathbb{W} \mathbf{w}, \quad q_{\mathbf{w}_J} q_{\mathbf{w}_J d \mathbf{w}} T_d^* = T_{\mathbf{w}_J} T_{\mathbf{w}_J d \mathbf{w}} T_{\mathbf{w}}^*.$$

The equality between the groups follows from the characterization of ${}^{\mathbb{W}_J}\mathbb{W}$ in \mathbb{W} : an element $d \in \mathbb{W}$ has minimal length in $\mathbb{W}_J d$ if and only if $\ell(ud) = \ell(u) + \ell(d)$ for all $u \in \mathbb{W}_J$. An easy computation shows that $\ell(u \mathbf{w}_J d \mathbf{w}) = \ell(u) + \ell(\mathbf{w}_J d \mathbf{w})$ for all $u \in \mathbb{W}_J, d \in {}^{\mathbb{W}_J}\mathbb{W}$ (both sides are equal to $\ell(u) + \ell(\mathbf{w}) - \ell(\mathbf{w}_J) - \ell(d)$). The second equality follows from $q_{\mathbf{w}_J} q_{\mathbf{w}_J d \mathbf{w}} = q_{d \mathbf{w}}$ because $(\mathbf{w}_J)^2 = 1$ and $\ell(\mathbf{w}_J) + \ell(\mathbf{w}_J d \mathbf{w}) = \ell(d \mathbf{w})$ (both sides are $\ell(\mathbf{w}) - \ell(d)$) and from $q_{d \mathbf{w}} T_d^* = T_{d \mathbf{w}} T_{\mathbf{w}_J d^{-1}}^* T_d^* = T_{d \mathbf{w}} T_{\mathbf{w}}^*$. We also have $T_{d \mathbf{w}} = T_{\mathbf{w}_J} T_{\mathbf{w}_J d \mathbf{w}}$.

Step 2. The multiplication by $q_{\mathbf{w}_J}$ on the quotient H/\mathcal{J} is injective (Lemma 4.32) and $q_{\mathbf{w}_J} \equiv T_{\mathbf{w}_J}$. By Step 1, $q_{\mathbf{w}_J} q_{\mathbf{w}_J d \mathbf{w}} T_d^* \equiv T_{\mathbf{w}_J d \mathbf{w}} T_{\mathbf{w}}^*$ and

$$\sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J} q_{\mathbf{w}_J d \mathbf{w}} T_d^* \equiv \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d T_{\mathbf{w}}^*.$$

The congruence

$$(4.28) \quad \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d T_s^*$$

for all $s \in S$ implies the lemma because $T_{\mathbf{w}}^* = T_{s_1}^* \dots T_{s_n}^*$ for any reduced decomposition $\mathbf{w} = s_1 \dots s_n$ with $s_i \in S$.

Step 3. When $J = \emptyset$, the congruence (4.28) is an equality

$$(4.29) \quad \sum_{w \in \mathbb{W}} T_w = \sum_{w \in \mathbb{W}} T_w T_s^*.$$

It holds true because $\sum_{w \in \mathbb{W}} T_w = \sum_{w < w_s} T_w (T_s + 1)$ and $(T_s + 1)T_s^* = T_s T_s^* + T_s^* = q_s + T_s^* = T_s + 1$.

Step 4. Conversely the congruence (4.28) follows from (4.29) because

$$\sum_{w \in \mathbb{W}} T_w = \left(\sum_{u \in W_J} T_u \right) \sum_{d \in {}^{\mathbb{W}J}\mathbb{W}} T_d \equiv \left(\sum_{u \in W_J} q_u \right) \sum_{d \in {}^{\mathbb{W}J}\mathbb{W}} T_d$$

(recall $q_u = T_{u^{-1}}^* T_u \equiv T_u$) and we can simplify by $\sum_{u \in W_J} q_u$ in H/J . \square

This ends the proof of Proposition 4.23.

5. UNIVERSAL REPRESENTATION $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$

The invariant functor $(-)^{\mathcal{U}}$ by the pro- p -Iwahori subgroup \mathcal{U} of G has a left adjoint

$$- \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] : \text{Mod}_R(\mathcal{H}) \rightarrow \text{Mod}_R^{\infty}(G).$$

The smooth R -representation $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G]$ of G constructed from the right \mathcal{H}_R -module \mathcal{V} is called universal. We write

$$R[\mathcal{U} \backslash G] = \mathbb{X}.$$

Question 5.1. Does $\mathcal{V} \neq 0$ imply $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$ or does $v \otimes 1_{\mathcal{U}} = 0$ for $v \in \mathcal{V}$ imply $v = 0$? We have no counterexample. If R is a field and the \mathcal{H}_R -module \mathcal{V} is simple, the two questions are equivalent: $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$ if and only if the map $v \mapsto v \otimes 1_{\mathcal{U}}$ is injective. When R is an algebraically closed field of characteristic p , $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$ for all simple \mathcal{H}_R -modules \mathcal{V} if this is true for \mathcal{V} simple supersingular (this is a consequence of Corollary 5.13).

The functor $- \otimes_{\mathcal{H}_R} \mathbb{X}$ satisfies a few good properties: it has a right adjoint and is compatible with the parabolic induction and the left adjoint (of the parabolic induction). Let $P = MN$ be a standard parabolic subgroup and $\mathbb{X}_M = R[\mathcal{U}_M \backslash M]$. We have functor isomorphisms

$$(5.1) \quad \left(- \otimes_{\mathcal{H}_R} \mathbb{X} \right) \circ \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \rightarrow \text{Ind}_P^G \circ \left(- \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M \right),$$

$$(5.2) \quad (-)_N \circ \left(- \otimes_{\mathcal{H}_R} \mathbb{X} \right) \rightarrow \left(- \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M \right) \circ \mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}.$$

The first one is [OV17, formula 4.15], the second one is obtained by left adjunction from the isomorphism $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \circ (-)^{\mathcal{U}_M} \rightarrow (-)^{\mathcal{U}} \circ \text{Ind}_P^G$ [OV17, formula (4.14)]. If \mathcal{V} is a right \mathcal{H}_R -supersingular module and p is nilpotent in \mathcal{V} , then $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$ if $M \neq G$ (Proposition 4.27). Applying (5.2) we deduce the following.

Proposition 5.2. *If p is nilpotent in \mathcal{V} and \mathcal{V} supersingular, then $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X}$ is left cuspidal.*

Remark 5.3. For a non-zero smooth R -representation τ of M , Δ_τ is orthogonal to Δ_P if τ is left cuspidal. Indeed, we recall from [AHHV17, II.7 Corollary 2] that Δ_τ is not orthogonal to Δ_P if and only if there exists a proper standard parabolic subgroup X of M such that σ is trivial on the unipotent radical of X ; moreover τ is a subrepresentation of $\text{Ind}_X^M(\tau|_X)$, so the image of τ by the left adjoint of Ind_X^M is not 0.

From now on, \mathcal{V} is a non-zero right $\mathcal{H}_{M,R}$ -module and

$$\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M.$$

In general, when $\sigma \neq 0$, let $P_\perp(\sigma)$ be the standard parabolic subgroup of G with $\Delta_{P_\perp(\sigma)} = \Delta_P \cup \Delta_{\perp,\sigma}$ where $\Delta_{\perp,\sigma}$ is the set of simple roots $\alpha \in \Delta_\sigma$ orthogonal to Δ_P .

Proposition 5.4.

- (1) $P(\mathcal{V}) \subset P_\perp(\sigma)$ if $\sigma \neq 0$.
- (2) $P(\mathcal{V}) = P_\perp(\sigma)$ if the map $v \mapsto v \otimes 1_{\mathcal{U}_M}$ is injective.
- (3) $P(\mathcal{V}) = P(\sigma)$ if the map $v \mapsto v \otimes 1_{\mathcal{U}_M}$ is injective, p nilpotent in \mathcal{V} and \mathcal{V} supersingular.
- (4) $P(\mathcal{V}) = P(\sigma)$ if $\sigma \neq 0$, R is a field of characteristic p and \mathcal{V} simple supersingular.

Proof.

(1) $P(\mathcal{V}) \subset P_\perp(\sigma)$ means that $Z \cap M'_\mathcal{V}$ acts trivially on $\mathcal{V} \otimes 1_{\mathcal{U}_M}$, where $M_\mathcal{V}$ is the standard Levi subgroup such that $\Delta_{M_\mathcal{V}} = \Delta_\mathcal{V}$. Let $z \in Z \cap M'_\mathcal{V}$, and $v \in \mathcal{V}$. As Δ_M and $\Delta_\mathcal{V}$ are orthogonal, we have $T^{M,*}(z) = T^M(z)$ and $\mathcal{U}_M z \mathcal{U}_M = \mathcal{U}_M z$. We have $v \otimes 1_{\mathcal{U}_M} = v T^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes T^M(z) 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M z} = v \otimes z^{-1} 1_{\mathcal{U}_M} = z^{-1}(v \otimes 1_{\mathcal{U}_M})$.

(2) If $v \otimes 1_{\mathcal{U}_M} = 0$ for $v \in \mathcal{V}$ implies $v = 0$, then $\sigma \neq 0$ because $\mathcal{V} \neq 0$. By (1) $P(\mathcal{V}) \subset P_\perp(\sigma)$. As in the proof of (1), for $z \in Z \cap M'_{\perp,\sigma}$ we have $v T^{M,*}(z) \otimes 1_{\mathcal{U}_M} = v T^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M}$ and our hypothesis implies $v T^{M,*}(z) = v$ hence $P(\mathcal{V}) \supset P_\perp(\sigma)$.

(3) Proposition 5.2, Remark 5.3, and (2).

(4) Question 5.1 and (3). □

Let Q be a parabolic subgroup of G with $P \subset Q \subset P(\mathcal{V})$. In this chapter we will compute $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ where $I_{\mathcal{H}}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\epsilon(\mathcal{V}) \otimes (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$ (Theorem 5.11). The smooth R -representation $I_G(P, \sigma, Q)$ of G is well defined: it is 0 if $\sigma = 0$ and $\text{Ind}_{P(\sigma)}^G(\epsilon(\sigma) \otimes \text{St}_Q^{P(\sigma)})$ if $\sigma \neq 0$ because (P, σ, Q) is an $R[G]$ -triple by Proposition 5.4. We will show that the universal representation $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ is isomorphic to $I_G(P, \sigma, Q)$, if $P(\mathcal{V}) = P(\sigma)$ and $p = 0$, or if $\sigma = 0$ (Corollary 5.12). In particular, $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$ when R is an algebraically closed field of characteristic p and \mathcal{V} is supersingular.

5.1. $Q = G$. We consider first the case $Q = G$. We are in the simple situation where \mathcal{V} is extensible to \mathcal{H} and $P(\mathcal{V}) = P(\sigma) = G$, $I_{\mathcal{H}}(P, \mathcal{V}, G) = \epsilon(\mathcal{V})$ and $I_G(P, \sigma, G) = \epsilon(\sigma)$. We recall that $\Delta \setminus \Delta_P$ is orthogonal to Δ_P and that M_2 denotes the standard Levi subgroup of G with $\Delta_{M_2} = \Delta \setminus \Delta_P$.

The \mathcal{H}_R -morphism $e(\mathcal{V}) \rightarrow e(\sigma)^{\mathcal{U}} = \sigma^{\mathcal{U}_M}$ sending v to $v \otimes 1_{\mathcal{U}_M}$ for $v \in \mathcal{V}$, gives by adjunction an $R[G]$ -homomorphism

$$v \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} : e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi^G} e(\sigma).$$

If Φ^G is an isomorphism, then $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ is the extension to G of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$, meaning that M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. The converse is true.

Lemma 5.5. *If M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$, then Φ^G is an isomorphism.*

Proof. Suppose that M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. Then $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ is the extension to G of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$, and by Theorem 3.13, $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}}$ is the extension of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$. Therefore, by (3.12),

$$(v \otimes 1_{\mathcal{U}})T_w^* = (v \otimes 1_{\mathcal{U}})T_w^{M,*} \quad \text{for all } v \in \mathcal{V}, w \in W_M(1).$$

As \mathcal{V} is extensible to \mathcal{H} , the natural map $v \mapsto v \otimes 1_{\mathcal{U}} : \mathcal{V} \xrightarrow{\Psi} (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$ is \mathcal{H}_M -equivariant, i.e.,

$$vT_w^{M,*} \otimes 1_{\mathcal{U}} = (v \otimes 1_{\mathcal{U}})T_w^{M,*} \quad \text{for all } v \in \mathcal{V}, w \in W_M(1)$$

because (3.12) $vT_w^{M,*} \otimes 1_{\mathcal{U}} = vT_w^* \otimes 1_{\mathcal{U}} = v \otimes T_w^* = (v \otimes 1_{\mathcal{U}})T_w^*$ in $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$.

We recall that $-\otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$ is the left adjoint of $(-)^{\mathcal{U}_M}$. The adjoint $R[M]$ -homomorphism $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M \rightarrow e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ sends $v \otimes 1_{\mathcal{U}_M}$ to $v \otimes 1_{\mathcal{U}}$ for all $v \in \mathcal{V}$. The $R[M]$ -module generated by the $v \otimes 1_{\mathcal{U}}$ for all $v \in \mathcal{V}$ is equal to $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ because M'_2 acts trivially. Hence we obtained an inverse of Φ^G . \square

Our next move is to determine if M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. It is equivalent to see if M'_2 acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$ as this set generates the representation $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ of G and M'_2 is a normal subgroup of G as M'_2 and M commute and $G = ZM'M'_2$. Obviously, $\mathcal{U} \cap M'_2$ acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$. The group of double classes $(\mathcal{U} \cap M'_2) \backslash M'_2 / (\mathcal{U} \cap M'_2)$ is generated by the lifts $\hat{s} \in \mathcal{N} \cap M'_2$ of the simple affine roots s of $W_{M'_2}$. Therefore, M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ if and only if for any simple affine root $s \in S_{M'_2}^{\text{aff}}$ of $W_{M'_2}$, any $\hat{s} \in \mathcal{N} \cap M'_2$ lifting s acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$.

Lemma 5.6. *Let $v \in \mathcal{V}$, $s \in S_{M'_2}^{\text{aff}}$ and $\hat{s} \in \mathcal{N} \cap M'_2$ lifting s . We have*

$$(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0.$$

Proof. We compute:

$$T_s(\hat{s}1_{\mathcal{U}}) = \hat{s}(T_s1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-1}} = \sum_u \hat{s}u(\hat{s})^{-1}1_{\mathcal{U}} = \sum_{u^{op}} u^{op}1_{\mathcal{U}},$$

$$T_s(\hat{s}^21_{\mathcal{U}}) = \hat{s}^2(T_s1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-2}} = 1_{\mathcal{U}(\hat{s})^{-1}\mathcal{U}} = \sum_u u\hat{s}1_{\mathcal{U}}$$

for u in the group $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U})$ and u^{op} in the group $\hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U})$; the reason is that \hat{s}^2 normalizes \mathcal{U} , $\mathcal{U}\hat{s}\mathcal{U}\hat{s}^{-1}$ is the disjoint union of the sets $\mathcal{U}\hat{s}u^{-1}(\hat{s})^{-1}$ and $\mathcal{U}(\hat{s})^{-1}\mathcal{U}$ is the disjoint union of the sets $\mathcal{U}(\hat{s})^{-1}u^{-1}$. We introduce now a natural bijection

$$(5.3) \quad u \rightarrow u^{op} : \mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U}) \rightarrow \hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U})$$

which is not a group homomorphism. We recall the finite reductive group $G_{k,s}$ quotient of the parahoric subgroup \mathfrak{K}_s of G fixing the face fixed by s of the alcove \mathcal{C} . The Iwahori groups $Z^0\mathcal{U}$ and $Z^0\hat{s}\mathcal{U}(\hat{s})^{-1}$ are contained in \mathfrak{K}_s and their images

in $G_{s,k}$ are opposite Borel subgroups $Z_k U_{s,k}$ and $Z_k U_{s,k}^{op}$. Via the surjective maps $u \mapsto \bar{u} : \mathcal{U} \rightarrow U_{s,k}$ and $u^{op} \mapsto \bar{u}^{op} : \hat{s}\mathcal{U}(\hat{s})^{-1} \rightarrow U_{s,k}^{op}$ we identify the groups $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U}) \simeq U_{s,k}$ and similarly $\hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U}) \simeq U_{s,k}^{op}$. Let $G'_{k,s}$ be the group generated by $U_{s,k}$ and $U_{s,k}^{op}$, and let $B'_{s,k} = G'_{k,s} \cap Z_k U_{s,k} = (G'_{k,s} \cap Z_k) U_{s,k}$. We suppose (as we can) that $\hat{s} \in \mathfrak{K}_s$ and that its image \hat{s}_k in $G_{s,k}$ lies in $G'_{k,s}$. We have $\hat{s}_k U_{s,k}(\hat{s}_k)^{-1} = U_{s,k}^{op}$ and the Bruhat decomposition $G'_{k,s} = B'_{k,s} \sqcup U_{k,s} \hat{s}_k B'_{k,s}$ implies the existence of a canonical bijection $\bar{u}^{op} \rightarrow \bar{u} : (U_{k,s}^{op} - \{1\}) \rightarrow (U_{k,s} - \{1\})$ respecting the cosets $\bar{u}^{op} B'_{k,s} = \bar{u} \hat{s}_k B'_{k,s}$. Via the preceding identifications we get the wanted bijection (5.3).

For $v \in e(\mathcal{V})$ and $z \in Z^0 \cap M'_2$ we have $vT_z = v$, $z1_{\mathcal{U}} = T_z 1_{\mathcal{U}}$ and $v \otimes T_z 1_{\mathcal{U}} = vT_z \otimes 1_{\mathcal{U}}$ therefore $Z^0 \cap M'_2$ acts trivially on $\mathcal{V} \otimes 1_{\mathcal{U}}$. The action of the group $(Z^0 \cap M'_2)\mathcal{U}$ on $\mathcal{V} \otimes 1_{\mathcal{U}}$ is also trivial. As the image of $Z^0 \cap M'_2$ in $G_{s,k}$ contains $Z_k \cap G'_{s,k}$,

$$u\hat{s}(v \otimes 1_{\mathcal{U}}) = u^{op}(v \otimes 1_{\mathcal{U}})$$

when u and u^{op} are not units and correspond via the bijection (5.3). So we have

$$(5.4) \quad v \otimes T_s(\hat{s}1_{\mathcal{U}}) - (v \otimes 1_{\mathcal{U}}) = v \otimes T_s(\hat{s}^2 1_{\mathcal{U}}) - v \otimes \hat{s}1_{\mathcal{U}}.$$

We can move T_s on the other side of \otimes and as $vT_s = q_s v$ (Corollary 3.9), we can replace T_s by q_s . We have $v \otimes \hat{s}^2 1_{\mathcal{U}} = v \otimes T_{s^{-2}} 1_{\mathcal{U}}$ because $\hat{s}^2 \in Z^0 \cap M'_2$ normalizes \mathcal{U} ; as we can move $T_{s^{-2}}$ on the other side of \otimes and as $vT_{s^{-2}} = v$ we can forget \hat{s}^2 . So (5.4) is equivalent to $(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0$. \square

Combining the two lemmas we obtain the following.

Proposition 5.7. *When \mathcal{V} is extensible to \mathcal{H} and has no $q_s + 1$ -torsion for any $s \in S_{M'_2}^{\text{aff}}$, then M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ and Φ^G is an $R[G]$ -isomorphism.*

Proposition 5.7 for the trivial character $\mathbf{1}_{\mathcal{H}}$, says that $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$ is the trivial representation $\mathbf{1}_G$ of G when $q_s + 1$ has no torsion in R for all $s \in S^{\text{aff}}$. This is proved in [OV17, Lemma 2.28] by a different method. The following counter-example shows that this is not true for all R .

Example 5.8. Let $G = GL(2, F)$ and let R be an algebraically closed field where $q_{s_0} + 1 = q_{s_1} + 1 = 0$ and $S_{\text{aff}} = \{s_0, s_1\}$. (Note that $q_{s_0} = q_{s_1}$ is the order of the residue field of F .) Then the dimension of $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$ is infinite, in particular $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \neq \mathbf{1}_G$.

Indeed, the Steinberg representation $\text{St}_G = (\text{Ind}_B^G \mathbf{1}_Z) / \mathbf{1}_G$ of G is an indecomposable representation of length 2 containing an irreducible infinite dimensional representation π with $\pi^{\mathcal{U}} = 0$ of quotient the character $(-1)^{\text{val} \circ \det}$. This follows from the proof of Theorem 3 and from Proposition 24 in [Vig89]. The kernel of the quotient map $\text{St}_G \otimes (-1)^{\text{val} \circ \det} \rightarrow \mathbf{1}_G$ is infinite dimensional without a non-zero \mathcal{U} -invariant vector. As the characteristic of R is not p , the functor of \mathcal{U} -invariants is exact hence $(\text{St}_G \otimes (-1)^{\text{val} \circ \det})^{\mathcal{U}} = \mathbf{1}_{\mathcal{H}}$. As $- \otimes_{\mathcal{H}_R} R[\mathcal{U} \setminus G]$ is the left adjoint of $(-)^{\mathcal{U}}$ there is a non-zero homomorphism

$$\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow \text{St}_G \otimes (-1)^{\text{val} \circ \det}$$

with image generated by its \mathcal{U} -invariants. The homomorphism is therefore surjective.

5.2. \mathcal{V} **extensible to \mathcal{H}** . Let $P = MN$ be a standard parabolic subgroup of G with Δ_P and $\Delta \setminus \Delta_P$ orthogonal. We still suppose that the $\mathcal{H}_{M,R}$ -module \mathcal{V} is extensible to \mathcal{H} , but now $P \subset Q \subset G$. So we have $I_{\mathcal{H}}(P, \mathcal{V}, Q) = e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$ and $I_G(P, \sigma, Q) = e(\sigma) \otimes_R \text{St}_Q^G$ where $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$. We compare the images by $-\otimes_{\mathcal{H}_R} \mathbb{X}$ of the \mathcal{H}_R -modules $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ and $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$ with the smooth R -representations $e(\sigma) \otimes \text{Ind}_Q^G \mathbf{1}$ and $e(\sigma) \otimes \text{St}_Q^G$ of G .

As $-\otimes_{\mathcal{H}_R} \mathbb{X}$ is left adjoint of $(-)^{\mathcal{U}}$, the \mathcal{H}_R -homomorphism $v \otimes f \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f : e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \rightarrow (e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ gives by adjunction an $R[G]$ -homomorphism

$$v \otimes f \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f : (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi_Q^G} e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}.$$

When $Q = G$ we have $\Phi_G^G = \Phi^G$. By Remark 4.10, Φ_Q^G is surjective. Proposition 5.7 applies with M_Q instead of G and gives the $R[M_Q]$ -homomorphism

$$v \otimes 1_{\mathcal{U}_{M_Q}} \mapsto v \otimes 1_{\mathcal{U}_M} : e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q} \xrightarrow{\Phi_Q^G} e_Q(\sigma).$$

Proposition 5.9. *The $R[G]$ -homomorphism Φ_Q^G is an isomorphism if Φ^Q is an isomorphism, in particular if \mathcal{V} has no $q_s + 1$ -torsion for any $s \in S_{M_2' \cap M_Q}^{\text{aff}}$.*

Proof. The proposition follows from another construction of Φ_Q^G that we now describe. Proposition 4.5 gives the \mathcal{H}_R -module isomorphism

$$v \otimes f_{QU} \mapsto v \otimes 1_{\mathcal{H}} : (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+, \theta}} \mathcal{H}.$$

We have the $R[G]$ -isomorphism [OV17, Corollary 4.7]

$$v \otimes 1_{\mathcal{H}} \otimes 1_{\mathcal{U}} \mapsto f_{QU, v \otimes 1_{\mathcal{U}_{M_Q}}} : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow \text{Ind}_Q^G(e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q})$$

and the $R[G]$ -isomorphism

$$f_{QU, v \otimes 1_{\mathcal{U}_M}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f_{QU} : \text{Ind}_Q^G(e_Q(\sigma)) \rightarrow e(\sigma) \otimes \text{Ind}_Q^G \mathbf{1}.$$

From Φ^Q and these three homomorphisms, there exists a unique $R[G]$ -homomorphism

$$(e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$$

sending $v \otimes f_{QU} \otimes 1_{\mathcal{U}}$ to $v \otimes 1_{\mathcal{U}_M} \otimes f_{QU}$. We deduce: this homomorphism is equal to Φ_Q^G , $\mathcal{V} \otimes 1_{QU} \otimes 1_{\mathcal{U}}$ generates $(e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X}$, if Φ^Q is an isomorphism, then Φ_Q^G is an isomorphism. By Proposition 5.7, if \mathcal{V} has no $q_s + 1$ -torsion for any $s \in S_{M_2' \cap M_Q}^{\text{aff}}$, then Φ^Q and Φ_Q^G are isomorphisms. \square

We recall that the $\mathcal{H}_{M,R}$ -module \mathcal{V} is extensible to \mathcal{H} .

Proposition 5.10. *The $R[G]$ -homomorphism Φ_Q^G induces an $R[G]$ -homomorphism*

$$(e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow e(\sigma) \otimes_R \text{St}_Q^G,$$

It is an isomorphism if Φ_Q^G is an $R[G]$ -isomorphism for all parabolic subgroups Q' of G containing Q , in particular if \mathcal{V} has no $q_s + 1$ -torsion for any $s \in S_{M_2'}^{\text{aff}}$.

Proof. The proof is straightforward, with the arguments already developed for Proposition 4.5 and Theorem 4.9. The representations $e(\sigma) \otimes_R \text{St}_Q^G$ and $(e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}$ of G are the cokernels of the natural $R[G]$ -homomorphisms

$$\begin{aligned} & \oplus_{Q \subsetneq Q'} e(\sigma) \otimes_R \text{Ind}_{Q'}^G \mathbf{1} \xrightarrow{\text{id} \otimes \alpha} e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}, \\ & \oplus_{Q \subsetneq Q'} (e(\mathcal{V}) \otimes_R (\text{Ind}_{Q'}^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\text{id} \otimes \alpha^\mathcal{U} \otimes \text{id}} (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}. \end{aligned}$$

These $R[G]$ -homomorphisms make a commutative diagram with the $R[G]$ -homomorphisms $\bigoplus_{Q \subsetneq Q'} \Phi_{Q'}^G$ and Φ_Q^G going from the lower line to the upper line. Indeed, let $v \otimes f_{Q'U} \otimes 1_U \in (e(\mathcal{V}) \otimes_R (\text{Ind}_{Q'}^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}$. On the one hand, it goes to $v \otimes f_{QU} \theta_{Q'}(e_{Q'}^{\mathcal{Q}'}) \otimes 1_U \in (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}$ by the horizontal map, and then to $v \otimes 1_{U_M} \otimes f_{QU} \theta_{Q'}(e_{Q'}^{\mathcal{Q}'})$ by the vertical map. On the other hand, it goes to $v \otimes 1_{U_M} \otimes f_{Q'U}$ by the vertical map, and then to $v \otimes 1_{U_M} \otimes f_{QU} \theta_{Q'}(e_{Q'}^{\mathcal{Q}'})$ by the horizontal map. One deduces that Φ_Q^G induces an $R[G]$ -homomorphism $(e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow e(\sigma) \otimes_R \text{St}_Q^G$, which is an isomorphism if $\Phi_{Q'}^G$ is an $R[G]$ -isomorphism for all $Q \subset Q'$. \square

5.3. General. We consider now the general case: let $P = MN \subset Q$ be two standard parabolic subgroups of G and let \mathcal{V} be a non-zero right $\mathcal{H}_{M,R}$ -module with $Q \subset P(\mathcal{V})$. We recall $I_{\mathcal{H}}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\text{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$ and $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$ (Proposition 5.4). There is a natural $R[G]$ -homomorphism

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi_I^G} \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})})$$

obtained by composition of the $R[G]$ -isomorphism [OV17, Corollary 4.7] (proof of Proposition 5.9):

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow \text{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathbb{X}_{M(\mathcal{V})}),$$

with the $R[G]$ -homomorphism

$$\text{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V}) \otimes_R (\text{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathbb{X}_{M(\mathcal{V})}) \rightarrow \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})}),$$

image by the parabolic induction $\text{Ind}_{P(\mathcal{V})}^G$ of the homomorphism

$$(e(\mathcal{V}) \otimes_R (\text{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathbb{X}_{M(\mathcal{V})} \rightarrow e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})}$$

induced by the $R[M(\mathcal{V})]$ -homomorphism $\Phi_Q^{P(\mathcal{V})} = \Phi_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})}$ of Proposition 5.10 applied to $M(\mathcal{V})$ instead of G .

This homomorphism Φ_I^G is an isomorphism if $\Phi_Q^{P(\mathcal{V})}$ is an isomorphism, in particular if \mathcal{V} has no $q_s + 1$ -torsion for any $s \in S_{M_2}^{\text{aff}}$ where $\Delta_{M_2} = \Delta_{M(\mathcal{V})} \setminus \Delta_M$ (Proposition 5.10). We get the main theorem of this section.

Theorem 5.11. *Let $(P = MN, \mathcal{V}, Q)$ be an \mathcal{H}_R -triple and $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$. Then, (P, σ, Q) is an $R[G]$ -triple. The $R[G]$ -homomorphism*

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \setminus G] \xrightarrow{\Phi_I^G} \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})})$$

is an isomorphism if $\Phi_Q^{P(\mathcal{V})}$ is an isomorphism. In particular Φ_I^G is an isomorphism if \mathcal{V} has no $q_s + 1$ -torsion for any $s \in S_{M_2}^{\text{aff}}$.

Recalling $I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \text{St}_Q^{P(\sigma)})$ when $\sigma \neq 0$, we deduce the following.

Corollary 5.12. *We have the following:*

$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$, if $\sigma \neq 0$, $P(\mathcal{V}) = P(\sigma)$ and \mathcal{V} has no $q_s + 1$ -torsion for any $s \in S_{M'_2}^{\text{aff}}$.

$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] = I_G(P, \sigma, Q) = 0$, if $\sigma = 0$.

Recalling $P(\mathcal{V}) = P(\sigma)$ if $\sigma \neq 0$, R is a field of characteristic p and \mathcal{V} simple supersingular (Proposition 5.4 (4)), we deduce the following.

Corollary 5.13. $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$ if R is a field of characteristic p and \mathcal{V} simple supersingular.

6. VANISHING OF THE SMOOTH DUAL

Let V be an $R[G]$ -module. The dual $\text{Hom}_R(V, R)$ of V is an $R[G]$ -module for the contragredient action: $gL(gv) = L(v)$ if $g \in G$, $L \in \text{Hom}_R(V, R)$ is a linear form and $v \in V$. When $V \in \text{Mod}_R^\infty(G)$ is a smooth R -representation of G , the dual of V is not necessarily smooth. A linear form L is smooth if there exists an open subgroup $H \subset G$ such that $L(hv) = L(v)$ for all $h \in H, v \in V$; the space $\text{Hom}_R(V, R)^\infty$ of smooth linear forms is a smooth R -representation of G , called the **smooth dual** (or smooth contragredient) of V . The smooth dual of V is contained in the dual of V .

Example 6.1. When R is a field and the dimension of V over R is finite, the dual of V is equal to the smooth dual of V because the kernel of the action of G on V is an open normal subgroup $H \subset G$; the action of G on the dual $\text{Hom}_R(V, R)$ is trivial on H .

We assume in this section that R is a field of characteristic p . Let $P = MN$ be a parabolic subgroup of G and $V \in \text{Mod}_R^\infty(M)$. Generalizing the proof given in [Vig07, 8.1] when $G = GL(2, F)$ and the dimension of V is 1, we show the following.

Proposition 6.2. *If $P \neq G$, the smooth dual of $\text{Ind}_P^G(V)$ is 0.*

Proof. Let L be a smooth linear form on $\text{Ind}_P^G(V)$ and let K be an open pro- p -subgroup of G which fixes L . Let J be an arbitrary open subgroup of K , $g \in G$ and $f \in (\text{Ind}_P^G(V))^J$ with support PgJ . We want to show that $L(f) = 0$. Let J' be any open normal subgroup of J and let φ denote the function in $(\text{Ind}_P^G(V))^{J'}$ with support PgJ' and value $\varphi(g) = f(g)$ at g . For $j \in J$ we have $L(j\varphi) = L(\varphi)$, and the support of $j\varphi(x) = \varphi(xj)$ is $PgJ'j^{-1}$. The function f is the sum of translates $j\varphi$, where j ranges through the left cosets of the image X of $g^{-1}Pg \cap J$ in J/J' , so that $L(f) = rL(\varphi)$ where r is the order of X in J/J' . We can certainly find J' such that $r \neq 1$, and then r is a positive power of p . As the characteristic of R is p we have $L(f) = 0$. \square

The module $R[\mathcal{U} \backslash G]$ is contained in the module $R^{\mathcal{U} \backslash G}$ of functions $f : \mathcal{U} \backslash G \rightarrow R$. The actions of \mathcal{H} and of G on $R[\mathcal{U} \backslash G]$ extend to $R^{\mathcal{U} \backslash G}$ by the same formulas. The pairing

$$(f, \varphi) \mapsto \langle f, \varphi \rangle = \sum_{g \in \mathcal{U} \backslash G} f(g)\varphi(g) : R^{\mathcal{U} \backslash G} \times R[\mathcal{U} \backslash G] \rightarrow R$$

identifies $R^{\mathcal{U}\setminus G}$ with the dual of $R[\mathcal{U}\setminus G]$. Let $h \in \mathcal{H}$ and $\check{h} \in \mathcal{H}$, $\check{h}(g) = h(g^{-1})$ for $g \in G$. We have

$$\langle f, h\varphi \rangle = \langle \check{h}f, \varphi \rangle.$$

Proposition 6.3. *When R is an algebraically closed field of characteristic p , G is not compact modulo the center and \mathcal{V} is a simple supersingular right \mathcal{H}_R -module, the smooth dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\setminus G]$ is 0.*

Proof. Let $\mathcal{H}_R^{\text{aff}}$ be the subalgebra of \mathcal{H}_R of basis $(T_w)_{w \in W'(1)}$ where $W'(1)$ is the inverse image of W' in $W(1)$. The dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\setminus G]$ is contained in the dual of $\mathcal{V} \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\setminus G]$; the $\mathcal{H}_R^{\text{aff}}$ -module $\mathcal{V}|_{\mathcal{H}_R^{\text{aff}}}$ is a finite sum of supersingular characters [Vig15a]. Let $\chi : \mathcal{H}_R^{\text{aff}} \rightarrow R$ be a supersingular character. The dual of $\chi \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\setminus G]$ is contained in the dual of $R[\mathcal{U}\setminus G]$ isomorphic to $R^{\mathcal{U}\setminus G}$. It is the space of $f \in R^{\mathcal{U}\setminus G}$ with $\check{h}f = \chi(h)f$ for all $h \in \mathcal{H}_R^{\text{aff}}$. The smooth dual of $\chi \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\setminus G]$ is 0 if the dual of $\chi \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\setminus G]$ has no non-zero element fixed by \mathcal{U} . Let us take $f \in R^{\mathcal{U}\setminus G/\mathcal{U}}$ with $\check{h}f = \chi(h)f$ for all $h \in \mathcal{H}_R^{\text{aff}}$. We shall prove that $f = 0$. We have $\check{T}_w = T_{w^{-1}}$ for $w \in W(1)$.

Let $<$ denote the Bruhat order of $W(1)$ associated to S^{aff} [Vig16]. The elements $(T_t)_{t \in Z_k}$ and $(T_{\tilde{s}})_{s \in S^{\text{aff}}}$ where \tilde{s} is an admissible lift of s in $W^{\text{aff}}(1)$, generate the algebra $\mathcal{H}_R^{\text{aff}}$ and

$$T_t T_w = T_{tw}, \quad T_{\tilde{s}} T_w = \begin{cases} T_{\tilde{s}w}, & \tilde{s}w > w, \\ c_{\tilde{s}} T_w, & \tilde{s}w < w, \end{cases}$$

with $c_{\tilde{s}} = -|Z'_{k,s}| \sum_{t \in Z'_{k,s}} T_t$ because the characteristic of R is p [Vig16, Proposition 4.4]. Expressing $f = \sum_{w \in W(1)} a_w T_w$, $a_w \in R$, as an infinite sum, we have

$$T_t f = \sum_{w \in W(1)} a_{t^{-1}w} T_w, \quad T_{\tilde{s}} f = \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w c_{\tilde{s}}) T_w.$$

A character χ of $\mathcal{H}_R^{\text{aff}}$ is associated to a character $\chi_k : Z_k \rightarrow R^*$ and a subset J of

$$S_{\chi_k}^{\text{aff}} = \{s \in S^{\text{aff}} \mid (\chi_k)|_{Z'_{k,s}} \text{ trivial} \}$$

[Vig15a, Definition 2.7]. We have

$$(6.1) \quad \begin{cases} \chi(T_t) = \chi_k(t), & t \in Z_k, \\ \chi(T_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\text{aff}} \setminus J, \\ -1, & s \in J. \end{cases} \end{cases} \quad (\chi_k)(c_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\text{aff}} \setminus S_{\chi_k}^{\text{aff}}, \\ -1, & s \in S_{\chi_k}^{\text{aff}}. \end{cases}$$

Therefore $\chi_k(t)f = \check{T}_t f = T_{t^{-1}} f$ hence $\chi_k(t)a_w = a_{tw}$. We have $\chi(T_{\tilde{s}})f = \check{T}_{\tilde{s}} f = T_{(\tilde{s})^{-1}} f = T_{\tilde{s}} T_{(\tilde{s})^{-2}} f = \chi_k((\tilde{s})^2) T_{\tilde{s}} f$; as $(\tilde{s})^2 \in Z'_{k,s}$ [Vig16, three lines before Proposition 4.4] and $J \subset S_{\chi_k}^{\text{aff}}$, we obtain

$$(6.2) \quad T_{\tilde{s}} f = \begin{cases} 0, & s \in S^{\text{aff}} \setminus J, \\ -f, & s \in J. \end{cases}$$

Introducing $\chi_k(t)a_w = a_{tw}$ in the formula for $T_{\tilde{s}}f$, we get

$$\begin{aligned}
\sum_{w \in W(1), \tilde{s}w < w} a_w c_{\tilde{s}} T_w &= -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_w T_{tw} \\
&= -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_{t^{-1}w} T_w \\
&= -|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} \chi_k(t^{-1}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w \\
&= \chi_k(c_{\tilde{s}}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w.
\end{aligned}$$

$$\begin{aligned}
T_{\tilde{s}}f &= \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w \chi_k(c_{\tilde{s}})) T_w \\
&= \begin{cases} \sum_{w \in W(1), \tilde{s}w < w} a_{(\tilde{s})^{-1}w} T_w, & s \in S^{\text{aff}} \setminus S_{\chi_k}^{\text{aff}}, \\ \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} - a_w) T_w, & s \in S_{\chi_k}^{\text{aff}}. \end{cases}
\end{aligned}$$

From the last equality and (6.2) for $T_{\tilde{s}}f$, we get:

$$(6.3) \quad a_{\tilde{s}w} = \begin{cases} 0, & s \in J \cup (S^{\text{aff}} \setminus S_{\chi_k}^{\text{aff}}), \tilde{s}w < w, \\ a_w, & s \in S_{\chi_k}^{\text{aff}} \setminus J. \end{cases}$$

Assume that $a_w \neq 0$. By the first condition, we know that $w > \tilde{s}w$ for $s \in J \cup (S^{\text{aff}} \setminus S_{\chi_k}^{\text{aff}})$. The character χ is supersingular if for each irreducible component X of S^{aff} , the intersection $X \cap J$ is not empty and different from X [Vig15a, Definition 2.7, Theorem 6.18]. This implies that the group generated by the $s \in S_{\chi_k}^{\text{aff}} \setminus J$ is finite. If χ is supersingular, by the second condition we can suppose $w > \tilde{s}w$ for any $s \in S^{\text{aff}}$. But there is no such element if S^{aff} is not empty. \square

Theorem 6.4. *Let π be an irreducible admissible R -representation of G with a non-zero smooth dual where R is an algebraically closed field of characteristic p . Then π is finite dimensional.*

Proof. Let (P, σ, Q) be an $R[G]$ -triple with σ supercuspidal such that $\pi \simeq I_G(P, \sigma, Q)$. The representation $I_G(P, \sigma, Q)$ is a quotient of $\text{Ind}_Q^G e_Q(\sigma)$ hence the smooth dual of $\text{Ind}_Q^G e_Q(\sigma)$ is not zero. From Proposition 6.2, $Q = G$. We have $I_G(P, \sigma, G) = e(\sigma)$. The smooth dual of σ contains the smooth linear dual of $e(\sigma)$ hence is not zero. As σ is supercuspidal, the \mathcal{H}_M -module $\sigma^{\mathcal{U}_M}$ contains a simple supersingular submodule \mathcal{V} [Vig15a, Proposition 7.10, Corollary 7.11]. The functor $- \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$ being the right adjoint of $(-)^{\mathcal{U}_M}$, the irreducible representation σ is a quotient of $\mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$, hence the smooth dual of $\mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$ is not zero. By Proposition 6.3, $M = Z$. Hence σ is finite dimensional and the same is true for $e(\sigma) = I_G(B, \sigma, G) \simeq \pi$. \square

Remark 6.5. When the characteristic of F is 0, Theorem 6.4 was proved by Kohlhaase for a field R of characteristic p . He gives two proofs [Koh, Proposition 3.9, Remark 3.10], but none of them extends to F of characteristic p . Our proof is valid without restriction on the characteristic of F and does not use the results of Kohlhaase. Our assumption that R is an algebraically closed field of characteristic p comes from the classification theorem in [AHHV17].

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