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ON PRO-$p$-IWAHORI INVARIANTS OF $R$-REPRESENTATIONS OF REDUCTIVE $p$-ADIC GROUPS

N. ABE, G. HENNIART, AND M.-F. VIGNÉRAS

Abstract. Let $F$ be a locally compact field with residue characteristic $p$, and let $G$ be a connected reductive $F$-group. Let $U$ be a pro- $p$ Iwahori subgroup of $G = G(F)$. Fix a commutative ring $R$. If $\pi$ is a smooth $R[G]$-representation, the space of invariants $\pi^U$ is a right module over the Hecke algebra $\mathcal{H}$ of $U$ in $G$.

Let $P$ be a parabolic subgroup of $G$ with a Levi decomposition $P = MN$ adapted to $U$. We complement a previous investigation of Olivier-Vignéras on the relation between taking $U$-invariants and various functor like Ind$_G^P$ and right and left adjoints. More precisely the authors’ previous work with Herzig introduced representations $I_G(P, \sigma, Q)$ where $\sigma$ is a smooth representation of $M$ extending, trivially on $N$, to a larger parabolic subgroup $P(\sigma)$, and $Q$ is a parabolic subgroup between $P$ and $P(\sigma)$. Here we relate $I_G(P, \sigma, Q)^U$ to an analogously defined $\mathcal{H}$-module $I_H(P, \sigma^{U_M}, Q)$, where $U_M = U \cap M$ and $\sigma^{U_M}$ is seen as a module over the Hecke algebra $\mathcal{H}_M$ of $U_M$ in $M$. In the reverse direction, if $V$ is a right $\mathcal{H}_M$-module, we relate $I_H(P, V, Q) \otimes_{c}\text{Ind}_{U_M}^{U} 1$ to $I_G(P, V \otimes_{\mathcal{H}_M} \text{c-Ind}_{U_M}^{U} 1, Q)$. As an application we prove that if $R$ is an algebraically closed field of characteristic $p$, and $\pi$ is an irreducible admissible representation of $G$, then the contragredient of $\pi$ is 0 unless $\pi$ has finite dimension.

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1. Introduction

1.1. The present paper is a companion to [AHV] and is similarly inspired by the classification results of [AHHV17]; however it can be read independently. We recall the setting. We have a non-archimedean locally compact field $F$ of residue characteristic $p$ and a connected reductive $F$-group $G$. We fix a commutative ring $R$ and study the smooth $R$-representations of $G = G(F)$.

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In [AHHV17] the irreducible admissible R-representations of G are classified in terms of supersingular ones when R is an algebraically closed field of characteristic p. That classification is expressed in terms of representations \( \text{Ind}_G^P(\rho, \sigma, Q) \), which make sense for any R. In that notation, P is a parabolic subgroup of G with a Levi decomposition \( P = MN \) and \( \sigma \) a smooth R-representation of the Levi subgroup M; there is a maximal parabolic subgroup \( P(\sigma) \) of G containing P to which \( \sigma \) inflated to P extends to a representation \( e_{P(\sigma)}(\sigma) \), and Q is a parabolic subgroup of G with \( P \subset Q \subset P(\sigma) \). Then

\[
\text{Ind}_G^P(\rho, \sigma, Q) = \text{Ind}_G^P(e_{P(\sigma)}(\sigma) \otimes \text{St}_Q^{P(\sigma)}),
\]

where \( \text{Ind} \) stands for parabolic induction and \( \text{St}_Q^{P(\sigma)} = \text{Ind}_Q^{P(\sigma)} R/\sum \text{Ind}_Q^{P(\sigma)} R \), the sum being over parabolic subgroups \( Q' \) of G with \( Q' \subsetneq Q' \subset P(\sigma) \). Alternatively, \( \text{Ind}_G^P(e_{Q}(\sigma)) \) is the quotient of \( \text{Ind}_Q^G(e_Q(\sigma)) \) by \( \sum \text{Ind}_Q^G e_Q(\sigma) \) with \( Q' \) as above, where \( e_Q(\sigma) \) is the restriction of \( e_{P(\sigma)}(\sigma) \) to Q, similarly for \( Q' \).

In [OV1] we mainly studied what happens to \( \text{Ind}_G^P(\rho, \sigma, Q) \) when we apply to it, for a parabolic subgroup \( P_1 \) of G, the left adjoint of \( \text{Ind}_G^P(\rho, \sigma, Q) \), or its right adjoint. Here we tackle a different question. We fix a pro-P Iwahori subgroup \( \mathcal{U} \) of G in good position with respect to P, so that in particular \( \mathcal{U}M = \mathcal{U} \cap M \) is a pro-P Iwahori subgroup of M. One of our main goals is to identify the \( R \)-module \( \text{Ind}_G^P(\rho, \sigma, Q)^\mathcal{U} \) of \( \mathcal{U} \)-invariants, as a right module over the Hecke algebra \( \mathcal{H} = \mathcal{H}_G \) of G in - the convolution algebra on the double coset space \( \mathcal{U} \backslash G \mathcal{U} \) - in terms of the module \( \sigma^{\mathcal{U}M} \) over the Hecke algebra \( \mathcal{H}_M \) of \( \mathcal{U}M \) in M. That goal is achieved in section 4, Theorem 4.17.

1.2. The initial work has been done in [OV17, §4] where \( \text{Ind}_G^P(\rho, \sigma)^\mathcal{U} \) is identified. Precisely, writing \( M^+ \) for the monoid of elements \( m \in M \) with \( m(\mathcal{U} \cap N) m^{-1} \subset \mathcal{U} \cap N \), the subalgebra \( \mathcal{H}_{M^+} \) of \( \mathcal{H}_M \) with support in \( M^+ \), has a natural algebra embedding \( \theta \) into \( \mathcal{H} \) and [OV17, Proposition 4.4] identifies \( \text{Ind}_G^P(\rho, \sigma)^\mathcal{U} \) with \( \text{Ind}_{\mathcal{H}_{M^+}}(\sigma^{\mathcal{U}M}) = \sigma^{\mathcal{U}M} \otimes \mathcal{H}_{M^+} \mathcal{H} \). So in a sense, this paper is a sequel to [OV17] although some of our results here are used in [OV17, §5].

As \( \text{Ind}_G^P(\rho, \sigma, Q) \) is only a subquotient of \( \text{Ind}_G^P(\rho, \sigma) \) and taking \( \mathcal{U} \)-invariants is only left exact, it is not straightforward to describe \( \text{Ind}_G^P(\rho, \sigma, Q)^\mathcal{U} \) from the previous result. However, that takes care of the parabolic induction step, so in a first approach we may assume \( P(\sigma) = G \) so that \( \text{Ind}_G^P(\rho, \sigma, Q) = e_G(\sigma) \otimes \text{St}_Q^{P(\sigma)} \). The crucial case is when moreover \( \sigma \) is c-minimal, that is, not an extension \( e_M(\tau) \) of a smooth \( R \)-representation \( \tau \) of a proper Levi subgroup of M. That case is treated first and the general case in section 4 only.

1.3. To explain our results, we need more notation. We choose a maximal F-split torus T in G and a minimal parabolic subgroup \( B = ZU \) with Levi component Z the G-centralizer of T. We assume that P = MN contains B and M contains Z, and that \( \mathcal{U} \) corresponds to an alcove in the apartment associated to T in the adjoint building of G. It turns out that when \( \sigma \) is c-minimal and \( P(\sigma) = G \), the set \( \Delta_M \) of simple roots of T in \( \text{Lie}(M \cap U) \) is orthogonal to its complement in the set \( \Delta \) of simple roots of T in \( \text{Lie}U \). We assume until the end of this section that \( \Delta_M \) and \( \Delta_2 = \Delta \setminus \Delta_M \) are orthogonal. If \( M_2 \) is the Levi subgroup containing Z corresponding to \( \Delta_2 \), both M and \( M_2 \) are normal in G, \( M \cap M_2 = Z \) and \( G = MM_2 \). Moreover, the normal subgroup \( M_2' \) of G generated by N is included in \( M_2 \) and \( G = MM_2' \).
We say that a right \( \mathcal{H}_M \)-module \( \mathcal{V} \) is extendible to \( \mathcal{H} \) if \( T_z^M \) acts trivially on \( \mathcal{V} \) for \( z \in Z \cap M'_z \) (section 3.3). In this case, we show that there is a natural structure of right \( \mathcal{H} \)-module \( e_{\mathcal{H}}(\mathcal{V}) \) on \( \mathcal{V} \) such that \( T_g^H \in \mathcal{H} \) corresponding to \( U_g\mathcal{H} \) for \( g \in M'_z \) acts as in the trivial character of \( G \) (section 3.4). We call \( e_{\mathcal{H}}(\mathcal{V}) \) the extension of \( \mathcal{V} \) to \( \mathcal{H} \) though \( \mathcal{H}_M \) is not a subalgebra of \( \mathcal{H} \). That notion is already present in [Abe] in the case where \( R \) has characteristic \( p \). Here we extend the construction to any \( R \) and prove some more properties. In particular we produce an \( \mathcal{H} \)-equivariant embedding \( e_{\mathcal{H}}(\mathcal{V}) \) into \( \text{Ind}^\mathcal{H}_{\mathcal{H}_M} \mathcal{V} \) (Lemma 3.10). If \( \mathcal{Q} \) is a parabolic subgroup of \( \mathcal{G} \) containing \( \mathcal{P} \), we go further and put on \( e_{\mathcal{H}}(\mathcal{V}) \otimes _{\mathcal{R}} (\text{Ind}^\mathcal{Q}_{\mathcal{G}} \mathcal{R})^{\mathcal{H}} \) and \( e_{\mathcal{H}}(\mathcal{V}) \otimes _{\mathcal{R}} (\text{St}_\mathcal{Q}^\mathcal{G})^{\mathcal{H}} \) structures of \( \mathcal{H} \)-modules (Proposition 3.15 and Corollary 3.17) - note that \( \mathcal{Q} \) is not a group algebra and there is no obvious notion of tensor product of \( \mathcal{H} \)-modules.

If \( \sigma \) is an \( R \)-representation of \( \mathcal{M} \) extendible to \( \mathcal{G} \), then its extension \( e_{\mathcal{G}}(\sigma) \) is simply obtained by letting \( M'_z \) act trivially on the space of \( \sigma \); moreover it is clear that \( \sigma^{\mathcal{M}} \) is extendible to \( \mathcal{H} \), and one shows easily that \( e_{\mathcal{G}}(\sigma)^{\mathcal{H}} = e_{\mathcal{H}}(\sigma^{\mathcal{M}}) \) as an \( \mathcal{H} \)-module (section 3.5). Moreover, the natural inclusion of \( e_{\mathcal{G}}(\sigma) \) into \( \text{Ind}^\mathcal{Q}_{\mathcal{G}} \sigma \) induces on taking pro-\( p \) Iwahori invariants an embedding \( e_{\mathcal{H}}(\sigma^{\mathcal{M}}) \rightarrow (\text{Ind}_\mathcal{Q}^\mathcal{G})^{\mathcal{H}} \) which, via the isomorphism of [OV17], yields exactly the above embedding of \( \mathcal{H} \)-modules of \( e_{\mathcal{H}}(\sigma^{\mathcal{M}}) \) into \( \text{Ind}^\mathcal{H}_{\mathcal{H}_M} (\sigma^{\mathcal{M}}) \).

Then we show the \( \mathcal{H} \)-modules \( (e_{\mathcal{G}}(\sigma) \otimes _{\mathcal{R}} \text{Ind}^\mathcal{Q}_{\mathcal{G}} \mathcal{R})^{\mathcal{H}} \) and \( e_{\mathcal{H}}(\sigma^{\mathcal{M}}) \otimes _{\mathcal{R}} (\text{Ind}^\mathcal{Q}_{\mathcal{G}} \mathcal{R})^{\mathcal{H}} \) are equal, and similarly \( (e_{\mathcal{G}}(\sigma) \otimes _{\mathcal{R}} \text{St}_\mathcal{Q}^\mathcal{G})^{\mathcal{H}} \) and \( e_{\mathcal{H}}(\sigma^{\mathcal{M}}) \otimes _{\mathcal{R}} (\text{St}_\mathcal{Q}^\mathcal{G})^{\mathcal{H}} \) are equal (Theorem 4.9).

1.4. We turn back to the general case where we do not assume that \( \Delta_M \) and \( \Delta \setminus \Delta_M \) are orthogonal. Nevertheless, given a right \( \mathcal{H}_M \)-module \( \mathcal{V} \), there exists a largest Levi subgroup \( \mathcal{M}(\mathcal{V}) \) of \( \mathcal{G} \) containing \( Z \) - corresponding to \( \Delta_M \cup \Delta_1 \) where \( \Delta_1 \) is a subset of \( \Delta \setminus \Delta_M \) orthogonal to \( \Delta_M \), such that \( \mathcal{V} \) extends to a right \( \mathcal{H}_M(\mathcal{V}) \)-module \( e_{\mathcal{M}(\mathcal{V})}(\mathcal{V}) \) with the notation of section 1.3. For any parabolic subgroup \( \mathcal{Q} \) between \( \mathcal{P} \) and \( \mathcal{P}(\mathcal{V}) = \mathcal{M}(\mathcal{V})\mathcal{U} \) we put (Definition 4.12)

\[
I_{\mathcal{H}}(\mathcal{P}, \mathcal{V}, \mathcal{Q}) = \text{Ind}^\mathcal{H}_{\mathcal{H}_M} (e_{\mathcal{M}(\mathcal{V})}(\mathcal{V}) \otimes _{\mathcal{R}} (\text{St}^\mathcal{M}(\mathcal{V}))^{\mathcal{H}})_{\mathcal{M}(\mathcal{V})}.
\]

We refer to Theorem 4.17 for the description of the right \( \mathcal{H} \)-module \( I_{\mathcal{H}}(\mathcal{P}, \sigma, \mathcal{Q})^{\mathcal{H}} \) for any smooth \( R \)-representation \( \sigma \) of \( \mathcal{H} \). As a special case, it says that when \( \sigma \) is \( \epsilon \)-minimal then \( P(\sigma) \supseteq P(\sigma^{\mathcal{M}}) \) and if moreover \( P(\sigma) = P(\sigma^{\mathcal{M}}) \), then \( I_{\mathcal{H}}(\mathcal{P}, \sigma, \mathcal{Q})^{\mathcal{H}} \) is isomorphic to \( I_{\mathcal{H}}(\mathcal{P}, \sigma^{\mathcal{M}}, \mathcal{Q}) \).

Remark 1.1. In [Abe] are attached similar \( \mathcal{H} \)-modules to \( (\mathcal{P}, \mathcal{V}, \mathcal{Q}) \); here we write them as \( CI_{\mathcal{H}}(\mathcal{P}, \mathcal{V}, \mathcal{Q}) \) because their definition uses, instead of \( \text{Ind}^\mathcal{H}_{\mathcal{H}_M} \), a different kind of induction, which we call coinduction. In [Abe] those modules are used to give, when \( R \) is an algebraically closed field of characteristic \( p \), a classification of simple \( \mathcal{H} \)-modules in terms of supersingular modules - that classification is similar to the classification of irreducible admissible \( R \)-representations of \( G \) in [AHHV17]. Using the comparison between induced and coinduced modules established in [Vig15b, 4.3] for any \( R \), our Corollary 4.24 expresses \( CI_{\mathcal{H}}(\mathcal{P}, \mathcal{V}, \mathcal{Q}) \) as a module \( I_{\mathcal{H}}(\mathcal{P}_1, \mathcal{V}_1, \mathcal{Q}_1) \); consequently we show in section 4.5 that the classification of [Abe] can also be expressed in terms of modules \( I_{\mathcal{H}}(\mathcal{P}, \mathcal{V}, \mathcal{Q}) \).

1.5. In a reverse direction one can associate to a right \( \mathcal{H} \)-module \( \mathcal{V} \) a smooth \( R \)-representation \( \mathcal{V} \otimes _{\mathcal{H}} R[\mathcal{U}\setminus \mathcal{G}] \) of \( G \) (seeing \( \mathcal{H} \) as the endomorphism ring of the \( R[G] \)-module \( R[\mathcal{U}\setminus \mathcal{G}] \)).
If $V$ is a right $\mathcal{H}_M$-module, we construct, again using $[OV17]$, a natural $R[G]$-map

$$I_H(P,V,Q) \otimes_R R[H(G)] \to \text{Ind}_{P(V)}^{H(V)}(e_{M(V)}(V \otimes_{\mathcal{H}_M} R[\mathcal{H}_M \setminus M])) \otimes_R S^{M(V)}_{\mathcal{H}_M \setminus M}(V),$$

with the notation of section 1.4. We show in section 5 that it is an isomorphism under a mild assumption on the $\mathbb{Z}$-torsion in $V$; in particular it is an isomorphism if $p = 0$ in $R$.

1.6. In the final section 6, we turn back to the case where $R$ is an algebraically closed field of characteristic $p$. We prove that the smooth dual of an irreducible admissible $R$-representation $V$ of $G$ is 0 unless $V$ is finite dimensional - that result is new if $F$ has positive characteristic, a case where the proof of Kohlhaase [Koh] for $\text{char}(F) = 0$ does not apply. Our proof first reduces to the case where $V$ is supercuspidal (by [AHHV17]) then uses again the $\mathcal{H}$-module $V^M$.

2. Notation, useful facts, and preliminaries

2.1. The group $G$ and its standard parabolic subgroups $P = MN$. In all that follows, $p$ is a prime number and $F$ is a local field with finite residue field $k$ of characteristic $p$. We denote an algebraic group over $F$ by a bold letter, like $H$, and use the same ordinary letter for the group of $F$-points, $H = H(F)$. We fix a connected reductive $F$-group $G$. We fix a maximal $F$-split torus $T$ and write $Z$ for its $G$-centralizer; we also fix a minimal parabolic subgroup $B$ of $G$ with Levi component $Z$, so that $B = ZU$ where $U$ is the unipotent radical of $B$. Let $X^*(T)$ be the group of $F$-rational characters of $T$ and let $\Phi$ be the subset of roots of $T$ in the Lie algebra of $G$. Then $B$ determines a subset $\Phi^+$ of positive roots - the roots of $T$ in the Lie algebra of $U$- and a subset of simple roots $\Delta$. The $G$-normalizer $N_G$ of $T$ acts on $X^*(T)$ and through that action, $N_G/Z$ identifies with the Weyl group of the root system $\Phi$. Set $N := N_G(F)$ and note that $N_G/Z \simeq N/Z$; we write $W$ for $N/Z$.

A standard parabolic subgroup of $G$ is a parabolic $F$-subgroup containing $B$. Such a parabolic subgroup $P$ has a unique Levi subgroup $M$ containing $Z$, so that $P = MN$ where $N$ is the unipotent radical of $P$ - we also call $M$ standard. By a common abuse of language to describe the preceding situation, we simply say “let $P = MN$ be a standard parabolic subgroup of $G$”; we sometimes write $N_P$ for $N$ and $M_P$ for $M$. The parabolic subgroup of $G$ opposite to $P$ will be written $P$ and its unipotent radical $N$, so that $P = MN$, but beware that $P$ is not standard! We write $W_M$ for the Weyl group $(M \cap N)/Z$.

If $P = MN$ is a standard parabolic subgroup of $G$, then $M \cap B$ is a minimal parabolic subgroup of $M$. If $\Phi_M$ denotes the set of roots of $T$ in the Lie algebra of $M$, with respect to $M \cap B$ we have $\Phi_M^+ = \Phi_M \cap \Phi^+$ and $\Delta_M = \Phi_M \cap \Delta$. We also write $\Delta_P$ for $\Delta_M$ as $P$ and $M$ determine each other, $P = MU$. Thus we obtain a bijection $P \mapsto \Delta_P$ from standard parabolic subgroups of $G$ to subsets of $\Delta$, with $B$ corresponding to $\Phi$ and $G$ to $\Delta$. If $I$ is a subset of $\Delta$, we sometimes denote by $P_I = M_I N_I$ the corresponding standard parabolic subgroup of $G$. If $I = \{\alpha\}$ is a singleton, we write $P_I = M_\alpha N_\alpha$. We note a few useful properties. If $P_I$ is another standard parabolic subgroup of $G$, then $P \subset P_I$ if and only if $\Delta_P \subset \Delta_P$; we have $\Delta_{P \cap P_I} = \Delta_P \cap \Delta_P$ and the parabolic subgroup corresponding to $\Delta_P \cup \Delta_P$ is the subgroup $(P, P_I)$ of $G$ generated by $P$ and $P_I$. The standard parabolic subgroup of $M$ associated to $\Delta_M \cap \Delta_M$ is $M \cap P_I = (M \cap M_I)(M \cap N_I)$ [Car85, Proposition 2.8.9]. It is convenient to write $G'$ for the subgroup of $G$ generated by the unipotent
radicals of the parabolic subgroups; it is also the normal subgroup of $G$ generated by $U$, and we have $G = ZG'$. For future reference, we now give a useful lemma extracted from [AHHV17].

**Lemma 2.1.** The group $Z \cap G'$ is generated by the $Z \cap M'_e$, $\alpha$ running through $\Delta$.

**Proof.** Take $I = \emptyset$ in [AHHV17, II.6.Proposition].

Let $v_F$ be the normalized valuation of $F$. For each $\alpha \in X^*(T)$, the homomorphism $x \mapsto v_F(\alpha(x)) : T \to \mathbb{Z}$ extends uniquely to a homomorphism $Z \to \mathbb{Q}$ that we denote in the same way. This defines a homomorphism $Z \cong X_+(T) \otimes \mathbb{Q}$ such that $\alpha(v(z)) = v_F(\alpha(z))$ for $z \in Z, \alpha \in X^*(T)$.

An interesting situation occurs when $\Delta = I \cup J$ is the union of two orthogonal subsets $I$ and $J$. In that case, $G' = M'_1M'_j$, $M'_j$ and $M'_j$ commute with each other, and their intersection is finite and central in $G$ [AHHV17, II.7 Remark 4].

2.2. $I_G(P, \sigma, Q)$ and minimality. We recall from [AHHV17] the construction of $I_G(P, \sigma, Q)$, our main object of study.

Let $\sigma$ be an $R$-representation of $M$ and let $P_\sigma$ be the standard parabolic subgroup with $\Delta_{P_\sigma} = \Delta_\sigma$ where

$$\Delta_\sigma = \{\alpha \in \Delta \setminus \Delta_P \mid Z \cap M'_\alpha \text{ acts trivially on } \sigma\}.$$ 

We also let $P(\sigma)$ be the standard parabolic subgroup with

$$\Delta_{P(\sigma)} = \Delta_\sigma \cup \Delta_P.$$ 

This is the largest parabolic subgroup $P(\sigma)$ containing $P$ to which $\sigma$ extends, here $N \subset P$ acts on $\sigma$ trivially. Clearly when $P \subset Q \subset P(\sigma)$, $\sigma$ extends to $Q$ and the extension is denoted by $e_Q(\sigma)$. The restriction of $e_P(\sigma)$ to $Q$ is $e_Q(\sigma)$. If there is no risk of ambiguity, we write

$$e(\sigma) = e_{P(\sigma)}(\sigma).$$

**Definition 2.2.** An $R[G]$-**triple** is a triple $(P, \sigma, Q)$ made out of a standard parabolic subgroup $P = MN$ of $G$, a smooth $R$-representation of $M$, and a parabolic subgroup $Q$ of $G$ with $P \subset Q \subset P(\sigma)$. To an $R[G]$-triple $(P, \sigma, Q)$ is associated a smooth $R$-representation of $G$:

$$I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^{G(\sigma)}(e(\sigma) \otimes \text{St}_{Q}^{P(\sigma)}),$$

where $\text{St}_{Q}^{P(\sigma)}$ is the quotient of $\text{Ind}_{Q}^{P(\sigma)} 1$, 1 denoting the trivial $R$-representation of $Q$, by the sum of its subrepresentations $\text{Ind}_{Q}^{P(\sigma)} 1$, the sum being over the set of parabolic subgroups $Q'$ of $G$ with $Q \subset Q' \subset P(\sigma)$.

Note that $I_G(P, \sigma, Q)$ is naturally isomorphic to the quotient of $\text{Ind}_{Q}^{G(\sigma)}(e_Q(\sigma))$ by the sum of its subrepresentations $\text{Ind}_{Q}^{G(\sigma)}(e_{Q'}(\sigma))$ for $Q \subset Q' \subset P(\sigma)$ by [AHV, Lemma 2.5].

It might happen that $\sigma$ itself has the form $e_P(\sigma_1)$ for some standard parabolic subgroup $P_1 = M_1N_1$ contained in $P$ and some $R$-representation $\sigma_1$ of $M_1$. In that case, $P(\sigma_1) = P(\sigma)$ and $e(\sigma) = e(\sigma_1)$. We say that $\sigma$ is $e$-**minimal** if $\sigma = e_P(\sigma_1)$ implies $P_1 = P, \sigma_1 = \sigma$. 
Lemma 2.3 ([AHV, Lemma 2.9]). Let \( P = MN \) be a standard parabolic subgroup of \( G \) and let \( \sigma \) be an \( R \)-representation of \( M \). There exists a unique standard parabolic subgroup \( P_{\text{min},\sigma} = M_{\text{min},\sigma} N_{\text{min},\sigma} \) of \( G \) and a unique \( e \)-minimal representation of \( \sigma_{\text{min}} \) of \( M_{\text{min},\sigma} \) with \( \sigma = e_P(\sigma_{\text{min}}) \). Moreover \( P(\sigma) = P(\sigma_{\text{min}}) \) and \( e(\sigma) = e(\sigma_{\text{min}}) \).

Lemma 2.4. Let \( P = MN \) be a standard parabolic subgroup of \( G \) and let \( \sigma \) be an \( e \)-minimal \( R \)-representation of \( M \). Then \( \Delta_P \) and \( \Delta_P(\sigma) \backslash \Delta_P \) are orthogonal.

That comes from ([AHV17, II.7 Corollary 2]). That corollary of [AHV17] also shows that when \( R \) is a field and \( \sigma \) is supercuspidal, then \( \sigma \) is \( e \)-minimal. Lemma 2.4 shows that \( \Delta_P(\sigma) \backslash \Delta_P \), \( P(\sigma) = P(\sigma_{\text{min}}) \) and \( e(\sigma) = e(\sigma_{\text{min}}) \).

Let \( \mathcal{P} \) be the Iwahori subgroup fixing a special vertex \( \mathcal{K} \) associated to \( \mathcal{H} \). We let \( U \) be the Iwahori subgroup fixing the alcove \( \mathcal{C} \) in \( \mathcal{A} \) with vertex \( x_0 \) contained in the Weyl chamber (of vertex \( x_0 \)) associated to \( B \). We let \( U \) be the pro-\( p \) radical of \( B \) (the pro-\( p \) Iwahori subgroup). The pro-\( p \) Iwahori Hecke ring \( \mathcal{H} = \mathcal{H}(\mathcal{G}, \mathcal{U}) \) is the convolution ring of compactly supported functions \( \mathcal{G} \to \mathcal{Z} \) constant on the double classes of \( \mathcal{G} \) modulo \( \mathcal{U} \). We denote by \( T(g) \) the characteristic function of \( \mathcal{U}g\mathcal{U} \) for \( g \in \mathcal{G} \), seen as an element of \( \mathcal{H} \). Let \( R \) be a commutative ring. The pro-\( p \) Iwahori Hecke \( R \)-algebra \( \mathcal{H}_R \) is \( R \otimes_{\mathcal{Z}} \mathcal{H} \). We will follow the custom to still denote by \( h \) the natural image \( 1 \otimes h \) of \( h \) in \( \mathcal{H}_R \).

For \( P = MN \) a standard parabolic subgroup of \( G \), the similar objects for \( M \) are indexed by \( P \), we have \( K_M = K \cap M, B_M = B \cap M, \mathcal{U}_M = \mathcal{U} \cap M \), the pro-\( p \) Iwahori Hecke ring \( \mathcal{H}_M = \mathcal{H}(M, \mathcal{U}_M) \), \( T^M(m) \) the characteristic function of \( \mathcal{U}_M m \mathcal{U}_M \) for \( m \in M \), seen as an element of \( \mathcal{H}_M \). The pro-\( p \) Iwahori subgroup \( \mathcal{U} \) of \( G \) satisfies the Iwahori decomposition with respect to \( P \):

\[
\mathcal{U} = \mathcal{U}_N \mathcal{U}_M \mathcal{U}_\sigma,
\]

where \( \mathcal{U}_N = \mathcal{U} \cap \mathcal{N}, \mathcal{U}_\sigma = \mathcal{U} \cap \overline{\mathcal{N}} \). The linear map
\[
\theta : \mathcal{H}_{M} \to \mathcal{H}_{\sigma}, \quad \theta(T^M(m)) = T(m) \quad (m \in M)
\]
does not respect the product. But if we introduce the monoid \( M^+ \) of elements \( m \in M \) contracting \( \mathcal{U}_N \), meaning \( m \mathcal{U}_N m^{-1} \subset \mathcal{U}_N \), and the submodule \( \mathcal{H}_{M^+} \subset \mathcal{H}_M \) of functions with support in \( M^+ \), we have [Vig15b, Theorem 1.4]:

\( \mathcal{H}_{M^+} \) is a subring of \( \mathcal{H}_M \) and \( \mathcal{H}_{M^+} \) is the localization of \( \mathcal{H}_{M^+} \) at an element \( \tau^M \in \mathcal{H}_{M^+} \) central and invertible in \( \mathcal{H}_M \), meaning \( \mathcal{H}_M = \bigcup_{n \in \mathbb{N}} \mathcal{H}_{M^+}((\tau^M)^{-n}) \). The map \( \mathcal{H}_{M^+} \to \mathcal{H}_M \) is injective and its restriction \( \theta \mid \mathcal{H}_{M^+} \) to \( \mathcal{H}_{M^+} \) respects the product.

These properties are also true when \( (M^+ : \tau^M) \) is replaced by its inverse \( (M^+, (\tau^M)^{-1}) \) where \( M^+ = \{ m^{-1} \in M \mid m \in M^+ \} \).

3. Pro-\( p \) Iwahori invariants of \( I_G(P, \sigma, Q) \)

3.1. Pro-\( p \) Iwahori Hecke algebras: Structures. Here we supplement the notation of sections 2.1 and 2.3. The subgroups \( Z_0 = Z \cap K = Z \cap B \) and \( Z_1 = Z \cap U \)
are normal in \( \mathcal{N} \) and we put

\[
W = \mathcal{N}/Z^0, \quad W(1) = \mathcal{N}/Z^1, \quad \Lambda = Z/Z^0, \quad \Lambda(1) = Z/Z^1, \quad Z_k = Z^0/Z^1.
\]

We have \( \mathcal{N} = (\mathcal{N} \cap K)Z \) so that we see the finite Weyl group \( \mathcal{W} = \mathcal{N}/Z \) as the subgroup \( (\mathcal{N} \cap K)/Z^0 \) of \( W \); in this way \( W \) is the semidirect product \( \Lambda \rtimes \mathcal{W} \). We put \( \mathcal{N}_G = \mathcal{N} \cap G^\circ \). The image \( W_G = W'' \) of \( \mathcal{N}_G \) in \( W \) is an affine Weyl group generated by the set \( S^a \) of affine reflections determined by the walls of the alcove \( \mathcal{C} \). The group \( W' \) is normal in \( W \) and \( W \) is the semidirect product \( W' \rtimes \Omega \) where \( \Omega \) is the image in \( W \) of the normalizer \( \mathcal{N}_G \) of \( \mathcal{C} \) in \( \mathcal{N} \). The length function \( \ell \) on the affine Weyl system \((W', S^a)\) extends to a length function on \( W \) such that \( \Omega \) is the set of elements of length 0. We also write \( \ell \) as a function of \( W(1) \) via the quotient map \( W(1) \to W \). We write

\[(3.1) \quad (\tilde{w}, \bar{w}) \in \mathcal{N} \times W(1) \times W \text{ corresponding via the quotient maps } \mathcal{N} \to W(1) \to W.\]

When \( w = s \) in \( S^a \) or more generally \( w \) in \( W_G \), we will most of the time choose \( \tilde{w} \) in \( \mathcal{N} \cap G^\circ \) and \( \bar{w} \) in the image \( sW_G \) of \( \mathcal{N} \cap G^\circ \) in \( W(1) \).

We are now ready to describe the pro-\( p \) Iwahori Hecke ring \( H = \mathcal{H}(G, U, \tilde{\mathcal{U}}) \) [Vig16]. We have \( \tilde{\mathcal{U}} = U \mathfrak{n}_U \mathfrak{U} \) and for \( n, n' \in \mathcal{N} \) we have \( \tilde{\mathcal{U}} n \tilde{\mathcal{U}} = U n' \tilde{\mathcal{U}} \) if and only if \( n \tilde{Z}^1 = n' \tilde{Z}^1 \). For \( n \in \mathcal{N} \) of image \( w \in W(1) \) and \( g \in \tilde{\mathcal{U}} \) we denote \( T_w = T(n) = T(g) \) in \( H \). The relations among the basis elements \((T_w)_{w \in W(1)}\) of \( H \) are:

1. Braid relations: \( T_w T_{w'} = T_{w w'} \) for \( w, w' \in W(1) \) with \( \ell(w w') = \ell(w) + \ell(w') \).
2. Quadratic relations: \( T_s^2 = q_s T_s + c_s T_{\tilde{s}} \) for \( s \in W(1) \) lifting \( s \in S^a \), where \( q_s = q_G(s) = [U/U \cap \tilde{s} U \tilde{s}^{-1}] \) depends only on \( s \), and \( c_s = \sum_{\alpha \in \Phi^+} c_s(\alpha) T_{\alpha} \) for integers \( c_s(\alpha) \in \mathbb{N} \) summing to \( q_s - 1 \).

We shall need the basis elements \((T^a_w)_{w \in W(1)}\) of \( H \) defined by:

1. \( T^a_w = T_w \) for \( w \in W(1) \) of length \( \ell(w) = 0 \).
2. \( T^a_{\tilde{s}} = T_{\tilde{s}} - c_{\tilde{s}} \) for \( \tilde{s} \in W(1) \) lifting \( s \in S^a \).
3. \( T^a_{w w'} = T_{w w'}^a T^a_w \) for \( w, w' \in W(1) \) with \( \ell(w w') = \ell(w) + \ell(w') \).

We need more notation for the definition of the admissible lifts of \( S^a \) in \( \mathcal{N}_G \). Let \( s \in S^a \) fixing a face \( \mathcal{C}_s \) of the alcove \( \mathcal{C} \) and \( \mathcal{K}_s \) the parahoric subgroup of \( G \) fixing \( \mathcal{C}_s \). The theory of Bruhat-Tits associates to \( \mathcal{C}_s \) a certain root \( \alpha_s \in \Phi^+ \) [Vig16, §4.2]. We consider the group \( G'_s \) generated by \( U_{\alpha_s} \cup U_{-\alpha_s} \), where \( U_{\pm \alpha_s} \) the root subgroup of \( \pm \alpha_s \) (if \( 2 \alpha_s \in \Phi \), then \( U_{2 \alpha_s} \subseteq U_{\alpha_s} \)) and the group \( G'_s \) generated by \( U_{\alpha_s} \cup U_{-\alpha_s} \), where \( U_{\alpha_s} = U_{\pm \alpha_s} \cap \mathcal{K}_s \). When \( u \in U_{\alpha_s} \cup \{1\} \), the intersection \( \mathcal{N}_G \cap U_{-\alpha_s} \cup U_{\alpha_s} \) (equal to \( \mathcal{N}_{G, U_{-\alpha_s} \cup U_{\alpha_s}} \) [BT72, 6.2.1 (V5)], [Vig16, §3.3 (19)]) possesses a single element \( n_u(s) \). The group \( Z'_s = Z \cap G'_s \) is contained in \( Z \cap \mathcal{K}_s = Z^0 \); its image in \( Z_k \) is denoted by \( Z'_{k, s} \).

The elements \( n_u(s) \) for \( u \in U_{\alpha_s} \cup \{1\} \) are the admissible lifts of \( s \) in \( \mathcal{N}_G \); their images in \( W(1) \) are the admissible lifts of \( s \) in \( W(1) \). By [Vig16, Theorem 2.2, Proposition 4.4], when \( s \in W(1) \) is an admissible lift of \( s \), \( c_s(t) = 0 \) if \( t \in Z_k \setminus Z'_{k, s} \), and

\[(3.2) \quad c_s \equiv (q_s - 1)|Z'_{k, s}|^{-1} \sum_{t \in Z'_{k, s}} T_t \mod p.\]

The admissible lifts of \( S \) in \( \mathcal{N}_G \) are contained in \( \mathcal{N}_G \cap \mathcal{K} \) because \( \mathcal{K}_s \subset \mathcal{K} \) when \( s \in S \).
Definition 3.1. An admissible lift of the finite Weyl group $W$ in $\mathcal{N}_G$ is a map $w \mapsto \hat{w} : W \to N_G \cap K$ such that $\hat{s}$ is admissible for all $s \in S$ and $\hat{w} = \hat{w}_1 \hat{w}_2$ for $w, w_1, w_2 \in W$ such that $w = w_1w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$.

Any choice of admissible lifts of $S$ in $\mathcal{N}_G \cap K$ extends uniquely to an admissible lift of $W$ ([AHHV17, IV.6], [OV17, Proposition 2.7]).

Let $P = MN$ be a standard parabolic subgroup of $G$. The groups $Z, Z^0 = Z \cap K_M = Z \cap B_M, Z^1 = Z \cap U_M$ are the same for $G$ and $M$, but $N_M = N \cap M$ and $M \cap G'$ are subgroups of $\mathcal{N}$ and $G'$. The monoid $M^+$ (section 2.3) contains $(N_M \cap K)$ and is equal to $M^+ = \delta M_N M_D + U_M$ where $N_M^+ = N \cap M^+$. An element $z \in Z$ belongs to $M^+$ if and only if $v_F(\alpha(z)) \geq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_M^+$ (see [Vig15b, Lemme 2.2]). Put $W_M = N_M / Z^0$ and $W_M(1) = N_M / Z^1$.

Let $\epsilon = +$ or $\epsilon = -$. We denote by $W_M, W_M(1)$ the images of $N_M^\epsilon$ in $W_M, W_M(1)$. We see the groups $W_M, W_M(1), W_M$ as subgroups of $W, W(1), W_G$. As $\theta$ (section 2.3), the linear injective map

$$(3.3) \quad \mathcal{H}_M \to \mathcal{H}, \quad \theta^*(T^{M,-}_w) = T^*_w, \quad (w \in W_M(1)),$$

respects the product on the subring $\mathcal{H}_M$. Here $T^*_w \in \mathcal{H}_M$ is defined in the same way as $T^*_w$ for $\mathcal{H}_M$. Note that $\theta$ and $\theta^*$ satisfy the obvious transitivity property with respect to a change of parabolic subgroups.

3.2. Orthogonal case. Let us examine the case where $\Delta_M$ and $\Delta_2 \setminus \Delta_M$ are orthogonal, writing $M_2 = M_2 \setminus \Delta_2$ as in section 1.3.

From $M \cap M_2 = Z$ we get $W_M \cap W_{M_2} = \Lambda, W_M(1) \cap W_{M_2}(1) = \Lambda(1)$, the semisimple building of $G$ is the product of those of $M$ and $M_2$. The set $S_M$ is the disjoint union of $S_M$ and $S_{M_2}$, the group $W_G$ is the direct product of $W_M$ and $W_{M_2}$. For $s \in W_M(1)$ lifting $\hat{s} \in S_M$, the elements $T^*_w \in \mathcal{H}_M$ and $T_2^* \in \mathcal{H}$ satisfy the same quadratic relations. A word of caution is necessary for the lengths $\ell_M$ of $W_M$ and $\ell_{M_2}$ of $W_{M_2}$ different from the restrictions of the length $\ell$ of $W_M$, for example $\ell_M(\lambda) = 0$ for $\lambda \in \Lambda \cap W_{M_2}$.

Lemma 3.2. We have $\Lambda = (W_M \cap \Lambda)(W_{M_2} \cap \Lambda)$.

Proof. We prove the lemma for $\epsilon = -$. The case $\epsilon = +$ is similar. The map $v : \mathbb{Z} \to X(X(T) \otimes \mathbb{Q}$ defined in section 2.1 is trivial on $Z^0$ and we also write $v$ for the resulting homomorphism on $\Lambda$. For $\lambda \in \Lambda$ there exists $\lambda_2 \in W_{M_2} \setminus \Lambda$ such that $\lambda_2 \in W_{M_2}$, or equivalently $v(\alpha(\lambda_2)) \leq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_M^+$. It suffices to have the inequality for $\alpha \in \Delta_M$. The matrix $(\alpha(\beta^*))_{\alpha, \beta \in \Delta_M}$ is invertible, hence there exists $n_\beta \in \mathbb{Z}$ such that $\sum_{\beta \in \Delta_M} n_\beta \alpha(\beta^*) \leq -\alpha(v(\lambda))$ for all $\alpha \in \Delta_M$. As $v(W_{M_2} \cap \Lambda)$ contains $\bigoplus_{\alpha \in \Delta_2} \mathbb{Z} \nu^\alpha$ where $\nu^\alpha$ is the coroot of $\alpha$ [Vig16, after formula (71)], there exists $\lambda_2 \in W_{M_2} \cap \Lambda$ with $v(\lambda_2) = \sum_{\beta \in \Delta_2} n_\beta \nu^\beta$. \hfill $\square$

The groups $N' \cap M'$ and $N' \cap M_2'$ are normal in $N'$, and

$$(N' \cap M')(N' \cap M_2') = Z(N' \cap M')(N' \cap M_2')$$

and

$$W = W_M \cdot W_{M_2} = W_M W_{M_2} = W_{M'} W_{M_2} = W_{M'} W_{M_2}.$$
We recall the function $q_0(n) = q(n) = \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)$ on $\mathcal{N}$ [Vig16, Proposition 3.38] and we extend to $\mathcal{N}$ the functions $q_M$ on $\mathcal{N} \cap M$ and $q_M$ on $\mathcal{N} \cap M_2$:

$$(3.5) \quad q_M(n) = |\mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_Mn)|, \quad q_M(n) = |\mathcal{U}_{M_2}/(\mathcal{U}_{M_2} \cap n^{-1}\mathcal{U}_{M_2}n)|. $$

The functions $q, q_M, q_{M_2}$ descend to functions on $W(1)$ and on $W$, also denoted by $q, q_M, q_{M_2}$.

**Lemma 3.3.** Let $n \in \mathcal{N}$ of image $w \in W$. We have

1. $q(n) = q_M(n)q_{M_2}(n)$.
2. $q_M(n) = q_M(nM)$ if $n = n_Mn_2$, $n_M \in \mathcal{N} \cap M$, $n_2 \in \mathcal{N} \cap M_2$ and similarly when $M$ and $M_2$ are permuted.
3. $q(w) = 1$ if $q_M(\lambda_\mathcal{W}M) = q_M(\lambda_\mathcal{W}M_2) = 1$, if $w = \lambda_\mathcal{W}M_2$, $w_M = \lambda_\mathcal{W}_2$, $w_{M_2} \in \Lambda \times \mathcal{W}_M \times \mathcal{W}_M$.
4. On the coset $(\mathcal{N} \cap M_2)\mathcal{N}_{Cn}$, $q_M$ is constant equal to $q_M(nM_2)$ for any element $nM_2 \in M_2' \cap (\mathcal{N} \cap M_2)\mathcal{N}_{Cn}$. A similar result is true when $M$ and $M_2$ are permuted.

**Proof.** We put $\mathcal{U}_M = \mathcal{U} \cap M$ and $\mathcal{U}_{M_2} = \mathcal{U} \cap M_2$. The product map

$$(3.6) \quad Z^1 \prod_{\alpha \in \Phi_{M,\text{red}}} \mathcal{U}_\alpha \prod_{\alpha \in \Phi_{M_2,\text{red}}} \mathcal{U}_\alpha \to \mathcal{U}$$

with $\mathcal{U}_\alpha = U_\alpha \cap \mathcal{U}_\alpha$ is a homeomorphism. We have $\mathcal{U}_M = Z^1\mathcal{Y}_M$, $\mathcal{U}_{M_2} = (Z^1 \cap M_2')\mathcal{Y}_{M_2}$ where $\mathcal{Y}_{M_2} = \prod_{\alpha \in \Phi_{M_2,\text{red}}} \mathcal{U}_\alpha$ and $\mathcal{N} \cap M_2'$ commutes with $\mathcal{Y}_{M_2}$, in particular $\mathcal{N} \cap M_2'$ normalizes $\mathcal{Y}_{M_2}$. Similar results are true when $M$ and $M_2$ are permuted, and $\mathcal{U} = \mathcal{U}_M\mathcal{U}_{M_2} = \mathcal{U}_M\mathcal{U}_{M_2}$.

Writing $\mathcal{N} = Z(\mathcal{N} \cap M_2')(\mathcal{N} \cap M_2')$ (in any order), we see that the product map

$$(3.7) \quad Z^1(\mathcal{Y}_{M_2} \cap n^{-1}\mathcal{Y}_{M_2}n)(\mathcal{Y}_{M_2} \cap n^{-1}\mathcal{Y}_{M_2}n) \to \mathcal{U} \cap n^{-1}\mathcal{U}n$$

is a homeomorphism. The inclusions induce bijections

$$(3.8) \quad \mathcal{Y}_{M_2}/(\mathcal{Y}_{M_2} \cap n^{-1}\mathcal{Y}_{M_2}n) \simeq \mathcal{U}_{M_2}/(\mathcal{U}_{M_2} \cap n^{-1}\mathcal{U}_{M_2}n) \simeq \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_Mn),$$

similarly for $M_2$, and also a bijection

$$(3.9) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n) \simeq (\mathcal{Y}_{M_2}/(\mathcal{Y}_{M_2} \cap n^{-1}\mathcal{Y}_{M_2}n)) \times (\mathcal{Y}_{M_2}/(\mathcal{Y}_{M_2} \cap n^{-1}\mathcal{Y}_{M_2}n)).$$

From (3.8) and (3.9), we get

$$(3.10) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n) \simeq (\mathcal{U}_M/(\mathcal{U}_M \cap n\mathcal{U}_Mn^{-1})) \times (\mathcal{U}_M/(\mathcal{U}_M \cap n\mathcal{U}_Mn^{-1}))$$

which implies the assertion (1) in the lemma.

The assertion (2) follows from (3.7) since $\mathcal{N} \cap M_2'$ normalizes $\mathcal{Y}_{M_2'}$; with (1), it implies the assertion (3).

A subgroup of $\mathcal{N}$ normalizes $\mathcal{U}_M$ if and only if it normalizes $\mathcal{Y}_{M_2'}$ by (3.8) if and only if $q_{M_2} = 1$ on this group. The group $\mathcal{N} \cap M_2'$ normalizes $\mathcal{Y}_{M_2'}$. Therefore the group $(\mathcal{N} \cap M_2')\mathcal{N}_{Cn}$ normalizes $\mathcal{U}_M$. The coset $(\mathcal{N} \cap M_2')\mathcal{N}_{Cn}$ contains an element $n_{M'} \in M'$. For $x \in (\mathcal{N} \cap M_2')\mathcal{N}_{Cn}$, $\mathcal{Y}_{M_2'}(n_{M'})^{-1}\mathcal{U}nx_{M'} = n_{M'}^{-1}\mathcal{U}nx_{M'}$, hence $q_M(nx_{M'}) = q_M(n_{M'})$. \qed
3.3. Extension of an $\mathcal{H}_M$-module to $\mathcal{H}$. This section is inspired by similar results for the pro-$p$ Iwahori Hecke algebras over an algebraically closed field of characteristic $p$ [Abe, Proposition 4.10]. We keep the setting of section 3.2 and we introduce ideals:

- $\mathcal{J}_r$ (resp., $\mathcal{J}_s$) the left (resp., right) ideal of $\mathcal{H}$ generated by $T_w^r - 1_H$ for all $w \in 1W_M$,
- $\mathcal{J}_{M,\ell}$ (resp., $\mathcal{J}_{M,r}$) the left (resp., right) ideal of $\mathcal{H}_M$ generated by $T_{\lambda}^{M,*} - 1_H$ for all $\lambda \in 1W_M \cap W_M(1) = 1W_M \cap \Lambda(1)$.

The next proposition shows that the ideals $\mathcal{J}_r = \mathcal{J}_s$ are equal and similarly $\mathcal{J}_{M,\ell} = \mathcal{J}_{M,r}$. After the proposition, we will drop the indices $\ell$ and $r$.

**Proposition 3.4.** The ideals $\mathcal{J}_r$ and $\mathcal{J}_s$ are equal to the submodule $\mathcal{J}'$ of $\mathcal{H}$ generated by $T_w^{r} - T_{w_2}^{r}$ for all $w \in W(1)$ and $w_2 \in 1W_M$.

The ideals $\mathcal{J}_{M,\ell}$ and $\mathcal{J}_{M,r}$ are equal to the submodule $\mathcal{J}'_M$ of $\mathcal{H}_M$ generated by $T_{w}^{M,*} - T_{w_2}^{M,*}$ for all $w \in W_M(1)$ and $w_2 \in 1W_M$.

**Proof.**

1) We prove $\mathcal{J}_r = \mathcal{J}'$. Let $w \in W(1), w_2 \in 1W_M$. We prove by induction on the length of $w_2$ that $T_w^r(T_{w_2}^r - 1) \in \mathcal{J}'$. This is obvious when $\ell(w_2) = 0$ because $T_w^r T_{w_2}^r = T_{w_2}^r$. Assume that $\ell(w_2) = 1$ and put $s = w_2$. If $\ell(ws) = \ell(w) + 1$, as before $T_w^r(T_{w_2}^r - 1) \in \mathcal{J}'$ because $T_w^r T_{s}^r = T_{w}^r$. Otherwise $\ell(ws) = \ell(w) - 1$ and $T_w^r T_{w_2}^r \subset T_{s}^r$. Hence

$$T_w^r(t^r - 1) = T_w^r(T_{w_2}^r - 1) = T_w^r(T_{w_2}^r - 1) = 1.$$

Assume now that $\ell(w_2) > 1$. Then, we factorize $w_2 = xy$ with $x, y \in 1W_M$ of length $\ell(x), \ell(y) < \ell(w_2)$ and $\ell(w_2) = \ell(x) + \ell(y)$. The element $T_w^r(T_{w_2}^r - 1) = T_w^r T_{w_2}^r(T_{w_2}^r - 1) = T_{w}^r(T_{w_2}^r - 1)$ lies in $\mathcal{J}'$ by induction.

Conversely, we prove $T_w^r - T_{w_2}^r \in \mathcal{J}_r$. We factorize $w = xy$ with $y \in 1W_M$ and $x \in 1W_M \cdot \Omega(1)$. Then, we have $\ell(w) = \ell(x) + \ell(y)$ and $\ell(w_2) = \ell(x) + \ell(yw_2)$. Hence

$$T_{w_2}^r - T_w^r = T_{w_2}^r(T_{w}^r - T_{w_2}^r) = T_{w_2}^r(T_{w_2}^r - 1) - T_w^r(T_{w_2}^r - 1) \in \mathcal{J}_r.$$

This ends the proof of $\mathcal{J}_r = \mathcal{J}'$.

By the same argument, the right ideal $\mathcal{J}_s$ of $\mathcal{H}$ is equal to the submodule of $\mathcal{H}$ generated by $T_w^s - T_{w_2}^s$ for all $w \in W(1)$ and $w_2 \in 1W_M$. But this latter submodule is equal to $\mathcal{J}'$ because $1W_M$ is normal in $W(1)$. Therefore we proved $\mathcal{J}' = \mathcal{J}_r = \mathcal{J}_s$.

2) (Proof of the second assertion.) We prove $\mathcal{J}_{M,\ell} = \mathcal{J}'_M$. The proof is easier than in (1) because for $w \in W_M(1)$ and $\lambda_2 \in 1W_M \cap \Lambda(1)$, we have $\ell(w) + \ell(\lambda_2) = \ell(ww_2)$ hence $T_w^{M,*}(T_{\lambda_2}^M - 1) = T_w^{M,*} - T_{w_2}^{M,*}$. We have also $\ell(\lambda_2w) = \ell(\lambda_2) + \ell(w)$ hence $T_w^{M,*}(T_{\lambda_2}^M - 1) T_w^{M,*} = T_w^{M,*} - T_{w_2}^{M,*}$ hence $\mathcal{J}_{M,\ell}$ is equal to the submodule of $\mathcal{H}_M$ generated by $T_{w}^{M,*} - T_{w}^{M,*}$ for all $w \in W_M(1)$ and $\lambda_2 \in 1W_M \cap \Lambda(1)$. This latter submodule is $\mathcal{J}'_M$ as $1W_M \cap \Lambda(1) = 1W_M \cap W_M(1)$ is normal in $W_M(1)$. Therefore $\mathcal{J}'_M = \mathcal{J}_{M,\ell} = \mathcal{J}_{M,r}$. 

$\square$
By Proposition 3.4, a basis of $\mathcal{J}$ is $T_w^* - T_{w\lambda_2}$ for $w$ in a system of representatives of $W(1)/1W_{M_2}$ and $w_2 \in 1W_{M_2} \setminus \{1\}$. Similarly a basis of $\mathcal{J}_M$ is $T_w^M - T_{w\lambda_2}$ for $w$ in a system of representatives of $W_M(1)/(\Lambda(1) \cap 1W_{M_2})$, and $\lambda_2 \in (\Lambda(1) \cap 1W_{M_2}) \setminus \{1\}$.

**Proposition 3.5.** The natural ring inclusion of $\mathcal{H}_{M^{-}}$ in $\mathcal{H}_M$ and the ring inclusion of $\mathcal{H}_{M^{-}}$ in $\mathcal{H}$ via $\theta^*$ induce ring isomorphisms

$$\mathcal{H}_M/\mathcal{J}_M \cong \mathcal{H}_{M^{-}}/(\mathcal{J}_M \cap \mathcal{H}_{M^{-}}) \cong \mathcal{H}/\mathcal{J}.$$ 

**Proof.**

(1) The left map is obviously injective. We prove the surjectivity. Let $w \in W_M(1)$. Let $\lambda_2 \in 1W_{M_2} \cap \Lambda(1)$ such that $w\lambda_2^{-1} \in W_{M^{-}}(1)$ (see (3.4)). We have $T_{w\lambda_2^{-1}} \in \mathcal{H}_{M^{-}}$ and $T_w^M = T_{w\lambda_2^{-1}} - T_{w\lambda_2}^M = T_{w\lambda_2^{-1}} - T_{w\lambda_2}(\lambda_2 - 1)$. Therefore $T_w^M \in \mathcal{H}_{M^{-}} + \mathcal{J}_M$. As $w$ is arbitrary, $\mathcal{H}_M = \mathcal{H}_{M^{-}} + \mathcal{J}_M$.

(2) The right map is surjective: let $w \in W(1)$ and $w_2 \in 1W_{M_2}$ such that $ww_2^{-1} \in W_{M^{-}}(1)$ (see (3.4)). Then $T_w^* - T_{ww_2^{-1}} \in \mathcal{J}_M$ with the same arguments as in (1), using Proposition 3.4. Therefore $\mathcal{H} = \theta^*(\mathcal{H}_{M^{-}}) + \mathcal{J}_M$.

We prove the injectivity: $\theta^*(\mathcal{H}_{M^{-}}) \cap \mathcal{J} = \theta^*(\mathcal{H}_{M^{-}} \cap \mathcal{J}_M)$. Let $\sum_{w \in W_{M^{-}}(1)} c_w T_w^M \lambda_2$, with $c_w \in 2$, be an element of $\mathcal{H}_{M^{-}}$. Its image by $\theta^*$ is $\sum_{w \in W(1)} c_w T_w^*$, where we have set $c_w = 0$ for $w \in W(1) \setminus W_{M^{-}}(1)$. We have $\sum_{w \in W(1)} c_w T_w \in \mathcal{J}$ if and only if $\sum_{w_2 \in 1W_{M_2}} c_w w_2 = 0$ for all $w \in W(1)$. If $c_{ww_2} \neq 0$, then $w_2 \in 1W_{M_2} \cap \Lambda(1)$. The sum $\sum_{w_2 \in 1W_{M_2}} c_w w_2$ is equal to $\sum_{\lambda_2 \in 1W_{M_2} \cap \Lambda(1)} c_\lambda_2 w_\lambda_2$. By Proposition 3.4, $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}_M$ if and only if $\sum_{w \in W_{M^{-}}(1)} c_w T_w^M \lambda_2 \in \mathcal{J}_M$. \hfill $\square$

We construct a ring isomorphism

$$e^* : \mathcal{H}_M/\mathcal{J}_M \cong \mathcal{H}/\mathcal{J}$$

by using Proposition 3.5. For any $w \in W(1)$, $T_w^* + \mathcal{J} = e^*(T_w^M \lambda_2 + \mathcal{J}_M)$ where $w_{M^{-}} \in W_{M^{-}}(1) \cap w_{1W_{M_2}}$ (see (3.4)). Therefore $T_w^* + \mathcal{J} = T_w^* + \mathcal{J}$ and $T_w^M + \mathcal{J} = e^*(T_w^M \lambda_2 + \mathcal{J}_M)$ by construction of $e^*$. We check that $e^*$ is induced by $\theta^*$.

**Theorem 3.6.** The linear map $\mathcal{H}_M \rightarrow \mathcal{H}$ induces a ring isomorphism

$$e^* : \mathcal{H}_M/\mathcal{J}_M \cong \mathcal{H}/\mathcal{J}.$$ 

**Proof.** Let $w \in W_M(1)$. We have to show that $T_w^* + \mathcal{J} = e^*(T_w^M \lambda_2 + \mathcal{J}_M)$. We saw above that $T_w^* + \mathcal{J} = e^*(T_w^M \lambda_2 + \mathcal{J}_M)$ with $w = w_{M^{-}} \lambda_2$ with $\lambda_2 \in 1W_{M_2} \cap \Lambda(1)$. As $\ell_M(\lambda_2) = 0$, $T_w^M = T_{w_{M^{-}}} \lambda_2 \in T_{w_{M^{-}}} + \mathcal{J}_M$. Therefore $T_{w_{M^{-}}} + \mathcal{J}_M = T_{w_{M^{-}}} + \mathcal{J}_M$. This ends the proof of the theorem. \hfill $\square$

We now wish to compute $e^*$ in terms of the $T_w$ instead of the $T_w^*$.  

**Proposition 3.7.** Let $w \in W(1)$. Then, $T_w + \mathcal{J} = e^*(T_w^M \Lambda_2(w) + \mathcal{J}_M)$ for any $w_{M} \in W_M(1) \cap w_{1W_{M_2}}$. 

Proof. The element $w_M$ is unique modulo right multiplication by an element $\lambda_2 \in W_M(1) \cap W_{M'}$ of length $\ell_M(\lambda_2) = 0$ and $T_{w_M} q_{M}(w) + J_M$ does not depend on the choice of $w_M$. We choose a decomposition (see (3.4)):

$$w = s_1 \ldots s_n u s_{n+1} \ldots s_{a+b}, \quad \ell(w) = a + b,$$

for $u \in \Omega(1)$, $s_i \in W_M$ lifting $s_i \in S_M^\text{eff}$ for $1 \leq i \leq a$ and $s_i \in W_M'$ lifting $s_i \in S_{M'}^\text{eff}$ for $a+1 \leq i \leq a+b$, and we choose $u_M \in W_M(1)$ such that $u \in u_M W_M$. Then

$$w_M = s_1 \ldots s_n u_M \in W_M(1) \cap w_M$$

and $q_{M}(w) = q_{M}(s_{a+1} \ldots s_{a+b})$ (Lemma 3.3 (i)). First we check the proposition in three simple cases:

Case 1. Let $w = s \in W_M$, lifting $s \in S_M^\text{eff}$; we have $T_s + J = e^* (T_M + J_M)$ because $T_s^2 = e^*(T_M^2)$ in $J$, $T_s = T_s^* + c_3$, $T_M = T_M^* + c_3$ and $1 = q_{M}(s)$.

Case 2. Let $w = u \in W(1)$ of length $\ell(u) = 0$ and $u_M \in W_M(1)$ such that $u \in u_M W_M$. We have $\ell_M(u_M) = 0$ and $q_{M}(u) = 1$ (Lemma 3.3). We deduce $T_u + J = e^* (T_{u_M} + J_M)$ because $T_u^* + J = T_{u_M}^* + J = e^*(T_{u_M} + J_M)$, and $T_u = T_{u_M}^*$.

Case 3. Let $w = s \in W_M'$, lifting $s \in S_{M'}^\text{eff}$; we have $T_s + J = e^* (q_{M}(s) + J_M)$ because $T_s^2 - 1, c_3 - (q_s - 1) \in J$, $T_s = T_s^* + c_3 \in J + J$ and $q_s = q_{M}(s)$.

In general, the braid relations $T_w = T_{s_1} \ldots T_{s_n} T_{s_{n+1}} \ldots T_{s_{a+b}}$ give a similar product decomposition of $T_w + J$; and the simple cases 1, 2, 3 imply that $T_w + J$ is equal to

$$e^*(T_{s_1}^* + J_M) \ldots e^*(T_{s_n}^* + J_M) e^*(T_{u_M}^* + J_M) e^*(q_{M}(s_{a+1}) + J_M) \ldots e^*(q_{M}(s_{a+b}) + J_M) = e^*(T_{w_M}^* q_{M}(w) + J_M).$$

The proposition is proved.

Propositions 3.4, 3.5, 3.7, and Theorem 3.6 are valid over any commutative ring $R$ (instead of $Z$).

The two-sided ideal of $\mathcal{H}_R$ generated by $T_w^* - 1$ for all $w \in W_M$ is $J_R = J \otimes Z R$, the two-sided ideal of $\mathcal{H}_{M,R}$ generated by $T_s^* - 1$ for all $\lambda \in W_M \cap \Lambda(1)$ is $J_{M,R} = J_M \otimes Z R$, and we get as in Proposition 3.5 isomorphisms

$$J_{M,R} \rightarrow J_{M,R} \rightarrow J_{M,R} / (J_{M,R} \cap J_{M,R}),$$

\begin{align*}
\mathcal{H}_{M,R} / J_{M,R} & \cong \mathcal{H}_{M,R} / (J_{M,R} \cap J_{M,R}), \\
\mathcal{H}_{M,R} & \cong \mathcal{H}_{M,R} / J_{M,R}.
\end{align*}

giving an isomorphism $\mathcal{H}_{M,R} / J_{M,R} \rightarrow \mathcal{H}/J_R$ induced by $\theta^*$. Therefore, we have an isomorphism from the category of right $\mathcal{H}_{M,R}$-modules where $J_M$ acts by 0 onto the category of right $\mathcal{H}_R$-modules where $J$ acts by 0.

Definition 3.8. A right $\mathcal{H}_{M,R}$-module $V$ where $J_M$ acts by 0 is called extensible to $\mathcal{H}$. The corresponding $\mathcal{H}_R$-module where $J$ acts by 0 is called its extension to $\mathcal{H}$ and denoted by $e_{\mathcal{H}}(V)$ or $e(V)$.

With the element basis $T_w$, $V$ is extensible to $\mathcal{H}$ if and only if

$$v T_{\lambda_2}^* = v \quad \text{for all} \quad v \in \mathcal{V} \text{ and } \lambda_2 \in W_{M'} \cap \Lambda(1).$$

The $\mathcal{H}$-module structure on the $R$-module $e(V) = \mathcal{V}$ is determined by

$$v T_{w_2} = v, \quad v T_{w_2} = v T_{w_2}^* \quad \text{for all} \quad v \in \mathcal{V}, w_2 \in W_{M'}, w \in W_{M}(1).$$
It is also determined by the action of $T_w^*$ for $w \in W_M \cup W_{M^-}(1)$ (or $w \in W_M \cup W_{M^-}(1)$). Conversely, a right $H$-module $W$ over $R$ is extended from an $H_M$-module if and only if

\begin{equation}
(3.13) \quad v T_w^* = v \quad \text{for all} \quad v \in W, w_2 \in W_{M'}^-.
\end{equation}

In terms of the basis elements $T_w$ instead of $T_w^*$, this says the following.

**Corollary 3.9.** A right $H_M$-module $V$ over $R$ is extensible to $H$ if and only if

\begin{equation}
(3.14) \quad v T^M_{w_2} = v \quad \text{for all} \quad v \in V \text{ and } \lambda_2 \in 1 W_{M'}^- \cap \Lambda(1).
\end{equation}

Then, the structure of an $H$-module on the $R$-module $e(\mathcal{V}) = V$ is determined by

\begin{equation}
(3.15) \quad v T_w = v q^M_{w_2}(w) \quad \text{for all} \quad v \in V, w_2 \in W_{M'}^- w \in W_M(1).
\end{equation}

$(W_{M^-}(1)$ or $W_{M^-}(1)$ instead of $W_{M'}(1)$ is enough.) A right $H$-module $W$ over $R$ is extended from an $H_M$-module if and only if

\begin{equation}
(3.16) \quad v T_w^* = v q_{w_2}^* \quad \text{for all} \quad v \in W, w_2 \in W_{M'}^-.
\end{equation}

3.4. $\sigma^{ext}$ is extensible to $H$ of extension $e(\mathcal{V}) = e(\sigma)^H$. Let $P = MN$ be a standard parabolic subgroup of $G$ such that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal, and let $\sigma$ be a smooth $R$-representation of $M$ extensible to $G$. Let $P_2 = M_2 N_2$ denote the standard parabolic subgroup of $G$ with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

Recall that $G = MM_2'$, that $M \cap M_2' = Z \cap M_2'$ acts trivially on $\sigma$, $c(\sigma)$ is the representation of $G$ equal to $\sigma$ on $M$ and trivial on $M_2'$. We will describe the $H$-module $e(\sigma)^H$ in this section. We first consider $e(\sigma)$ as a subrepresentation of $\text{Ind}^G_P \sigma$. For $v \in \sigma$, let $f_v \in (\text{Ind}^G_P \sigma)^M_{M_2'}$ be the unique function with value $v$ on $M_2'$.

Then, the map

\begin{equation}
(3.17) \quad v \mapsto f_v : \sigma \to \text{Ind}^G_P \sigma
\end{equation}

is the natural $G$-equivariant embedding of $e(\sigma)$ in $\text{Ind}^G_P \sigma$. As $\sigma^{ext} = e(\sigma)^H$ as $R$-modules, the image of $e(\sigma)^H$ in $\text{Ind}^G_P \sigma$ is made out of the $f_v$ for $v \in \sigma^{ext}$.

We now recall the explicit description of $(\text{Ind}^G_P \sigma)^H$. For each $d \in \mathcal{W}_M$, we fix a lift $\hat{d} \in 1 W_{M_2'}$ and for $v \in \sigma^{ext}$ let $f_{P\hat{d}^* v} \in (\text{Ind}^G_P \sigma)^H$ for the function with support contained in $P\hat{d}^*$ and value $v$ on $\hat{d}$. As $Z \cap M_2'$ acts trivially on $\sigma$, the function $f_{P\hat{d}^* v}$ does not depend on the choice of the lift $\hat{d} \in 1 W_{M_2'}$ of $d$. By [OV17, Lemma 4.3], recalling that $w \in \mathcal{W}_{M_2'}$ is of minimal length in its coset $w \mathcal{W}_M = \mathcal{W}_M w$ as $\Delta_M$ and $\Delta_{M_2'}$ are orthogonal to each other:

The map $\bigoplus_{d \in \mathcal{W}_{M_2'}}^{\sigma^{ext}} \to (\text{Ind}^G_P \sigma)^H$ given on each $d$-component by $v \mapsto f_{P\hat{d}^* v}$, is an $H_{M_2'}$-equivariant isomorphism where $H_{M_2'}$ is seen as a subring of $H$ via $\theta$, and induces an $H_R$-module isomorphism

\begin{equation}
(3.18) \quad v \otimes h \mapsto f_{P\hat{d}^* v} : \sigma^{ext} \otimes_{H_{M_2'}} \mathcal{H} \to (\text{Ind}^G_P \sigma)^H.
\end{equation}

In particular for $v \in \sigma^{ext}$, $v \otimes T(\hat{d})$ does not depend on the choice of the lift $\hat{d} \in 1 W_{M_2'}$ of $d$ and

\begin{equation}
(3.19) \quad f_{P\hat{d}^* v} = f_{P\hat{d}^* v} T(\hat{d}).
\end{equation}
As $G$ is the disjoint union of $P_u d t$ for $d \in \mathbb{W}_M$, we have $f_v = \sum_{d \in \mathbb{W}_M} f_P u d t$ and $f_n$ is the image of $v \otimes e_M$ in (3.18), where

\[(3.20)\]

\[e_M = \sum_{d \in \mathbb{W}_M} T(d).\]

Recalling (3.17) we get the following.

**Lemma 3.10.** The map $v \mapsto v \otimes e_M : e(\sigma)^d \to \sigma^{d t u} \otimes \mathcal{H}_{G + \theta} \mathcal{H}$ is an $\mathcal{H}_R$-equivariant embedding.

**Remark 3.11.** The trivial map $v \mapsto v \otimes 1_\mathcal{H}$ is not an $\mathcal{H}_R$-equivariant embedding.

We describe the action of $T(n)$ on $e(\sigma)^d$ for $n \in N$. By definition for $v \in e(\sigma)^d$,

\[(3.21)\]

\[vT(n) = \sum_{y \in \mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} n)} y n^{-1} v.\]

**Proposition 3.12.** We have $vT(n) = v T^M(n M)_q M(n)$ for any $n M \in N \cap M$ is such that $n = n M (N \cap M)$.

**Proof.** The description (3.10) of $\mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} n)$ gives

\[vT(n) = \sum_{y \in \mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} n)} y n^{-1} v.\]

As $M$ acts trivially on $e(\sigma)$, we obtain

\[vT(n) = q M(n) \sum_{y \in \mathcal{U}/(\mathcal{U} \cap n^{-1} \mathcal{U} n)} y n^{-1} v = q M(n) v T^M(n M).\]

**Theorem 3.13.** Let $\sigma$ be a smooth $R$-representation of $M$. If $P(\sigma) = G$, then $\sigma^{d t u}$ is extendible to $\mathcal{H}$ of extension $e(\sigma^{d t u}) = e(\sigma)^d$. Conversely, if $\sigma^{d t u}$ is extendible to $\mathcal{H}$ and generates $\sigma$, then $P(\sigma) = G$.

**Proof.**

(1) The $\mathcal{H}_M$-module $\sigma^{d t u}$ is extendible to $\mathcal{H}$ if and only if $Z \cap M'$ acts trivially on $\sigma^{d t u}$. Indeed, for $v \in \sigma^{d t u}$, $z_2 \in Z \cap M'$,

\[v T^M(z_2) = \sum_{y \in \mathcal{U}/(\mathcal{U} \cap z_2^{-1} \mathcal{U} n)} y z_2^{-1} v = \sum_{y \in \mathcal{U}/(\mathcal{U} \cap z_2^{-1} \mathcal{U} n)} y z_2^{-1} v = z_2^{-1} v,\]

by (3.21), then (3.8), then the fact that $z_2^{-1}$ commutes with the elements of $\mathcal{Y}_M$.

(2) $P(\sigma) = G$ if and only if $Z \cap M'$ acts trivially on $\sigma$ (the group $Z \cap M'$ is generated by $Z \cap M'_\alpha$ for $\alpha \in \Delta_M$ by Lemma 2.1). The $R$-submodule $\sigma^{d t u} M'_{G+\theta}$ of elements fixed by $Z \cap M'$ is stable by $M$, because $M = Z M'$, the elements of $M'$ commute with those of $Z \cap M'$ and $Z$ normalizes $Z \cap M'$.

(3) Apply (1) and (2) to get the theorem except the equality $e(\sigma^{d t u}) = e(\sigma)^d$ when $P(\sigma) = G$ which follows from Propositions 3.12 and 3.7.

Let $1_M$ denote the trivial representation of $M$ over $R$ (or $1$ when there is no ambiguity of $M$). The right $\mathcal{H}_R$-module $(1_M)^d = 1_\mathcal{H}$ (or $1$ if there is no ambiguity) is the trivial right $\mathcal{H}_R$-module: for $w \in W_M(1), T_w = q_w id$ and $T_w^* = id$ on $1_\mathcal{H}$.
Example 3.14. The $\mathcal{H}$-module $(\text{Ind}_G^F 1)^{\mathcal{H}}$ is the extension of the $\mathcal{H}_{M_2}$-module $(\text{Ind}_M^{M_2} 1)^{\mathcal{H}_{M_2}}$. Indeed, the representation $\text{Ind}_G^F 1$ of $G$ is trivial on $N_2$, as $G = M M_2$ and $N_2 \subseteq M'$ (as $\Phi = \Phi_M \cup \Phi_{M_2}$). For $g = mm'w$ with $m \in M, m' \in M_2$ and $w \in N_2$, we have $Pnm = Pn'mw = n'n'mw = Pm'w = Pg$. The group $M_2 \cap B = M_2 \cap P$ is the standard minimal parabolic subgroup of $M_2$ and $(\text{Ind}_G^F 1)|_{M_2} = (\text{Ind}_M^{M_2} 1)|_{M_2}$. Apply Theorem 3.13 as follows.

3.5. The $\mathcal{H}_R$-module $e(\mathcal{V}) \otimes_R (\text{Ind}_G^F 1)^{\mathcal{H}}$. Let $P = MN$ be a standard parabolic subgroup of $G$ such that $\Delta_P$ and $\Delta \triangleleft \Delta_P$ are orthogonal, let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module which is extendible to $\mathcal{H}_R$ of extension $e(\mathcal{V})$, and let $Q$ be a parabolic subgroup of $G$ containing $P$. Let $P_2 = M_2 N_2$ denote the standard parabolic subgroup of $G$ with $\Delta_{P_2} = \Delta \triangleleft \Delta_P$.

We define on the $R$-module $e(\mathcal{V}) \otimes_R (\text{Ind}_G^F 1)^{\mathcal{H}}$ a structure of a right $\mathcal{H}_R$-module as follows.

Proposition 3.15.

1. The diagonal action of $T_w^*$ for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\text{Ind}_G^F 1)^{\mathcal{H}}$ defines a structure of a right $\mathcal{H}_R$-module.

2. The action of the $T_w^*$ is also diagonal and satisfies:

$((v \otimes f)T_w^*, (v \otimes f)T_w^* u) = (vT_{uwM} \otimes fT_{uwM}^*, vT_{uwM}^* \otimes fT_{uwM}^*)$,

where $w = uwM \cdot wM_2'$ with $u \in W(1), \ell(u) = 0, wM' \in \text{I}_{M'}, wM_2' \in \text{I}_{M_2'}$.

Proof. If the lemma is true for $P$ it is also true for $Q$, because the $R$-module $e(\mathcal{V}) \otimes_R (\text{Ind}_G^F 1)^{\mathcal{H}}$ naturally embedded in $e(\mathcal{V}) \otimes_R (\text{Ind}_G^F 1)^{\mathcal{H}}$ is stable by the action of $\mathcal{H}$ defined in the lemma. So, we suppose $Q = P$.

For each element in $1 S^\text{aff}$ we fix an admissible lift and denote the set of admissible lifts by $\mathcal{S}^\text{aff}$. We also use the obvious notation $1 S^\text{aff}_{M_2'}$ and $1 S^\text{aff}_{M_2}$. Suppose that $T_w^*$ for $w \in W(1)$ acts on $e(\mathcal{V}) \otimes_R (\text{Ind}_G^F 1)^{\mathcal{H}}$ as in (1). The braid relations obviously hold. The quadratic relations hold because $T_s^*$ for $s \in 1 S^\text{aff}$ acts trivially either on $e(\mathcal{V})$ or on $(\text{Ind}_G^F 1)^{\mathcal{H}}$. Indeed, $1 S^\text{aff} = 1 S^\text{aff}_{M} \cup 1 S^\text{aff}_{M_2}$. $T_s^*$ for $s \in 1 S^\text{aff}$ acts trivially on $(\text{Ind}_G^F 1)^{\mathcal{H}}$ which is extended from an $\mathcal{H}_{M,R}$-module (Example 3.14), and $T_s^*$ for $s \in 1 S^\text{aff}_{M_2}$ acts trivially on $e(\mathcal{V})$ which is extended from an $\mathcal{H}_{M_2}$-module.

This proves (1).

We describe now the action of $T_w^*$ instead of $T_w^*$ on the $\mathcal{H}$-module $e(\mathcal{V}) \otimes_R (\text{Ind}_G^F 1)^{\mathcal{H}}$. Let $w \in W(1)$. We write $w = uwM \cdot wM_2'$ with $u \in W(1), \ell(u) = 0, wM' \in \text{I}_{M'}, wM_2' \in \text{I}_{M_2'}$. We have $\ell(w) = \ell(wM') + \ell(wM_2')$.

Hence $T_w = T_{uwM} T_{wM_2'}$.

For $w = u$, we have $T_u = T_{uwM}^*$ and $(v \otimes f)T_u = (v \otimes f)T_{uwM}^* = vT_{uwM}^* \otimes fT_{uwM}^* = vT_u \otimes fT_u$.

For $w = wM'$, $(v \otimes f)T_w^* = vT_w^* \otimes f$; for $s \in 1 S^\text{aff}_M, c_s = \sum_{i \in Z} c_i W_M, c_s(t)T_w^*$.

In particular, we have $(v \otimes f)T_s = (v \otimes f)(T_s^* + c_s) = v(T_s^* + c_s) \otimes f = vT_s^* \otimes f$. Hence $(v \otimes f)T_s = vT_s^* \otimes f$.

For $w = wM_2'$, we have similarly $(v \otimes f)T_w^* = v \otimes fT_w^*$ and $(v \otimes f)T_w = v \otimes fT_w$. □

Example 3.16. Let $\mathcal{X}$ be a right $\mathcal{H}_R$-module. Then $1_{\mathcal{H}} \otimes_R \mathcal{X}$ where the $T_w^*$ acts diagonally is an $\mathcal{H}_R$-module isomorphic to $\mathcal{X}$. But the action of the $T_w$ on $1_{\mathcal{H}} \otimes_R \mathcal{X}$ is not diagonal.
It is known [Ly15] that \((\text{Ind}_Q^G 1)^{1d}\) and \((\text{St}_Q^G)^{1d}\) are free \(R\)-modules and that 
\((\text{St}_Q^G)^{1d}\) is the cokernel of the natural \(H_R\)-map

\[(3.22) \bigoplus_{Q \subseteq Q'} (\text{Ind}_Q^G 1)^{1d} \rightarrow (\text{Ind}_Q^G 1)^{1d}\]

although the invariant functor \((-)^{1d}\) is only left exact.

**Corollary 3.17.** The diagonal action of \(T_w\) for \(w \in W(1)\) on \(e(V) \otimes_R (\text{St}_Q^G)^{1d}\) defines a structure of a right \(H_R\)-module satisfying Proposition 3.15(2).

### 4. Hecke module \(I_H(P,V,Q)\)

#### 4.1. Case \(V\) extensible to \(H\)

Let \(P = MN\) be a standard parabolic subgroup of \(G\) such that \(\Delta_P\) and \(\Delta \setminus \Delta_P\) are orthogonal, let \(V\) be a right \(H_{M,R}\)-module extensible to \(H_R\) of extension \(e(V)\), and let \(Q\) be a parabolic subgroup of \(G\) containing \(P\).

As \(Q\) and \(M_Q\) determine each other: \(Q = M_Q U\), we denote also \(H_{M_Q} = H_Q\) and \(H_{M_Q,R} = H_{Q,R}\) when \(Q \neq P, G\). When \(Q = G\) we drop \(G\) and we denote \(e_H(V) = e(V)\).

**Lemma 4.1.** \(V\) is extensible to an \(H_{Q,R}\)-module \(e_{H_Q}(V)\).

**Proof.** This is straightforward. By Corollary 3.9, \(V\) extensible to \(H\) means that \(T^M(z)\) acts trivially on \(V\) for all \(z \in N_{M_Q} \cap Z\). We have \(M_Q = MM_{Z,Q}\) with \(M_{Z,Q} \subset M_Q \cap M_Z\) and \(N_{M_{Z,Q}} \subset N_{M_Q}\); hence \(T^M(z)\) acts trivially on \(V\) for all \(z \in N_{M_{Z,Q}} \cap Z\) meaning that \(V\) is extensible to \(H_Q\). \(\square\)

**Remark 4.2.** We cannot say that \(e_{H_Q}(V)\) is extensible to \(H\) of extension \(e(V)\) when the set of roots \(\Delta_Q\) and \(\Delta \setminus \Delta_Q\) are not orthogonal (Definition 3.8).

Let \(Q'\) be an arbitrary parabolic subgroup of \(G\) containing \(Q\). We are going to define an \(H_R\)-embedding \(\text{Ind}_{H_{Q'}}^H(e_{H_{Q'}}(V)) \xrightarrow{\iota(Q',Q)} \text{Ind}_{H_Q}^H(e_{H_Q}(V)) = e_{H_Q}(V) \otimes_{H_{M_Q}} \theta\).

Define \(H\)-homomorphism

\[\bigoplus_{Q \subseteq Q' \subset G} \text{Ind}_{H_{Q'}}^H(e_{H_{Q'}}(V)) \rightarrow \text{Ind}_{H_Q}^H(e_{H_Q}(V))\]

of cokernel isomorphic to \(e(V) \otimes_R (\text{St}_Q^G)^{1d}\). In the extreme case \((Q, Q') = (P, G)\), the \(H_R\)-embedding \(e(V) \xrightarrow{\iota(P,G)} \text{Ind}_{H_P}^H(V)\) is given in the following lemma where \(f_G\) and \(f_{PU} \in (\text{Ind}_P^G 1)^{1d}\) denote the characteristic functions of \(G\) and \(PU\), \(f_G = f_{PU} e_{M_2}\) (see (3.20)).

**Lemma 4.3.** There is a natural \(H_R\)-isomorphism

\[v \otimes 1_H \mapsto v \otimes f_{PU} : \text{Ind}_{H_{M,+}}^H(V) \otimes H_{M,+} \xrightarrow{\iota} e(V) \otimes_R (\text{Ind}_P^G 1)^{1d},\]

and compatible \(H_R\)-embeddings

\[(4.1) \quad v \mapsto v \otimes f_G : e(V) \otimes_R (\text{Ind}_P^G 1)^{1d},\]

\[(4.2) \quad v \mapsto v \otimes e_{M_2} : e(V) \xrightarrow{\iota(P,G)} \text{Ind}_{H_P}^H(V).\]

**Proof.** We show first that the map

\[(4.3) \quad v \mapsto v \otimes f_{PU} : V \rightarrow e(V) \otimes_R (\text{Ind}_P^G 1)^{1d}\]
is $\mathcal{H}_M$-equivariant. Let $w \in W_M(1)$. We write $w = uw_Mw_M'$ as in Proposition 3.15 (2), so that $f_{\mu M}T_w = f_{\mu M}T_{uw_M}$ because $1W_M \subset W_M(1)\cap W_M^{-1}$ (hence $uw_M' = uw_M^{-1} \in W_M(1)$ and in $1\mathcal{H}_M \otimes \mathcal{H}_M$, we have $(1 \otimes 1)T_{uw_M} = 1T_{uw_M} \otimes 1$, and $T_{uw_M}M$ acts trivially in $1\mathcal{H}_M$ because $\ell_M(uw_M') = 0$. We deduce $(v \otimes f_{\mu M})T_w = vT_w \otimes f_{\mu M}T_w = vT_w \otimes f_{\mu M}$.

By adjunction (4.3) gives an $\mathcal{H}_R$-equivariant linear map

$$v \otimes 1_H \mapsto v \otimes f_{\mu M} : V \otimes \mathcal{H}_M \otimes \mathcal{H}_R \xrightarrow{\kappa_P} e(V) \otimes_R (\text{Ind}_P^G 1)^M.$$  

We prove that $\kappa_P$ is an isomorphism. Recalling $d \in \mathcal{N} \cap M$, $\tilde{d} \in 1W_M'$ lift $d$, one knows that

$$V \otimes \mathcal{H}_M \otimes \mathcal{H}_R \xrightarrow{\kappa_P} e(V) \otimes_R (\text{Ind}_P^G 1)^M = \bigoplus_{d \in \mathcal{W}_M} V \otimes f_{P,d},$$

where each summand is isomorphic to $V$. The left equality follows from section 4.1 and Remark 3.7 in [Vig15b] recalling that $w \in \mathcal{W}_M$ is of minimal length in its coset $\mathcal{W}_Mw = w\mathcal{W}_M$ as $\Delta_M$ and $\Delta_M'$ are orthogonal; for the second equality see section 3.4 (3.19). We have $\kappa_P(v \otimes T_d) = (v \otimes f_{\mu M})T_d = v \otimes f_{\mu M}T_d$ (Proposition 3.15). Hence $\kappa_P$ is an isomorphism.

We consider the composite map

$$v \mapsto v \otimes 1 \mapsto v \otimes f_{\mu M} : e(V) \rightarrow e(V) \otimes_R 1_H \rightarrow e(V) \otimes_R (\text{Ind}_P^G 1)^M,$$

where the right map is the tensor product $e(V) \otimes_R -$ of the $\mathcal{H}_R$-equivariant embedding $1_H \rightarrow (\text{Ind}_P^G 1)^M$ sending $1_R$ to $f_{\mu M}e_M$ (Lemma 3.10); this map is injective because $(\text{Ind}_P^G 1)^M/1$ is a free $R$-module; it is $\mathcal{H}_R$-equivariant for the diagonal action of the $T_w$ on the tensor products (Example 3.16 for the first map). By compatibility with (4.4), we get the $\mathcal{H}_R$-equivariant embedding $v \mapsto v \otimes e_{M_2} : e(V) \xrightarrow{\text{Ind}_P^G} \text{Ind}_{\mathcal{H}_M}(V)$.

For a general $(Q, Q')$ the $\mathcal{H}_R$-embedding $\text{Ind}_{\mathcal{H}_M(Q', e_{\mathcal{H}_M}(V))} \xrightarrow{\text{Ind}_{\mathcal{H}_M(Q, e_{\mathcal{H}_M}(V))}} \text{Ind}_{\mathcal{H}_M}(V)$ is given in the next proposition generalizing Lemma 4.3. The element $e_{M_2}$ of $\mathcal{H}_R$ appearing in the definition of $\iota(P, Q)$ is replaced in the definition of $\iota(Q, Q')$ by an element $\theta_{Q'}(e_{M_2}) \in \mathcal{H}_R$ that we define first.

Until the end of section 4, we fix an admissible lift $w \rightarrow \tilde{w} : \mathcal{W} \rightarrow \mathcal{N} \cap K$ (Definition 3.1) and $\tilde{w}$ denotes the image of $\tilde{w}$ in $W(1)$. We denote $\mathcal{W}_M = \mathcal{W}_Q$ and by $\mathcal{W}_W$ the set of $w \in \mathcal{W}$ of minimal length in their coset $\mathcal{W}_Qw$. The group $G$ is the disjoint union of $\mathcal{Qd} \setminus d$ for $d$ running through $\mathcal{W}_Q\mathcal{W}$ [OV17, Lemma 2.15 (2)]:

$$G = \bigsqcup_{d \in \mathcal{W}_Q} \mathcal{Qd}. $$

Since $\mathcal{Qd} \subset \mathcal{QU}$ if and only if $d \in \mathcal{Q'}$, namely $d \in \mathcal{W}_Q\mathcal{W}_{Q'}$, we have

$$Q'U = \bigsqcup_{d \in \mathcal{W}_Q\mathcal{W}_{Q'}} \mathcal{Qd}. $$

Set

$$e_{Q'} = \sum_{d \in \mathcal{W}_Q\mathcal{W}_{Q'}} T_{dQ'}. $$

We write $e_Q = e_{Q'}$. We have $e_{Q'} = \sum_{d \in \mathcal{W}_M \cap Q} T_{dQ'}$. 

Remark 4.4. Note that \( \mathbb{W}' \mathbb{W} = \mathbb{W}_M \) and \( e_P = e_{M_2} \), where \( M_2 \) is the standard Levi subgroup of \( G \) with \( \Delta_{M_2} = \Delta \setminus \Delta_M \), as \( \Delta_M \) and \( \Delta \setminus \Delta_M \) are orthogonal. More generally, \( \mathbb{W}_q \mathbb{W}_{M_Q'} = \mathbb{W}_{M_2 \cap Q} \) where \( M_2 \cap Q' = M_2 \cap M_{Q'} \).

Note that \( e_Q^Q \in \mathcal{H}_{M_2^+} \cap \mathcal{H}_{M_2^-} \). We consider the linear map
\[
\theta_Q^Q : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'} \quad \text{for} \quad w \in W_{M_Q}(1).
\]
We write \( \theta_Q^Q = \theta_Q \) so that \( \theta_Q(T_w^{M_Q}) = T_w \). When \( Q = P \) this is the map \( \theta \) defined earlier. Similarly we denote by \( \theta_Q^{Q'} \) the linear map sending the \( T_w^{M_{Q'}} \) to \( T_w^{M_{Q'}} \).

\[
\theta_Q^Q(e_Q^Q) = \sum_{\kappa \in \mathcal{H}_{M_2^+} \cap \mathcal{H}_{M_2^-}} T_{\kappa}, \quad \theta_Q^Q(e_Q^Q) = \theta_Q(e_Q^Q) \theta_Q(e_Q^Q).
\]

**Proposition 4.5.** There exists an \( \mathcal{H}_R \)-isomorphism
\[
\theta_Q^Q : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'} \quad \text{for} \quad w \in W_{M_Q}(1)
\]
and compatible \( \mathcal{H}_R \)-embeddings
\[
\theta_Q^Q : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'} \quad \text{for} \quad w \in W_{M_Q}(1)
\]

\[
\theta_Q^Q : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'} \quad \text{for} \quad w \in W_{M_Q}(1)
\]

**Proof.** We have the \( \mathcal{H}_{M_2^+,R} \)-embedding
\[
v \mapsto v \otimes e_Q^Q : e_Q^Q(V) \rightarrow V \otimes \mathcal{H}_{M_2^+} \quad \mathcal{H} \quad \text{to} \quad e(V) \otimes_R (\text{Ind}_{Q}^{G} 1)^{\mu},
\]
and compatible \( \mathcal{H}_R \)-embeddings
\[
v \otimes f_{QU} : \mathcal{H}_Q \rightarrow \mathcal{H}_Q \quad \text{for} \quad w \in W_{M_Q}(1)
\]

Note that \( T_{\sigma}^{M_Q} \in \mathcal{H}_{M_2^+} \) for \( d \in \mathbb{W}_{M_Q} \). By transitivity of the parabolic induction, it is equal to the \( \mathcal{H}_R \)-embedding
\[
\theta_Q^Q : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'} \quad \text{for} \quad w \in W_{M_Q}(1)
\]

On the other hand we have the \( \mathcal{H}_R \)-embedding
\[
v \otimes 1_H \mapsto v \otimes \theta_Q(e_Q^Q) \quad : \text{Ind}_{H_Q}^{H} (e_{H_Q}(V)) \rightarrow \text{Ind}_{H_{Q'}}^{H}(V)
\]

given by the restriction to \( e(V) \otimes_R (\text{Ind}_{Q}^{G} 1)^{\mu} \) of the \( \mathcal{H}_R \)-isomorphism given in Lemma 4.3 (4.1), from \( e(V) \otimes_R (\text{Ind}_{Q}^{G} 1)^{\mu} \) to \( V \otimes _{\mathcal{H}_{M_2^+} \otimes \mathcal{H}} \) sending \( v \otimes f_{PU} \) to \( v \otimes 1_H \), noting that \( v \otimes f_{QU} = (v \otimes f_{PU}) \theta_Q(e_Q^Q) \) by Proposition 3.15, \( f_{QU} = f_{PU} \theta_Q(e_P^Q) \) and \( \theta_Q(e_P^Q) \) acts trivially on \( e(V) \) (this is true for \( T_d \) for \( d \in W_{M_2} \)). Comparing the embeddings (4.12) and (4.13), we get the \( \mathcal{H}_R \)-isomorphism (4.9).

We can replace \( Q \) by \( Q' \) in the \( \mathcal{H}_R \)-isomorphisms (4.9), (4.12), and (4.13). With (4.12) we see \( \text{Ind}_{H_Q}^{H} (e_{H_Q}(V)) \) and \( \text{Ind}_{H_{Q'}}^{H}(V) \) as \( \mathcal{H}_R \)-submodules of \( \text{Ind}_{H_Q}^{H} (V) \). As seen in (4.8) we have \( \theta_Q(e_P^Q) = \theta_Q(e_P^Q) \theta_Q(e_Q^Q) \). We deduce the \( \mathcal{H}_R \)-isomorphism (4.11).
By (3.19) for $Q$ and (4.6),

$$f_{Q\mu} = \sum_{d \in \mathcal{Q}_{\mu}, \mathcal{W}_{\mu}} f_{Q\mu} T_d = f_{Q\mu} \theta_Q(e_{\mathcal{Q}_{\mu}})$$

in $(\text{Ind}_{\mathcal{Q}}^{G} 1)^{\mu}$. We deduce that the $H_R$-embedding corresponding to (4.11) via $\kappa_Q$ and $\kappa_{Q'}$ is the $H_R$-embedding (4.10).

We recall that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal and that $V$ is extensible to $G$ of extension $e(V)$.

**Corollary 4.6.** The cokernel of the $H_R$-map

$$\bigoplus_{Q \subseteq Q' \subseteq G} \text{Ind}^H_{H_Q}(e_{H_Q}(V)) \to \text{Ind}^H_{H_Q}(e_{H_Q}(V))$$

defined by the $\iota(Q, Q')$, is isomorphic to $e(V) \otimes_R (\text{St}^G_{\mathcal{Q}})^{\mu}$ via $\kappa_Q$.

### 4.2. Invariants in the tensor product.

We return to the setting where $P = MN$ is a standard parabolic subgroup of $G$, $\sigma$ is a smooth $R$-representation of $M$ with $P(\sigma) = G$ of extension $e(\sigma)$ to $G$, and $Q$ a parabolic subgroup of $G$ containing $P$. We still assume that $\Delta_P$ and $\Delta \setminus \Delta_P$ are orthogonal.

The $H_R$-modules $e(\rho_{\mathcal{Q}}) = e(\sigma)^{\mu}$ are equal (Theorem 3.13). We compute

$$I_G(P, \sigma, Q)^{\mu} = (e(\sigma) \otimes_R \text{St}_{\mathcal{Q}})$$

**Theorem 4.7.** The natural linear maps $e(\sigma)^{\mu} \otimes_R (\text{Ind}_{\mathcal{Q}}^{G} 1)^{\mu} \to (e(\sigma) \otimes_R \text{Ind}_{\mathcal{Q}}^{G} 1)^{\mu}$ and $e(\sigma)^{\mu} \otimes_R (\text{St}^G_{\mathcal{Q}})^{\mu} \to (e(\sigma) \otimes_R \text{St}^G_{\mathcal{Q}})^{\mu}$ are isomorphisms.

**Proof.** We need some preliminaries. In [GK14, Ly15], are introduced a finite free $\mathcal{Z}$-module $\mathfrak{M}$ (depending on $\Delta_{\mathcal{Q}}$) and a $\mathcal{B}$-equivariant embedding $\text{St}_{\mathcal{Q}} \hookrightarrow C^\infty(\mathcal{B}, \mathfrak{M})$ (we indicate the coefficient ring in the Steinberg representation) which induces an isomorphism $(\text{St}_{\mathcal{Q}})^{\mathcal{Z}} \cong C^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$.

**Lemma 4.8.**

1. $(\text{Ind}_{\mathcal{Q}}^{G} \mathcal{Z})^{\mathcal{B}}$ is a direct factor of $\text{Ind}_{\mathcal{Q}}^{G} \mathcal{Z}$.
2. $(\text{St}_{\mathcal{Q}}^{G})^{\mathcal{B}}$ is a direct factor of $\text{St}_{\mathcal{Q}}^{G} \mathcal{Z}$.

**Proof.**

1. [AHV, Example 2.2].
2. As $\mathfrak{M}$ is a free $\mathcal{Z}$-module, $C^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$ is a direct factor of $C^\infty(\mathcal{B}, \mathfrak{M})$. Consequently, $\iota(\text{Ind}_{\mathcal{Q}}^{G} \mathcal{Z})^{\mathcal{B}} = C^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$ is a direct factor of $\text{Ind}_{\mathcal{Q}}^{G} \mathcal{Z}$. As $\iota$ is injective, we get (2).

We now prove Theorem 4.7. We may and do assume that $\sigma$ is $e$-minimal (because $P(\sigma) = P(\sigma_{\text{min}})$, $e(\sigma) = e(\sigma_{\text{min}})$) so that $\Delta_M$ and $\Delta \setminus \Delta_M$ are orthogonal and we use the same notation as in section 3.2 in particular $M_2 = M_{\Delta \setminus \Delta_M}$. Let $V$ be the space of $e(\sigma)$ on which $M_2$ acts trivially. The restriction of $\text{Ind}_{\mathcal{Q}}^{G} \mathcal{Z}$ to $M_2$ is $\text{Ind}_{Q \cap M_2}^{M_2} \mathcal{Z}$, that of $\text{St}_{\mathcal{Q}}^{G} \mathcal{Z}$ in $\text{St}_{Q \cap M_2}^{M_2} \mathcal{Z}$.

As in [AHV, Example 2.2], $(\text{Ind}_{Q \cap M_2}^{M_2} \mathcal{Z} \otimes V)^{\mu_{M_2}} \cong (\text{Ind}_{Q \cap M_2}^{M_2} \mathcal{Z})^{\mu_{M_2}} \otimes V$. We have

$$(\text{Ind}_{Q \cap M_2}^{M_2} \mathcal{Z})^{\mu_{M_2}} = (\text{Ind}_{Q \cap M_2}^{M_2} \mathcal{Z})^{\mu_{M_2}} = (\text{Ind}_{Q \cap M_2}^{M_2} \mathcal{Z})^{\mu_{M_2}}.$$
Consider $\mathcal{U}_M \cup \mathcal{U}_G$, and $\text{Ind}_Q^G \mathbb{Z}$ is trivial on $M'$. Therefore, $((\text{Ind}_Q^G \mathbb{Z}) \otimes V)^{\mathcal{U}_M} \simeq (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}_M} \otimes V$.

Now taking fixed points under $\mathcal{U}_M$, as $\mathcal{U} = \mathcal{U}_M \cup \mathcal{U}_G$.

$$((\text{Ind}_Q^G \mathbb{Z}) \otimes V)^{\mathcal{U}_M} \simeq (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}_M} \otimes V^{\mathcal{U}_M}.$$  

The equality uses that the $\mathbb{Z}$-module $\text{Ind}_Q^G \mathbb{Z}$ is free. We get the first part of the theorem as $(\text{Ind}_Q^G \mathbb{Z})^G \otimes V^{\mathcal{U}_M} \simeq (\text{Ind}_Q^G \mathcal{R})^G \otimes V^{\mathcal{U}_M}$.

Tensoring with $R$ the usual exact sequence defining $\text{St}_Q^G \mathcal{Z}$ gives an isomorphism $\text{St}_Q^G \mathcal{Z} \otimes R \simeq \text{St}_Q^G \mathcal{R}$ and in [GK14, Ly15], it is proved that the resulting map $\text{St}_Q^G \mathcal{R} \rightarrow C^\infty(\mathcal{B}, \mathcal{M} \otimes R)$ is also injective. Their proof in no way uses the ring structure of $R$, and for any $\mathcal{Z}$-module $V$, tensoring with $V$ gives a $\mathcal{B}$-equivariant embedding $\text{St}_Q^G \mathcal{Z} \otimes V \rightarrow C^\infty(\mathcal{B}, \mathcal{M} \otimes V)$. The natural map $((\text{Ind}_Q^G \mathbb{Z})^G \otimes V \rightarrow \text{St}_Q^G \mathcal{Z} \otimes V$ is also injective by Lemma 4.8 (2). Taking $\mathcal{B}$-fixed points we get inclusions

$$(4.14) \quad (\text{St}_Q^G \mathbb{Z})^G \otimes V \rightarrow (\text{St}_Q^G \mathcal{Z} \otimes V) \rightarrow C^\infty(\mathcal{B}, \mathcal{M} \otimes V)^G \simeq \mathcal{M} \otimes V.$$  

The composite map is surjective, so the inclusions are isomorphisms. The image of $\mathcal{I}V$ consists of functions which are left $\mathbb{Z}$-invariant, and $\mathcal{B} = \mathbb{Z}^{\mathcal{U}_M}$ where $\mathcal{U}' = \mathcal{G}' \cap \mathcal{U}$. It follows that $\mathcal{I}$ yields an isomorphism $(\text{St}_Q^G \mathbb{Z})^G \simeq C^\infty(\mathcal{B}, \mathcal{M} \otimes \mathcal{I}'$ and consisting of the constant functions. So that in particular $(\text{St}_Q^G \mathbb{Z})^G = (\text{St}_Q^G \mathcal{Z})^G$ and reasoning as previously we get isomorphisms

$$(4.15) \quad (\text{St}_Q^G \mathbb{Z})^G \otimes V \simeq (\text{St}_Q^G \mathcal{Z} \otimes V)^G \simeq \mathcal{M} \otimes V.$$  

The equality $(\text{St}_Q^G \mathcal{Z})^G = (\text{St}_Q^G \mathcal{Z})^G$ and the isomorphisms remain true when we replace $\mathcal{U}'$ by any group between $\mathcal{B}$ and $\mathcal{U}$. We apply these results to $\text{St}_Q^G \mathcal{Z} \otimes V$ to get that the natural map $(\text{St}_Q^G \mathcal{Z})^{\mathcal{U}_M} \otimes V \rightarrow (\text{St}_Q^G \mathcal{Z})^{\mathcal{U}_G} \otimes V$ is an isomorphism and also that $(\text{St}_Q^G \mathcal{Z})^{\mathcal{U}_G} \otimes V \rightarrow (\text{St}_Q^G \mathcal{Z} \otimes V)^{\mathcal{U}_G}$ is an isomorphism. The $\mathcal{Z}$-module $\text{St}_Q^G \mathcal{Z}^{\mathcal{U}_G}$ is free and the $V^{\mathcal{U}_M} = V^{\mathcal{U}_G}$, so taking fixed points under $\mathcal{U}_M$, we get $\text{St}_Q^G \mathcal{Z}^{\mathcal{U}_G} \otimes V^{\mathcal{U}_M} \simeq (\text{St}_Q^G \mathcal{Z} \otimes V)^{\mathcal{U}_G}$. We have $\text{St}_Q^G \mathcal{Z} \otimes \mathcal{V} = \text{St}_Q^G \mathcal{R} \otimes \mathcal{V}$ and $(\text{St}_Q^G \mathcal{Z})^{\mathcal{U}_G} \otimes V^{\mathcal{U}_G} = (\text{St}_Q^G \mathcal{R})^{\mathcal{U}_G} \otimes V^{\mathcal{U}_G}$. This ends the proof of the theorem.

**Theorem 4.9.** The $\mathcal{R}$-modules $(e(\sigma) \otimes_R \text{Ind}_Q^G 1)^{\mathcal{U}_G} = e(\sigma)^{\mathcal{U}_G} \otimes_R (\text{Ind}_Q^G 1)^{\mathcal{U}_G}$ are equal. The $\mathcal{R}$-modules $(e(\sigma) \otimes_R \text{St}_Q^G 1)^{\mathcal{U}_G} = e(\sigma)^{\mathcal{U}_G} \otimes_R (\text{St}_Q^G 1)^{\mathcal{U}_G}$ are also equal.

**Proof.** We already know that the $\mathcal{R}$-modules are equal (Theorem 4.7). We show that they are equal as $\mathcal{H}$-modules. The $\mathcal{H}$-modules $e(\sigma)^{\mathcal{U}_G} \otimes_R (\text{Ind}_Q^G 1)^{\mathcal{U}_G}$ are equal (Theorem 3.13), they are isomorphic to $\text{Ind}_Q^G (e(\mathcal{H}_Q(\sigma)^{\mathcal{U}_G}))$ (Proposition 4.5), to $(\text{Ind}_Q^G (e(\mathcal{Q}(\sigma)))^{\mathcal{U}_G}$ [OV17, Proposition 4.4], and to $(e(\sigma) \otimes_R \text{Ind}_Q^G 1)^{\mathcal{U}_G}$ [AHIV, Lemma 2.5]). We deduce that the $\mathcal{H}$-modules $e(\sigma)^{\mathcal{U}_G} \otimes_R (\text{Ind}_Q^G 1)^{\mathcal{U}_G}$ are equal. The same is true when $Q$ is replaced by a parabolic subgroup $Q'$ of $G$ containing $Q$. The representation $e(\sigma) \otimes_R \text{St}_Q^G$ is the cokernel of the natural $\mathcal{R}[G]$-map

$$\bigoplus_{Q \subset Q'} (e(\sigma) \otimes_R \text{Ind}_Q^G 1) \overset{\alpha_Q}{\longrightarrow} e(\sigma) \otimes_R \text{Ind}_Q^G 1$$
and the $\mathcal{H}_R$-module $e(\sigma)^{\ell} \otimes_R (\text{St}_{Q}^G)^{\ell}$ is the cokernel of the natural $\mathcal{H}_R$-map

$$\bigoplus_{\sigma \in Q} e(\sigma)^{\ell} \otimes_R (\text{Ind}_{Q}^G 1)^{\ell} \otimes_R (\text{Ind}_{Q}^G 1)^{\ell}$$

obtained by tensoring (3.22) by $e(\sigma)^{\ell}$ over $R$, because the tensor product is right exact. The maps $\beta_Q = \alpha_Q^{\ell}$ are equal and the $R$-modules $e(\sigma)^{\ell} \otimes_R (\text{St}_{Q}^G)^{\ell} = (e(\sigma) \otimes_R \text{St}_{Q}^G)^{\ell}$ are equal. This implies that the $\mathcal{H}_R$-modules $e(\sigma)^{\ell} \otimes_R (\text{St}_{Q}^G)^{\ell} = (e(\sigma) \otimes_R \text{St}_{Q}^G)^{\ell}$ are equal.

Remark 4.10. The proof shows that the representations $e(\sigma) \otimes_R \text{Ind}_{Q}^G 1$ and $e(\sigma) \otimes \text{St}_{Q}^G$ of $G$ are generated by their $\mathcal{U}$-fixed vectors if the representation $\sigma$ of $M$ is generated by its $\mathcal{U}_M$-fixed vectors. Indeed, the $R$-modules $e(\sigma)\ell = \sigma^{\ell}$, $(\text{Ind}_{Q}^G 1)\ell = (\text{St}_{Q}^G 1)^{\ell}$ are equal. If $\sigma^{\ell}$ generates $\sigma$, then $e(\sigma)$ is generated by $e(\sigma)\ell$. The representation $\text{Ind}_{Q}^G 1|_{M}$ is generated by $(\text{Ind}_{Q}^G 1)^{\ell}$ (this follows from the lemma below), we have $G = MM'$ and $M'$ acts trivially on $e(\sigma)$. Therefore the $R[G]$-module generated by $\sigma^{\ell} \otimes_R (\text{Ind}_{Q}^G 1)^{\ell}$ is $e(\sigma) \otimes_R \text{Ind}_{Q}^G 1$. As $e(\sigma) \otimes_R \text{St}_{Q}^G$ is a quotient of $e(\sigma) \otimes_R \text{Ind}_{Q}^G 1$, the $R[G]$-module generated by $\sigma^{\ell} \otimes_R (\text{St}_{Q}^G)^{\ell}$ is $e(\sigma) \otimes_R \text{St}_{Q}^G$.

Lemma 4.11. For any standard parabolic subgroup $P$ of $G$, the representation $\text{Ind}_{P}^G 1|_{G'}$ is generated by its $\mathcal{U}$-fixed vectors.

Proof. Because $G = PG'$ it suffices to prove that if $J$ is an open compact subgroup of $G$ the characteristic function $\chi_{J}$ on $G$ is a finite sum of translates of $\chi_{PJ} = 1_{P|_{G'}}$ by $G'$. For $t \in T$ we have $P|T = P^{-1}T|_{G'}$ and we can choose $t \in T \cap J$ such that $t^{-1}T_{G'} \subseteq J$.

4.3. General triples. Let $P = MN$ be a standard parabolic subgroup of $G$. We now investigate situations where $\Delta_P$ and $\Delta \setminus \Delta_P$ are not necessarily orthogonal. Let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module.

Definition 4.12. Let $P(\mathcal{V}) = M(\mathcal{V})N(\mathcal{V})$ be the standard parabolic subgroup of $G$ with $P(\mathcal{V}) = M(\mathcal{V}) \cup \Delta_P$ and $\Delta_P = \{ \alpha \in \Delta \mid$ orthogonal to $\Delta_M \mid T^{\mathcal{U}}(z)$ acts trivially on $\mathcal{V}$ for all $z \in \mathcal{Z} \cap M'_\alpha \}$. If $Q$ is a parabolic subgroup of $G$ between $P$ and $P(\mathcal{V})$, the triple $(P, \mathcal{V}, Q)$ called an $\mathcal{H}_R$-triple, defines a right $\mathcal{H}_R$-module $I_R(P, \mathcal{V}, Q)$ equal to\

$$\text{Ind}_{M(\mathcal{V})}^R(e(\mathcal{V}) \otimes_R (\text{St}_{Q,M(\mathcal{V})}^M)^{\mathcal{H}_R(\mathcal{V})}) (e(\mathcal{V}) \otimes_R (\text{St}_{Q,M(\mathcal{V})}^M)^{\mathcal{H}_R(\mathcal{V})}) \otimes_{\mathcal{H}_{M(\mathcal{V})},N(\mathcal{V})} \mathcal{H}_R,$$

where $e(\mathcal{V})$ is the extension of $\mathcal{V}$ to $\mathcal{H}_M(\mathcal{V})$.

This definition is justified by the fact that $M(\mathcal{V})$ is the maximal standard Levi subgroup of $G$ such that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}_M(\mathcal{V})$.

Lemma 4.13. $\Delta_P$ is the maximal subset of $\Delta \setminus \Delta_P$ orthogonal to $\Delta_P$ such that $T^{\mathcal{U}}_\alpha$ acts trivially on $\mathcal{V}$ for all $\alpha \in \Lambda(1) \cap \mathcal{W}_{M'_\alpha}$.

Proof. For $J \subseteq \Delta$ let $M_J$ denote the standard Levi subgroup of $G$ with $\Delta_M = J$. The group $Z \cap M'_J$ is generated by the $Z \cap M'_\alpha$ for all $\alpha \in J$ (Lemma 2.1). When $J$ is orthogonal to $\Delta_M$ and $\lambda \not\in \Lambda_M(1)$, $\ell_M(\lambda) = 0$ where $\ell_M$ is the length associated to $S^\text{aff}_M$ and the map $\lambda \mapsto T^{\mathcal{U}}_\alpha = T^{\mathcal{U}}_\alpha : \Lambda_M(1) \to \mathcal{H}_M$ is multiplicative.
The following is the natural generalization of Proposition 4.5 and Corollary 4.6. Let \( Q' \) be a parabolic subgroup of \( G \) with \( Q \subset Q' \subset P(V) \). Applying the results of section 4.1 to \( M(V) \) and its standard parabolic subgroups \( Q \cap M(V) \subset Q' \cap M(V) \), we have an \( \mathcal{H}_M(V,R) \)-isomorphism

\[
\text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_Q(V)}(e_{\mathcal{H}_Q}(V)) = e_{\mathcal{H}_Q}(V) \otimes_{\mathcal{H}_M} e_{\mathcal{H}_Q(V)} \otimes_R (\text{Ind}_{Q \cap M(V)}^{M(V)} 1_{M(V)})
\]

and an \( \mathcal{H}_M(V,R) \)-embedding

\[
\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}_{Q'}}(e_{\mathcal{H}_Q}(V)) \rightarrow \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}_{Q'}}(e_{\mathcal{H}_Q}(V)),
\]

Applying the parabolic induction \( \text{Ind}_{\mathcal{H}_M(V)}^H \) which is exact and transitive, we obtain an \( \mathcal{H}_R \)-isomorphism \( \kappa_Q = \text{Ind}_H^V(\kappa_Q \cap M(V)) \).

(4.16) \[
\text{Ind}_{\mathcal{H}_Q}^H(e_{\mathcal{H}_Q}(V)) \overset{\kappa_Q}{\rightarrow} \text{Ind}_{\mathcal{H}_M(V)}^H(e(V) \otimes_R (\text{Ind}_{Q \cap M(V)}^{M(V)} 1_{M(V)}))
\]

\[
v \otimes 1_H \mapsto v \otimes f_{Q \cap M(V)} \otimes 1_H.
\]

and an \( \mathcal{H}_R \)-embedding \( \iota(Q,Q') = \text{Ind}_{\mathcal{H}_M(V)}^H(\iota(Q,Q') M(V)) \)

(4.17) \[
v \otimes 1_H \mapsto v \otimes \theta_{Q'}(e_{\mathcal{H}_Q}(V)) \rightarrow \text{Ind}_{\mathcal{H}_Q}^H(e_{\mathcal{H}_Q}(V)).
\]

Applying Corollary 4.6 we obtain:

**Theorem 4.14.** Let \((P,V,Q)\) be an \( \mathcal{H}_R \)-triple. Then, the cokernel of the \( \mathcal{H}_R \)-map

\[
\oplus_{Q \subseteq Q' \subset P(V)} \text{Ind}_{\mathcal{H}_Q}^H(e_{\mathcal{H}_Q}(V)) \rightarrow \text{Ind}_{\mathcal{H}_Q}(e_{\mathcal{H}_Q}(V)),
\]

defined by the \( \iota(Q,Q') \) is isomorphic to \( I_H(P,V,Q) \) via the \( \mathcal{H}_R \)-isomorphism \( \kappa_Q \).

Let \( \sigma \) be a smooth \( R \)-representation of \( M \) and let \( Q \) be a parabolic subgroup of \( G \) with \( P \subset Q \subset P(\sigma) \).

**Remark 4.15.** The \( \mathcal{H}_R \)-module \( I_H(P,\sigma^{M_{\text{min}}},Q) \) is defined if \( \Delta_Q \setminus \Delta_P \) and \( \Delta_P \) are orthogonal because \( Q \subset P(\sigma) \subset P(\sigma^{M_{\text{max}}}) \) (Theorem 3.13).

We denote here by \( P_{\text{min}} = M_{\text{min}}N_{\text{min}} \) the minimal standard parabolic subgroup of \( G \) contained in \( P \) such that \( \sigma = e_P(\sigma|_{M_{\text{min}}}) \) (Lemma 2.3), we drop the index \( \sigma \). The sets of roots \( \Delta_{P_{\text{min}}} \) and \( \Delta_P(\sigma|_{M_{\text{min}}}) \setminus \Delta_{P_{\text{min}}} \) are orthogonal (Lemma 2.4).

The groups \( P(\sigma|_{M_{\text{min}}}) = I_G(P_{\text{min}},\sigma|_{M_{\text{min}}},Q) = \text{Ind}_{P_{\text{min}}}^G(\sigma|_{M_{\text{min}}}) \otimes_R \text{St}_Q^{(\sigma|_{M_{\text{min}}})} \) of \( G \), and the \( R \)-modules \( \sigma^{M_{\text{min}}} = \sigma^{M_{\text{min}}}|_{M_{\text{min}}} \) are equal. From Theorem 3.13,

\[
P(\sigma) \subset P(\sigma^{M_{\text{min}}}), \quad e_{M(\sigma)}(\sigma^{M_{\text{min}}}) = e_{M(\sigma)}(\sigma^{M_{\text{min}}}),
\]

and \( P(\sigma^{M_{\text{min}}}) = P(\sigma) \) if \( \sigma^{M_{\text{min}}} \) generates the representation \( \sigma|_{M_{\text{min}}} \). The \( \mathcal{H}_R \)-module

\[
I_H(P_{\text{min}},\sigma^{M_{\text{min}}},Q) = \text{Ind}_{M(\sigma)^{M_{\text{min}}}}^H(e(\sigma^{M_{\text{min}}}) \otimes_R (\text{St}_Q^{(\sigma^{M_{\text{min}}})} 1_{M(\sigma)^{M_{\text{min}}}}))
\]

is defined because \( \Delta_{P_{\text{min}}} \) and \( \Delta_{P(\sigma^{M_{\text{min}}}) \setminus \Delta_{P_{\text{min}}} \setminus \Delta_{P_{\text{min}}} \} \Delta_{P_{\text{min}}} \) are orthogonal and \( P \subset Q \subset P(\sigma) \subset P(\sigma^{M_{\text{min}}}). \)
Remark 4.16. If $\sigma^{M_{\text{min}}}$ generates the representation $\sigma|_{M_{\text{min}}}$ (in particular if $R$ is an algebraically closed field of characteristic $p$ and $\sigma$ is irreducible), then $P(\sigma) = P(\sigma^{M_{\text{min}}})$ hence

$$I_{H}(P_{\text{min}}, \sigma^{M_{\text{min}}}, Q) = \text{Ind}_{H_{M_{\text{min}}}}^{H_{\sigma}}(e_{H_{M_{\text{min}}}}(\sigma^{M_{\text{min}}}) \otimes_R (\text{St}_{Q^\text{min}M_{\text{min}}}(\sigma))_{M_{\text{min}}}).$$

Applying Theorem 4.9 to $(P_{\text{min}} \cap M(\sigma), \sigma|_{M_{\text{min}}} : Q \cap M(\sigma))$, we have $(\text{Ind}_{H(\sigma)}^{H})_{R}$-modules are equal. We have the $H_{R}$-isomorphism [OV17, Proposition 4.4]:

$$I_{G}(P, \sigma, Q)^{H} = \text{Ind}_{H_{M(\sigma)}}^{H}(e_{H_{M(\sigma)}}(\sigma^{M(\sigma)})_{M(\sigma)}).$$

Remark 4.16

We deduce the following.

Theorem 4.17. Let $(P, \sigma, Q)$ be an $R[G]$-triple. Then, we have the $H_{R}$-isomorphism

$$I_{G}(P, \sigma, Q)^{H} \cong \text{Ind}_{H_{M(\sigma)}}^{H}(e_{H_{M(\sigma)}}(\sigma^{M_{\text{min}}})_{M(\sigma)}).$$

In particular,

$$I_{G}(P, \sigma, Q)^{H} \cong \begin{cases} I_{H}(P_{\text{min}}, \sigma^{M_{\text{min}}}, Q) & \text{if } P(\sigma) = P(\sigma^{M_{\text{min}}}), \\ I_{H}(P, \sigma^{M_{\text{min}}}, Q) & \text{if } P(\sigma) = P_{\text{min}}, P(\sigma) = P(\sigma^{M_{\text{min}}}). \end{cases}$$

4.4. Comparison of the parabolic induction and coinduction. Let $P = MN$ be a standard parabolic subgroup of $G$, let $V$ be a right $H_{R}$-module, and let $Q$ be a parabolic subgroup of $G$ with $Q \subset P(V)$. When $R$ is an algebraically closed field of characteristic $p$, in [Abe], we associated to $(P, V, Q)$ an $H_{R}$-module using the parabolic coinduction

$$\text{Coind}_{H_{M_{Q}}}^{H_{R}}(-) = \text{Hom}_{H_{M_{Q}, R}}(\mathcal{H}, -) : \text{Mod}_{R}(H_{M_{Q}}) \rightarrow \text{Mod}_{R}(H_{R})$$

instead of the parabolic induction $\text{Ind}_{H_{M_{Q}}}^{H_{R}}(-) = \otimes_{H_{M_{Q}}} \mathcal{H}$. The index $\theta^{*}$ in the parabolic coinduction means that $H_{M_{Q}}$ embeds in $H_{R}$ by $\theta_{Q}$. Our terminology is different from the one in [Abe] where the parabolic coinduction is called induction. For a parabolic subgroup $Q'$ of $G$ with $Q \subset Q' \subset P(V)$, there is a natural inclusion of $H_{R}$-modules

$$\text{Hom}_{H_{M_{Q'}, R}}(\mathcal{H}, e_{H_{Q'}}(V)) \xrightarrow{i(Q, Q')} \text{Hom}_{H_{M_{Q}, R}}(\mathcal{H}, e_{H_{Q}}(V))$$

because $\theta^{*}(H_{M_{Q}}) \subset \theta^{*}(H_{M_{Q'}})$ as $W^{-1}_{M_{Q}}(1) \subset W^{-1}_{M_{Q'}}(1)$, and $\iota_{M_{Q'}}^{w} = \iota_{M_{Q}}^{w}$ for $w \in W_{M_{Q}}(1)$ and $v \in V$. (This is [Abe, Proposition 4.19] when $R$ is an algebraically closed field of characteristic $p$. This follows from our formulation of the extension for any $R$.)

Definition 4.18. Let $CI_{K}(P, V, Q)$ denote the cokernel of the map

$$\bigoplus_{Q \subset Q' \subset P(V)} \text{Hom}_{H_{M_{Q'}, R}}(\mathcal{H}, e_{H_{Q'}}(V)) \rightarrow \text{Hom}_{H_{M_{Q}, R}}(\mathcal{H}, e_{H_{Q}}(V))$$

defined by the $H_{R}$-embeddings $i(Q, Q').$
When $R$ is an algebraically closed field of characteristic $p$, we showed that the $H_R$-module $CI_q(P, V, Q)$ is simple when $V$ is simple and supersingular (Definition 4.25), and that any simple $H_R$-module is of this form for an $H_R$-triple $(P, V, Q)$ where $V$ is simple and supersingular, $P$ and $Q$ and the isomorphism class of $V$ are unique [Abe]. The aim of this section is to compare the $H_R$-modules $I(R, V, Q)$ with the $H_R$-modules $CI_q(P, V, Q)$ and to show that the classification is also valid with the $H_R$-modules $I_q(P, V, Q)$.

It is already known that a parabolically coinduced module is a parabolically induced module and vice versa [Abe, Proposition 4.15], [Vig15b, Theorem 1.8]. To make it more precise we need to introduce notation.

We lift the elements $w$ of the finite Weyl group $W$ to $\tilde{w} \in N_G \cap K$ as in [AHIV17, IV.6], [OV17, Proposition 2.7]; they satisfy the braid relations $\tilde{w}_1 \tilde{w}_2 = (w_1 w_2)\ell$ when $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ and when $s \in S$, $\tilde{s}$ is admissible, in particular lies in $1 W_G$.

Let $w, w_M, w^M$ denote, respectively, the longest elements in $W, W_M$ and $w w_M$. We have $w = w^{-1} = w^M w_M, w_M = w^M 1, w = w^M w_M$.

$$w^M(\Delta_M) = -w(\Delta_M) \subset \Delta, \quad w^M(\Phi^+ \setminus \Phi^+_M) = w(\Phi^+ \setminus \Phi^+_M).$$

Let $w M$ be the standard Levi subgroup of $G$ with $\Delta_{w M} = w^M(\Delta_M)$ and $w P$ the standard parabolic subgroup of $G$ with Levi $w M$. We have

$$w M = w^M M(w^M)^{-1} = w M(\tilde{w})^{-1}, \quad w^M M = w M w = (w^M)^{-1}.$$ The conjugation $w \mapsto w^M w (w^M)^{-1}$ in $W$ gives a group isomorphism $W_M \to W_{w M}$ sending $S^aff_M$ onto $S^aff_{w M}$, respecting the finite Weyl subgroups $w^M M W_M(w^M)^{-1} = W_{w M} = w M w^{-1}$, and exchanging $W_{M^+}$ and $W_{(w M)^-} = w W_{M^+} w^{-1}$. The conjugation by $w^M$ restricts to a group isomorphism $W_M(1) \to W_{w M}(1)$ sending $W_M(1)$ onto $W_{w M}(1)$.

The linear isomorphism

$$(4.20) \quad H_M \xrightarrow{(w^M)} H_{w M} \quad T^M_w \mapsto T^M_{w^M w(\tilde{w})^{-1}} \text{ for } w \in W_M(1),$$

is a ring isomorphism between the pro-$p$-Iwahori Hecke rings of $M$ and $w M$. It sends the positive part $H_{M^+}$ of $H_M$ onto the negative part $H_{(w M)^-}$ of $H_{w M}$ [Vig15b, Proposition 2.20]. We have $w = w_M w w_M = w^M w_M, (w^M)^{-1} = w w_M w M$, where $t M = w M w^{-2} \in Z_k$.

**Definition 4.19.** The twist $w^M, V$ of $V$ by $w^M$ is the right $H_{w M}$-module deduced from the right $H_M$-module $V$ by functoriality: as $R$-modules $w^M, V = V$ and for $v \in V, w \in W_M(1)$ we have $v T^M_w = v T^M_{w^M w(\tilde{w})^{-1}} = v T^M_w$.

We can define the twist $w^M, V$ of $V$ with the $T^M_w$ instead of $T^M_w$.

**Lemma 4.20.** For $v \in V, w \in W_M(1)$ we have $v T^M_{w^M w(\tilde{w})^{-1}} = v T^M_w$ in $w^M, V$.

**Proof.** By the ring isomorphism $H_M \xrightarrow{(w^M)} H_{w M}$, we have $v^M_{w^M w(\tilde{w})^{-1}} = c_k^M$ when $\tilde{w} \in W_M(1)$ lifts $s \in S^aff_M$. So the equality of the lemma is true for $w = \tilde{w}$. Apply the braid relations to get the equality for all $w \in W_M(1)$.

We return to the $H_R$-module $H_{M^-}\otimes H(V, \bar{V})$ parabolically coinduced from $V$. It has a natural direct decomposition indexed by the set $W^I_M$ of elements $d$ in
the finite Weyl group $\mathbb{W}$ of minimal length in the coset $d\mathbb{W}_M$. Indeed it is known that the linear map

$$f \mapsto (f(T_d))_{d \in \mathbb{W}_M} : \text{Hom}_{\mathbb{H}_{M^*}, \theta^*}(\mathcal{H}, \mathcal{V}) \to \bigoplus_{d \in \mathbb{W}_M} \mathcal{V}$$

is an isomorphism. For $v \in \mathcal{V}$ and $d \in \mathbb{W}_M$, there is a unique element $f_d,v \in \text{Hom}_{\mathbb{H}_{M^*}, \theta^*}(\mathcal{H}, \mathcal{V})$ satisfying $f(T_d) = v$ and $f(T_{d'}) = 0$ for $d' \in \mathbb{W}_M \setminus \{d\}$.

It is known that the map $v \mapsto f_{d,v} : \mathbb{w}^M, \mathcal{V} \to \text{Hom}_{\mathbb{H}_{M^*}, \theta^*}(\mathcal{H}, \mathcal{V})$ is $\mathbb{H}_{M^*}$-equivariant: $f_{d,w,v:T_{d^2}} = f_{d,w,v}$ for all $v \in \mathcal{V}, w \in \mathbb{W}_{M^*}(1)$. By adjunction, this $\mathbb{H}_{M^*}$-equivariant map gives an $\mathbb{H}_R$-homomorphism from an induced module to a conduced module:

$$\mathbb{w}^M \mathcal{V} \otimes_{\mathbb{H}_{M^*}^*, \theta} \mathbb{H} \xrightarrow{\theta^\#} \text{Hom}_{\mathbb{H}_{M^*}, \theta^*}(\mathcal{H}, \mathcal{V}).$$

This is an isomorphism [Abe, Proposition 4.15], [Vig15b, Theorem 1.8].

The naive guess that a variant $\mu_Q$ of $\mu_P$ induces an $\mathbb{H}_R$-isomorphism between the $\mathbb{H}_R$-modules $I_\mathbb{H}(\mathbb{w}, \mathbb{P}, \mathbb{w}^M, \mathcal{V}, \mathcal{W}, Q)$ and $CI_\mathbb{H}(P, \mathcal{V}, Q)$ turns out to be true. The proof is the aim of the rest of this section.

The $\mathbb{H}_R$-module $I_\mathbb{H}(\mathbb{w}, \mathbb{P}, \mathbb{w}^M, \mathcal{V}, \mathcal{W}, Q)$ is well defined because the parabolic subgroups of $G$ containing $\mathbb{w}, \mathbb{P}$ and contained in $P(\mathbb{w}^M, \mathcal{V})$ are $\mathbb{w}Q$ for $P \subset Q \subset P(\mathcal{V})$, as follows from Lemma 4.21.

**Lemma 4.21.** $\Delta_{\mathbb{w}^M, \mathcal{V}} = -\mathbb{w}(\Delta_{\mathcal{V}})$.

**Proof.** Recall that $\Delta_{\mathcal{V}}$ is the set of simple roots $\alpha \in \Delta \setminus \Delta_M$ orthogonal to $\Delta_M$ and $T^M(z)$ acts trivially on $\mathcal{V}$ for all $z \in Z \cap M'_\alpha$, and the corresponding standard parabolic subgroup $P_{\mathcal{V}} = M_N \mathcal{V}$. The $Z \cap M'_{\alpha}$ for $\alpha \in \Delta_{\mathcal{V}}$ generate the group $Z \cap M'_\mathcal{V}$. A root $\alpha \in \Delta \setminus \Delta_M$ orthogonal to $\Delta_M$ is fixed by $\mathbb{w}_M$ so $\mathbb{w}^M(\alpha) = \mathbb{w}(\alpha)$ and

$$\mathbb{w}^M M_{\mathcal{V}}(\mathbb{w}^M)^{-1} = \mathbb{w} M_{\mathcal{V}}(\mathbb{w})^{-1}.$$

The proof of Lemma 4.21 is straightforward as $\Delta = -\mathbb{w}(\Delta), \Delta_{\mathbb{w}, M} = -\mathbb{w}(\Delta_M)$. □

Before going further, we check the commutativity of the extension with the twist. As $Q = M_Q U$ and $M_Q$ determine each other we denote $\mathbb{w}_{M_Q} = \mathbb{w}Q, \mathbb{w}_M = \mathbb{w}Q$ when $Q \neq P, G$.

**Lemma 4.22.** $e_{\mathbb{H}_{M^*}, \mathbb{w}^M, \mathcal{V}} = \mathbb{w}^Q e_{\mathbb{H}_{M^*}, \mathcal{V}}$.

**Proof.** As $\mathbb{R}$-modules $\mathcal{V} = e_{\mathbb{H}_{M^*}, \mathbb{w}^M, \mathcal{V}} = \mathbb{w}^Q e_{\mathbb{H}_{M^*}, \mathcal{V}}$. A direct computation shows that the Hecke element $T^Q_D$ acts in the $\mathbb{H}_{M^*}$-module $e_{\mathbb{H}_{M^*}, \mathbb{w}^M, \mathcal{V}}$ by the identity if $w \in \mathbb{w}^Q W_{M_Q}^D(\mathbb{w}Q)^{-1}$ and by $T^Q_D$ if $w \in \mathbb{w}^Q W_{M_Q}^D(\mathbb{w}Q)^{-1}$ where $M_Q$ denotes the standard Levi subgroup with $\Delta_M = \Delta_Q \setminus \Delta_P$. Whereas in the $\mathbb{H}_R$-module $\mathbb{w}^Q e_{\mathbb{H}_{M^*}, \mathcal{V}}$, the Hecke element $T^Q_D$ acts by the identity if $w \in \mathbb{w}^Q W_{M_Q}^D$ and by $T^Q_D$ if $w \in W_{M}^Q(1)$. So the lemma means that

$$1 W_{M^*}^D = \mathbb{w}^Q \mathbb{w}^Q W_{M_Q}^D(\mathbb{w}Q)^{-1}, \quad (\mathbb{w}Q)^{-1} \mathbb{w}^Q = (\mathbb{w}^M)^{-1} \mathbb{w}^M \mathbb{w}^Q.$$

These properties are easily proved using that $1 W_{M_Q}$ is normal in $W(1)$ and that the sets of roots $\Delta_P$ and $\Delta_Q \setminus \Delta_P$ are orthogonal: $\mathbb{w}Q = \mathbb{w}M Q \mathbb{w}M$; the elements $\mathbb{w}_{M_Q}$ and $\mathbb{w}_M$ normalize $W_M$ and $W_{M_Q}$, the elements of $W_{M_Q}$ commutes with the elements of $W_M$. □
We return to our guess. The variant $\mu_Q$ of $\mu_P$ is obtained by combining the commutativity of the extension with the twist and the isomorphism (4.21) applied to $(Q, e_{H_Q}(V))$ instead of $(P, V)$. The $H_R$-isomorphism $\mu_Q$ is

$$v \otimes 1_R \mapsto f_{w.M} : \text{Ind}_{H_w.Q}^H(e_{H_w.Q}(w.M,V)) \to \text{Hom}_{H_{Q}.P}(\mathcal{H}, e_{H.Q}(V)).$$

Our guess is that $\mu_Q$ induces an $H_R$-isomorphism from the cokernel of the $H_R$-map

$$\bigoplus_{Q \subseteq Q' \subseteq P(V)} \text{Ind}_{H_{w.Q}.Q}^H(e_{H_w.Q}(w.M,V)) \to \text{Ind}_{H_{w.Q}}^H(e_{H_w.Q}(w.M,V))$$

defined by the $H_R$-embeddings $i(w.Q, w.Q')$, isomorphic to $I_H(w.P, w.M.V, w.Q)$ via $\kappa_{w.Q}$ (Theorem 4.14), onto the cokernel $CI_R(P, V, Q)$ the $H_R$-map

$$\bigoplus_{Q \subseteq Q' \subseteq P(V)} \text{Hom}_{H_{Q}.Q}(\mathcal{H}, e_{H.Q}(V)) \to \text{Hom}_{H_{Q}.Q}(\mathcal{H}, e_{H.Q}(V))$$

defined by the $H_R$-embeddings $i(Q, Q')$. This is true if $i(Q, Q')$ corresponds to $i(w.Q, w.Q')$ via the isomorphisms $\mu_{Q'}$ and $\mu_Q$. This is the content of the next proposition.

**Proposition 4.23.** For all $Q \subseteq Q' \subseteq P(V)$ we have

$$i(Q, Q') \circ \mu_{Q'} = \mu_Q \circ i(w.Q, w.Q').$$

We postpone to section 4.6 the rather long proof of the proposition.

**Corollary 4.24.** The $H_R$-isomorphism $\mu_Q \circ \kappa_{w.Q}^{-1}$ induces an $H_R$-isomorphism

$$I_H(w.P, w.M.V, w.Q) \to CI_R(P, V, Q).$$

### 4.5. Supersingular $H_R$-modules, classification of simple $H_R$-modules

We recall first the notion of supersingularity based on the action of the center of $H$.

The center of $H$ [Vig14, Theorem 1.3] contains a subalgebra $\mathbb{Z}_{T'}$ isomorphic to $\mathbb{Z}[T^+/T_1]$ where $T^+$ is the monoid of dominant elements of $T$ and $T_1$ is the pro-$p$-Sylow subgroup of the maximal compact subgroup of $T$.

Let $t \in T$ of image $\mu_t \in W(1)$ and let $(E_a(w))_{w \in W(1)}$ denote the alcove walk basis of $H$ associated to a closed Weyl chamber $\alpha$ of $W$. The element

$$E_a(C(\mu_t)) = \sum_{\mu'} E_a(\mu')$$

is the sum over the elements in $\mu'$ in the conjugacy class $C(\mu_t)$ of $\mu_t$ in $W(1)$. It is a central element of $H$ and does not depend on the choice of $\alpha$. We write also $z(t) = E_a(C(\mu_t))$.

**Definition 4.25.** A non-zero right $H_R$-module $V$ is called supersingular when, for any $v \in V$ and any non-invertible $t \in T^+$, there exists a positive integer $n \in \mathbb{N}$ such that $v(z(t))^n = 0$. If one can choose $n$ independent on $(v, t)$, then $V$ is called uniformly supersingular.

**Remark 4.26.** One can choose $n$ independent on $(v, t)$ when $V$ is finitely generated as a right $H_R$-module. If $R$ is a field and $V$ is simple we can take $n = 1$.

When $G$ is compact modulo the center, $T^+ = T$, and any non-zero $H_R$-module is supersingular.
such that and applying (3) and (2) for \( w \in \mathcal{M} \).

Proposition 4.27. Assume that \( V \) is a supersingular right \( \mathcal{H}_R \)-module and that \( p \) is nilpotent in \( V \). Then \( L_{\mathcal{H}_R}(V) = 0 \), and if \( V \) is uniformly supersingular \( R_{\mathcal{H}_R}(V) = 0 \).

Proof. This is a consequence of three known properties:

1. \( \mathcal{H}_M \) is the localization of \( \mathcal{H}_M^+ \) (resp., \( \mathcal{H}_M^- \)) at \( T^+_M \) for any element \( \mu \in \Lambda^+(1) \), central in \( W^+(1) \) and strictly \( N \)-positive (resp., \( N \)-negative), and \( T^+_M = T^{\mu^+}_M \). See [Vig15b, Theorem 1.4].
2. When \( \sigma \) is anti-dominant, \( E_\sigma(\mu) = T^\mu \) if \( \mu \in \Lambda^+(1) \) and \( E_\sigma(\mu) = T^\mu_\mu \) if \( \mu \in \Lambda^-(1) \).
3. Let an integer \( n > 0 \) and \( \mu \in \Lambda(1) \) such that the \( W \)-orbit of \( v_\mu \in X_\Lambda(T) \otimes \mathbb{Q} \) (definition in section 2.1) and of \( \mu \) have the same number of elements.

Then 

\[
(E_\sigma(C(\mu)))^n E_\sigma(\mu) - E_\sigma(\mu)^{n+1} \in \mathfrak{p} \mathcal{H}.
\]

See [Vig15a, Lemma 6.5.1], where the hypotheses are given in the proof (but not written in the lemma).

Let \( \mu \in \Lambda_T^+(1) \) satisfying (1) for \( \mathcal{M}^+ \) and (3), similarly let \( w, \mu \in \Lambda_T^+(1) \) satisfying (1) for \( (w, \mathcal{M}) \) and (3). For \( (R, V) \) as in the proposition, let \( v \in V \) and \( n > 0 \) such that \( vE_\sigma(C(\mu))^n = vE_\sigma(C(\mu))n = 0 \). Multiplying by \( E_\sigma(\mu) \) or \( E_\sigma(w, \mu) \), and applying (3) and (2) for \( \sigma \) anti-dominant we get:

\[
vE_\sigma(\mu)^{n+1} = vT^\mu_\mu^{n+1} \in pV, \quad vE_\sigma((w, \mu)^{n+1}) = v(T^\mu_{w, \mu})^{n+1} \in pV.
\]

The proposition follows from: \( vT^\mu_{w, \mu} \in P^\mu_{w, \mu} \) in \( pV \) (as explained in [Abe16, Proposition 5.17] when \( p = 0 \) in \( R \)). From \( v(T^\mu_{w, \mu})^{n+1} \) in \( pV \), we get \( v(T^\mu_{w, \mu})^{n+1} \) in \( pV \). As \( T_{w, \mathcal{M}}^\mu = T_{w, \mathcal{M}}^\mu \) is invertible in \( \mathcal{H}_{w, \mathcal{M}} \), we get \( v1_{1_{w, \mathcal{M}}} \) in \( pV \in \mathcal{H}_{w, \mathcal{M}} \). As we arbitrary, \( \mathcal{V} \mathcal{H}_{w, \mathcal{M}} = \mathcal{H}_{w, \mathcal{M}} \). If \( p \) is nilpotent in \( V \), then \( \mathcal{V} \mathcal{H}_{w, \mathcal{M}} = \mathcal{H}_{w, \mathcal{M}} \). We deduce \( \mathcal{V} \mathcal{H}_{w, \mathcal{M}} = \mathcal{H}_{w, \mathcal{M}} \). If \( p \) is nilpotent in \( V \), then \( \mathcal{V} \mathcal{H}_{w, \mathcal{M}} = \mathcal{H}_{w, \mathcal{M}} \).}

Recalling that \( \tilde{w}_M \mathcal{V} \) is obtained by functoriality from \( \mathcal{V} \) and the ring isomorphism \( \tilde{w}_M \mathcal{V} \) defined in (4.20), the equivalence between \( \mathcal{V} \) supersingular and \( \tilde{w}_M \mathcal{V} \) supersingular follows from Lemma 4.28.

Lemma 4.28.

1. Let \( t \in T \). Then \( t \) is dominant for \( U_M \) if and only if \( \tilde{w}_M t(\tilde{w}_M)^{-1} \in T \) is dominant for \( U_{w, \mathcal{M}} \).

(2) The $R$-algebra isomorphism $\mathcal{H}_{M,R} \overset{i(\tilde{w}^M)}{\rightarrow} \mathcal{H}_{w,M,R}$, $T^M_{\tilde{w}} \mapsto T^M_{w,M\tilde{w}(\tilde{w}^M)^{-1}}$ for $w \in W_M(1)$ sends $z^M(t)$ to $z^M_{w,M}(w^M t(\tilde{w}^M)^{-1})$ for $t \in T$ dominant for $U_M$.

Proof. The conjugation by $\tilde{w}^M$ stabilizes $T$, sends $U_M$ to $U_{w,M}$, and sends the $\mathbb{W}_M$-orbit of $t \in T$ to the $\mathbb{W}_{w,M}$-orbit of $\tilde{w}^M t(\tilde{w}^M)^{-1}$, as $\tilde{w}^M \mathbb{W}_M(\tilde{w}^M)^{-1} = \mathbb{W}_{w,M}$. It is known that $i(\tilde{w}^M)$ respects the anti-dominant alcove walk bases [Vig15b, Proposition 2.20]: it sends $E^M(w)$ to $E^M(\tilde{w}^M_{w,M}(w^M)^{-1})$ for $w \in W_M(1)$. $\square$

We deduce the following.

**Corollary 4.29.** Let $V$ be a right $\mathcal{H}_{M,R}$-module. Then $V$ is supersingular if and only if the right $\mathcal{H}_{w,M,R}$-module $\tilde{w}^M V$ is supersingular.

Assume $R$ is an algebraically close field of characteristic $p$. The supersingular simple $\mathcal{H}_{M,R}$-modules are classified in [Vig15a]. By Corollaries 4.24 and 4.29, the classification of the simple $\mathcal{H}_R$-modules in [Abe] remains valid with the $\mathcal{H}_R$-modules $I_H(P,V,Q)$ instead of $CI_H(P,V,Q)$.

**Corollary 4.30** (Classification of simple $\mathcal{H}_R$-modules). Assume $R$ is an algebraically closed field of characteristic $p$. Let $(P,V,Q)$ be an $\mathcal{H}_R$-triple where $\mathcal{V}$ is simple and supersingular. Then, the $\mathcal{H}_R$-module $I_H(P,V,Q)$ is simple. A simple $\mathcal{H}_R$-module is isomorphic to $I_H(P,V,Q)$ for an $\mathcal{H}_R$-triple $(P,V,Q)$ where $\mathcal{V}$ is simple and supersingular, $P,Q$ and the isomorphism class of $\mathcal{V}$ are unique.

### 4.6. A commutative diagram.

We prove in this section Proposition 4.23. For $Q \subset Q' \subset P(V)$ we show by an explicit computation that

$$\mu_Q^{-1} \circ i(Q,Q') \circ \mu_{Q'} \text{Ind}_{\mathcal{H}_{w,q'}}^\mathcal{H}(e_{\mathcal{H}_{w,q'}}(\tilde{w}^M,\mathcal{V})) \to \text{Ind}_{\mathcal{H}_{w,q'}}^\mathcal{H}(e_{\mathcal{H}_{w,q'}}(\tilde{w}^M,\mathcal{V}))$$

is equal to $i(w.Q,w.Q')$. The $R$-module $e_{\mathcal{H}_{w,q'}}(\tilde{w}^M,\mathcal{V}) \otimes I_{\mathcal{H}_{w,q'}}$ generates the $\mathcal{H}_R$-module $e_{\mathcal{H}_{w,q'}}(\tilde{w}^M,\mathcal{V}) \otimes_{\mathcal{H}_{w,q'}}, \theta^{\mathcal{H}_R}$, and $\mathcal{H}_R = \text{Ind}_{\mathcal{H}_{w,q'}}^\mathcal{H}(e_{\mathcal{H}_{w,q'}}(\tilde{w}^M,\mathcal{V}))$ and by (4.17)

$$i(w.Q,w.Q')(v \otimes 1_{\mathcal{H}_{w,q'}}) = v \otimes \sum_{d \in \mathbb{W}_{w,q'} \mathbb{W}_{\mathcal{H}_{w,q'}}} t^d$$

for $v \in \mathcal{V}$ seen as an element of $e_{\mathcal{H}_{w,q'}}(\tilde{w}^M,\mathcal{V})$ in the LHS and an element of $e_{\mathcal{H}_{w,q'}}(\tilde{w}^M,\mathcal{V})$ in the RHS.

**Lemma 4.31.** $(\mu_Q^{-1} \circ i(Q,Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in \mathbb{W}_{w,q'}} q_d T^d_{\mathbb{W}_{w,q'}}(w'.q)^{-1}$.

Proof. $\mu_{Q'}(v \otimes 1_{\mathcal{H}_{w,q'}})$ is the unique homomorphism $f_{\mathbb{W}_{w,q'},v} \in \text{Hom}_{\mathcal{H}_{w,q'},\mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{w,q'}}(\mathcal{V}))$ sending $T_{w.q'}$ to $v$ and vanishing on $T_{\hat{d}}$ for $d' \in \mathbb{W}_{w,q'} \setminus \{w.q'\}$ by (4.22). By (4.19), $i(Q,Q')$ is the natural embedding of $\text{Hom}_{\mathcal{H}_{w,q'},\mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{w,q'}}(\mathcal{V}))$ in $\text{Hom}_{\mathcal{H}_{w,q'},\mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{w,q'}}(\mathcal{V}))$ therefore $i(Q,Q')(f_{\mathbb{W}_{w,q'},v})$ is the unique homomorphism $\text{Hom}_{\mathcal{H}_{w,q'},\mathcal{V}}(\mathcal{H}, e_{\mathcal{H}_{w,q'}}(\mathcal{V}))$ sending $T_{w.q'}$ to $v$ and vanishing on $T_{\hat{d}}$ for $d' \in \mathbb{W}_{w,q'} \setminus \{w.q'\}$. As $\mathbb{W}_{w,q'} = \mathbb{W}_{w,q'} \mathbb{W}_{w.q'}$, this homomorphism vanishes on $T_{\hat{d}}$ for $w$ not
in \(w_{M,q'}W_{M,q}^{W_{M,q}}\). By [Abe16, Lemma 2.22], the inverse of \(\mu_Q\) is the \(H_R\)-isomorphism

\[
(4.25) \quad \text{Hom}_{H_{M,q}}(H, e_{H_{M,q}}(V)) \xrightarrow{\mu_Q^{-1}} \text{Ind}_{H_{M,q}}^{H} (e_{H_{M,q}}(w^M,V))
\]

where \(W_W\) is the set of \(d \in W\) with minimal length in the coset \(dW_M\). We deduce the explicit formula

\[
(\mu_Q^{-1} \circ i(Q,Q') \circ \mu_Q)(v \otimes 1_h) = \sum_{w \in W_{M,q}} i(Q,Q') (f_{w_{M,q}',v})(T_w) \otimes T_{w_{M,q}w^{-1}}.
\]

Some terms are zero: the terms for \(w \in W_{M,q} \setminus W_{M,q'}\) not in \(W_{M,q'}W_{M,q}^{W_{M,q}}\). We analyze the other terms for \(w \in W_{M,q} \cap W_{M,q'}W_{M,q}^{W_{M,q}'}\); this is \(W_{M,q'}W_{M,q}^{W_{M,q}'}\). Let \(w = w_{M,q}'d, d \in \mathbb{W}_{M,q}^{W_{M,q}'}\), and \(d = \tilde{w} \in W_{M,q}d\) with \(d \in W_{M,q}^W\), lifting \(d\). By the braid relations \(T_{\tilde{w}} = T_{W_{M,q}'}T_d\). We have \(T_d = \theta^*(T_{M,q'})\) by the braid relations because \(d \in W_{M,q'}\), \(S_{M,q'} \subset S_{\text{aff}}\) and \(\theta^*(e_M^{\alpha}) = c_s\) for \(s \in S_{M,q'}\). As \(W_{M,q'} \subset W_{M,q'} \cap W_{M,q}^{W_{M,q}'}\), we deduce:

\[
i(Q,Q')(f_{w_{M,q}',v})(T_{\tilde{w}}) = i(Q,Q')(f_{w_{M,q}',v})(T_{W_{M,q}'}T_d)
\]

\[
= i(Q,Q')(f_{w_{M,q}',v})(T_{W_{M,q}'}T_{d'})
\]

\[
= vT_{d'} = q_d v,
\]

Corollary 3.9 gives the last equality. \(\square\)

The formula for \((\mu_Q^{-1} \circ i(Q,Q') \circ \mu_Q)(v \otimes 1_h)\) given in Lemma 4.31 is different from the formula \((4.24)\) for \(i(w,Q,w,Q')(v \otimes 1_h)\). It needs some work to prove that they are equal.

A first reassuring remark is that \(W_{M,q}W_{M,q} = \{wd^{-1}w \mid d \in W_{M,q}\}\), so the two summation sets have the same number. But better,

\[
W_{M,q}'W_{M,q}W_{M,q} = \{wQ(wQ')^{-1}d \mid d \in W_{M,q}'\}
\]

because \(W_{M,q}'W_{M,q}W_{M,q} = W_{M,q}W_{M,q}^{W_{M,q}}\). To prove the latter equality, we apply the criterion: \(w \in W_{M,q}\), lies in \(W_{M,q}W_{M,q}W_{M,q}\) if and only if \(w(a) > 0\) for all \(a \in A_Q\) noticing that \(d \in W_{M,q}W_{M,q}\) implies \(w(a) \in -\Delta_Q\), \(dw(a) \in -\Phi_{\text{aff}}\), \(wQ'dw(a) > 0\).

Let \(x_d = wQ(wQ')^{-1}\). We have \(wQ(wQ')^{-1} = \tilde{x}_d\) because the lifts \(\tilde{w}\) of the elements \(w \in W\) satisfy the braid relations and \(\ell(x_d) = \ell(wQ'd^{-1}W_Q) = \ell(wQ) - \ell(wQ'd^{-1}) = \ell(wQ) - \ell(d) - \ell(wQ') + \ell(wQ') - \ell(d)\). We have \(q_d = q_{wQd}x_dw_{w,Q}'\) because \(wQ'd^{-1}\) and \(q_d = q_{wQd}x_dw_{w,Q}'\), and \(q_d = q_{wQd}x_dw_{w,Q}'\). So

\[
\sum_{d \in W_{M,q}^{W_{M,q}}q_dT_{wQ}(q_d^{-1}d) = \sum_{x_d \in W_{M,q}W_{M,q}^W} q_{wQ}x_dw_{w,Q}T_{x_d}'}
\]
In the RHS, only $\tilde{w}^M \cdot \mathcal{V}, w \cdot Q, w \cdot Q'$ appear. The same holds true in the formula (4.24). The map $(P, \mathcal{V}, Q, Q') \mapsto (w \cdot P, w^M \cdot \mathcal{V}, w \cdot Q, w \cdot Q')$ is a bijection of the set of triples $(P, \mathcal{V}, Q, Q')$ where $P = MN, Q, Q'$ are standard parabolic subgroups of $G$, $\mathcal{V}$ a right $\mathcal{H}_R$-module, $Q \subseteq Q' \subseteq P(\mathcal{V})$ by Lemma 4.21. So we can replace $(w \cdot P, w^M \cdot \mathcal{V}, w \cdot Q, w \cdot Q')$ by $(P, \mathcal{V}, Q, Q')$. Our task is reduced to prove in $e_{H_Q}(\mathcal{V}) \otimes_{H^t_{Q'}} \mathcal{H}_R$:

$$v \otimes \sum_{d \in \mathcal{W}_{M_{Q'}}} T_d = v \otimes \sum_{d \in \mathcal{W}_{M_{Q'}}} q_{w_Q} dw_{Q'} T_d^*.$$  

A second simplification is possible: we can replace $Q \subseteq Q'$ by the standard parabolic subgroups $Q_2 \subseteq Q_2'$ of $G$ with $\Delta_{Q_2} = \Delta_Q \setminus \Delta_P$ and $\Delta_{Q_2'} = \Delta_{Q'} \setminus \Delta_P$, because $\Delta_P$ and $\Delta_{P(\mathcal{V})} \setminus \Delta_P$ are orthogonal. Indeed, $\mathcal{W}_{M_{Q_2}} = \mathcal{W}_M \times \mathcal{W}_{M_{Q_2}}$ and $\mathcal{W}_{M_{Q_2'}} = \mathcal{W}_M \times \mathcal{W}_{M_{Q_2'}}$ are direct products, the longest elements $w_{Q'} = w_M w_{Q_2'}, w_Q = w_M w_{Q_2}$ are direct products and

$$\mathcal{W}_{M_{Q_2}} \mathcal{W}_{M_{Q_2'}} = \mathcal{W}_{M_{Q_2}} \mathcal{W}_{M_{Q_2'}}, \quad w_Q dw_{Q'} = w_{Q_2'} dw_{Q'_2}.$$  

Once this is done, we use the properties of $e_{H_Q}(\mathcal{V})$: $vh \otimes 1_H = v \otimes \theta_q(h)$ for $h \in \mathcal{H}_{M_{Q_2}}$, and $T_{\mathcal{V}_w}^* \cdot h$ acts trivially on $e_{H_Q}(\mathcal{V})$ for $w \in \mathcal{W}_{M_{Q_2}} \cup (\Lambda(1) \cap \mathcal{W}_{M_{Q_2'}})$. Set $i\mathcal{W}_{M_{Q_2}} = \{ w \in \mathcal{W}_{M_{Q_2}} \mid w \text{ is a lift of some element in } \mathcal{W}_{M_{Q_2}} \}$ and $i\mathcal{W}_{M_{Q_2'}}$ similarly. Then $Z_k \cap i\mathcal{W}_{M_{Q_2}} \subseteq (\Lambda(1) \cap \mathcal{W}_{M_{Q_2'}}) \cap i\mathcal{W}_{M_{Q_2}}$ and $i\mathcal{W}_{M_{Q_2'}} \subseteq i\mathcal{W}_{M_{Q_2'}} \cap i\mathcal{W}_{M_{Q_2}}$. This implies that (4.26) where $Q \subseteq Q'$ has been replaced by $Q_2 \subseteq Q_2'$ follows from a congruence

$$\sum_{d \in \mathcal{W}_{M_{Q_2}} \mathcal{W}_{M_{Q_2'}}} T_d \equiv \sum_{d \in \mathcal{W}_{M_{Q_2}} \mathcal{W}_{M_{Q_2'}}} q_{w_{Q_2}} dw_{Q'_2} T_d^*$$  

in the finite subring $H(i\mathcal{W}_{M_{Q_2}})$ of $H$ generated by $\{ T_w \mid w \in \mathcal{W}_{M_{Q_2}} \}$ modulo the right ideal $J_2$ with generators $\{ \theta_q(T_{\mathcal{V}_w}) - 1 \mid w \in (Z_k \cap i\mathcal{W}_{M_{Q_2}}) \cup i\mathcal{W}_{M_{Q_2}} \}$.

Another simplification concerns $T_d^*$ modulo $J_2$ for $d \in \mathcal{W}_{M_{Q_2'}}$. We recall that for any reduced decomposition $d = s_1 \cdots s_n$ with $s_i \in S \cap M_{Q_2}$ we have $T_d^* = (T_{s_1} - c_{s_1}) \cdots (T_{s_n} - c_{s_n})$ where the $c_i$ are admissible. For $s$ admissible, by (3.2)

$$c_s \equiv q_s - 1.$$  

Therefore

$$T_d^* \equiv (T_{s_1} - q_{s_1} + 1) \cdots (T_{s_n} - q_{s_n} + 1).$$  

Let $J' \subseteq J_2$ be the ideal of $H(i\mathcal{W}_{M_{Q_2}})$ generated by $\{ T_t - 1 \mid t \in Z_k \cap i\mathcal{W}_{M_{Q_2}} \}$. Then the ring $H(i\mathcal{W}_{M_{Q_2}})/J'$ and its right ideal $J'_2/\mathcal{J'}$ are the specialization of the generic finite ring $H(\mathcal{W}_{M_{Q_2}})^g$ over $\mathbb{Z}[(q_s)_{s \in S_{M_{Q_2}}}]$ where the $q_s$ for $s \in S_{M_{Q_2}} = S \cap \mathcal{W}_{M_{Q_2}}$ are indeterminates, and of its right ideal $\mathcal{J}_2'$ with the same generators. The similar congruence modulo $\mathcal{J}_2'$ in $H(\mathcal{W}_{M_{Q_2}})^g$ (the generic congruence) implies the congruence (4.27) by specialization.

We will prove the generic congruence in a more general setting where $H$ is the generic Hecke ring of a finite Coxeter system $(\mathcal{W}, S)$ and parameters $(q_s)_{s \in S}$ such that $q_s = q_{s'}$ when $s, s'$ are conjugate in $\mathcal{W}$. The Hecke ring $H$ is a $\mathbb{Z}[(q_s)_{s \in S}]$-free
module of basis \((T_w)_{w \in W}\) satisfying the braid relations and the quadratic relations 
\[ T_s^2 = q_s + (q_s - 1)T_s \] for \( s \in S \). The other basis \((T_w^*)_{w \in W}\) satisfies the braid relations and the quadratic relations 
\[ (T_s^*)^2 = q_s - (q_s - 1)T_s^* \] for \( s \in S \), and is related to the first basis by \( T_s^* = T_s - (q_s - 1) \) for \( s \in S \), and more generally \( T_wT_{w'} = T_{w'}T_w^* \) for \( w \in W \) [Vig16, Proposition 4.13].

Let \( J \subset S \) and \( J \) is the right ideal of \( H \) with generators \( T^*_w - 1 \) for all \( w \) in the group \( W_J \) generated by \( J \).

**Lemma 4.32.** A basis of \( J \) is \((T^*_{w_1} - 1)T^*_{w_2} \) for \( w_1 \in W_J \setminus \{1\}, w_2 \in W_J \), and adding \( T^*_{w_3} \) for \( w_3 \in W_J \) gives a basis of \( H \). In particular, \( J \) is a direct factor of \( H \).

**Proof.** The elements \((T^*_{w_1} - 1)T^*_{w_2} \) for \( w \in W_J \), \( w \in W \) generate \( J \). We write 
\[ w = u_1w_2 \] with unique elements \( u_1 \in W_J, w_2 \in W_J \), and \( T^*_{w_1} = T^*_{u_1}T^*_{w_2} \). Therefore, 
\((T^*_{w_1} - 1)T^*_{w_2} \) by an induction on the length of \( u_1 \), one proves that \((T^*_{w_1} - 1)T^*_{w_2} \) is a linear combination of \((T^*_{v_1} - 1)\) for \( v_1 \in W_J \) as in the proof of Proposition 3.4. It is clear that the elements \((T^*_{w_1} - 1)T^*_{w_2} \) and \( T^*_{w_3} \) for \( w_1 \in W_J \setminus \{1\}, w_2 \in W_J \) form a basis of \( H \). \( \square \)

Let \( w_J \) denote the longest element of \( W_J \) and \( w = w_S \).

**Lemma 4.33.** In the generic Hecke ring \( H \), the congruence modulo \( J \)
\[ \sum_{d \in W_J} T_d \equiv \sum_{d \in W_J} q_{w,J} q_{w,J} T_d^* \]
holds true.

**Proof.**

**Step 1.** We show
\[ W_J \cap W = w_J W_J w, \quad q_{w,J} q_{w,J} T_d^* = T_{w_J} T_{w_J} T_d^* \]
The equality between the groups follows from the characterization of \( W \cap W \). an element \( d \in W \) has minimal length in \( W_J \) if and only if \( \ell(ud) = \ell(u) + \ell(d) \) for all \( u \in W_J \). An easy computation shows that \( \ell(uw_J d) = \ell(u) + \ell(w_J d) \) for all \( u \in W_J, d \in \tilde{W} \) (both sides are equal to \( \ell(u) + \ell(w_J - \ell(d)) \). The second equality follows from \( q_{w,J} q_{w,J} = q_{w,J} \) because \( (w_J)^2 = 1 \) and \( \ell(w_J d) = \ell(d) \) (both sides are \( \ell(w) - \ell(d) \)) from \( q_{w,J} T_d^* = T_{w} T_{w_J} T_d^* \). We also have \( T_{w_J} = T_{w_J} T_{w_J} \).

**Step 2.** The multiplication by \( q_{w,J} \) on the quotient \( H/J \) is injective (Lemma 4.32) and \( q_{w,J} T_d^* = T_{w_J} \). By Step 1, \( q_{w,J} T_d^* \equiv T_{w_J} T_d^* \) and
\[ \sum_{d \in W_J} q_{w,J} T_d^* \equiv \sum_{d \in W_J} T_d T_{w_J} \]
The congruence
\[ (4.28) \quad \sum_{d \in W_J} T_d \equiv \sum_{d \in W_J} T_d T_{w_J} \]
for all \( s \in S \) implies the lemma because \( T_w^* = T_{s_1}^* \cdots T_{s_n}^* \) for any reduced decomposition \( w = s_1 \ldots s_n \) with \( s_i \in S \).
Step 3. When $J = \emptyset$, the congruence (4.28) is an equality

\[(4.29) \quad \sum_{w \in W} T_w = \sum_{w \in W} T_w T_s^*.
\]

It holds true because $\sum_{w \in W} T_w = \sum_{w \in W} T_w (T_s + 1)$ and $(T_s + 1) T_s^* = T_s T_s^* + T_s^* = q_s + T_s^* = T_s + 1$.

Step 4. Conversely the congruence (4.28) follows from (4.29) because

\[
\sum_{w \in W} T_w = (\sum_{u \in W_J} T_u) \sum_{d \in W_J W} T_d \equiv (\sum_{u \in W_J} q_u) \sum_{d \in W_J W} T_d \quad \text{(recall \quad q_u = T_u^* T_u - 1 \equiv T_u)}
\]

and we can simplify by $\sum_{w \in W_J} q_u$ in $H/\mathcal{J}$. □

This ends the proof of Proposition 4.23.

5. Universal representation $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \setminus G]$\]

The invariant functor $(-)^{\mathcal{H}}$ by the pro-$p$-Iwahori subgroup $\mathcal{U}$ of $G$ has a left adjoint

\[- \bigotimes_{\mathcal{H}} R[\mathcal{U} \setminus G] : \text{Mod}_{\mathcal{H}}(\mathcal{U}) \to \text{Mod}_{\mathcal{H}}(G).
\]

The smooth $R$-representation $\mathcal{V} \otimes_{\mathcal{H}} R[\mathcal{U} \setminus G]$ of $G$ constructed from the right $\mathcal{H}_R$-module $\mathcal{V}$ is called universal. We write

\[R[\mathcal{U} \setminus G] = X.
\]

**Question 5.1.** Does $\mathcal{V} \not= 0$ imply $\mathcal{V} \otimes_{\mathcal{H}_R} X \not= 0$ or does $\mathcal{V} \otimes_{\mathcal{H}_R} 1_{\mathcal{U}} = 0$ for $v \in \mathcal{V}$ imply $v = 0$? We have no counterexample. If $R$ is a field and the $\mathcal{H}_R$-module $\mathcal{V}$ is simple, the two questions are equivalent: $\mathcal{V} \otimes_{\mathcal{H}_R} X \not= 0$ if and only if the map $v \mapsto v \otimes 1_{\mathcal{U}}$ is injective. When $R$ is an algebraically closed field of characteristic $p$, $\mathcal{V} \otimes_{\mathcal{H}_R} X \not= 0$ for all simple $\mathcal{H}_R$-modules $\mathcal{V}$ if this is true for $\mathcal{V}$ simple supersingular (this is a consequence of Corollary 5.13).

The functor $- \bigotimes_{\mathcal{H}_R} X$ satisfies a few good properties: it has a right adjoint and is compatible with the parabolic induction and the left adjoint (of the parabolic induction). Let $P = MN$ be a standard parabolic subgroup and $X_M = R[\mathcal{U}_M \setminus M]$. We have functor isomorphisms

\[
(5.1) \quad (- \bigotimes X) \circ \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}_R} \to \text{Ind}_{\mathcal{H}_R}^{\mathcal{H}_{M,R}} \circ (- \bigotimes X_M),
\]

\[
(5.2) \quad (-)_{\mathcal{H}_R} \circ (- \bigotimes X) \to (- \bigotimes X_M) \circ \mathcal{L}_{M,R}^{\mathcal{H}}.
\]

The first one is [OV17, formula 4.15], the second one is obtained by left adjunction from the isomorphism $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}_R} \circ (-)^{\mathcal{H}_M} \to (-)^{\mathcal{H}_R} \circ \text{Ind}_{\mathcal{H}_R}^{\mathcal{H}_{M,R}}$ [OV17, formula (4.14)]. If $\mathcal{V}$ is a right $\mathcal{H}_R$-supersingular module and $p$ is nilpotent in $\mathcal{V}$, then $\mathcal{L}_{M,R}^{\mathcal{H}}(\mathcal{V}) = 0$ if $M \not= G$ (Proposition 4.27). Applying (5.2) we deduce the following.

**Proposition 5.2.** If $p$ is nilpotent in $\mathcal{V}$ and $\mathcal{V}$ supersingular, then $\mathcal{V} \otimes_{\mathcal{H}_R} X$ is left cuspidal.
**Remark 5.3.** For a non-zero smooth $R$-representation $\tau$ of $M$, $\Delta_\tau$ is orthogonal to $\Delta_\rho$ if $\tau$ is left cuspidal. Indeed, we recall from [AHIV17, II.7 Corollary 2] that $\Delta_\tau$ is not orthogonal to $\Delta_\rho$ if and only if there exists a proper standard parabolic subgroup $X$ of $M$ such that $\sigma$ is trivial on the unipotent radical of $X$; moreover $\tau$ is a subrepresentation of $\text{Ind}_X^M(\tau|_X)$, so the image of $\tau$ by the left adjoint of $\text{Ind}_X^M$ is not 0.

From now on, $V$ is a non-zero right $H_{M,R}$-module and

$$\sigma = V \otimes_{H_{M,R}} X_M.$$  

In general, when $\sigma \neq 0$, let $P_\perp(\sigma)$ be the standard parabolic subgroup of $G$ with $\Delta_{P_\perp(\sigma)} = \Delta_\rho \cup \Delta_{\perp,\sigma}$ where $\Delta_{\perp,\sigma}$ is the set of simple roots $\alpha \in \Delta_\sigma$ orthogonal to $\Delta_\rho$.

**Proposition 5.4.**

1. $P(V) \subset P_\perp(\sigma)$ if $\sigma \neq 0$.
2. $P(V) = P_\perp(\sigma)$ if the map $v \mapsto v \otimes 1_{\text{I}M}$ is injective.
3. $P(V) = P(\sigma)$ if the map $v \mapsto v \otimes 1_{\text{I}M}$ is injective, $p$ nilpotent in $V$ and $V$ supersingular.
4. $P(V) = P(\sigma)$ if $\sigma \neq 0$, $R$ is a field of characteristic $p$ and $V$ simple supersingular.

**Proof.**

1. $P(V) \subset P_\perp(\sigma)$ means that $Z \cap M'$ acts trivially on $V \otimes 1_{\text{I}M}$, where $M'$ is the standard Levi subgroup such that $\Delta_{M'} = \Delta_\sigma$. Let $z \in Z \cap M'$ and $v \in V$. As $\Delta_\rho$ and $\Delta_\rho$ are orthogonal, we have $T_{M',\sigma}(z) = T_{M}(z)$ and $U_M \otimes_{M'} = U_M \otimes_{M'\sigma}$. We have $v \otimes 1_{\text{I}M} = v T_{M}(z) \otimes 1_{\text{I}M} = v \otimes z^{-1} \text{I}M = z^{-1} (v \otimes 1_{\text{I}M})$.
2. If $v \otimes 1_{\text{I}M} = 0$ for $v \in V$ implies $v = 0$, then $\sigma \neq 0$ because $V \neq 0$. By (1) $P(V) \subset P_\perp(\sigma)$. As in the proof of (1), for $z \in Z \cap M_\perp$, we have $v T_{M',\sigma}(z) \otimes 1_{\text{I}M} = v T_{M}(z) \otimes 1_{\text{I}M} = v \otimes z^{-1} \text{I}M$ and our hypothesis implies $v T_{M',\sigma}(z) = v$ hence $P(V) \supset P_\perp(\sigma)$.
3. Proposition 5.2, Remark 5.3, and (2).
4. Question 5.1 and (3).

Let $Q$ be a parabolic subgroup of $G$ with $P \subset Q \subset P(V)$. In this chapter we will compute $I_H(P,V,Q) \otimes_R R[U|G]$ where $I_H(P,V,Q) = \text{Ind}_H^M(V,P) \otimes (S_{Q;M'',\text{H}_M(V)} P_{\text{I}M}(V))$ (Theorem 5.11). The smooth $R$-representation $I_G(P,\sigma, Q)$ of $G$ is well defined: it is 0 if $\sigma = 0$ and $\text{Ind}_H^M(\sigma) \otimes S_{Q;M'',\text{H}_M(V)}$ if $\sigma \neq 0$ because $(P,\sigma, Q)$ is an $R[G]$-triple by Proposition 5.4. We will show that the universal representation $I_H(P,V,Q) \otimes_R R[U|G]$ is isomorphic to $I_G(P,\sigma, Q)$, if $P(V) = P(\sigma)$ and $p = 0$, or if $\sigma = 0$ (Corollary 5.12). In particular, $I_H(P,V,Q) \otimes_R R[U|G] \simeq I_G(P,\sigma, Q)$ when $R$ is an algebraically closed field of characteristic $p$ and $V$ is supersingular.

**5.1.** $Q = G$. We consider first the case $Q = G$. We are in the simple situation where $V$ is extensible to $\text{H}$ and $P(V) = P(\sigma) = G$. $I_H(P,V,G) = e(V)$ and $I_G(P,\sigma, G) = e(\sigma )$. We recall that $\Delta \setminus \Delta_\rho$ is orthogonal to $\Delta_\rho$ and that $M_2$ denotes the standard Levi subgroup of $G$ with $\Delta_{M_2} = \Delta \setminus \Delta_\rho$. 


The $\mathcal{H}_R$-morphism $e(V) \to e(\sigma)^{\text{fl}} = \sigma^{\text{fl}}$ sending $v$ to $v \otimes 1_{U_M}$ for $v \in V$, gives by adjunction an $R[G]$-homomorphism

$$v \otimes 1_{U} \mapsto v \otimes 1_{U_M} : e(V) \otimes_{\mathcal{H}_R} X \xrightarrow{\Phi^G} e(\sigma).$$

If $\Phi^G$ is an isomorphism, then $e(V) \otimes_{\mathcal{H}_R} X$ is the extension to $G$ of $(e(V) \otimes_{\mathcal{H}_R} X)|_{M}$, meaning that $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$. The converse is true.

**Lemma 5.5.** If $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$, then $\Phi^G$ is an isomorphism.

**Proof.** Suppose that $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$. Then $e(V) \otimes_{\mathcal{H}_R} X$ is the extension to $G$ of $(e(V) \otimes_{\mathcal{H}_R} X)|_{M}$, and by Theorem 3.13, $(e(V) \otimes_{\mathcal{H}_R} X)^{\text{fl}}$ is the extension of $(e(V) \otimes_{\mathcal{H}_R} X)^{\text{fl}}|_{M}$. Therefore, by (3.12),

$$(v \otimes 1_{U})T^M_w = (v \otimes 1_{U})T^M_{w,1} \quad \text{for all } v \in V, w \in W_M(1).$$

As $V$ is extensible to $\mathcal{H}$, the natural map $v \mapsto v \otimes 1_{U} : V \xrightarrow{\Phi} (e(V) \otimes_{\mathcal{H}_R} X)^{\text{fl}}$ is $\mathcal{H}_M$-equivariant, i.e.,

$$vT^M_{w,1} \otimes 1_{U} = (v \otimes 1_{U})T^M_{w,1} \quad \text{for all } v \in V, w \in W_M(1)$$

because (3.12) $vT^M_{w,1} \otimes 1_{U} = vT^M_w \otimes 1_{U} = v \otimes T^M_w = (v \otimes 1_{U})T^M_w$ in $e(V) \otimes_{\mathcal{H}_R} X$.

We recall that $-\otimes_{\mathcal{H}_{M,\text{op}}} X_{M}$ is the left adjoint of $(-)^{\text{fl}}$. The adjoint $R[M]$-homomorphism $\sigma = V \otimes_{\mathcal{H}_{M,\text{op}}} X_{M} \to e(V) \otimes_{\mathcal{H}_R} X$ sends $v \otimes 1_{U_M}$ to $v \otimes 1_{U}$ for all $v \in V$. The $R[M]$-module generated by the $v \otimes 1_{U}$ for all $v \in V$ is equal to $e(V) \otimes_{\mathcal{H}_R} X$ because $M'_2$ acts trivially. Hence we obtained an inverse of $\Phi^G$. $\square$

Our next move is to determine if $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$. It is equivalent to see if $M'_2$ acts trivially on $e(V) \otimes 1_{U_M}$ as this set generates the representation $e(V) \otimes_{\mathcal{H}_R} X$ of $G$ and $M'_2$ is a normal subgroup of $G$ as $M'_2$ and $M$ commute and $G = ZM'_2M'_2$. Obviously, $U \cap M'_2$ acts trivially on $e(V) \otimes 1_{U}$. The group of double classes $(U \cap M'_2) \setminus M'_2/(U \cap M'_2)$ is generated by the lifts $\tilde{s} \in N \cap M'_2$ of the simple affine roots $s$ of $W_{M'_2}$. Therefore, $M'_2$ acts trivially on $e(V) \otimes_{\mathcal{H}_R} X$ if and only if for any simple affine root $s \in S^{\text{aff}}_{M'_2}$ of $W_{M'_2}$, any $\tilde{s} \in N \cap M'_2$ lifting $s$ acts trivially on $e(V) \otimes 1_{U_M}$.

**Lemma 5.6.** Let $v \in V, s \in S^{\text{aff}}_{M'_2}$ and $\tilde{s} \in N \cap M'_2$ lifting $s$. We have

$$(q_{s} + 1)(v \otimes 1_{U} - \tilde{s}(v \otimes 1_{U})) = 0.$$  

**Proof.** We compute:

$$T_{s}(\tilde{s}1_{U}) = \tilde{s}(T_{s}1_{U}) = 1_{U \setminus U_{s}}^{-1}1_{U} = \sum_{u} u(\tilde{s})^{-1}1_{U},$$

$$T_{s}(\tilde{s}^{2}1_{U}) = \tilde{s}^{2}(T_{s}1_{U}) = 1_{U \setminus U_{s}}^{-2}1_{U} = \sum_{u} u\tilde{s}1_{U}$$

for $u$ in the group $U/(\tilde{s}^{-1}U\tilde{s} \cap U)$ and $u^{\text{op}}$ in the group $\tilde{s}U\tilde{s}^{-1}/(\tilde{s}U\tilde{s}^{-1} \cap U)$; the reason is that $\tilde{s}^{2}$ normalizes $U$, $U\tilde{s}U\tilde{s}^{-1}$ is the disjoint union of the sets $U\tilde{s}U\tilde{s}^{-1}$ and $U\tilde{s}^{-1}U\tilde{s}^{-1}$. We introduce now a natural bijection

$$(5.3) \quad u \to u^{\text{op}} : U/(\tilde{s}^{-1}U\tilde{s} \cap U) \to \tilde{s}U\tilde{s}^{-1}/(\tilde{s}U\tilde{s}^{-1} \cap U)$$

which is not a group homomorphism. We recall the finite reductive group $G_{b',s}$ quotient of the parahoric subgroup $\mathcal{R}_{b'}$ of $G$ fixing the face fixed by $s$ of the alcove $C$. The Iwahori groups $Z_{0}U$ and $Z_{0}\tilde{s}U\tilde{s}^{-1}$ are contained in $\mathcal{R}_{b'}$ and their images
in \( G_{s,k} \) are opposite Borel subgroups \( Z_{s,k}U_{s,k} \) and \( Z_{s,k}U_{s,k}^{\text{op}} \). Via the surjective maps 
\( u \mapsto \pi : \mathcal{U} \to U_{s,k} \) and \( u^{\text{op}} \mapsto \pi'^{\text{op}} : \mathcal{U}(\hat{s})^{-1} \to U_{s,k}^{\text{op}} \) we identify the groups 
\( \mathcal{U}(\hat{s}^{-1}\mathcal{U}s \cap \mathcal{U}) \cong U_{s,k} \) and similarly \( \mathcal{U}(\hat{s}^{-1}\mathcal{U}s \cap \mathcal{U}) \cong U_{s,k}^{\text{op}} \). Let \( G'_{k,s} \) be the group generated by \( U_{s,k} \) and \( U_{s,k}^{\text{op}} \), and let \( B'_{s,k} = G'_{k,s} \cap Z_{s,k}U_{s,k} = (G'_{k,s} \cap Z_{s,k})U_{s,k} \).

We suppose (as we can) that \( \hat{s} \in R_1 \) and that its image \( \hat{s}_k \) in \( G_{s,k} \) lies in \( G_{k,s} \). We have \( \hat{s}_k U_{s,k}(\hat{s}_k)^{-1} = U_{s,k}^{\text{op}} \) and the Bruhat decomposition \( G_{k,s}^{\prime} = B'_{k,s} \sqcup U_{k,s} \hat{s}_k B'_{k,s} \) implies the existence of a canonical bijection \( \pi'^{\text{op}} \to \pi : (U_{k,s} - \{1\}) \to (U_{k,s} - \{1\}) \) respecting the cosets \( \pi'^{\text{op}}B'_{k,s} = \pi B'_{k,s} \). Via the preceding identifications we get the wanted bijection (5.3).

For \( v \in \mathcal{e}(V) \) and \( z \in Z_0 \cap M'_2 \) we have \( vT_z = v, z1_{\mathcal{U}} = T_z1_{\mathcal{U}} \) and \( v \otimes T_z1_{\mathcal{U}} = vT_z \otimes 1_{\mathcal{U}} \) therefore \( Z_0 \cap M'_2 \) acts trivially on \( V \otimes 1_{\mathcal{U}} \). The action of the group \( (Z_0 \cap M'_2)\mathcal{U} \) on \( V \otimes 1_{\mathcal{U}} \) is also trivial. As the image of \( Z_0 \cap M'_2 \) in \( G_{s,k} \) contains \( Z_k \cap G_{s,k} \),

\[
us(v \otimes 1_{\mathcal{U}}) = u^{\text{op}}(v \otimes 1_{\mathcal{U}})
\]

when \( u \) and \( u^{\text{op}} \) are not units and correspond via the bijection (5.3). So we have

\[
(5.4) \quad v \otimes T_z(\hat{s}1_{\mathcal{U}}) - (v \otimes 1_{\mathcal{U}}) = v \otimes T_z(\hat{s}^21_{\mathcal{U}}) - v \otimes 1_{\mathcal{U}}.
\]

We can move \( T_z \) on the other side of \( \otimes \) and as \( vT_z = q_zv \) (Corollary 3.9), we can replace \( T_z \) by \( q_z \). We have \( v \otimes \hat{s}^21_{\mathcal{U}} = v \otimes T_z^{-1}1_{\mathcal{U}} \) because \( \hat{s}^2 \in Z_0 \cap M'_2 \) normalizes \( \mathcal{U} \); as we can move \( T_z^{-1} \) on the other side of \( \otimes \) and as \( vT_z^{-1} = v \) we can forget \( \hat{s}^2 \). So (5.4) is equivalent to \( (q_z+1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0 \). \( \square \)

Combining the two lemmas we obtain the following.

**Proposition 5.7.** When \( V \) is extensible to \( \mathcal{H} \) and has no \( q_k + 1 \)-torsion for any \( s \in S_\text{aff} \), then \( M'_2 \) acts trivially on \( e(V) \otimes \mathcal{H} \), and \( \Phi^G \) is an \( R[G] \)-isomorphism.

Proposition 5.7 for the trivial character \( 1_{\mathcal{H}} \), says that \( 1_{\mathcal{H}} \otimes \mathcal{H} \) is the trivial representation \( 1_G \) of \( G \) when \( q_k + 1 \) has no torsion in \( R \) for all \( s \in S_\text{aff} \). This is proved in [OV17, Lemma 2.28] by a different method. The following counterexample shows that this is not true for all \( R \).

**Example 5.8.** Let \( G = GL(2,F) \) and let \( R \) be an algebraically closed field where \( q_0 + 1 = q_1 + 1 = 0 \) and \( S_\text{aff} = \{ s_0, s_1 \} \). (Note that \( q_{s_0} = q_{s_1} \) is the order of the residue field of \( F \).) Then the dimension of \( 1_{\mathcal{H}} \otimes \mathcal{H} \) is infinite, in particular

\[
1_{\mathcal{H}} \otimes \mathcal{H}, \mathcal{X} \neq 1_G.
\]

Indeed, the Steinberg representation \( St_G = (\text{Ind}_{GL(2,F)}^G 1_Z)/1_G \) of \( G \) is an indecomposable representation of length 2 containing an irreducible infinite dimensional representation \( \pi \) with \( \pi^{\mathcal{U}} = 0 \) of quotient the character \(-1)^{\text{val} \cdot \text{det}} \). This follows from the proof of Theorem 3 and from Proposition 24 in [Vig89]. The kernel of the quotient map \( St_G \otimes (-1)^{\text{val} \cdot \text{det}} \to 1_G \) is infinite dimensional without a non-zero \( \mathcal{U} \)-invariant vector. As the characteristic of \( R \) is not \( p \), the functor of \( \mathcal{U} \)-invariants is exact hence \( (St_G \otimes (-1)^{\text{val} \cdot \text{det}})^{\mathcal{U}} = 1_{\mathcal{H}} \). As \(- \otimes \mathcal{H} R[G] \mathcal{U} \) is the left adjoint of \((-)^{\mathcal{U}} \) there is a non-zero homomorphism

\[
1_{\mathcal{H}} \otimes \mathcal{H}, \mathcal{X} : St_G \otimes (-1)^{\text{val} \cdot \text{det}} \to 1_{\mathcal{H}}
\]

with image generated by its \( \mathcal{U} \)-invariants. The homomorphism is therefore surjective.
5.2. $V$ extensible to $H$. Let $P = MN$ be a standard parabolic subgroup of $G$ with $\Delta_P$ and $\Delta \setminus \Delta_P$ orthogonal. We still suppose that the $H_{M,R}$-module $V$ is extensible to $H$, but now $P \subset Q \subset G$. So we have $I_H(P,V,Q) = e(V) \otimes_R (S^G_1)^H$ and $I_G(P,\sigma,Q) = e(\sigma) \otimes_R S^G_1$ where $\sigma = V \otimes_{H_{M,R}} X_M$. We compare the images by $- \otimes_{H_{M}} X$ of the $H_R$-modules $e(V) \otimes_R (S^G_1)^H$ and $e(V) \otimes_R (S^G_1)^H$ with the smooth $R$-representations $e(\sigma) \otimes Ind^G_1$ and $e(\sigma) \otimes S^G_1$ of $G$.

As $- \otimes_{H_{M}} X$ is left adjoint of $(-)^H$, the $H_R$-homomorphism $v \otimes f \mapsto v \otimes 1_{ut} \otimes f : e(V) \otimes_R (Ind^G_1)^H \to e(\sigma) \otimes_R Ind^G_1$ gives by adjunction an $R[G]$-homomorphism

$$v \otimes f \mapsto 1_{ut} \otimes : e(V) \otimes_R (Ind^G_1)^H \otimes_{H_{M}} X \xrightarrow{\Phi_Q^G} e(\sigma) \otimes_R Ind^G_1.$$

When $Q = G$ we have $\Phi_Q^G = \Phi$. By Remark 4.10, $\Phi_Q^G$ is surjective. Proposition 5.7 applies with $M_Q$ instead of $G$ and gives the $R[M_Q]$-isomorphism

$$v \otimes 1_{ut} \otimes : e(V) \otimes_{H_{M}} X \xrightarrow{\Phi_Q^G} e(\sigma) \otimes_R Ind^G_1.$$

**Proposition 5.9.** The $R[G]$-homomorphism $\Phi_Q^G$ is an isomorphism if $\Phi^Q$ is an isomorphism, in particular if $V$ has no $q_s + 1$-torsion for any $s \in S^G_{M_Q \cap M_Q}$.

**Proof.** The proposition follows from another construction of $\Phi_Q^G$ that we now describe. Proposition 4.5 gives the $H_R$-module isomorphism

$$v \otimes f_{\text{ut}} \mapsto v \otimes 1_H : e(V) \otimes_R (Ind^G_1)^H \to Ind^H_{H_R}(e_{H_R}(V)) = e_{H_R}(V) \otimes_{H_{M_Q \cap M_Q}} H.$$

We have the $R[G]$-isomorphism [OV17, Corollary 4.7]

$$v \otimes 1_H \otimes : f_{\text{ut}} \mapsto f_{\text{ut}} \otimes : e_{H_R}(V) \otimes_{H_{M_Q \cap M_Q}} H \xrightarrow{\Phi_Q^G} e(\sigma) \otimes_R Ind^G_1.$$

From $\Phi^Q$ and these three homomorphisms, there exists a unique $R[G]$-homomorphism

$$(e(V) \otimes_R (Ind^G_1)^H) \otimes_{H_{M}} X \to e(\sigma) \otimes_R Ind^G_1.$$

sending $v \otimes f_{\text{ut}} \otimes 1_H$ to $v \otimes 1_{ut} \otimes \otimes f_{\text{ut}}$. We deduce: this homomorphism is equal to $\Phi_Q^G$. $V \otimes 1_{ut} \otimes 1_H$ generates $(e(V) \otimes_R (Ind^G_1)^H) \otimes_{H_{M}} X$, if $\Phi^Q$ is an isomorphism, then $\Phi_Q^G$ is an isomorphism. By Proposition 5.7, if $V$ has no $q_s + 1$-torsion for any $s \in S^G_{M_Q \cap M_Q}$, then $\Phi_Q^G$ and $\Phi^G_Q$ are isomorphisms.

We recall that the $H_{M,R}$-module $V$ is extensible to $H$.

**Proposition 5.10.** The $R[G]$-homomorphism $\Phi_Q^G$ induces an $R[G]$-homomorphism

$$(e(V) \otimes_R (S^G_1)^H) \otimes_{H_{M}} X \to e(\sigma) \otimes_R S^G_1.$$

It is an isomorphism if $\Phi_Q^G$ is an $R[G]$-isomorphism for all parabolic subgroups $Q'$ of $G$ containing $Q$, in particular if $V$ has no $q_s + 1$-torsion for any $s \in S^G_{M_Q}$. 
Proof. The proof is straightforward, with the arguments already developed for Proposition 4.5 and Theorem 4.9. The representations \( e(\sigma) \otimes_R \text{St}_Q^G \) and \( (\text{St}_Q^G)^{\text{d}j} \otimes_{\mathcal{H}_n} \mathbb{X} \) of \( G \) are the cokernels of the natural \( R[G]^v \)-homomorphisms

\[
\oplus_{Q \subseteq \mathcal{Q}} e(\sigma) \otimes_R \text{Ind}_Q^G 1 \xrightarrow{\oplus \otimes \phi} e(\sigma) \otimes_R \text{Ind}_Q^G 1,
\]

\[
\oplus_{Q \subseteq \mathcal{Q}} (e(V) \otimes_R (\text{Ind}_Q^G)\text{d}j) \otimes_{\mathcal{H}_n} \mathbb{X} \xrightarrow{\oplus \otimes \phi} (e(V) \otimes_R (\text{Ind}_Q^G)\text{d}j) \otimes_{\mathcal{H}_n} \mathbb{X}.
\]

These \( R[G]^v \)-homomorphisms make a commutative diagram with the \( R[G]^v \)-homomorphisms \( \oplus_{Q \subseteq \mathcal{Q}} \phi^G_Q \) and \( \Phi^Q_G \) going from the lower to the upper line. Indeed, let \( v \otimes f_Q \otimes \mathbf{1} \subseteq (e(V) \otimes_R (\text{Ind}_Q^G)\text{d}j) \otimes_{\mathcal{H}_n} \mathbb{X} \). On the one hand, it goes to \( v \otimes f_Q \otimes \mathbf{1} \subseteq (e(V) \otimes_R (\text{Ind}_Q^G)\text{d}j) \otimes_{\mathcal{H}_n} \mathbb{X} \) by the horizontal map, and then to \( v \otimes \mathbf{1} \otimes f_Q \otimes \mathbf{1} \) by the vertical map. On the other hand, it goes to \( v \otimes \mathbf{1} \otimes f_Q \otimes \mathbf{1} \) by the vertical map, and then to \( v \otimes \mathbf{1} \otimes f_Q \otimes \mathbf{1} \) by the horizontal map. One deduces that \( \Phi^Q_G \) induces an \( R[G]^v \)-homomorphism \( (e(V) \otimes_R (\text{St}_Q^G)\text{d}j) \otimes_{\mathcal{H}_n} \mathbb{X} \rightarrow e(\sigma) \otimes_R \text{St}_Q^G \), which is an isomorphism if \( \Phi^Q_G \) is an \( R[G]^v \)-isomorphism for all \( Q \subseteq \mathcal{Q} \).

5.3. General. We consider now the general case: let \( P = MN \subseteq \mathcal{Q} \) be two standard parabolic subgroups of \( G \) and let \( V \) be a non-zero right \( \mathcal{H}_{M,R} \)-module with \( Q \subseteq P(V) \). We recall \( I_H(P, V, Q) = \text{Ind}_{M \setminus M(V)}^H (e(V) \otimes_R (\text{St}_Q^P(V))\text{d}M(V)) \) and \( \sigma = V \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M \) (Proposition 5.4). There is a natural \( R[G]^v \)-homomorphism

\[
I_H(P, V, Q) \otimes_{\mathcal{H}_n} \mathbb{X} \xrightarrow{\Phi^G_P} \text{Ind}_{P(V)}^G (e(M(V)) \otimes_R \text{St}_Q^P(V))
\]

obtained by composition of the \( R[G]^v \)-isomorphism [OV17, Corollary 4.7] (proof of Proposition 5.9):

\[
I_H(P, V, Q) \otimes_{\mathcal{H}_n} \mathbb{X} \xrightarrow{\text{Ind}_P^G} \text{Ind}_M^G (e(M(V)) \otimes_R \text{St}_Q^P(V)),
\]

with the \( R[G]^v \)-homomorphism

\[
\text{Ind}_P^G ((e(V) \otimes_R (\text{St}_Q^P(V))\text{d}M(V)) \otimes_{\mathcal{H}_{M,V},n} \mathbb{X}_M(V)) \rightarrow \text{Ind}_{P(V)}^G (e(M(V)) \otimes_R \text{St}_Q^P(V)),
\]

image by the parabolic induction \( \text{Ind}_P^G \) of the homomorphism

\[
(e(V) \otimes_R (\text{St}_Q^P(V))\text{d}M(V)) \otimes_{\mathcal{H}_{M,V},n} \mathbb{X}_M(V) \rightarrow e(M(V)) \otimes_R \text{St}_Q^P(V)
\]

induced by the \( R[M(V)]^v \)-homomorphism \( \Phi^P(V) = \Phi^M(V) \) of Proposition 5.10 applied to \( M(V) \) instead of \( G \).

This homomorphism \( \Phi^P(V) \) is an isomorphism if \( \Phi^P(V) \) is an isomorphism, in particular if \( V \) has no \( q_s + 1 \)-torsion for any \( s \in S_{M_2}^\text{aff} \) where \( \Delta_{M_2} = M(V) \setminus M \) (Proposition 5.10). We get the main theorem of this section.

Theorem 5.11. Let \( (P = MN, V, Q) \) be an \( \mathcal{H}_{M,R} \)-triple and \( \sigma = V \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}(M \setminus M)] \). Then, \( (P, V, Q, \sigma) \) is an \( R[G]^v \)-triple. The \( R[G]^v \)-homomorphism

\[
I_H(P, V, Q) \otimes_{\mathcal{H}_n} R[\mathcal{U}(G)] \xrightarrow{\Phi^P(V)} \text{Ind}_{P(V)}^G (e(M(V)) \otimes_R \text{St}_Q^P(V))
\]

is an isomorphism if \( \Phi^P(V) \) is an isomorphism. In particular \( \Phi^P(V) \) is an isomorphism if \( V \) has no \( q_s + 1 \)-torsion for any \( s \in S_{M_2}^\text{aff} \).
Recalling \( I_G(P, \sigma, Q) = \text{Ind}_P^G(\varepsilon(\sigma) \otimes_R S_{t, Q}^{P(\sigma)}) \) when \( \sigma \neq 0 \), we deduce the following.

**Corollary 5.12.** We have the following:

- \( I_H(P, V, Q) \otimes_{\mathcal{H}} \mathbb{R}[\mathcal{U}\backslash G] \simeq I_G(P, \sigma, Q) \), if \( \sigma \neq 0 \), \( P(V) = P(\sigma) \) and \( V \) has no \( q_i + 1 \)-torsion for any \( s \in S_{t, G}^P \).
- \( I_H(P, V, Q) \otimes_{\mathcal{H}} \mathbb{R}[\mathcal{U}\backslash G] = I_G(P, \sigma, Q) = 0 \), if \( \sigma = 0 \).

Recalling \( P(V) = P(\sigma) \) if \( \sigma \neq 0 \), \( R \) is a field of characteristic \( p \) and \( V \) simple supersingular (Proposition 5.4 (4)), we deduce the following.

**Corollary 5.13.** \( I_H(P, V, Q) \otimes_{\mathcal{H}} \mathbb{R}[\mathcal{U}\backslash G] \simeq I_G(P, \sigma, Q) \) if \( R \) is a field of characteristic \( p \) and \( V \) simple supersingular.

6. **Vanishing of the smooth dual**

Let \( V \) be an \( \mathbb{R}[G] \)-module. The dual \( \text{Hom}_R(V, R) \) of \( V \) is an \( \mathbb{R}[G] \)-module for the contragredient action: \( gL(gv) = L(v) \) if \( g \in G, L \in \text{Hom}_R(V, R) \) is a linear form and \( v \in V \). When \( V \in \text{Mod}_R^\infty(G) \) is a smooth \( R \)-representation of \( G \), the dual of \( V \) is not necessarily smooth. A linear form \( L \) is smooth if there exists an open subgroup \( H \subset G \) such that \( L(hv) = L(v) \) for all \( h \in H, v \in V \); the space \( \text{Hom}_R(V, R) \) of smooth linear forms is a smooth \( R \)-representation of \( G \), called the **smooth dual** (or smooth contragredient) of \( V \). The smooth dual of \( V \) is contained in the dual of \( V \).

**Example 6.1.** When \( R \) is a field and the dimension of \( V \) over \( R \) is finite, the dual of \( V \) is equal to the smooth dual of \( V \) because the kernel of the action of \( G \) on \( V \) is an open normal subgroup \( H \subset G \); the action of \( G \) on the dual \( \text{Hom}_R(V, R) \) is trivial on \( H \).

We assume in this section that \( R \) is a field of characteristic \( p \). Let \( P = MN \) be a parabolic subgroup of \( G \) and \( V \in \text{Mod}_R^\infty(M) \). Generalizing the proof given in [Vig07, 8.1] when \( G = GL(2, F) \) and the dimension of \( V \) is 1, we show the following.

**Proposition 6.2.** If \( P \neq G \), the smooth dual of \( \text{Ind}_P^G(V) \) is 0.

**Proof.** Let \( L \) be a smooth linear form on \( \text{Ind}_P^G(V) \) and let \( K \) be an open pro-\( p \)-subgroup of \( G \) which fixes \( L \). Let \( J \) be an arbitrary open subgroup of \( K, g \in G \) and \( f \in (\text{Ind}_P^G(V))^J \) with support \( PgJ \). We want to show that \( L(f) = 0 \). Let \( J' \) be any open normal subgroup of \( J \) and let \( \varphi \) denote the function in \( (\text{Ind}_P^G(V))^{J'} \) with support \( PgJ' \) and value \( \varphi(g) = f(g) \) at \( g \). For \( j \in J \) we have \( L(j\varphi) = L(\varphi) \), and the support of \( j\varphi(x) = \varphi(xj) \) is \( PgJ'^{-1} \). The function \( f \) is the sum of translates \( j\varphi \), where \( j \) ranges through the left cosets of the image \( X \) of \( g^{-1}Pg \cap J \) in \( J/J' \), so that \( L(f) = rL(\varphi) \) where \( r \) is the order of \( X \) in \( J/J' \). We can certainly find \( J' \) such that \( r \neq 1 \), and then \( r \) is a positive power of \( p \). As the characteristic of \( R \) is \( p \), we have \( L(f) = 0 \). \( \square \)

The module \( \mathbb{R}[\mathcal{U}\backslash G] \) is contained in the module \( R^\mathcal{U}\backslash G \) of functions \( f : \mathcal{U}\backslash G \to R \). The actions of \( \mathcal{H} \) and of \( G \) on \( \mathbb{R}[\mathcal{U}\backslash G] \) extend to \( R^\mathcal{U}\backslash G \) by the same formulas. The pairing

\[
(f, \varphi) \mapsto \langle f, \varphi \rangle = \sum_{(g) \in \mathcal{U}\backslash G} f(g)\varphi(g) : R^\mathcal{U}\backslash G \times R[\mathcal{U}\backslash G] \to R
\]
identifies $R^{H \setminus G}$ with the dual of $R[U \setminus G]$. Let $h \in H$ and $\tilde{h} \in H$, $\tilde{h}(g) = h(g^{-1})$ for $g \in G$. We have 

$$\langle f, h\varphi \rangle = \langle \tilde{h}f, \varphi \rangle.$$ 

**Proposition 6.3.** When $R$ is an algebraically closed field of characteristic $p$, $G$ is not compact modulo the center and $V$ is a simple supersingular right $H_R$-module, the smooth dual of $V \otimes_{H_R} R[U \setminus G]$ is 0.

**Proof.** Let $H^\text{aff}_R$ be the subalgebra of $H_R$ of basis $(T_w)_{w \in W'(1)}$ where $W'(1)$ is the inverse image of $W'$ in $W(1)$. The dual of $V \otimes H^\text{aff}_R R[U \setminus G]$ is contained in the dual of $V \otimes H^\text{aff}_R R[U \setminus G]$; the $H^\text{aff}_R$-module $V|_{H^\text{aff}_R}$ is a finite sum of supersingular characters [Vig15a]. Let $\chi : H^\text{aff}_R \to R$ be a supersingular character. The dual of $\chi \otimes H^\text{aff}_R R[U \setminus G]$ is contained in the dual of $R[U \setminus G]$ isomorphic to $R^{H \setminus G}$. It is the space of $f \in R^{H \setminus G}$ with $\tilde{h}f = \chi(h)f$ for all $h \in H^\text{aff}_R$. The smooth dual of $\chi \otimes H^\text{aff}_R R[U \setminus G]$ is 0 if the dual of $\chi \otimes H^\text{aff}_R R[U \setminus G]$ has no non-zero element fixed by $U$. Let us take $f \in R^{H \setminus G}/U$ with $\tilde{h}f = \chi(h)f$ for all $h \in H^\text{aff}_R$. We shall prove that $f = 0$. We have $\tilde{T}_w = T_{w^{-1}}$ for $w \in W(1)$.

Let $\prec$ denote the Bruhat order of $W(1)$ associated to $S^\text{aff}$ [Vig16]. The elements $(T_t)_{t \in Z_k}$ and $(T_s) s \in S^\text{aff}$ where $\tilde{s}$ is an admissible lift of $s$ in $W^\text{aff}(1)$, generate the algebra $H^\text{aff}_R$ and 

$$T_s T_w = T_{tw}, \quad T_s T_w \begin{cases} T_{\tilde{w}}, & \tilde{w} > w, \\ c_s T_{sw}, & \tilde{w} < w, \end{cases}$$

with $c_s = -|Z'_k| \sum_{t \in Z'_k} |s_t| T_t$ because the characteristic of $R$ is $p$ [Vig16, Proposition 4.4]. Expressing $f = \sum_{w \in W(1)} a_w T_w$, $a_w \in R$, as an infinite sum, we have

$$T_t f = \sum_{w \in W(1)} a_{tw} T_w, \quad T_s f = \sum_{w \in W(1), \tilde{w} < w} (a_{\tilde{w}} + a_w c_s) T_{sw}.$$ 

A character $\chi$ of $H^\text{aff}_R$ is associated to a character $\chi_k : Z_k \to R^*$ and a subset $J$ of 

$$S^\text{aff}_{\chi_k} = \{ s \in S^\text{aff} \mid (\chi_k)|_{Z'_k, s} \text{ trivial} \}$$ 

[Vig15a, Definition 2.7]. We have

$$\begin{cases} \chi(T_t) = \chi(t), & t \in Z_k, \\ \chi(T_s) = \begin{cases} 0, & s \in S^\text{aff} \setminus J, \\ -1, & s \in J. \end{cases}, & (\chi_k)(c_s) = \begin{cases} 0, & s \in S^\text{aff} \setminus S_{\chi_k}, \\ -1, & s \in S_{\chi_k}. \end{cases} \end{cases}$$

Therefore $\chi_k(t)f = \tilde{T}_tf = T_{-t}f$ hence $\chi_k(t)a_w = a_{tw}$. We have $\chi(T_t)f = \tilde{T}_tf = T_{\tilde{s}}T_{\tilde{s}}f = \chi_k((\tilde{s})^2)T_s f$; as $(\tilde{s})^2 \in Z'_k$ [Vig16, three lines before Proposition 4.4] and $J \subseteq S_{\chi_k}$, we obtain

$$T_s f = \begin{cases} 0, & s \in S^\text{aff} \setminus J, \\ -f, & s \in J. \end{cases}$$

(6.2)
Introducing $\chi_k(t)a_w = a_{tw}$ in the formula for $T_{sf}$, we get
\[
\sum_{w \in W(1), \tilde{s}w < w} a_w \epsilon_tw_w = -|Z_{k,s}^r|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z_{k,s}^r} a_{tw} T_w
\]
\[
= -|Z_{k,s}^r|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z_{k,s}^r} a_{t^{-1}w} T_w
\]
\[
= -|Z_{k,s}^r|^{-1} \sum_{t \in Z_{k,s}^r} \chi_k(t^{-1}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w
\]
\[
= \chi_k(c_3) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w.
\]

Thus $T_{sf} = \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w \chi_k(c_3))T_w = \sum_{w \in W(1), \tilde{s}w < w} a_w T_w$.

From the last equality and (6.2) for $T_{sf}$, we get:
\[
(6.3)
\]
\[
a_{\tilde{s}w} = \begin{cases} 
0, & s \in J \cup (S_{\text{aff}} \setminus S_{\chi_k}), \tilde{s}w < w, \\
2a_w, & s \in S_{\text{aff}} \setminus \chi_k \setminus J.
\end{cases}
\]

Assume that $a_w \neq 0$. By the first condition, we know that $w > \tilde{s}w$ for $s \in J \cup (S_{\text{aff}} \setminus S_{\chi_k})$. The character $\chi$ is supersingular if for each irreducible component $X$ of $S_{\text{aff}}$, the intersection $X \cap J$ is not empty and different from $X$ [Vig15a, Definition 2.7, Theorem 6.18]. This implies that the group generated by the $s \in S_{\text{aff}} \setminus \chi_k$ is finite. If $\chi$ is supersingular, by the second condition we can suppose $w > \tilde{s}w$ for any $s \in S_{\text{aff}}$. But there is no such element if $S_{\text{aff}}$ is not empty.

**Theorem 6.4.** Let $\pi$ be an irreducible admissible $R$-representation of $G$ with a non-zero smooth dual where $R$ is an algebraically closed field of characteristic $p$. Then $\pi$ is finite dimensional.

**Proof.** Let $(P, \sigma, Q)$ be an $R[G]$-triple with $\sigma$ supercuspidal such that $\pi \simeq I_G(P, \sigma, Q)$. The representation $I_G(P, \sigma, Q)$ is a quotient of $\text{Ind}_Q^G e_Q(\sigma)$ hence the smooth dual of $\text{Ind}_Q^G e_Q(\sigma)$ is not zero. From Proposition 6.2, $Q = G$. We have $I_G(P, \sigma, G) = e(\sigma)$. The smooth dual of $\sigma$ contains the smooth linear dual of $e(\sigma)$ hence is not zero. As $\sigma$ is supercuspidal, the $H_M$-module $\sigma^{\text{aff}}$ contains a simple supersingular submodule $V$ [Vig15a, Proposition 7.10, Corollary 7.11]. The functor $- \otimes_{H_M, R} R[U_M \setminus M]$ being the right adjoint of $(-)^{\text{aff}}$, the irreducible representation $\sigma$ is a quotient of $V \otimes_{H_M, R} R[U_M \setminus M]$, hence the smooth dual of $V \otimes_{H_M, R} R[U_M \setminus M]$ is not zero. By Proposition 6.3, $M = Z$. Hence $\sigma$ is finite dimensional and the same is true for $e(\sigma) = I_G(B, \sigma, G) \simeq \pi$.

**Remark 6.5.** When the characteristic of $F$ is 0, Theorem 6.4 was proved by Kohlhaase for a field $R$ of characteristic $p$. He gives two proofs [Koh, Proposition 3.9, Remark 3.10], but none of them extends to $F$ of characteristic $p$. Our proof is valid without restriction on the characteristic of $F$ and does not use the results of Kohlhaase. Our assumption that $R$ is an algebraically closed field of characteristic $p$ comes from the classification theorem in [AHHV17].
References


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