HAL
open science

On pro-p-Iwahori invariants of R-representations of reductive p-adic groups

N. Abe, G. Henniart, M.-F. Vignéras

▶ **To cite this version:**

N. Abe, G. Henniart, M.-F. Vignéras. On pro-p-Iwahori invariants of R-representations of reductive p-adic groups. American Mathematical Society, 2018, 22 (5), pp.119 - 159. 10.1090/ert/518 . hal-01919780

HAL Id: hal-01919780

https://hal.science/hal-01919780v1

Submitted on 12 Nov 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON PRO-p-IWAHORI INVARIANTS OF R-REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

N. ABE, G. HENNIART, AND M.-F. VIGNÉRAS

Abstract. Let F be a locally compact field with residue characteristic p, and let $\mathbf{G}$ be a connected reductive F-group. Let $\mathcal{U}$ be a pro-p Iwahori subgroup of $G = \mathbf{G}(F)$. Fix a commutative ring R. If π is a smooth $R[G]$-representation, the space of invariants $\pi^{\mathcal{U}}$ is a right module over the Hecke algebra $\mathcal{H}$ of $\mathcal{U}$ in G.

Let P be a parabolic subgroup of G with a Levi decomposition $P = MN$ adapted to $\mathcal{U}$. We complement a previous investigation of Ollivier-Vignéras on the relation between taking $\mathcal{U}$-invariants and various functor like Ind_P^G and right and left adjoints. More precisely the authors' previous work with Herzig introduced representations $I_G(P, \sigma, Q)$ where σ is a smooth representation of M extending, trivially on N, to a larger parabolic subgroup $P(\sigma)$, and Q is a parabolic subgroup between P and $P(\sigma)$. Here we relate $I_G(P, \sigma, Q)^{\mathcal{U}}$ to an analogously defined $\mathcal{H}$-module $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$, where $\mathcal{U}_M = \mathcal{U} \cap M$ and $\sigma^{\mathcal{U}_M}$ is seen as a module over the Hecke algebra $\mathcal{H}_M$ of $\mathcal{U}_M$ in M. In the reverse direction, if $\mathcal{V}$ is a right $\mathcal{H}_M$-module, we relate $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes \text{c-Ind}_{\mathcal{U}}^G \mathbf{1}$ to $I_G(P, \mathcal{V} \otimes_{\mathcal{H}_M} \text{c-Ind}_{\mathcal{U}_M}^M \mathbf{1}, Q)$. As an application we prove that if R is an algebraically closed field of characteristic p, and π is an irreducible admissible representation of G, then the contragredient of π is 0 unless π has finite dimension.

Contents

1. Introduction 119
2. Notation, useful facts, and preliminaries 122
3. Pro-p Iwahori invariants of $I_G(P, \sigma, Q)$ 124
4. Hecke module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ 134
5. Universal representation $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U}\backslash G]$ 150
6. Vanishing of the smooth dual 156
References 159

1. Introduction

1.1. The present paper is a companion to [AHV] and is similarly inspired by the classification results of [AHHV17]; however it can be read independently. We recall the setting. We have a non-archimedean locally compact field F of residue characteristic p and a connected reductive F-group $\mathbf{G}$. We fix a commutative ring R and study the smooth R-representations of $G = \mathbf{G}(F)$.

2010 *Mathematics Subject Classification.* Primary 20C08; Secondary 11F70.
Key words and phrases. Parabolic induction, pro-p Iwahori Hecke algebra.
The first-named author was supported by JSPS KAKENHI Grant Number 26707001.

In [AHHV17] the irreducible admissible R-representations of G are classified in terms of supersingular ones when R is an algebraically closed field of characteristic p. That classification is expressed in terms of representations $I_G(P,\sigma,Q)$, which make sense for any R. In that notation, P is a parabolic subgroup of G with a Levi decomposition $P = MN$ and σ a smooth R-representation of the Levi subgroup M; there is a maximal parabolic subgroup $P(\sigma)$ of G containing P to which σ inflated to P extends to a representation $e_{P(\sigma)}(\sigma)$, and Q is a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$. Then

$$I_G(P,\sigma,Q) = \mathrm{Ind}_{P(\sigma)}^G(e_{P(\sigma)}(\sigma) \otimes \mathrm{St}_Q^{P(\sigma)}),$$

where Ind stands for parabolic induction and $\mathrm{St}_Q^{P(\sigma)} = \mathrm{Ind}_Q^{P(\sigma)} R / \sum \mathrm{Ind}_{Q'}^{P(\sigma)} R$, the sum being over parabolic subgroups Q' of G with $Q \subsetneq Q' \subset P(\sigma)$. Alternatively, $I_G(P,\sigma,Q)$ is the quotient of $Ind_Q^G(e_Q(\sigma))$ by $\sum \mathrm{Ind}_{Q'}^G e_{Q'}(\sigma)$ with Q' as above, where $e_Q(\sigma)$ is the restriction of $e_{P(\sigma)}(\sigma)$ to Q, similarly for Q'.

In [AHV] we mainly studied what happens to $I_G(P,\sigma,Q)$ when we apply to it, for a parabolic subgroup P_1 of G, the left adjoint of $\mathrm{Ind}_{P_1}^G$, or its right adjoint. Here we tackle a different question. We fix a pro-p Iwahori subgroup $\mathcal{U}$ of G in good position with respect to P, so that in particular $\mathcal{U}_M = \mathcal{U} \cap M$ is a pro-p Iwahori subgroup of M. One of our main goals is to identify the R-module $I_G(P,\sigma,Q)^{\mathcal{U}}$ of $\mathcal{U}$-invariants, as a right module over the Hecke algebra $\mathcal{H} = \mathcal{H}_G$ of $\mathcal{U}$ in G - the convolution algebra on the double coset space $\mathcal{U}\backslash G/\mathcal{U}$ - in terms of the module $\sigma^{\mathcal{U}_M}$ over the Hecke algebra $\mathcal{H}_M$ of $\mathcal{U}_M$ in M. That goal is achieved in section 4, Theorem 4.17.

1.2. The initial work has been done in [OV17, §4] where $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ is identified. Precisely, writing M^+ for the monoid of elements $m \in M$ with $m(\mathcal{U} \cap N)m^{-1} \subset \mathcal{U} \cap N$, the subalgebra $\mathcal{H}_{M^+}$ of $\mathcal{H}_M$ with support in M^+, has a natural algebra embedding θ into $\mathcal{H}$ and [OV17, Proposition 4.4] identifies $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ with $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma^{\mathcal{U}_M}) = \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+}} \mathcal{H}$. So in a sense, this paper is a sequel to [OV17] although some of our results here are used in [OV17, §5].

As $I_G(P,\sigma,Q)$ is only a subquotient of $\mathrm{Ind}_P^G \sigma$ and taking $\mathcal{U}$-invariants is only left exact, it is not straightforward to describe $I_G(P,\sigma,Q)^{\mathcal{U}}$ from the previous result. However, that takes care of the parabolic induction step, so in a first approach we may assume $P(\sigma) = G$ so that $I_G(P,\sigma,Q) = e_G(\sigma) \otimes \mathrm{St}_Q^G$. The crucial case is when moreover σ is e-minimal, that is, not an extension $e_M(\tau)$ of a smooth R-representation τ of a proper Levi subgroup of M. That case is treated first and the general case in section 4 only.

1.3. To explain our results, we need more notation. We choose a maximal F-split torus T in G and a minimal parabolic subgroup $B = ZU$ with Levi component Z the G-centralizer of T. We assume that $P = MN$ contains B and M contains Z, and that $\mathcal{U}$ corresponds to an alcove in the apartment associated to T in the adjoint building of G. It turns out that when σ is e-minimal and $P(\sigma) = G$, the set Δ_M of simple roots of T in $\mathrm{Lie}(M \cap U)$ is orthogonal to its complement in the set Δ of simple roots of T in $\mathrm{Lie}\, U$. We assume until the end of this section that Δ_M and $\Delta_2 = \Delta \setminus \Delta_M$ are orthogonal. If M_2 is the Levi subgroup - containing Z - corresponding to Δ_2, both M and M_2 are normal in G, $M \cap M_2 = Z$ and $G = MM_2$. Moreover the normal subgroup M_2' of G generated by N is included in M_2 and $G = MM_2'$.

We say that a right $\mathcal{H}_M$-module $\mathcal{V}$ is extensible to $\mathcal{H}$ if T_z^M acts trivially on $\mathcal{V}$ for $z \in Z \cap M_2'$ (section 3.3). In this case, we show that there is a natural structure of right $\mathcal{H}$-module $e_{\mathcal{H}}(\mathcal{V})$ on $\mathcal{V}$ such that $T_g \in \mathcal{H}$ corresponding to $\mathcal{U}g\mathcal{U}$ for $g \in M_2'$ acts as in the trivial character of G (section 3.4). We call $e_{\mathcal{H}}(\mathcal{V})$ the extension of $\mathcal{V}$ to $\mathcal{H}$ though $\mathcal{H}_M$ is not a subalgebra of $\mathcal{H}$. That notion is already present in [Abe] in the case where R has characteristic p. Here we extend the construction to any R and prove some more properties. In particular we produce an $\mathcal{H}$-equivariant embedding $e_{\mathcal{H}}(\mathcal{V})$ into $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \mathcal{V}$ (Lemma 3.10). If Q is a parabolic subgroup of G containing P, we go further and put on $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G R)^{\mathcal{U}}$ and $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ structures of $\mathcal{H}$-modules (Proposition 3.15 and Corollary 3.17) - note that $\mathcal{H}$ is not a group algebra and there is no obvious notion of tensor product of $\mathcal{H}$-modules.

If σ is an R-representation of M extensible to G, then its extension $e_G(\sigma)$ is simply obtained by letting M_2' act trivially on the space of σ; moreover it is clear that $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$, and one shows easily that $e_G(\sigma)^{\mathcal{U}} = e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$ as an $\mathcal{H}$-module (section 3.5). Moreover, the natural inclusion of $e_G(\sigma)$ into $\mathrm{Ind}_P^G \sigma$ induces on taking pro-p Iwahori invariants an embedding $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \to (\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ which, via the isomorphism of [OV17], yields exactly the above embedding of $\mathcal{H}$-modules of $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$ into $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma^{\mathcal{U}_M})$.

Then we show the $\mathcal{H}$-modules $(e_G(\sigma) \otimes_R \mathrm{Ind}_Q^G R)^{\mathcal{U}}$ and $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\mathrm{Ind}_Q^G R)^{\mathcal{U}}$ are equal, and similarly $(e_G(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$and $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ are equal (Theorem 4.9).

1.4. We turn back to the general case where we do not assume that Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. Nevertheless, given a right $\mathcal{H}_M$-module $\mathcal{V}$, there exists a largest Levi subgroup $M(\mathcal{V})$ of G - containing Z - corresponding to $\Delta_M \cup \Delta_1$ where Δ_1 is a subset of $\Delta \setminus \Delta_M$ orthogonal to Δ_M, such that $\mathcal{V}$ extends to a right $\mathcal{H}_{M(\mathcal{V})}$-module $e_{M(\mathcal{V})}(\mathcal{V})$ with the notation of section 1.3. For any parabolic subgroup Q between P and $P(\mathcal{V}) = M(\mathcal{V})U$ we put (Definition 4.12)

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) = \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(e_{M(\mathcal{V})}(\mathcal{V}) \otimes_R (\mathrm{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}).$$

We refer to Theorem 4.17 for the description of the right $\mathcal{H}$-module $I_G(P, \sigma, Q)^{\mathcal{U}}$ for any smooth R-representation σ of $\mathcal{U}$. As a special case, it says that when σ is e-minimal then $P(\sigma) \supset P(\sigma^{\mathcal{U}_M})$ and if moreover $P(\sigma) = P(\sigma^{\mathcal{U}_M})$, then $I_G(P, \sigma, Q)^{\mathcal{U}}$ is isomorphic to $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$.

Remark 1.1. In [Abe] are attached similar $\mathcal{H}$-modules to $(P, \mathcal{V}, Q)$; here we write them as $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ because their definition uses, instead of $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}$ a different kind of induction, which we call coinduction. In [Abe] those modules are used to give, when R is an algebraically closed field of characteristic p, a classification of simple $\mathcal{H}$-modules in terms of supersingular modules - that classification is similar to the classification of irreducible admissible R-representations of G in [AHHV17]. Using the comparison between induced and coinduced modules established in [Vig15b, 4.3] for any R, our Corollary 4.24 expresses $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ as a module $I_{\mathcal{H}}(P_1, \mathcal{V}_1, Q_1)$; consequently we show in section 4.5 that the classification of [Abe] can also be expressed in terms of modules $I_{\mathcal{H}}(P, \mathcal{V}, Q)$.

1.5. In a reverse direction one can associate to a right $\mathcal{H}$-module $\mathcal{V}$ a smooth R-representation $\mathcal{V} \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$ of G (seeing $\mathcal{H}$ as the endomorphism ring of the $R[G]$-module $R[\mathcal{U} \backslash G]$).

If $\mathcal{V}$ is a right $\mathcal{H}_M$-module, we construct, again using [OV17], a natural $R[G]$-map

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}} R[\mathcal{U}\backslash G]\to \mathrm{Ind}_{P(\mathcal{V})}^{G}(e_{M(\mathcal{V})}(\mathcal{V}\otimes_{\mathcal{H}_M} R[\mathcal{U}_M\backslash M])\otimes_R \mathrm{St}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})}),$$

with the notation of section 1.4. We show in section 5 that it is an isomorphism under a mild assumption on the $\mathbb{Z}$-torsion in $\mathcal{V}$; in particular it is an isomorphism if $p=0$ in R.

1.6. In the final section 6, we turn back to the case where R is an algebraically closed field of characteristic p. We prove that the smooth dual of an irreducible admissible R-representation V of G is 0 unless V is finite dimensional - that result is new if F has positive characteristic, a case where the proof of Kohlhaase [Koh] for $\mathrm{char}(F)=0$ does not apply. Our proof first reduces to the case where V is supercuspidal (by [AHHV17]) then uses again the $\mathcal{H}$-module $V^{\mathcal{U}}$.

2. Notation, useful facts, and preliminaries

2.1. **The group G and its standard parabolic subgroups $P=MN$.** In all that follows, p is a prime number and F is a local field with finite residue field k of characteristic p. We denote an algebraic group over F by a bold letter, like $\mathbf{H}$, and use the same ordinary letter for the group of F-points, $H=\mathbf{H}(F)$. We fix a connected reductive F-group $\mathbf{G}$. We fix a maximal F-split subtorus $\mathbf{T}$ and write $\mathbf{Z}$ for its $\mathbf{G}$-centralizer; we also fix a minimal parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with Levi component $\mathbf{Z}$, so that $\mathbf{B}=\mathbf{ZU}$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Let $X^*(\mathbf{T})$ be the group of F-rational characters of $\mathbf{T}$ and let Φ be the subset of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{G}$. Then $\mathbf{B}$ determines a subset Φ^+ of positive roots - the roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$- and a subset of simple roots Δ. The $\mathbf{G}$-normalizer $\mathbf{N_G}$ of $\mathbf{T}$ acts on $X^*(\mathbf{T})$ and through that action, $\mathbf{N_G}/\mathbf{Z}$ identifies with the Weyl group of the root system Φ. Set $\mathcal{N}:=\mathbf{N_G}(F)$ and note that $\mathbf{N_G}/\mathbf{Z}\simeq\mathcal{N}/Z$; we write $\mathbb{W}$ for $\mathcal{N}/Z$.

A standard parabolic subgroup of $\mathbf{G}$ is a parabolic F-subgroup containing $\mathbf{B}$. Such a parabolic subgroup $\mathbf{P}$ has a unique Levi subgroup $\mathbf{M}$ containing $\mathbf{Z}$, so that $\mathbf{P}=\mathbf{MN}$ where $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$ - we also call $\mathbf{M}$ standard. By a common abuse of language to describe the preceding situation, we simply say "let $P=MN$ be a standard parabolic subgroup of G"; we sometimes write N_P for N and M_P for M. The parabolic subgroup of G opposite to P will be written $\overline{P}$ and its unipotent radical $\overline{N}$, so that $\overline{P}=M\overline{N}$, but beware that $\overline{P}$ is not standard! We write $\mathbb{W}_M$ for the Weyl group $(M\cap\mathcal{N})/Z$.

If $\mathbf{P}=\mathbf{MN}$ is a standard parabolic subgroup of G, then $\mathbf{M}\cap\mathbf{B}$ is a minimal parabolic subgroup of $\mathbf{M}$. If Φ_M denotes the set of roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{M}$, with respect to $\mathbf{M}\cap\mathbf{B}$ we have $\Phi_M^+=\Phi_M\cap\Phi^+$ and $\Delta_M=\Phi_M\cap\Delta$. We also write Δ_P for Δ_M as P and M determine each other, $P=MU$. Thus we obtain a bijection $P\mapsto\Delta_P$ from standard parabolic subgroups of G to subsets of Δ, with B corresponding to Φ and G to Δ. If I is a subset of Δ, we sometimes denote by $P_I=M_IN_I$ the corresponding standard parabolic subgroup of G. If $I=\{\alpha\}$ is a singleton, we write $P_\alpha=M_\alpha N_\alpha$. We note a few useful properties. If P_1 is another standard parabolic subgroup of G, then $P\subset P_1$ if and only if $\Delta_P\subset\Delta_{P_1}$; we have $\Delta_{P\cap P_1}=\Delta_P\cap\Delta_{P_1}$ and the parabolic subgroup corresponding to $\Delta_P\cup\Delta_{P_1}$ is the subgroup $\langle P,P_1\rangle$ of G generated by P and P_1. The standard parabolic subgroup of M associated to $\Delta_M\cap\Delta_{M_1}$ is $M\cap P_1=(M\cap M_1)(M\cap N_1)$ [Car85, Proposition 2.8.9]. It is convenient to write G' for the subgroup of G generated by the unipotent

radicals of the parabolic subgroups; it is also the normal subgroup of G generated by U, and we have $G = ZG'$. For future reference, we now give a useful lemma extracted from [AHHV17].

Lemma 2.1. *The group $Z \cap G'$ is generated by the $Z \cap M'_\alpha$, α running through Δ.*

Proof. Take $I = \emptyset$ in [AHHV17, II.6.Proposition]. □

Let v_F be the normalized valuation of F. For each $\alpha \in X^*(T)$, the homomorphism $x \mapsto v_F(\alpha(x)) : T \to \mathbb{Z}$ extends uniquely to a homomorphism $Z \to \mathbb{Q}$ that we denote in the same way. This defines a homomorphism $Z \xrightarrow{v} X_*(T) \otimes \mathbb{Q}$ such that $\alpha(v(z)) = v_F(\alpha(z))$ for $z \in Z, \alpha \in X^*(T)$.

An interesting situation occurs when $\Delta = I \sqcup J$ is the union of two orthogonal subsets I and J. In that case, $G' = M'_I M'_J$, M'_I and M'_J commute with each other, and their intersection is finite and central in G [AHHV17, II.7 Remark 4].

2.2. $I_G(P, \sigma, Q)$ **and minimality.** We recall from [AHHV17] the construction of $I_G(P, \sigma, Q)$, our main object of study.

Let σ be an R-representation of M and let P_σ be the standard parabolic subgroup with $\Delta_{P_\sigma} = \Delta_\sigma$ where

$$\Delta_\sigma = \{\alpha \in \Delta \setminus \Delta_P \mid Z \cap M'_\alpha \text{ acts trivially on } \sigma\}.$$

We also let $P(\sigma)$ be the standard parabolic subgroup with

$$\Delta_{P(\sigma)} = \Delta_\sigma \cup \Delta_P.$$

This is the largest parabolic subgroup $P(\sigma)$ containing P to which σ extends, here $N \subset P$ acts on σ trivially. Clearly when $P \subset Q \subset P(\sigma)$, σ extends to Q and the extension is denoted by $e_Q(\sigma)$. The restriction of $e_{P(\sigma)}(\sigma)$ to Q is $e_Q(\sigma)$. If there is no risk of ambiguity, we write

$$e(\sigma) = e_{P(\sigma)}(\sigma).$$

Definition 2.2. An $R[G]$**-triple** is a triple (P, σ, Q) made out of a standard parabolic subgroup $P = MN$ of G, a smooth R-representation of M, and a parabolic subgroup Q of G with $P \subset Q \subset P(\sigma)$. To an $R[G]$-triple (P, σ, Q) is associated a smooth R-representation of G:

$$I_G(P, \sigma, Q) = \mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \mathrm{St}_Q^{P(\sigma)}),$$

where $\mathrm{St}_Q^{P(\sigma)}$ is the quotient of $\mathrm{Ind}_Q^{P(\sigma)} \mathbf{1}$, $\mathbf{1}$ denoting the trivial R-representation of Q, by the sum of its subrepresentations $\mathrm{Ind}_{Q'}^{P(\sigma)} \mathbf{1}$, the sum being over the set of parabolic subgroups Q' of G with $Q \subsetneq Q' \subset P(\sigma)$.

Note that $I_G(P, \sigma, Q)$ is naturally isomorphic to the quotient of $\mathrm{Ind}_Q^G(e_Q(\sigma))$ by the sum of its subrepresentations $\mathrm{Ind}_{Q'}^G(e_{Q'}(\sigma))$ for $Q \subsetneq Q' \subset P(\sigma)$ by [AHV, Lemma 2.5].

It might happen that σ itself has the form $e_P(\sigma_1)$ for some standard parabolic subgroup $P_1 = M_1 N_1$ contained in P and some R-representation σ_1 of M_1. In that case, $P(\sigma_1) = P(\sigma)$ and $e(\sigma) = e(\sigma_1)$. We say that σ is e**-minimal** if $\sigma = e_P(\sigma_1)$ implies $P_1 = P, \sigma_1 = \sigma$.

Lemma 2.3 ([AHV, Lemma 2.9])**.** *Let $P = MN$ be a standard parabolic subgroup of G and let σ be an R-representation of M. There exists a unique standard parabolic subgroup $P_{\min,\sigma} = M_{\min,\sigma}N_{\min,\sigma}$ of G and a unique e-minimal representation of $\sigma_{\min}$ of $M_{\min,\sigma}$ with $\sigma = e_P(\sigma_{\min})$. Moreover $P(\sigma) = P(\sigma_{\min})$ and $e(\sigma) = e(\sigma_{\min})$.*

Lemma 2.4. *Let $P = MN$ be a standard parabolic subgroup of G and let σ be an e-minimal R-representation of M. Then Δ_P and $\Delta_{P(\sigma)} \setminus \Delta_P$ are orthogonal.*

That comes from [AHHV17, II.7 Corollary 2]. That corollary of [AHHV17] also shows that when R is a field and σ is supercuspidal, then σ is e-minimal. Lemma 2.4 shows that $\Delta_{P_{\min,\sigma}}$ and $\Delta_{P(\sigma_{\min})} \setminus \Delta_{P_{\min,\sigma}}$ are orthogonal.

Note that when Δ_P and Δ_σ are orthogonal of union $\Delta = \Delta_P \sqcup \Delta_\sigma$, then $G = P(\sigma) = MM'_\sigma$ and $e(\sigma)$ is the R-representation of G simply obtained by extending σ trivially on M'_σ.

Lemma 2.5 ([AHV, Lemma 2.11])**.** *Let (P, σ, Q) be an $R[G]$-triple. Then we have that $(P_{\min,\sigma}, \sigma_{\min}, Q)$ is an $R[G]$-triple and $I_G(P, \sigma, Q) = I_G(P_{\min,\sigma}, \sigma_{\min}, Q)$.*

2.3. **Pro-p Iwahori Hecke algebras.** We fix a special parahoric subgroup $\mathcal{K}$ of G fixing a special vertex x_0 in the apartment $\mathcal{A}$ associated to T in the Bruhat-Tits building of the adjoint group of G. We let $\mathcal{B}$ be the Iwahori subgroup fixing the alcove $\mathcal{C}$ in $\mathcal{A}$ with vertex x_0 contained in the Weyl chamber (of vertex x_0) associated to B. We let $\mathcal{U}$ be the pro-p radical of $\mathcal{B}$ (the pro-p Iwahori subgroup). The pro-p Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$ is the convolution ring of compactly supported functions $G \to \mathbb{Z}$ constant on the double classes of G modulo $\mathcal{U}$. We denote by $T(g)$ the characteristic function of $\mathcal{U}g\mathcal{U}$ for $g \in G$, seen as an element of $\mathcal{H}$. Let R be a commutative ring. The pro-p Iwahori Hecke R-algebra $\mathcal{H}_R$ is $R \otimes_{\mathbb{Z}} \mathcal{H}$. We will follow the custom to still denote by h the natural image $1 \otimes h$ of $h \in \mathcal{H}$ in $\mathcal{H}_R$.

For $P = MN$ a standard parabolic subgroup of G, the similar objects for M are indexed by M, we have $\mathcal{K}_M = \mathcal{K} \cap M, \mathcal{B}_M = \mathcal{B} \cap M, \mathcal{U}_M = \mathcal{U} \cap M$, the pro-$p$ Iwahori Hecke ring $\mathcal{H}_M = \mathcal{H}(M, \mathcal{U}_M)$, $T^M(m)$ the characteristic function of $\mathcal{U}_M m \mathcal{U}_M$ for $m \in M$, seen as an element of $\mathcal{H}_M$. The pro-p Iwahori subgroup $\mathcal{U}$ of G satisfies the Iwahori decomposition with respect to P:

$$\mathcal{U} = \mathcal{U}_N \mathcal{U}_M \mathcal{U}_{\overline{N}},$$

where $\mathcal{U}_N = \mathcal{U} \cap N, \mathcal{U}_{\overline{N}} = \mathcal{U} \cap \overline{N}$. The linear map

$$\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}, \quad \theta(T^M(m)) = T(m) \quad (m \in M) \tag{2.1}$$

does not respect the product. But if we introduce the monoid M^+ of elements $m \in M$ contracting $\mathcal{U}_N$, meaning $m\mathcal{U}_N m^{-1} \subset \mathcal{U}_N$, and the submodule $\mathcal{H}_{M^+} \subset \mathcal{H}_M$ of functions with support in M^+, we have [Vig15b, Theorem 1.4]:

$\mathcal{H}_{M^+}$ is a subring of $\mathcal{H}_M$ and $\mathcal{H}_M$ is the localization of $\mathcal{H}_{M^+}$ at an element $\tau^M \in \mathcal{H}_{M^+}$ central and invertible in $\mathcal{H}_M$, meaning $\mathcal{H}_M = \bigcup_{n \in \mathbb{N}} \mathcal{H}_{M^+}(\tau^M)^{-n}$. The map $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$ is injective and its restriction $\theta|_{\mathcal{H}_{M^+}}$ to $\mathcal{H}_{M^+}$ respects the product.

These properties are also true when (M^+, τ^M) is replaced by its inverse $(M^-, (\tau^M)^{-1})$ where $M^- = \{m^{-1} \in M \mid m \in M^+\}$.

3. Pro-p Iwahori invariants of $I_G(P, \sigma, Q)$

3.1. **Pro-p Iwahori Hecke algebras: Structures.** Here we supplement the notation of sections 2.1 and 2.3. The subgroups $Z^0 = Z \cap \mathcal{K} = Z \cap \mathcal{B}$ and $Z^1 = Z \cap \mathcal{U}$

are normal in $\mathcal{N}$ and we put

$$W = \mathcal{N}/Z^0,\ W(1) = \mathcal{N}/Z^1,\ \Lambda = Z/Z^0,\ \Lambda(1) = Z/Z^1,\ Z_k = Z^0/Z^1.$$

We have $\mathcal{N} = (\mathcal{N} \cap \mathcal{K})Z$ so that we see the finite Weyl group $\mathbb{W} = \mathcal{N}/Z$ as the subgroup $(\mathcal{N} \cap \mathcal{K})/Z^0$ of W; in this way W is the semidirect product $\Lambda \rtimes \mathbb{W}$. We put $\mathcal{N}_{G'} = \mathcal{N} \cap G'$. The image $W_{G'} = W'$ of $\mathcal{N}_{G'}$ in W is an affine Weyl group generated by the set S^{aff} of affine reflections determined by the walls of the alcove $\mathcal{C}$. The group W' is normal in W and W is the semidirect product $W' \rtimes \Omega$ where Ω is the image in W of the normalizer $\mathcal{N}_{\mathcal{C}}$ of $\mathcal{C}$ in $\mathcal{N}$. The length function ℓ on the affine Weyl system (W', S^{aff}) extends to a length function on W such that Ω is the set of elements of length 0. We also view ℓ as a function of $W(1)$ via the quotient map $W(1) \to W$. We write

(3.1)

$(\hat{w}, \tilde{w}, w) \in \mathcal{N} \times W(1) \times W$ corresponding via the quotient maps $\mathcal{N} \to W(1) \to W$.

When $w = s$ in S^{aff} or more generally w in $W_{G'}$, we will most of the time choose $\hat{w}$ in $\mathcal{N} \cap G'$ and $\tilde{w}$ in the image ${}_1W_{G'}$ of $\mathcal{N} \cap G'$ in $W(1)$.

We are now ready to describe the pro-p Iwahori Hecke ring $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$ [Vig16]. We have $G = \mathcal{U}\mathcal{N}\mathcal{U}$ and for $n, n' \in \mathcal{N}$ we have $\mathcal{U}n\mathcal{U} = \mathcal{U}n'\mathcal{U}$ if and only if $nZ^1 = n'Z^1$. For $n \in \mathcal{N}$ of image $w \in W(1)$ and $g \in \mathcal{U}n\mathcal{U}$ we denote $T_w = T(n) = T(g)$ in $\mathcal{H}$. The relations among the basis elements $(T_w)_{w \in W(1)}$ of $\mathcal{H}$ are:

(1) Braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.
(2) Quadratic relations: $T_{\tilde{s}}^2 = q_s T_{\widetilde{s^2}} + c_{\tilde{s}} T_{\tilde{s}}$

for $\tilde{s} \in W(1)$ lifting $s \in S^{\mathrm{aff}}$, where $q_s = q_G(s) = |\mathcal{U}/\mathcal{U} \cap \hat{s}\mathcal{U}(\hat{s})^{-1}|$ depends only on s, and $c_{\tilde{s}} = \sum_{t \in Z_k} c_{\tilde{s}}(t) T_t$ for integers $c_{\tilde{s}}(t) \in \mathbb{N}$ summing to $q_s - 1$.

We shall need the basis elements $(T_w^*)_{w \in W(1)}$ of $\mathcal{H}$ defined by:

(1) $T_w^* = T_w$ for $w \in W(1)$ of length $\ell(w) = 0$.
(2) $T_{\tilde{s}}^* = T_{\tilde{s}} - c_{\tilde{s}}$ for $\tilde{s} \in W(1)$ lifting $s \in S^{\mathrm{aff}}$.
(3) $T_{ww'}^* = T_w^* T_{w'}^*$ for $w, w' \in W(1)$ with $\ell(ww') = \ell(w) + \ell(w')$.

We need more notation for the definition of the admissible lifts of S^{aff} in $\mathcal{N}_G$. Let $s \in S^{\mathrm{aff}}$ fixing a face $\mathcal{C}_s$ of the alcove $\mathcal{C}$ and $\mathcal{K}_s$ the parahoric subgroup of G fixing $\mathcal{C}_s$. The theory of Bruhat-Tits associates to $\mathcal{C}_s$ a certain root $\alpha_s \in \Phi^+$ [Vig16, §4.2]. We consider the group G'_s generated by $U_{\alpha_s} \cup U_{-\alpha_s}$ where $U_{\pm\alpha_s}$ the root subgroup of $\pm\alpha_s$ (if $2\alpha_s \in \Phi$, then $U_{2\alpha_s} \subset U_{\alpha_s}$) and the group $\mathcal{G}'_s$ generated by $\mathcal{U}_{\alpha_s} \cup \mathcal{U}_{-\alpha_s}$ where $\mathcal{U}_{\pm\alpha_s} = U_{\pm\alpha_s} \cap \mathcal{K}_s$. When $u \in \mathcal{U}_{\alpha_s} - \{1\}$, the intersection $\mathcal{N}_G \cap \mathcal{U}_{-\alpha_s} u \mathcal{U}_{-\alpha_s}$ (equal to $\mathcal{N}_G \cap U_{-\alpha_s} u U_{-\alpha_s}$ [BT72, 6.2.1 (V5)], [Vig16, §3.3 (19)]) possesses a single element $n_s(u)$. The group $Z'_s = Z \cap \mathcal{G}'_s$ is contained in $Z \cap \mathcal{K}_s = Z^0$; its image in Z_k is denoted by $Z'_{k,s}$.

The elements $n_s(u)$ for $u \in \mathcal{U}_{\alpha_s} - \{1\}$ are the admissible lifts of s in $\mathcal{N}_G$; their images in $W(1)$ are the admissible lifts of s in $W(1)$. By [Vig16, Theorem 2.2, Proposition 4.4], when $\tilde{s} \in W(1)$ is an admissible lift of s, $c_{\tilde{s}}(t) = 0$ if $t \in Z_k \setminus Z'_{k,s}$, and

$$c_{\tilde{s}} \equiv (q_s - 1)|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} T_t \qquad \bmod p. \tag{3.2}$$

The admissible lifts of S in $\mathcal{N}_G$ are contained in $\mathcal{N}_G \cap \mathcal{K}$ because $\mathcal{K}_s \subset \mathcal{K}$ when $s \in S$.

Definition 3.1. An admissible lift of the finite Weyl group $\mathbb{W}$ in $\mathcal{N}_G$ is a map $w \mapsto \hat{w} : \mathbb{W} \to \mathcal{N}_G \cap \mathcal{K}$ such that $\hat{s}$ is admissible for all $s \in S$ and $\hat{w} = \hat{w}_1\hat{w}_2$ for $w_1, w_2 \in \mathbb{W}$ such that $w = w_1w_2$ and $\ell(w) = \ell(w_1) + \ell(w_2)$.

Any choice of admissible lifts of S in $\mathcal{N}_G \cap \mathcal{K}$ extends uniquely to an admissible lift of $\mathbb{W}$ ([AHHV17, IV.6], [OV17, Proposition 2.7]).

Let $P = MN$ be a standard parabolic subgroup of G. The groups $Z, Z^0 = Z \cap \mathcal{K}_M = Z \cap \mathcal{B}_M, Z^1 = Z \cap \mathcal{U}_M$ are the same for G and M, but $\mathcal{N}_M = \mathcal{N} \cap M$ and $M \cap G'$ are subgroups of $\mathcal{N}$ and G'. The monoid M^+ (section 2.3) contains $(\mathcal{N}_M \cap \mathcal{K})$ and is equal to $M^+ = \mathcal{U}_M\mathcal{N}_{M^+}\mathcal{U}_M$ where $\mathcal{N}_{M^+} = \mathcal{N} \cap M^+$. An element $z \in Z$ belongs to M^+ if and only if $v_F(\alpha(z)) \geq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_M^+$ (see [Vig15b, Lemme 2.2]). Put $W_M = \mathcal{N}_M/Z^0$ and $W_M(1) = \mathcal{N}_M/Z^1$.

Let $\epsilon = +$ or $\epsilon = -$. We denote by $W_{M^\epsilon}, W_{M^\epsilon}(1)$ the images of $\mathcal{N}_{M^\epsilon}$ in $W_M, W_M(1)$. We see the groups $W_M, W_M(1), {}_1W_{M'}$ as subgroups of $W, W(1), {}_1W_{G'}$. As θ (section 2.3), the linear injective map

$$\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}, \quad \theta^*(T_w^{M,*}) = T_w^*, \quad (w \in W_M(1)), \tag{3.3}$$

respects the product on the subring $\mathcal{H}_{M^\epsilon}$. Here $T_w^{M,*} \in \mathcal{H}_M$ is defined in the same way as T_w^* for $\mathcal{H}_M$. Note that θ and θ^* satisfy the obvious transitivity property with respect to a change of parabolic subgroups.

3.2. **Orthogonal case.** Let us examine the case where Δ_M and $\Delta \setminus \Delta_M$ are orthogonal, writing $M_2 = M_{\Delta\setminus\Delta_M}$ as in section 1.3.

From $M \cap M_2 = Z$ we get $W_M \cap W_{M_2} = \Lambda, W_M(1) \cap W_{M_2}(1) = \Lambda(1)$, the semisimple building of G is the product of those of M and M_2. The set S^{aff} is the disjoint union of S_M^{aff} and $S_{M_2}^{\mathrm{aff}}$, the group $W_{G'}$ is the direct product of $W_{M'}$ and $W_{M_2'}$. For $\tilde{s} \in W_M(1)$ lifting $s \in S_M^{\mathrm{aff}}$, the elements $T_{\tilde{s}}^M \in \mathcal{H}_M$ and $T_{\tilde{s}} \in \mathcal{H}$ satisfy the same quadratic relations. A word of caution is necessary for the lengths ℓ_M of W_M and ℓ_{M_2} of W_{M_2} different from the restrictions of the length ℓ of W_M, for example $\ell_M(\lambda) = 0$ for $\lambda \in \Lambda \cap W_{M_2'}$.

Lemma 3.2. *We have* $\Lambda = (W_{M^\epsilon} \cap \Lambda)(W_{M_2'} \cap \Lambda)$.

Proof. We prove the lemma for $\epsilon = -$. The case $\epsilon = +$ is similar. The map $v : Z \to X_*(T) \otimes \mathbb{Q}$ defined in section 2.1 is trivial on Z^0 and we also write v for the resulting homomorphism on Λ. For $\lambda \in \Lambda$ there exists $\lambda_2 \in W_{M_2'} \cap \Lambda$ such that $\lambda\lambda_2 \in W_{M^-}$, or equivalently $\alpha(v(\lambda\lambda_2)) \leq 0$ for all $\alpha \in \Phi^+ \setminus \Phi_M^+ = \Phi_{M_2}^+$. It suffices to have the inequality for $\alpha \in \Delta_{M_2}$. The matrix $(\alpha(\beta^\vee))_{\alpha,\beta\in\Delta_{M_2}}$ is invertible, hence there exists $n_\beta \in \mathbb{Z}$ such that $\sum_{\beta\in\Delta_{M_2}} n_\beta\alpha(\beta^\vee) \leq -\alpha(v(\lambda))$ for all $\alpha \in \Delta_{M_2}$. As $v(W_{M_2'} \cap \Lambda)$ contains $\bigoplus_{\alpha\in\Delta_{M_2}} \mathbb{Z}\alpha^\vee$ where $\alpha^\vee$ is the coroot of α [Vig16, after formula (71)], there exists $\lambda_2 \in W_{M_2'} \cap \Lambda$ with $v(\lambda_2) = \sum_{\beta\in\Delta_{M_2}} n_\beta\beta^\vee$. $\square$

The groups $\mathcal{N} \cap M'$ and $\mathcal{N} \cap M_2'$ are normal in $\mathcal{N}$, and

$$\mathcal{N} = (\mathcal{N} \cap M')\mathcal{N}_{\mathcal{C}}(\mathcal{N} \cap M_2') = Z(\mathcal{N} \cap M')(\mathcal{N} \cap M_2'),$$

and

$$W = W_{M'}\Omega W_{M_2'} = W_M W_{M_2'} = W_{M^+}W_{M_2'} = W_{M^-}W_{M_2'}.$$

The first two equalities are clear, the equality $W_M W_{M_2'} = W_{M^\epsilon}W_{M_2'}$ follows from $W_M = \mathbb{W}_M\Lambda$, $\mathbb{W}_M \subset W_{M^\epsilon}$ and the lemma. The inverse image in $W(1)$ of these

groups are

(3.4)
$$W(1) = {}_1W_{M'}\,\Omega(1)\,{}_1W_{M_2'} = W_M(1)\,{}_1W_{M_2'} = W_{M^+}(1)\,{}_1W_{M_2'} = W_{M^-}(1)\,{}_1W_{M_2'}.$$

We recall the function $q_G(n) = q(n) = |\mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)|$ on $\mathcal{N}$ [Vig16, Proposition 3.38] and we extend to $\mathcal{N}$ the functions q_M on $\mathcal{N}\cap M$ and q_{M_2} on $\mathcal{N}\cap M_2$:

$$q_M(n) = |\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n)|, \quad q_{M_2}(n) = |\mathcal{U}_{M_2}/(\mathcal{U}_{M_2}\cap n^{-1}\mathcal{U}_{M_2} n)|. \tag{3.5}$$

The functions q, q_M, q_{M_2} descend to functions on $W(1)$ and on W, also denoted by q, q_M, q_{M_2}.

Lemma 3.3. *Let $n\in\mathcal{N}$ of image $w\in W$. We have*

1. $q(n) = q_M(n)q_{M_2}(n)$.
2. $q_M(n) = q_M(n_M)$ *if* $n = n_M n_2$, $n_M\in\mathcal{N}\cap M, n_2\in\mathcal{N}\cap M_2'$ *and similarly when M and M_2 are permuted.*
3. $q(w) = 1 \Leftrightarrow q_M(\lambda w_M) = q_{M_2}(\lambda w_{M_2}) = 1$, *if* $w = \lambda w_M w_{M_2}, (\lambda, w_M, w_{M_2})\in\Lambda\times\mathbb{W}_M\times\mathbb{W}_{M_2}$.
4. *On the coset $(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}n$, q_M is constant equal to $q_M(n_{M'})$ for any element $n_{M'}\in M'\cap(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}n$. A similar result is true when M and M_2 are permuted.*

Proof. We put $\mathcal{U}_{M'} = \mathcal{U}\cap M'$ and $\mathcal{U}_{M_2'} = \mathcal{U}\cap M_2'$. The product map

$$Z^1\prod_{\alpha\in\Phi_{M,red}}\mathcal{U}_\alpha\prod_{\alpha\in\Phi_{M_2,red}}\mathcal{U}_\alpha\to\mathcal{U} \tag{3.6}$$

with $\mathcal{U}_\alpha = U_\alpha\cap\mathcal{U}$, is a homeomorphism. We have $\mathcal{U}_M = Z^1\mathcal{Y}_{M'}$, $\mathcal{U}_{M'} = (Z^1\cap M')\mathcal{Y}_{M'}$ where $\mathcal{Y}_{M'} = \prod_{\alpha\in\Phi_{M,red}}\mathcal{U}_\alpha$ and $\mathcal{N}\cap M_2'$ commutes with $\mathcal{Y}_{M'}$, in particular $\mathcal{N}\cap M_2'$ normalizes $\mathcal{Y}_{M'}$. Similar results are true when M and M_2 are permuted, and $\mathcal{U} = \mathcal{U}_{M'}\mathcal{U}_{M_2} = \mathcal{U}_M\mathcal{U}_{M_2'}$.

Writing $\mathcal{N} = Z(\mathcal{N}\cap M')(\mathcal{N}\cap M_2')$ (in any order), we see that the product map

$$Z^1(\mathcal{Y}_{M'}\cap n^{-1}\mathcal{Y}_{M'}n)(\mathcal{Y}_{M_2'}\cap n^{-1}\mathcal{Y}_{M_2'}n)\to\mathcal{U}\cap n^{-1}\mathcal{U}n \tag{3.7}$$

is a homeomorphism. The inclusions induce bijections

$$\mathcal{Y}_{M'}/(\mathcal{Y}_{M'}\cap n^{-1}\mathcal{Y}_{M'}n)\simeq\mathcal{U}_{M'}/(\mathcal{U}_{M'}\cap n^{-1}\mathcal{U}_{M'}n)\simeq\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n), \tag{3.8}$$

similarly for M_2, and also a bijection

$$\mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)\simeq(\mathcal{Y}_{M_2'}/(\mathcal{Y}_{M_2'}\cap n^{-1}\mathcal{Y}_{M_2'}n))\times(\mathcal{Y}_{M'}/(\mathcal{Y}_{M'}\cap n^{-1}\mathcal{Y}_{M'}n)). \tag{3.9}$$

From (3.8) and (3.9), we get

$$\mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)\simeq(\mathcal{U}_{M_2'}/(\mathcal{U}_{M_2'}\cap n\mathcal{U}_{M_2'}n^{-1}))\times(\mathcal{U}_{M'}/(\mathcal{U}_{M'}\cap n\mathcal{U}_{M'}n^{-1})) \tag{3.10}$$

which implies the assertion (1) in the lemma.

The assertion (2) follows from (3.7) since $\mathcal{N}\cap M_2'$ normalizes $\mathcal{Y}_{M'}$; with (1), it implies the assertion (3).

A subgroup of $\mathcal{N}$ normalizes $\mathcal{U}_M$ if and only if it normalizes $\mathcal{Y}_{M'}$ by (3.8) if and only if $q_M = 1$ on this group. The group $\mathcal{N}\cap M_2'$ normalizes $\mathcal{Y}_{M'}$. Therefore the group $(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}$ normalizes $\mathcal{U}_M$. The coset $(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}n$ contains an element $n_{M'}\in M'$. For $x\in(\mathcal{N}\cap M_2')\mathcal{N}_{\mathcal{C}}$, $(xn_{M'})^{-1}\mathcal{U}xn_{M'} = n_{M'}^{-1}\mathcal{U}n_{M'}$ hence $q_M(xn_{M'}) = q_M(n_{M'})$. $\square$

3.3. **Extension of an $\mathcal{H}_M$-module to $\mathcal{H}$.** This section is inspired by similar results for the pro-p Iwahori Hecke algebras over an algebraically closed field of characteristic p [Abe, Proposition 4.16]. We keep the setting of section 3.2 and we introduce ideals:

- $\mathcal{J}_\ell$ (resp., $\mathcal{J}_r$) the left (resp., right) ideal of $\mathcal{H}$ generated by $T_w^* - 1_{\mathcal{H}}$ for all $w \in {}_1W_{M_2'}$,
- $\mathcal{J}_{M,\ell}$ (resp., $\mathcal{J}_{M,r}$) the left (resp., right) ideal of $\mathcal{H}_M$ generated by $T_\lambda^{M,*} - 1_{\mathcal{H}_M}$ for all λ in ${}_1W_{M_2'} \cap W_M(1) = {}_1W_{M_2'} \cap \Lambda(1)$.

The next proposition shows that the ideals $\mathcal{J}_\ell = \mathcal{J}_r$ are equal and similarly $\mathcal{J}_{M,\ell} = \mathcal{J}_{M,r}$. After the proposition, we will drop the indices ℓ and r.

Proposition 3.4. *The ideals $\mathcal{J}_\ell$ and $\mathcal{J}_r$ are equal to the submodule $\mathcal{J}'$ of $\mathcal{H}$ generated by $T_w^* - T_{ww_2}^*$ for all $w \in W(1)$ and $w_2 \in {}_1W_{M_2'}$.*

The ideals $\mathcal{J}_{M,\ell}$ and $\mathcal{J}_{M,r}$ are equal to the submodule $\mathcal{J}_M'$ of $\mathcal{H}_M$ generated by $T_w^{M,*} - T_{w\lambda_2}^{M,*}$ for all $w \in W_M(1)$ and $\lambda_2 \in \Lambda(1) \cap {}_1W_{M_2'}$.

Proof.

(1) We prove $\mathcal{J}_\ell = \mathcal{J}'$. Let $w \in W(1), w_2 \in {}_1W_{M_2'}$. We prove by induction on the length of w_2 that $T_w^*(T_{w_2}^* - 1) \in \mathcal{J}'$. This is obvious when $\ell(w_2) = 0$ because $T_w^* T_{w_2}^* = T_{ww_2}^*$. Assume that $\ell(w_2) = 1$ and put $s = w_2$. If $\ell(ws) = \ell(w) + 1$, as before $T_w^*(T_s^* - 1) \in \mathcal{J}'$ because $T_w^* T_s^* = T_{ws}^*$. Otherwise $\ell(ws) = \ell(w) - 1$ and $T_w^* = T_{ws^{-1}}^* T_s^*$ hence

$$T_w^*(T_s^* - 1) = T_{ws^{-1}}^*(T_s^*)^2 - T_w^* = T_{ws^{-1}}^*(q_s T_{s^2}^* - T_s^* c_s) - T_w^* = q_s T_{ws}^* - T_w^*(c_s + 1).$$

Since $c_s + 1 = \sum_{t \in Z_k'} c_s(t) T_t$ with $c_s(t) \in \mathbb{N}$ and $\sum_{t \in Z_k'} c_s(t) = q_s$ [Vig16, Proposition 4.4],

$$q_s T_{ws}^* - T_w^*(c_s + 1) = \sum_{t \in Z_k'} c_s(t)(T_{ws}^* - T_w^* T_t^*) = \sum_{t \in Z_k'} c_s(t)(T_{ws}^* - T_{wss^{-1}t}^*) \in \mathcal{J}'.$$

Assume now that $\ell(w_2) > 1$. Then, we factorize $w_2 = xy$ with $x, y \in {}_1W_{M_2}$ of length $\ell(x), \ell(y) < \ell(w_2)$ and $\ell(w_2) = \ell(x) + \ell(y)$. The element $T_w^*(T_{w_2}^* - 1) = T_w^* T_x^*(T_y^* - 1) + T_w^*(T_x^* - 1)$ lies in $\mathcal{J}'$ by induction.

Conversely, we prove $T_{ww_2}^* - T_w^* \in \mathcal{J}_\ell$. We factorize $w = xy$ with $y \in {}_1W_{M_2'}$ and $x \in {}_1W_{M'}\Omega(1)$. Then, we have $\ell(w) = \ell(x) + \ell(y)$ and $\ell(ww_2) = \ell(x) + \ell(yw_2)$. Hence

$$T_{ww_2}^* - T_w^* = T_x^*(T_{yw_2}^* - T_y^*) = T_x^*(T_{yw_2}^* - 1) - T_x^*(T_y^* - 1) \in \mathcal{J}_\ell.$$

This ends the proof of $\mathcal{J}_\ell = \mathcal{J}'$.

By the same argument, the right ideal $\mathcal{J}_r$ of $\mathcal{H}$ is equal to the submodule of $\mathcal{H}$ generated by $T_{w_2 w}^* - T_w^*$ for all $w \in W(1)$ and $w_2 \in {}_1W_{M_2'}$. But this latter submodule is equal to $\mathcal{J}'$ because ${}_1W_{M_2'}$ is normal in $W(1)$. Therefore we proved $\mathcal{J}' = \mathcal{J}_r = \mathcal{J}_\ell$.

(2) Proof of the second assertion. We prove $\mathcal{J}_{M,\ell} = \mathcal{J}_M'$. The proof is easier than in (1) because for $w \in W_M(1)$ and $\lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1)$, we have $\ell(w\lambda_2) = \ell(w) + \ell(\lambda_2)$ hence $T_w^{M,*}(T_{\lambda_2}^{M,*} - 1) = T_{w\lambda_2}^{M,*} - T_w^{M,*}$. We have also $\ell(\lambda_2 w) = \ell(\lambda_2) + \ell(w)$ hence $(T_{\lambda_2}^{M,*} - 1)T_w^{M,*} = T_{\lambda_2 w}^{M,*} - T_w^{M,*}$ hence $\mathcal{J}_{M,r}$ is equal to the submodule of $\mathcal{H}_M$ generated by $T_{\lambda_2 w}^{M,*} - T_w^{M,*}$ for all $w \in W_M(1)$ and $\lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1)$. This latter submodule is $\mathcal{J}_M'$, as ${}_1W_{M_2'} \cap \Lambda(1) = {}_1W_{M_2'} \cap W_M(1)$ is normal in $W_M(1)$. Therefore $\mathcal{J}_M' = \mathcal{J}_{M,r} = \mathcal{J}_{M,\ell}$. $\square$

By Proposition 3.4, a basis of $\mathcal{J}$ is $T_w^* - T_{ww_2}^*$ for w in a system of representatives of $W(1)/{}_1W_{M_2'}$, and $w_2 \in {}_1W_{M_2'} \setminus \{1\}$. Similarly a basis of $\mathcal{J}_M$ is $T_w^{M,*} - T_{w\lambda_2}^{M,*}$ for w in a system of representatives of $W_M(1)/(\Lambda(1) \cap {}_1W_{M_2'})$. and $\lambda_2 \in (\Lambda(1) \cap {}_1W_{M_2'}) \setminus \{1\}$.

Proposition 3.5. *The natural ring inclusion of $\mathcal{H}_{M^-}$ in $\mathcal{H}_M$ and the ring inclusion of $\mathcal{H}_{M^-}$ in $\mathcal{H}$ via θ^* induce ring isomorphisms*

$$\mathcal{H}_M/\mathcal{J}_M \xleftarrow{\sim} \mathcal{H}_{M^-}/(\mathcal{J}_M \cap \mathcal{H}_{M^-}) \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

Proof.

(1) The left map is obviously injective. We prove the surjectivity. Let $w \in W_M(1)$. Let $\lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1)$ such that $w\lambda_2^{-1} \in W_{M^-}(1)$ (see (3.4)). We have $T_{w\lambda_2^{-1}}^{M,*} \in \mathcal{H}_{M^-}$ and $T_w^{M,*} = T_{w\lambda_2^{-1}}^{M,*} T_{\lambda_2}^{M,*} = T_{w\lambda_2^{-1}}^{M,*} + T_{w\lambda_2^{-1}}^{M,*}(T_{\lambda_2}^{M,*} - 1)$. Therefore $T_w^{M,*} \in \mathcal{H}_{M^-} + \mathcal{J}_M$. As w is arbitrary, $\mathcal{H}_M = \mathcal{H}_{M^-} + \mathcal{J}_M$.

(2) The right map is surjective: let $w \in W(1)$ and $w_2 \in {}_1W_{M_2'}$ such that $ww_2^{-1} \in W_{M^-}(1)$ (see (3.4)). Then $T_w^* - T_{ww_2^{-1}}^* \in \mathcal{J}$ with the same arguments as in (1), using Proposition 3.4. Therefore $\mathcal{H} = \theta^*(\mathcal{H}_{M^-}) + \mathcal{J}$.

We prove the injectivity: $\theta^*(\mathcal{H}_{M^-}) \cap \mathcal{J} = \theta^*(\mathcal{H}_{M^-} \cap \mathcal{J}_M)$. Let $\sum_{w \in W_{M^-}(1)} c_w T_w^{M,*}$, with $c_w \in \mathbb{Z}$, be an element of $\mathcal{H}_{M^-}$. Its image by θ^* is $\sum_{w \in W(1)} c_w T_w^*$ where we have set $c_w = 0$ for $w \in W(1) \setminus W_{M^-}(1)$. We have $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}$ if and only if $\sum_{w_2 \in {}_1W_{M_2'}} c_{ww_2} = 0$ for all $w \in W(1)$. If $c_{ww_2} \neq 0$, then $w_2 \in {}_1W_{M_2'} \cap W_M(1)$, that is, $w_2 \in {}_1W_{M_2'} \cap \Lambda(1)$. The sum $\sum_{w_2 \in {}_1W_{M_2'}} c_{ww_2}$ is equal to $\sum_{\lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1)} c_{w\lambda_2}$. By Proposition 3.4, $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}$ if and only if $\sum_{w \in W_{M^-}(1)} c_w T_w^{M,*} \in \mathcal{J}_M$. □

We construct a ring isomorphism

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}$$

by using Proposition 3.5. For any $w \in W(1)$, $T_w^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$ where $w_{M^-} \in W_{M^-}(1) \cap w\ {}_1W_{M_2'}$ (see (3.4)), because by Proposition 3.4, $T_w^* + \mathcal{J} = T_{w_{M^-}}^* + \mathcal{J}$ and $T_{w_{M^-}}^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$ by construction of e^*. We check that e^* is induced by θ^*.

Theorem 3.6. *The linear map $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$ induces a ring isomorphism*

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

Proof. Let $w \in W_M(1)$. We have to show that $T_w^* + \mathcal{J} = e^*(T_w^{M,*} + \mathcal{J}_M)$. We saw above that $T_w^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$ with $w = w_{M^-}\lambda_2$ with $\lambda_2 \in {}_1W_{M_2'} \cap W_M(1)$. As $\ell_M(\lambda_2) = 0$, $T_w^{M,*} = T_{w_{M^-}}^{M,*} T_{\lambda_2}^{M,*} \in T_{w_{M^-}}^{M,*} + \mathcal{J}_M$. Therefore $T_{w_{M^-}}^{M,*} + \mathcal{J}_M = T_w^{M,*} + \mathcal{J}_M$. This ends the proof of the theorem. □

We now wish to compute e^* in terms of the T_w instead of the T_w^*.

Proposition 3.7. *Let $w \in W(1)$. Then, $T_w + \mathcal{J} = e^*(T_{w_M}^M q_{M_2}(w) + \mathcal{J}_M)$ for any $w_M \in W_M(1) \cap w\ {}_1W_{M_2'}$.*

Proof. The element w_M is unique modulo right multiplication by an element $\lambda_2 \in W_M(1) \cap {}_1W_{M_2'}$ of length $\ell_M(\lambda_2) = 0$ and $T^M_{w_M} q_{M_2}(w) + \mathcal{J}_M$ does not depend on the choice of w_M. We choose a decomposition (see (3.4)):

$$w = \tilde{s}_1 \dots \tilde{s}_a u \tilde{s}_{a+1} \dots \tilde{s}_{a+b}, \quad \ell(w) = a + b,$$

for $u \in \Omega(1)$, $\tilde{s}_i \in {}_1W_{M'}$ lifting $s_i \in S^{\mathrm{aff}}_M$ for $1 \le i \le a$ and $\tilde{s}_i \in {}_1W_{M_2'}$ lifting $s_i \in S^{\mathrm{aff}}_{M_2}$ for $a+1 \le i \le a+b$, and we choose $u_M \in W_M(1)$ such that $u \in u_M\, {}_1W_{M_2'}$. Then

$$w_M = \tilde{s}_1 \dots \tilde{s}_a u_M \in W_M(1) \cap w\ {}_1W_{M_2'}$$

and $q_{M_2}(w) = q_{M_2}(\tilde{s}_{a+1} \dots \tilde{s}_{a+b})$ (Lemma 3.3 (4)). First we check the proposition in three simple cases:

Case 1. Let $w = \tilde{s} \in {}_1W_{M'}$ lifting $s \in S^{\mathrm{aff}}_M$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(T^M_{\tilde{s}} + \mathcal{J}_M)$ because $T^*_{\tilde{s}} - e^*(T^{M,*}_{\tilde{s}}) \in \mathcal{J}$, $T_{\tilde{s}} = T^*_{\tilde{s}} + c_{\tilde{s}}$, $T^M_{\tilde{s}} = T^{M,*}_{\tilde{s}} + c_{\tilde{s}}$ and $1 = q_{M_2}(\tilde{s})$.

Case 2. Let $w = u \in W(1)$ of length $\ell(u) = 0$ and $u_M \in W_M(1)$ such that $u \in u_M\, {}_1W_{M_2'}$. We have $\ell_M(u_M) = 0$ and $q_{M_2}(u) = 1$ (Lemma 3.3). We deduce $T_u + \mathcal{J} = e^*(T^M_{u_M} + \mathcal{J}_M)$ because $T^*_u + \mathcal{J} = T^*_{u_M} + \mathcal{J} = e^*(T^{M,*}_{u_M} + \mathcal{J}_M)$, and $T_u = T^*_u, T^M_{u_M} = T^{M,*}_{u_M}$.

Case 3. Let $w = \tilde{s} \in {}_1W_{M_2'}$ lifting $s \in S^{\mathrm{aff}}_{M_2}$; we have $T_{\tilde{s}} + \mathcal{J} = e^*(q_{M_2}(\tilde{s}) + \mathcal{J}_M)$ because $T^*_{\tilde{s}} - 1, c_{\tilde{s}} - (q_s - 1) \in \mathcal{J}$, $T_{\tilde{s}} = T^*_{\tilde{s}} + c_{\tilde{s}} \in q_s + \mathcal{J}$ and $q_s = q_{M_2}(\tilde{s})$.

In general, the braid relations $T_w = T_{\tilde{s}_1} \dots T_{\tilde{s}_a} T_u T_{\tilde{s}_{a+1}} \dots T_{\tilde{s}_{a+b}}$ give a similar product decomposition of $T_w + \mathcal{J}$, and the simple cases 1, 2, 3 imply that $T_w + \mathcal{J}$ is equal to

$$\begin{aligned} e^*(T^M_{\tilde{s}_1} + \mathcal{J}_M) \dots e^*(T^M_{\tilde{s}_a} + \mathcal{J}_M) e^*(T^M_{u_M} + \mathcal{J}_M) e^*(q_{M_2}(\tilde{s}_{a+1}) \\ + \mathcal{J}_M) \dots e^*(q_{M_2}(\tilde{s}_{a+b}) + \mathcal{J}_M) \\ = e^*(T^M_{w_M} q_{M_2}(w) + \mathcal{J}_M). \end{aligned}$$

The proposition is proved. □

Propositions 3.4, 3.5, 3.7, and Theorem 3.6 are valid over any commutative ring R (instead of $\mathbb{Z}$).

The two-sided ideal of $\mathcal{H}_R$ generated by $T^*_w - 1$ for all $w \in {}_1W_{M_2'}$ is $\mathcal{J}_R = \mathcal{J} \otimes_{\mathbb{Z}} R$, the two-sided ideal of $\mathcal{H}_{M,R}$ generated by $T^*_\lambda - 1$ for all $\lambda \in {}_1W_{M_2'} \cap \Lambda(1)$ is $\mathcal{J}_{M,R} = \mathcal{J}_M \otimes_{\mathbb{Z}} R$, and we get as in Proposition 3.5 isomorphisms

$$\mathcal{H}_{M,R}/\mathcal{J}_{M,R} \xleftarrow{\sim} \mathcal{H}_{M^-,R}/(\mathcal{J}_{M,R} \cap \mathcal{H}_{M^-,R}) \xrightarrow{\sim} \mathcal{H}_R/\mathcal{J}_R,$$

giving an isomorphism $\mathcal{H}_{M,R}/\mathcal{J}_{M,R} \to \mathcal{H}_R/\mathcal{J}_R$ induced by θ^*. Therefore, we have an isomorphism from the category of right $\mathcal{H}_{M,R}$-modules where $\mathcal{J}_M$ acts by 0 onto the category of right $\mathcal{H}_R$-modules where $\mathcal{J}$ acts by 0.

Definition 3.8. A right $\mathcal{H}_{M,R}$-module $\mathcal{V}$ where $\mathcal{J}_M$ acts by 0 is called extensible to $\mathcal{H}$. The corresponding $\mathcal{H}_R$-module where $\mathcal{J}$ acts by 0 is called its extension to $\mathcal{H}$ and denoted by $e_{\mathcal{H}}(\mathcal{V})$ or $e(\mathcal{V})$.

With the element basis T^*_w, $\mathcal{V}$ is extensible to $\mathcal{H}$ if and only if

$$vT^{M,*}_{\lambda_2} = v \text{ for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1). \tag{3.11}$$

The $\mathcal{H}$-module structure on the R-module $e(\mathcal{V}) = \mathcal{V}$ is determined by

$$vT^*_{w_2} = v, \quad vT^*_w = vT^{M,*}_w \quad \text{for all } v \in \mathcal{V}, w_2 \in {}_1W_{M_2'}, w \in W_M(1). \tag{3.12}$$

It is also determined by the action of T_w^* for $w \in {}_1W_{M_2'} \cup W_{M^+}(1)$ (or $w \in {}_1W_{M_2'} \cup W_{M^-}(1)$). Conversely, a right $\mathcal{H}$-module $\mathcal{W}$ over R is extended from an $\mathcal{H}_M$-module if and only if

$$vT_{w_2}^* = v \quad \text{for all} \quad v \in \mathcal{W}, w_2 \in {}_1W_{M_2'}. \tag{3.13}$$

In terms of the basis elements T_w instead of T_w^*, this says the following.

Corollary 3.9. *A right $\mathcal{H}_M$-module $\mathcal{V}$ over R is extensible to $\mathcal{H}$ if and only if*

$$vT_{\lambda_2}^M = v \text{ for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M_2'} \cap \Lambda(1). \tag{3.14}$$

Then, the structure of an $\mathcal{H}$-module on the R-module $e(\mathcal{V}) = \mathcal{V}$ is determined by

$$vT_{w_2} = vq_{w_2}, \quad vT_w = vT_w^M q_{M_2}(w) \quad \text{for all} \quad v \in \mathcal{V}, w_2 \in {}_1W_{M_2'}, w \in W_M(1). \tag{3.15}$$

($W_{M^+}(1)$ or $W_{M^-}(1)$ instead of $W_M(1)$ is enough.) A right $\mathcal{H}$-module $\mathcal{W}$ over R is extended from an $\mathcal{H}_M$-module if and only if

$$vT_{w_2} = vq_{w_2} \quad \text{for all} \quad v \in \mathcal{W}, w_2 \in {}_1W_{M_2'}. \tag{3.16}$$

3.4. $\sigma^{\mathcal{U}_M}$ **is extensible to $\mathcal{H}$ of extension** $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$**.** Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal, and let σ be a smooth R-representation of M extensible to G. Let $P_2 = M_2N_2$ denote the standard parabolic subgroup of G with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

Recall that $G = MM_2'$, that $M \cap M_2' = Z \cap M_2'$ acts trivially on σ, $e(\sigma)$ is the representation of G equal to σ on M and trivial on M_2'. We will describe the $\mathcal{H}$-module $e(\sigma)^{\mathcal{U}}$ in this section. We first consider $e(\sigma)$ as a subrepresentation of $\mathrm{Ind}_P^G \sigma$. For $v \in \sigma$, let $f_v \in (\mathrm{Ind}_P^G \sigma)^{M_2'}$ be the unique function with value v on M_2'. Then, the map

$$v \mapsto f_v : \sigma \to \mathrm{Ind}_P^G \sigma \tag{3.17}$$

is the natural G-equivariant embedding of $e(\sigma)$ in $\mathrm{Ind}_P^G \sigma$. As $\sigma^{\mathcal{U}_M} = e(\sigma)^{\mathcal{U}}$ as R-modules, the image of $e(\sigma)^{\mathcal{U}}$ in $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ is made out of the f_v for $v \in \sigma^{\mathcal{U}_M}$.

We now recall the explicit description of $(\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$. For each $d \in \mathbb{W}_{M_2}$, we fix a lift $\hat{d} \in {}_1W_{M_2'}$ and for $v \in \sigma^{\mathcal{U}_M}$ let $f_{P\hat{d}\mathcal{U},v} \in (\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ for the function with support contained in $P\hat{d}\mathcal{U}$ and value v on $\hat{d}\mathcal{U}$. As $Z \cap M_2'$ acts trivially on σ, the function $f_{P\hat{d}\mathcal{U},v}$ does not depend on the choice of the lift $\hat{d} \in {}_1W_{M_2'}$ of d. By [OV17, Lemma 4.5], recalling that $w \in \mathbb{W}_{M_2}$ is of minimal length in its coset $w\mathbb{W}_M = \mathbb{W}_M w$ as Δ_M and Δ_{M_2} are orthogonal to each other:

The map $\bigoplus_{d \in \mathbb{W}_{M_2}} \sigma^{\mathcal{U}_M} \to (\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}$ given on each d-component by $v \mapsto f_{P\hat{d}\mathcal{U},v}$, is an $\mathcal{H}_{M^+}$-equivariant isomorphism where $\mathcal{H}_{M^+}$ is seen as a subring of $\mathcal{H}$ via θ, and induces an $\mathcal{H}_R$-module isomorphism

$$v \otimes h \mapsto f_{P\mathcal{U},v} h : \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} \to (\mathrm{Ind}_P^G \sigma)^{\mathcal{U}}. \tag{3.18}$$

In particular for $v \in \sigma^{\mathcal{U}_M}$, $v \otimes T(\hat{d})$ does not depend on the choice of the lift $\hat{d} \in {}_1W_{M_2'}$ of d and

$$f_{P\hat{d}\mathcal{U},v} = f_{P\mathcal{U},v} T(\hat{d}). \tag{3.19}$$

As G is the disjoint union of $P\hat{d}\mathcal{U}$ for $d \in \mathbb{W}_{M_2}$, we have $f_v = \sum_{d\in\mathbb{W}_{M_2}} f_{P\hat{d}\mathcal{U},v}$ and f_v is the image of $v \otimes e_{M_2}$ in (3.18), where

$$e_{M_2} = \sum_{d\in\mathbb{W}_{M_2}} T(\hat{d}). \tag{3.20}$$

Recalling (3.17) we get the following.

Lemma 3.10. *The map $v \mapsto v \otimes e_{M_2} : e(\sigma)^{\mathcal{U}} \to \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ is an $\mathcal{H}_R$-equivariant embedding.*

Remark 3.11. The trivial map $v \mapsto v \otimes 1_{\mathcal{H}}$ is not an $\mathcal{H}_R$-equivariant embedding.

We describe the action of $T(n)$ on $e(\sigma)^{\mathcal{U}}$ for $n \in \mathcal{N}$. By definition for $v \in e(\sigma)^{\mathcal{U}}$,

$$vT(n) = \sum_{y\in\mathcal{U}/(\mathcal{U}\cap n^{-1}\mathcal{U}n)} yn^{-1}v. \tag{3.21}$$

Proposition 3.12. *We have $vT(n) = vT^M(n_M)q_{M_2}(n)$ for any $n_M \in \mathcal{N} \cap M$ is such that $n = n_M(\mathcal{N} \cap M_2')$.*

Proof. The description (3.10) of $\mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)$ gives

$$vT(n) = \sum_{y_1\in\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n)} y_1 \sum_{y_2\in\mathcal{U}_{M_2'}/(\mathcal{U}_{M_2'}\cap n^{-1}\mathcal{U}_{M_2'} n)} y_2 n^{-1}v.$$

As M_2' acts trivially on $e(\sigma)$, we obtain

$$vT(n) = q_{M_2}(n) \sum_{y_1\in\mathcal{U}_M/(\mathcal{U}_M\cap n^{-1}\mathcal{U}_M n)} y_1 n_M^{-1} v = q_{M_2}(n)\, vT^M(n_M).$$

□

Theorem 3.13. *Let σ be a smooth R-representation of M. If $P(\sigma) = G$, then $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$ of extension $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$. Conversely, if $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$ and generates σ, then $P(\sigma) = G$.*

Proof.

(1) The $\mathcal{H}_M$-module $\sigma^{\mathcal{U}_M}$ is extensible to $\mathcal{H}$ if and only if $Z \cap M_2'$ acts trivially on $\sigma^{\mathcal{U}_M}$. Indeed, for $v \in \sigma^{\mathcal{U}_M}, z_2 \in Z \cap M_2'$,

$$vT^M(z_2) = \sum_{y\in\mathcal{U}_M/(\mathcal{U}_M\cap z_2^{-1}\mathcal{U}_M z_2)} yz_2^{-1}v = \sum_{y\in\mathcal{Y}_{M'}/(\mathcal{Y}_{M'}\cap z_2^{-1}\mathcal{Y}_{M'} z_2)} yz_2^{-1}v = z_2^{-1}v,$$

by (3.21), then (3.8), then the fact that z_2^{-1} commutes with the elements of $\mathcal{Y}_M$.

(2) $P(\sigma) = G$ if and only if $Z \cap M_2'$ acts trivially on σ (the group $Z \cap M_2'$ is generated by $Z \cap M_\alpha'$ for $\alpha \in \Delta_{M_2}$ by Lemma 2.1). The R-submodule $\sigma^{Z\cap M_2'}$ of elements fixed by $Z \cap M_2'$ is stable by M, because $M = ZM'$, the elements of M' commute with those of $Z \cap M_2'$ and Z normalizes $Z \cap M_2'$.

(3) Apply (1) and (2) to get the theorem except the equality $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$ when $P(\sigma) = G$ which follows from Propositions 3.12 and 3.7. □

Let $\mathbf{1}_M$ denote the trivial representation of M over R (or $\mathbf{1}$ when there is no ambiguity on M). The right $\mathcal{H}_R$-module $(\mathbf{1}_G)^{\mathcal{U}} = \mathbf{1}_{\mathcal{H}}$ (or $\mathbf{1}$ if there is no ambiguity) is the trivial right $\mathcal{H}_R$-module: for $w \in W_M(1)$, $T_w = q_w\mathrm{id}$ and $T_w^* = \mathrm{id}$ on $\mathbf{1}_{\mathcal{H}}$.

Example 3.14. The $\mathcal{H}$-module $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ is the extension of the $\mathcal{H}_{M_2}$-module $(\mathrm{Ind}_{M_2\cap B}^{M_2} \mathbf{1})^{\mathcal{U}_{M_2}}$. Indeed, the representation $\mathrm{Ind}_P^G \mathbf{1}$ of G is trivial on N_2, as $G = MM_2'$ and $N_2 \subset M'$ (as $\Phi = \Phi_M \cup \Phi_{M_2}$). For $g = mm_2'$ with $m \in M, m_2' \in M_2'$ and $n_2 \in N_2$, we have $Pgn_2 = Pm_2'n_2 = Pn_2m_2' = Pm_2' = Pg$. The group $M_2 \cap B = M_2 \cap P$ is the standard minimal parabolic subgroup of M_2 and $(\mathrm{Ind}_P^G \mathbf{1})|_{M_2} = \mathrm{Ind}_{M_2\cap B}^{M_2} \mathbf{1}$. Apply Theorem 3.13 as follows.

3.5. **The $\mathcal{H}_R$-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$.** Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal, let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module which is extensible to $\mathcal{H}_R$ of extension $e(\mathcal{V})$, and let Q be a parabolic subgroup of G containing P. Let $P_2 = M_2N_2$ denote the standard parabolic subgroup of G with $\Delta_{P_2} = \Delta \setminus \Delta_P$.

We define on the R-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ a structure of a right $\mathcal{H}_R$-module as follows.

Proposition 3.15.

(1) *The diagonal action of T_w^* for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ defines a structure of a right $\mathcal{H}_R$-module.*

(2) *The action of the T_w is also diagonal and satisfies:*

$$((v \otimes f)T_w, (v \otimes f)T_w^*) = (vT_{uw_{M'}} \otimes fT_{uw_{M_2'}}, vT^*_{uw_{M'}} \otimes fT^*_{uw_{M_2'}}),$$

where $w = uw_{M'}w_{M_2'}$ with $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M_2'} \in {}_1W_{M_2'}$.

Proof. If the lemma is true for P it is also true for Q, because the R-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ naturally embedded in $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ is stable by the action of $\mathcal{H}$ defined in the lemma. So, we suppose $Q = P$.

For each element in ${}_1S^{\mathrm{aff}}$ we fix an admissible lift and denote the set of admissible lifts by ${}_1S^{\mathrm{aff}}$. We also use the obvious notation ${}_1S_M^{\mathrm{aff}}$ and ${}_1S_{M_2}^{\mathrm{aff}}$. Suppose that T_w^* for $w \in W(1)$ acts on $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ as in (1). The braid relations obviously hold. The quadratic relations hold because T_s^* with $s \in {}_1S^{\mathrm{aff}}$, acts trivially either on $e(\mathcal{V})$ or on $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$. Indeed, ${}_1S^{\mathrm{aff}} = {}_1S_M^{\mathrm{aff}} \cup {}_1S_{M_2}^{\mathrm{aff}}$, T_s^* for $s \in {}_1S_M^{\mathrm{aff}}$, acts trivially on $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ which is extended from an $\mathcal{H}_{M_2}$-module (Example 3.14), and T_s^* for $s \in {}_1S_{M_2}^{\mathrm{aff}}$, acts trivially on $e(\mathcal{V})$ which is extended from an $\mathcal{H}_M$-module. This proves (1).

We describe now the action of T_w instead of T_w^* on the $\mathcal{H}$-module $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$. Let $w \in W(1)$. We write $w = uw_{M'}w_{M_2'} = uw_{M_2'}w_{M'}$ with $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M_2'} \in {}_1W_{M_2'}$. We have $\ell(w) = \ell(w_{M'}) + \ell(w_{M_2'})$ hence $T_w = T_uT_{w_{M'}}T_{w_{M_2'}}$.

For $w = u$, we have $T_u = T_u^*$ and $(v \otimes f)T_u = (v \otimes f)T_u^* = vT_u^* \otimes fT_u^* = vT_u \otimes fT_u$.

For $w = w_{M'}$, $(v \otimes f)T_w^* = vT_w^* \otimes f$; for $s \in {}_1S_M^{\mathrm{aff}}$, $c_s = \sum_{t \in Z_k \cap {}_1W_{M'}} c_s(t)T_t^*$ in particular, we have $(v \otimes f)T_s = (v \otimes f)(T_s^* + c_s) = v(T_s^* + c_s) \otimes f = vT_s \otimes f$. Hence $(v \otimes f)T_w = vT_w \otimes f$.

For $w = w_{M_2'}$, we have similarly $(v \otimes f)T_w^* = v \otimes fT_w^*$ and $(v \otimes f)T_w = v \otimes fT_w$. $\square$

Example 3.16. Let $\mathcal{X}$ be a right $\mathcal{H}_R$-module. Then $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$ where the T_w^* acts diagonally is an $\mathcal{H}_R$-module isomorphic to $\mathcal{X}$. But the action of the T_w on $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$ is not diagonal.

It is known [Ly15] that $(\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}}$ and $(\mathrm{St}_Q^G)^{\mathcal{U}}$ are free R-modules and that $(\mathrm{St}_Q^G)^{\mathcal{U}}$ is the cokernel of the natural $\mathcal{H}_R$-map

$$\bigoplus_{Q \subsetneq Q'} (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \to (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \tag{3.22}$$

although the invariant functor $(-)^{\mathcal{U}}$ is only left exact.

Corollary 3.17. *The diagonal action of T_w^* for $w \in W(1)$ on $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ defines a structure of a right $\mathcal{H}_R$-module satisfying Proposition* 3.15(2).

4. Hecke module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$

4.1. **Case $\mathcal{V}$ extensible to $\mathcal{H}$.** Let $P = MN$ be a standard parabolic subgroup of G such that Δ_P and $\Delta \backslash \Delta_P$ are orthogonal, let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module extensible to $\mathcal{H}_R$ of extension $e(\mathcal{V})$, and let Q be a parabolic subgroup of G containing P. As Q and M_Q determine each other: $Q = M_Q U$, we denote also $\mathcal{H}_{M_Q} = \mathcal{H}_Q$ and $\mathcal{H}_{M_Q,R} = \mathcal{H}_{Q,R}$ when $Q \neq P, G$. When $Q = G$ we drop G and we denote $e_{\mathcal{H}}(\mathcal{V}) = e(\mathcal{V})$.

Lemma 4.1. *$\mathcal{V}$ is extensible to an $\mathcal{H}_{Q,R}$-module $e_{\mathcal{H}_Q}(\mathcal{V})$.*

Proof. This is straightforward. By Corollary 3.9, $\mathcal{V}$ extensible to $\mathcal{H}$ means that $T^M(z)$ acts trivially on $\mathcal{V}$ for all $z \in \mathcal{N}_{M_2'} \cap Z$. We have $M_Q = M M_{2,Q}'$ with $M_{2,Q}' \subset M_Q \cap M_2'$ and $\mathcal{N}_{M_{2,Q}'} \subset \mathcal{N}_{M_2'}$; hence $T^M(z)$ acts trivially on $\mathcal{V}$ for all $z \in \mathcal{N}_{M_{2,Q}'} \cap Z$ meaning that $\mathcal{V}$ is extensible to $\mathcal{H}_Q$. □

Remark 4.2. We cannot say that $e_{\mathcal{H}_Q}(\mathcal{V})$ is extensible to $\mathcal{H}$ of extension $e(\mathcal{V})$ when the set of roots Δ_Q and $\Delta \setminus \Delta_Q$ are not orthogonal (Definition 3.8).

Let Q' be an arbitrary parabolic subgroup of G containing Q. We are going to define an $\mathcal{H}_R$-embedding $\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+},\theta} \mathcal{H}$ defining an $\mathcal{H}_R$-homomorphism

$$\oplus_{Q \subsetneq Q' \subset G} \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

of cokernel isomorphic to $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$. In the extreme case $(Q, Q') = (P, G)$, the $\mathcal{H}_R$-embedding $e(\mathcal{V}) \xrightarrow{\iota(P,G)} \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$ is given in the following lemma where f_G and $f_{P\mathcal{U}} \in (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ denote the characteristic functions of G and $P\mathcal{U}$, $f_G = f_{P\mathcal{U}} e_{M_2}$ (see (3.20)).

Lemma 4.3. *There is a natural $\mathcal{H}_R$-isomorphism*

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{P\mathcal{U}} : \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}},$$

and compatible $\mathcal{H}_R$-embeddings

$$v \mapsto v \otimes f_G : e(\mathcal{V}) \to e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}, \tag{4.1}$$

$$v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P,G)} \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}). \tag{4.2}$$

Proof. We show first that the map

$$v \mapsto v \otimes f_{P\mathcal{U}} : \mathcal{V} \to e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}} \tag{4.3}$$

is $\mathcal{H}_{M^+}$-equivariant. Let $w \in W_{M^+}(1)$. We write $w = uw_{M'}w_{M_2'}$ as in Proposition 3.15 (2), so that $f_{P\mathcal{U}}T_w = f_{P\mathcal{U}}T_{uw_{M_2'}}$. We have $f_{P\mathcal{U}}T_{uw_{M_2'}} = f_{P\mathcal{U}}$ because ${}_1W_{M'} \subset W_{M^+}(1) \cap W_{M^-}(1)$ hence $uw_{M_2'} = ww_{M'}^{-1} \in W_{M^+}(1)$ and in $\mathbf{1}_{\mathcal{H}_M} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ we have $(1 \otimes 1_{\mathcal{H}})T_{uw_{M_2'}} = 1T^M_{uw_{M_2'}} \otimes 1_{\mathcal{H}}$, and $T^M_{uw_{M_2'}}$ acts trivially in $\mathbf{1}_{\mathcal{H}_M}$ because $\ell_M(uw_{M_2'}) = 0$. We deduce $(v \otimes f_{P\mathcal{U}})T_w = vT_w \otimes f_{P\mathcal{U}}T_w = vT^M_w \otimes f_{P\mathcal{U}}$.

By adjunction (4.3) gives an $\mathcal{H}_R$-equivariant linear map

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{P\mathcal{U}} : \mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}. \tag{4.4}$$

We prove that κ_P is an isomorphism. Recalling $\hat{d} \in \mathcal{N} \cap M_2'$, $\tilde{d} \in {}_1W_{M_2'}$ lift d, one knows that

$$\mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes T_{\tilde{d}}, \quad e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes f_{P\hat{d}\mathcal{U}}, \tag{4.5}$$

where each summand is isomorphic to $\mathcal{V}$. The left equality follows from section 4.1 and Remark 3.7 in [Vig15b] recalling that $w \in \mathbb{W}_{M_2}$ is of minimal length in its coset $\mathbb{W}_M w = w\mathbb{W}_M$ as Δ_M and Δ_{M_2} are orthogonal; for the second equality see section 3.4 (3.19). We have $\kappa_P(v \otimes T_{\tilde{d}}) = (v \otimes f_{P\mathcal{U}})T_{\tilde{d}} = v \otimes f_{P\mathcal{U}}T_{\tilde{d}}$ (Proposition 3.15). Hence κ_P is an isomorphism.

We consider the composite map

$$v \mapsto v \otimes 1 \mapsto v \otimes f_{P\mathcal{U}}e_{M_2} : e(\mathcal{V}) \to e(\mathcal{V}) \otimes_R \mathbf{1}_{\mathcal{H}} \to e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}},$$

where the right map is the tensor product $e(\mathcal{V}) \otimes_R -$ of the $\mathcal{H}_R$-equivariant embedding $\mathbf{1}_{\mathcal{H}} \to (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ sending 1_R to $f_{P\mathcal{U}}e_{M_2}$ (Lemma 3.10); this map is injective because $(\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}/\mathbf{1}$ is a free R-module; it is $\mathcal{H}_R$-equivariant for the diagonal action of the T_w^* on the tensor products (Example 3.16 for the first map). By compatibility with (4.4), we get the $\mathcal{H}_R$-equivariant embedding $v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P,G)} \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$. □

For a general (Q, Q') the $\mathcal{H}_R$-embedding $\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$ is given in the next proposition generalizing Lemma 4.3. The element e_{M_2} of $\mathcal{H}_R$ appearing in the definition of $\iota(P, G)$ is replaced in the definition of $\iota(Q, Q')$ by an element $\theta_{Q'}(e_Q^{Q'}) \in \mathcal{H}_R$ that we define first.

Until the end of section 4, we fix an admissible lift $w \mapsto \hat{w} : \mathbb{W} \to \mathcal{N} \cap \mathcal{K}$ (Definition 3.1) and $\tilde{w}$ denotes the image of $\hat{w}$ in $W(1)$. We denote $\mathbb{W}_{M_Q} = \mathbb{W}_Q$ and by ${}^{\mathbb{W}_Q}\mathbb{W}$ the set of $w \in \mathbb{W}$ of minimal length in their coset $\mathbb{W}_Q w$. The group G is the disjoint union of $Q\hat{d}\mathcal{U}$ for d running through ${}^{\mathbb{W}_Q}\mathbb{W}$ [OV17, Lemma 2.15 (2)]: $G = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}} Q\hat{d}\mathcal{U}$. Since $Q\hat{d}\mathcal{U} \subset Q'\mathcal{U}$ if and only if $\hat{d} \in Q'$, namely $d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}$, we have

$$Q'\mathcal{U} = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} Q\hat{d}\mathcal{U}. \tag{4.6}$$

Set

$$e_Q^{Q'} = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} T_{\tilde{d}}^{M_{Q'}}. \tag{4.7}$$

We write $e_Q^G = e_Q$. We have $e_P^Q = \sum_{d \in \mathbb{W}_{M_{2,Q}}} T_{\tilde{d}}^{M_Q}$.

Remark 4.4. Note that ${}^{\mathbb{W}_M}\mathbb{W} = \mathbb{W}_{M_2}$ and $e_P = e_{M_2}$, where M_2 is the standard Levi subgroup of G with $\Delta_{M_2} = \Delta \setminus \Delta_M$, as Δ_M and $\Delta \setminus \Delta_M$ are orthogonal. More generally, ${}^{\mathbb{W}_Q}\mathbb{W}_{M_{Q'}} = {}^{\mathbb{W}_{M_{2,Q}}}\mathbb{W}_{M_{2,Q'}}$ where $M_{2,Q'} = M_2 \cap M_{Q'}$.

Note that $e_Q^{Q'} \in \mathcal{H}_{M^+} \cap \mathcal{H}_{M^-}$. We consider the linear map

$$\theta_Q^{Q'} : \mathcal{H}_Q \to \mathcal{H}_{Q'} \quad T_w^{M_Q} \mapsto T_w^{M_{Q'}} \quad (w \in W_{M_Q}(1)).$$

We write $\theta_Q^G = \theta_Q$ so that $\theta_Q(T_w^{M_Q}) = T_w$. When $Q = P$ this is the map θ defined earlier. Similarly we denote by $\theta_Q^{Q',*}$ the linear map sending the $T_w^{M_Q,*}$ to $T_w^{M_{Q'},*}$ and $\theta_Q^{G,*} = \theta_Q^*$. We have

$$\theta_{Q'}(e_Q^{Q'}) = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} T_{\tilde{d}}, \quad \theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q)\theta_{Q'}(e_Q^{Q'}). \tag{4.8}$$

Proposition 4.5. *There exists an $\mathcal{H}_R$-isomorphism*

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{Q\mathcal{U}} : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+},\theta} \mathcal{H} \xrightarrow{\kappa_Q} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}, \tag{4.9}$$

and compatible $\mathcal{H}_R$-embeddings

$$v \otimes f_{Q'\mathcal{U}} \mapsto v \otimes f_{Q'\mathcal{U}} : e_{\mathcal{H}_{Q'}}(\mathcal{V}) \otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \to e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}, \tag{4.10}$$

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}(e_Q^{Q'}) : \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})). \tag{4.11}$$

Proof. We have the $\mathcal{H}_{M_Q,R}$-embedding

$$v \mapsto v \otimes e_P^Q : e_{\mathcal{H}_Q}(\mathcal{V}) \to \mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}_Q = \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})$$

by Lemma 4.3 (4.2) as Δ_M is orthogonal to $\Delta_{M_Q} \setminus \Delta_M$. Applying the parabolic induction which is exact, we get the $\mathcal{H}$-embedding

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes e_P^Q \otimes 1_{\mathcal{H}} : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})).$$

Note that $T_{\tilde{d}}^{M_Q} \in \mathcal{H}_{M_Q^+}$ for $d \in \mathbb{W}_{M_Q}$. By transitivity of the parabolic induction, it is equal to the $\mathcal{H}_R$-embedding

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_Q(e_P^Q) : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}). \tag{4.12}$$

On the other hand we have the $\mathcal{H}_R$-embedding

$$v \otimes f_{Q\mathcal{U}} \mapsto v \otimes \theta_Q(e_P^Q) : e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \to \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) \tag{4.13}$$

given by the restriction to $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ of the $\mathcal{H}_R$-isomorphism given in Lemma 4.3 (4.1), from $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_P^G \mathbf{1})^{\mathcal{U}}$ to $\mathcal{V} \otimes_{\mathcal{H}_{M^+},\theta} \mathcal{H}$ sending $v \otimes f_{P\mathcal{U}}$ to $v \otimes 1_{\mathcal{H}}$, noting that $v \otimes f_{Q\mathcal{U}} = (v \otimes f_{P\mathcal{U}})\theta_Q(e_P^Q)$ by Proposition 3.15, $f_{Q\mathcal{U}} = f_{P\mathcal{U}}\theta_Q(e_P^Q)$ and $\theta_Q(e_P^Q)$ acts trivially on $e(\mathcal{V})$ (this is true for $T_{\tilde{d}}$ for $\tilde{d} \in {}_1W_{M_2'}$). Comparing the embeddings (4.12) and (4.13), we get the $\mathcal{H}_R$-isomorphism (4.9).

We can replace Q by Q' in the $\mathcal{H}_R$-homomorphisms (4.9), (4.12), and (4.13). With (4.12) we see $\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ and $\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$ as $\mathcal{H}_R$-submodules of $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$. As seen in (4.8) we have $\theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q)\theta_{Q'}(e_Q^{Q'})$. We deduce the $\mathcal{H}_R$-embedding (4.11).

By (3.19) for Q and (4.6),

$$f_{Q'\mathcal{U}} = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} f_{Q\mathcal{U}} T_{\tilde{d}} = f_{Q\mathcal{U}} \theta_{Q'}(e_Q^{Q'})$$

in $(\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$. We deduce that the $\mathcal{H}_R$-embedding corresponding to (4.11) via κ_Q and $\kappa_{Q'}$ is the $\mathcal{H}_R$-embedding (4.10). □

We recall that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal and that $\mathcal{V}$ is extensible to $\mathcal{H}$ of extension $e(\mathcal{V})$.

Corollary 4.6. *The cokernel of the $\mathcal{H}_R$-map*

$$\bigoplus_{Q \subsetneq Q' \subset G} \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the $\iota(Q, Q')$, is isomorphic to $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ via κ_Q.

4.2. **Invariants in the tensor product.** We return to the setting where $P = MN$ is a standard parabolic subgroup of G, σ is a smooth R-representation of M with $P(\sigma) = G$ of extension $e(\sigma)$ to G, and Q a parabolic subgroup of G containing P. We still assume that Δ_P and $\Delta \setminus \Delta_P$ are orthogonal.

The $\mathcal{H}_R$-modules $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$ are equal (Theorem 3.13). We compute $I_G(P, \sigma, Q)^{\mathcal{U}} = (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$.

Theorem 4.7. *The natural linear maps $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \to (e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ and $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}} \to (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ are isomorphisms.*

Proof. We need some preliminaries. In [GK14, Ly15], are introduced a finite free $\mathbb{Z}$-module $\mathfrak{M}$ (depending on Δ_Q) and a $\mathcal{B}$-equivariant embedding $\mathrm{St}_Q^G\mathbb{Z} \xrightarrow{\iota} C_c^\infty(\mathcal{B}, \mathfrak{M})$ (we indicate the coefficient ring in the Steinberg representation) which induces an isomorphism $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}} \simeq C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$.

Lemma 4.8.

(1) $(\mathrm{Ind}_Q^G \mathbb{Z})^{\mathcal{B}}$ *is a direct factor of* $\mathrm{Ind}_Q^G \mathbb{Z}$.

(2) $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}$ *is a direct factor of* $\mathrm{St}_Q^G\mathbb{Z}$.

Proof.

(1) [AHV, Example 2.2].

(2) As $\mathfrak{M}$ is a free $\mathbb{Z}$-module, $C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$ is a direct factor of $C_c^\infty(\mathcal{B}, \mathfrak{M})$. Consequently, $\iota((\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}) = C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$ is a direct factor of $\iota(\mathrm{St}_Q^G\mathbb{Z})$. As ι is injective, we get (2). □

We now prove Theorem 4.7. We may and do assume that σ is e-minimal (because $P(\sigma) = P(\sigma_{\min}), e(\sigma) = e(\sigma_{\min})$) so that Δ_M and $\Delta \setminus \Delta_M$ are orthogonal and we use the same notation as in section 3.2 in particular $M_2 = M_{\Delta \setminus \Delta_M}$. Let V be the space of $e(\sigma)$ on which M_2' acts trivially. The restriction of $\mathrm{Ind}_Q^G \mathbb{Z}$ to M_2 is $\mathrm{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z}$, that of $\mathrm{St}_Q^G\mathbb{Z}$ is $\mathrm{St}_{Q \cap M_2}^{M_2}\mathbb{Z}$.

As in [AHV, Example 2.2], $((\mathrm{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z}) \otimes V)^{\mathcal{U}_{M_2'}} \simeq (\mathrm{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2'}} \otimes V$. We have

$$(\mathrm{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2'}} = (\mathrm{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}} = (\mathrm{Ind}_Q^G \mathbb{Z})^{\mathcal{U}}.$$

The first equality follows from $M_2 = (Q \cap M_2)\mathbb{W}_{M_2}\mathcal{U}_{M_2}$, $\mathcal{U}_{M_2} = Z^1\mathcal{U}_{M_2'}$ and Z^1 normalizes $\mathcal{U}_{M_2'}$ and is normalized by $\mathbb{W}_{M_2}$. The second equality follows from $\mathcal{U} =$

$\mathcal{U}_{M'}\mathcal{U}_{M_2}$ and $\mathrm{Ind}_Q^G\,\mathbb{Z}$ is trivial on M'. Therefore $((\mathrm{Ind}_Q^G\,\mathbb{Z})\otimes V)^{\mathcal{U}_{M_2'}} \simeq (\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V$. Now taking fixed points under $\mathcal{U}_M$, as $\mathcal{U}=\mathcal{U}_{M_2'}\mathcal{U}_M$,

$$((\mathrm{Ind}_Q^G\,\mathbb{Z})\otimes V)^{\mathcal{U}} \simeq ((\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V)^{\mathcal{U}_M} = (\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}_M}.$$

The equality uses that the $\mathbb{Z}$-module $\mathrm{Ind}_Q^G\,\mathbb{Z}$ is free. We get the first part of the theorem as $(\mathrm{Ind}_Q^G\,\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}_M} \simeq (\mathrm{Ind}_Q^G\,R)^{\mathcal{U}}\otimes_R V^{\mathcal{U}_M}$.

Tensoring with R the usual exact sequence defining $\mathrm{St}_Q^G\mathbb{Z}$ gives an isomorphism $\mathrm{St}_Q^G\mathbb{Z}\otimes R\simeq \mathrm{St}_Q^G R$ and in [GK14,Ly15], it is proved that the resulting map $\mathrm{St}_Q^G R \xrightarrow{\iota_R} C^\infty(\mathcal{B},\mathfrak{M}\otimes R)$ is also injective. Their proof in no way uses the ring structure of R, and for any $\mathbb{Z}$-module V, tensoring with V gives a $\mathcal{B}$-equivariant embedding $\mathrm{St}_Q^G\mathbb{Z}\otimes V \xrightarrow{\iota_V} C_c^\infty(\mathcal{B},\mathfrak{M}\otimes V)$. The natural map $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}\otimes V\to \mathrm{St}_Q^G\mathbb{Z}\otimes V$ is also injective by Lemma 4.8 (2). Taking $\mathcal{B}$-fixed points we get inclusions

$$(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}\otimes V \to (\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{B}} \to C_c^\infty(\mathcal{B},\mathfrak{M}\otimes V)^{\mathcal{B}} \simeq \mathfrak{M}\otimes V. \tag{4.14}$$

The composite map is surjective, so the inclusions are isomorphisms. The image of ι_V consists of functions which are left Z^0-invariant, and $\mathcal{B}=Z^0\mathcal{U}'$ where $\mathcal{U}'=G'\cap\mathcal{U}$. It follows that ι yields an isomorphism $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}\simeq C_c^\infty(Z^0\backslash\mathcal{B},\mathfrak{M})^{\mathcal{U}'}$ again consisting of the constant functions. So that in particular $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}=(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}$ and reasoning as previously we get isomorphisms

$$(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}\otimes V \simeq (\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{U}'} \simeq \mathfrak{M}\otimes V. \tag{4.15}$$

The equality $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}'}=(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{B}}$ and the isomorphisms remain true when we replace $\mathcal{U}'$ by any group between $\mathcal{B}$ and $\mathcal{U}'$. We apply these results to $\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z}\otimes V$ to get that the natural map $(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2'}}\otimes V\to(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z}\otimes V)^{\mathcal{U}_{M_2'}}$ is an isomorphism and also that $(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2'}}=(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2}}$. We have $\mathcal{U}=\mathcal{U}_{M'}\mathcal{U}_{M_2}$ so $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}=(\mathrm{St}_{Q\cap M_2}^{M_2}\mathbb{Z})^{\mathcal{U}_{M_2}}$ and the natural map $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}\otimes V\to(\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{U}_{M_2'}}$ is an isomorphism. The $\mathbb{Z}$-module $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}$ is free and the $V^{\mathcal{U}_M}=V^{\mathcal{U}}$, so taking fixed points under $\mathcal{U}_M$, we get $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}}\simeq(\mathrm{St}_Q^G\mathbb{Z}\otimes V)^{\mathcal{U}}$. We have $\mathrm{St}_Q^G\mathbb{Z}\otimes V=\mathrm{St}_Q^G R\otimes_R V$ and $(\mathrm{St}_Q^G\mathbb{Z})^{\mathcal{U}}\otimes V^{\mathcal{U}}=(\mathrm{St}_Q^G R)^{\mathcal{U}}\otimes_R V^{\mathcal{U}}$. This ends the proof of the theorem. □

Theorem 4.9. *The $\mathcal{H}_R$-modules $(e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}=e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ are equal. The $\mathcal{H}_R$-modules $(e(\sigma)\otimes_R\mathrm{St}_Q^G)^{\mathcal{U}}=e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{St}_Q^G)^{\mathcal{U}}$ are also equal.*

Proof. We already know that the R-modules are equal (Theorem 4.7). We show that they are equal as $\mathcal{H}$-modules. The $\mathcal{H}_R$-modules $e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}=e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ are equal (Theorem 3.13), they are isomorphic to $\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\sigma^{\mathcal{U}_M}))$ (Proposition 4.5), to $(\mathrm{Ind}_Q^G(e_Q(\sigma)))^{\mathcal{U}}$ [OV17, Proposition 4.4], and to $(e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ [AHV, Lemma 2.5]). We deduce that the $\mathcal{H}_R$-modules $e(\sigma)^{\mathcal{U}}\otimes_R(\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}=(e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1})^{\mathcal{U}}$ are equal. The same is true when Q is replaced by a parabolic subgroup Q' of G containing Q. The representation $e(\sigma)\otimes_R\mathrm{St}_Q^G$ is the cokernel of the natural $R[G]$-map

$$\bigoplus_{Q\subsetneq Q'} e(\sigma)\otimes_R\mathrm{Ind}_{Q'}^G\,\mathbf{1}\xrightarrow{\alpha_Q} e(\sigma)\otimes_R\mathrm{Ind}_Q^G\,\mathbf{1}$$

and the $\mathcal{H}_R$-module $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ is the cokernel of the natural $\mathcal{H}_R$-map

$$\bigoplus_{Q \subsetneq Q'} e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \xrightarrow{\beta_Q} e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$$

obtained by tensoring (3.22) by $e(\sigma)^{\mathcal{U}}$ over R, because the tensor product is right exact. The maps $\beta_Q = \alpha_Q^{\mathcal{U}}$ are equal and the R-modules $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}} = (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ are equal. This implies that the $\mathcal{H}_R$-modules $e(\sigma)^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}} = (e(\sigma) \otimes_R \mathrm{St}_Q^G)^{\mathcal{U}}$ are equal. □

Remark 4.10. The proof shows that the representations $e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$ and $e(\sigma) \otimes \mathrm{St}_Q^G$ of G are generated by their $\mathcal{U}$-fixed vectors if the representation σ of M is generated by its $\mathcal{U}_M$-fixed vectors. Indeed, the R-modules $e(\sigma)^{\mathcal{U}} = \sigma^{\mathcal{U}_M}, (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}_{M_2'}} = (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ are equal. If $\sigma^{\mathcal{U}_M}$ generates σ, then $e(\sigma)$ is generated by $e(\sigma)^{\mathcal{U}}$. The representation $\mathrm{Ind}_Q^G \mathbf{1}|_{M_2'}$ is generated by $(\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ (this follows from the lemma below), we have $G = MM_2'$ and M_2' acts trivially on $e(\sigma)$. Therefore the $R[G]$-module generated by $\sigma^{\mathcal{U}} \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ is $e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$. As $e(\sigma) \otimes_R \mathrm{St}_Q^G$ is a quotient of $e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$, the $R[G]$-module generated by $\sigma^{\mathcal{U}} \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ is $e(\sigma) \otimes_R \mathrm{St}_Q^G$.

Lemma 4.11. *For any standard parabolic subgroup P of G, the representation $\mathrm{Ind}_P^G \mathbf{1}|_{G'}$ is generated by its $\mathcal{U}$-fixed vectors.*

Proof. Because $G = PG'$ it suffices to prove that if J is an open compact subgroup of $\overline{N}$ the characteristic function 1_{PJ} of PJ is a finite sum of translates of $1_{P\mathcal{U}} = 1_{P\mathcal{U}_{\overline{N}}}$ by G'. For $t \in T$ we have $P\mathcal{U}t = Pt^{-1}\mathcal{U}_{\overline{N}}t$ and we can choose $t \in T \cap J'$ such that $t^{-1}\mathcal{U}_{\overline{N}}t \subset J$. □

4.3. **General triples.** Let $P = MN$ be a standard parabolic subgroup of G. We now investigate situations where Δ_P and $\Delta \setminus \Delta_P$ are not necessarily orthogonal. Let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module.

Definition 4.12. Let $P(\mathcal{V}) = M(\mathcal{V})N(\mathcal{V})$ be the standard parabolic subgroup of G with $\Delta_{P(\mathcal{V})} = \Delta_P \cup \Delta_{\mathcal{V}}$ and

$$\Delta_{\mathcal{V}} = \{\alpha \in \Delta \ \text{ orthogonal to } \Delta_M,\ T^M(z) \text{ acts trivially on } \mathcal{V} \text{ for all } z \in Z \cap M_\alpha'\}.$$

If Q is a parabolic subgroup of G between P and $P(\mathcal{V})$, the triple $(P, \mathcal{V}, Q)$ called an $\mathcal{H}_R$-triple, defines a right $\mathcal{H}_R$-module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ equal to

$$\mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}} (e(\mathcal{V}) \otimes_R (\mathrm{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) = (e(\mathcal{V}) \otimes_R (\mathrm{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V})^+,R},\theta} \mathcal{H}_R,$$

where $e(\mathcal{V})$ is the extension of $\mathcal{V}$ to $\mathcal{H}_{M(\mathcal{V})}$.

This definition is justified by the fact that $M(\mathcal{V})$ is the maximal standard Levi subgroup of G such that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}_{M(\mathcal{V})}$.

Lemma 4.13. *$\Delta_{\mathcal{V}}$ is the maximal subset of $\Delta \setminus \Delta_P$ orthogonal to Δ_P such that $T_\lambda^{M,*}$ acts trivially on $\mathcal{V}$ for all $\lambda \in \Lambda(1) \cap {}_1W_{M_{\mathcal{V}}'}$.*

Proof. For $J \subset \Delta$ let M_J denote the standard Levi subgroup of G with $\Delta_{M_J} = J$. The group $Z \cap M_J'$ is generated by the $Z \cap M_\alpha'$ for all $\alpha \in J$ (Lemma 2.1). When J is orthogonal to Δ_M and $\lambda \in \Lambda_{M_J'}(1)$, $\ell_M(\lambda) = 0$ where ℓ_M is the length associated to S_M^{aff}, and the map $\lambda \mapsto T_\lambda^{M,*} = T_\lambda^M : \Lambda_{M_J'}(1) \to \mathcal{H}_M$ is multiplicative. □

The following is the natural generalization of Proposition 4.5 and Corollary 4.6. Let Q' be a parabolic subgroup of G with $Q \subset Q' \subset P(\mathcal{V})$. Applying the results of section 4.1 to $M(\mathcal{V})$ and its standard parabolic subgroups $Q \cap M(\mathcal{V}) \subset Q' \cap M(\mathcal{V})$, we have an $\mathcal{H}_{M(\mathcal{V}),R}$-isomorphism

$$\begin{aligned}
&\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V})) \\
&\quad = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+},\theta} \mathcal{H}_{M(\mathcal{V}),R} \xrightarrow{\kappa_{Q\cap M(\mathcal{V})}} e(\mathcal{V}) \otimes_R (\mathrm{Ind}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1})^{\mathcal{U}_{M(\mathcal{V})}} \\
&\qquad\qquad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{Q\mathcal{U}\cap M(\mathcal{V})} :
\end{aligned}$$

and an $\mathcal{H}_{M(\mathcal{V}),R}$-embedding

$$\begin{aligned}
\mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) &\xrightarrow{\iota(Q\cap M(\mathcal{V}),Q'\cap M(\mathcal{V}))} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V})) \\
v \otimes 1_{\mathcal{H}_{M(\mathcal{V})}} &\mapsto v \otimes \theta_{Q'}^{P(\mathcal{V})}(e_Q^{Q'}).
\end{aligned}$$

Applying the parabolic induction $\mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}$ which is exact and transitive, we obtain an $\mathcal{H}_R$-isomorphism $\kappa_Q = \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\kappa_{Q\cap M(\mathcal{V})})$,

$$\begin{aligned}
\mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) &\xrightarrow{\kappa_Q} \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\mathrm{Ind}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1}_{M_Q})^{\mathcal{U}_{M(\mathcal{V})}}) \\
v \otimes 1_{\mathcal{H}} &\mapsto v \otimes f_{Q\mathcal{U}_{M(\mathcal{V})}} \otimes 1_{\mathcal{H}}
\end{aligned} \tag{4.16}$$

and an $\mathcal{H}_R$-embedding $\iota(Q,Q') = \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\iota(Q,Q')^{M(\mathcal{V})})$

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}(e_Q^{Q'}) : \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})). \tag{4.17}$$

Applying Corollary 4.6 we obtain:

Theorem 4.14. *Let $(P,\mathcal{V},Q)$ be an $\mathcal{H}_R$-triple. Then, the cokernel of the $\mathcal{H}_R$-map*

$$\oplus_{Q\subsetneq Q'\subset P(\mathcal{V})} \mathrm{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})),$$

defined by the $\iota(Q,Q')$ is isomorphic to $I_{\mathcal{H}}(P,\mathcal{V},Q)$ via the $\mathcal{H}_R$-isomorphism κ_Q.

Let σ be a smooth R-representation of M and let Q be a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$.

Remark 4.15. The $\mathcal{H}_R$-module $I_{\mathcal{H}}(P,\sigma^{\mathcal{U}_M},Q)$ is defined if $\Delta_Q \setminus \Delta_P$ and Δ_P are orthogonal because $Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_M})$ (Theorem 3.13).

We denote here by $P_{\min} = M_{\min}N_{\min}$ the minimal standard parabolic subgroup of G contained in P such that $\sigma = e_P(\sigma|_{M_{\min}})$ (Lemma 2.3, we drop the index σ). The sets of roots $\Delta_{P_{\min}}$ and $\Delta_{P(\sigma|_{M_{\min}})} \setminus \Delta_{P_{\min}}$ are orthogonal (Lemma 2.4). The groups $P(\sigma) = P(\sigma|_{M_{\min}})$, the representations $e(\sigma) = e(\sigma|_{M_{\min}})$ of $M(\sigma)$, the representations $I_G(P,\sigma,Q) = I_G(P_{\min},\sigma|_{M_{\min}},Q) = \mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \mathrm{St}_Q^{P(\sigma)})$ of G, and the R-modules $\sigma^{\mathcal{U}_{M_{\min}}} = \sigma^{\mathcal{U}_M}$ are equal. From Theorem 3.13,

$$P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}}), \quad e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) = e(\sigma)^{\mathcal{U}_{M(\sigma)}},$$

and $P(\sigma^{\mathcal{U}_{M_{\min}}}) = P(\sigma)$ if $\sigma^{\mathcal{U}_{M_{\min}}}$ generates the representation $\sigma|_{M_{\min}}$. The $\mathcal{H}_R$-module

$$I_{\mathcal{H}}(P_{\min},\sigma^{\mathcal{U}_{M_{\min}}},Q) = \mathrm{Ind}_{\mathcal{H}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}}^{\mathcal{H}}(e(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_Q^{P(\sigma^{\mathcal{U}_{M_{\min}}})})^{\mathcal{U}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}})$$

is defined because $\Delta_{P_{\min}}$ and $\Delta_{P(\sigma^{\mathcal{U}_{M_{\min}}})} \setminus \Delta_{P_{\min}}$ are orthogonal and $P \subset Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}})$.

Remark 4.16. If $\sigma^{\mathcal{U}_{M_{\min}}}$ generates the representation $\sigma|_{M_{\min}}$ (in particular if R is an algebraically closed field of characteristic p and σ is irrreducible), then $P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}})$ hence

$$I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) = \mathrm{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

Applying Theorem 4.9 to $(P_{\min}\cap M(\sigma), \sigma|_{M_{\min}}, Q\cap M(\sigma))$, the $\mathcal{H}_{M(\sigma),R}$-modules

$$e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}} = (e_{M(\sigma)}(\sigma) \otimes_R \mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}} \tag{4.18}$$

are equal. We have the $\mathcal{H}_R$-isomorphism [OV17, Proposition 4.4]:

$$\begin{aligned} I_G(P,\sigma,Q)^{\mathcal{U}} &= (\mathrm{Ind}_{P(\sigma)}^{G}(e(\sigma)\otimes_R \mathrm{St}_Q^{P(\sigma)}))^{\mathcal{U}} \\ &\xrightarrow{ov} \mathrm{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}((e(\sigma)\otimes_R \mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}) \\ f_{P(\sigma)\mathcal{U},x} &\mapsto x\otimes 1_{\mathcal{H}} \quad (x\in (e(\sigma)\otimes_R \mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}). \end{aligned}$$

We deduce the following.

Theorem 4.17. *Let (P,σ,Q) be an $R[G]$-triple. Then, we have the $\mathcal{H}_R$-isomorphism*

$$I_G(P,\sigma,Q)^{\mathcal{U}} \xrightarrow{ov} \mathrm{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\mathrm{St}_{Q\cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

In particular,

$$I_G(P,\sigma,Q)^{\mathcal{U}} \simeq \begin{cases} I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) & \text{if } P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}}), \\ I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q) & \text{if } P = P_{\min}, P(\sigma) = P(\sigma^{\mathcal{U}_M}). \end{cases}$$

4.4. **Comparison of the parabolic induction and coinduction.** Let $P = MN$ be a standard parabolic subgroup of G, let $\mathcal{V}$ be a right $\mathcal{H}_R$-module, and let Q be a parabolic subgroup of G with $Q \subset P(\mathcal{V})$. When R is an algebraically closed field of characteristic p, in [Abe], we associated to $(P,\mathcal{V},Q)$ an $\mathcal{H}_R$-module using the parabolic coinduction

$$\mathrm{Coind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = \mathrm{Hom}_{\mathcal{H}_{M_Q^-},\theta^*}(\mathcal{H},-) : \mathrm{Mod}_R(\mathcal{H}_{M_Q}) \to \mathrm{Mod}_R(\mathcal{H})$$

instead of the parabolic induction $\mathrm{Ind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = - \bigotimes_{\mathcal{H}_{M_Q^+},\theta} \mathcal{H}$. The index θ^* in the parabolic coinduction means that $\mathcal{H}_{M_Q^-}$ embeds in $\mathcal{H}$ by θ_Q^*. Our terminology is different from the one in [Abe] where the parabolic coinduction is called induction. For a parabolic subgroup Q' of G with $Q\subset Q'\subset P(\mathcal{V})$, there is a natural inclusion of $\mathcal{H}_R$-modules

$$\mathrm{Hom}_{\mathcal{H}_{M_{Q'}^-},\theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{i(Q,Q')} \mathrm{Hom}_{\mathcal{H}_{M_Q^-},\theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})) \tag{4.19}$$

because $\theta^*(\mathcal{H}_{M_Q^-}) \subset \theta^*(\mathcal{H}_{M_{Q'}^-})$ as $W_{M_Q^-}(1) \subset W_{M_{Q'}^-}(1)$, and $vT_w^{M_{Q'},*} = vT_w^{M_Q,*}$ for $w\in W_{M_Q^-}(1)$ and $v\in\mathcal{V}$. (This is [Abe, Proposition 4.19] when R is an algebraically closed field of characteristic p. This follows from our formulation of the extension for any R.)

Definition 4.18. Let $CI_{\mathcal{H}}(P,\mathcal{V},Q)$ denote the cokernel of the map

$$\bigoplus_{Q\subsetneq Q'\subset P(\mathcal{V})} \mathrm{Hom}_{\mathcal{H}_{M_{Q'}^-},\theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Hom}_{\mathcal{H}_{M_Q^-},\theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the $\mathcal{H}_R$-embeddings $i(Q,Q')$.

When R is an algebraically closed field of characteristic p, we showed that the $\mathcal{H}_R$-module $CI_{\mathcal{H}}(P,\mathcal{V},Q)$ is simple when $\mathcal{V}$ is simple and supersingular (Definition 4.25), and that any simple $\mathcal{H}_R$-module is of this form for an $\mathcal{H}_R$-triple $(P,\mathcal{V},Q)$ where $\mathcal{V}$ is simple and supersingular, P,Q and the isomorphism class of $\mathcal{V}$ are unique [Abe]. The aim of this section is to compare the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P,\mathcal{V},Q)$ with the $\mathcal{H}_R$-modules $CI_{\mathcal{H}}(P,\mathcal{V},Q)$ and to show that the classification is also valid with the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P,\mathcal{V},Q)$.

It is already known that a parabolically coinduced module is a parabolically induced module and vice versa [Abe, Proposition 4.15], [Vig15b, Theorem 1.8]. To make it more precise we need to introduce notation.

We lift the elements w of the finite Weyl group $\mathbb{W}$ to $\hat{w}\in\mathcal{N}_G\cap\mathcal{K}$ as in [AHHV17, IV.6], [OV17, Proposition 2.7]: they satisfy the braid relations $\hat{w}_1\hat{w}_2=(w_1w_2)\hat{}$ when $\ell(w_1)+\ell(w_2)=\ell(w_1w_2)$ and when $s\in S$, $\hat{s}$ is admissible, in particular lies in ${}_1W_{G'}$.

Let $\mathbf{w},\mathbf{w}_M,\mathbf{w}^M$ denote, respectively, the longest elements in $\mathbb{W},\mathbb{W}_M$ and $\mathbf{w}\mathbf{w}_M$. We have $\mathbf{w}=\mathbf{w}^{-1}=\mathbf{w}^M\mathbf{w}_M,\mathbf{w}_M=\mathbf{w}_M^{-1}$, $\hat{\mathbf{w}}=\hat{\mathbf{w}}^M\hat{\mathbf{w}}_M$,

$$\mathbf{w}^M(\Delta_M)=-\mathbf{w}(\Delta_M)\subset\Delta,\quad \mathbf{w}^M(\Phi^+\setminus\Phi_M^+)=\mathbf{w}(\Phi^+\setminus\Phi_M^+).$$

Let $\mathbf{w}.M$ be the standard Levi subgroup of G with $\Delta_{\mathbf{w}.M}=\mathbf{w}^M(\Delta_M)$ and $\mathbf{w}.P$ the standard parabolic subgroup of G with Levi $\mathbf{w}.M$. We have

$$\mathbf{w}.M=\hat{\mathbf{w}}^M M(\hat{\mathbf{w}}^M)^{-1}=\hat{\mathbf{w}}M(\hat{\mathbf{w}})^{-1},\quad \mathbf{w}^{\mathbf{w}.M}=\mathbf{w}_M\mathbf{w}=(\mathbf{w}^M)^{-1}.$$

The conjugation $w\mapsto\mathbf{w}^M w(\mathbf{w}^M)^{-1}$ in W gives a group isomorphism $W_M\to W_{\mathbf{w}.M}$ sending S_M^{aff} onto $S_{\mathbf{w}.M}^{\mathrm{aff}}$, respecting the finite Weyl subgroups $\mathbf{w}^M\mathbb{W}_M(\mathbf{w}^M)^{-1}=\mathbb{W}_{\mathbf{w}.M}=\mathbf{w}\mathbb{W}_M\mathbf{w}^{-1}$, and exchanging W_{M^+} and $W_{(\mathbf{w}.M)^-}=\mathbf{w}W_{M^+}\mathbf{w}^{-1}$. The conjugation by $\tilde{\mathbf{w}}^M$ restricts to a group isomorphism $W_M(1)\to W_{\mathbf{w}.M}(1)$ sending $W_{M^+}(1)$ onto $W_{(\mathbf{w}.M)^-}(1)$. The linear isomorphism

$$\mathcal{H}_M\xrightarrow{\iota(\tilde{\mathbf{w}}^M)}\mathcal{H}_{\mathbf{w}.M}\quad T_w^M\mapsto T^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}\text{ for }w\in W_M(1),\tag{4.20}$$

is a ring isomorphism between the pro-p-Iwahori Hecke rings of M and $\mathbf{w}.M$. It sends the positive part $\mathcal{H}_{M^+}$ of $\mathcal{H}_M$ onto the negative part $\mathcal{H}_{(\mathbf{w}.M)^-}$ of $\mathcal{H}_{\mathbf{w}.M}$ [Vig15b, Proposition 2.20]. We have $\tilde{\mathbf{w}}=\tilde{\mathbf{w}}_M\tilde{\mathbf{w}}^{\mathbf{w}.M}=\tilde{\mathbf{w}}^M\tilde{\mathbf{w}}_M$, $(\tilde{\mathbf{w}}^M)^{-1}=\tilde{\mathbf{w}}^{\mathbf{w}.M}t_M$ where $t_M=\tilde{\mathbf{w}}^2\tilde{\mathbf{w}}_M^{-2}\in Z_k$.

Definition 4.19. The **twist $\tilde{\mathbf{w}}^M.\mathcal{V}$ of $\mathcal{V}$ by $\tilde{\mathbf{w}}^M$** is the right $\mathcal{H}_{\mathbf{w}.M}$-module deduced from the right $\mathcal{H}_M$-module $\mathcal{V}$ by functoriality: as R-modules $\tilde{\mathbf{w}}^M.\mathcal{V}=\mathcal{V}$ and for $v\in\mathcal{V},w\in W_M(1)$ we have $vT^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}=vT_w^M$.

We can define the twist $\tilde{\mathbf{w}}^M.\mathcal{V}$ of $\mathcal{V}$ with the $T_w^{M,*}$ instead of T_w^M.

Lemma 4.20. *For $v\in\mathcal{V},w\in W_M(1)$ we have $vT^{\mathbf{w}.M,*}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}=vT_w^{M,*}$ in $\tilde{\mathbf{w}}^M.\mathcal{V}$.*

Proof. By the ring isomorphism $\mathcal{H}_M\xrightarrow{\iota(\tilde{\mathbf{w}}^M)}\mathcal{H}_{\mathbf{w}.M}$, we have $c^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M\tilde{s}(\tilde{\mathbf{w}}^M)^{-1}}=c_{\tilde{s}}^M$ when $\tilde{s}\in W_M(1)$ lifts $s\in S_M^{\mathrm{aff}}$. So the equality of the lemma is true for $w=\tilde{s}$. Apply the braid relations to get the equality for all $w\in W_M(1)$. □

We return to the $\mathcal{H}_R$-module $\mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},V)$ parabolically coinduced from $\mathcal{V}$. It has a natural direct decomposition indexed by the set $\mathbb{W}^{\mathbb{W}_M}$ of elements d in

the finite Weyl group $\mathbb{W}$ of minimal length in the coset $d\mathbb{W}_M$. Indeed it is known that the linear map

$$f \mapsto (f(T_{\tilde{d}}))_{d\in\mathbb{W}^{\mathbb{W}_M}} : \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V}) \to \bigoplus_{d\in\mathbb{W}^{\mathbb{W}_M}} \mathcal{V}$$

is an isomorphism. For $v \in \mathcal{V}$ and $d \in \mathbb{W}^{\mathbb{W}_M}$, there is a unique element

$f_{\tilde{d},v} \in \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V})$ satisfying $f(T_{\tilde{d}}) = v$ and $f(T_{\tilde{d}'}) = 0$ for $d' \in \mathbb{W}^{\mathbb{W}_M} \setminus \{d\}$.

It is known that the map $v \mapsto f_{\tilde{\mathbf{w}}^M,v} : \tilde{\mathbf{w}}^M.\mathcal{V} \to \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V})$ is $\mathcal{H}_{(\mathbf{w}.M)^+}$-equivariant: $f_{\tilde{\mathbf{w}}^M,v}T_w^{\mathbf{w}.M} = f_{\tilde{\mathbf{w}}^M,v}T_w$ for all $v \in \mathcal{V}, w \in W_{\mathbf{w}.M^+}(1)$. By adjunction, this $\mathcal{H}_{(\mathbf{w}.M)^+}$-equivariant map gives an $\mathcal{H}_R$-homomorphism from an induced module to a coinduced module:

$$v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M,v} : \tilde{\mathbf{w}}^M.\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^+},\theta} \mathcal{H} \xrightarrow{\mu_P} \mathrm{Hom}_{\mathcal{H}_{M^-},\theta^*}(\mathcal{H},\mathcal{V}). \tag{4.21}$$

This is an isomorphism [Abe, Proposition 4.15], [Vig15b, Theorem 1.8].

The naive guess that a variant μ_Q of μ_P induces an $\mathcal{H}_R$-isomorphism between the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q)$ and $CI_{\mathcal{H}}(P,\mathcal{V},Q)$ turns out to be true. The proof is the aim of the rest of this section.

The $\mathcal{H}_R$-module $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q)$ is well defined because the parabolic subgroups of G containing $\mathbf{w}.P$ and contained in $P(\tilde{\mathbf{w}}^M.\mathcal{V})$ are $\mathbf{w}.Q$ for $P \subset Q \subset P(\mathcal{V})$, as follows from Lemma 4.21.

Lemma 4.21. $\Delta_{\tilde{\mathbf{w}}^M.\mathcal{V}} = -\mathbf{w}(\Delta_{\mathcal{V}})$.

Proof. Recall that $\Delta_{\mathcal{V}}$ is the set of simple roots $\alpha \in \Delta \setminus \Delta_M$ orthogonal to Δ_M and $T^M(z)$ acts trivially on $\mathcal{V}$ for all $z \in Z \cap M'_\alpha$, and the corresponding standard parabolic subgroup $P_{\mathcal{V}} = M_{\mathcal{V}}N_{\mathcal{V}}$. The $Z \cap M'_\alpha$ for $\alpha \in \Delta_{\mathcal{V}}$ generate the group $Z \cap M'_{\mathcal{V}}$. A root $\alpha \in \Delta \setminus \Delta_M$ orthogonal to Δ_M is fixed by $\mathbf{w}_M$ so $\mathbf{w}^M(\alpha) = \mathbf{w}(\alpha)$ and

$$\hat{\mathbf{w}}^M M_{\mathcal{V}}(\hat{\mathbf{w}}^M)^{-1} = \hat{\mathbf{w}} M_{\mathcal{V}}(\hat{\mathbf{w}})^{-1}.$$

The proof of Lemma 4.21 is straightforward as $\Delta = -\mathbf{w}(\Delta)$, $\Delta_{\mathbf{w}.M} = -\mathbf{w}(\Delta_M)$. □

Before going further, we check the commutativity of the extension with the twist. As $Q = M_Q U$ and M_Q determine each other we denote $\mathbf{w}_{M_Q} = \mathbf{w}_Q, \mathbf{w}^{M_Q} = \mathbf{w}^Q$ when $Q \neq P, G$.

Lemma 4.22. $e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}) = \tilde{\mathbf{w}}^Q.e_{\mathcal{H}_Q}(\mathcal{V})$.

Proof. As R-modules $\mathcal{V} = e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}) = \tilde{\mathbf{w}}^Q.e_{\mathcal{H}_Q}(\mathcal{V})$. A direct computation shows that the Hecke element $T_w^{\mathbf{w}.Q,*}$ acts in the $\mathcal{H}_R$-module $e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V})$, by the identity if $w \in \tilde{\mathbf{w}}^Q{}_1W_{M_2'}(\mathbf{w}^Q)^{-1}$ and by $T^{M,*}_{(\tilde{\mathbf{w}}^Q)^{-1}w\tilde{\mathbf{w}}^Q}$ if $w \in \tilde{\mathbf{w}}^Q{}_1W_{M_2'}(\mathbf{w}^Q)^{-1}$ where M_2 denotes the standard Levi subgroup with $\Delta_{M_2} = \Delta_Q \setminus \Delta_P$. Whereas in the $\mathcal{H}_R$-module $\tilde{\mathbf{w}}^Q.e_{\mathcal{H}_Q}(\mathcal{V})$, the Hecke element $T_w^{\mathbf{w}.Q,*}$ acts by the identity if $w \in {}_1W_{\mathbf{w}.M_2'}$ and by $T^{M,*}_{(\tilde{\mathbf{w}}^M)^{-1}w\tilde{\mathbf{w}}^M}$ if $w \in W_{\mathbf{w}.M}(1)$. So the lemma means that

$${}_1W_{\mathbf{w}.M_2'} = \tilde{\mathbf{w}}^Q{}_1W_{M_2'}(\mathbf{w}^Q)^{-1}, \quad (\tilde{\mathbf{w}}^Q)^{-1}w\tilde{\mathbf{w}}^Q = (\tilde{\mathbf{w}}^M)^{-1}w\tilde{\mathbf{w}}^M \text{ if } w \in W_{\mathbf{w}.M}(1).$$

These properties are easily proved using that ${}_1W_{G'}$ is normal in $W(1)$ and that the sets of roots Δ_P and $\Delta_Q \setminus \Delta_P$ are orthogonal: $\mathbf{w}_Q = \mathbf{w}_{M_2}\mathbf{w}_M$, the elements $\mathbf{w}_{M_2}$ and $\mathbf{w}_M$ normalize W_M and W_{M_2}, the elements of $\mathbb{W}_{M_2}$ commutes with the elements of $\mathbb{W}_M$. □

We return to our guess. The variant μ_Q of μ_P is obtained by combining the commutativity of the extension with the twist and the isomorphism (4.21) applied to $(Q, e_{\mathcal{H}_Q}(\mathcal{V}))$ instead of $(P, \mathcal{V})$. The $\mathcal{H}_R$-isomorphism μ_Q is

$$v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M, v} : \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V})) \xrightarrow{\mu_Q} \mathrm{Hom}_{\mathcal{H}_{M_Q^-}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})). \tag{4.22}$$

Our guess is that μ_Q induces an $\mathcal{H}_R$-isomorphism from the cokernel of the $\mathcal{H}_R$-map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V})) \to \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}))$$

defined by the $\mathcal{H}_R$-embeddings $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$, isomorphic to $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M\mathcal{V}, \mathbf{w}.\overline{Q})$ via $\kappa_{\mathbf{w}.Q}$ (Theorem 4.14), onto the cokernel $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ the $\mathcal{H}_R$-map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \mathrm{Hom}_{\mathcal{H}_{M_{Q'}^-, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \to \mathrm{Hom}_{\mathcal{H}_{M_Q^-, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the $\mathcal{H}_R$-embeddings $i(Q, Q')$. This is true if $i(Q, Q')$ corresponds to $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$ via the isomorphisms $\mu_{Q'}$ and μ_Q. This is the content of the next proposition.

Proposition 4.23. *For all $Q \subsetneq Q' \subset P(\mathcal{V})$ we have*

$$i(Q, Q') \circ \mu_{Q'} = \mu_Q \circ \iota(\mathbf{w}.Q, \mathbf{w}.Q').$$

We postpone to section 4.6 the rather long proof of the proposition.

Corollary 4.24. *The $\mathcal{H}_R$-isomorphism $\mu_Q \circ \kappa_{\mathbf{w}.Q}^{-1}$ induces an $\mathcal{H}_R$-isomorphism*

$$I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M\mathcal{V}, \mathbf{w}.\overline{Q}) \to CI_{\mathcal{H}}(P, \mathcal{V}, Q).$$

4.5. **Supersingular $\mathcal{H}_R$-modules, classification of simple $\mathcal{H}_R$-modules.** We recall first the notion of supersingularity based on the action of the center of $\mathcal{H}$.

The center of $\mathcal{H}$ [Vig14, Theorem 1.3] contains a subalgebra $\mathcal{Z}_{T^+}$ isomorphic to $\mathbb{Z}[T^+/T_1]$ where T^+ is the monoid of dominant elements of T and T_1 is the pro-p-Sylow subgroup of the maximal compact subgroup of T.

Let $t \in T$ of image $\mu_t \in W(1)$ and let $(E_o(w))_{w \in W(1)}$ denote the alcove walk basis of $\mathcal{H}$ associated to a closed Weyl chamber o of $\mathbb{W}$. The element

$$E_o(C(\mu_t)) = \sum_{\mu'} E_o(\mu')$$

is the sum over the elements in μ' in the conjugacy class $C(\mu_t)$ of μ_t in $W(1)$. It is a central element of $\mathcal{H}$ and does not depend on the choice of o. We write also $z(t) = E_o(C(\mu_t))$.

Definition 4.25. A non-zero right $\mathcal{H}_R$-module $\mathcal{V}$ is called supersingular when, for any $v \in \mathcal{V}$ and any non-invertible $t \in T^+$, there exists a positive integer $n \in \mathbb{N}$ such that $v(z(t))^n = 0$. If one can choose n independent on (v, t), then $\mathcal{V}$ is called uniformly supersingular.

Remark 4.26. One can choose n independent on (v, t) when $\mathcal{V}$ is finitely generated as a right $\mathcal{H}_R$-module. If R is a field and $\mathcal{V}$ is simple we can take $n = 1$.

When G is compact modulo the center, $T^+ = T$, and any non-zero $\mathcal{H}_R$-module is supersingular.

The induction functor $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} : \mathrm{Mod}(\mathcal{H}_{M,R}) \to \mathrm{Mod}(\mathcal{H}_R)$ has a left adjoint $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}$ and a right adjoint $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}$ [Vig15b]: for $\mathcal{V} \in \mathrm{Mod}(\mathcal{H}_R)$,

(4.23)
$$\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \tilde{\mathbf{w}}^{\mathbf{w}.M} \circ (\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}), \quad \mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V}).$$

In the left adjoint, $\mathcal{V}$ is seen as a right $\mathcal{H}_{(\mathbf{w}.M)^-}$-module via the ring homomorphism $\theta_{\mathbf{w}.M}^* \colon \mathcal{H}_{(\mathbf{w}.M)^-} \to \mathcal{H}$; in the right adjoint, $\mathcal{V}$ is seen as a right $\mathcal{H}_{M^+}$-module via the ring homomorphism $\theta_M \colon \mathcal{H}_{M^+} \to \mathcal{H}$ (section 2.3).

Proposition 4.27. *Assume that $\mathcal{V}$ is a supersingular right $\mathcal{H}_R$-module and that p is nilpotent in $\mathcal{V}$. Then $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$, and if $\mathcal{V}$ is uniformly supersingular $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$.*

Proof. This is a consequence of three known properties:

(1) $\mathcal{H}_M$ is the localization of $\mathcal{H}_{M^+}$ (resp., $\mathcal{H}_{M^-}$) at T_μ^M for any element $\mu \in \Lambda_T(1)$, central in $W_M(1)$ and strictly N-positive (resp., N-negative), and $T_\mu^M = T_\mu^{M,*}$. See [Vig15b, Theorem 1.4].
(2) When o is anti-dominant, $E_o(\mu) = T_\mu$ if $\mu \in \Lambda^+(1)$ and $E_o(\mu) = T_\mu^*$ if $\mu \in \Lambda^-(1)$.
(3) Let an integer $n > 0$ and $\mu \in \Lambda(1)$ such that the $\mathbb{W}$-orbit of $v(\mu) \in X_*(T) \otimes \mathbb{Q}$ (definition in section 2.1) and of μ have the same number of elements. Then
$$(E_o(C(\mu)))^n E_o(\mu) - E_o(\mu)^{n+1} \in p\mathcal{H}.$$
See [Vig15a, Lemma 6.5], where the hypotheses are given in the proof (but not written in the lemma).

Let $\mu \in \Lambda_T^+(1)$ satisfying (1) for M^+ and (3), similarly let $\mathbf{w}.\mu \in \Lambda_T^-(1)$ satisfying (1) for $(\mathbf{w}.M)^-$ and (3). For $(R, \mathcal{V})$ as in the proposition, let $v \in \mathcal{V}$ and $n > 0$ such that $vE_o(C(\mu))^n = vE_o(C(\mathbf{w}.\mu))^n = 0$. Multiplying by $E_o(\mu)$ or $E_o(\mathbf{w}.\mu)$, and applying (3) and (2) for o anti-dominant we get:
$$vE_o(\mu^{n+1}) = vT_\mu^{n+1} \in p\mathcal{V}, \quad vE_o((\mathbf{w}.\mu)^{n+1}) = v(T_{\mathbf{w}.\mu}^*)^{n+1} \in p\mathcal{V}.$$
The proposition follows from: $vT_\mu^{n+1}, v(T_{\mathbf{w}.\mu}^*)^{n+1}$ in $p\mathcal{V}$ (as explained in [Abe16, Proposition 5.17] when $p = 0$ in R). From $v(T_{\mathbf{w}.\mu}^*)^{n+1}$ in $p\mathcal{V}$, we get $v \otimes (T_{\mathbf{w}.\mu}^{\mathbf{w}.M,*})^{n+1} = v(T_{\mathbf{w}.\mu}^*)^{n+1} \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$ in $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}$. As $T^{\mathbf{w}.M,*} = T^{\mathbf{w}.M}$ is invertible in $\mathcal{H}_{\mathbf{w}.M}$ we get $v \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$ in $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}$. As v was arbitrary, $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M} \subset p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M}$. If p is nilpotent in $\mathcal{V}$, then $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-},\theta^*} \mathcal{H}_{\mathbf{w}.M} = 0$. Suppose now that there exists $n > 0$ such that $\mathcal{V}(z(t))^n = 0$ for any non-invertible $t \in T^+$; then $\mathcal{V}T_\mu^{n+1} \subset p\mathcal{V}$ where $\mu = \mu_t$ and hence $\varphi(h) = \varphi(hT_{\mu^{-n-1}}^M)T_\mu^{n+1}$ in $p\mathcal{V}$ for an arbitrary $\varphi \in \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V})$ and an arbitrary $h \in \mathcal{H}_M$. We deduce $\mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V}) \subset \mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, p\mathcal{V})$. If p is nilpotent in $\mathcal{V}$, then $\mathrm{Hom}_{\mathcal{H}_{M^+},\theta}(\mathcal{H}_M, \mathcal{V}) = 0$. □

Recalling that $\tilde{\mathbf{w}}^M.\mathcal{V}$ is obtained by functoriality from $\mathcal{V}$ and the ring isomorphism $\iota(\tilde{\mathbf{w}}^M)$ defined in (4.20), the equivalence between $\mathcal{V}$ supersingular and $\tilde{\mathbf{w}}^M\mathcal{V}$ supersingular follows from Lemma 4.28

Lemma 4.28.

(1) *Let $t \in T$. Then t is dominant for U_M if and only if $\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1} \in T$ is dominant for $U_{\mathbf{w}.M}$.*

(2) *The R-algebra isomorphism* $\mathcal{H}_{M,R} \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w}.M,R}$, $T_w^M \mapsto T^{\mathbf{w}.M}_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}$ *for* $w \in W_M(1)$ *sends* $z^M(t)$ *to* $z^{\mathbf{w}.M}(\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1})$ *for* $t \in T$ *dominant for* U_M.

Proof. The conjugation by $\hat{\mathbf{w}}^M$ stabilizes T, sends U_M to $U_{\mathbf{w}.M}$, and sends the $\mathbb{W}_M$-orbit of $t \in T$ to the $\mathbb{W}_{\mathbf{w}.M}$-orbit of $\hat{\mathbf{w}}^M t(\hat{\mathbf{w}}^M)^{-1}$, as $\mathbf{w}^M \mathbb{W}_M (\mathbf{w}^M)^{-1} = \mathbb{W}_{\mathbf{w}.M}$. It is known that $\iota(\tilde{\mathbf{w}}^M)$ respects the anti-dominant alcove walk bases [Vig15b, Proposition 2.20]: it sends $E^M(w)$ to $E^{\mathbf{w}.M}(\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1})$ for $w \in W_M(1)$. □

We deduce the following.

Corollary 4.29. *Let $\mathcal{V}$ be a right $\mathcal{H}_{M,R}$-module. Then $\mathcal{V}$ is supersingular if and only if the right $\mathcal{H}_{\mathbf{w}.M,R}$-module $\tilde{\mathbf{w}}^M \mathcal{V}$ is supersingular.*

Assume R is an algebraically close field of characteristic p. The supersingular simple $\mathcal{H}_{M,R}$-modules are classified in [Vig15a]. By Corollaries 4.24 and 4.29, the classification of the simple $\mathcal{H}_R$-modules in [Abe] remains valid with the $\mathcal{H}_R$-modules $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ instead of $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$:

Corollary 4.30 (Classification of simple $\mathcal{H}_R$-modules)**.** *Assume R is an algebraically closed field of characteristic p. Let $(P, \mathcal{V}, Q)$ be an $\mathcal{H}_R$-triple where $\mathcal{V}$ is simple and supersingular. Then, the $\mathcal{H}_R$-module $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ is simple. A simple $\mathcal{H}_R$-module is isomorphic to $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ for an $\mathcal{H}_R$-triple $(P, \mathcal{V}, Q)$ where $\mathcal{V}$ is simple and supersingular, P, Q and the isomorphism class of $\mathcal{V}$ are unique.*

4.6. **A commutative diagram.** We prove in this section Proposition 4.23. For $Q \subset Q' \subset P(\mathcal{V})$ we show by an explicit computation that

$$\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'} \operatorname{Ind}^{\mathcal{H}}_{\mathcal{H}_{\mathbf{w}.Q'}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V})) \to \operatorname{Ind}^{\mathcal{H}}_{\mathcal{H}_{\mathbf{w}.Q}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V}))$$

is equal to $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$. The R-module $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V}) \otimes 1_{\mathcal{H}}$ generates the $\mathcal{H}_R$-module $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V}) \otimes_{\mathcal{H}_{\mathbf{w}.Q',R},\theta^+} \mathcal{H}_R = \operatorname{Ind}^{\mathcal{H}}_{\mathcal{H}_{\mathbf{w}.Q'}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V}))$ and by (4.17)

$$\iota(\mathbf{w}.Q, \mathbf{w}.Q')(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in {}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}}} T_{\tilde{d}} \tag{4.24}$$

for $v \in \mathcal{V}$ seen as an element of $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M.\mathcal{V})$ in the LHS and an element of $e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V})$ in the RHS.

Lemma 4.31. $(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in \mathbb{W}^{\mathbb{W}_{M_Q}}_{M_{Q'}}} q_d \, T^*_{\tilde{\mathbf{w}}^Q (\tilde{\mathbf{w}}^{Q'} \tilde{d})^{-1}}$.

Proof. $\mu_{Q'}(v \otimes 1_{\mathcal{H}})$ is the unique homomorphism $f_{\tilde{\mathbf{w}}^{M_{Q'}}, v} \in \operatorname{Hom}_{\mathcal{H}_{M^-_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ sending $T_{\tilde{\mathbf{w}}^{Q'}}$ to v and vanishing on $T_{\tilde{d}'}$ for $d' \in \mathbb{W}^{\mathbb{W}_{M_{Q'}}} \setminus \{\mathbf{w}^{Q'}\}$ by (4.22). By (4.19), $i(Q, Q')$ is the natural embedding of $\operatorname{Hom}_{\mathcal{H}_{M^-_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$ in $\operatorname{Hom}_{\mathcal{H}_{M^-_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$ therefore $i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v})$ is the unique homomorphism $\operatorname{Hom}_{\mathcal{H}_{M^-_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$ sending $T_{\tilde{\mathbf{w}}^{Q'}}$ to v and vanishing on $T_{\tilde{d}'}$ for $d' \in \mathbb{W}^{\mathbb{W}_{M_{Q'}}} \setminus \{\mathbf{w}^{Q'}\}$. As $\mathbb{W}^{\mathbb{W}_{M_Q}} = \mathbb{W}^{\mathbb{W}_{Q'}} \mathbb{W}^{\mathbb{W}_{M_Q}}_{M_{Q'}}$, this homomorphism vanishes on $T_{\tilde{w}}$ for w not

in $\mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. By [Abe16, Lemma 2.22], the inverse of μ_Q is the $\mathcal{H}_R$-isomorphism

$$\mathrm{Hom}_{\mathcal{H}_{M_Q^-},\theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\mu_Q^{-1}} \mathrm{Ind}_{\mathcal{H}_{\mathbf{w}.M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M.\mathcal{V})) \tag{4.25}$$

$$f \mapsto \sum_{d\in\mathbb{W}^{\mathbb{W}_M}} f(T_{\tilde{d}}) \otimes T^*_{\tilde{\mathbf{w}}^M\tilde{d}^{-1}},$$

where $\mathbb{W}^{\mathbb{W}_M}$ is the set of $d \in \mathbb{W}$ with minimal length in the coset $d\mathbb{W}_M$. We deduce the explicit formula

$$(\mu_Q^{-1} \circ i(Q,Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = \sum_{w\in\mathbb{W}^{\mathbb{W}_{M_Q}}} i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde{w}}) \otimes T^*_{\tilde{\mathbf{w}}^{M_Q}\tilde{w}^{-1}}.$$

Some terms are zero: the terms for $w \in \mathbb{W}^{\mathbb{W}_{M_Q}}$ not in $\mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. We analyze the other terms for w in $\mathbb{W}^{\mathbb{W}_{M_Q}} \cap \mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$; this set is $\mathbf{w}^{M_{Q'}}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. Let $w = \mathbf{w}^{M_{Q'}}d, d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$, and $\tilde{w} = \tilde{\mathbf{w}}^{M_{Q'}}\tilde{d}$ with $\tilde{d} \in {}_1W_{G'}$ lifting d. By the braid relations $T_{\tilde{w}} = T_{\tilde{\mathbf{w}}^{M_{Q'}}}T_{\tilde{d}}$. We have $T_{\tilde{d}} = \theta^*(T_{\tilde{d}}^{M_{Q'}})$ by the braid relations because $d \in \mathbb{W}_{M_{Q'}}$, $S_{M_{Q'}} \subset S^{\mathrm{aff}}$ and $\theta^*(c_{\tilde{s}}^{M_{Q'}}) = c_{\tilde{s}}$ for $s \in S_{M_{Q'}}$. As $\mathbb{W}_{M_{Q'}} \subset W_{M_{Q'}^-} \cap W_{M_{Q'}^+}$, we deduce:

$$\begin{aligned} i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde{w}}) &= i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde{\mathbf{w}}^{M_{Q'}}}T_{\tilde{d}}) \\ &= i(Q,Q')(f^{Q'}_{\tilde{\mathbf{w}}^{M_{Q'}},v})(T_{\tilde{\mathbf{w}}^{M_{Q'}}})T_{\tilde{d}}^{M_{Q'}} \\ &= vT_{\tilde{d}}^{M_{Q'}} = q_d v. \end{aligned}$$

Corollary 3.9 gives the last equality. □

The formula for $(\mu_Q^{-1} \circ i(Q,Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}})$ given in Lemma 4.31 is different from the formula (4.24) for $\iota(\mathbf{w}.Q, \mathbf{w}.Q')(v \otimes 1_{\mathcal{H}})$. It needs some work to prove that they are equal.

A first reassuring remark is that ${}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}} = \{\mathbf{w}d^{-1}\mathbf{w} \mid d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$, so the two summation sets have the same number of elements. But better,

$${}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}} = \{\mathbf{w}^Q(\mathbf{w}^{Q'}d)^{-1} \mid d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$$

because $\mathbf{w}_{Q'}\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\mathbf{w}_Q = \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$. To prove the latter equality, we apply the criterion: $w \in \mathbb{W}_{M_{Q'}}$ lies in ${\mathbb{W}_{M_{Q'}}}^{\mathbb{W}_{M_Q}}$ if and only if $w(\alpha) > 0$ for all $\alpha \in \Delta_Q$ noticing that $d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ implies $\mathbf{w}_Q(\alpha) \in -\Delta_Q$, $d\mathbf{w}_Q(\alpha) \in -\Phi_{M_{Q'}}$, $\mathbf{w}_{Q'}d\mathbf{w}_Q(\alpha) > 0$. Let $x_d = \mathbf{w}^Q(\mathbf{w}^{Q'}d)^{-1}$. We have $\tilde{\mathbf{w}}^{M_Q}(\tilde{\mathbf{w}}^{M_{Q'}}\tilde{d})^{-1} = \tilde{x}_d$ because the lifts $\tilde{w}$ of the elements $w \in \mathbb{W}$ satisfy the braid relations and $\ell(x_d) = \ell(\mathbf{w}_Q d^{-1}\mathbf{w}_{Q'}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d) = -\ell(\mathbf{w}^{Q'}) + \ell(\mathbf{w}^Q) - \ell(d)$. We have $q_d = q_{\mathbf{w}_{\mathbf{w}.Q}x_d\mathbf{w}_{\mathbf{w}.Q'}}$ because $\mathbf{w}d^{-1}\mathbf{w} = \mathbf{w}_{\mathbf{w}.Q}x_d\mathbf{w}_{\mathbf{w}.Q'}$, and $q_d = q_{d^{-1}} = q_{\mathbf{w}d^{-1}\mathbf{w}}$. So

$$\sum_{d\in\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}} q_d T^*_{\tilde{\mathbf{w}}^Q(\tilde{\mathbf{w}}^{Q'}\tilde{d})^{-1}} = \sum_{x_d\in{}^{\mathbb{W}_{M_{\mathbf{w}.Q}}}\mathbb{W}_{M_{\mathbf{w}.Q'}}} q_{\mathbf{w}_{\mathbf{w}.Q}x_d\mathbf{w}_{\mathbf{w}.Q'}} T^*_{\tilde{x}_d}.$$

In the RHS, only $\tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q, \mathbf{w}.Q'$ appear. The same holds true in the formula (4.24). The map $(P, \mathcal{V}, Q, Q') \mapsto (\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q, \mathbf{w}.Q')$ is a bijection of the set of triples $(P, \mathcal{V}, Q, Q')$ where $P = MN, Q, Q'$ are standard parabolic subgroups of G, $\mathcal{V}$ a right $\mathcal{H}_R$-module, $Q \subset Q' \subset P(\mathcal{V})$ by Lemma 4.21. So we can replace $(\mathbf{w}.P, \tilde{\mathbf{w}}^M.\mathcal{V}, \mathbf{w}.Q, \mathbf{w}.Q')$ by $(P, \mathcal{V}, Q, Q')$. Our task is reduced to prove in $e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+}, \theta} \mathcal{H}_R$:

$$v \otimes \sum_{d \in {}^{\mathbb{W}_{M_Q}}\mathbb{W}_{M_{Q'}}} T_{\tilde{d}} = v \otimes \sum_{d \in {}^{\mathbb{W}_{M_Q}}\mathbb{W}_{M_{Q'}}} q_{\mathbf{w}_Q d \mathbf{w}_{Q'}} T^*_{\tilde{d}}. \tag{4.26}$$

A second simplification is possible: we can replace $Q \subset Q'$ by the standard parabolic subgroups $Q_2 \subset Q'_2$ of G with $\Delta_{Q_2} = \Delta_Q \setminus \Delta_P$ and $\Delta_{Q'_2} = \Delta_{Q'} \setminus \Delta_P$, because Δ_P and $\Delta_{P(\mathcal{V})} \setminus \Delta_P$ are orthogonal. Indeed, $\mathbb{W}_{M_{Q'}} = \mathbb{W}_M \times \mathbb{W}_{M_{Q'_2}}$ and $\mathbb{W}_{M_Q} = \mathbb{W}_M \times \mathbb{W}_{M_{Q_2}}$ are direct products, the longest elements $\mathbf{w}_{Q'} = \mathbf{w}_M \mathbf{w}_{Q'_2}, \mathbf{w}_Q = \mathbf{w}_M \mathbf{w}_{Q_2}$ are direct products and

$${}^{\mathbb{W}_{M_Q}}\mathbb{W}_{M_{Q'}} = {}^{\mathbb{W}_{M_{Q_2}}}\mathbb{W}_{M_{Q'_2}}, \quad \mathbf{w}_Q d \mathbf{w}_{Q'} = \mathbf{w}_{Q_2} d \mathbf{w}_{Q'_2}.$$

Once this is done, we use the properties of $e_{\mathcal{H}_Q}(\mathcal{V})$: $vh \otimes 1_{\mathcal{H}} = v \otimes \theta_Q(h)$ for $h \in \mathcal{H}_{M^+_{Q_2}}$, and $T^{Q,*}_w$ acts trivially on $e_{\mathcal{H}_Q}(\mathcal{V})$ for $w \in {}_1W_{M'_{Q_2}} \cup (\Lambda(1) \cap {}_1W_{M'_{Q'_2}})$. Set ${}_1\mathbb{W}_{M'_{Q'_2}} = \{w \in {}_1W_{M'_{Q'_2}} \mid w \text{ is a lift of some element in } \mathbb{W}_{M_{Q'_2}}\}$ and ${}_1\mathbb{W}_{M'_{Q_2}}$ similarly. Then $Z_k \cap {}_1\mathbb{W}_{M'_{Q'_2}} \subset (\Lambda(1) \cap {}_1W_{M'_{Q'_2}}) \cap {}_1W_{M^+_{Q_2}}$ and ${}_1\mathbb{W}_{M'_{Q_2}} \subset {}_1W_{M'_{Q_2}} \cap {}_1W_{M^+_{Q_2}}$. This implies that (4.26) where $Q \subset Q'$ has been replaced by $Q_2 \subset Q'_2$ follows from a congruence

$$\sum_{d \in {}^{\mathbb{W}_{M_{Q_2}}}\mathbb{W}_{M_{Q'_2}}} T_{\tilde{d}} \equiv \sum_{d \in {}^{\mathbb{W}_{M_{Q_2}}}\mathbb{W}_{M_{Q'_2}}} q_{\mathbf{w}_{Q_2} d \mathbf{w}_{Q'_2}} T^*_{\tilde{d}} \tag{4.27}$$

in the finite subring $H({}_1\mathbb{W}_{M_{Q'_2}})$ of $\mathcal{H}$ generated by $\{T_w \mid w \in {}_1\mathbb{W}_{M'_{Q'_2}}\}$ modulo the right ideal $\mathcal{J}_2$ with generators $\{\theta_Q(T^{Q,*}_w) - 1 \mid w \in (Z_k \cap {}_1\mathbb{W}_{M'_{Q'_2}}) \cup {}_1\mathbb{W}_{M'_{Q_2}}\}$.

Another simplification concerns $T^*_{\tilde{d}}$ modulo $\mathcal{J}_2$ for $d \in \mathbb{W}_{M_{Q'_2}}$. We recall that for any reduced decomposition $d = s_1 \dots s_n$ with $s_i \in S \cap \mathbb{W}_{M_{Q'_2}}$ we have $T^*_{\tilde{d}} = (T_{\tilde{s}_1} - c_{\tilde{s}_1}) \dots (T_{\tilde{s}_n} - c_{\tilde{s}_n})$ where the $\tilde{s}_i$ are admissible. For $\tilde{s}$ admissible, by (3.2)

$$c_{\tilde{s}} \equiv q_s - 1.$$

Therefore

$$T^*_d \equiv (T_{\tilde{s}_1} - q_{s_1} + 1) \cdots (T_{\tilde{s}_n} - q_{s_n} + 1).$$

Let $\mathcal{J}' \subset \mathcal{J}_2$ be the ideal of $H({}_1\mathbb{W}_{M'_{Q'_2}})$ generated by $\{T_t - 1 \mid t \in Z_k \cap {}_1W_{M'_{Q'_2}}\}$. Then the ring $H({}_1\mathbb{W}_{M'_{Q'_2}})/\mathcal{J}'$ and its right ideal $\mathcal{J}_2/\mathcal{J}'$ are the specialization of the generic finite ring $H(\mathbb{W}_{M_{Q'_2}})^g$ over $\mathbb{Z}[(q_s)_{s \in S_{M_{Q'_2}}}]$ where the q_s for $s \in S_{M_{Q'_2}} = S \cap \mathbb{W}_{M_{Q'_2}}$ are indeterminates, and of its right ideal $\mathcal{J}^g_2$ with the same generators. The similar congruence modulo $\mathcal{J}^g_2$ in $H(\mathbb{W}_{M_{Q'_2}})^g$ (the generic congruence) implies the congruence (4.27) by specialization.

We will prove the generic congruence in a more general setting where H is the generic Hecke ring of a finite Coxeter system$(\mathbb{W}, S)$ and parameters $(q_s)_{s \in S}$ such that $q_s = q_{s'}$ when s, s' are conjugate in $\mathbb{W}$. The Hecke ring H is a $\mathbb{Z}[(q_s)_{s \in S}]$-free

module of basis $(T_w)_{w\in\mathbb{W}}$ satisfying the braid relations and the quadratic relations $T_s^2 = q_s + (q_s - 1)T_s$ for $s \in S$. The other basis $(T_w^*)_{w\in\mathbb{W}}$ satisfies the braid relations and the quadratic relations $(T_s^*)^2 = q_s - (q_s - 1)T_s^*$ for $s \in S$, and is related to the first basis by $T_s^* = T_s - (q_s - 1)$ for $s \in S$, and more generally $T_w T_{w^{-1}}^* = T_{w^{-1}}^* T_w = q_w$ for $w \in \mathbb{W}$ [Vig16, Proposition 4.13].

Let $J \subset S$ and $\mathcal{J}$ is the right ideal of H with generators $T_w^* - 1$ for all w in the group $\mathbb{W}_J$ generated by J.

Lemma 4.32. *A basis of $\mathcal{J}$ is $(T_{w_1}^* - 1)T_{w_2}^*$ for $w_1 \in \mathbb{W}_J \setminus \{1\}, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$, and adding $T_{w_2}^*$ for $w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$ gives a basis of H. In particular, $\mathcal{J}$ is a direct factor of $\mathcal{H}$.*

Proof. The elements $(T_{w_1}^* - 1)T_w^*$ for $w_1 \in \mathbb{W}_J, w \in \mathbb{W}$ generate $\mathcal{J}$. We write $w = u_1 w_2$ with unique elements $u_1 \in \mathbb{W}_J, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$, and $T_w^* = T_{u_1}^* T_{w_2}^*$. Therefore, $(T_{w_1}^* - 1)T_{u_1}^* T_{w_2}^*$. By an induction on the length of u_1, one proves that $(T_{w_1}^* - 1)T_{u_1}^*$ is a linear combination of $(T_{v_1}^* - 1)$ for $v_1 \in \mathbb{W}_J$ as in the proof of Proposition 3.4. It is clear that the elements $(T_{w_1}^* - 1)T_{w_2}^*$ and $T_{w_2}^*$ for $w_1 \in \mathbb{W}_J \setminus \{1\}, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$ form a basis of H. □

Let $\mathbf{w}_J$ denote the longest element of $\mathbb{W}_J$ and $\mathbf{w} = \mathbf{w}_S$.

Lemma 4.33. *In the generic Hecke ring H, the congruence modulo $\mathcal{J}$*

$$\sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J d\mathbf{w}} T_d^*$$

holds true.

Proof.

Step 1. We show

$${}^{\mathbb{W}_J}\mathbb{W} = \mathbf{w}_J\,{}^{\mathbb{W}_J}\mathbb{W}\,\mathbf{w}, \quad q_{\mathbf{w}_J} q_{\mathbf{w}_J d\mathbf{w}} T_d^* = T_{\mathbf{w}_J} T_{\mathbf{w}_J d\mathbf{w}} T_{\mathbf{w}}^*.$$

The equality between the groups follows from the characterization of ${}^{\mathbb{W}_J}\mathbb{W}$ in $\mathbb{W}$: an element $d \in \mathbb{W}$ has minimal length in $\mathbb{W}_J d$ if and only if $\ell(ud) = \ell(u) + \ell(d)$ for all $u \in \mathbb{W}_J$. An easy computation shows that $\ell(u\mathbf{w}_J d\mathbf{w}) = \ell(u) + \ell(\mathbf{w}_J d\mathbf{w})$ for all $u \in \mathbb{W}_J, d \in {}^{\mathbb{W}_J}\mathbb{W}$ (both sides are equal to $\ell(u) + \ell(\mathbf{w}) - \ell(\mathbf{w}_J) - \ell(d)$). The second equality follows from $q_{\mathbf{w}_J} q_{\mathbf{w}_J d\mathbf{w}} = q_{d\mathbf{w}}$ because $(\mathbf{w}_J)^2 = 1$ and $\ell(\mathbf{w}_J) + \ell(\mathbf{w}_J d\mathbf{w}) = \ell(d\mathbf{w})$ (both sides are $\ell(\mathbf{w}) - \ell(d)$) and from $q_{d\mathbf{w}} T_d^* = T_{d\mathbf{w}} T_{\mathbf{w}d^{-1}}^* T_d^* = T_{d\mathbf{w}} T_{\mathbf{w}}^*$. We also have $T_{d\mathbf{w}} = T_{\mathbf{w}_J} T_{\mathbf{w}_J d\mathbf{w}}$.

Step 2. The multiplication by $q_{\mathbf{w}_J}$ on the quotient $H/\mathcal{J}$ is injective (Lemma 4.32) and $q_{\mathbf{w}_J} \equiv T_{\mathbf{w}_J}$. By Step 1, $q_{\mathbf{w}_J d\mathbf{w}} T_d^* \equiv T_{\mathbf{w}_J d\mathbf{w}} T_{\mathbf{w}}^*$ and

$$\sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J d\mathbf{w}} T_d^* \equiv \sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} T_d T_{\mathbf{w}}^*.$$

The congruence

$$\sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} T_d T_s^* \tag{4.28}$$

for all $s \in S$ implies the lemma because $T_{\mathbf{w}}^* = T_{s_1}^* \dots T_{s_n}^*$ for any reduced decomposition $\mathbf{w} = s_1 \dots s_n$ with $s_i \in S$.

Step 3. When $J=\emptyset$, the congruence (4.28) is an equality

$$\sum_{w\in\mathbb{W}} T_w = \sum_{w\in\mathbb{W}} T_w T_s^*. \tag{4.29}$$

It holds true because $\sum_{w\in\mathbb{W}} T_w = \sum_{w<ws} T_w(T_s+1)$ and $(T_s+1)T_s^* = T_sT_s^*+T_s^* = q_s + T_s^* = T_s+1$.

Step 4. Conversely the congruence (4.28) follows from (4.29) because

$$\sum_{w\in\mathbb{W}} T_w = (\sum_{u\in W_J} T_u) \sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv (\sum_{u\in W_J} q_u) \sum_{d\in{}^{\mathbb{W}_J}\mathbb{W}} T_d$$

(recall $q_u = T_{u^{-1}}^* T_u \equiv T_u$) and we can simplify by $\sum_{u\in W_J} q_u$ in $H/\mathcal{J}$. □

This ends the proof of Proposition 4.23.

5. Universal representation $I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}} R[\mathcal{U}\backslash G]$

The invariant functor $(-)^{\mathcal{U}}$ by the pro-p-Iwahori subgroup $\mathcal{U}$ of G has a left adjoint

$$-\bigotimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G] : \mathrm{Mod}_R(\mathcal{H}) \to \mathrm{Mod}_R^\infty(G).$$

The smooth R-representation $\mathcal{V}\otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ of G constructed from the right $\mathcal{H}_R$-module $\mathcal{V}$ is called universal. We write

$$R[\mathcal{U}\backslash G] = \mathbb{X}.$$

Question 5.1. Does $\mathcal{V}\neq 0$ imply $\mathcal{V}\otimes_{\mathcal{H}_R}\mathbb{X}\neq 0$ or does $v\otimes 1_{\mathcal{U}} = 0$ for $v\in\mathcal{V}$ imply $v=0$? We have no counterexample. If R is a field and the $\mathcal{H}_R$-module $\mathcal{V}$ is simple, the two questions are equivalent: $\mathcal{V}\otimes_{\mathcal{H}_R}\mathbb{X}\neq 0$ if and only if the map $v\mapsto v\otimes 1_{\mathcal{U}}$ is injective. When R is an algebraically closed field of characteristic p, $\mathcal{V}\otimes_{\mathcal{H}_R}\mathbb{X}\neq 0$ for all simple $\mathcal{H}_R$-modules $\mathcal{V}$ if this is true for $\mathcal{V}$ simple supersingular (this is a consequence of Corollary 5.13).

The functor $-\bigotimes_{\mathcal{H}_R}\mathbb{X}$ satisfies a few good properties: it has a right adjoint and is compatible with the parabolic induction and the left adjoint (of the parabolic induction). Let $P=MN$ be a standard parabolic subgroup and $\mathbb{X}_M = R[\mathcal{U}_M\backslash M]$. We have functor isomorphisms

$$(-\bigotimes_{\mathcal{H}_R}\mathbb{X})\circ \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \to \mathrm{Ind}_P^G\circ(-\bigotimes_{\mathcal{H}_{M,R}}\mathbb{X}_M), \tag{5.1}$$

$$(-)_N\circ(-\bigotimes_{\mathcal{H}_R}\mathbb{X}) \to (-\bigotimes_{\mathcal{H}_{M,R}}\mathbb{X}_M)\circ\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}. \tag{5.2}$$

The first one is [OV17, formula 4.15], the second one is obtained by left adjunction from the isomorphism $\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}\circ(-)^{\mathcal{U}_M} \to (-)^{\mathcal{U}}\circ\mathrm{Ind}_P^G$ [OV17, formula (4.14)]. If $\mathcal{V}$ is a right $\mathcal{H}_R$-supersingular module and p is nilpotent in $\mathcal{V}$, then $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})=0$ if $M\neq G$ (Proposition 4.27). Applying (5.2) we deduce the following.

Proposition 5.2. *If p is nilpotent in $\mathcal{V}$ and $\mathcal{V}$ supersingular, then $\mathcal{V}\otimes_{\mathcal{H}_R}\mathbb{X}$ is left cuspidal.*

Remark 5.3. For a non-zero smooth R-representation τ of M, Δ_τ is orthogonal to Δ_P if τ is left cuspidal. Indeed, we recall from [AHHV17, II.7 Corollary 2] that Δ_τ is not orthogonal to Δ_P if and only if there exists a proper standard parabolic subgroup X of M such that σ is trivial on the unipotent radical of X; moreover τ is a subrepresentation of $\mathrm{Ind}_X^M(\tau|_X)$, so the image of τ by the left adjoint of Ind_X^M is not 0.

From now on, $\mathcal{V}$ is a non-zero right $\mathcal{H}_{M,R}$-module and

$$\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M.$$

In general, when $\sigma \neq 0$, let $P_\perp(\sigma)$ be the standard parabolic subgroup of G with $\Delta_{P_\perp(\sigma)} = \Delta_P \cup \Delta_{\perp,\sigma}$ where $\Delta_{\perp,\sigma}$ is the set of simple roots $\alpha \in \Delta_\sigma$ orthogonal to Δ_P.

Proposition 5.4.

(1) $P(\mathcal{V}) \subset P_\perp(\sigma)$ *if* $\sigma \neq 0$.
(2) $P(\mathcal{V}) = P_\perp(\sigma)$ *if the map* $v \mapsto v \otimes 1_{\mathcal{U}_M}$ *is injective.*
(3) $P(\mathcal{V}) = P(\sigma)$ *if the map* $v \mapsto v \otimes 1_{\mathcal{U}_M}$ *is injective,* p *nilpotent in* $\mathcal{V}$ *and* $\mathcal{V}$ *supersingular.*
(4) $P(\mathcal{V}) = P(\sigma)$ *if* $\sigma \neq 0$, R *is a field of characteristic* p *and* $\mathcal{V}$ *simple supersingular.*

Proof.
(1) $P(\mathcal{V}) \subset P_\perp(\sigma)$ means that $Z \cap M'_{\mathcal{V}}$ acts trivially on $\mathcal{V} \otimes 1_{\mathcal{U}_M}$, where $M_{\mathcal{V}}$ is the standard Levi subgroup such that $\Delta_{M_{\mathcal{V}}} = \Delta_{\mathcal{V}}$. Let $z \in Z \cap M'_{\mathcal{V}}$ and $v \in \mathcal{V}$. As Δ_M and $\Delta_{\mathcal{V}}$ are orthogonal, we have $T^{M,*}(z) = T^M(z)$ and $\mathcal{U}_M z \mathcal{U}_M = \mathcal{U}_M z$. We have $v \otimes 1_{\mathcal{U}_M} = vT^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes T^M(z) 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M z} = v \otimes z^{-1} 1_{\mathcal{U}_M} = z^{-1}(v \otimes 1_{\mathcal{U}_M})$.
(2) If $v \otimes 1_{\mathcal{U}_M} = 0$ for $v \in \mathcal{V}$ implies $v = 0$, then $\sigma \neq 0$ because $\mathcal{V} \neq 0$. By (1) $P(\mathcal{V}) \subset P_\perp(\sigma)$. As in the proof of (1), for $z \in Z \cap M'_{\perp,\sigma}$ we have $vT^{M,*}(z) \otimes 1_{\mathcal{U}_M} = vT^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M}$ and our hypothesis implies $vT^{M,*}(z) = v$ hence $P(\mathcal{V}) \supset P_\perp(\sigma)$.
(3) Proposition 5.2, Remark 5.3, and (2).
(4) Question 5.1 and (3). □

Let Q be a parabolic subgroup of G with $P \subset Q \subset P(\mathcal{V})$. In this chapter we will compute $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U}\backslash G]$ where $I_{\mathcal{H}}(P, \mathcal{V}, Q) = \mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes (\mathrm{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$ (Theorem 5.11). The smooth R-representation $I_G(P, \sigma, Q)$ of G is well defined: it is 0 if $\sigma = 0$ and $\mathrm{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \mathrm{St}_Q^{P(\sigma)})$ if $\sigma \neq 0$ because (P, σ, Q) is an $R[G]$-triple by Proposition 5.4. We will show that the universal representation $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U}\backslash G]$ is isomorphic to $I_G(P, \sigma, Q)$, if $P(\mathcal{V}) = P(\sigma)$ and $p = 0$, or if $\sigma = 0$ (Corollary 5.12). In particular, $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U}\backslash G] \simeq I_G(P, \sigma, Q)$ when R is an algebraically closed field of characteristic p and $\mathcal{V}$ is supersingular.

5.1. $Q = G$**.** We consider first the case $Q = G$. We are in the simple situation where $\mathcal{V}$ is extensible to $\mathcal{H}$ and $P(\mathcal{V}) = P(\sigma) = G$, $I_{\mathcal{H}}(P, \mathcal{V}, G) = e(\mathcal{V})$ and $I_G(P, \sigma, G) = e(\sigma)$. We recall that $\Delta \setminus \Delta_P$ is orthogonal to Δ_P and that M_2 denotes the standard Levi subgroup of G with $\Delta_{M_2} = \Delta \setminus \Delta_P$.

The $\mathcal{H}_R$-morphism $e(\mathcal{V}) \to e(\sigma)^{\mathcal{U}} = \sigma^{\mathcal{U}_M}$ sending v to $v \otimes 1_{\mathcal{U}_M}$ for $v \in \mathcal{V}$, gives by adjunction an $R[G]$-homomorphism

$$v \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} : e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi^G} e(\sigma).$$

If Φ^G is an isomorphism, then $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ is the extension to G of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$, meaning that M_2' acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. The converse is true.

Lemma 5.5. *If M_2' acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$, then Φ^G is an isomorphism.*

Proof. Suppose that M_2' acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. Then $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ is the extension to G of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$, and by Theorem 3.13, $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}}$ is the extension of $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$. Therefore, by (3.12),

$$(v \otimes 1_{\mathcal{U}})T_w^* = (v \otimes 1_{\mathcal{U}})T_w^{M,*} \quad \text{for all } v \in \mathcal{V}, w \in W_M(1).$$

As $\mathcal{V}$ is extensible to $\mathcal{H}$, the natural map $v \mapsto v \otimes 1_{\mathcal{U}} : \mathcal{V} \xrightarrow{\Psi} (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$ is $\mathcal{H}_M$-equivariant, i.e.,

$$vT_w^{M,*} \otimes 1_{\mathcal{U}} = (v \otimes 1_{\mathcal{U}})T_w^{M,*} \quad \text{for all } v \in \mathcal{V}, w \in W_M(1)$$

because (3.12) $vT_w^{M,*} \otimes 1_{\mathcal{U}} = vT_w^* \otimes 1_{\mathcal{U}} = v \otimes T_w^* = (v \otimes 1_{\mathcal{U}})T_w^*$ in $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$.

We recall that $-\bigotimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$ is the left adjoint of $(-)^{\mathcal{U}_M}$. The adjoint $R[M]$-homomorphism $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M \to e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ sends $v \otimes 1_{\mathcal{U}_M}$ to $v \otimes 1_{\mathcal{U}}$ for all $v \in \mathcal{V}$. The $R[M]$-module generated by the $v \otimes 1_{\mathcal{U}}$ for all $v \in \mathcal{V}$ is equal to $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ because M_2' acts trivially. Hence we obtained an inverse of Φ^G. $\square$

Our next move is to determine if M_2' acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$. It is equivalent to see if M_2' acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$ as this set generates the representation $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ of G and M_2' is a normal subgroup of G as M_2' and M commute and $G = ZM'M_2'$. Obviously, $\mathcal{U} \cap M_2'$ acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$. The group of double classes $(\mathcal{U} \cap M_2')\backslash M_2'/(\mathcal{U} \cap M_2')$ is generated by the lifts $\hat{s} \in \mathcal{N} \cap M_2'$ of the simple affine roots s of $W_{M_2'}$. Therefore, M_2' acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ if and only if for any simple affine root $s \in S_{M_2'}^{\mathrm{aff}}$ of $W_{M_2'}$, any $\hat{s} \in \mathcal{N} \cap M_2'$ lifting s acts trivially on $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$.

Lemma 5.6. *Let $v \in \mathcal{V}, s \in S_{M_2'}^{\mathrm{aff}}$ and $\hat{s} \in \mathcal{N} \cap M_2'$ lifting s. We have*

$$(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0.$$

Proof. We compute:

$$T_s(\hat{s}1_{\mathcal{U}}) = \hat{s}(T_s 1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-1}} = \sum_u \hat{s}u(\hat{s})^{-1}1_{\mathcal{U}} = \sum_{u^{op}} u^{op}1_{\mathcal{U}},$$

$$T_s(\hat{s}^2 1_{\mathcal{U}}) = \hat{s}^2(T_s 1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-2}} = 1_{\mathcal{U}(\hat{s})^{-1}\mathcal{U}} = \sum_u u\hat{s}1_{\mathcal{U}}$$

for u in the group $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U})$ and u^{op} in the group $\hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U})$; the reason is that $\hat{s}^2$ normalizes $\mathcal{U}$, $\mathcal{U}\hat{s}\mathcal{U}\hat{s}^{-1}$ is the disjoint union of the sets $\mathcal{U}\hat{s}u^{-1}(\hat{s})^{-1}$ and $\mathcal{U}(\hat{s})^{-1}\mathcal{U}$ is the disjoint union of the sets $\mathcal{U}(\hat{s})^{-1}u^{-1}$. We introduce now a natural bijection

$$u \to u^{op} : \mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U}) \to \hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U}) \tag{5.3}$$

which is not a group homomorphism. We recall the finite reductive group $G_{k,s}$ quotient of the parahoric subgroup $\mathfrak{K}_s$ of G fixing the face fixed by s of the alcove $\mathcal{C}$. The Iwahori groups $Z^0\mathcal{U}$ and $Z^0\hat{s}\mathcal{U}(\hat{s})^{-1}$ are contained in $\mathfrak{K}_s$ and their images

in $G_{s,k}$ are opposite Borel subgroups $Z_kU_{s,k}$ and $Z_kU_{s,k}^{op}$. Via the surjective maps $u \mapsto \overline{u} : \mathcal{U} \to U_{s,k}$ and $u^{op} \mapsto \overline{u}^{op} : \hat{s}\mathcal{U}(\hat{s})^{-1} \to U_{s,k}^{op}$ we identify the groups $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s}\cap\mathcal{U}) \simeq U_{s,k}$ and similarly $\hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1}\cap\mathcal{U}) \simeq U_{s,k}^{op}$. Let $G'_{k,s}$ be the group generated by $U_{s,k}$ and $U_{s,k}^{op}$, and let $B'_{s,k} = G'_{k,s} \cap Z_kU_{s,k} = (G'_{k,s} \cap Z_k)U_{s,k}$. We suppose (as we can) that $\hat{s} \in \mathfrak{K}_s$ and that its image $\hat{s}_k$ in $G_{s,k}$ lies in $G'_{k,s}$. We have $\hat{s}_kU_{s,k}(\hat{s}_k)^{-1} = U_{s,k}^{op}$ and the Bruhat decomposition $G'_{k,s} = B'_{k,s} \sqcup U_{k,s}\hat{s}_kB'_{k,s}$ implies the existence of a canonical bijection $\overline{u}^{op} \to \overline{u} : (U_{k,s}^{op} - \{1\}) \to (U_{k,s} - \{1\})$ respecting the cosets $\overline{u}^{op}B'_{k,s} = \overline{u}\hat{s}_kB'_{k,s}$. Via the preceding identifications we get the wanted bijection (5.3).

For $v \in e(\mathcal{V})$ and $z \in Z^0 \cap M'_2$ we have $vT_z = v$, $z1_{\mathcal{U}} = T_z1_{\mathcal{U}}$ and $v \otimes T_z1_{\mathcal{U}} = vT_z \otimes 1_{\mathcal{U}}$ therefore $Z^0 \cap M'_2$ acts trivially on $\mathcal{V} \otimes 1_{\mathcal{U}}$. The action of the group $(Z^0 \cap M'_2)\mathcal{U}$ on $\mathcal{V} \otimes 1_{\mathcal{U}}$ is also trivial. As the image of $Z^0 \cap M'_2$ in $G_{s,k}$ contains $Z_k \cap G'_{s,k}$,

$$u\hat{s}(v \otimes 1_{\mathcal{U}}) = u^{op}(v \otimes 1_{\mathcal{U}})$$

when u and u^{op} are not units and correspond via the bijection (5.3). So we have

$$v \otimes T_s(\hat{s}1_{\mathcal{U}}) - (v \otimes 1_{\mathcal{U}}) = v \otimes T_s(\hat{s}^2 1_{\mathcal{U}}) - v \otimes \hat{s}1_{\mathcal{U}}. \tag{5.4}$$

We can move T_s on the other side of $\otimes$ and as $vT_s = q_sv$ (Corollary 3.9), we can replace T_s by q_s. We have $v \otimes \hat{s}^2 1_{\mathcal{U}} = v \otimes T_{s^{-2}}1_{\mathcal{U}}$ because $\hat{s}^2 \in Z^0 \cap M'_2$ normalizes $\mathcal{U}$; as we can move $T_{s^{-2}}$ on the other side of $\otimes$ and as $vT_{s^{-2}} = v$ we can forget $\hat{s}^2$. So (5.4) is equivalent to $(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0$. □

Combining the two lemmas we obtain the following.

Proposition 5.7. *When $\mathcal{V}$ is extensible to $\mathcal{H}$ and has no $q_s + 1$-torsion for any $s \in S^{\mathrm{aff}}_{M'_2}$, then M'_2 acts trivially on $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ and Φ^G is an $R[G]$-isomorphism.*

Proposition 5.7 for the trivial character $\mathbf{1}_{\mathcal{H}}$, says that $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$ is the trivial representation $\mathbf{1}_G$ of G when $q_s + 1$ has no torsion in R for all $s \in S^{\mathrm{aff}}$. This is proved in [OV17, Lemma 2.28] by a different method. The following counter-example shows that this is not true for all R.

Example 5.8. Let $G = GL(2, F)$ and let R be an algebraically closed field where $q_{s_0} + 1 = q_{s_1} + 1 = 0$ and $S_{\mathrm{aff}} = \{s_0, s_1\}$. (Note that $q_{s_0} = q_{s_1}$ is the order of the residue field of F.) Then the dimension of $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$ is infinite, in particular $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \neq \mathbf{1}_G$.

Indeed, the Steinberg representation $\mathrm{St}_G = (\mathrm{Ind}_B^G \mathbf{1}_Z)/\mathbf{1}_G$ of G is an indecomposable representation of length 2 containing an irreducible infinite dimensional representation π with $\pi^{\mathcal{U}} = 0$ of quotient the character $(-1)^{\mathrm{val} \circ \det}$. This follows from the proof of Theorem 3 and from Proposition 24 in [Vig89]. The kernel of the quotient map $\mathrm{St}_G \otimes (-1)^{\mathrm{val} \circ \det} \to \mathbf{1}_G$ is infinite dimensional without a non-zero $\mathcal{U}$-invariant vector. As the characteristic of R is not p, the functor of $\mathcal{U}$-invariants is exact hence $(\mathrm{St}_G \otimes (-1)^{\mathrm{val} \circ \det})^{\mathcal{U}} = \mathbf{1}_{\mathcal{H}}$. As $- \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is the left adjoint of $(-)^{\mathcal{U}}$ there is a non-zero homomorphism

$$\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \to \mathrm{St}_G \otimes (-1)^{\mathrm{val} \circ \det}$$

with image generated by its $\mathcal{U}$-invariants. The homomorphism is therefore surjective.

5.2. $\mathcal{V}$ **extensible to** $\mathcal{H}$**.** Let $P = MN$ be a standard parabolic subgroup of G with Δ_P and $\Delta \setminus \Delta_P$ orthogonal. We still suppose that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}$, but now $P \subset Q \subset G$. So we have $I_{\mathcal{H}}(P, \mathcal{V}, Q) = e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ and $I_G(P, \sigma, Q) = e(\sigma) \otimes_R \mathrm{St}_Q^G$ where $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$. We compare the images by $-\bigotimes_{\mathcal{H}_R} \mathbb{X}$ of the $\mathcal{H}_R$-modules $e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ and $e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}$ with the smooth R-representations $e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}$ and $e(\sigma) \otimes \mathrm{St}_Q^G$ of G.

As $-\bigotimes_{\mathcal{H}_R} \mathbb{X}$ is left adjoint of $(-)^{\mathcal{U}}$, the $\mathcal{H}_R$-homomorphism $v \otimes f \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f : e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \to (e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ gives by adjunction an $R[G]$-homomorphism

$$v \otimes f \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f : (e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi_Q^G} e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}.$$

When $Q = G$ we have $\Phi_G^G = \Phi^G$. By Remark 4.10, Φ_Q^G is surjective. Proposition 5.7 applies with M_Q instead of G and gives the $R[M_Q]$-homomorphism

$$v \otimes 1_{\mathcal{U}_{M_Q}} \mapsto v \otimes 1_{\mathcal{U}_M} : e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q} \xrightarrow{\Phi^Q} e_Q(\sigma).$$

Proposition 5.9. *The $R[G]$-homomorphism Φ_Q^G is an isomorphism if Φ^Q is an isomorphism, in particular if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S_{M_2' \cap M_Q}^{\mathrm{aff}}$.*

Proof. The proposition follows from another construction of Φ_Q^G that we now describe. Proposition 4.5 gives the $\mathcal{H}_R$-module isomorphism

$$v \otimes f_{Q\mathcal{U}} \mapsto v \otimes 1_{\mathcal{H}} : (e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \to \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_{Q,R}^+}, \vartheta} \mathcal{H}.$$

We have the $R[G]$-isomorphism [OV17, Corollary 4.7]

$$v \otimes 1_{\mathcal{H}} \otimes 1_{\mathcal{U}} \mapsto f_{Q\mathcal{U}, v \otimes 1_{\mathcal{U}_{M_Q}}} : \mathrm{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \otimes_{\mathcal{H}_R} \mathbb{X} \to \mathrm{Ind}_Q^G(e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q})$$

and the $R[G]$-isomorphism

$$f_{Q\mathcal{U}, v \otimes 1_{\mathcal{U}_M}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f_{Q\mathcal{U}} : \mathrm{Ind}_Q^G(e_Q(\sigma)) \to e(\sigma) \otimes \mathrm{Ind}_Q^G \mathbf{1}.$$

From Φ^Q and these three homomorphisms, there exists a unique $R[G]$-homomorphism

$$(e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \to e(\sigma) \otimes_R \mathrm{Ind}_Q^G \mathbf{1}$$

sending $v \otimes f_{Q\mathcal{U}} \otimes 1_{\mathcal{U}}$ to $v \otimes 1_{\mathcal{U}_M} \otimes f_{Q\mathcal{U}}$. We deduce: this homomorphism is equal to Φ_Q^G, $\mathcal{V} \otimes 1_{Q\mathcal{U}} \otimes 1_{\mathcal{U}}$ generates $(e(\mathcal{V}) \otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X}$, if Φ^Q is an isomorphism, then Φ_Q^G is an isomorphism. By Proposition 5.7, if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S_{M_2' \cap M_Q}^{\mathrm{aff}}$, then Φ^Q and Φ_Q^G are isomorphisms. $\square$

We recall that the $\mathcal{H}_{M,R}$-module $\mathcal{V}$ is extensible to $\mathcal{H}$.

Proposition 5.10. *The $R[G]$-homomorphism Φ_Q^G induces an $R[G]$-homomorphism*

$$(e(\mathcal{V}) \otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \to e(\sigma) \otimes_R \mathrm{St}_Q^G,$$

It is an isomorphism if $\Phi_{Q'}^G$ is an $R[G]$-isomorphism for all parabolic subgroups Q' of G containing Q, in particular if $\mathcal{V}$ has no $q_s + 1$-torsion for any $s \in S_{M_2'}^{\mathrm{aff}}$.

Proof. The proof is straightforward, with the arguments already developed for Proposition 4.5 and Theorem 4.9. The representations $e(\sigma)\otimes_R \mathrm{St}_Q^G$ and $(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}$ of G are the cokernels of the natural $R[G]$-homomorphisms

$$\oplus_{Q\subsetneq Q'} e(\sigma)\otimes_R \mathrm{Ind}_{Q'}^G \mathbf{1} \xrightarrow{\mathrm{id}\otimes\alpha} e(\sigma)\otimes_R \mathrm{Ind}_Q^G \mathbf{1},$$

$$\oplus_{Q\subsetneq Q'} (e(\mathcal{V})\otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X} \xrightarrow{\mathrm{id}\otimes\alpha^{\mathcal{U}}\otimes\mathrm{id}} (e(\mathcal{V})\otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}.$$

These $R[G]$-homomorphisms make a commutative diagram with the $R[G]$-homomorphisms $\bigoplus_{Q\subsetneq Q'}\Phi_{Q'}^G$ and Φ_Q^G going from the lower line to the upper line. Indeed, let $v\otimes f_{Q'\mathcal{U}}\otimes 1_{\mathcal{U}}\in (e(\mathcal{V})\otimes_R (\mathrm{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}$. On the one hand, it goes to $v\otimes f_{Q\mathcal{U}}\theta_{Q'}(e_Q^{Q'})\otimes 1_{\mathcal{U}}\in (e(\mathcal{V})\otimes_R (\mathrm{Ind}_Q^G \mathbf{1})^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}$ by the horizontal map, and then to $v\otimes 1_{\mathcal{U}_M}\otimes f_{Q\mathcal{U}}\theta_{Q'}(e_Q^{Q'})$ by the vertical map. On the other hand, it goes to $v\otimes 1_{\mathcal{U}_M}\otimes f_{Q'\mathcal{U}}$ by the vertical map, and then to $v\otimes 1_{\mathcal{U}_M}\otimes f_{Q\mathcal{U}}\theta_{Q'}(e_Q^{Q'})$ by the horizontal map. One deduces that Φ_Q^G induces an $R[G]$-homomorphism $(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^G)^{\mathcal{U}})\otimes_{\mathcal{H}_R}\mathbb{X}\to e(\sigma)\otimes_R \mathrm{St}_Q^G$, which is an isomorphism if $\Phi_{Q'}^G$ is an $R[G]$-isomorphism for all $Q\subset Q'$. □

5.3. **General.** We consider now the general case: let $P=MN\subset Q$ be two standard parabolic subgroups of G and let $\mathcal{V}$ be a non-zero right $\mathcal{H}_{M,R}$-module with $Q\subset P(\mathcal{V})$. We recall $I_{\mathcal{H}}(P,\mathcal{V},Q)=\mathrm{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$ and $\sigma=\mathcal{V}\otimes_{\mathcal{H}_{M,R}}\mathbb{X}_M$ (Proposition 5.4). There is a natural $R[G]$-homomorphism

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R}\mathbb{X}\xrightarrow{\Phi_I^G}\mathrm{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})})$$

obtained by composition of the $R[G]$-isomorphism [OV17, Corollary 4.7] (proof of Proposition 5.9):

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R}\mathbb{X}\to\mathrm{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V})\otimes_R (\mathrm{St}_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})\otimes_{\mathcal{H}_{M(\mathcal{V}),R}}\mathbb{X}_{M(\mathcal{V})}),$$

with the $R[G]$-homomorphism

$$\mathrm{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V})\otimes_R (\mathrm{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})\otimes_{\mathcal{H}_{M(\mathcal{V}),R}}\mathbb{X}_{M(\mathcal{V})})\to\mathrm{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})}),$$

image by the parabolic induction $\mathrm{Ind}_{P(\mathcal{V})}^G$ of the homomorphism

$$(e(\mathcal{V})\otimes_R (\mathrm{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})\otimes_{\mathcal{H}_{M(\mathcal{V}),R}}\mathbb{X}_{M(\mathcal{V})}\to e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})}$$

induced by the $R[M(\mathcal{V})]$-homomorphism $\Phi_Q^{P(\mathcal{V})}=\Phi_{Q\cap M(\mathcal{V})}^{M(\mathcal{V})}$ of Proposition 5.10 applied to $M(\mathcal{V})$ instead of G.

This homomorphism Φ_I^G is an isomorphism if $\Phi_Q^{P(\mathcal{V})}$ is an isomorphism, in particular if $\mathcal{V}$ has no q_s+1-torsion for any $s\in S_{M_2}^{\mathrm{aff}}$ where $\Delta_{M_2}=\Delta_{M(\mathcal{V})}\setminus\Delta_M$ (Proposition 5.10). We get the main theorem of this section.

Theorem 5.11. *Let $(P=MN,\mathcal{V},Q)$ be an $\mathcal{H}_R$-triple and $\sigma=\mathcal{V}\otimes_{\mathcal{H}_{M,R}}R[\mathcal{U}_M\backslash M]$. Then, (P,σ,Q) is an $R[G]$-triple. The $R[G]$-homomorphism*

$$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R}R[\mathcal{U}\backslash G]\xrightarrow{\Phi_I^G}\mathrm{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma)\otimes_R \mathrm{St}_Q^{P(\mathcal{V})})$$

is an isomorphism if $\Phi_Q^{P(\mathcal{V})}$ is an isomorphism. In particular Φ_I^G is an isomorphism if $\mathcal{V}$ has no q_s+1-torsion for any $s\in S_{M_2'}^{\mathrm{aff}}$.

Recalling $I_G(P,\sigma,Q) = \mathrm{Ind}_{P(\sigma)}^G(e(\sigma)\otimes_R \mathrm{St}_Q^{P(\sigma)})$ when $\sigma \neq 0$, we deduce the following.

Corollary 5.12. *We have the following:*
$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G] \simeq I_G(P,\sigma,Q)$, if $\sigma\neq 0$, $P(\mathcal{V}) = P(\sigma)$ and $\mathcal{V}$ has no q_s+1-torsion for any $s\in S_{M_2'}^{\mathrm{aff}}$.
$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G] = I_G(P,\sigma,Q) = 0$, if $\sigma = 0$.

Recalling $P(\mathcal{V}) = P(\sigma)$ if $\sigma\neq 0$, R is a field of characteristic p and $\mathcal{V}$ simple supersingular (Proposition 5.4 (4)), we deduce the following.

Corollary 5.13. *$I_{\mathcal{H}}(P,\mathcal{V},Q)\otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G] \simeq I_G(P,\sigma,Q)$ if R is a field of characteristic p and $\mathcal{V}$ simple supersingular.*

6. Vanishing of the smooth dual

Let V be an $R[G]$-module. The dual $\mathrm{Hom}_R(V,R)$ of V is an $R[G]$-module for the contragredient action: $gL(gv) = L(v)$ if $g\in G$, $L\in \mathrm{Hom}_R(V,R)$ is a linear form and $v\in V$. When $V\in \mathrm{Mod}_R^\infty(G)$ is a smooth R-representation of G, the dual of V is not necessarily smooth. A linear form L is smooth if there exists an open subgroup $H\subset G$ such that $L(hv) = L(v)$ for all $h\in H, v\in V$; the space $\mathrm{Hom}_R(V,R)^\infty$of smooth linear forms is a smooth R-representation of G, called the **smooth dual** (or smooth contragredient) of V. The smooth dual of V is contained in the dual of V.

Example 6.1. When R is a field and the dimension of V over R is finite, the dual of V is equal to the smooth dual of V because the kernel of the action of G on V is an open normal subgroup $H\subset G$; the action of G on the dual $\mathrm{Hom}_R(V,R)$ is trivial on H.

We assume in this section that R is a field of characteristic p. Let $P = MN$ be a parabolic subgroup of G and $V\in \mathrm{Mod}_R^\infty(M)$. Generalizing the proof given in [Vig07, 8.1] when $G = GL(2,F)$ and the dimension of V is 1, we show the following.

Proposition 6.2. *If $P\neq G$, the smooth dual of $\mathrm{Ind}_P^G(V)$ is 0.*

Proof. Let L be a smooth linear form on $\mathrm{Ind}_P^G(V)$ and let K be an open pro-p-subgroup of G which fixes L. Let J be an arbitrary open subgroup of K, $g\in G$ and $f\in(\mathrm{Ind}_P^G(V))^J$ with support PgJ. We want to show that $L(f) = 0$. Let J' be any open normal subgroup of J and let φ denote the function in $(\mathrm{Ind}_P^G(V))^{J'}$ with support PgJ' and value $\varphi(g) = f(g)$ at g. For $j\in J$ we have $L(j\varphi) = L(\varphi)$, and the support of $j\varphi(x) = \varphi(xj)$ is $PgJ'j^{-1}$. The function f is the sum of translates $j\varphi$, where j ranges through the left cosets of the image X of $g^{-1}Pg\cap J$ in J/J', so that $L(f) = rL(\varphi)$ where r is the order of X in J/J'. We can certainly find J' such that $r\neq 1$, and then r is a positive power of p. As the characteristic of R is p we have $L(f) = 0$. □

The module $R[\mathcal{U}\backslash G]$ is contained in the module $R^{\mathcal{U}\backslash G}$ of functions $f:\mathcal{U}\backslash G\to R$. The actions of $\mathcal{H}$ and of G on $R[\mathcal{U}\backslash G]$ extend to $R^{\mathcal{U}\backslash G}$ by the same formulas. The pairing

$$(f,\varphi)\mapsto \langle f,\varphi\rangle = \sum_{g\in\mathcal{U}\backslash G} f(g)\varphi(g) : R^{\mathcal{U}\backslash G}\times R[\mathcal{U}\backslash G]\to R$$

identifies $R^{\mathcal{U}\backslash G}$ with the dual of $R[\mathcal{U}\backslash G]$. Let $h \in \mathcal{H}$ and $\check{h} \in \mathcal{H}$, $\check{h}(g) = h(g^{-1})$ for $g \in G$. We have

$$\langle f, h\varphi \rangle = \langle \check{h} f, \varphi \rangle.$$

Proposition 6.3. *When R is an algebraically closed field of characteristic p, G is not compact modulo the center and $\mathcal{V}$ is a simple supersingular right $\mathcal{H}_R$-module, the smooth dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is 0.*

Proof. Let $\mathcal{H}_R^{\mathrm{aff}}$ be the subalgebra of $\mathcal{H}_R$ of basis $(T_w)_{w \in W'(1)}$ where $W'(1)$ is the inverse image of W' in $W(1)$. The dual of $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$ is contained in the dual of $\mathcal{V} \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$; the $\mathcal{H}_R^{\mathrm{aff}}$-module $\mathcal{V}|_{\mathcal{H}_R^{\mathrm{aff}}}$ is a finite sum of supersingular characters [Vig15a]. Let $\chi : \mathcal{H}_R^{\mathrm{aff}} \to R$ be a supersingular character. The dual of $\chi \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$ is contained in the dual of $R[\mathcal{U}\backslash G]$ isomorphic to $R^{\mathcal{U}\backslash G}$. It is the space of $f \in R^{\mathcal{U}\backslash G}$ with $\check{h} f = \chi(h) f$ for all $h \in \mathcal{H}_R^{\mathrm{aff}}$. The smooth dual of $\chi \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$ is 0 if the dual of $\chi \otimes_{\mathcal{H}_R^{\mathrm{aff}}} R[\mathcal{U}\backslash G]$ has no non-zero element fixed by $\mathcal{U}$. Let us take $f \in R^{\mathcal{U}\backslash G/\mathcal{U}}$ with $\check{h} f = \chi(h) f$ for all $h \in \mathcal{H}_R^{\mathrm{aff}}$. We shall prove that $f = 0$. We have $\check{T}_w = T_{w^{-1}}$ for $w \in W(1)$.

Let $<$ denote the Bruhat order of $W(1)$ associated to S^{aff} [Vig16]. The elements $(T_t)_{t \in Z_k}$ and $(T_{\tilde{s}})_{s \in S^{\mathrm{aff}}}$ where $\tilde{s}$ is an admissible lift of s in $W^{\mathrm{aff}}(1)$, generate the algebra $\mathcal{H}_R^{\mathrm{aff}}$ and

$$T_t T_w = T_{tw}, \quad T_{\tilde{s}} T_w = \begin{cases} T_{\tilde{s}w}, & \tilde{s}w > w, \\ c_{\tilde{s}} T_w, & \tilde{s}w < w, \end{cases}$$

with $c_{\tilde{s}} = -|Z'_{k,s}| \sum_{t \in Z'_{k,s}} T_t$ because the characteristic of R is p [Vig16, Proposition 4.4]. Expressing $f = \sum_{w \in W(1)} a_w T_w$, $a_w \in R$, as an infinite sum, we have

$$T_t f = \sum_{w \in W(1)} a_{t^{-1}w} T_w, \quad T_{\tilde{s}} f = \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w c_{\tilde{s}}) T_w.$$

A character χ of $\mathcal{H}_R^{\mathrm{aff}}$ is associated to a character $\chi_k : Z_k \to R^*$ and a subset J of

$$S_{\chi_k}^{\mathrm{aff}} = \{ s \in S^{\mathrm{aff}} \mid (\chi_k)|_{Z'_{k,s}} \text{ trivial } \}$$

[Vig15a, Definition 2.7]. We have

$$\begin{cases} \chi(T_t) = \chi_k(t), & t \in Z_k, \\ \chi(T_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\mathrm{aff}} \setminus J, \\ -1, & s \in J. \end{cases} \end{cases} \qquad (\chi_k)(c_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\mathrm{aff}} \setminus S_{\chi_k}^{\mathrm{aff}}, \\ -1, & s \in S_{\chi_k}^{\mathrm{aff}}. \end{cases} \tag{6.1}$$

Therefore $\chi_k(t) f = \check{T}_t f = T_{t^{-1}} f$ hence $\chi_k(t) a_w = a_{tw}$. We have $\chi(T_{\tilde{s}}) f = \check{T}_{\tilde{s}} f = T_{(\tilde{s})^{-1}} f = T_{\tilde{s}} T_{(\tilde{s})^{-2}} f = \chi_k((\tilde{s})^2) T_{\tilde{s}} f$; as $(\tilde{s})^2 \in Z'_{k,s}$ [Vig16, three lines before Proposition 4.4] and $J \subset S_{\chi_k}^{\mathrm{aff}}$, we obtain

$$T_{\tilde{s}} f = \begin{cases} 0, & s \in S^{\mathrm{aff}} \setminus J, \\ -f, & s \in J. \end{cases} \tag{6.2}$$

Introducing $\chi_k(t)a_w = a_{tw}$ in the formula for $T_{\tilde{s}}f$, we get

$$\begin{aligned}\sum_{w\in W(1),\tilde{s}w<w} a_w c_{\tilde{s}} T_w &= -|Z'_{k,s}|^{-1} \sum_{w\in W(1),\tilde{s}w<w,t\in Z'_{k,s}} a_w T_{tw} \\ &= -|Z'_{k,s}|^{-1} \sum_{w\in W(1),\tilde{s}w<w,t\in Z'_{k,s}} a_{t^{-1}w} T_w \\ &= -|Z'_{k,s}|^{-1} \sum_{t\in Z'_{k,s}} \chi_k(t^{-1}) \sum_{w\in W(1),\tilde{s}w<w} a_w T_w \\ &= \chi_k(c_{\tilde{s}}) \sum_{w\in W(1),\tilde{s}w<w} a_w T_w.\end{aligned}$$

$$\begin{aligned}T_{\tilde{s}}f &= \sum_{w\in W(1),\tilde{s}w<w} (a_{(\tilde{s})^{-1}w} + a_w\chi_k(c_{\tilde{s}}))T_w \\ &= \begin{cases}\sum_{w\in W(1),\tilde{s}w<w} a_{(\tilde{s})^{-1}w}T_w, & s\in S^{\mathrm{aff}}\setminus S^{\mathrm{aff}}_{\chi_k}, \\ \sum_{w\in W(1),\tilde{s}w<w}(a_{(\tilde{s})^{-1}w} - a_w)T_w, & s\in S^{\mathrm{aff}}_{\chi_k}.\end{cases}\end{aligned}$$

From the last equality and (6.2) for $T_{\tilde{s}}f$, we get:

$$a_{\tilde{s}w} = \begin{cases}0, & s\in J\cup(S^{\mathrm{aff}}\setminus S^{\mathrm{aff}}_{\chi_k}), \tilde{s}w<w, \\ a_w, & s\in S^{\mathrm{aff}}_{\chi_k}\setminus J.\end{cases} \tag{6.3}$$

Assume that $a_w \neq 0$. By the first condition, we know that $w > \tilde{s}w$ for $s \in J\cup(S^{\mathrm{aff}}\setminus S^{\mathrm{aff}}_{\chi_k})$. The character χ is supersingular if for each irreducible component X of S^{aff}, the intersection $X\cap J$ is not empty and different from X [Vig15a, Definition 2.7, Theorem 6.18]. This implies that the group generated by the $s \in S^{\mathrm{aff}}_{\chi_k}\setminus J$ is finite. If χ is supersingular, by the second condition we can suppose $w > \tilde{s}w$ for any $s \in S^{\mathrm{aff}}$. But there is no such element if S^{aff} is not empty. □

Theorem 6.4. *Let π be an irreducible admissible R-representation of G with a non-zero smooth dual where R is an algebraically closed field of characteristic p. Then π is finite dimensional.*

Proof. Let (P,σ,Q) be an $R[G]$-triple with σ supercuspidal such that $\pi \simeq I_G(P,\sigma,Q)$. The representation $I_G(P,\sigma,Q)$ is a quotient of $\mathrm{Ind}_Q^G\, e_Q(\sigma)$ hence the smooth dual of $\mathrm{Ind}_Q^G\, e_Q(\sigma)$ is not zero. From Proposition 6.2, $Q = G$. We have $I_G(P,\sigma,G) = e(\sigma)$. The smooth dual of σ contains the smooth linear dual of $e(\sigma)$ hence is not zero. As σ is supercuspidal, the $\mathcal{H}_M$-module $\sigma^{\mathcal{U}_M}$ contains a simple supersingular submodule $\mathcal{V}$ [Vig15a, Proposition 7.10, Corollary 7.11]. The functor $-\bigotimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M\backslash M]$ being the right adjoint of $(-)^{\mathcal{U}_M}$, the irreducible representation σ is a quotient of $\mathcal{V}\otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M\backslash M]$, hence the smooth dual of $\mathcal{V}\otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M\backslash M]$ is not zero. By Proposition 6.3, $M = Z$. Hence σ is finite dimensional and the same is true for $e(\sigma) = I_G(B,\sigma,G) \simeq \pi$. □

Remark 6.5. When the characteristic of F is 0, Theorem 6.4 was proved by Kohlhaase for a field R of characteristic p. He gives two proofs [Koh, Proposition 3.9, Remark 3.10], but none of them extends to F of characteristic p. Our proof is valid without restriction on the characteristic of F and does not use the results of Kohlhaase. Our assumption that R is an algebraically closed field of characteristic p comes from the classification theorem in [AHHV17].

References

[Abe] N. Abe, *Modulo p parabolic induction of pro-p-Iwahori Hecke algebra*, J. Reine Angew. Math., DOI:10.1515/crelle-2016-0043.

[Abe16] N. Abe, *Parabolic inductions for pro-p-Iwahori Hecke algebras*, arXiv:1612.01312.

[AHHV17] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, *A classification of irreducible admissible* mod p *representations of p-adic reductive groups*, J. Amer. Math. Soc. **30** (2017), no. 2, 495–559, DOI 10.1090/jams/862. MR3600042

[AHV] N. Abe, G. Henniart, and M.-F. Vignéras, *Modulo p representations of reductive p-adic groups: Functorial properties*, to appear in Transaction of AMS.

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local* (French), Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251. MR0327923

[Car85] Roger W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985. Conjugacy classes and complex characters; A Wiley-Interscience Publication. MR794307

[GK14] Elmar Grosse-Klönne, *On special representations of p-adic reductive groups*, Duke Math. J. **163** (2014), no. 12, 2179–2216, DOI 10.1215/00127094-2785697. MR3263032

[Koh] Jan Kohlhaase, *Smooth duality in natural characteristic*, Adv. Math. **317** (2017), 1–49, DOI 10.1016/j.aim.2017.06.038. MR3682662

[Ly15] Tony Ly, *Représentations de Steinberg modulo p pour un groupe réductif sur un corps local* (French, with English and French summaries), Pacific J. Math. **277** (2015), no. 2, 425–462, DOI 10.2140/pjm.2015.277.425. MR3402357

[OV17] R. Ollivier and M.-F. Vignéras, *Parabolic induction in characteristic p*, arXiv:1703.04921.

[Vig89] Marie-France Vignéras, *Représentations modulaires de* $\mathrm{GL}(2, F)$ *en caractéristique* l, *F corps p-adique,* $p \neq l$ (French), Compositio Math. **72** (1989), no. 1, 33–66. MR1026328

[Vig07] Marie-France Vignéras, *Représentations irréductibles de* $\mathrm{GL}(2, F)$ *modulo p* (French, with English summary), L-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, pp. 548–563, DOI 10.1017/CBO9780511721267.015. MR2392364

[Vig14] Marie-France Vignéras, *The pro-p-Iwahori-Hecke algebra of a reductive p-adic group, II*, Münster J. Math. **7** (2014), no. 1, 363–379. MR3271250

[Vig15a] Marie-France Vignéras, *The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction)*, Pacific J. Math. **279** (2015), no. 1-2, 499–529, DOI 10.2140/pjm.2015.279.499. MR3437789

[Vig15b] Marie-France Vignéras, *The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction)*, Pacific J. Math. **279** (2015), no. 1-2, 499–529, DOI 10.2140/pjm.2015.279.499. MR3437789

[Vig16] Marie-France Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group I*, Compos. Math. **152** (2016), no. 4, 693–753, DOI 10.1112/S0010437X15007666. MR3484112

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

Email address: abenori@math.sci.hokudai.ac.jp

Université de Paris-Sud, Laboratoire de Mathématiques d'Orsay, Orsay cedex F-91405 France; CNRS, Orsay cedex F-91405 France

Email address: Guy.Henniart@math.u-psud.fr

Institut de Mathématiques de Jussieu, 175 rue du Chevaleret, Paris 75013 France

Email address: vigneras@math.jussieu.fr