



# On pro- $p$ -Iwahori invariants of $R$ -representations of reductive $p$ -adic groups

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# ON PRO- $p$ -IWAHORI INVARIANTS OF $R$ -REPRESENTATIONS OF REDUCTIVE $p$ -ADIC GROUPS

N. ABE, G. HENNIART, AND M.-F. VIGNÉRAS

ABSTRACT. Let  $F$  be a locally compact field with residue characteristic  $p$ , and let  $\mathbf{G}$  be a connected reductive  $F$ -group. Let  $\mathcal{U}$  be a pro- $p$  Iwahori subgroup of  $G = \mathbf{G}(F)$ . Fix a commutative ring  $R$ . If  $\pi$  is a smooth  $R[G]$ -representation, the space of invariants  $\pi^{\mathcal{U}}$  is a right module over the Hecke algebra  $\mathcal{H}$  of  $\mathcal{U}$  in  $G$ .

Let  $P$  be a parabolic subgroup of  $G$  with a Levi decomposition  $P = MN$  adapted to  $\mathcal{U}$ . We complement a previous investigation of Ollivier-Vignéras on the relation between taking  $\mathcal{U}$ -invariants and various functor like  $\mathrm{Ind}_P^G$  and right and left adjoints. More precisely the authors' previous work with Herzig introduced representations  $I_G(P, \sigma, Q)$  where  $\sigma$  is a smooth representation of  $M$  extending, trivially on  $N$ , to a larger parabolic subgroup  $P(\sigma)$ , and  $Q$  is a parabolic subgroup between  $P$  and  $P(\sigma)$ . Here we relate  $I_G(P, \sigma, Q)^{\mathcal{U}}$  to an analogously defined  $\mathcal{H}$ -module  $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$ , where  $\mathcal{U}_M = \mathcal{U} \cap M$  and  $\sigma^{\mathcal{U}_M}$  is seen as a module over the Hecke algebra  $\mathcal{H}_M$  of  $\mathcal{U}_M$  in  $M$ . In the reverse direction, if  $\mathcal{V}$  is a right  $\mathcal{H}_M$ -module, we relate  $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes \mathrm{c}\text{-Ind}_{\mathcal{U}}^G \mathbf{1}$  to  $I_G(P, \mathcal{V} \otimes_{\mathcal{H}_M} \mathrm{c}\text{-Ind}_{\mathcal{U}_M}^M \mathbf{1}, Q)$ . As an application we prove that if  $R$  is an algebraically closed field of characteristic  $p$ , and  $\pi$  is an irreducible admissible representation of  $G$ , then the contragredient of  $\pi$  is 0 unless  $\pi$  has finite dimension.

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## 1. INTRODUCTION

1.1. The present paper is a companion to [AHV] and is similarly inspired by the classification results of [AHHV17]; however it can be read independently. We recall the setting. We have a non-archimedean locally compact field  $F$  of residue characteristic  $p$  and a connected reductive  $F$ -group  $\mathbf{G}$ . We fix a commutative ring  $R$  and study the smooth  $R$ -representations of  $G = \mathbf{G}(F)$ .

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In [AHHV17] the irreducible admissible  $R$ -representations of  $G$  are classified in terms of supersingular ones when  $R$  is an algebraically closed field of characteristic  $p$ . That classification is expressed in terms of representations  $I_G(P, \sigma, Q)$ , which make sense for any  $R$ . In that notation,  $P$  is a parabolic subgroup of  $G$  with a Levi decomposition  $P = MN$  and  $\sigma$  a smooth  $R$ -representation of the Levi subgroup  $M$ ; there is a maximal parabolic subgroup  $P(\sigma)$  of  $G$  containing  $P$  to which  $\sigma$  inflated to  $P$  extends to a representation  $e_{P(\sigma)}(\sigma)$ , and  $Q$  is a parabolic subgroup of  $G$  with  $P \subset Q \subset P(\sigma)$ . Then

$$I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e_{P(\sigma)}(\sigma) \otimes \text{St}_Q^{P(\sigma)}),$$

where  $\text{Ind}$  stands for parabolic induction and  $\text{St}_Q^{P(\sigma)} = \text{Ind}_{Q'}^{P(\sigma)} R / \sum \text{Ind}_{Q''}^{P(\sigma)} R$ , the sum being over parabolic subgroups  $Q'$  of  $G$  with  $Q \subsetneq Q' \subset P(\sigma)$ . Alternatively,  $I_G(P, \sigma, Q)$  is the quotient of  $\text{Ind}_Q^G(e_Q(\sigma))$  by  $\sum \text{Ind}_{Q'}^G e_{Q'}(\sigma)$  with  $Q'$  as above, where  $e_Q(\sigma)$  is the restriction of  $e_{P(\sigma)}(\sigma)$  to  $Q$ , similarly for  $Q'$ .

In [AHV] we mainly studied what happens to  $I_G(P, \sigma, Q)$  when we apply to it, for a parabolic subgroup  $P_1$  of  $G$ , the left adjoint of  $\text{Ind}_{P_1}^G$ , or its right adjoint. Here we tackle a different question. We fix a pro- $p$  Iwahori subgroup  $\mathcal{U}$  of  $G$  in good position with respect to  $P$ , so that in particular  $\mathcal{U}_M = \mathcal{U} \cap M$  is a pro- $p$  Iwahori subgroup of  $M$ . One of our main goals is to identify the  $R$ -module  $I_G(P, \sigma, Q)^{\mathcal{U}}$  of  $\mathcal{U}$ -invariants, as a right module over the Hecke algebra  $\mathcal{H} = \mathcal{H}_G$  of  $\mathcal{U}$  in  $G$  - the convolution algebra on the double coset space  $\mathcal{U} \backslash G / \mathcal{U}$  - in terms of the module  $\sigma^{\mathcal{U}_M}$  over the Hecke algebra  $\mathcal{H}_M$  of  $\mathcal{U}_M$  in  $M$ . That goal is achieved in section 4, Theorem 4.17.

1.2. The initial work has been done in [OV17, §4] where  $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$  is identified. Precisely, writing  $M^+$  for the monoid of elements  $m \in M$  with  $m(\mathcal{U} \cap N)m^{-1} \subset \mathcal{U} \cap N$ , the subalgebra  $\mathcal{H}_{M^+}$  of  $\mathcal{H}_M$  with support in  $M^+$ , has a natural algebra embedding  $\theta$  into  $\mathcal{H}$  and [OV17, Proposition 4.4] identifies  $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$  with  $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma^{\mathcal{U}_M}) = \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+}} \mathcal{H}$ . So in a sense, this paper is a sequel to [OV17] although some of our results here are used in [OV17, §5].

As  $I_G(P, \sigma, Q)$  is only a subquotient of  $\text{Ind}_P^G \sigma$  and taking  $\mathcal{U}$ -invariants is only left exact, it is not straightforward to describe  $I_G(P, \sigma, Q)^{\mathcal{U}}$  from the previous result. However, that takes care of the parabolic induction step, so in a first approach we may assume  $P(\sigma) = G$  so that  $I_G(P, \sigma, Q) = e_G(\sigma) \otimes \text{St}_Q^G$ . The crucial case is when moreover  $\sigma$  is  $e$ -minimal, that is, not an extension  $e_M(\tau)$  of a smooth  $R$ -representation  $\tau$  of a proper Levi subgroup of  $M$ . That case is treated first and the general case in section 4 only.

1.3. To explain our results, we need more notation. We choose a maximal  $F$ -split torus  $T$  in  $G$  and a minimal parabolic subgroup  $B = ZU$  with Levi component  $Z$  the  $G$ -centralizer of  $T$ . We assume that  $P = MN$  contains  $B$  and  $M$  contains  $Z$ , and that  $\mathcal{U}$  corresponds to an alcove in the apartment associated to  $T$  in the adjoint building of  $G$ . It turns out that when  $\sigma$  is  $e$ -minimal and  $P(\sigma) = G$ , the set  $\Delta_M$  of simple roots of  $T$  in  $\text{Lie}(M \cap U)$  is orthogonal to its complement in the set  $\Delta$  of simple roots of  $T$  in  $\text{Lie} U$ . We assume until the end of this section that  $\Delta_M$  and  $\Delta_2 = \Delta \setminus \Delta_M$  are orthogonal. If  $M_2$  is the Levi subgroup - containing  $Z$  - corresponding to  $\Delta_2$ , both  $M$  and  $M_2$  are normal in  $G$ ,  $M \cap M_2 = Z$  and  $G = MM_2$ . Moreover the normal subgroup  $M'_2$  of  $G$  generated by  $N$  is included in  $M_2$  and  $G = MM'_2$ .

We say that a right  $\mathcal{H}_M$ -module  $\mathcal{V}$  is extensible to  $\mathcal{H}$  if  $T_z^M$  acts trivially on  $\mathcal{V}$  for  $z \in Z \cap M'_2$  (section 3.3). In this case, we show that there is a natural structure of right  $\mathcal{H}$ -module  $e_{\mathcal{H}}(\mathcal{V})$  on  $\mathcal{V}$  such that  $T_g \in \mathcal{H}$  corresponding to  $\mathcal{U}g\mathcal{U}$  for  $g \in M'_2$  acts as in the trivial character of  $G$  (section 3.4). We call  $e_{\mathcal{H}}(\mathcal{V})$  the extension of  $\mathcal{V}$  to  $\mathcal{H}$  though  $\mathcal{H}_M$  is not a subalgebra of  $\mathcal{H}$ . That notion is already present in [Abe] in the case where  $R$  has characteristic  $p$ . Here we extend the construction to any  $R$  and prove some more properties. In particular we produce an  $\mathcal{H}$ -equivariant embedding  $e_{\mathcal{H}}(\mathcal{V})$  into  $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \mathcal{V}$  (Lemma 3.10). If  $Q$  is a parabolic subgroup of  $G$  containing  $P$ , we go further and put on  $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\text{Ind}_Q^G R)^{\mathcal{U}}$  and  $e_{\mathcal{H}}(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$  structures of  $\mathcal{H}$ -modules (Proposition 3.15 and Corollary 3.17) - note that  $\mathcal{H}$  is not a group algebra and there is no obvious notion of tensor product of  $\mathcal{H}$ -modules.

If  $\sigma$  is an  $R$ -representation of  $M$  extensible to  $G$ , then its extension  $e_G(\sigma)$  is simply obtained by letting  $M'_2$  act trivially on the space of  $\sigma$ ; moreover it is clear that  $\sigma^{\mathcal{U}_M}$  is extensible to  $\mathcal{H}$ , and one shows easily that  $e_G(\sigma)^{\mathcal{U}} = e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$  as an  $\mathcal{H}$ -module (section 3.5). Moreover, the natural inclusion of  $e_G(\sigma)$  into  $\text{Ind}_P^G \sigma$  induces on taking pro- $p$  Iwahori invariants an embedding  $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \rightarrow (\text{Ind}_P^G \sigma)^{\mathcal{U}}$  which, via the isomorphism of [OV17], yields exactly the above embedding of  $\mathcal{H}$ -modules of  $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M})$  into  $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\sigma^{\mathcal{U}_M})$ .

Then we show the  $\mathcal{H}$ -modules  $(e_G(\sigma) \otimes_R \text{Ind}_Q^G R)^{\mathcal{U}}$  and  $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\text{Ind}_Q^G R)^{\mathcal{U}}$  are equal, and similarly  $(e_G(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}}$  and  $e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$  are equal (Theorem 4.9).

1.4. We turn back to the general case where we do not assume that  $\Delta_M$  and  $\Delta \setminus \Delta_M$  are orthogonal. Nevertheless, given a right  $\mathcal{H}_M$ -module  $\mathcal{V}$ , there exists a largest Levi subgroup  $M(\mathcal{V})$  of  $G$  - containing  $Z$  - corresponding to  $\Delta_M \cup \Delta_1$  where  $\Delta_1$  is a subset of  $\Delta \setminus \Delta_M$  orthogonal to  $\Delta_M$ , such that  $\mathcal{V}$  extends to a right  $\mathcal{H}_{M(\mathcal{V})}$ -module  $e_{M(\mathcal{V})}(\mathcal{V})$  with the notation of section 1.3. For any parabolic subgroup  $Q$  between  $P$  and  $P(\mathcal{V}) = M(\mathcal{V})U$  we put (Definition 4.12)

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} (e_{M(\mathcal{V})}(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}).$$

We refer to Theorem 4.17 for the description of the right  $\mathcal{H}$ -module  $I_G(P, \sigma, Q)^{\mathcal{U}}$  for any smooth  $R$ -representation  $\sigma$  of  $\mathcal{U}$ . As a special case, it says that when  $\sigma$  is  $e$ -minimal then  $P(\sigma) \supset P(\sigma^{\mathcal{U}_M})$  and if moreover  $P(\sigma) = P(\sigma^{\mathcal{U}_M})$ , then  $I_G(P, \sigma, Q)^{\mathcal{U}}$  is isomorphic to  $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$ .

*Remark 1.1.* In [Abe] are attached similar  $\mathcal{H}$ -modules to  $(P, \mathcal{V}, Q)$ ; here we write them as  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$  because their definition uses, instead of  $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}$  a different kind of induction, which we call coinduction. In [Abe] those modules are used to give, when  $R$  is an algebraically closed field of characteristic  $p$ , a classification of simple  $\mathcal{H}$ -modules in terms of supersingular modules - that classification is similar to the classification of irreducible admissible  $R$ -representations of  $G$  in [AHHV17]. Using the comparison between induced and coinduced modules established in [Vig15b, 4.3] for any  $R$ , our Corollary 4.24 expresses  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$  as a module  $I_{\mathcal{H}}(P_1, \mathcal{V}_1, Q_1)$ ; consequently we show in section 4.5 that the classification of [Abe] can also be expressed in terms of modules  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ .

1.5. In a reverse direction one can associate to a right  $\mathcal{H}$ -module  $\mathcal{V}$  a smooth  $R$ -representation  $\mathcal{V} \otimes_{\mathcal{H}} R[\mathcal{U} \setminus G]$  of  $G$  (seeing  $\mathcal{H}$  as the endomorphism ring of the  $R[G]$ -module  $R[\mathcal{U} \setminus G]$ ).

If  $\mathcal{V}$  is a right  $\mathcal{H}_M$ -module, we construct, again using [OV17], a natural  $R[G]$ -map

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G] \rightarrow \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\mathcal{V} \otimes_{\mathcal{H}_M} R[\mathcal{U}_M \backslash M]) \otimes_R \text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})}),$$

with the notation of section 1.4. We show in section 5 that it is an isomorphism under a mild assumption on the  $\mathbb{Z}$ -torsion in  $\mathcal{V}$ ; in particular it is an isomorphism if  $p = 0$  in  $R$ .

1.6. In the final section 6, we turn back to the case where  $R$  is an algebraically closed field of characteristic  $p$ . We prove that the smooth dual of an irreducible admissible  $R$ -representation  $V$  of  $G$  is 0 unless  $V$  is finite dimensional - that result is new if  $F$  has positive characteristic, a case where the proof of Kohlhaase [Koh] for  $\text{char}(F) = 0$  does not apply. Our proof first reduces to the case where  $V$  is supercuspidal (by [AHHV17]) then uses again the  $\mathcal{H}$ -module  $V^{\mathcal{U}}$ .

## 2. NOTATION, USEFUL FACTS, AND PRELIMINARIES

2.1. **The group  $G$  and its standard parabolic subgroups  $P = MN$ .** In all that follows,  $p$  is a prime number and  $F$  is a local field with finite residue field  $k$  of characteristic  $p$ . We denote an algebraic group over  $F$  by a bold letter, like  $\mathbf{H}$ , and use the same ordinary letter for the group of  $F$ -points,  $H = \mathbf{H}(F)$ . We fix a connected reductive  $F$ -group  $\mathbf{G}$ . We fix a maximal  $F$ -split subtorus  $\mathbf{T}$  and write  $\mathbf{Z}$  for its  $\mathbf{G}$ -centralizer; we also fix a minimal parabolic subgroup  $\mathbf{B}$  of  $\mathbf{G}$  with Levi component  $\mathbf{Z}$ , so that  $\mathbf{B} = \mathbf{Z}\mathbf{U}$  where  $\mathbf{U}$  is the unipotent radical of  $\mathbf{B}$ . Let  $X^*(\mathbf{T})$  be the group of  $F$ -rational characters of  $\mathbf{T}$  and let  $\Phi$  be the subset of roots of  $\mathbf{T}$  in the Lie algebra of  $\mathbf{G}$ . Then  $\mathbf{B}$  determines a subset  $\Phi^+$  of positive roots - the roots of  $\mathbf{T}$  in the Lie algebra of  $\mathbf{U}$ - and a subset of simple roots  $\Delta$ . The  $\mathbf{G}$ -normalizer  $\mathbf{N}_{\mathbf{G}}$  of  $\mathbf{T}$  acts on  $X^*(\mathbf{T})$  and through that action,  $\mathbf{N}_{\mathbf{G}}/\mathbf{Z}$  identifies with the Weyl group of the root system  $\Phi$ . Set  $\mathcal{N} := \mathbf{N}_{\mathbf{G}}(F)$  and note that  $\mathbf{N}_{\mathbf{G}}/\mathbf{Z} \simeq \mathcal{N}/Z$ ; we write  $\mathbb{W}$  for  $\mathcal{N}/Z$ .

A standard parabolic subgroup of  $\mathbf{G}$  is a parabolic  $F$ -subgroup containing  $\mathbf{B}$ . Such a parabolic subgroup  $\mathbf{P}$  has a unique Levi subgroup  $\mathbf{M}$  containing  $\mathbf{Z}$ , so that  $\mathbf{P} = \mathbf{M}\mathbf{N}$  where  $\mathbf{N}$  is the unipotent radical of  $\mathbf{P}$  - we also call  $\mathbf{M}$  standard. By a common abuse of language to describe the preceding situation, we simply say “let  $P = MN$  be a standard parabolic subgroup of  $G$ ”; we sometimes write  $N_P$  for  $N$  and  $M_P$  for  $M$ . The parabolic subgroup of  $G$  opposite to  $P$  will be written  $\overline{P}$  and its unipotent radical  $\overline{N}$ , so that  $\overline{P} = M\overline{N}$ , but beware that  $\overline{P}$  is not standard! We write  $\mathbb{W}_M$  for the Weyl group  $(M \cap \mathcal{N})/Z$ .

If  $\mathbf{P} = \mathbf{M}\mathbf{N}$  is a standard parabolic subgroup of  $G$ , then  $\mathbf{M} \cap \mathbf{B}$  is a minimal parabolic subgroup of  $\mathbf{M}$ . If  $\Phi_M$  denotes the set of roots of  $\mathbf{T}$  in the Lie algebra of  $\mathbf{M}$ , with respect to  $\mathbf{M} \cap \mathbf{B}$  we have  $\Phi_M^+ = \Phi_M \cap \Phi^+$  and  $\Delta_M = \Phi_M \cap \Delta$ . We also write  $\Delta_P$  for  $\Delta_M$  as  $P$  and  $M$  determine each other,  $P = MU$ . Thus we obtain a bijection  $P \mapsto \Delta_P$  from standard parabolic subgroups of  $G$  to subsets of  $\Delta$ , with  $B$  corresponding to  $\Phi$  and  $G$  to  $\Delta$ . If  $I$  is a subset of  $\Delta$ , we sometimes denote by  $P_I = M_I N_I$  the corresponding standard parabolic subgroup of  $G$ . If  $I = \{\alpha\}$  is a singleton, we write  $P_{\alpha} = M_{\alpha} N_{\alpha}$ . We note a few useful properties. If  $P_1$  is another standard parabolic subgroup of  $G$ , then  $P \subset P_1$  if and only if  $\Delta_P \subset \Delta_{P_1}$ ; we have  $\Delta_{P \cap P_1} = \Delta_P \cap \Delta_{P_1}$  and the parabolic subgroup corresponding to  $\Delta_P \cup \Delta_{P_1}$  is the subgroup  $\langle P, P_1 \rangle$  of  $G$  generated by  $P$  and  $P_1$ . The standard parabolic subgroup of  $M$  associated to  $\Delta_M \cap \Delta_{M_1}$  is  $M \cap P_1 = (M \cap M_1)(M \cap N_1)$  [Car85, Proposition 2.8.9]. It is convenient to write  $G'$  for the subgroup of  $G$  generated by the unipotent

radicals of the parabolic subgroups; it is also the normal subgroup of  $G$  generated by  $U$ , and we have  $G = ZG'$ . For future reference, we now give a useful lemma extracted from [AHHV17].

**Lemma 2.1.** *The group  $Z \cap G'$  is generated by the  $Z \cap M'_\alpha$ ,  $\alpha$  running through  $\Delta$ .*

*Proof.* Take  $I = \emptyset$  in [AHHV17, II.6.Proposition].  $\square$

Let  $v_F$  be the normalized valuation of  $F$ . For each  $\alpha \in X^*(T)$ , the homomorphism  $x \mapsto v_F(\alpha(x)) : T \rightarrow \mathbb{Z}$  extends uniquely to a homomorphism  $Z \rightarrow \mathbb{Q}$  that we denote in the same way. This defines a homomorphism  $Z \xrightarrow{v} X_*(T) \otimes \mathbb{Q}$  such that  $\alpha(v(z)) = v_F(\alpha(z))$  for  $z \in Z, \alpha \in X^*(T)$ .

An interesting situation occurs when  $\Delta = I \sqcup J$  is the union of two orthogonal subsets  $I$  and  $J$ . In that case,  $G' = M'_I M'_J$ ,  $M'_I$  and  $M'_J$  commute with each other, and their intersection is finite and central in  $G$  [AHHV17, II.7 Remark 4].

**2.2.  $I_G(P, \sigma, Q)$  and minimality.** We recall from [AHHV17] the construction of  $I_G(P, \sigma, Q)$ , our main object of study.

Let  $\sigma$  be an  $R$ -representation of  $M$  and let  $P_\sigma$  be the standard parabolic subgroup with  $\Delta_{P_\sigma} = \Delta_\sigma$  where

$$\Delta_\sigma = \{\alpha \in \Delta \setminus \Delta_P \mid Z \cap M'_\alpha \text{ acts trivially on } \sigma\}.$$

We also let  $P(\sigma)$  be the standard parabolic subgroup with

$$\Delta_{P(\sigma)} = \Delta_\sigma \cup \Delta_P.$$

This is the largest parabolic subgroup  $P(\sigma)$  containing  $P$  to which  $\sigma$  extends, here  $N \subset P$  acts on  $\sigma$  trivially. Clearly when  $P \subset Q \subset P(\sigma)$ ,  $\sigma$  extends to  $Q$  and the extension is denoted by  $e_Q(\sigma)$ . The restriction of  $e_{P(\sigma)}(\sigma)$  to  $Q$  is  $e_Q(\sigma)$ . If there is no risk of ambiguity, we write

$$e(\sigma) = e_{P(\sigma)}(\sigma).$$

**Definition 2.2.** An  $R[G]$ -triple is a triple  $(P, \sigma, Q)$  made out of a standard parabolic subgroup  $P = MN$  of  $G$ , a smooth  $R$ -representation of  $M$ , and a parabolic subgroup  $Q$  of  $G$  with  $P \subset Q \subset P(\sigma)$ . To an  $R[G]$ -triple  $(P, \sigma, Q)$  is associated a smooth  $R$ -representation of  $G$ :

$$I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \text{St}_Q^{P(\sigma)}),$$

where  $\text{St}_Q^{P(\sigma)}$  is the quotient of  $\text{Ind}_Q^{P(\sigma)} \mathbf{1}$ ,  $\mathbf{1}$  denoting the trivial  $R$ -representation of  $Q$ , by the sum of its subrepresentations  $\text{Ind}_{Q'}^{P(\sigma)} \mathbf{1}$ , the sum being over the set of parabolic subgroups  $Q'$  of  $G$  with  $Q \subsetneq Q' \subset P(\sigma)$ .

Note that  $I_G(P, \sigma, Q)$  is naturally isomorphic to the quotient of  $\text{Ind}_Q^G(e_Q(\sigma))$  by the sum of its subrepresentations  $\text{Ind}_{Q'}^G(e_{Q'}(\sigma))$  for  $Q \subsetneq Q' \subset P(\sigma)$  by [AHV, Lemma 2.5].

It might happen that  $\sigma$  itself has the form  $e_P(\sigma_1)$  for some standard parabolic subgroup  $P_1 = M_1 N_1$  contained in  $P$  and some  $R$ -representation  $\sigma_1$  of  $M_1$ . In that case,  $P(\sigma_1) = P(\sigma)$  and  $e(\sigma) = e(\sigma_1)$ . We say that  $\sigma$  is  **$e$ -minimal** if  $\sigma = e_P(\sigma_1)$  implies  $P_1 = P, \sigma_1 = \sigma$ .

**Lemma 2.3** ([AHV, Lemma 2.9]). *Let  $P = MN$  be a standard parabolic subgroup of  $G$  and let  $\sigma$  be an  $R$ -representation of  $M$ . There exists a unique standard parabolic subgroup  $P_{\min, \sigma} = M_{\min, \sigma} N_{\min, \sigma}$  of  $G$  and a unique  $e$ -minimal representation of  $\sigma_{\min}$  of  $M_{\min, \sigma}$  with  $\sigma = e_P(\sigma_{\min})$ . Moreover  $P(\sigma) = P(\sigma_{\min})$  and  $e(\sigma) = e(\sigma_{\min})$ .*

**Lemma 2.4.** *Let  $P = MN$  be a standard parabolic subgroup of  $G$  and let  $\sigma$  be an  $e$ -minimal  $R$ -representation of  $M$ . Then  $\Delta_P$  and  $\Delta_{P(\sigma)} \setminus \Delta_P$  are orthogonal.*

That comes from [AHHV17, II.7 Corollary 2]. That corollary of [AHHV17] also shows that when  $R$  is a field and  $\sigma$  is supercuspidal, then  $\sigma$  is  $e$ -minimal. Lemma 2.4 shows that  $\Delta_{P_{\min, \sigma}}$  and  $\Delta_{P(\sigma_{\min})} \setminus \Delta_{P_{\min, \sigma}}$  are orthogonal.

Note that when  $\Delta_P$  and  $\Delta_\sigma$  are orthogonal of union  $\Delta = \Delta_P \sqcup \Delta_\sigma$ , then  $G = P(\sigma) = MM'_\sigma$  and  $e(\sigma)$  is the  $R$ -representation of  $G$  simply obtained by extending  $\sigma$  trivially on  $M'_\sigma$ .

**Lemma 2.5** ([AHV, Lemma 2.11]). *Let  $(P, \sigma, Q)$  be an  $R[G]$ -triple. Then we have that  $(P_{\min, \sigma}, \sigma_{\min}, Q)$  is an  $R[G]$ -triple and  $I_G(P, \sigma, Q) = I_G(P_{\min, \sigma}, \sigma_{\min}, Q)$ .*

**2.3. Pro- $p$  Iwahori Hecke algebras.** We fix a special parahoric subgroup  $\mathcal{K}$  of  $G$  fixing a special vertex  $x_0$  in the apartment  $\mathcal{A}$  associated to  $T$  in the Bruhat-Tits building of the adjoint group of  $G$ . We let  $\mathcal{B}$  be the Iwahori subgroup fixing the alcove  $\mathcal{C}$  in  $\mathcal{A}$  with vertex  $x_0$  contained in the Weyl chamber (of vertex  $x_0$ ) associated to  $B$ . We let  $\mathcal{U}$  be the pro- $p$  radical of  $\mathcal{B}$  (the pro- $p$  Iwahori subgroup). The pro- $p$  Iwahori Hecke ring  $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$  is the convolution ring of compactly supported functions  $G \rightarrow \mathbb{Z}$  constant on the double classes of  $G$  modulo  $\mathcal{U}$ . We denote by  $T(g)$  the characteristic function of  $\mathcal{U}g\mathcal{U}$  for  $g \in G$ , seen as an element of  $\mathcal{H}$ . Let  $R$  be a commutative ring. The pro- $p$  Iwahori Hecke  $R$ -algebra  $\mathcal{H}_R$  is  $R \otimes_{\mathbb{Z}} \mathcal{H}$ . We will follow the custom to still denote by  $h$  the natural image  $1 \otimes h$  of  $h \in \mathcal{H}$  in  $\mathcal{H}_R$ .

For  $P = MN$  a standard parabolic subgroup of  $G$ , the similar objects for  $M$  are indexed by  $M$ , we have  $\mathcal{K}_M = \mathcal{K} \cap M$ ,  $\mathcal{B}_M = \mathcal{B} \cap M$ ,  $\mathcal{U}_M = \mathcal{U} \cap M$ , the pro- $p$  Iwahori Hecke ring  $\mathcal{H}_M = \mathcal{H}(M, \mathcal{U}_M)$ ,  $T^M(m)$  the characteristic function of  $\mathcal{U}_M m \mathcal{U}_M$  for  $m \in M$ , seen as an element of  $\mathcal{H}_M$ . The pro- $p$  Iwahori subgroup  $\mathcal{U}$  of  $G$  satisfies the Iwahori decomposition with respect to  $P$ :

$$\mathcal{U} = \mathcal{U}_N \mathcal{U}_M \mathcal{U}_{\overline{N}},$$

where  $\mathcal{U}_N = \mathcal{U} \cap N$ ,  $\mathcal{U}_{\overline{N}} = \mathcal{U} \cap \overline{N}$ . The linear map

$$(2.1) \quad \mathcal{H}_M \xrightarrow{\theta} \mathcal{H}, \quad \theta(T^M(m)) = T(m) \quad (m \in M)$$

does not respect the product. But if we introduce the monoid  $M^+$  of elements  $m \in M$  contracting  $\mathcal{U}_N$ , meaning  $m\mathcal{U}_N m^{-1} \subset \mathcal{U}_N$ , and the submodule  $\mathcal{H}_{M^+} \subset \mathcal{H}_M$  of functions with support in  $M^+$ , we have [Vig15b, Theorem 1.4]:

$\mathcal{H}_{M^+}$  is a subring of  $\mathcal{H}_M$  and  $\mathcal{H}_M$  is the localization of  $\mathcal{H}_{M^+}$  at an element  $\tau^M \in \mathcal{H}_{M^+}$  central and invertible in  $\mathcal{H}_M$ , meaning  $\mathcal{H}_M = \bigcup_{n \in \mathbb{N}} \mathcal{H}_{M^+} (\tau^M)^{-n}$ . The map  $\mathcal{H}_M \xrightarrow{\theta} \mathcal{H}$  is injective and its restriction  $\theta|_{\mathcal{H}_{M^+}}$  to  $\mathcal{H}_{M^+}$  respects the product.

These properties are also true when  $(M^+, \tau^M)$  is replaced by its inverse  $(M^-, (\tau^M)^{-1})$  where  $M^- = \{m^{-1} \in M \mid m \in M^+\}$ .

### 3. PRO- $p$ IWAHORI INVARIANTS OF $I_G(P, \sigma, Q)$

**3.1. Pro- $p$  Iwahori Hecke algebras: Structures.** Here we supplement the notation of sections 2.1 and 2.3. The subgroups  $Z^0 = Z \cap \mathcal{K} = Z \cap \mathcal{B}$  and  $Z^1 = Z \cap \mathcal{U}$

are normal in  $\mathcal{N}$  and we put

$$W = \mathcal{N}/Z^0, \quad W(1) = \mathcal{N}/Z^1, \quad \Lambda = Z/Z^0, \quad \Lambda(1) = Z/Z^1, \quad Z_k = Z^0/Z^1.$$

We have  $\mathcal{N} = (\mathcal{N} \cap \mathcal{K})Z$  so that we see the finite Weyl group  $\mathbb{W} = \mathcal{N}/Z$  as the subgroup  $(\mathcal{N} \cap \mathcal{K})/Z^0$  of  $W$ ; in this way  $W$  is the semidirect product  $\Lambda \rtimes \mathbb{W}$ . We put  $\mathcal{N}_{G'} = \mathcal{N} \cap G'$ . The image  $W_{G'} = W'$  of  $\mathcal{N}_{G'}$  in  $W$  is an affine Weyl group generated by the set  $S^{\text{aff}}$  of affine reflections determined by the walls of the alcove  $\mathcal{C}$ . The group  $W'$  is normal in  $W$  and  $W$  is the semidirect product  $W' \rtimes \Omega$  where  $\Omega$  is the image in  $W$  of the normalizer  $\mathcal{N}_{\mathcal{C}}$  of  $\mathcal{C}$  in  $\mathcal{N}$ . The length function  $\ell$  on the affine Weyl system  $(W', S^{\text{aff}})$  extends to a length function on  $W$  such that  $\Omega$  is the set of elements of length 0. We also view  $\ell$  as a function of  $W(1)$  via the quotient map  $W(1) \rightarrow W$ . We write

(3.1)

$(\hat{w}, \tilde{w}, w) \in \mathcal{N} \times W(1) \times W$  corresponding via the quotient maps  $\mathcal{N} \rightarrow W(1) \rightarrow W$ .

When  $w = s$  in  $S^{\text{aff}}$  or more generally  $w$  in  $W_{G'}$ , we will most of the time choose  $\hat{w}$  in  $\mathcal{N} \cap G'$  and  $\tilde{w}$  in the image  ${}_1W_{G'}$  of  $\mathcal{N} \cap G'$  in  $W(1)$ .

We are now ready to describe the pro- $p$  Iwahori Hecke ring  $\mathcal{H} = \mathcal{H}(G, \mathcal{U})$  [Vig16]. We have  $G = \mathcal{U}\mathcal{N}\mathcal{U}$  and for  $n, n' \in \mathcal{N}$  we have  $\mathcal{U}n\mathcal{U} = \mathcal{U}n'\mathcal{U}$  if and only if  $nZ^1 = n'Z^1$ . For  $n \in \mathcal{N}$  of image  $w \in W(1)$  and  $g \in \mathcal{U}n\mathcal{U}$  we denote  $T_w = T(n) = T(g)$  in  $\mathcal{H}$ . The relations among the basis elements  $(T_w)_{w \in W(1)}$  of  $\mathcal{H}$  are:

- (1) Braid relations:  $T_w T_{w'} = T_{ww'}$  for  $w, w' \in W(1)$  with  $\ell(ww') = \ell(w) + \ell(w')$ .
- (2) Quadratic relations:  $T_{\tilde{s}}^2 = q_s T_{\tilde{s}^2} + c_{\tilde{s}} T_{\tilde{s}}$

for  $\tilde{s} \in W(1)$  lifting  $s \in S^{\text{aff}}$ , where  $q_s = q_G(s) = |\mathcal{U}/\mathcal{U} \cap \hat{s}\mathcal{U}(\hat{s})^{-1}|$  depends only on  $s$ , and  $c_{\tilde{s}} = \sum_{t \in Z_k} c_{\tilde{s}}(t) T_t$  for integers  $c_{\tilde{s}}(t) \in \mathbb{N}$  summing to  $q_s - 1$ .

We shall need the basis elements  $(T_w^*)_{w \in W(1)}$  of  $\mathcal{H}$  defined by:

- (1)  $T_w^* = T_w$  for  $w \in W(1)$  of length  $\ell(w) = 0$ .
- (2)  $T_{\tilde{s}}^* = T_{\tilde{s}} - c_{\tilde{s}}$  for  $\tilde{s} \in W(1)$  lifting  $s \in S^{\text{aff}}$ .
- (3)  $T_{ww'}^* = T_w^* T_{w'}^*$  for  $w, w' \in W(1)$  with  $\ell(ww') = \ell(w) + \ell(w')$ .

We need more notation for the definition of the admissible lifts of  $S^{\text{aff}}$  in  $\mathcal{N}_G$ . Let  $s \in S^{\text{aff}}$  fixing a face  $\mathcal{C}_s$  of the alcove  $\mathcal{C}$  and  $\mathcal{K}_s$  the parahoric subgroup of  $G$  fixing  $\mathcal{C}_s$ . The theory of Bruhat-Tits associates to  $\mathcal{C}_s$  a certain root  $\alpha_s \in \Phi^+$  [Vig16, §4.2]. We consider the group  $G'_s$  generated by  $U_{\alpha_s} \cup U_{-\alpha_s}$  where  $U_{\pm\alpha_s}$  the root subgroup of  $\pm\alpha_s$  (if  $2\alpha_s \in \Phi$ , then  $U_{2\alpha_s} \subset U_{\alpha_s}$ ) and the group  $\mathcal{G}'_s$  generated by  $\mathcal{U}_{\alpha_s} \cup \mathcal{U}_{-\alpha_s}$  where  $\mathcal{U}_{\pm\alpha_s} = U_{\pm\alpha_s} \cap \mathcal{K}_s$ . When  $u \in \mathcal{U}_{\alpha_s} - \{1\}$ , the intersection  $\mathcal{N}_G \cap \mathcal{U}_{-\alpha_s} u \mathcal{U}_{-\alpha_s}$  (equal to  $\mathcal{N}_G \cap U_{-\alpha_s} u U_{-\alpha_s}$  [BT72, 6.2.1 (V5)], [Vig16, §3.3 (19)]) possesses a single element  $n_s(u)$ . The group  $Z'_s = Z \cap \mathcal{G}'_s$  is contained in  $Z \cap \mathcal{K}_s = Z^0$ ; its image in  $Z_k$  is denoted by  $Z'_{k,s}$ .

The elements  $n_s(u)$  for  $u \in \mathcal{U}_{\alpha_s} - \{1\}$  are the admissible lifts of  $s$  in  $\mathcal{N}_G$ ; their images in  $W(1)$  are the admissible lifts of  $s$  in  $W(1)$ . By [Vig16, Theorem 2.2, Proposition 4.4], when  $\tilde{s} \in W(1)$  is an admissible lift of  $s$ ,  $c_{\tilde{s}}(t) = 0$  if  $t \in Z_k \setminus Z'_{k,s}$ , and

$$(3.2) \quad c_{\tilde{s}} \equiv (q_s - 1) |Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} T_t \pmod{p}.$$

The admissible lifts of  $S$  in  $\mathcal{N}_G$  are contained in  $\mathcal{N}_G \cap \mathcal{K}$  because  $\mathcal{K}_s \subset \mathcal{K}$  when  $s \in S$ .



**Definition 3.1.** An admissible lift of the finite Weyl group  $\mathbb{W}$  in  $\mathcal{N}_G$  is a map  $w \mapsto \hat{w} : \mathbb{W} \rightarrow \mathcal{N}_G \cap \mathcal{K}$  such that  $\hat{s}$  is admissible for all  $s \in S$  and  $\hat{w} = \hat{w}_1 \hat{w}_2$  for  $w_1, w_2 \in \mathbb{W}$  such that  $w = w_1 w_2$  and  $\ell(w) = \ell(w_1) + \ell(w_2)$ .

Any choice of admissible lifts of  $S$  in  $\mathcal{N}_G \cap \mathcal{K}$  extends uniquely to an admissible lift of  $\mathbb{W}$  ([AHHV17, IV.6], [OV17, Proposition 2.7]).

Let  $P = MN$  be a standard parabolic subgroup of  $G$ . The groups  $Z, Z^0 = Z \cap \mathcal{K}_M = Z \cap \mathcal{B}_M, Z^1 = Z \cap \mathcal{U}_M$  are the same for  $G$  and  $M$ , but  $\mathcal{N}_M = \mathcal{N} \cap M$  and  $M \cap G'$  are subgroups of  $\mathcal{N}$  and  $G'$ . The monoid  $M^+$  (section 2.3) contains  $(\mathcal{N}_M \cap \mathcal{K})$  and is equal to  $M^+ = \mathcal{U}_M \mathcal{N}_{M^+} \mathcal{U}_M$  where  $\mathcal{N}_{M^+} = \mathcal{N} \cap M^+$ . An element  $z \in Z$  belongs to  $M^+$  if and only if  $v_F(\alpha(z)) \geq 0$  for all  $\alpha \in \Phi^+ \setminus \Phi_M^+$  (see [Vig15b, Lemme 2.2]). Put  $W_M = \mathcal{N}_M / Z^0$  and  $W_M(1) = \mathcal{N}_M / Z^1$ .

Let  $\epsilon = +$  or  $\epsilon = -$ . We denote by  $W_{M^\epsilon}, W_{M^\epsilon}(1)$  the images of  $\mathcal{N}_{M^\epsilon}$  in  $W_M, W_M(1)$ . We see the groups  $W_M, W_M(1), {}_1 W_{M'}$  as subgroups of  $W, W(1), {}_1 W_{G'}$ . As  $\theta$  (section 2.3), the linear injective map

$$(3.3) \quad \mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}, \quad \theta^*(T_w^{M,*}) = T_w^*, \quad (w \in W_M(1)),$$

respects the product on the subring  $\mathcal{H}_{M^\epsilon}$ . Here  $T_w^{M,*} \in \mathcal{H}_M$  is defined in the same way as  $T_w^*$  for  $\mathcal{H}_M$ . Note that  $\theta$  and  $\theta^*$  satisfy the obvious transitivity property with respect to a change of parabolic subgroups.

**3.2. Orthogonal case.** Let us examine the case where  $\Delta_M$  and  $\Delta \setminus \Delta_M$  are orthogonal, writing  $M_2 = M_{\Delta \setminus \Delta_M}$  as in section 1.3.

From  $M \cap M_2 = Z$  we get  $W_M \cap W_{M_2} = \Lambda, W_M(1) \cap W_{M_2}(1) = \Lambda(1)$ , the semisimple building of  $G$  is the product of those of  $M$  and  $M_2$ . The set  $S^{\text{aff}}$  is the disjoint union of  $S_M^{\text{aff}}$  and  $S_{M_2}^{\text{aff}}$ , the group  $W_{G'}$  is the direct product of  $W_{M'}$  and  $W_{M'_2}$ . For  $\tilde{s} \in W_M(1)$  lifting  $s \in S_M^{\text{aff}}$ , the elements  $T_{\tilde{s}}^M \in \mathcal{H}_M$  and  $T_{\tilde{s}} \in \mathcal{H}$  satisfy the same quadratic relations. A word of caution is necessary for the lengths  $\ell_M$  of  $W_M$  and  $\ell_{M_2}$  of  $W_{M_2}$  different from the restrictions of the length  $\ell$  of  $W_M$ , for example  $\ell_M(\lambda) = 0$  for  $\lambda \in \Lambda \cap W_{M'_2}$ .

**Lemma 3.2.** *We have  $\Lambda = (W_{M^\epsilon} \cap \Lambda)(W_{M'_2} \cap \Lambda)$ .*

*Proof.* We prove the lemma for  $\epsilon = -$ . The case  $\epsilon = +$  is similar. The map  $v : Z \rightarrow X_*(T) \otimes \mathbb{Q}$  defined in section 2.1 is trivial on  $Z^0$  and we also write  $v$  for the resulting homomorphism on  $\Lambda$ . For  $\lambda \in \Lambda$  there exists  $\lambda_2 \in W_{M'_2} \cap \Lambda$  such that  $\lambda \lambda_2 \in W_{M^-}$ , or equivalently  $\alpha(v(\lambda \lambda_2)) \leq 0$  for all  $\alpha \in \Phi^+ \setminus \Phi_M^+ = \Phi_{M_2}^+$ . It suffices to have the inequality for  $\alpha \in \Delta_{M_2}$ . The matrix  $(\alpha(\beta^\vee))_{\alpha, \beta \in \Delta_{M_2}}$  is invertible, hence there exists  $n_\beta \in \mathbb{Z}$  such that  $\sum_{\beta \in \Delta_{M_2}} n_\beta \alpha(\beta^\vee) \leq -\alpha(v(\lambda))$  for all  $\alpha \in \Delta_{M_2}$ . As  $v(W_{M'_2} \cap \Lambda)$  contains  $\bigoplus_{\alpha \in \Delta_{M_2}} \mathbb{Z} \alpha^\vee$  where  $\alpha^\vee$  is the coroot of  $\alpha$  [Vig16, after formula (71)], there exists  $\lambda_2 \in W_{M'_2} \cap \Lambda$  with  $v(\lambda_2) = \sum_{\beta \in \Delta_{M_2}} n_\beta \beta^\vee$ .  $\square$

The groups  $\mathcal{N} \cap M'$  and  $\mathcal{N} \cap M'_2$  are normal in  $\mathcal{N}$ , and

$$\mathcal{N} = (\mathcal{N} \cap M') \mathcal{N}_C(\mathcal{N} \cap M'_2) = Z(\mathcal{N} \cap M')(\mathcal{N} \cap M'_2),$$

and

$$W = W_{M'} \Omega W_{M'_2} = W_M W_{M'_2} = W_{M^+} W_{M'_2} = W_{M^-} W_{M'_2}.$$

The first two equalities are clear, the equality  $W_M W_{M'_2} = W_{M^\epsilon} W_{M'_2}$  follows from  $W_M = \mathbb{W}_M \Lambda$ ,  $\mathbb{W}_M \subset W_{M^\epsilon}$  and the lemma. The inverse image in  $W(1)$  of these

groups are

(3.4)

$$W(1) = {}_1W_{M'} \Omega(1) {}_1W_{M'_2} = W_M(1) {}_1W_{M'_2} = W_{M^+}(1) {}_1W_{M'_2} = W_{M^-}(1) {}_1W_{M'_2}.$$

We recall the function  $q_G(n) = q(n) = |\mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)|$  on  $\mathcal{N}$  [Vig16, Proposition 3.38] and we extend to  $\mathcal{N}$  the functions  $q_M$  on  $\mathcal{N} \cap M$  and  $q_{M_2}$  on  $\mathcal{N} \cap M_2$ :

$$(3.5) \quad q_M(n) = |\mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_Mn)|, \quad q_{M_2}(n) = |\mathcal{U}_{M_2}/(\mathcal{U}_{M_2} \cap n^{-1}\mathcal{U}_{M_2}n)|.$$

The functions  $q, q_M, q_{M_2}$  descend to functions on  $W(1)$  and on  $W$ , also denoted by  $q, q_M, q_{M_2}$ .

**Lemma 3.3.** *Let  $n \in \mathcal{N}$  of image  $w \in W$ . We have*

- (1)  $q(n) = q_M(n)q_{M_2}(n)$ .
- (2)  $q_M(n) = q_M(n_M)$  if  $n = n_M n_2$ ,  $n_M \in \mathcal{N} \cap M$ ,  $n_2 \in \mathcal{N} \cap M'_2$  and similarly when  $M$  and  $M_2$  are permuted.
- (3)  $q(w) = 1 \Leftrightarrow q_M(\lambda w_M) = q_{M_2}(\lambda w_{M_2}) = 1$ , if  $w = \lambda w_M w_{M_2}$ ,  $(\lambda, w_M, w_{M_2}) \in \Lambda \times \mathbb{W}_M \times \mathbb{W}_{M_2}$ .
- (4) On the coset  $(\mathcal{N} \cap M'_2)\mathcal{N}_C n$ ,  $q_M$  is constant equal to  $q_M(n_{M'})$  for any element  $n_{M'} \in M' \cap (\mathcal{N} \cap M'_2)\mathcal{N}_C n$ . A similar result is true when  $M$  and  $M_2$  are permuted.

*Proof.* We put  $\mathcal{U}_{M'} = \mathcal{U} \cap M'$  and  $\mathcal{U}_{M'_2} = \mathcal{U} \cap M'_2$ . The product map

$$(3.6) \quad Z^1 \prod_{\alpha \in \Phi_{M, red}} \mathcal{U}_\alpha \prod_{\alpha \in \Phi_{M_2, red}} \mathcal{U}_\alpha \rightarrow \mathcal{U}$$

with  $\mathcal{U}_\alpha = U_\alpha \cap \mathcal{U}$ , is a homeomorphism. We have  $\mathcal{U}_M = Z^1 \mathcal{Y}_{M'}$ ,  $\mathcal{U}_{M'} = (Z^1 \cap M')\mathcal{Y}_{M'}$  where  $\mathcal{Y}_{M'} = \prod_{\alpha \in \Phi_{M, red}} \mathcal{U}_\alpha$  and  $\mathcal{N} \cap M'_2$  commutes with  $\mathcal{Y}_{M'}$ , in particular  $\mathcal{N} \cap M'_2$  normalizes  $\mathcal{Y}_{M'}$ . Similar results are true when  $M$  and  $M_2$  are permuted, and  $\mathcal{U} = \mathcal{U}_{M'}\mathcal{U}_{M_2} = \mathcal{U}_M\mathcal{U}_{M'_2}$ .

Writing  $\mathcal{N} = Z(\mathcal{N} \cap M')(\mathcal{N} \cap M'_2)$  (in any order), we see that the product map

$$(3.7) \quad Z^1(\mathcal{Y}_{M'} \cap n^{-1}\mathcal{Y}_{M'}n)(\mathcal{Y}_{M'_2} \cap n^{-1}\mathcal{Y}_{M'_2}n) \rightarrow \mathcal{U} \cap n^{-1}\mathcal{U}n$$

is a homeomorphism. The inclusions induce bijections

$$(3.8) \quad \mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap n^{-1}\mathcal{Y}_{M'}n) \simeq \mathcal{U}_{M'}/(\mathcal{U}_{M'} \cap n^{-1}\mathcal{U}_{M'}n) \simeq \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_Mn),$$

similarly for  $M_2$ , and also a bijection

$$(3.9) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n) \simeq (\mathcal{Y}_{M'_2}/(\mathcal{Y}_{M'_2} \cap n^{-1}\mathcal{Y}_{M'_2}n)) \times (\mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap n^{-1}\mathcal{Y}_{M'}n)).$$

From (3.8) and (3.9), we get

$$(3.10) \quad \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n) \simeq (\mathcal{U}_{M'_2}/(\mathcal{U}_{M'_2} \cap n\mathcal{U}_{M'_2}n^{-1})) \times (\mathcal{U}_{M'}/(\mathcal{U}_{M'} \cap n\mathcal{U}_{M'}n^{-1}))$$

which implies the assertion (1) in the lemma.

The assertion (2) follows from (3.7) since  $\mathcal{N} \cap M'_2$  normalizes  $\mathcal{Y}_{M'}$ ; with (1), it implies the assertion (3).

A subgroup of  $\mathcal{N}$  normalizes  $\mathcal{U}_M$  if and only if it normalizes  $\mathcal{Y}_{M'}$  by (3.8) if and only if  $q_M = 1$  on this group. The group  $\mathcal{N} \cap M'_2$  normalizes  $\mathcal{Y}_{M'}$ . Therefore the group  $(\mathcal{N} \cap M'_2)\mathcal{N}_C$  normalizes  $\mathcal{U}_M$ . The coset  $(\mathcal{N} \cap M'_2)\mathcal{N}_C n$  contains an element  $n_{M'} \in M'$ . For  $x \in (\mathcal{N} \cap M'_2)\mathcal{N}_C$ ,  $(xn_{M'})^{-1}\mathcal{U}xn_{M'} = n_{M'}^{-1}\mathcal{U}n_{M'}$  hence  $q_M(xn_{M'}) = q_M(n_{M'})$ .  $\square$

**3.3. Extension of an  $\mathcal{H}_M$ -module to  $\mathcal{H}$ .** This section is inspired by similar results for the pro- $p$  Iwahori Hecke algebras over an algebraically closed field of characteristic  $p$  [Abe, Proposition 4.16]. We keep the setting of section 3.2 and we introduce ideals:

- $\mathcal{J}_\ell$  (resp.,  $\mathcal{J}_r$ ) the left (resp., right) ideal of  $\mathcal{H}$  generated by  $T_w^* - 1_{\mathcal{H}}$  for all  $w \in {}_1W_{M'_2}$ ,
- $\mathcal{J}_{M,\ell}$  (resp.,  $\mathcal{J}_{M,r}$ ) the left (resp., right) ideal of  $\mathcal{H}_M$  generated by  $T_\lambda^{M,*} - 1_{\mathcal{H}_M}$  for all  $\lambda$  in  ${}_1W_{M'_2} \cap W_M(1) = {}_1W_{M'_2} \cap \Lambda(1)$ .

The next proposition shows that the ideals  $\mathcal{J}_\ell = \mathcal{J}_r$  are equal and similarly  $\mathcal{J}_{M,\ell} = \mathcal{J}_{M,r}$ . After the proposition, we will drop the indices  $\ell$  and  $r$ .

**Proposition 3.4.** *The ideals  $\mathcal{J}_\ell$  and  $\mathcal{J}_r$  are equal to the submodule  $\mathcal{J}'$  of  $\mathcal{H}$  generated by  $T_w^* - T_{ww_2}^*$  for all  $w \in W(1)$  and  $w_2 \in {}_1W_{M'_2}$ .*

*The ideals  $\mathcal{J}_{M,\ell}$  and  $\mathcal{J}_{M,r}$  are equal to the submodule  $\mathcal{J}'_M$  of  $\mathcal{H}_M$  generated by  $T_w^{M,*} - T_{w\lambda_2}^{M,*}$  for all  $w \in W_M(1)$  and  $\lambda_2 \in \Lambda(1) \cap {}_1W_{M'_2}$ .*

*Proof.*

(1) We prove  $\mathcal{J}_\ell = \mathcal{J}'$ . Let  $w \in W(1), w_2 \in {}_1W_{M'_2}$ . We prove by induction on the length of  $w_2$  that  $T_w^*(T_{w_2}^* - 1) \in \mathcal{J}'$ . This is obvious when  $\ell(w_2) = 0$  because  $T_w^*T_{w_2}^* = T_{ww_2}^*$ . Assume that  $\ell(w_2) = 1$  and put  $s = w_2$ . If  $\ell(ws) = \ell(w) + 1$ , as before  $T_w^*(T_s^* - 1) \in \mathcal{J}'$  because  $T_w^*T_s^* = T_{ws}^*$ . Otherwise  $\ell(ws) = \ell(w) - 1$  and  $T_w^* = T_{ws^{-1}}^*T_s^*$  hence

$$T_w^*(T_s^* - 1) = T_{ws^{-1}}^*(T_s^*)^2 - T_w^* = T_{ws^{-1}}^*(q_s T_{s^2}^* - T_s^* c_s) - T_w^* = q_s T_{ws}^* - T_w^*(c_s + 1).$$

Since  $c_s + 1 = \sum_{t \in Z'_k} c_s(t) T_t$  with  $c_s(t) \in \mathbb{N}$  and  $\sum_{t \in Z'_k} c_s(t) = q_s$  [Vig16, Proposition 4.4],

$$q_s T_{ws}^* - T_w^*(c_s + 1) = \sum_{t \in Z'_k} c_s(t) (T_{ws}^* - T_w^* T_t^*) = \sum_{t \in Z'_k} c_s(t) (T_{ws}^* - T_{ws s^{-1}t}^*) \in \mathcal{J}'.$$

Assume now that  $\ell(w_2) > 1$ . Then, we factorize  $w_2 = xy$  with  $x, y \in {}_1W_{M'_2}$  of length  $\ell(x), \ell(y) < \ell(w_2)$  and  $\ell(w_2) = \ell(x) + \ell(y)$ . The element  $T_w^*(T_{w_2}^* - 1) = T_w^*T_x^*(T_y^* - 1) + T_w^*(T_x^* - 1)$  lies in  $\mathcal{J}'$  by induction.

Conversely, we prove  $T_{ww_2}^* - T_w^* \in \mathcal{J}_\ell$ . We factorize  $w = xy$  with  $y \in {}_1W_{M'_2}$  and  $x \in {}_1W_{M'} \Omega(1)$ . Then, we have  $\ell(w) = \ell(x) + \ell(y)$  and  $\ell(ww_2) = \ell(x) + \ell(yw_2)$ . Hence

$$T_{ww_2}^* - T_w^* = T_x^*(T_{yw_2}^* - T_y^*) = T_x^*(T_{yw_2}^* - 1) - T_x^*(T_y^* - 1) \in \mathcal{J}_\ell.$$

This ends the proof of  $\mathcal{J}_\ell = \mathcal{J}'$ .

By the same argument, the right ideal  $\mathcal{J}_r$  of  $\mathcal{H}$  is equal to the submodule of  $\mathcal{H}$  generated by  $T_{w_2w}^* - T_w^*$  for all  $w \in W(1)$  and  $w_2 \in {}_1W_{M'_2}$ . But this latter submodule is equal to  $\mathcal{J}'$  because  ${}_1W_{M'_2}$  is normal in  $W(1)$ . Therefore we proved  $\mathcal{J}' = \mathcal{J}_r = \mathcal{J}_\ell$ .

(2) Proof of the second assertion. We prove  $\mathcal{J}_{M,\ell} = \mathcal{J}'_M$ . The proof is easier than in (1) because for  $w \in W_M(1)$  and  $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$ , we have  $\ell(w\lambda_2) = \ell(w) + \ell(\lambda_2)$  hence  $T_w^{M,*}(T_{\lambda_2}^{M,*} - 1) = T_{w\lambda_2}^{M,*} - T_w^{M,*}$ . We have also  $\ell(\lambda_2 w) = \ell(\lambda_2) + \ell(w)$  hence  $(T_{\lambda_2}^{M,*} - 1)T_w^{M,*} = T_{\lambda_2 w}^{M,*} - T_w^{M,*}$  hence  $\mathcal{J}_{M,r}$  is equal to the submodule of  $\mathcal{H}_M$  generated by  $T_{\lambda_2 w}^{M,*} - T_w^{M,*}$  for all  $w \in W_M(1)$  and  $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$ . This latter submodule is  $\mathcal{J}'_M$ , as  ${}_1W_{M'_2} \cap \Lambda(1) = {}_1W_{M'_2} \cap W_M(1)$  is normal in  $W_M(1)$ . Therefore  $\mathcal{J}'_M = \mathcal{J}_{M,r} = \mathcal{J}_{M,\ell}$ .  $\square$

By Proposition 3.4, a basis of  $\mathcal{J}$  is  $T_w^* - T_{ww_2}^*$  for  $w$  in a system of representatives of  $W(1)/{}_1W_{M'_2}$ , and  $w_2 \in {}_1W_{M'_2} \setminus \{1\}$ . Similarly a basis of  $\mathcal{J}_M$  is  $T_w^{M,*} - T_{w\lambda_2}^{M,*}$  for  $w$  in a system of representatives of  $W_M(1)/(\Lambda(1) \cap {}_1W_{M'_2})$ , and  $\lambda_2 \in (\Lambda(1) \cap {}_1W_{M'_2}) \setminus \{1\}$ .

**Proposition 3.5.** *The natural ring inclusion of  $\mathcal{H}_{M^-}$  in  $\mathcal{H}_M$  and the ring inclusion of  $\mathcal{H}_{M^-}$  in  $\mathcal{H}$  via  $\theta^*$  induce ring isomorphisms*

$$\mathcal{H}_M/\mathcal{J}_M \xleftarrow{\sim} \mathcal{H}_{M^-}/(\mathcal{J}_M \cap \mathcal{H}_{M^-}) \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

*Proof.*

(1) The left map is obviously injective. We prove the surjectivity. Let  $w \in W_M(1)$ . Let  $\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)$  such that  $w\lambda_2^{-1} \in W_{M^-}(1)$  (see (3.4)). We have  $T_{w\lambda_2^{-1}}^{M,*} \in \mathcal{H}_{M^-}$  and  $T_w^{M,*} = T_{w\lambda_2^{-1}}^{M,*} T_{\lambda_2}^{M,*} = T_{w\lambda_2^{-1}}^{M,*} + T_{w\lambda_2^{-1}}^{M,*} (T_{\lambda_2}^{M,*} - 1)$ . Therefore  $T_w^{M,*} \in \mathcal{H}_{M^-} + \mathcal{J}_M$ . As  $w$  is arbitrary,  $\mathcal{H}_M = \mathcal{H}_{M^-} + \mathcal{J}_M$ .

(2) The right map is surjective: let  $w \in W(1)$  and  $w_2 \in {}_1W_{M'_2}$  such that  $ww_2^{-1} \in W_{M^-}(1)$  (see (3.4)). Then  $T_w^* - T_{ww_2^{-1}}^* \in \mathcal{J}$  with the same arguments as in (1), using Proposition 3.4. Therefore  $\mathcal{H} = \theta^*(\mathcal{H}_{M^-}) + \mathcal{J}$ .

We prove the injectivity:  $\theta^*(\mathcal{H}_{M^-}) \cap \mathcal{J} = \theta^*(\mathcal{H}_{M^-} \cap \mathcal{J}_M)$ . Let  $\sum_{w \in W_{M^-}(1)} c_w T_w^{M,*}$ , with  $c_w \in \mathbb{Z}$ , be an element of  $\mathcal{H}_{M^-}$ . Its image by  $\theta^*$  is  $\sum_{w \in W(1)} c_w T_w^*$  where we have set  $c_w = 0$  for  $w \in W(1) \setminus W_{M^-}(1)$ . We have  $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}$  if and only if  $\sum_{w_2 \in {}_1W_{M'_2}} c_{ww_2} = 0$  for all  $w \in W(1)$ . If  $c_{ww_2} \neq 0$ , then  $w_2 \in {}_1W_{M'_2} \cap W_M(1)$ , that is,  $w_2 \in {}_1W_{M'_2} \cap \Lambda(1)$ . The sum  $\sum_{w_2 \in {}_1W_{M'_2}} c_{ww_2}$  is equal to  $\sum_{\lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1)} c_{w\lambda_2}$ . By Proposition 3.4,  $\sum_{w \in W(1)} c_w T_w^* \in \mathcal{J}$  if and only if  $\sum_{w \in W_{M^-}(1)} c_w T_w^{M,*} \in \mathcal{J}_M$ .  $\square$

We construct a ring isomorphism

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}$$

by using Proposition 3.5. For any  $w \in W(1)$ ,  $T_w^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$  where  $w_{M^-} \in W_{M^-}(1) \cap {}_1W_{M'_2}$  (see (3.4)), because by Proposition 3.4,  $T_w^* + \mathcal{J} = T_{w_{M^-}}^* + \mathcal{J}$  and  $T_{w_{M^-}}^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$  by construction of  $e^*$ . We check that  $e^*$  is induced by  $\theta^*$ .

**Theorem 3.6.** *The linear map  $\mathcal{H}_M \xrightarrow{\theta^*} \mathcal{H}$  induces a ring isomorphism*

$$e^* : \mathcal{H}_M/\mathcal{J}_M \xrightarrow{\sim} \mathcal{H}/\mathcal{J}.$$

*Proof.* Let  $w \in W_M(1)$ . We have to show that  $T_w^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$ . We saw above that  $T_w^* + \mathcal{J} = e^*(T_{w_{M^-}}^{M,*} + \mathcal{J}_M)$  with  $w = w_{M^-}\lambda_2$  with  $\lambda_2 \in {}_1W_{M'_2} \cap W_M(1)$ . As  $\ell_M(\lambda_2) = 0$ ,  $T_w^{M,*} = T_{w_{M^-}}^{M,*} T_{\lambda_2}^{M,*} \in T_{w_{M^-}}^{M,*} + \mathcal{J}_M$ . Therefore  $T_{w_{M^-}}^{M,*} + \mathcal{J}_M = T_w^{M,*} + \mathcal{J}_M$ . This ends the proof of the theorem.  $\square$

We now wish to compute  $e^*$  in terms of the  $T_w$  instead of the  $T_w^*$ .

**Proposition 3.7.** *Let  $w \in W(1)$ . Then,  $T_w + \mathcal{J} = e^*(T_{w_M}^M q_{M_2}(w) + \mathcal{J}_M)$  for any  $w_M \in W_M(1) \cap {}_1W_{M'_2}$ .*

*Proof.* The element  $w_M$  is unique modulo right multiplication by an element  $\lambda_2 \in W_M(1) \cap {}_1W_{M'_2}$  of length  $\ell_M(\lambda_2) = 0$  and  $T_{w_M}^M q_{M_2}(w) + \mathcal{J}_M$  does not depend on the choice of  $w_M$ . We choose a decomposition (see (3.4)):

$$w = \tilde{s}_1 \dots \tilde{s}_a u \tilde{s}_{a+1} \dots \tilde{s}_{a+b}, \quad \ell(w) = a + b,$$

for  $u \in \Omega(1)$ ,  $\tilde{s}_i \in {}_1W_{M'}$  lifting  $s_i \in S_M^{\text{aff}}$  for  $1 \leq i \leq a$  and  $\tilde{s}_i \in {}_1W_{M'_2}$  lifting  $s_i \in S_{M_2}^{\text{aff}}$  for  $a+1 \leq i \leq a+b$ , and we choose  $u_M \in W_M(1)$  such that  $u \in u_M {}_1W_{M'_2}$ . Then

$$w_M = \tilde{s}_1 \dots \tilde{s}_a u_M \in W_M(1) \cap w {}_1W_{M'_2}$$

and  $q_{M_2}(w) = q_{M_2}(\tilde{s}_{a+1} \dots \tilde{s}_{a+b})$  (Lemma 3.3 (4)). First we check the proposition in three simple cases:

*Case 1.* Let  $w = \tilde{s} \in {}_1W_{M'}$  lifting  $s \in S_M^{\text{aff}}$ ; we have  $T_{\tilde{s}} + \mathcal{J} = e^*(T_{\tilde{s}}^M + \mathcal{J}_M)$  because  $T_{\tilde{s}}^* - e^*(T_{\tilde{s}}^{M,*}) \in \mathcal{J}$ ,  $T_{\tilde{s}} = T_{\tilde{s}}^* + c_{\tilde{s}}$ ,  $T_{\tilde{s}}^M = T_{\tilde{s}}^{M,*} + c_{\tilde{s}}$  and  $1 = q_{M_2}(\tilde{s})$ .

*Case 2.* Let  $w = u \in W(1)$  of length  $\ell(u) = 0$  and  $u_M \in W_M(1)$  such that  $u \in u_M {}_1W_{M'_2}$ . We have  $\ell_M(u_M) = 0$  and  $q_{M_2}(u) = 1$  (Lemma 3.3). We deduce  $T_u + \mathcal{J} = e^*(T_{u_M}^M + \mathcal{J}_M)$  because  $T_u^* + \mathcal{J} = T_{u_M}^* + \mathcal{J} = e^*(T_{u_M}^{M,*} + \mathcal{J}_M)$ , and  $T_u = T_u^*, T_{u_M}^M = T_{u_M}^{M,*}$ .

*Case 3.* Let  $w = \tilde{s} \in {}_1W_{M'_2}$  lifting  $s \in S_{M_2}^{\text{aff}}$ ; we have  $T_{\tilde{s}} + \mathcal{J} = e^*(q_{M_2}(\tilde{s}) + \mathcal{J}_M)$  because  $T_{\tilde{s}}^* - 1, c_{\tilde{s}} - (q_s - 1) \in \mathcal{J}$ ,  $T_{\tilde{s}} = T_{\tilde{s}}^* + c_{\tilde{s}} \in q_s + \mathcal{J}$  and  $q_s = q_{M_2}(\tilde{s})$ .

In general, the braid relations  $T_w = T_{\tilde{s}_1} \dots T_{\tilde{s}_a} T_u T_{\tilde{s}_{a+1}} \dots T_{\tilde{s}_{a+b}}$  give a similar product decomposition of  $T_w + \mathcal{J}$ , and the simple cases 1, 2, 3 imply that  $T_w + \mathcal{J}$  is equal to

$$\begin{aligned} & e^*(T_{\tilde{s}_1}^M + \mathcal{J}_M) \dots e^*(T_{\tilde{s}_a}^M + \mathcal{J}_M) e^*(T_{u_M}^M + \mathcal{J}_M) e^*(q_{M_2}(\tilde{s}_{a+1}) \\ & \quad + \mathcal{J}_M) \dots e^*(q_{M_2}(\tilde{s}_{a+b}) + \mathcal{J}_M) \\ & = e^*(T_{w_M}^M q_{M_2}(w) + \mathcal{J}_M). \end{aligned}$$

The proposition is proved.  $\square$

Propositions 3.4, 3.5, 3.7, and Theorem 3.6 are valid over any commutative ring  $R$  (instead of  $\mathbb{Z}$ ).

The two-sided ideal of  $\mathcal{H}_R$  generated by  $T_w^* - 1$  for all  $w \in {}_1W_{M'_2}$  is  $\mathcal{J}_R = \mathcal{J} \otimes_{\mathbb{Z}} R$ , the two-sided ideal of  $\mathcal{H}_{M,R}$  generated by  $T_{\lambda}^* - 1$  for all  $\lambda \in {}_1W_{M'_2} \cap \Lambda(1)$  is  $\mathcal{J}_{M,R} = \mathcal{J}_M \otimes_{\mathbb{Z}} R$ , and we get as in Proposition 3.5 isomorphisms

$$\mathcal{H}_{M,R}/\mathcal{J}_{M,R} \xleftarrow{\sim} \mathcal{H}_{M^-,R}/(\mathcal{J}_{M,R} \cap \mathcal{H}_{M^-,R}) \xrightarrow{\sim} \mathcal{H}_R/\mathcal{J}_R,$$

giving an isomorphism  $\mathcal{H}_{M,R}/\mathcal{J}_{M,R} \rightarrow \mathcal{H}_R/\mathcal{J}_R$  induced by  $\theta^*$ . Therefore, we have an isomorphism from the category of right  $\mathcal{H}_{M,R}$ -modules where  $\mathcal{J}_M$  acts by 0 onto the category of right  $\mathcal{H}_R$ -modules where  $\mathcal{J}$  acts by 0.

**Definition 3.8.** A right  $\mathcal{H}_{M,R}$ -module  $\mathcal{V}$  where  $\mathcal{J}_M$  acts by 0 is called extensible to  $\mathcal{H}$ . The corresponding  $\mathcal{H}_R$ -module where  $\mathcal{J}$  acts by 0 is called its extension to  $\mathcal{H}$  and denoted by  $e_{\mathcal{H}}(\mathcal{V})$  or  $e(\mathcal{V})$ .

With the element basis  $T_w^*$ ,  $\mathcal{V}$  is extensible to  $\mathcal{H}$  if and only if

$$(3.11) \quad v T_{\lambda_2}^{M,*} = v \text{ for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1).$$

The  $\mathcal{H}$ -module structure on the  $R$ -module  $e(\mathcal{V}) = \mathcal{V}$  is determined by

$$(3.12) \quad v T_{w_2}^* = v, \quad v T_w^* = v T_w^{M,*} \quad \text{for all } v \in \mathcal{V}, w_2 \in {}_1W_{M'_2}, w \in W_M(1).$$

It is also determined by the action of  $T_w^*$  for  $w \in {}_1W_{M'_2} \cup W_{M^+}(1)$  (or  $w \in {}_1W_{M'_2} \cup W_{M^-}(1)$ ). Conversely, a right  $\mathcal{H}$ -module  $\mathcal{W}$  over  $R$  is extended from an  $\mathcal{H}_M$ -module if and only if

$$(3.13) \quad vT_{w_2}^* = v \quad \text{for all } v \in \mathcal{W}, w_2 \in {}_1W_{M'_2}.$$

In terms of the basis elements  $T_w$  instead of  $T_w^*$ , this says the following.

**Corollary 3.9.** *A right  $\mathcal{H}_M$ -module  $\mathcal{V}$  over  $R$  is extensible to  $\mathcal{H}$  if and only if*

$$(3.14) \quad vT_{\lambda_2}^M = v \quad \text{for all } v \in \mathcal{V} \text{ and } \lambda_2 \in {}_1W_{M'_2} \cap \Lambda(1).$$

*Then, the structure of an  $\mathcal{H}$ -module on the  $R$ -module  $e(\mathcal{V}) = \mathcal{V}$  is determined by*

$$(3.15) \quad vT_{w_2} = vq_{w_2}, \quad vT_w = vT_w^M q_{M_2}(w) \quad \text{for all } v \in \mathcal{V}, w_2 \in {}_1W_{M'_2}, w \in W_M(1).$$

*( $W_{M^+}(1)$  or  $W_{M^-}(1)$  instead of  $W_M(1)$  is enough.) A right  $\mathcal{H}$ -module  $\mathcal{W}$  over  $R$  is extended from an  $\mathcal{H}_M$ -module if and only if*

$$(3.16) \quad vT_{w_2} = vq_{w_2} \quad \text{for all } v \in \mathcal{W}, w_2 \in {}_1W_{M'_2}.$$

**3.4.  $\sigma^{\mathcal{U}_M}$  is extensible to  $\mathcal{H}$  of extension  $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$ .** Let  $P = MN$  be a standard parabolic subgroup of  $G$  such that  $\Delta_P$  and  $\Delta \setminus \Delta_P$  are orthogonal, and let  $\sigma$  be a smooth  $R$ -representation of  $M$  extensible to  $G$ . Let  $P_2 = M_2N_2$  denote the standard parabolic subgroup of  $G$  with  $\Delta_{P_2} = \Delta \setminus \Delta_P$ .

Recall that  $G = MM'_2$ , that  $M \cap M'_2 = Z \cap M'_2$  acts trivially on  $\sigma$ ,  $e(\sigma)$  is the representation of  $G$  equal to  $\sigma$  on  $M$  and trivial on  $M'_2$ . We will describe the  $\mathcal{H}$ -module  $e(\sigma)^{\mathcal{U}}$  in this section. We first consider  $e(\sigma)$  as a subrepresentation of  $\text{Ind}_P^G \sigma$ . For  $v \in \sigma$ , let  $f_v \in (\text{Ind}_P^G \sigma)^{M'_2}$  be the unique function with value  $v$  on  $M'_2$ . Then, the map

$$(3.17) \quad v \mapsto f_v : \sigma \rightarrow \text{Ind}_P^G \sigma$$

is the natural  $G$ -equivariant embedding of  $e(\sigma)$  in  $\text{Ind}_P^G \sigma$ . As  $\sigma^{\mathcal{U}_M} = e(\sigma)^{\mathcal{U}}$  as  $R$ -modules, the image of  $e(\sigma)^{\mathcal{U}}$  in  $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$  is made out of the  $f_v$  for  $v \in \sigma^{\mathcal{U}_M}$ .

We now recall the explicit description of  $(\text{Ind}_P^G \sigma)^{\mathcal{U}}$ . For each  $d \in \mathbb{W}_{M_2}$ , we fix a lift  $\hat{d} \in {}_1W_{M'_2}$  and for  $v \in \sigma^{\mathcal{U}_M}$  let  $f_{P\hat{d}\mathcal{U},v} \in (\text{Ind}_P^G \sigma)^{\mathcal{U}}$  for the function with support contained in  $P\hat{d}\mathcal{U}$  and value  $v$  on  $\hat{d}\mathcal{U}$ . As  $Z \cap M'_2$  acts trivially on  $\sigma$ , the function  $f_{P\hat{d}\mathcal{U},v}$  does not depend on the choice of the lift  $\hat{d} \in {}_1W_{M'_2}$  of  $d$ . By [OV17, Lemma 4.5], recalling that  $w \in \mathbb{W}_{M_2}$  is of minimal length in its coset  $w\mathbb{W}_M = \mathbb{W}_M w$  as  $\Delta_M$  and  $\Delta_{M_2}$  are orthogonal to each other:

*The map  $\bigoplus_{d \in \mathbb{W}_{M_2}} \sigma^{\mathcal{U}_M} \rightarrow (\text{Ind}_P^G \sigma)^{\mathcal{U}}$  given on each  $d$ -component by  $v \mapsto f_{P\hat{d}\mathcal{U},v}$ , is an  $\mathcal{H}_{M^+}$ -equivariant isomorphism where  $\mathcal{H}_{M^+}$  is seen as a subring of  $\mathcal{H}$  via  $\theta$ , and induces an  $\mathcal{H}_R$ -module isomorphism*

$$(3.18) \quad v \otimes h \mapsto f_{P\mathcal{U},v}h : \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H} \rightarrow (\text{Ind}_P^G \sigma)^{\mathcal{U}}.$$

In particular for  $v \in \sigma^{\mathcal{U}_M}$ ,  $v \otimes T(\hat{d})$  does not depend on the choice of the lift  $\hat{d} \in {}_1W_{M'_2}$  of  $d$  and

$$(3.19) \quad f_{P\hat{d}\mathcal{U},v} = f_{P\mathcal{U},v}T(\hat{d}).$$

As  $G$  is the disjoint union of  $Pd\mathcal{U}$  for  $d \in \mathbb{W}_{M_2}$ , we have  $f_v = \sum_{d \in \mathbb{W}_{M_2}} f_{Pd\mathcal{U},v}$  and  $f_v$  is the image of  $v \otimes e_{M_2}$  in (3.18), where

$$(3.20) \quad e_{M_2} = \sum_{d \in \mathbb{W}_{M_2}} T(\hat{d}).$$

Recalling (3.17) we get the following.

**Lemma 3.10.** *The map  $v \mapsto v \otimes e_{M_2} : e(\sigma)^\mathcal{U} \rightarrow \sigma^{\mathcal{U}_M} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H}$  is an  $\mathcal{H}_R$ -equivariant embedding.*

*Remark 3.11.* The trivial map  $v \mapsto v \otimes 1_{\mathcal{H}}$  is not an  $\mathcal{H}_R$ -equivariant embedding.

We describe the action of  $T(n)$  on  $e(\sigma)^\mathcal{U}$  for  $n \in \mathcal{N}$ . By definition for  $v \in e(\sigma)^\mathcal{U}$ ,

$$(3.21) \quad vT(n) = \sum_{y \in \mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)} yn^{-1}v.$$

**Proposition 3.12.** *We have  $vT(n) = vT^M(n_M)q_{M_2}(n)$  for any  $n_M \in \mathcal{N} \cap M$  is such that  $n = n_M(\mathcal{N} \cap M'_2)$ .*

*Proof.* The description (3.10) of  $\mathcal{U}/(\mathcal{U} \cap n^{-1}\mathcal{U}n)$  gives

$$vT(n) = \sum_{y_1 \in \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_M n)} y_1 \sum_{y_2 \in \mathcal{U}_{M'_2}/(\mathcal{U}_{M'_2} \cap n^{-1}\mathcal{U}_{M'_2} n)} y_2 n^{-1}v.$$

As  $M'_2$  acts trivially on  $e(\sigma)$ , we obtain

$$vT(n) = q_{M_2}(n) \sum_{y_1 \in \mathcal{U}_M/(\mathcal{U}_M \cap n^{-1}\mathcal{U}_M n)} y_1 n_M^{-1}v = q_{M_2}(n) vT^M(n_M).$$

□

**Theorem 3.13.** *Let  $\sigma$  be a smooth  $R$ -representation of  $M$ . If  $P(\sigma) = G$ , then  $\sigma^{\mathcal{U}_M}$  is extensible to  $\mathcal{H}$  of extension  $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^\mathcal{U}$ . Conversely, if  $\sigma^{\mathcal{U}_M}$  is extensible to  $\mathcal{H}$  and generates  $\sigma$ , then  $P(\sigma) = G$ .*

*Proof.*

(1) The  $\mathcal{H}_M$ -module  $\sigma^{\mathcal{U}_M}$  is extensible to  $\mathcal{H}$  if and only if  $Z \cap M'_2$  acts trivially on  $\sigma^{\mathcal{U}_M}$ . Indeed, for  $v \in \sigma^{\mathcal{U}_M}$ ,  $z_2 \in Z \cap M'_2$ ,

$$vT^M(z_2) = \sum_{y \in \mathcal{U}_M/(\mathcal{U}_M \cap z_2^{-1}\mathcal{U}_M z_2)} yz_2^{-1}v = \sum_{y \in \mathcal{Y}_{M'}/(\mathcal{Y}_{M'} \cap z_2^{-1}\mathcal{Y}_{M'} z_2)} yz_2^{-1}v = z_2^{-1}v,$$

by (3.21), then (3.8), then the fact that  $z_2^{-1}$  commutes with the elements of  $\mathcal{Y}_M$ .

(2)  $P(\sigma) = G$  if and only if  $Z \cap M'_2$  acts trivially on  $\sigma$  (the group  $Z \cap M'_2$  is generated by  $Z \cap M'_\alpha$  for  $\alpha \in \Delta_{M_2}$  by Lemma 2.1). The  $R$ -submodule  $\sigma^{Z \cap M'_2}$  of elements fixed by  $Z \cap M'_2$  is stable by  $M$ , because  $M = ZM'$ , the elements of  $M'$  commute with those of  $Z \cap M'_2$  and  $Z$  normalizes  $Z \cap M'_2$ .

(3) Apply (1) and (2) to get the theorem except the equality  $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^\mathcal{U}$  when  $P(\sigma) = G$  which follows from Propositions 3.12 and 3.7. □

Let  $\mathbf{1}_M$  denote the trivial representation of  $M$  over  $R$  (or  $\mathbf{1}$  when there is no ambiguity on  $M$ ). The right  $\mathcal{H}_R$ -module  $(\mathbf{1}_G)^\mathcal{U} = \mathbf{1}_{\mathcal{H}}$  (or  $\mathbf{1}$  if there is no ambiguity) is the trivial right  $\mathcal{H}_R$ -module: for  $w \in W_M(1)$ ,  $T_w = q_w \text{id}$  and  $T_w^* = \text{id}$  on  $\mathbf{1}_{\mathcal{H}}$ .

**Example 3.14.** The  $\mathcal{H}$ -module  $(\text{Ind}_P^G \mathbf{1})^\mathcal{U}$  is the extension of the  $\mathcal{H}_{M_2}$ -module  $(\text{Ind}_{M_2 \cap B}^{M_2} \mathbf{1})^{\mathcal{U}_{M_2}}$ . Indeed, the representation  $\text{Ind}_P^G \mathbf{1}$  of  $G$  is trivial on  $N_2$ , as  $G = MM'_2$  and  $N_2 \subset M'$  (as  $\Phi = \Phi_M \cup \Phi_{M_2}$ ). For  $g = mm'_2$  with  $m \in M, m'_2 \in M'_2$  and  $n_2 \in N_2$ , we have  $Pgn_2 = Pm'_2n_2 = Pn_2m'_2 = Pm'_2 = Pg$ . The group  $M_2 \cap B = M_2 \cap P$  is the standard minimal parabolic subgroup of  $M_2$  and  $(\text{Ind}_P^G \mathbf{1})|_{M_2} = \text{Ind}_{M_2 \cap B}^{M_2} \mathbf{1}$ . Apply Theorem 3.13 as follows.

**3.5. The  $\mathcal{H}_R$ -module  $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ .** Let  $P = MN$  be a standard parabolic subgroup of  $G$  such that  $\Delta_P$  and  $\Delta \setminus \Delta_P$  are orthogonal, let  $\mathcal{V}$  be a right  $\mathcal{H}_{M,R}$ -module which is extensible to  $\mathcal{H}_R$  of extension  $e(\mathcal{V})$ , and let  $Q$  be a parabolic subgroup of  $G$  containing  $P$ . Let  $P_2 = M_2N_2$  denote the standard parabolic subgroup of  $G$  with  $\Delta_{P_2} = \Delta \setminus \Delta_P$ .

We define on the  $R$ -module  $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$  a structure of a right  $\mathcal{H}_R$ -module as follows.

**Proposition 3.15.**

- (1) *The diagonal action of  $T_w^*$  for  $w \in W(1)$  on  $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$  defines a structure of a right  $\mathcal{H}_R$ -module.*
- (2) *The action of the  $T_w$  is also diagonal and satisfies:*

$$((v \otimes f)T_w, (v \otimes f)T_w^*) = (vT_{uw_{M'}}, \otimes fT_{uw_{M'_2}}, vT_{uw_{M'}}^* \otimes fT_{uw_{M'_2}}^*),$$

where  $w = uw_{M'}w_{M'_2}$  with  $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M'_2} \in {}_1W_{M'_2}$ .

*Proof.* If the lemma is true for  $P$  it is also true for  $Q$ , because the  $R$ -module  $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$  naturally embedded in  $e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$  is stable by the action of  $\mathcal{H}$  defined in the lemma. So, we suppose  $Q = P$ .

For each element in  ${}_1S^{\text{aff}}$  we fix an admissible lift and denote the set of admissible lifts by  ${}_1S^{\text{aff}}$ . We also use the obvious notation  ${}_1S_M^{\text{aff}}$  and  ${}_1S_{M_2}^{\text{aff}}$ . Suppose that  $T_w^*$  for  $w \in W(1)$  acts on  $e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$  as in (1). The braid relations obviously hold. The quadratic relations hold because  $T_s^*$  with  $s \in {}_1S^{\text{aff}}$ , acts trivially either on  $e(\mathcal{V})$  or on  $(\text{Ind}_P^G \mathbf{1})^\mathcal{U}$ . Indeed,  ${}_1S^{\text{aff}} = {}_1S_M^{\text{aff}} \cup {}_1S_{M_2}^{\text{aff}}$ ,  $T_s^*$  for  $s \in {}_1S_M^{\text{aff}}$ , acts trivially on  $(\text{Ind}_P^G \mathbf{1})^\mathcal{U}$  which is extended from an  $\mathcal{H}_{M_2}$ -module (Example 3.14), and  $T_s^*$  for  $s \in {}_1S_{M_2}^{\text{aff}}$ , acts trivially on  $e(\mathcal{V})$  which is extended from an  $\mathcal{H}_M$ -module. This proves (1).

We describe now the action of  $T_w$  instead of  $T_w^*$  on the  $\mathcal{H}$ -module  $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$ . Let  $w \in W(1)$ . We write  $w = uw_{M'}w_{M'_2} = uw_{M'_2}w_{M'}$  with  $u \in W(1), \ell(u) = 0, w_{M'} \in {}_1W_{M'}, w_{M'_2} \in {}_1W_{M'_2}$ . We have  $\ell(w) = \ell(w_{M'}) + \ell(w_{M'_2})$  hence  $T_w = T_uT_{w_{M'}}T_{w_{M'_2}}$ .

For  $w = u$ , we have  $T_u = T_u^*$  and  $(v \otimes f)T_u = (v \otimes f)T_u^* = vT_u^* \otimes fT_u^* = vT_u \otimes fT_u$ .

For  $w = w_{M'}$ ,  $(v \otimes f)T_w^* = vT_w^* \otimes f$ ; for  $s \in {}_1S_M^{\text{aff}}$ ,  $c_s = \sum_{t \in Z_k \cap {}_1W_{M'}} c_s(t)T_t^*$  in particular, we have  $(v \otimes f)T_s = (v \otimes f)(T_s^* + c_s) = v(T_s^* + c_s) \otimes f = vT_s \otimes f$ . Hence  $(v \otimes f)T_w = vT_w \otimes f$ .

For  $w = w_{M'_2}$ , we have similarly  $(v \otimes f)T_w^* = v \otimes fT_w^*$  and  $(v \otimes f)T_w = v \otimes fT_w$ .  $\square$

**Example 3.16.** Let  $\mathcal{X}$  be a right  $\mathcal{H}_R$ -module. Then  $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$  where the  $T_w^*$  acts diagonally is an  $\mathcal{H}_R$ -module isomorphic to  $\mathcal{X}$ . But the action of the  $T_w$  on  $\mathbf{1}_{\mathcal{H}} \otimes_R \mathcal{X}$  is not diagonal.



It is known [Ly15] that  $(\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$  and  $(\text{St}_Q^G)^\mathcal{U}$  are free  $R$ -modules and that  $(\text{St}_Q^G)^\mathcal{U}$  is the cokernel of the natural  $\mathcal{H}_R$ -map

$$(3.22) \quad \bigoplus_{Q \subsetneq Q'} (\text{Ind}_Q^G \mathbf{1})^\mathcal{U} \rightarrow (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$$

although the invariant functor  $(-)^{\mathcal{U}}$  is only left exact.

**Corollary 3.17.** *The diagonal action of  $T_w^*$  for  $w \in W(1)$  on  $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}$  defines a structure of a right  $\mathcal{H}_R$ -module satisfying Proposition 3.15(2).*

#### 4. HECKE MODULE $I_{\mathcal{H}}(P, \mathcal{V}, Q)$

**4.1. Case  $\mathcal{V}$  extensible to  $\mathcal{H}$ .** Let  $P = MN$  be a standard parabolic subgroup of  $G$  such that  $\Delta_P$  and  $\Delta \setminus \Delta_P$  are orthogonal, let  $\mathcal{V}$  be a right  $\mathcal{H}_{M,R}$ -module extensible to  $\mathcal{H}_R$  of extension  $e(\mathcal{V})$ , and let  $Q$  be a parabolic subgroup of  $G$  containing  $P$ . As  $Q$  and  $M_Q$  determine each other:  $Q = M_Q U$ , we denote also  $\mathcal{H}_{M_Q} = \mathcal{H}_Q$  and  $\mathcal{H}_{M_Q,R} = \mathcal{H}_{Q,R}$  when  $Q \neq P, G$ . When  $Q = G$  we drop  $G$  and we denote  $e_{\mathcal{H}}(\mathcal{V}) = e(\mathcal{V})$ .

**Lemma 4.1.**  *$\mathcal{V}$  is extensible to an  $\mathcal{H}_{Q,R}$ -module  $e_{\mathcal{H}_Q}(\mathcal{V})$ .*

*Proof.* This is straightforward. By Corollary 3.9,  $\mathcal{V}$  extensible to  $\mathcal{H}$  means that  $T^M(z)$  acts trivially on  $\mathcal{V}$  for all  $z \in \mathcal{N}_{M'_2} \cap Z$ . We have  $M_Q = MM'_{2,Q}$  with  $M'_{2,Q} \subset M_Q \cap M'_2$  and  $\mathcal{N}_{M'_{2,Q}} \subset \mathcal{N}_{M'_2}$ ; hence  $T^M(z)$  acts trivially on  $\mathcal{V}$  for all  $z \in \mathcal{N}_{M'_{2,Q}} \cap Z$  meaning that  $\mathcal{V}$  is extensible to  $\mathcal{H}_Q$ .  $\square$

*Remark 4.2.* We cannot say that  $e_{\mathcal{H}_Q}(\mathcal{V})$  is extensible to  $\mathcal{H}$  of extension  $e(\mathcal{V})$  when the set of roots  $\Delta_Q$  and  $\Delta \setminus \Delta_Q$  are not orthogonal (Definition 3.8).

Let  $Q'$  be an arbitrary parabolic subgroup of  $G$  containing  $Q$ . We are going to define an  $\mathcal{H}_R$ -embedding  $\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q,Q')} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q}^+, \theta} \mathcal{H}$  defining an  $\mathcal{H}_R$ -homomorphism

$$\bigoplus_{Q \subsetneq Q' \subset G} \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

of cokernel isomorphic to  $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}$ . In the extreme case  $(Q, Q') = (P, G)$ , the  $\mathcal{H}_R$ -embedding  $e(\mathcal{V}) \xrightarrow{\iota(P,G)} \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$  is given in the following lemma where  $f_G$  and  $f_{PU} \in (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$  denote the characteristic functions of  $G$  and  $PU$ ,  $f_G = f_{PU} e_{M_2}$  (see (3.20)).

**Lemma 4.3.** *There is a natural  $\mathcal{H}_R$ -isomorphism*

$$v \otimes \mathbf{1}_{\mathcal{H}} \mapsto v \otimes f_{PU} : \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \mathcal{V} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U},$$

and compatible  $\mathcal{H}_R$ -embeddings

$$(4.1) \quad v \mapsto v \otimes f_G : e(\mathcal{V}) \rightarrow e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U},$$

$$(4.2) \quad v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P,G)} \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}).$$

*Proof.* We show first that the map

$$(4.3) \quad v \mapsto v \otimes f_{PU} : \mathcal{V} \rightarrow e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^\mathcal{U}$$

is  $\mathcal{H}_{M^+}$ -equivariant. Let  $w \in W_{M^+}(1)$ . We write  $w = uw_{M'}w_{M'_2}$  as in Proposition 3.15 (2), so that  $f_{PU}T_w = f_{PU}T_{uw_{M'_2}}$ . We have  $f_{PU}T_{uw_{M'_2}} = f_{PU}$  because  ${}_1W_{M'} \subset W_{M^+}(1) \cap W_{M^-}(1)$  hence  $uw_{M'_2} = ww_{M'}^{-1} \in W_{M^+}(1)$  and in  $\mathbf{1}_{\mathcal{H}_M} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H}$  we have  $(1 \otimes 1_{\mathcal{H}})T_{uw_{M'_2}} = 1T_{uw_{M'_2}}^M \otimes 1_{\mathcal{H}}$ , and  $T_{uw_{M'_2}}^M$  acts trivially in  $\mathbf{1}_{\mathcal{H}_M}$  because  $\ell_M(uw_{M'_2}) = 0$ . We deduce  $(v \otimes f_{PU})T_w = vT_w \otimes f_{PU}T_w = vT_w^M \otimes f_{PU}$ .

By adjunction (4.3) gives an  $\mathcal{H}_R$ -equivariant linear map

$$(4.4) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{PU} : \mathcal{V} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H} \xrightarrow{\kappa_P} e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}.$$

We prove that  $\kappa_P$  is an isomorphism. Recalling  $\hat{d} \in \mathcal{N} \cap M'_2$ ,  $\tilde{d} \in {}_1W_{M'_2}$  lift  $d$ , one knows that

$$(4.5) \quad \mathcal{V} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes T_{\tilde{d}}, \quad e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}} = \bigoplus_{d \in \mathbb{W}_{M_2}} \mathcal{V} \otimes f_{P\hat{d}\mathcal{U}},$$

where each summand is isomorphic to  $\mathcal{V}$ . The left equality follows from section 4.1 and Remark 3.7 in [Vig15b] recalling that  $w \in \mathbb{W}_{M_2}$  is of minimal length in its coset  $\mathbb{W}_M w = w\mathbb{W}_M$  as  $\Delta_M$  and  $\Delta_{M_2}$  are orthogonal; for the second equality see section 3.4 (3.19). We have  $\kappa_P(v \otimes T_{\tilde{d}}) = (v \otimes f_{PU})T_{\tilde{d}} = v \otimes f_{PU}T_{\tilde{d}}$  (Proposition 3.15). Hence  $\kappa_P$  is an isomorphism.

We consider the composite map

$$v \mapsto v \otimes 1 \mapsto v \otimes f_{PU}e_{M_2} : e(\mathcal{V}) \rightarrow e(\mathcal{V}) \otimes_R \mathbf{1}_{\mathcal{H}} \rightarrow e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}},$$

where the right map is the tensor product  $e(\mathcal{V}) \otimes_R -$  of the  $\mathcal{H}_R$ -equivariant embedding  $\mathbf{1}_{\mathcal{H}} \rightarrow (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}$  sending  $1_R$  to  $f_{PU}e_{M_2}$  (Lemma 3.10); this map is injective because  $(\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}/\mathbf{1}$  is a free  $R$ -module; it is  $\mathcal{H}_R$ -equivariant for the diagonal action of the  $T_w^*$  on the tensor products (Example 3.16 for the first map). By compatibility with (4.4), we get the  $\mathcal{H}_R$ -equivariant embedding  $v \mapsto v \otimes e_{M_2} : e(\mathcal{V}) \xrightarrow{\iota(P, G)} \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$ .  $\square$

For a general  $(Q, Q')$  the  $\mathcal{H}_R$ -embedding  $\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q, Q')} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$  is given in the next proposition generalizing Lemma 4.3. The element  $e_{M_2}$  of  $\mathcal{H}_R$  appearing in the definition of  $\iota(P, G)$  is replaced in the definition of  $\iota(Q, Q')$  by an element  $\theta_{Q'}(e_Q^{Q'}) \in \mathcal{H}_R$  that we define first.

Until the end of section 4, we fix an admissible lift  $w \mapsto \hat{w} : \mathbb{W} \rightarrow \mathcal{N} \cap \mathcal{K}$  (Definition 3.1) and  $\tilde{w}$  denotes the image of  $\hat{w}$  in  $W(1)$ . We denote  $\mathbb{W}_{M_Q} = \mathbb{W}_Q$  and by  ${}^{\mathbb{W}_Q}\mathbb{W}$  the set of  $w \in \mathbb{W}$  of minimal length in their coset  $\mathbb{W}_Q w$ . The group  $G$  is the disjoint union of  $Q\hat{d}\mathcal{U}$  for  $d$  running through  ${}^{\mathbb{W}_Q}\mathbb{W}$  [OV17, Lemma 2.15 (2)]:  $G = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}} Q\hat{d}\mathcal{U}$ . Since  $Q\hat{d}\mathcal{U} \subset Q'\mathcal{U}$  if and only if  $\hat{d} \in Q'$ , namely  $d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}$ , we have

$$(4.6) \quad Q'\mathcal{U} = \bigsqcup_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} Q\hat{d}\mathcal{U}.$$

Set

$$(4.7) \quad e_Q^{Q'} = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} T_{\tilde{d}}^{M_{Q'}}.$$

We write  $e_P^G = e_Q$ . We have  $e_P^Q = \sum_{d \in \mathbb{W}_{M_2, Q}} T_{\tilde{d}}^{M_Q}$ .

*Remark 4.4.* Note that  ${}^{\mathbb{W}_M}\mathbb{W} = \mathbb{W}_{M_2}$  and  $e_P = e_{M_2}$ , where  $M_2$  is the standard Levi subgroup of  $G$  with  $\Delta_{M_2} = \Delta \setminus \Delta_M$ , as  $\Delta_M$  and  $\Delta \setminus \Delta_M$  are orthogonal. More generally,  ${}^{\mathbb{W}_Q}\mathbb{W}_{M_{Q'}} = {}^{\mathbb{W}_{M_2, Q}}\mathbb{W}_{M_2, Q'}$  where  $M_{2, Q'} = M_2 \cap M_{Q'}$ .

Note that  $e_Q^{Q'} \in \mathcal{H}_{M^+} \cap \mathcal{H}_{M^-}$ . We consider the linear map

$$\theta_Q^{Q'} : \mathcal{H}_Q \rightarrow \mathcal{H}_{Q'} \quad T_w^{M_Q} \mapsto T_w^{M_{Q'}} \quad (w \in W_{M_Q}(1)).$$

We write  $\theta_Q^G = \theta_Q$  so that  $\theta_Q(T_w^{M_Q}) = T_w$ . When  $Q = P$  this is the map  $\theta$  defined earlier. Similarly we denote by  $\theta_Q^{Q',*}$  the linear map sending the  $T_w^{M_Q,*}$  to  $T_w^{M_{Q'},*}$  and  $\theta_Q^{G,*} = \theta_Q^*$ . We have

$$(4.8) \quad \theta_{Q'}(e_Q^{Q'}) = \sum_{d \in {}^{\mathbb{W}_Q}\mathbb{W}_{Q'}} T_{\tilde{d}}, \quad \theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q) \theta_{Q'}(e_Q^{Q'}).$$

**Proposition 4.5.** *There exists an  $\mathcal{H}_R$ -isomorphism*

$$(4.9) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{QU} : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H} \xrightarrow{\kappa_Q} e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}},$$

and compatible  $\mathcal{H}_R$ -embeddings

$$(4.10) \quad v \otimes f_{QU} \mapsto v \otimes f_{Q'U} : e_{\mathcal{H}_{Q'}}(\mathcal{V}) \otimes_R (\text{Ind}_{Q'}^G \mathbf{1})^{\mathcal{U}} \rightarrow e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}},$$

$$(4.11) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}(e_Q^{Q'}) : \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota_{(Q, Q')}} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})).$$

*Proof.* We have the  $\mathcal{H}_{M_Q, R}$ -embedding

$$v \mapsto v \otimes e_P^Q : e_{\mathcal{H}_Q}(\mathcal{V}) \rightarrow \mathcal{V} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H}_Q = \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})$$

by Lemma 4.3 (4.2) as  $\Delta_M$  is orthogonal to  $\Delta_{M_Q} \setminus \Delta_M$ . Applying the parabolic induction which is exact, we get the  $\mathcal{H}$ -embedding

$$v \otimes 1_{\mathcal{H}} \mapsto v \otimes e_P^Q \otimes 1_{\mathcal{H}} : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}_Q}(\mathcal{V})).$$

Note that  $T_{\tilde{d}}^{M_Q} \in \mathcal{H}_{M_Q^+}$  for  $d \in \mathbb{W}_{M_Q}$ . By transitivity of the parabolic induction, it is equal to the  $\mathcal{H}_R$ -embedding

$$(4.12) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_Q(e_P^Q) : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}).$$

On the other hand we have the  $\mathcal{H}_R$ -embedding

$$(4.13) \quad v \otimes f_{QU} \mapsto v \otimes \theta_Q(e_P^Q) : e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \rightarrow \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$$

given by the restriction to  $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  of the  $\mathcal{H}_R$ -isomorphism given in Lemma 4.3 (4.1), from  $e(\mathcal{V}) \otimes_R (\text{Ind}_P^G \mathbf{1})^{\mathcal{U}}$  to  $\mathcal{V} \otimes_{\mathcal{H}_{M^+}, \theta} \mathcal{H}$  sending  $v \otimes f_{PU}$  to  $v \otimes 1_{\mathcal{H}}$ , noting that  $v \otimes f_{QU} = (v \otimes f_{PU}) \theta_Q(e_P^Q)$  by Proposition 3.15,  $f_{QU} = f_{PU} \theta_Q(e_P^Q)$  and  $\theta_Q(e_P^Q)$  acts trivially on  $e(\mathcal{V})$  (this is true for  $T_{\tilde{d}}$  for  $\tilde{d} \in {}_1W_{M_2'}$ ). Comparing the embeddings (4.12) and (4.13), we get the  $\mathcal{H}_R$ -isomorphism (4.9).

We can replace  $Q$  by  $Q'$  in the  $\mathcal{H}_R$ -homomorphisms (4.9), (4.12), and (4.13). With (4.12) we see  $\text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V}))$  and  $\text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$  as  $\mathcal{H}_R$ -submodules of  $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V})$ . As seen in (4.8) we have  $\theta_{Q'}(e_P^{Q'}) = \theta_Q(e_P^Q) \theta_{Q'}(e_Q^{Q'})$ . We deduce the  $\mathcal{H}_R$ -embedding (4.11).

By (3.19) for  $Q$  and (4.6),

$$f_{Q'}u = \sum_{d \in {}^{\mathbb{W}Q}\mathbb{W}_{Q'}} f_{Qu}T_{\bar{d}} = f_{Qu}\theta_{Q'}(e_Q^{Q'})$$

in  $(\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$ . We deduce that the  $\mathcal{H}_R$ -embedding corresponding to (4.11) via  $\kappa_Q$  and  $\kappa_{Q'}$  is the  $\mathcal{H}_R$ -embedding (4.10).  $\square$

We recall that  $\Delta_P$  and  $\Delta \setminus \Delta_P$  are orthogonal and that  $\mathcal{V}$  is extensible to  $\mathcal{H}$  of extension  $e(\mathcal{V})$ .

**Corollary 4.6.** *The cokernel of the  $\mathcal{H}_R$ -map*

$$\bigoplus_{Q \subseteq Q' \subset G} \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V}))$$

*defined by the  $\iota(Q, Q')$ , is isomorphic to  $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$  via  $\kappa_Q$ .*

**4.2. Invariants in the tensor product.** We return to the setting where  $P = MN$  is a standard parabolic subgroup of  $G$ ,  $\sigma$  is a smooth  $R$ -representation of  $M$  with  $P(\sigma) = G$  of extension  $e(\sigma)$  to  $G$ , and  $Q$  a parabolic subgroup of  $G$  containing  $P$ . We still assume that  $\Delta_P$  and  $\Delta \setminus \Delta_P$  are orthogonal.

The  $\mathcal{H}_R$ -modules  $e(\sigma^{\mathcal{U}_M}) = e(\sigma)^{\mathcal{U}}$  are equal (Theorem 3.13). We compute  $I_G(P, \sigma, Q)^{\mathcal{U}} = (e(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}}$ .

**Theorem 4.7.** *The natural linear maps  $e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \rightarrow (e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  and  $e(\sigma)^{\mathcal{U}} \otimes_R (\text{St}_Q^G)^{\mathcal{U}} \rightarrow (e(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}}$  are isomorphisms.*

*Proof.* We need some preliminaries. In [GK14, Ly15], are introduced a finite free  $\mathbb{Z}$ -module  $\mathfrak{M}$  (depending on  $\Delta_Q$ ) and a  $\mathcal{B}$ -equivariant embedding  $\text{St}_Q^G \mathbb{Z} \xrightarrow{\iota} C_c^\infty(\mathcal{B}, \mathfrak{M})$  (we indicate the coefficient ring in the Steinberg representation) which induces an isomorphism  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{B}} \simeq C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$ .

**Lemma 4.8.**

- (1)  $(\text{Ind}_Q^G \mathbb{Z})^{\mathcal{B}}$  is a direct factor of  $\text{Ind}_Q^G \mathbb{Z}$ .
- (2)  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}$  is a direct factor of  $\text{St}_Q^G \mathbb{Z}$ .

*Proof.*

- (1) [AHV, Example 2.2].

(2) As  $\mathfrak{M}$  is a free  $\mathbb{Z}$ -module,  $C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$  is a direct factor of  $C_c^\infty(\mathcal{B}, \mathfrak{M})$ . Consequently,  $\iota((\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}) = C_c^\infty(\mathcal{B}, \mathfrak{M})^{\mathcal{B}}$  is a direct factor of  $\iota(\text{St}_Q^G \mathbb{Z})$ . As  $\iota$  is injective, we get (2).  $\square$

We now prove Theorem 4.7. We may and do assume that  $\sigma$  is  $e$ -minimal (because  $P(\sigma) = P(\sigma_{\min})$ ,  $e(\sigma) = e(\sigma_{\min})$ ) so that  $\Delta_M$  and  $\Delta \setminus \Delta_M$  are orthogonal and we use the same notation as in section 3.2 in particular  $M_2 = M_{\Delta \setminus \Delta_M}$ . Let  $V$  be the space of  $e(\sigma)$  on which  $M_2'$  acts trivially. The restriction of  $\text{Ind}_Q^G \mathbb{Z}$  to  $M_2$  is  $\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z}$ , that of  $\text{St}_Q^G \mathbb{Z}$  is  $\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z}$ .

As in [AHV, Example 2.2],  $((\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z}) \otimes V)^{\mathcal{U}_{M_2'}} \simeq (\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2'}} \otimes V$ . We have

$$(\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2'}} = (\text{Ind}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}} = (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}}.$$

The first equality follows from  $M_2 = (Q \cap M_2)\mathbb{W}_{M_2}\mathcal{U}_{M_2}$ ,  $\mathcal{U}_{M_2} = Z^1\mathcal{U}_{M_2'}$  and  $Z^1$  normalizes  $\mathcal{U}_{M_2'}$  and is normalized by  $\mathbb{W}_{M_2}$ . The second equality follows from  $\mathcal{U} =$

$\mathcal{U}_{M'}\mathcal{U}_{M_2}$  and  $\text{Ind}_Q^G \mathbb{Z}$  is trivial on  $M'$ . Therefore  $((\text{Ind}_Q^G \mathbb{Z}) \otimes V)^{\mathcal{U}_{M'_2}} \simeq (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V$ . Now taking fixed points under  $\mathcal{U}_M$ , as  $\mathcal{U} = \mathcal{U}_{M'_2}\mathcal{U}_M$ ,

$$((\text{Ind}_Q^G \mathbb{Z}) \otimes V)^{\mathcal{U}} \simeq ((\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V)^{\mathcal{U}_M} = (\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}_M}.$$

The equality uses that the  $\mathbb{Z}$ -module  $\text{Ind}_Q^G \mathbb{Z}$  is free. We get the first part of the theorem as  $(\text{Ind}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}_M} \simeq (\text{Ind}_Q^G R)^{\mathcal{U}} \otimes_R V^{\mathcal{U}_M}$ .

Tensoring with  $R$  the usual exact sequence defining  $\text{St}_Q^G \mathbb{Z}$  gives an isomorphism  $\text{St}_Q^G \mathbb{Z} \otimes R \simeq \text{St}_Q^G R$  and in [GK14, Ly15], it is proved that the resulting map  $\text{St}_Q^G R \xrightarrow{\iota_R} C^\infty(\mathcal{B}, \mathfrak{M} \otimes R)$  is also injective. Their proof in no way uses the ring structure of  $R$ , and for any  $\mathbb{Z}$ -module  $V$ , tensoring with  $V$  gives a  $\mathcal{B}$ -equivariant embedding  $\text{St}_Q^G \mathbb{Z} \otimes V \xrightarrow{\iota_V} C_c^\infty(\mathcal{B}, \mathfrak{M} \otimes V)$ . The natural map  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{B}} \otimes V \rightarrow \text{St}_Q^G \mathbb{Z} \otimes V$  is also injective by Lemma 4.8 (2). Taking  $\mathcal{B}$ -fixed points we get inclusions

$$(4.14) \quad (\text{St}_Q^G \mathbb{Z})^{\mathcal{B}} \otimes V \rightarrow (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{B}} \rightarrow C_c^\infty(\mathcal{B}, \mathfrak{M} \otimes V)^{\mathcal{B}} \simeq \mathfrak{M} \otimes V.$$

The composite map is surjective, so the inclusions are isomorphisms. The image of  $\iota_V$  consists of functions which are left  $Z^0$ -invariant, and  $\mathcal{B} = Z^0 \mathcal{U}'$  where  $\mathcal{U}' = G' \cap \mathcal{U}$ . It follows that  $\iota$  yields an isomorphism  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} \simeq C_c^\infty(Z^0 \backslash \mathcal{B}, \mathfrak{M})^{\mathcal{U}'}$  again consisting of the constant functions. So that in particular  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} = (\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}$  and reasoning as previously we get isomorphisms

$$(4.15) \quad (\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} \otimes V \simeq (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{U}'} \simeq \mathfrak{M} \otimes V.$$

The equality  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}'} = (\text{St}_Q^G \mathbb{Z})^{\mathcal{B}}$  and the isomorphisms remain true when we replace  $\mathcal{U}'$  by any group between  $\mathcal{B}$  and  $\mathcal{U}'$ . We apply these results to  $\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z} \otimes V$  to get that the natural map  $(\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M'_2}} \otimes V \rightarrow (\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z} \otimes V)^{\mathcal{U}_{M'_2}}$  is an isomorphism and also that  $(\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M'_2}} = (\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}}$ . We have  $\mathcal{U} = \mathcal{U}_M \mathcal{U}_{M_2}$  so  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} = (\text{St}_{Q \cap M_2}^{M_2} \mathbb{Z})^{\mathcal{U}_{M_2}}$  and the natural map  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V \rightarrow (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{U}_{M'_2}}$  is an isomorphism. The  $\mathbb{Z}$ -module  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}}$  is free and the  $V^{\mathcal{U}_M} = V^{\mathcal{U}}$ , so taking fixed points under  $\mathcal{U}_M$ , we get  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}} \simeq (\text{St}_Q^G \mathbb{Z} \otimes V)^{\mathcal{U}}$ . We have  $\text{St}_Q^G \mathbb{Z} \otimes V = \text{St}_Q^G R \otimes_R V$  and  $(\text{St}_Q^G \mathbb{Z})^{\mathcal{U}} \otimes V^{\mathcal{U}} = (\text{St}_Q^G R)^{\mathcal{U}} \otimes_R V^{\mathcal{U}}$ . This ends the proof of the theorem.  $\square$

**Theorem 4.9.** *The  $\mathcal{H}_R$ -modules  $(e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} = e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  are equal. The  $\mathcal{H}_R$ -modules  $(e(\sigma) \otimes_R \text{St}_Q^G)^{\mathcal{U}} = e(\sigma)^{\mathcal{U}} \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$  are also equal.*

*Proof.* We already know that the  $R$ -modules are equal (Theorem 4.7). We show that they are equal as  $\mathcal{H}$ -modules. The  $\mathcal{H}_R$ -modules  $e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} = e_{\mathcal{H}}(\sigma^{\mathcal{U}_M}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  are equal (Theorem 3.13), they are isomorphic to  $\text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\sigma^{\mathcal{U}_M}))$  (Proposition 4.5), to  $(\text{Ind}_Q^G(e_Q(\sigma)))^{\mathcal{U}}$  [OV17, Proposition 4.4], and to  $(e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  [AHV, Lemma 2.5]). We deduce that the  $\mathcal{H}_R$ -modules  $e(\sigma)^{\mathcal{U}} \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} = (e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  are equal. The same is true when  $Q$  is replaced by a parabolic subgroup  $Q'$  of  $G$  containing  $Q$ . The representation  $e(\sigma) \otimes_R \text{St}_Q^G$  is the cokernel of the natural  $R[G]$ -map

$$\bigoplus_{Q \subsetneq Q'} e(\sigma) \otimes_R \text{Ind}_{Q'}^G \mathbf{1} \xrightarrow{\alpha_Q} e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$$

and the  $\mathcal{H}_R$ -module  $e(\sigma)^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U}$  is the cokernel of the natural  $\mathcal{H}_R$ -map

$$\bigoplus_{Q \subsetneq Q'} e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U} \xrightarrow{\beta_Q} e(\sigma)^\mathcal{U} \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$$

obtained by tensoring (3.22) by  $e(\sigma)^\mathcal{U}$  over  $R$ , because the tensor product is right exact. The maps  $\beta_Q = \alpha_Q^\mathcal{U}$  are equal and the  $R$ -modules  $e(\sigma)^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U} = (e(\sigma) \otimes_R \text{St}_Q^G)^\mathcal{U}$  are equal. This implies that the  $\mathcal{H}_R$ -modules  $e(\sigma)^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U} = (e(\sigma) \otimes_R \text{St}_Q^G)^\mathcal{U}$  are equal.  $\square$

*Remark 4.10.* The proof shows that the representations  $e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$  and  $e(\sigma) \otimes \text{St}_Q^G$  of  $G$  are generated by their  $\mathcal{U}$ -fixed vectors if the representation  $\sigma$  of  $M$  is generated by its  $\mathcal{U}_M$ -fixed vectors. Indeed, the  $R$ -modules  $e(\sigma)^\mathcal{U} = \sigma^{\mathcal{U}_M}, (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}_{M'_2}} = (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$  are equal. If  $\sigma^{\mathcal{U}_M}$  generates  $\sigma$ , then  $e(\sigma)$  is generated by  $e(\sigma)^\mathcal{U}$ . The representation  $\text{Ind}_Q^G \mathbf{1}|_{M'_2}$  is generated by  $(\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$  (this follows from the lemma below), we have  $G = MM'_2$  and  $M'_2$  acts trivially on  $e(\sigma)$ . Therefore the  $R[G]$ -module generated by  $\sigma^\mathcal{U} \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}$  is  $e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$ . As  $e(\sigma) \otimes_R \text{St}_Q^G$  is a quotient of  $e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$ , the  $R[G]$ -module generated by  $\sigma^\mathcal{U} \otimes_R (\text{St}_Q^G)^\mathcal{U}$  is  $e(\sigma) \otimes_R \text{St}_Q^G$ .

**Lemma 4.11.** *For any standard parabolic subgroup  $P$  of  $G$ , the representation  $\text{Ind}_P^G \mathbf{1}|_{G'}$  is generated by its  $\mathcal{U}$ -fixed vectors.*

*Proof.* Because  $G = PG'$  it suffices to prove that if  $J$  is an open compact subgroup of  $\overline{N}$  the characteristic function  $1_{PJ}$  of  $PJ$  is a finite sum of translates of  $1_{PU} = 1_{P\mathcal{U}_{\overline{N}}}$  by  $G'$ . For  $t \in T$  we have  $P\mathcal{U}t = Pt^{-1}\mathcal{U}_{\overline{N}}t$  and we can choose  $t \in T \cap J'$  such that  $t^{-1}\mathcal{U}_{\overline{N}}t \subset J$ .  $\square$

**4.3. General triples.** Let  $P = MN$  be a standard parabolic subgroup of  $G$ . We now investigate situations where  $\Delta_P$  and  $\Delta \setminus \Delta_P$  are not necessarily orthogonal. Let  $\mathcal{V}$  be a right  $\mathcal{H}_{M,R}$ -module.

**Definition 4.12.** Let  $P(\mathcal{V}) = M(\mathcal{V})N(\mathcal{V})$  be the standard parabolic subgroup of  $G$  with  $\Delta_{P(\mathcal{V})} = \Delta_P \cup \Delta_{\mathcal{V}}$  and

$$\Delta_{\mathcal{V}} = \{\alpha \in \Delta \text{ orthogonal to } \Delta_M, T^M(z) \text{ acts trivially on } \mathcal{V} \text{ for all } z \in Z \cap M'_\alpha\}.$$

If  $Q$  is a parabolic subgroup of  $G$  between  $P$  and  $P(\mathcal{V})$ , the triple  $(P, \mathcal{V}, Q)$  called an  $\mathcal{H}_R$ -triple, defines a right  $\mathcal{H}_R$ -module  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$  equal to

$$\text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}} (e(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) = (e(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V})+}, R, \theta} \mathcal{H}_R,$$

where  $e(\mathcal{V})$  is the extension of  $\mathcal{V}$  to  $\mathcal{H}_{M(\mathcal{V})}$ .

This definition is justified by the fact that  $M(\mathcal{V})$  is the maximal standard Levi subgroup of  $G$  such that the  $\mathcal{H}_{M,R}$ -module  $\mathcal{V}$  is extensible to  $\mathcal{H}_{M(\mathcal{V})}$ .

**Lemma 4.13.**  $\Delta_{\mathcal{V}}$  is the maximal subset of  $\Delta \setminus \Delta_P$  orthogonal to  $\Delta_P$  such that  $T_\lambda^{M,*}$  acts trivially on  $\mathcal{V}$  for all  $\lambda \in \Lambda(1) \cap {}_1W_{M'_\mathcal{V}}$ .

*Proof.* For  $J \subset \Delta$  let  $M_J$  denote the standard Levi subgroup of  $G$  with  $\Delta_{M_J} = J$ . The group  $Z \cap M'_J$  is generated by the  $Z \cap M'_\alpha$  for all  $\alpha \in J$  (Lemma 2.1). When  $J$  is orthogonal to  $\Delta_M$  and  $\lambda \in \Lambda_{M'_J}(1)$ ,  $\ell_M(\lambda) = 0$  where  $\ell_M$  is the length associated to  $S_M^{\text{aff}}$ , and the map  $\lambda \mapsto T_\lambda^{M,*} = T_\lambda^M : \Lambda_{M'_J}(1) \rightarrow \mathcal{H}_M$  is multiplicative.  $\square$

The following is the natural generalization of Proposition 4.5 and Corollary 4.6. Let  $Q'$  be a parabolic subgroup of  $G$  with  $Q \subset Q' \subset P(\mathcal{V})$ . Applying the results of section 4.1 to  $M(\mathcal{V})$  and its standard parabolic subgroups  $Q \cap M(\mathcal{V}) \subset Q' \cap M(\mathcal{V})$ , we have an  $\mathcal{H}_{M(\mathcal{V}),R}$ -isomorphism

$$\begin{aligned} & \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V})) \\ &= e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+}, \theta} \mathcal{H}_{M(\mathcal{V}),R} \xrightarrow{\kappa_{Q \cap M(\mathcal{V})}} e(\mathcal{V}) \otimes_R (\text{Ind}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1})^{\mathcal{U}_{M(\mathcal{V})}} \\ & \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{QU \cap M(\mathcal{V})} : \end{aligned}$$

and an  $\mathcal{H}_{M(\mathcal{V}),R}$ -embedding

$$\begin{aligned} & \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q \cap M(\mathcal{V}), Q' \cap M(\mathcal{V}))} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}_{M(\mathcal{V})}}(e_{\mathcal{H}_Q}(\mathcal{V})) \\ & \quad v \otimes 1_{\mathcal{H}_{M(\mathcal{V})}} \mapsto v \otimes \theta_{Q'}^{P(\mathcal{V})}(e_{\mathcal{H}_Q}^{Q'}). \end{aligned}$$

Applying the parabolic induction  $\text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}$  which is exact and transitive, we obtain an  $\mathcal{H}_R$ -isomorphism  $\kappa_Q = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\kappa_{Q \cap M(\mathcal{V})})$ ,

$$(4.16) \quad \begin{aligned} & \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\kappa_Q} \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\text{Ind}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})} \mathbf{1}_{M_Q})^{\mathcal{U}_{M(\mathcal{V})}}) \\ & \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes f_{QU \cap M(\mathcal{V})} \otimes 1_{\mathcal{H}} \end{aligned}$$

and an  $\mathcal{H}_R$ -embedding  $\iota(Q, Q') = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(\iota(Q, Q')^{M(\mathcal{V})})$

$$(4.17) \quad v \otimes 1_{\mathcal{H}} \mapsto v \otimes \theta_{Q'}^{Q'} : \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{\iota(Q, Q')} \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})).$$

Applying Corollary 4.6 we obtain:

**Theorem 4.14.** *Let  $(P, \mathcal{V}, Q)$  be an  $\mathcal{H}_R$ -triple. Then, the cokernel of the  $\mathcal{H}_R$ -map*

$$\oplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Ind}_{\mathcal{H}_{Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})),$$

*defined by the  $\iota(Q, Q')$  is isomorphic to  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$  via the  $\mathcal{H}_R$ -isomorphism  $\kappa_Q$ .*

Let  $\sigma$  be a smooth  $R$ -representation of  $M$  and let  $Q$  be a parabolic subgroup of  $G$  with  $P \subset Q \subset P(\sigma)$ .

*Remark 4.15.* The  $\mathcal{H}_R$ -module  $I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q)$  is defined if  $\Delta_Q \setminus \Delta_P$  and  $\Delta_P$  are orthogonal because  $Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_M})$  (Theorem 3.13).

We denote here by  $P_{\min} = M_{\min}N_{\min}$  the minimal standard parabolic subgroup of  $G$  contained in  $P$  such that  $\sigma = e_P(\sigma|_{M_{\min}})$  (Lemma 2.3, we drop the index  $\sigma$ ). The sets of roots  $\Delta_{P_{\min}}$  and  $\Delta_{P(\sigma|_{M_{\min}})} \setminus \Delta_{P_{\min}}$  are orthogonal (Lemma 2.4). The groups  $P(\sigma) = P(\sigma|_{M_{\min}})$ , the representations  $e(\sigma) = e(\sigma|_{M_{\min}})$  of  $M(\sigma)$ , the representations  $I_G(P, \sigma, Q) = I_G(P_{\min}, \sigma|_{M_{\min}}, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \text{St}_Q^{P(\sigma)})$  of  $G$ , and the  $R$ -modules  $\sigma^{\mathcal{U}_{M_{\min}}} = \sigma^{\mathcal{U}_M}$  are equal. From Theorem 3.13,

$$P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}}), \quad e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) = e(\sigma)^{\mathcal{U}_{M(\sigma)}},$$

and  $P(\sigma^{\mathcal{U}_{M_{\min}}}) = P(\sigma)$  if  $\sigma^{\mathcal{U}_{M_{\min}}}$  generates the representation  $\sigma|_{M_{\min}}$ . The  $\mathcal{H}_R$ -module

$$I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) = \text{Ind}_{\mathcal{H}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}}^{\mathcal{H}}(e(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_Q^{P(\sigma^{\mathcal{U}_{M_{\min}}})})^{\mathcal{U}_{M(\sigma^{\mathcal{U}_{M_{\min}}})}})$$

is defined because  $\Delta_{P_{\min}}$  and  $\Delta_{P(\sigma^{\mathcal{U}_{M_{\min}}})} \setminus \Delta_{P_{\min}}$  are orthogonal and  $P \subset Q \subset P(\sigma) \subset P(\sigma^{\mathcal{U}_{M_{\min}}})$ .

*Remark 4.16.* If  $\sigma^{\mathcal{U}_{M_{\min}}}$  generates the representation  $\sigma|_{M_{\min}}$  (in particular if  $R$  is an algebraically closed field of characteristic  $p$  and  $\sigma$  is irreducible), then  $P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}})$  hence

$$I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) = \text{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

Applying Theorem 4.9 to  $(P_{\min} \cap M(\sigma), \sigma|_{M_{\min}}, Q \cap M(\sigma))$ , the  $\mathcal{H}_{M(\sigma), R}$ -modules

$$(4.18) \quad e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}} = (e_{M(\sigma)}(\sigma) \otimes_R \text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}$$

are equal. We have the  $\mathcal{H}_R$ -isomorphism [OV17, Proposition 4.4]:

$$\begin{aligned} I_G(P, \sigma, Q)^{\mathcal{U}} &= (\text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \text{St}_Q^{P(\sigma)}))^{\mathcal{U}} \\ &\xrightarrow{ov} \text{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}((e(\sigma) \otimes_R \text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}) \\ &f_{P(\sigma)\mathcal{U}, x} \mapsto x \otimes 1_{\mathcal{H}} \quad (x \in (e(\sigma) \otimes_R \text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}). \end{aligned}$$

We deduce the following.

**Theorem 4.17.** *Let  $(P, \sigma, Q)$  be an  $R[G]$ -triple. Then, we have the  $\mathcal{H}_R$ -isomorphism*

$$I_G(P, \sigma, Q)^{\mathcal{U}} \xrightarrow{ov} \text{Ind}_{\mathcal{H}_{M(\sigma)}}^{\mathcal{H}}(e_{\mathcal{H}_{M(\sigma)}}(\sigma^{\mathcal{U}_{M_{\min}}}) \otimes_R (\text{St}_{Q \cap M(\sigma)}^{M(\sigma)})^{\mathcal{U}_{M(\sigma)}}).$$

In particular,

$$I_G(P, \sigma, Q)^{\mathcal{U}} \simeq \begin{cases} I_{\mathcal{H}}(P_{\min}, \sigma^{\mathcal{U}_{M_{\min}}}, Q) & \text{if } P(\sigma) = P(\sigma^{\mathcal{U}_{M_{\min}}}), \\ I_{\mathcal{H}}(P, \sigma^{\mathcal{U}_M}, Q) & \text{if } P = P_{\min}, P(\sigma) = P(\sigma^{\mathcal{U}_M}). \end{cases}$$

**4.4. Comparison of the parabolic induction and coinduction.** Let  $P = MN$  be a standard parabolic subgroup of  $G$ , let  $\mathcal{V}$  be a right  $\mathcal{H}_R$ -module, and let  $Q$  be a parabolic subgroup of  $G$  with  $Q \subset P(\mathcal{V})$ . When  $R$  is an algebraically closed field of characteristic  $p$ , in [Abe], we associated to  $(P, \mathcal{V}, Q)$  an  $\mathcal{H}_R$ -module using the parabolic coinduction

$$\text{Coind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = \text{Hom}_{\mathcal{H}_{M_Q}^-, \theta^*}(\mathcal{H}, -) : \text{Mod}_R(\mathcal{H}_{M_Q}) \rightarrow \text{Mod}_R(\mathcal{H})$$

instead of the parabolic induction  $\text{Ind}_{\mathcal{H}_{M_Q}}^{\mathcal{H}}(-) = - \otimes_{\mathcal{H}_{M_Q}^+, \theta} \mathcal{H}$ . The index  $\theta^*$  in the parabolic coinduction means that  $\mathcal{H}_{M_Q}^-$  embeds in  $\mathcal{H}$  by  $\theta_Q^*$ . Our terminology is different from the one in [Abe] where the parabolic coinduction is called induction. For a parabolic subgroup  $Q'$  of  $G$  with  $Q \subset Q' \subset P(\mathcal{V})$ , there is a natural inclusion of  $\mathcal{H}_R$ -modules

$$(4.19) \quad \text{Hom}_{\mathcal{H}_{M_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \xrightarrow{i(Q, Q')} \text{Hom}_{\mathcal{H}_{M_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

because  $\theta^*(\mathcal{H}_{M_Q}^-) \subset \theta^*(\mathcal{H}_{M_{Q'}}^-)$  as  $W_{M_Q}^-(1) \subset W_{M_{Q'}}^-(1)$ , and  $vT_w^{M_{Q'}, *}=vT_w^{M_Q, *}$  for  $w \in W_{M_Q}^-(1)$  and  $v \in \mathcal{V}$ . (This is [Abe, Proposition 4.19] when  $R$  is an algebraically closed field of characteristic  $p$ . This follows from our formulation of the extension for any  $R$ .)

**Definition 4.18.** Let  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$  denote the cokernel of the map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Hom}_{\mathcal{H}_{M_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Hom}_{\mathcal{H}_{M_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the  $\mathcal{H}_R$ -embeddings  $i(Q, Q')$ .



When  $R$  is an algebraically closed field of characteristic  $p$ , we showed that the  $\mathcal{H}_R$ -module  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$  is simple when  $\mathcal{V}$  is simple and supersingular (Definition 4.25), and that any simple  $\mathcal{H}_R$ -module is of this form for an  $\mathcal{H}_R$ -triple  $(P, \mathcal{V}, Q)$  where  $\mathcal{V}$  is simple and supersingular,  $P, Q$  and the isomorphism class of  $\mathcal{V}$  are unique [Abe]. The aim of this section is to compare the  $\mathcal{H}_R$ -modules  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$  with the  $\mathcal{H}_R$ -modules  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$  and to show that the classification is also valid with the  $\mathcal{H}_R$ -modules  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$ .

It is already known that a parabolically coinduced module is a parabolically induced module and vice versa [Abe, Proposition 4.15], [Vig15b, Theorem 1.8]. To make it more precise we need to introduce notation.

We lift the elements  $w$  of the finite Weyl group  $\mathbb{W}$  to  $\hat{w} \in \mathcal{N}_G \cap \mathcal{K}$  as in [AHHV17, IV.6], [OV17, Proposition 2.7]: they satisfy the braid relations  $\hat{w}_1 \hat{w}_2 = (\hat{w}_1 \hat{w}_2)$  when  $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$  and when  $s \in S$ ,  $\hat{s}$  is admissible, in particular lies in  ${}_1 W_{G'}$ .

Let  $\mathbf{w}, \mathbf{w}_M, \mathbf{w}^M$  denote, respectively, the longest elements in  $\mathbb{W}, \mathbb{W}_M$  and  $\mathbf{w}\mathbf{w}_M$ . We have  $\mathbf{w} = \mathbf{w}^{-1} = \mathbf{w}^M \mathbf{w}_M$ ,  $\mathbf{w}_M = \mathbf{w}_M^{-1}$ ,  $\hat{\mathbf{w}} = \hat{\mathbf{w}}^M \hat{\mathbf{w}}_M$ ,

$$\mathbf{w}^M(\Delta_M) = -\mathbf{w}(\Delta_M) \subset \Delta, \quad \mathbf{w}^M(\Phi^+ \setminus \Phi_M^+) = \mathbf{w}(\Phi^+ \setminus \Phi_M^+).$$

Let  $\mathbf{w}.M$  be the standard Levi subgroup of  $G$  with  $\Delta_{\mathbf{w}.M} = \mathbf{w}^M(\Delta_M)$  and  $\mathbf{w}.P$  the standard parabolic subgroup of  $G$  with Levi  $\mathbf{w}.M$ . We have

$$\mathbf{w}.M = \hat{\mathbf{w}}^M M (\hat{\mathbf{w}}^M)^{-1} = \hat{\mathbf{w}} M (\hat{\mathbf{w}})^{-1}, \quad \mathbf{w}^{\mathbf{w}.M} = \mathbf{w}_M \mathbf{w} = (\mathbf{w}^M)^{-1}.$$

The conjugation  $w \mapsto \mathbf{w}^M w (\mathbf{w}^M)^{-1}$  in  $W$  gives a group isomorphism  $W_M \rightarrow W_{\mathbf{w}.M}$  sending  $S_M^{\text{aff}}$  onto  $S_{\mathbf{w}.M}^{\text{aff}}$ , respecting the finite Weyl subgroups  $\mathbf{w}^M \mathbb{W}_M (\mathbf{w}^M)^{-1} = \mathbb{W}_{\mathbf{w}.M} = \mathbf{w} \mathbb{W}_M \mathbf{w}^{-1}$ , and exchanging  $W_{M^+}$  and  $W_{(\mathbf{w}.M)^-} = \mathbf{w} W_{M^+} \mathbf{w}^{-1}$ . The conjugation by  $\tilde{\mathbf{w}}^M$  restricts to a group isomorphism  $W_M(1) \rightarrow W_{\mathbf{w}.M}(1)$  sending  $W_{M^+}(1)$  onto  $W_{(\mathbf{w}.M)^-}(1)$ . The linear isomorphism

$$(4.20) \quad \mathcal{H}_M \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w}.M} \quad T_w^M \mapsto T_{\tilde{\mathbf{w}}^M w (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M} \text{ for } w \in W_M(1),$$

is a ring isomorphism between the pro- $p$ -Iwahori Hecke rings of  $M$  and  $\mathbf{w}.M$ . It sends the positive part  $\mathcal{H}_{M^+}$  of  $\mathcal{H}_M$  onto the negative part  $\mathcal{H}_{(\mathbf{w}.M)^-}$  of  $\mathcal{H}_{\mathbf{w}.M}$  [Vig15b, Proposition 2.20]. We have  $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}_M \tilde{\mathbf{w}}^{\mathbf{w}.M} = \tilde{\mathbf{w}}^M \tilde{\mathbf{w}}_M$ ,  $(\tilde{\mathbf{w}}^M)^{-1} = \tilde{\mathbf{w}}^{\mathbf{w}.M} t_M$  where  $t_M = \tilde{\mathbf{w}}^2 \tilde{\mathbf{w}}_M^{-2} \in Z_k$ .

**Definition 4.19.** The twist  $\tilde{\mathbf{w}}^M.\mathcal{V}$  of  $\mathcal{V}$  by  $\tilde{\mathbf{w}}^M$  is the right  $\mathcal{H}_{\mathbf{w}.M}$ -module deduced from the right  $\mathcal{H}_M$ -module  $\mathcal{V}$  by functoriality: as  $R$ -modules  $\tilde{\mathbf{w}}^M.\mathcal{V} = \mathcal{V}$  and for  $v \in \mathcal{V}, w \in W_M(1)$  we have  $v T_{\tilde{\mathbf{w}}^M w (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M} = v T_w^M$ .

We can define the twist  $\tilde{\mathbf{w}}^M.\mathcal{V}$  of  $\mathcal{V}$  with the  $T_w^{M,*}$  instead of  $T_w^M$ .

**Lemma 4.20.** For  $v \in \mathcal{V}, w \in W_M(1)$  we have  $v T_{\tilde{\mathbf{w}}^M w (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M,*} = v T_w^{M,*}$  in  $\tilde{\mathbf{w}}^M.\mathcal{V}$ .

*Proof.* By the ring isomorphism  $\mathcal{H}_M \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w}.M}$ , we have  $c_{\tilde{\mathbf{w}}^M \tilde{s} (\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M} = c_{\tilde{s}}^M$  when  $\tilde{s} \in W_M(1)$  lifts  $s \in S_M^{\text{aff}}$ . So the equality of the lemma is true for  $w = \tilde{s}$ . Apply the braid relations to get the equality for all  $w \in W_M(1)$ .  $\square$

We return to the  $\mathcal{H}_R$ -module  $\text{Hom}_{\mathcal{H}_{M^-, \theta^*}}(\mathcal{H}, V)$  parabolically coinduced from  $\mathcal{V}$ . It has a natural direct decomposition indexed by the set  $\mathbb{W}^{\mathbb{W}_M}$  of elements  $d$  in

the finite Weyl group  $\mathbb{W}$  of minimal length in the coset  $d\mathbb{W}_M$ . Indeed it is known that the linear map

$$f \mapsto (f(T_{\tilde{d}}))_{d \in \mathbb{W}^{\mathbb{W}_M}} : \text{Hom}_{\mathcal{H}_{M-}, \theta^*}(\mathcal{H}, \mathcal{V}) \rightarrow \bigoplus_{d \in \mathbb{W}^{\mathbb{W}_M}} \mathcal{V}$$

is an isomorphism. For  $v \in \mathcal{V}$  and  $d \in \mathbb{W}^{\mathbb{W}_M}$ , there is a unique element

$$f_{\tilde{d}, v} \in \text{Hom}_{\mathcal{H}_{M-}, \theta^*}(\mathcal{H}, \mathcal{V}) \text{ satisfying } f(T_{\tilde{d}}) = v \text{ and } f(T_{\tilde{d}'} ) = 0 \text{ for } d' \in \mathbb{W}^{\mathbb{W}_M} \setminus \{d\}.$$

It is known that the map  $v \mapsto f_{\tilde{\mathbf{w}}^M, v} : \tilde{\mathbf{w}}^M \cdot \mathcal{V} \rightarrow \text{Hom}_{\mathcal{H}_{M-}, \theta^*}(\mathcal{H}, \mathcal{V})$  is  $\mathcal{H}_{(\mathbf{w}, M)^+}$ -equivariant:  $f_{\tilde{\mathbf{w}}^M, v} T_w^{\mathbf{w}, M} = f_{\tilde{\mathbf{w}}^M, v} T_w$  for all  $v \in \mathcal{V}, w \in W_{\mathbf{w}, M^+}(1)$ . By adjunction, this  $\mathcal{H}_{(\mathbf{w}, M)^+}$ -equivariant map gives an  $\mathcal{H}_R$ -homomorphism from an induced module to a coinduced module:

$$(4.21) \quad v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M, v} : \tilde{\mathbf{w}}^M \cdot \mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}, M)^+}, \theta} \mathcal{H} \xrightarrow{\mu_P} \text{Hom}_{\mathcal{H}_{M-}, \theta^*}(\mathcal{H}, \mathcal{V}).$$

This is an isomorphism [Abe, Proposition 4.15], [Vig15b, Theorem 1.8].

The naive guess that a variant  $\mu_Q$  of  $\mu_P$  induces an  $\mathcal{H}_R$ -isomorphism between the  $\mathcal{H}_R$ -modules  $I_{\mathcal{H}}(\mathbf{w}, P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w}, Q)$  and  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$  turns out to be true. The proof is the aim of the rest of this section.

The  $\mathcal{H}_R$ -module  $I_{\mathcal{H}}(\mathbf{w}, P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w}, Q)$  is well defined because the parabolic subgroups of  $G$  containing  $\mathbf{w}, P$  and contained in  $P(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$  are  $\mathbf{w}, Q$  for  $P \subset Q \subset P(\mathcal{V})$ , as follows from Lemma 4.21.

**Lemma 4.21.**  $\Delta_{\tilde{\mathbf{w}}^M \cdot \mathcal{V}} = -\mathbf{w}(\Delta_{\mathcal{V}})$ .

*Proof.* Recall that  $\Delta_{\mathcal{V}}$  is the set of simple roots  $\alpha \in \Delta \setminus \Delta_M$  orthogonal to  $\Delta_M$  and  $T^M(z)$  acts trivially on  $\mathcal{V}$  for all  $z \in Z \cap M'_{\alpha}$ , and the corresponding standard parabolic subgroup  $P_{\mathcal{V}} = M_{\mathcal{V}} N_{\mathcal{V}}$ . The  $Z \cap M'_{\alpha}$  for  $\alpha \in \Delta_{\mathcal{V}}$  generate the group  $Z \cap M'_{\mathcal{V}}$ . A root  $\alpha \in \Delta \setminus \Delta_M$  orthogonal to  $\Delta_M$  is fixed by  $\mathbf{w}_M$  so  $\mathbf{w}^M(\alpha) = \mathbf{w}(\alpha)$  and

$$\hat{\mathbf{w}}^M M_{\mathcal{V}}(\hat{\mathbf{w}}^M)^{-1} = \hat{\mathbf{w}} M_{\mathcal{V}}(\hat{\mathbf{w}})^{-1}.$$

The proof of Lemma 4.21 is straightforward as  $\Delta = -\mathbf{w}(\Delta)$ ,  $\Delta_{\mathbf{w}, M} = -\mathbf{w}(\Delta_M)$ .  $\square$

Before going further, we check the commutativity of the extension with the twist. As  $Q = M_Q U$  and  $M_Q$  determine each other we denote  $\mathbf{w}_{M_Q} = \mathbf{w}_Q, \mathbf{w}^{M_Q} = \mathbf{w}^Q$  when  $Q \neq P, G$ .

**Lemma 4.22.**  $e_{\mathcal{H}_{\mathbf{w}, Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) = \tilde{\mathbf{w}}^Q \cdot e_{\mathcal{H}_Q}(\mathcal{V})$ .

*Proof.* As  $R$ -modules  $\mathcal{V} = e_{\mathcal{H}_{\mathbf{w}, Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) = \tilde{\mathbf{w}}^Q \cdot e_{\mathcal{H}_Q}(\mathcal{V})$ . A direct computation shows that the Hecke element  $T_w^{\mathbf{w}, Q, *}$  acts in the  $\mathcal{H}_R$ -module  $e_{\mathcal{H}_{\mathbf{w}, Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$ , by the identity if  $w \in \tilde{\mathbf{w}}^Q {}_1 W_{M'_2}(\mathbf{w}^Q)^{-1}$  and by  $T_{(\tilde{\mathbf{w}}^Q)^{-1} w \tilde{\mathbf{w}}^Q}^{M, *}$  if  $w \in \tilde{\mathbf{w}}^Q {}_1 W_{M'_2}(\mathbf{w}^Q)^{-1}$  where  $M_2$  denotes the standard Levi subgroup with  $\Delta_{M_2} = \Delta_Q \setminus \Delta_P$ . Whereas in the  $\mathcal{H}_R$ -module  $\tilde{\mathbf{w}}^Q \cdot e_{\mathcal{H}_Q}(\mathcal{V})$ , the Hecke element  $T_w^{\mathbf{w}, Q, *}$  acts by the identity if  $w \in {}_1 W_{\mathbf{w}, M'_2}$  and by  $T_{(\tilde{\mathbf{w}}^M)^{-1} w \tilde{\mathbf{w}}^M}^{M, *}$  if  $w \in W_{\mathbf{w}, M}(1)$ . So the lemma means that

$${}_1 W_{\mathbf{w}, M'_2} = \tilde{\mathbf{w}}^Q {}_1 W_{M'_2}(\mathbf{w}^Q)^{-1}, \quad (\tilde{\mathbf{w}}^Q)^{-1} w \tilde{\mathbf{w}}^Q = (\tilde{\mathbf{w}}^M)^{-1} w \tilde{\mathbf{w}}^M \text{ if } w \in W_{\mathbf{w}, M}(1).$$

These properties are easily proved using that  ${}_1 W_{G'}$  is normal in  $W(1)$  and that the sets of roots  $\Delta_P$  and  $\Delta_Q \setminus \Delta_P$  are orthogonal:  $\mathbf{w}_Q = \mathbf{w}_{M_2} \mathbf{w}_M$ , the elements  $\mathbf{w}_{M_2}$  and  $\mathbf{w}_M$  normalize  $W_M$  and  $W_{M_2}$ , the elements of  $\mathbb{W}_{M_2}$  commutes with the elements of  $\mathbb{W}_M$ .  $\square$

We return to our guess. The variant  $\mu_Q$  of  $\mu_P$  is obtained by combining the commutativity of the extension with the twist and the isomorphism (4.21) applied to  $(Q, e_{\mathcal{H}_Q}(\mathcal{V}))$  instead of  $(P, \mathcal{V})$ . The  $\mathcal{H}_R$ -isomorphism  $\mu_Q$  is

$$(4.22) \quad v \otimes 1_{\mathcal{H}} \mapsto f_{\tilde{\mathbf{w}}^M, v} : \text{Ind}_{\mathcal{H}_{\mathbf{w}.M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})) \xrightarrow{\mu_Q} \text{Hom}_{\mathcal{H}_{M_Q}^-, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})).$$

Our guess is that  $\mu_Q$  induces an  $\mathcal{H}_R$ -isomorphism from the cokernel of the  $\mathcal{H}_R$ -map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Ind}_{\mathcal{H}_{\mathbf{w}.Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_{\mathbf{w}.Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}))$$

defined by the  $\mathcal{H}_R$ -embeddings  $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$ , isomorphic to  $I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M \mathcal{V}, \mathbf{w}.\overline{Q})$  via  $\kappa_{\mathbf{w}.Q}$  (Theorem 4.14), onto the cokernel  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$  the  $\mathcal{H}_R$ -map

$$\bigoplus_{Q \subsetneq Q' \subset P(\mathcal{V})} \text{Hom}_{\mathcal{H}_{M_{Q'}^-, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V})) \rightarrow \text{Hom}_{\mathcal{H}_{M_Q^-, \theta^*}}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$$

defined by the  $\mathcal{H}_R$ -embeddings  $i(Q, Q')$ . This is true if  $i(Q, Q')$  corresponds to  $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$  via the isomorphisms  $\mu_{Q'}$  and  $\mu_Q$ . This is the content of the next proposition.

**Proposition 4.23.** *For all  $Q \subsetneq Q' \subset P(\mathcal{V})$  we have*

$$i(Q, Q') \circ \mu_{Q'} = \mu_Q \circ \iota(\mathbf{w}.Q, \mathbf{w}.Q').$$

We postpone to section 4.6 the rather long proof of the proposition.

**Corollary 4.24.** *The  $\mathcal{H}_R$ -isomorphism  $\mu_Q \circ \kappa_{\mathbf{w}.Q}^{-1}$  induces an  $\mathcal{H}_R$ -isomorphism*

$$I_{\mathcal{H}}(\mathbf{w}.P, \tilde{\mathbf{w}}^M \mathcal{V}, \mathbf{w}.\overline{Q}) \rightarrow CI_{\mathcal{H}}(P, \mathcal{V}, Q).$$

**4.5. Supersingular  $\mathcal{H}_R$ -modules, classification of simple  $\mathcal{H}_R$ -modules.** We recall first the notion of supersingularity based on the action of the center of  $\mathcal{H}$ .

The center of  $\mathcal{H}$  [Vig14, Theorem 1.3] contains a subalgebra  $\mathcal{Z}_{T^+}$  isomorphic to  $\mathbb{Z}[T^+/T_1]$  where  $T^+$  is the monoid of dominant elements of  $T$  and  $T_1$  is the pro- $p$ -Sylow subgroup of the maximal compact subgroup of  $T$ .

Let  $t \in T$  of image  $\mu_t \in W(1)$  and let  $(E_o(w))_{w \in W(1)}$  denote the alcove walk basis of  $\mathcal{H}$  associated to a closed Weyl chamber  $o$  of  $\mathbb{W}$ . The element

$$E_o(C(\mu_t)) = \sum_{\mu'} E_o(\mu')$$

is the sum over the elements in  $\mu'$  in the conjugacy class  $C(\mu_t)$  of  $\mu_t$  in  $W(1)$ . It is a central element of  $\mathcal{H}$  and does not depend on the choice of  $o$ . We write also  $z(t) = E_o(C(\mu_t))$ .

**Definition 4.25.** A non-zero right  $\mathcal{H}_R$ -module  $\mathcal{V}$  is called supersingular when, for any  $v \in \mathcal{V}$  and any non-invertible  $t \in T^+$ , there exists a positive integer  $n \in \mathbb{N}$  such that  $v(z(t))^n = 0$ . If one can choose  $n$  independent on  $(v, t)$ , then  $\mathcal{V}$  is called uniformly supersingular.

*Remark 4.26.* One can choose  $n$  independent on  $(v, t)$  when  $\mathcal{V}$  is finitely generated as a right  $\mathcal{H}_R$ -module. If  $R$  is a field and  $\mathcal{V}$  is simple we can take  $n = 1$ .

When  $G$  is compact modulo the center,  $T^+ = T$ , and any non-zero  $\mathcal{H}_R$ -module is supersingular.

The induction functor  $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} : \text{Mod}(\mathcal{H}_{M,R}) \rightarrow \text{Mod}(\mathcal{H}_R)$  has a left adjoint  $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}$  and a right adjoint  $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}$  [Vig15b]: for  $\mathcal{V} \in \text{Mod}(\mathcal{H}_R)$ ,

$$(4.23) \quad \mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \tilde{\mathbf{w}}^{\mathbf{w}.M} \circ (\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}), \quad \mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = \text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V}).$$

In the left adjoint,  $\mathcal{V}$  is seen as a right  $\mathcal{H}_{(\mathbf{w}.M)^-}$ -module via the ring homomorphism  $\theta_{\mathbf{w}.M}^* : \mathcal{H}_{(\mathbf{w}.M)^-} \rightarrow \mathcal{H}$ ; in the right adjoint,  $\mathcal{V}$  is seen as a right  $\mathcal{H}_{M^+}$ -module via the ring homomorphism  $\theta_M : \mathcal{H}_{M^+} \rightarrow \mathcal{H}$  (section 2.3).

**Proposition 4.27.** *Assume that  $\mathcal{V}$  is a supersingular right  $\mathcal{H}_R$ -module and that  $p$  is nilpotent in  $\mathcal{V}$ . Then  $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$ , and if  $\mathcal{V}$  is uniformly supersingular  $\mathcal{R}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$ .*

*Proof.* This is a consequence of three known properties:

- (1)  $\mathcal{H}_M$  is the localization of  $\mathcal{H}_{M^+}$  (resp.,  $\mathcal{H}_{M^-}$ ) at  $T_{\mu}^M$  for any element  $\mu \in \Lambda_T(1)$ , central in  $W_M(1)$  and strictly  $N$ -positive (resp.,  $N$ -negative), and  $T_{\mu}^M = T_{\mu}^{M,*}$ . See [Vig15b, Theorem 1.4].
- (2) When  $o$  is anti-dominant,  $E_o(\mu) = T_{\mu}$  if  $\mu \in \Lambda^+(1)$  and  $E_o(\mu) = T_{\mu}^*$  if  $\mu \in \Lambda^-(1)$ .
- (3) Let an integer  $n > 0$  and  $\mu \in \Lambda(1)$  such that the  $\mathbb{W}$ -orbit of  $v(\mu) \in X_*(T) \otimes \mathbb{Q}$  (definition in section 2.1) and of  $\mu$  have the same number of elements. Then

$$(E_o(C(\mu)))^n E_o(\mu) - E_o(\mu)^{n+1} \in p\mathcal{H}.$$

See [Vig15a, Lemma 6.5], where the hypotheses are given in the proof (but not written in the lemma).

Let  $\mu \in \Lambda_T^+(1)$  satisfying (1) for  $M^+$  and (3), similarly let  $\mathbf{w}.\mu \in \Lambda_T^-(1)$  satisfying (1) for  $(\mathbf{w}.M)^-$  and (3). For  $(R, \mathcal{V})$  as in the proposition, let  $v \in \mathcal{V}$  and  $n > 0$  such that  $vE_o(C(\mu))^n = vE_o(C(\mathbf{w}.\mu))^n = 0$ . Multiplying by  $E_o(\mu)$  or  $E_o(\mathbf{w}.\mu)$ , and applying (3) and (2) for  $o$  anti-dominant we get:

$$vE_o(\mu^{n+1}) = vT_{\mu}^{n+1} \in p\mathcal{V}, \quad vE_o((\mathbf{w}.\mu)^{n+1}) = v(T_{\mathbf{w}.\mu}^*)^{n+1} \in p\mathcal{V}.$$

The proposition follows from:  $vT_{\mu}^{n+1}, v(T_{\mathbf{w}.\mu}^*)^{n+1}$  in  $p\mathcal{V}$  (as explained in [Abe16, Proposition 5.17] when  $p = 0$  in  $R$ ). From  $v(T_{\mathbf{w}.\mu}^*)^{n+1}$  in  $p\mathcal{V}$ , we get  $v \otimes (T_{\mathbf{w}.\mu}^{\mathbf{w}.M,*})^{n+1} = v(T_{\mathbf{w}.\mu}^*)^{n+1} \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$  in  $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}$ . As  $T^{\mathbf{w}.M,*} = T^{\mathbf{w}.M}$  is invertible in  $\mathcal{H}_{\mathbf{w}.M}$  we get  $v \otimes 1_{\mathcal{H}_{\mathbf{w}.M}}$  in  $p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}$ . As  $v$  was arbitrary,  $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M} \subset p\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M}$ . If  $p$  is nilpotent in  $\mathcal{V}$ , then  $\mathcal{V} \otimes_{\mathcal{H}_{(\mathbf{w}.M)^-}, \theta^*} \mathcal{H}_{\mathbf{w}.M} = 0$ . Suppose now that there exists  $n > 0$  such that  $\mathcal{V}(z(t))^n = 0$  for any non-invertible  $t \in T^+$ ; then  $\mathcal{V}T_{\mu}^{n+1} \subset p\mathcal{V}$  where  $\mu = \mu_t$  and hence  $\varphi(h) = \varphi(hT_{\mu}^{M_{\mu-n-1}})T_{\mu}^{n+1}$  in  $p\mathcal{V}$  for an arbitrary  $\varphi \in \text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V})$  and an arbitrary  $h \in \mathcal{H}_M$ . We deduce  $\text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V}) \subset \text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, p\mathcal{V})$ . If  $p$  is nilpotent in  $\mathcal{V}$ , then  $\text{Hom}_{\mathcal{H}_{M^+}, \theta}(\mathcal{H}_M, \mathcal{V}) = 0$ .  $\square$

Recalling that  $\tilde{\mathbf{w}}^M \mathcal{V}$  is obtained by functoriality from  $\mathcal{V}$  and the ring isomorphism  $\iota(\tilde{\mathbf{w}}^M)$  defined in (4.20), the equivalence between  $\mathcal{V}$  supersingular and  $\tilde{\mathbf{w}}^M \mathcal{V}$  supersingular follows from Lemma 4.28

**Lemma 4.28.**

- (1) *Let  $t \in T$ . Then  $t$  is dominant for  $U_M$  if and only if  $\tilde{\mathbf{w}}^M t (\tilde{\mathbf{w}}^M)^{-1} \in T$  is dominant for  $U_{\mathbf{w}.M}$ .*

- (2) The  $R$ -algebra isomorphism  $\mathcal{H}_{M,R} \xrightarrow{\iota(\tilde{\mathbf{w}}^M)} \mathcal{H}_{\mathbf{w}.M,R}$ ,  $T_w^M \mapsto T_{\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1}}^{\mathbf{w}.M}$  for  $w \in W_M(1)$  sends  $z^M(t)$  to  $z^{\mathbf{w}.M}(\tilde{\mathbf{w}}^M t(\tilde{\mathbf{w}}^M)^{-1})$  for  $t \in T$  dominant for  $U_M$ .

*Proof.* The conjugation by  $\tilde{\mathbf{w}}^M$  stabilizes  $T$ , sends  $U_M$  to  $U_{\mathbf{w}.M}$ , and sends the  $\mathbb{W}_M$ -orbit of  $t \in T$  to the  $\mathbb{W}_{\mathbf{w}.M}$ -orbit of  $\tilde{\mathbf{w}}^M t(\tilde{\mathbf{w}}^M)^{-1}$ , as  $\mathbf{w}^M \mathbb{W}_M(\mathbf{w}^M)^{-1} = \mathbb{W}_{\mathbf{w}.M}$ . It is known that  $\iota(\tilde{\mathbf{w}}^M)$  respects the anti-dominant alcove walk bases [Vig15b, Proposition 2.20]: it sends  $E^M(w)$  to  $E^{\mathbf{w}.M}(\tilde{\mathbf{w}}^M w(\tilde{\mathbf{w}}^M)^{-1})$  for  $w \in W_M(1)$ .  $\square$

We deduce the following.

**Corollary 4.29.** *Let  $\mathcal{V}$  be a right  $\mathcal{H}_{M,R}$ -module. Then  $\mathcal{V}$  is supersingular if and only if the right  $\mathcal{H}_{\mathbf{w}.M,R}$ -module  $\tilde{\mathbf{w}}^M \mathcal{V}$  is supersingular.*

Assume  $R$  is an algebraically close field of characteristic  $p$ . The supersingular simple  $\mathcal{H}_{M,R}$ -modules are classified in [Vig15a]. By Corollaries 4.24 and 4.29, the classification of the simple  $\mathcal{H}_R$ -modules in [Abe] remains valid with the  $\mathcal{H}_R$ -modules  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$  instead of  $CI_{\mathcal{H}}(P, \mathcal{V}, Q)$ :

**Corollary 4.30** (Classification of simple  $\mathcal{H}_R$ -modules). *Assume  $R$  is an algebraically closed field of characteristic  $p$ . Let  $(P, \mathcal{V}, Q)$  be an  $\mathcal{H}_R$ -triple where  $\mathcal{V}$  is simple and supersingular. Then, the  $\mathcal{H}_R$ -module  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$  is simple. A simple  $\mathcal{H}_R$ -module is isomorphic to  $I_{\mathcal{H}}(P, \mathcal{V}, Q)$  for an  $\mathcal{H}_R$ -triple  $(P, \mathcal{V}, Q)$  where  $\mathcal{V}$  is simple and supersingular,  $P, Q$  and the isomorphism class of  $\mathcal{V}$  are unique.*

**4.6. A commutative diagram.** We prove in this section Proposition 4.23. For  $Q \subset Q' \subset P(\mathcal{V})$  we show by an explicit computation that

$$\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'} \text{Ind}_{\mathcal{H}_{\mathbf{w}.Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})) \rightarrow \text{Ind}_{\mathcal{H}_{\mathbf{w}.Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}))$$

is equal to  $\iota(\mathbf{w}.Q, \mathbf{w}.Q')$ . The  $R$ -module  $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) \otimes 1_{\mathcal{H}}$  generates the  $\mathcal{H}_R$ -module  $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}) \otimes_{\mathcal{H}_{\mathbf{w}.Q'}, R, \theta^+} \mathcal{H}_R = \text{Ind}_{\mathcal{H}_{\mathbf{w}.Q'}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}))$  and by (4.17)

$$(4.24) \quad \iota(\mathbf{w}.Q, \mathbf{w}.Q')(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in {}^{\mathbb{W}_{M_{\mathbf{w}.Q}}} \mathbb{W}_{M_{\mathbf{w}.Q'}}} T_{\tilde{d}}$$

for  $v \in \mathcal{V}$  seen as an element of  $e_{\mathcal{H}_{\mathbf{w}.Q'}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$  in the LHS and an element of  $e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V})$  in the RHS.

**Lemma 4.31.**  $(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = v \otimes \sum_{d \in {}^{\mathbb{W}_{M_Q}} \mathbb{W}_{M_{Q'}}} qd T_{\tilde{\mathbf{w}}^Q(\tilde{\mathbf{w}}^{Q'} \tilde{d})^{-1}}^*$ .

*Proof.*  $\mu_{Q'}(v \otimes 1_{\mathcal{H}})$  is the unique homomorphism  $f_{\tilde{\mathbf{w}}^{M_{Q'}}, v} \in \text{Hom}_{\mathcal{H}_{M_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$  sending  $T_{\tilde{\mathbf{w}}^{Q'}}$  to  $v$  and vanishing on  $T_{\tilde{d}'}$  for  $d' \in {}^{\mathbb{W}_{M_{Q'}}} \mathbb{W}_{M_{Q'}} \setminus \{\mathbf{w}^{Q'}\}$  by (4.22). By (4.19),  $i(Q, Q')$  is the natural embedding of  $\text{Hom}_{\mathcal{H}_{M_{Q'}}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_{Q'}}(\mathcal{V}))$  in  $\text{Hom}_{\mathcal{H}_{M_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$  therefore  $i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v})$  is the unique homomorphism  $\text{Hom}_{\mathcal{H}_{M_Q}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V}))$  sending  $T_{\tilde{\mathbf{w}}^{Q'}}$  to  $v$  and vanishing on  $T_{\tilde{d}'}$  for  $d' \in {}^{\mathbb{W}_{M_{Q'}}} \mathbb{W}_{M_{Q'}} \setminus \{\mathbf{w}^{Q'}\}$ . As  $\mathbb{W}^{\mathbb{W}_{M_Q}} = \mathbb{W}^{\mathbb{W}_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ , this homomorphism vanishes on  $T_{\tilde{w}}$  for  $w$  not

in  $\mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ . By [Abe16, Lemma 2.22], the inverse of  $\mu_Q$  is the  $\mathcal{H}_R$ -isomorphism

$$(4.25) \quad \text{Hom}_{\mathcal{H}_{M_Q^-}, \theta^*}(\mathcal{H}, e_{\mathcal{H}_Q}(\mathcal{V})) \xrightarrow{\mu_Q^{-1}} \text{Ind}_{\mathcal{H}_{\mathbf{w}.M_Q}}^{\mathcal{H}}(e_{\mathcal{H}_{\mathbf{w}.Q}}(\tilde{\mathbf{w}}^M \cdot \mathcal{V}))$$

$$f \mapsto \sum_{d \in \mathbb{W}^{\mathbb{W}_M}} f(T_{\tilde{d}}) \otimes T_{\tilde{\mathbf{w}}^M \tilde{d}^{-1}}^*,$$

where  $\mathbb{W}^{\mathbb{W}_M}$  is the set of  $d \in \mathbb{W}$  with minimal length in the coset  $d\mathbb{W}_M$ . We deduce the explicit formula

$$(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}}) = \sum_{w \in \mathbb{W}^{\mathbb{W}_{M_Q}}} i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{w}}) \otimes T_{\tilde{\mathbf{w}}^{M_Q} \tilde{w}^{-1}}^*.$$

Some terms are zero: the terms for  $w \in \mathbb{W}^{\mathbb{W}_{M_Q}}$  not in  $\mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ . We analyze the other terms for  $w$  in  $\mathbb{W}^{\mathbb{W}_{M_Q}} \cap \mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ ; this set is  $\mathbf{w}^{M_{Q'}} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ . Let  $w = \mathbf{w}^{M_{Q'}} d, d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ , and  $\tilde{w} = \tilde{\mathbf{w}}^{M_{Q'}} \tilde{d}$  with  $\tilde{d} \in {}_1W_{G'}$  lifting  $d$ . By the braid relations  $T_{\tilde{w}} = T_{\tilde{\mathbf{w}}^{M_{Q'}}} T_{\tilde{d}}$ . We have  $T_{\tilde{d}} = \theta^*(T_{\tilde{d}}^{M_{Q'}})$  by the braid relations because  $d \in \mathbb{W}_{M_{Q'}}$ ,  $S_{M_{Q'}} \subset S^{\text{aff}}$  and  $\theta^*(c_{\tilde{s}}^{M_{Q'}}) = c_{\tilde{s}}$  for  $s \in S_{M_{Q'}}$ . As  $\mathbb{W}_{M_{Q'}} \subset W_{M_{Q'}}^- \cap W_{M_{Q'}}^+$ , we deduce:

$$\begin{aligned} i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{w}}) &= i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{\mathbf{w}}^{M_{Q'}}} T_{\tilde{d}}) \\ &= i(Q, Q')(f_{\tilde{\mathbf{w}}^{M_{Q'}}, v}^{Q'})(T_{\tilde{\mathbf{w}}^{M_{Q'}}}) T_{\tilde{d}}^{M_{Q'}} \\ &= v T_{\tilde{d}}^{M_{Q'}} = q_d v. \end{aligned}$$

Corollary 3.9 gives the last equality.  $\square$

The formula for  $(\mu_Q^{-1} \circ i(Q, Q') \circ \mu_{Q'})(v \otimes 1_{\mathcal{H}})$  given in Lemma 4.31 is different from the formula (4.24) for  $\iota(\mathbf{w}.Q, \mathbf{w}.Q')(v \otimes 1_{\mathcal{H}})$ . It needs some work to prove that they are equal.

A first reassuring remark is that  ${}^{\mathbb{W}_{M_{\mathbf{w}.Q}}} \mathbb{W}_{M_{\mathbf{w}.Q'}} = \{\mathbf{w} d^{-1} \mathbf{w} \mid d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$ , so the two summation sets have the same number of elements. But better,

$${}^{\mathbb{W}_{M_{\mathbf{w}.Q}}} \mathbb{W}_{M_{\mathbf{w}.Q'}} = \{\mathbf{w}^Q (\mathbf{w}^{Q'} d)^{-1} \mid d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}\}$$

because  $\mathbf{w}_{Q'} \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}} \mathbf{w}_Q = \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$ . To prove the latter equality, we apply the criterion:  $w \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$  lies in  $\mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$  if and only if  $w(\alpha) > 0$  for all  $\alpha \in \Delta_Q$  noticing that  $d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}$  implies  $\mathbf{w}_Q(\alpha) \in -\Delta_Q$ ,  $d\mathbf{w}_Q(\alpha) \in -\Phi_{M_{Q'}}$ ,  $\mathbf{w}_{Q'} d\mathbf{w}_Q(\alpha) > 0$ . Let  $x_d = \mathbf{w}^Q (\mathbf{w}^{Q'} d)^{-1}$ . We have  $\tilde{\mathbf{w}}^{M_Q} (\tilde{\mathbf{w}}^{M_{Q'}} \tilde{d})^{-1} = \tilde{x}_d$  because the lifts  $\tilde{w}$  of the elements  $w \in \mathbb{W}$  satisfy the braid relations and  $\ell(x_d) = \ell(\mathbf{w}_Q d^{-1} \mathbf{w}_{Q'}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d^{-1}) = \ell(\mathbf{w}_{Q'}) - \ell(\mathbf{w}_Q) - \ell(d) = -\ell(\mathbf{w}^{Q'}) + \ell(\mathbf{w}^Q) - \ell(d)$ . We have  $q_d = q_{\mathbf{w}_{\mathbf{w}.Q} x_d \mathbf{w}_{\mathbf{w}.Q'}}$  because  $\mathbf{w} d^{-1} \mathbf{w} = \mathbf{w}_{\mathbf{w}.Q} x_d \mathbf{w}_{\mathbf{w}.Q'}$ , and  $q_d = q_{d^{-1}} = q_{\mathbf{w} d^{-1} \mathbf{w}}$ . So

$$\sum_{d \in \mathbb{W}_{M_{Q'}}^{\mathbb{W}_{M_Q}}} q_d T_{\tilde{\mathbf{w}}^Q (\tilde{\mathbf{w}}^{Q'} \tilde{d})^{-1}}^* = \sum_{x_d \in {}^{\mathbb{W}_{M_{\mathbf{w}.Q}}} \mathbb{W}_{M_{\mathbf{w}.Q'}}} q_{\mathbf{w}_{\mathbf{w}.Q} x_d \mathbf{w}_{\mathbf{w}.Q'}} T_{\tilde{x}_d}^*.$$

In the RHS, only  $\tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w} \cdot Q, \mathbf{w} \cdot Q'$  appear. The same holds true in the formula (4.24). The map  $(P, \mathcal{V}, Q, Q') \mapsto (\mathbf{w} \cdot P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w} \cdot Q, \mathbf{w} \cdot Q')$  is a bijection of the set of triples  $(P, \mathcal{V}, Q, Q')$  where  $P = MN, Q, Q'$  are standard parabolic subgroups of  $G$ ,  $\mathcal{V}$  a right  $\mathcal{H}_R$ -module,  $Q \subset Q' \subset P(\mathcal{V})$  by Lemma 4.21. So we can replace  $(\mathbf{w} \cdot P, \tilde{\mathbf{w}}^M \cdot \mathcal{V}, \mathbf{w} \cdot Q, \mathbf{w} \cdot Q')$  by  $(P, \mathcal{V}, Q, Q')$ . Our task is reduced to prove in  $e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q^+}, \theta} \mathcal{H}_R$ :

$$(4.26) \quad v \otimes \sum_{d \in {}^{\mathbb{W}}M_Q {}^{\mathbb{W}}M_{Q'}} T_{\bar{d}} = v \otimes \sum_{d \in {}^{\mathbb{W}}M_Q {}^{\mathbb{W}}M_{Q'}} q_{\mathbf{w}_Q d \mathbf{w}_{Q'}} T_{\bar{d}}^*.$$

A second simplification is possible: we can replace  $Q \subset Q'$  by the standard parabolic subgroups  $Q_2 \subset Q'_2$  of  $G$  with  $\Delta_{Q_2} = \Delta_Q \setminus \Delta_P$  and  $\Delta_{Q'_2} = \Delta_{Q'} \setminus \Delta_P$ , because  $\Delta_P$  and  $\Delta_{P(\mathcal{V})} \setminus \Delta_P$  are orthogonal. Indeed,  $\mathbb{W}_{M_{Q'}} = \mathbb{W}_M \times \mathbb{W}_{M_{Q'_2}}$  and  $\mathbb{W}_{M_Q} = \mathbb{W}_M \times \mathbb{W}_{M_{Q_2}}$  are direct products, the longest elements  $\mathbf{w}_{Q'} = \mathbf{w}_M \mathbf{w}_{Q'_2}, \mathbf{w}_Q = \mathbf{w}_M \mathbf{w}_{Q_2}$  are direct products and

$${}^{\mathbb{W}}M_Q {}^{\mathbb{W}}M_{Q'} = {}^{\mathbb{W}}M_{Q_2} {}^{\mathbb{W}}M_{Q'_2}, \quad \mathbf{w}_Q d \mathbf{w}_{Q'} = \mathbf{w}_{Q_2} d \mathbf{w}_{Q'_2}.$$

Once this is done, we use the properties of  $e_{\mathcal{H}_Q}(\mathcal{V})$ :  $vh \otimes 1_{\mathcal{H}} = v \otimes \theta_Q(h)$  for  $h \in \mathcal{H}_{M_{Q_2}^+}$ , and  $T_w^{Q,*}$  acts trivially on  $e_{\mathcal{H}_Q}(\mathcal{V})$  for  $w \in {}_1W_{M_{Q'_2}'} \cup (\Lambda(1) \cap {}_1W_{M_{Q'_2}'})$ . Set  ${}_1\mathbb{W}_{M_{Q'_2}'} = \{w \in {}_1W_{M_{Q'_2}'} \mid w \text{ is a lift of some element in } \mathbb{W}_{M_{Q'_2}'}\}$  and  ${}_1\mathbb{W}_{M_{Q_2}'}^+$  similarly. Then  $Z_k \cap {}_1\mathbb{W}_{M_{Q'_2}'} \subset (\Lambda(1) \cap {}_1W_{M_{Q'_2}'}^+) \cap {}_1W_{M_{Q_2}'}^+$  and  ${}_1\mathbb{W}_{M_{Q_2}'}^+ \subset {}_1W_{M_{Q_2}'}^+ \cap {}_1W_{M_{Q_2}'}^+$ . This implies that (4.26) where  $Q \subset Q'$  has been replaced by  $Q_2 \subset Q'_2$  follows from a congruence

$$(4.27) \quad \sum_{d \in {}^{\mathbb{W}}M_{Q_2} {}^{\mathbb{W}}M_{Q'_2}} T_{\bar{d}} \equiv \sum_{d \in {}^{\mathbb{W}}M_{Q_2} {}^{\mathbb{W}}M_{Q'_2}} q_{\mathbf{w}_{Q_2} d \mathbf{w}_{Q'_2}} T_{\bar{d}}^*$$

in the finite subring  $H({}_1\mathbb{W}_{M_{Q'_2}'})$  of  $\mathcal{H}$  generated by  $\{T_w \mid w \in {}_1\mathbb{W}_{M_{Q'_2}'}\}$  modulo the right ideal  $\mathcal{J}_2$  with generators  $\{\theta_Q(T_w^{Q,*}) - 1 \mid w \in (Z_k \cap {}_1\mathbb{W}_{M_{Q'_2}'}) \cup {}_1\mathbb{W}_{M_{Q'_2}'}\}$ .

Another simplification concerns  $T_{\bar{d}}^*$  modulo  $\mathcal{J}_2$  for  $d \in \mathbb{W}_{M_{Q'_2}'}$ . We recall that for any reduced decomposition  $d = s_1 \dots s_n$  with  $s_i \in S \cap \mathbb{W}_{M_{Q'_2}'}$  we have  $T_{\bar{d}}^* = (T_{\tilde{s}_1} - c_{\tilde{s}_1}) \dots (T_{\tilde{s}_n} - c_{\tilde{s}_n})$  where the  $\tilde{s}_i$  are admissible. For  $\tilde{s}$  admissible, by (3.2)

$$c_{\tilde{s}} \equiv q_s - 1.$$

Therefore

$$T_{\bar{d}}^* \equiv (T_{\tilde{s}_1} - q_{s_1} + 1) \dots (T_{\tilde{s}_n} - q_{s_n} + 1).$$

Let  $\mathcal{J}' \subset \mathcal{J}_2$  be the ideal of  $H({}_1\mathbb{W}_{M_{Q'_2}'})$  generated by  $\{T_t - 1 \mid t \in Z_k \cap {}_1W_{M_{Q'_2}'}\}$ . Then the ring  $H({}_1\mathbb{W}_{M_{Q'_2}'})/\mathcal{J}'$  and its right ideal  $\mathcal{J}_2/\mathcal{J}'$  are the specialization of the generic finite ring  $H(\mathbb{W}_{M_{Q'_2}'})^g$  over  $\mathbb{Z}[(q_s)_{s \in S_{M_{Q'_2}'}}]$  where the  $q_s$  for  $s \in S_{M_{Q'_2}'} = S \cap \mathbb{W}_{M_{Q'_2}'}$  are indeterminates, and of its right ideal  $\mathcal{J}_2^g$  with the same generators. The similar congruence modulo  $\mathcal{J}_2^g$  in  $H(\mathbb{W}_{M_{Q'_2}'})^g$  (the generic congruence) implies the congruence (4.27) by specialization.

We will prove the generic congruence in a more general setting where  $H$  is the generic Hecke ring of a finite Coxeter system  $(\mathbb{W}, S)$  and parameters  $(q_s)_{s \in S}$  such that  $q_s = q_{s'}$  when  $s, s'$  are conjugate in  $\mathbb{W}$ . The Hecke ring  $H$  is a  $\mathbb{Z}[(q_s)_{s \in S}]$ -free

module of basis  $(T_w)_{w \in \mathbb{W}}$  satisfying the braid relations and the quadratic relations  $T_s^2 = q_s + (q_s - 1)T_s$  for  $s \in S$ . The other basis  $(T_w^*)_{w \in \mathbb{W}}$  satisfies the braid relations and the quadratic relations  $(T_s^*)^2 = q_s - (q_s - 1)T_s^*$  for  $s \in S$ , and is related to the first basis by  $T_s^* = T_s - (q_s - 1)$  for  $s \in S$ , and more generally  $T_w T_{w^{-1}}^* = T_{w^{-1}}^* T_w = q_w$  for  $w \in \mathbb{W}$  [Vig16, Proposition 4.13].

Let  $J \subset S$  and  $\mathcal{J}$  is the right ideal of  $H$  with generators  $T_w^* - 1$  for all  $w$  in the group  $\mathbb{W}_J$  generated by  $J$ .

**Lemma 4.32.** *A basis of  $\mathcal{J}$  is  $(T_{w_1}^* - 1)T_{w_2}^*$  for  $w_1 \in \mathbb{W}_J \setminus \{1\}, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$ , and adding  $T_{w_2}^*$  for  $w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$  gives a basis of  $H$ . In particular,  $\mathcal{J}$  is a direct factor of  $\mathcal{H}$ .*

*Proof.* The elements  $(T_{w_1}^* - 1)T_w^*$  for  $w_1 \in \mathbb{W}_J, w \in \mathbb{W}$  generate  $\mathcal{J}$ . We write  $w = u_1 w_2$  with unique elements  $u_1 \in \mathbb{W}_J, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$ , and  $T_w^* = T_{u_1}^* T_{w_2}^*$ . Therefore,  $(T_{w_1}^* - 1)T_{u_1}^* T_{w_2}^*$ . By an induction on the length of  $u_1$ , one proves that  $(T_{w_1}^* - 1)T_{u_1}^*$  is a linear combination of  $(T_{v_1}^* - 1)$  for  $v_1 \in \mathbb{W}_J$  as in the proof of Proposition 3.4. It is clear that the elements  $(T_{w_1}^* - 1)T_{w_2}^*$  and  $T_{w_2}^*$  for  $w_1 \in \mathbb{W}_J \setminus \{1\}, w_2 \in {}^{\mathbb{W}_J}\mathbb{W}$  form a basis of  $H$ .  $\square$

Let  $\mathbf{w}_J$  denote the longest element of  $\mathbb{W}_J$  and  $\mathbf{w} = \mathbf{w}_S$ .

**Lemma 4.33.** *In the generic Hecke ring  $H$ , the congruence modulo  $\mathcal{J}$*

$$\sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J d \mathbf{w}} T_d^*$$

*holds true.*

*Proof.*

*Step 1.* We show

$${}^{\mathbb{W}_J}\mathbb{W} = \mathbf{w}_J {}^{\mathbb{W}_J}\mathbb{W} \mathbf{w}, \quad q_{\mathbf{w}_J} q_{\mathbf{w}_J d \mathbf{w}} T_d^* = T_{\mathbf{w}_J} T_{\mathbf{w}_J d \mathbf{w}} T_{\mathbf{w}}^*.$$

The equality between the groups follows from the characterization of  ${}^{\mathbb{W}_J}\mathbb{W}$  in  $\mathbb{W}$ : an element  $d \in \mathbb{W}$  has minimal length in  $\mathbb{W}_J d$  if and only if  $\ell(ud) = \ell(u) + \ell(d)$  for all  $u \in \mathbb{W}_J$ . An easy computation shows that  $\ell(u \mathbf{w}_J d \mathbf{w}) = \ell(u) + \ell(\mathbf{w}_J d \mathbf{w})$  for all  $u \in \mathbb{W}_J, d \in {}^{\mathbb{W}_J}\mathbb{W}$  (both sides are equal to  $\ell(u) + \ell(\mathbf{w}) - \ell(\mathbf{w}_J) - \ell(d)$ ). The second equality follows from  $q_{\mathbf{w}_J} q_{\mathbf{w}_J d \mathbf{w}} = q_{d \mathbf{w}}$  because  $(\mathbf{w}_J)^2 = 1$  and  $\ell(\mathbf{w}_J) + \ell(\mathbf{w}_J d \mathbf{w}) = \ell(d \mathbf{w})$  (both sides are  $\ell(\mathbf{w}) - \ell(d)$ ) and from  $q_{d \mathbf{w}} T_d^* = T_{d \mathbf{w}} T_{\mathbf{w}_J d^{-1}}^* T_d^* = T_{d \mathbf{w}} T_{\mathbf{w}}^*$ . We also have  $T_{d \mathbf{w}} = T_{\mathbf{w}_J} T_{\mathbf{w}_J d \mathbf{w}}$ .

*Step 2.* The multiplication by  $q_{\mathbf{w}_J}$  on the quotient  $H/\mathcal{J}$  is injective (Lemma 4.32) and  $q_{\mathbf{w}_J} \equiv T_{\mathbf{w}_J}$ . By Step 1,  $q_{\mathbf{w}_J d \mathbf{w}} T_d^* \equiv T_{\mathbf{w}_J d \mathbf{w}} T_{\mathbf{w}}^*$  and

$$\sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} q_{\mathbf{w}_J d \mathbf{w}} T_d^* \equiv \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d T_{\mathbf{w}}^*.$$

The congruence

$$(4.28) \quad \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d \equiv \sum_{d \in {}^{\mathbb{W}_J}\mathbb{W}} T_d T_s^*$$

for all  $s \in S$  implies the lemma because  $T_{\mathbf{w}}^* = T_{s_1}^* \dots T_{s_n}^*$  for any reduced decomposition  $\mathbf{w} = s_1 \dots s_n$  with  $s_i \in S$ .



*Step 3.* When  $J = \emptyset$ , the congruence (4.28) is an equality

$$(4.29) \quad \sum_{w \in \mathbb{W}} T_w = \sum_{w \in \mathbb{W}} T_w T_s^*.$$

It holds true because  $\sum_{w \in \mathbb{W}} T_w = \sum_{w < ws} T_w (T_s + 1)$  and  $(T_s + 1)T_s^* = T_s T_s^* + T_s^* = q_s + T_s^* = T_s + 1$ .

*Step 4.* Conversely the congruence (4.28) follows from (4.29) because

$$\sum_{w \in \mathbb{W}} T_w = \left( \sum_{u \in W_J} T_u \right) \sum_{d \in {}^{\mathbb{W}J}\mathbb{W}} T_d \equiv \left( \sum_{u \in W_J} q_u \right) \sum_{d \in {}^{\mathbb{W}J}\mathbb{W}} T_d$$

(recall  $q_u = T_{u^{-1}}^* T_u \equiv T_u$ ) and we can simplify by  $\sum_{u \in W_J} q_u$  in  $H/\mathcal{J}$ .  $\square$

This ends the proof of Proposition 4.23.

## 5. UNIVERSAL REPRESENTATION $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}} R[\mathcal{U} \backslash G]$

The invariant functor  $(-)^{\mathcal{U}}$  by the pro- $p$ -Iwahori subgroup  $\mathcal{U}$  of  $G$  has a left adjoint

$$- \bigotimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] : \text{Mod}_R(\mathcal{H}) \rightarrow \text{Mod}_R^{\infty}(G).$$

The smooth  $R$ -representation  $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G]$  of  $G$  constructed from the right  $\mathcal{H}_R$ -module  $\mathcal{V}$  is called universal. We write

$$R[\mathcal{U} \backslash G] = \mathbb{X}.$$

**Question 5.1.** Does  $\mathcal{V} \neq 0$  imply  $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$  or does  $v \otimes 1_{\mathcal{U}} = 0$  for  $v \in \mathcal{V}$  imply  $v = 0$ ? We have no counterexample. If  $R$  is a field and the  $\mathcal{H}_R$ -module  $\mathcal{V}$  is simple, the two questions are equivalent:  $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$  if and only if the map  $v \mapsto v \otimes 1_{\mathcal{U}}$  is injective. When  $R$  is an algebraically closed field of characteristic  $p$ ,  $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X} \neq 0$  for all simple  $\mathcal{H}_R$ -modules  $\mathcal{V}$  if this is true for  $\mathcal{V}$  simple supersingular (this is a consequence of Corollary 5.13).

The functor  $- \bigotimes_{\mathcal{H}_R} \mathbb{X}$  satisfies a few good properties: it has a right adjoint and is compatible with the parabolic induction and the left adjoint (of the parabolic induction). Let  $P = MN$  be a standard parabolic subgroup and  $\mathbb{X}_M = R[\mathcal{U}_M \backslash M]$ . We have functor isomorphisms

$$(5.1) \quad (- \bigotimes_{\mathcal{H}_R} \mathbb{X}) \circ \text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \rightarrow \text{Ind}_P^G \circ (- \bigotimes_{\mathcal{H}_{M,R}} \mathbb{X}_M),$$

$$(5.2) \quad (-)_N \circ (- \bigotimes_{\mathcal{H}_R} \mathbb{X}) \rightarrow (- \bigotimes_{\mathcal{H}_{M,R}} \mathbb{X}_M) \circ \mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}.$$

The first one is [OV17, formula 4.15], the second one is obtained by left adjunction from the isomorphism  $\text{Ind}_{\mathcal{H}_M}^{\mathcal{H}} \circ (-)^{\mathcal{U}_M} \rightarrow (-)^{\mathcal{U}} \circ \text{Ind}_P^G$  [OV17, formula (4.14)]. If  $\mathcal{V}$  is a right  $\mathcal{H}_R$ -supersingular module and  $p$  is nilpotent in  $\mathcal{V}$ , then  $\mathcal{L}_{\mathcal{H}_M}^{\mathcal{H}}(\mathcal{V}) = 0$  if  $M \neq G$  (Proposition 4.27). Applying (5.2) we deduce the following.

**Proposition 5.2.** *If  $p$  is nilpotent in  $\mathcal{V}$  and  $\mathcal{V}$  supersingular, then  $\mathcal{V} \otimes_{\mathcal{H}_R} \mathbb{X}$  is left cuspidal.*

*Remark 5.3.* For a non-zero smooth  $R$ -representation  $\tau$  of  $M$ ,  $\Delta_\tau$  is orthogonal to  $\Delta_P$  if  $\tau$  is left cuspidal. Indeed, we recall from [AHHV17, II.7 Corollary 2] that  $\Delta_\tau$  is not orthogonal to  $\Delta_P$  if and only if there exists a proper standard parabolic subgroup  $X$  of  $M$  such that  $\sigma$  is trivial on the unipotent radical of  $X$ ; moreover  $\tau$  is a subrepresentation of  $\text{Ind}_X^M(\tau|_X)$ , so the image of  $\tau$  by the left adjoint of  $\text{Ind}_X^M$  is not 0.

From now on,  $\mathcal{V}$  is a non-zero right  $\mathcal{H}_{M,R}$ -module and

$$\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M.$$

In general, when  $\sigma \neq 0$ , let  $P_\perp(\sigma)$  be the standard parabolic subgroup of  $G$  with  $\Delta_{P_\perp(\sigma)} = \Delta_P \cup \Delta_{\perp,\sigma}$  where  $\Delta_{\perp,\sigma}$  is the set of simple roots  $\alpha \in \Delta_\sigma$  orthogonal to  $\Delta_P$ .

**Proposition 5.4.**

- (1)  $P(\mathcal{V}) \subset P_\perp(\sigma)$  if  $\sigma \neq 0$ .
- (2)  $P(\mathcal{V}) = P_\perp(\sigma)$  if the map  $v \mapsto v \otimes 1_{\mathcal{U}_M}$  is injective.
- (3)  $P(\mathcal{V}) = P(\sigma)$  if the map  $v \mapsto v \otimes 1_{\mathcal{U}_M}$  is injective,  $p$  nilpotent in  $\mathcal{V}$  and  $\mathcal{V}$  supersingular.
- (4)  $P(\mathcal{V}) = P(\sigma)$  if  $\sigma \neq 0$ ,  $R$  is a field of characteristic  $p$  and  $\mathcal{V}$  simple supersingular.

*Proof.*

(1)  $P(\mathcal{V}) \subset P_\perp(\sigma)$  means that  $Z \cap M'_\mathcal{V}$  acts trivially on  $\mathcal{V} \otimes 1_{\mathcal{U}_M}$ , where  $M_\mathcal{V}$  is the standard Levi subgroup such that  $\Delta_{M_\mathcal{V}} = \Delta_\mathcal{V}$ . Let  $z \in Z \cap M'_\mathcal{V}$  and  $v \in \mathcal{V}$ . As  $\Delta_M$  and  $\Delta_\mathcal{V}$  are orthogonal, we have  $T^{M,*}(z) = T^M(z)$  and  $\mathcal{U}_M z \mathcal{U}_M = \mathcal{U}_M z$ . We have  $v \otimes 1_{\mathcal{U}_M} = v T^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes T^M(z) 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M} z = v \otimes z^{-1} 1_{\mathcal{U}_M} = z^{-1}(v \otimes 1_{\mathcal{U}_M})$ .

(2) If  $v \otimes 1_{\mathcal{U}_M} = 0$  for  $v \in \mathcal{V}$  implies  $v = 0$ , then  $\sigma \neq 0$  because  $\mathcal{V} \neq 0$ . By (1)  $P(\mathcal{V}) \subset P_\perp(\sigma)$ . As in the proof of (1), for  $z \in Z \cap M'_{\perp,\sigma}$  we have  $v T^{M,*}(z) \otimes 1_{\mathcal{U}_M} = v T^M(z) \otimes 1_{\mathcal{U}_M} = v \otimes 1_{\mathcal{U}_M}$  and our hypothesis implies  $v T^{M,*}(z) = v$  hence  $P(\mathcal{V}) \supset P_\perp(\sigma)$ .

(3) Proposition 5.2, Remark 5.3, and (2).

(4) Question 5.1 and (3). □

Let  $Q$  be a parabolic subgroup of  $G$  with  $P \subset Q \subset P(\mathcal{V})$ . In this chapter we will compute  $I_\mathcal{H}(P, \mathcal{V}, Q) \otimes_\mathcal{H} R[\mathcal{U} \backslash G]$  where  $I_\mathcal{H}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^\mathcal{H}(e(\mathcal{V}) \otimes (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$  (Theorem 5.11). The smooth  $R$ -representation  $I_G(P, \sigma, Q)$  of  $G$  is well defined: it is 0 if  $\sigma = 0$  and  $\text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes \text{St}_Q^{P(\sigma)})$  if  $\sigma \neq 0$  because  $(P, \sigma, Q)$  is an  $R[G]$ -triple by Proposition 5.4. We will show that the universal representation  $I_\mathcal{H}(P, \mathcal{V}, Q) \otimes_\mathcal{H} R[\mathcal{U} \backslash G]$  is isomorphic to  $I_G(P, \sigma, Q)$ , if  $P(\mathcal{V}) = P(\sigma)$  and  $p = 0$ , or if  $\sigma = 0$  (Corollary 5.12). In particular,  $I_\mathcal{H}(P, \mathcal{V}, Q) \otimes_\mathcal{H} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$  when  $R$  is an algebraically closed field of characteristic  $p$  and  $\mathcal{V}$  is supersingular.

**5.1.  $Q = G$ .** We consider first the case  $Q = G$ . We are in the simple situation where  $\mathcal{V}$  is extensible to  $\mathcal{H}$  and  $P(\mathcal{V}) = P(\sigma) = G$ ,  $I_\mathcal{H}(P, \mathcal{V}, G) = e(\mathcal{V})$  and  $I_G(P, \sigma, G) = e(\sigma)$ . We recall that  $\Delta \setminus \Delta_P$  is orthogonal to  $\Delta_P$  and that  $M_2$  denotes the standard Levi subgroup of  $G$  with  $\Delta_{M_2} = \Delta \setminus \Delta_P$ .

The  $\mathcal{H}_R$ -morphism  $e(\mathcal{V}) \rightarrow e(\sigma)^{\mathcal{U}} = \sigma^{\mathcal{U}_M}$  sending  $v$  to  $v \otimes 1_{\mathcal{U}_M}$  for  $v \in \mathcal{V}$ , gives by adjunction an  $R[G]$ -homomorphism

$$v \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} : e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi^G} e(\sigma).$$

If  $\Phi^G$  is an isomorphism, then  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$  is the extension to  $G$  of  $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$ , meaning that  $M'_2$  acts trivially on  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ . The converse is true.

**Lemma 5.5.** *If  $M'_2$  acts trivially on  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ , then  $\Phi^G$  is an isomorphism.*

*Proof.* Suppose that  $M'_2$  acts trivially on  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ . Then  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$  is the extension to  $G$  of  $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})|_M$ , and by Theorem 3.13,  $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}}$  is the extension of  $(e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$ . Therefore, by (3.12),

$$(v \otimes 1_{\mathcal{U}})T_w^* = (v \otimes 1_{\mathcal{U}})T_w^{M,*} \quad \text{for all } v \in \mathcal{V}, w \in W_M(1).$$

As  $\mathcal{V}$  is extensible to  $\mathcal{H}$ , the natural map  $v \mapsto v \otimes 1_{\mathcal{U}} : \mathcal{V} \xrightarrow{\Psi} (e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X})^{\mathcal{U}_M}$  is  $\mathcal{H}_M$ -equivariant, i.e.,

$$vT_w^{M,*} \otimes 1_{\mathcal{U}} = (v \otimes 1_{\mathcal{U}})T_w^{M,*} \quad \text{for all } v \in \mathcal{V}, w \in W_M(1)$$

because (3.12)  $vT_w^{M,*} \otimes 1_{\mathcal{U}} = vT_w^* \otimes 1_{\mathcal{U}} = v \otimes T_w^* = (v \otimes 1_{\mathcal{U}})T_w^*$  in  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ .

We recall that  $-\otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$  is the left adjoint of  $(-)^{\mathcal{U}_M}$ . The adjoint  $R[M]$ -homomorphism  $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M \rightarrow e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$  sends  $v \otimes 1_{\mathcal{U}_M}$  to  $v \otimes 1_{\mathcal{U}}$  for all  $v \in \mathcal{V}$ . The  $R[M]$ -module generated by the  $v \otimes 1_{\mathcal{U}}$  for all  $v \in \mathcal{V}$  is equal to  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$  because  $M'_2$  acts trivially. Hence we obtained an inverse of  $\Phi^G$ .  $\square$

Our next move is to determine if  $M'_2$  acts trivially on  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$ . It is equivalent to see if  $M'_2$  acts trivially on  $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$  as this set generates the representation  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$  of  $G$  and  $M'_2$  is a normal subgroup of  $G$  as  $M'_2$  and  $M$  commute and  $G = ZM'M'_2$ . Obviously,  $\mathcal{U} \cap M'_2$  acts trivially on  $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$ . The group of double classes  $(\mathcal{U} \cap M'_2) \backslash M'_2 / (\mathcal{U} \cap M'_2)$  is generated by the lifts  $\hat{s} \in \mathcal{N} \cap M'_2$  of the simple affine roots  $s$  of  $W_{M'_2}$ . Therefore,  $M'_2$  acts trivially on  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$  if and only if for any simple affine root  $s \in S_{M'_2}^{\text{aff}}$  of  $W_{M'_2}$ , any  $\hat{s} \in \mathcal{N} \cap M'_2$  lifting  $s$  acts trivially on  $e(\mathcal{V}) \otimes 1_{\mathcal{U}}$ .

**Lemma 5.6.** *Let  $v \in \mathcal{V}$ ,  $s \in S_{M'_2}^{\text{aff}}$  and  $\hat{s} \in \mathcal{N} \cap M'_2$  lifting  $s$ . We have*

$$(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0.$$

*Proof.* We compute:

$$T_s(\hat{s}1_{\mathcal{U}}) = \hat{s}(T_s1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-1}} = \sum_u \hat{s}u(\hat{s})^{-1}1_{\mathcal{U}} = \sum_{u^{op}} u^{op}1_{\mathcal{U}},$$

$$T_s(\hat{s}^21_{\mathcal{U}}) = \hat{s}^2(T_s1_{\mathcal{U}}) = 1_{\mathcal{U}\hat{s}\mathcal{U}(\hat{s})^{-2}} = 1_{\mathcal{U}(\hat{s})^{-1}\mathcal{U}} = \sum_u u\hat{s}1_{\mathcal{U}}$$

for  $u$  in the group  $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U})$  and  $u^{op}$  in the group  $\hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U})$ ; the reason is that  $\hat{s}^2$  normalizes  $\mathcal{U}$ ,  $\mathcal{U}\hat{s}\mathcal{U}\hat{s}^{-1}$  is the disjoint union of the sets  $\mathcal{U}\hat{s}u^{-1}(\hat{s})^{-1}$  and  $\mathcal{U}(\hat{s})^{-1}\mathcal{U}$  is the disjoint union of the sets  $\mathcal{U}(\hat{s})^{-1}u^{-1}$ . We introduce now a natural bijection

$$(5.3) \quad u \rightarrow u^{op} : \mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U}) \rightarrow \hat{s}\mathcal{U}(\hat{s})^{-1}/(\hat{s}\mathcal{U}(\hat{s})^{-1} \cap \mathcal{U})$$

which is not a group homomorphism. We recall the finite reductive group  $G_{k,s}$  quotient of the parahoric subgroup  $\mathfrak{K}_s$  of  $G$  fixing the face fixed by  $s$  of the alcove  $\mathcal{C}$ . The Iwahori groups  $Z^0\mathcal{U}$  and  $Z^0\hat{s}\mathcal{U}(\hat{s})^{-1}$  are contained in  $\mathfrak{K}_s$  and their images

in  $G_{s,k}$  are opposite Borel subgroups  $Z_k U_{s,k}$  and  $Z_k U_{s,k}^{op}$ . Via the surjective maps  $u \mapsto \bar{u} : \mathcal{U} \rightarrow U_{s,k}$  and  $u^{op} \mapsto \bar{u}^{op} : \hat{\mathcal{U}}(\hat{s})^{-1} \rightarrow U_{s,k}^{op}$  we identify the groups  $\mathcal{U}/(\hat{s}^{-1}\mathcal{U}\hat{s} \cap \mathcal{U}) \simeq U_{s,k}$  and similarly  $\hat{\mathcal{U}}(\hat{s})^{-1}/(\hat{\mathcal{U}}(\hat{s})^{-1} \cap \mathcal{U}) \simeq U_{s,k}^{op}$ . Let  $G'_{k,s}$  be the group generated by  $U_{s,k}$  and  $U_{s,k}^{op}$ , and let  $B'_{s,k} = G'_{k,s} \cap Z_k U_{s,k} = (G'_{k,s} \cap Z_k) U_{s,k}$ . We suppose (as we can) that  $\hat{s} \in \mathfrak{K}_s$  and that its image  $\hat{s}_k$  in  $G_{s,k}$  lies in  $G'_{k,s}$ . We have  $\hat{s}_k U_{s,k}(\hat{s}_k)^{-1} = U_{s,k}^{op}$  and the Bruhat decomposition  $G'_{k,s} = B'_{k,s} \sqcup U_{k,s} \hat{s}_k B'_{k,s}$  implies the existence of a canonical bijection  $\bar{u}^{op} \rightarrow \bar{u} : (U_{k,s}^{op} - \{1\}) \rightarrow (U_{k,s} - \{1\})$  respecting the cosets  $\bar{u}^{op} B'_{k,s} = \bar{u} \hat{s}_k B'_{k,s}$ . Via the preceding identifications we get the wanted bijection (5.3).

For  $v \in e(\mathcal{V})$  and  $z \in Z^0 \cap M'_2$  we have  $vT_z = v$ ,  $z1_{\mathcal{U}} = T_z 1_{\mathcal{U}}$  and  $v \otimes T_z 1_{\mathcal{U}} = vT_z \otimes 1_{\mathcal{U}}$  therefore  $Z^0 \cap M'_2$  acts trivially on  $\mathcal{V} \otimes 1_{\mathcal{U}}$ . The action of the group  $(Z^0 \cap M'_2)\mathcal{U}$  on  $\mathcal{V} \otimes 1_{\mathcal{U}}$  is also trivial. As the image of  $Z^0 \cap M'_2$  in  $G_{s,k}$  contains  $Z_k \cap G'_{s,k}$ ,

$$u\hat{s}(v \otimes 1_{\mathcal{U}}) = u^{op}(v \otimes 1_{\mathcal{U}})$$

when  $u$  and  $u^{op}$  are not units and correspond via the bijection (5.3). So we have

$$(5.4) \quad v \otimes T_s(\hat{s}1_{\mathcal{U}}) - (v \otimes 1_{\mathcal{U}}) = v \otimes T_s(\hat{s}^2 1_{\mathcal{U}}) - v \otimes \hat{s}1_{\mathcal{U}}.$$

We can move  $T_s$  on the other side of  $\otimes$  and as  $vT_s = q_s v$  (Corollary 3.9), we can replace  $T_s$  by  $q_s$ . We have  $v \otimes \hat{s}^2 1_{\mathcal{U}} = v \otimes T_{s^{-2}} 1_{\mathcal{U}}$  because  $\hat{s}^2 \in Z^0 \cap M'_2$  normalizes  $\mathcal{U}$ ; as we can move  $T_{s^{-2}}$  on the other side of  $\otimes$  and as  $vT_{s^{-2}} = v$  we can forget  $\hat{s}^2$ . So (5.4) is equivalent to  $(q_s + 1)(v \otimes 1_{\mathcal{U}} - \hat{s}(v \otimes 1_{\mathcal{U}})) = 0$ .  $\square$

Combining the two lemmas we obtain the following.

**Proposition 5.7.** *When  $\mathcal{V}$  is extensible to  $\mathcal{H}$  and has no  $q_s + 1$ -torsion for any  $s \in S_{M'_2}^{\text{aff}}$ , then  $M'_2$  acts trivially on  $e(\mathcal{V}) \otimes_{\mathcal{H}_R} \mathbb{X}$  and  $\Phi^G$  is an  $R[G]$ -isomorphism.*

Proposition 5.7 for the trivial character  $\mathbf{1}_{\mathcal{H}}$ , says that  $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$  is the trivial representation  $\mathbf{1}_G$  of  $G$  when  $q_s + 1$  has no torsion in  $R$  for all  $s \in S^{\text{aff}}$ . This is proved in [OV17, Lemma 2.28] by a different method. The following counter-example shows that this is not true for all  $R$ .

**Example 5.8.** Let  $G = GL(2, F)$  and let  $R$  be an algebraically closed field where  $q_{s_0} + 1 = q_{s_1} + 1 = 0$  and  $S_{\text{aff}} = \{s_0, s_1\}$ . (Note that  $q_{s_0} = q_{s_1}$  is the order of the residue field of  $F$ .) Then the dimension of  $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X}$  is infinite, in particular  $\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \neq \mathbf{1}_G$ .

Indeed, the Steinberg representation  $\text{St}_G = (\text{Ind}_B^G \mathbf{1}_Z)/\mathbf{1}_G$  of  $G$  is an indecomposable representation of length 2 containing an irreducible infinite dimensional representation  $\pi$  with  $\pi^{\mathcal{U}} = 0$  of quotient the character  $(-1)^{\text{val} \circ \det}$ . This follows from the proof of Theorem 3 and from Proposition 24 in [Vig89]. The kernel of the quotient map  $\text{St}_G \otimes (-1)^{\text{val} \circ \det} \rightarrow \mathbf{1}_G$  is infinite dimensional without a non-zero  $\mathcal{U}$ -invariant vector. As the characteristic of  $R$  is not  $p$ , the functor of  $\mathcal{U}$ -invariants is exact hence  $(\text{St}_G \otimes (-1)^{\text{val} \circ \det})^{\mathcal{U}} = \mathbf{1}_{\mathcal{H}}$ . As  $-\otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G]$  is the left adjoint of  $(-)^{\mathcal{U}}$  there is a non-zero homomorphism

$$\mathbf{1}_{\mathcal{H}} \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow \text{St}_G \otimes (-1)^{\text{val} \circ \det}$$

with image generated by its  $\mathcal{U}$ -invariants. The homomorphism is therefore surjective.

**5.2.  $\mathcal{V}$  extensible to  $\mathcal{H}$ .** Let  $P = MN$  be a standard parabolic subgroup of  $G$  with  $\Delta_P$  and  $\Delta \setminus \Delta_P$  orthogonal. We still suppose that the  $\mathcal{H}_{M,R}$ -module  $\mathcal{V}$  is extensible to  $\mathcal{H}$ , but now  $P \subset Q \subset G$ . So we have  $I_{\mathcal{H}}(P, \mathcal{V}, Q) = e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$  and  $I_G(P, \sigma, Q) = e(\sigma) \otimes_R \text{St}_Q^G$  where  $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$ . We compare the images by  $-\otimes_{\mathcal{H}_R} \mathbb{X}$  of the  $\mathcal{H}_R$ -modules  $e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  and  $e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}$  with the smooth  $R$ -representations  $e(\sigma) \otimes \text{Ind}_Q^G \mathbf{1}$  and  $e(\sigma) \otimes \text{St}_Q^G$  of  $G$ .

As  $-\otimes_{\mathcal{H}_R} \mathbb{X}$  is left adjoint of  $(-)^{\mathcal{U}}$ , the  $\mathcal{H}_R$ -homomorphism  $v \otimes f \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f : e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}} \rightarrow (e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}$  gives by adjunction an  $R[G]$ -homomorphism

$$v \otimes f \otimes 1_{\mathcal{U}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f : (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi_Q^G} e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}.$$

When  $Q = G$  we have  $\Phi_G^G = \Phi^G$ . By Remark 4.10,  $\Phi_Q^G$  is surjective. Proposition 5.7 applies with  $M_Q$  instead of  $G$  and gives the  $R[M_Q]$ -homomorphism

$$v \otimes 1_{\mathcal{U}_{M_Q}} \mapsto v \otimes 1_{\mathcal{U}_M} : e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q} \xrightarrow{\Phi_Q^Q} e_Q(\sigma).$$

**Proposition 5.9.** *The  $R[G]$ -homomorphism  $\Phi_Q^G$  is an isomorphism if  $\Phi^Q$  is an isomorphism, in particular if  $\mathcal{V}$  has no  $q_s + 1$ -torsion for any  $s \in S_{M'_2 \cap M_Q}^{\text{aff}}$ .*

*Proof.* The proposition follows from another construction of  $\Phi_Q^G$  that we now describe. Proposition 4.5 gives the  $\mathcal{H}_R$ -module isomorphism

$$v \otimes f_{QU} \mapsto v \otimes 1_{\mathcal{H}} : (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \rightarrow \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) = e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{M_Q,R}^+, \theta} \mathcal{H}.$$

We have the  $R[G]$ -isomorphism [OV17, Corollary 4.7]

$$v \otimes 1_{\mathcal{H}} \otimes 1_{\mathcal{U}} \mapsto f_{QU, v \otimes 1_{\mathcal{U}_{M_Q}}} : \text{Ind}_{\mathcal{H}_Q}^{\mathcal{H}}(e_{\mathcal{H}_Q}(\mathcal{V})) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow \text{Ind}_Q^G(e_{\mathcal{H}_Q}(\mathcal{V}) \otimes_{\mathcal{H}_{Q,R}} \mathbb{X}_{M_Q})$$

and the  $R[G]$ -isomorphism

$$f_{QU, v \otimes 1_{\mathcal{U}_M}} \mapsto v \otimes 1_{\mathcal{U}_M} \otimes f_{QU} : \text{Ind}_Q^G(e_Q(\sigma)) \rightarrow e(\sigma) \otimes \text{Ind}_Q^G \mathbf{1}.$$

From  $\Phi^Q$  and these three homomorphisms, there exists a unique  $R[G]$ -homomorphism

$$(e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}$$

sending  $v \otimes f_{QU} \otimes 1_{\mathcal{U}}$  to  $v \otimes 1_{\mathcal{U}_M} \otimes f_{QU}$ . We deduce: this homomorphism is equal to  $\Phi_Q^G$ ,  $\mathcal{V} \otimes 1_{QU} \otimes 1_{\mathcal{U}}$  generates  $(e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X}$ , if  $\Phi^Q$  is an isomorphism, then  $\Phi_Q^G$  is an isomorphism. By Proposition 5.7, if  $\mathcal{V}$  has no  $q_s + 1$ -torsion for any  $s \in S_{M'_2 \cap M_Q}^{\text{aff}}$ , then  $\Phi^Q$  and  $\Phi_Q^G$  are isomorphisms.  $\square$

We recall that the  $\mathcal{H}_{M,R}$ -module  $\mathcal{V}$  is extensible to  $\mathcal{H}$ .

**Proposition 5.10.** *The  $R[G]$ -homomorphism  $\Phi_Q^G$  induces an  $R[G]$ -homomorphism*

$$(e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^{\mathcal{U}}) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow e(\sigma) \otimes_R \text{St}_Q^G,$$

*It is an isomorphism if  $\Phi_Q^G$  is an  $R[G]$ -isomorphism for all parabolic subgroups  $Q'$  of  $G$  containing  $Q$ , in particular if  $\mathcal{V}$  has no  $q_s + 1$ -torsion for any  $s \in S_{M'_2}^{\text{aff}}$ .*

*Proof.* The proof is straightforward, with the arguments already developed for Proposition 4.5 and Theorem 4.9. The representations  $e(\sigma) \otimes_R \text{St}_Q^G$  and  $(e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}$  of  $G$  are the cokernels of the natural  $R[G]$ -homomorphisms

$$\begin{aligned} \oplus_{Q \subsetneq Q'} e(\sigma) \otimes_R \text{Ind}_{Q'}^G \mathbf{1} &\xrightarrow{\text{id} \otimes \alpha} e(\sigma) \otimes_R \text{Ind}_Q^G \mathbf{1}, \\ \oplus_{Q \subsetneq Q'} (e(\mathcal{V}) \otimes_R (\text{Ind}_{Q'}^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X} &\xrightarrow{\text{id} \otimes \alpha^\mathcal{U} \otimes \text{id}} (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}. \end{aligned}$$

These  $R[G]$ -homomorphisms make a commutative diagram with the  $R[G]$ -homomorphisms  $\oplus_{Q \subsetneq Q'} \Phi_{Q'}^G$  and  $\Phi_Q^G$  going from the lower line to the upper line. Indeed, let  $v \otimes f_{Q'\mathcal{U}} \otimes 1_{\mathcal{U}} \in (e(\mathcal{V}) \otimes_R (\text{Ind}_{Q'}^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}$ . On the one hand, it goes to  $v \otimes f_{Q\mathcal{U}} \theta_{Q'}(e_{Q'}^{\mathcal{Q}'}) \otimes 1_{\mathcal{U}} \in (e(\mathcal{V}) \otimes_R (\text{Ind}_Q^G \mathbf{1})^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X}$  by the horizontal map, and then to  $v \otimes 1_{\mathcal{U}_M} \otimes f_{Q\mathcal{U}} \theta_{Q'}(e_{Q'}^{\mathcal{Q}'})$  by the vertical map. On the other hand, it goes to  $v \otimes 1_{\mathcal{U}_M} \otimes f_{Q'\mathcal{U}}$  by the vertical map, and then to  $v \otimes 1_{\mathcal{U}_M} \otimes f_{Q\mathcal{U}} \theta_{Q'}(e_{Q'}^{\mathcal{Q}'})$  by the horizontal map. One deduces that  $\Phi_Q^G$  induces an  $R[G]$ -homomorphism  $(e(\mathcal{V}) \otimes_R (\text{St}_Q^G)^\mathcal{U}) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow e(\sigma) \otimes_R \text{St}_Q^G$ , which is an isomorphism if  $\Phi_{Q'}^G$  is an  $R[G]$ -isomorphism for all  $Q \subset Q'$ .  $\square$

**5.3. General.** We consider now the general case: let  $P = MN \subset Q$  be two standard parabolic subgroups of  $G$  and let  $\mathcal{V}$  be a non-zero right  $\mathcal{H}_{M,R}$ -module with  $Q \subset P(\mathcal{V})$ . We recall  $I_{\mathcal{H}}(P, \mathcal{V}, Q) = \text{Ind}_{\mathcal{H}_{M(\mathcal{V})}}^{\mathcal{H}}(e(\mathcal{V}) \otimes_R (\text{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}})$  and  $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} \mathbb{X}_M$  (Proposition 5.4). There is a natural  $R[G]$ -homomorphism

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} \mathbb{X} \xrightarrow{\Phi_I^G} \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})})$$

obtained by composition of the  $R[G]$ -isomorphism [OV17, Corollary 4.7] (proof of Proposition 5.9):

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} \mathbb{X} \rightarrow \text{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathbb{X}_{M(\mathcal{V})}),$$

with the  $R[G]$ -homomorphism

$$\text{Ind}_{P(\mathcal{V})}^G((e(\mathcal{V}) \otimes_R (\text{St}_{Q \cap M(\mathcal{V})}^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathbb{X}_{M(\mathcal{V})}) \rightarrow \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})}),$$

image by the parabolic induction  $\text{Ind}_{P(\mathcal{V})}^G$  of the homomorphism

$$(e(\mathcal{V}) \otimes_R (\text{St}_Q^{P(\mathcal{V})})^{\mathcal{U}_{M(\mathcal{V})}}) \otimes_{\mathcal{H}_{M(\mathcal{V}),R}} \mathbb{X}_{M(\mathcal{V})} \rightarrow e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})}$$

induced by the  $R[M(\mathcal{V})]$ -homomorphism  $\Phi_Q^{P(\mathcal{V})} = \Phi_{Q \cap M(\mathcal{V})}^{M(\mathcal{V})}$  of Proposition 5.10 applied to  $M(\mathcal{V})$  instead of  $G$ .

This homomorphism  $\Phi_I^G$  is an isomorphism if  $\Phi_Q^{P(\mathcal{V})}$  is an isomorphism, in particular if  $\mathcal{V}$  has no  $q_s + 1$ -torsion for any  $s \in S_{M_2'}^{\text{aff}}$  where  $\Delta_{M_2} = \Delta_{M(\mathcal{V})} \setminus \Delta_M$  (Proposition 5.10). We get the main theorem of this section.

**Theorem 5.11.** *Let  $(P = MN, \mathcal{V}, Q)$  be an  $\mathcal{H}_R$ -triple and  $\sigma = \mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$ . Then,  $(P, \sigma, Q)$  is an  $R[G]$ -triple. The  $R[G]$ -homomorphism*

$$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \setminus G] \xrightarrow{\Phi_I^G} \text{Ind}_{P(\mathcal{V})}^G(e_{M(\mathcal{V})}(\sigma) \otimes_R \text{St}_Q^{P(\mathcal{V})})$$

*is an isomorphism if  $\Phi_Q^{P(\mathcal{V})}$  is an isomorphism. In particular  $\Phi_I^G$  is an isomorphism if  $\mathcal{V}$  has no  $q_s + 1$ -torsion for any  $s \in S_{M_2'}^{\text{aff}}$ .*

Recalling  $I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(e(\sigma) \otimes_R \text{St}_Q^{P(\sigma)})$  when  $\sigma \neq 0$ , we deduce the following.

**Corollary 5.12.** *We have the following:*

$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$ , if  $\sigma \neq 0$ ,  $P(\mathcal{V}) = P(\sigma)$  and  $\mathcal{V}$  has no  $q_s + 1$ -torsion for any  $s \in S_{M'_2}^{\text{aff}}$ .

$I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] = I_G(P, \sigma, Q) = 0$ , if  $\sigma = 0$ .

Recalling  $P(\mathcal{V}) = P(\sigma)$  if  $\sigma \neq 0$ ,  $R$  is a field of characteristic  $p$  and  $\mathcal{V}$  simple supersingular (Proposition 5.4 (4)), we deduce the following.

**Corollary 5.13.**  $I_{\mathcal{H}}(P, \mathcal{V}, Q) \otimes_{\mathcal{H}_R} R[\mathcal{U} \backslash G] \simeq I_G(P, \sigma, Q)$  if  $R$  is a field of characteristic  $p$  and  $\mathcal{V}$  simple supersingular.

## 6. VANISHING OF THE SMOOTH DUAL

Let  $V$  be an  $R[G]$ -module. The dual  $\text{Hom}_R(V, R)$  of  $V$  is an  $R[G]$ -module for the contragredient action:  $gL(gv) = L(v)$  if  $g \in G$ ,  $L \in \text{Hom}_R(V, R)$  is a linear form and  $v \in V$ . When  $V \in \text{Mod}_R^\infty(G)$  is a smooth  $R$ -representation of  $G$ , the dual of  $V$  is not necessarily smooth. A linear form  $L$  is smooth if there exists an open subgroup  $H \subset G$  such that  $L(hv) = L(v)$  for all  $h \in H, v \in V$ ; the space  $\text{Hom}_R(V, R)^\infty$  of smooth linear forms is a smooth  $R$ -representation of  $G$ , called the **smooth dual** (or smooth contragredient) of  $V$ . The smooth dual of  $V$  is contained in the dual of  $V$ .

**Example 6.1.** When  $R$  is a field and the dimension of  $V$  over  $R$  is finite, the dual of  $V$  is equal to the smooth dual of  $V$  because the kernel of the action of  $G$  on  $V$  is an open normal subgroup  $H \subset G$ ; the action of  $G$  on the dual  $\text{Hom}_R(V, R)$  is trivial on  $H$ .

We assume in this section that  $R$  is a field of characteristic  $p$ . Let  $P = MN$  be a parabolic subgroup of  $G$  and  $V \in \text{Mod}_R^\infty(M)$ . Generalizing the proof given in [Vig07, 8.1] when  $G = GL(2, F)$  and the dimension of  $V$  is 1, we show the following.

**Proposition 6.2.** *If  $P \neq G$ , the smooth dual of  $\text{Ind}_P^G(V)$  is 0.*

*Proof.* Let  $L$  be a smooth linear form on  $\text{Ind}_P^G(V)$  and let  $K$  be an open pro- $p$ -subgroup of  $G$  which fixes  $L$ . Let  $J$  be an arbitrary open subgroup of  $K$ ,  $g \in G$  and  $f \in (\text{Ind}_P^G(V))^J$  with support  $PgJ$ . We want to show that  $L(f) = 0$ . Let  $J'$  be any open normal subgroup of  $J$  and let  $\varphi$  denote the function in  $(\text{Ind}_P^G(V))^{J'}$  with support  $PgJ'$  and value  $\varphi(g) = f(g)$  at  $g$ . For  $j \in J$  we have  $L(j\varphi) = L(\varphi)$ , and the support of  $j\varphi(x) = \varphi(xj)$  is  $PgJ'j^{-1}$ . The function  $f$  is the sum of translates  $j\varphi$ , where  $j$  ranges through the left cosets of the image  $X$  of  $g^{-1}Pg \cap J$  in  $J/J'$ , so that  $L(f) = rL(\varphi)$  where  $r$  is the order of  $X$  in  $J/J'$ . We can certainly find  $J'$  such that  $r \neq 1$ , and then  $r$  is a positive power of  $p$ . As the characteristic of  $R$  is  $p$  we have  $L(f) = 0$ .  $\square$

The module  $R[\mathcal{U} \backslash G]$  is contained in the module  $R^{\mathcal{U} \backslash G}$  of functions  $f : \mathcal{U} \backslash G \rightarrow R$ . The actions of  $\mathcal{H}$  and of  $G$  on  $R[\mathcal{U} \backslash G]$  extend to  $R^{\mathcal{U} \backslash G}$  by the same formulas. The pairing

$$(f, \varphi) \mapsto \langle f, \varphi \rangle = \sum_{g \in \mathcal{U} \backslash G} f(g) \varphi(g) : R^{\mathcal{U} \backslash G} \times R[\mathcal{U} \backslash G] \rightarrow R$$

identifies  $R^{\mathcal{U}\backslash G}$  with the dual of  $R[\mathcal{U}\backslash G]$ . Let  $h \in \mathcal{H}$  and  $\check{h} \in \mathcal{H}$ ,  $\check{h}(g) = h(g^{-1})$  for  $g \in G$ . We have

$$\langle f, h\varphi \rangle = \langle \check{h}f, \varphi \rangle.$$

**Proposition 6.3.** *When  $R$  is an algebraically closed field of characteristic  $p$ ,  $G$  is not compact modulo the center and  $\mathcal{V}$  is a simple supersingular right  $\mathcal{H}_R$ -module, the smooth dual of  $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$  is 0.*

*Proof.* Let  $\mathcal{H}_R^{\text{aff}}$  be the subalgebra of  $\mathcal{H}_R$  of basis  $(T_w)_{w \in W'(1)}$  where  $W'(1)$  is the inverse image of  $W'$  in  $W(1)$ . The dual of  $\mathcal{V} \otimes_{\mathcal{H}_R} R[\mathcal{U}\backslash G]$  is contained in the dual of  $\mathcal{V} \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\backslash G]$ ; the  $\mathcal{H}_R^{\text{aff}}$ -module  $\mathcal{V}|_{\mathcal{H}_R^{\text{aff}}}$  is a finite sum of supersingular characters [Vig15a]. Let  $\chi : \mathcal{H}_R^{\text{aff}} \rightarrow R$  be a supersingular character. The dual of  $\chi \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\backslash G]$  is contained in the dual of  $R[\mathcal{U}\backslash G]$  isomorphic to  $R^{\mathcal{U}\backslash G}$ . It is the space of  $f \in R^{\mathcal{U}\backslash G}$  with  $\check{h}f = \chi(h)f$  for all  $h \in \mathcal{H}_R^{\text{aff}}$ . The smooth dual of  $\chi \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\backslash G]$  is 0 if the dual of  $\chi \otimes_{\mathcal{H}_R^{\text{aff}}} R[\mathcal{U}\backslash G]$  has no non-zero element fixed by  $\mathcal{U}$ . Let us take  $f \in R^{\mathcal{U}\backslash G/\mathcal{U}}$  with  $\check{h}f = \chi(h)f$  for all  $h \in \mathcal{H}_R^{\text{aff}}$ . We shall prove that  $f = 0$ . We have  $\tilde{T}_w = T_{w^{-1}}$  for  $w \in W(1)$ .

Let  $<$  denote the Bruhat order of  $W(1)$  associated to  $S^{\text{aff}}$  [Vig16]. The elements  $(T_t)_{t \in Z_k}$  and  $(T_{\tilde{s}})_{s \in S^{\text{aff}}}$  where  $\tilde{s}$  is an admissible lift of  $s$  in  $W^{\text{aff}}(1)$ , generate the algebra  $\mathcal{H}_R^{\text{aff}}$  and

$$T_t T_w = T_{tw}, \quad T_{\tilde{s}} T_w = \begin{cases} T_{\tilde{s}w}, & \tilde{s}w > w, \\ c_{\tilde{s}} T_w, & \tilde{s}w < w, \end{cases}$$

with  $c_{\tilde{s}} = -|Z'_{k,s}| \sum_{t \in Z'_{k,s}} T_t$  because the characteristic of  $R$  is  $p$  [Vig16, Proposition 4.4]. Expressing  $f = \sum_{w \in W(1)} a_w T_w$ ,  $a_w \in R$ , as an infinite sum, we have

$$T_t f = \sum_{w \in W(1)} a_{t^{-1}w} T_w, \quad T_{\tilde{s}} f = \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w c_{\tilde{s}}) T_w.$$

A character  $\chi$  of  $\mathcal{H}_R^{\text{aff}}$  is associated to a character  $\chi_k : Z_k \rightarrow R^*$  and a subset  $J$  of

$$S_{\chi_k}^{\text{aff}} = \{s \in S^{\text{aff}} \mid (\chi_k)|_{Z'_{k,s}} \text{ trivial} \}$$

[Vig15a, Definition 2.7]. We have

$$(6.1) \quad \begin{cases} \chi(T_t) = \chi_k(t), & t \in Z_k, \\ \chi(T_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\text{aff}} \setminus J, \\ -1, & s \in J. \end{cases} \end{cases} \quad (\chi_k)(c_{\tilde{s}}) = \begin{cases} 0, & s \in S^{\text{aff}} \setminus S_{\chi_k}^{\text{aff}}, \\ -1, & s \in S_{\chi_k}^{\text{aff}}. \end{cases}$$

Therefore  $\chi_k(t)f = \tilde{T}_t f = T_{t^{-1}} f$  hence  $\chi_k(t)a_w = a_{tw}$ . We have  $\chi(T_{\tilde{s}})f = \tilde{T}_{\tilde{s}} f = T_{(\tilde{s})^{-1}} f = T_{\tilde{s}} T_{(\tilde{s})^{-2}} f = \chi_k((\tilde{s})^2) T_{\tilde{s}} f$ ; as  $(\tilde{s})^2 \in Z'_{k,s}$  [Vig16, three lines before Proposition 4.4] and  $J \subset S_{\chi_k}^{\text{aff}}$ , we obtain

$$(6.2) \quad T_{\tilde{s}} f = \begin{cases} 0, & s \in S^{\text{aff}} \setminus J, \\ -f, & s \in J. \end{cases}$$



Introducing  $\chi_k(t)a_w = a_{tw}$  in the formula for  $T_{\tilde{s}}f$ , we get

$$\begin{aligned}
\sum_{w \in W(1), \tilde{s}w < w} a_w c_{\tilde{s}} T_w &= -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_w T_{tw} \\
&= -|Z'_{k,s}|^{-1} \sum_{w \in W(1), \tilde{s}w < w, t \in Z'_{k,s}} a_{t^{-1}w} T_w \\
&= -|Z'_{k,s}|^{-1} \sum_{t \in Z'_{k,s}} \chi_k(t^{-1}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w \\
&= \chi_k(c_{\tilde{s}}) \sum_{w \in W(1), \tilde{s}w < w} a_w T_w.
\end{aligned}$$

$$\begin{aligned}
T_{\tilde{s}}f &= \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} + a_w \chi_k(c_{\tilde{s}})) T_w \\
&= \begin{cases} \sum_{w \in W(1), \tilde{s}w < w} a_{(\tilde{s})^{-1}w} T_w, & s \in S^{\text{aff}} \setminus S^{\text{aff}}_{\chi_k}, \\ \sum_{w \in W(1), \tilde{s}w < w} (a_{(\tilde{s})^{-1}w} - a_w) T_w, & s \in S^{\text{aff}}_{\chi_k}. \end{cases}
\end{aligned}$$

From the last equality and (6.2) for  $T_{\tilde{s}}f$ , we get:

$$(6.3) \quad a_{\tilde{s}w} = \begin{cases} 0, & s \in J \cup (S^{\text{aff}} \setminus S^{\text{aff}}_{\chi_k}), \tilde{s}w < w, \\ a_w, & s \in S^{\text{aff}}_{\chi_k} \setminus J. \end{cases}$$

Assume that  $a_w \neq 0$ . By the first condition, we know that  $w > \tilde{s}w$  for  $s \in J \cup (S^{\text{aff}} \setminus S^{\text{aff}}_{\chi_k})$ . The character  $\chi$  is supersingular if for each irreducible component  $X$  of  $S^{\text{aff}}$ , the intersection  $X \cap J$  is not empty and different from  $X$  [Vig15a, Definition 2.7, Theorem 6.18]. This implies that the group generated by the  $s \in S^{\text{aff}}_{\chi_k} \setminus J$  is finite. If  $\chi$  is supersingular, by the second condition we can suppose  $w > \tilde{s}w$  for any  $s \in S^{\text{aff}}$ . But there is no such element if  $S^{\text{aff}}$  is not empty.  $\square$

**Theorem 6.4.** *Let  $\pi$  be an irreducible admissible  $R$ -representation of  $G$  with a non-zero smooth dual where  $R$  is an algebraically closed field of characteristic  $p$ . Then  $\pi$  is finite dimensional.*

*Proof.* Let  $(P, \sigma, Q)$  be an  $R[G]$ -triple with  $\sigma$  supercuspidal such that  $\pi \simeq I_G(P, \sigma, Q)$ . The representation  $I_G(P, \sigma, Q)$  is a quotient of  $\text{Ind}_Q^G e_Q(\sigma)$  hence the smooth dual of  $\text{Ind}_Q^G e_Q(\sigma)$  is not zero. From Proposition 6.2,  $Q = G$ . We have  $I_G(P, \sigma, G) = e(\sigma)$ . The smooth dual of  $\sigma$  contains the smooth linear dual of  $e(\sigma)$  hence is not zero. As  $\sigma$  is supercuspidal, the  $\mathcal{H}_M$ -module  $\sigma^{\mathcal{U}_M}$  contains a simple supersingular submodule  $\mathcal{V}$  [Vig15a, Proposition 7.10, Corollary 7.11]. The functor  $- \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$  being the right adjoint of  $(-)^{\mathcal{U}_M}$ , the irreducible representation  $\sigma$  is a quotient of  $\mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$ , hence the smooth dual of  $\mathcal{V} \otimes_{\mathcal{H}_{M,R}} R[\mathcal{U}_M \setminus M]$  is not zero. By Proposition 6.3,  $M = Z$ . Hence  $\sigma$  is finite dimensional and the same is true for  $e(\sigma) = I_G(B, \sigma, G) \simeq \pi$ .  $\square$

*Remark 6.5.* When the characteristic of  $F$  is 0, Theorem 6.4 was proved by Kohlhaase for a field  $R$  of characteristic  $p$ . He gives two proofs [Koh, Proposition 3.9, Remark 3.10], but none of them extends to  $F$  of characteristic  $p$ . Our proof is valid without restriction on the characteristic of  $F$  and does not use the results of Kohlhaase. Our assumption that  $R$  is an algebraically closed field of characteristic  $p$  comes from the classification theorem in [AHHV17].

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