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On homomorphisms of planar signed graphs and absolute cliques

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Abstract

A simple signed graph (G, Σ) is a simple graph with a $+$ or a $-$ sign assigned to each of its edges where Σ denotes the set of negative edges. A closed-walk is unbalanced if it has an odd number of negative edges, it is balanced otherwise. Homomorphisms of simple signed graphs are adjacency and balance of a closed-walk preserving vertex mappings. Naserasr, Rollova and Sopena (Journal of Graph Theory 2015) posed the important question of finding out the minimum $|V(H)|$ such that any planar signed graph (G, Σ) admits a homomorphism to (H, Π) . It is known that if this minimum value is equal to 10, then every planar signed graph must admit homomorphism to a particular unique signed graph (P_9^+, Γ^+) on 10 vertices.

A graph G is an underlying absolute signed clique if there exists a signed graph (G, Σ) which does not admit any homomorphism to any signed graph (H, Π) with $|V(H)| < |V(G)|$. We characterize all underlying absolute signed planar cliques upto spanning subgraph inclusion. Furthermore, we show that every signed planar graph having underlying graphs obtained by (repeated, finite) k -clique sums ($k \leq 3$) of underlying absolute signed planar cliques admits a homomorphism to (P_9^+, Γ^+) . Based on this evidence, we conjecture that every planar signed graph admits a homomorphism to (P_9^+, Γ^+) .

Keywords: signed graphs, graph homomorphisms, absolute signed cliques, planar graphs.

1 Introduction and main results

The work of Naserasr, Rollová and Sopena [10], based on the work of Zaslavsky [13], have generated attention to the topic of homomorphisms of signed graphs in recent times [12, 7, 6, 2, 11, 9].

A *signed graph* (G, Σ) is a graph with set of vertices $V(G)$ and set of edges $E(G)$ where each edge is assigned one of the two possible signs, $+$ or $-$, and the *signature* Σ is its set of negative edges.

Furthermore, $N(u)$ denotes the set of neighbors of u . Also the set of vertices adjacent to u using a positive (or a negative) edge, that is, the set of *+neighbors* (or *-neighbors*) of u , is denoted by $N^+(u)$ (or $N^-(u)$).

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Moreover, *resigning* a vertex v of (G, Σ) is to switch the signs of the edges incident to v . One may do this on a set S of vertices and that is the same as resigning the signs of all edges of an edge-cut (edges with exactly one endpoint in S) of (G, Σ) .

One may regard the signs $+$ and $-$ as the elements $+1$ and -1 , respectively, of the multiplicative group \mathbb{F}_2 of order 2. Thus, resigning a vertex v of (G, Σ) is to multiply the signs of the edges incident to v by -1 .

A closed-walk C of a signed graph (G, Σ) is *unbalanced* if the product of the signs of $E(C)$ (repetition considered) is negative and C is *balanced* otherwise. Informally speaking, homomorphisms of simple signed graphs are adjacency and balance of a closed-walk preserving vertex mappings.

As we are only interested in simple signed graphs in this article, we provide only the definition of homomorphisms of simple signed graphs instead of presenting the generalized definition (available in the website of HOSIGRA¹).

Formally, a *homomorphism* of a signed graph (G, Σ) to a signed graph (H, Π) is a function $f : V(G) \rightarrow V(H)$ such that the following conditions are satisfied:

- (i) if $x, y \in V(G)$ are adjacent, then $f(x), f(y)$ are also adjacent,
- (ii) if $C_G = (x_1, x_2, \dots, x_k)$ is a closed walk in (G, Σ) , then $C_H = (f(x_1), f(x_2), \dots, f(x_k))$ is a closed-walk in (H, Π) having the same balance as C_G .

Furthermore, we will use the notation $(G, \Sigma) \rightarrow (H, \Pi)$ to mean that (G, Σ) admits a homomorphism to (H, Π) .

The *chromatic number* $\chi_s(G, \Sigma)$ of the signed graph (G, Σ) is the minimum cardinality of $|V(H)|$ such that $(G, \Sigma) \rightarrow (H, \Pi)$. Furthermore, for a family \mathcal{F} of simple signed graphs the chromatic number is $\chi_s(\mathcal{F}) = \max\{\chi_s(G, \Sigma) | G \in \mathcal{F}\}$.

One of the important open problems in this area is to find the chromatic number of planar signed graphs. The best known bounds regarding this problem is the following.

Theorem 1.1 (Ochem, Pinlou and Sen 2017 [12]). *For the family \mathcal{P}_3 of planar graphs we have $10 \leq \chi_s(\mathcal{P}_3) \leq 40$.*

It is known [12] that if $\chi_s(\mathcal{P}) = k$, then there exists a signed graph (H, Π) on k vertices such that $G \in \mathcal{P}_3$ implies $(G, \Sigma) \rightarrow (H, \Pi)$ for any signature Σ of G . Such an (H, Π) is called an *optimized universal target* of \mathcal{P}_3 .

We are going to describe an alleged such optimized universal target graph. Let (H, Π) be a signed graph. Let (H^+, Π^+) be the signed graph obtained by adding one vertex ∞ to (H, Π) and positive edges joining each vertex of (H, Π) to the new vertex ∞ . Let $C_3 \square C_3$ be the Cartesian product of two 3-cycles. The signed Paley graph (P_9, Γ) on 9 vertices is the signed graph having $V(P_9) = V(C_3 \square C_3)$, $E(P_9) \setminus \Gamma = \{xy | xy \in E(C_3 \square C_3)\}$ and $\Gamma = \{xy | xy \notin E(C_3 \square C_3)\}$. The graph of our interest is the graph (P_9^+, Γ^+) .

Theorem 1.2 (Ochem, Pinlou and Sen 2017 [12]). *If $\chi_s(\mathcal{P}_3) = 10$, then (P_9^+, Γ^+) is the unique optimized universal target of \mathcal{P}_3 .*

An integrally important concept related to chromatic number of signed graphs is the notion of absolute signed cliques [10]. A signed graph (C, Λ) is an *absolute signed clique* if $\chi_s(C, \Lambda) = |V(C)|$.

¹<https://www.irif.fr/hosigra/pmwiki/pmwiki.php>

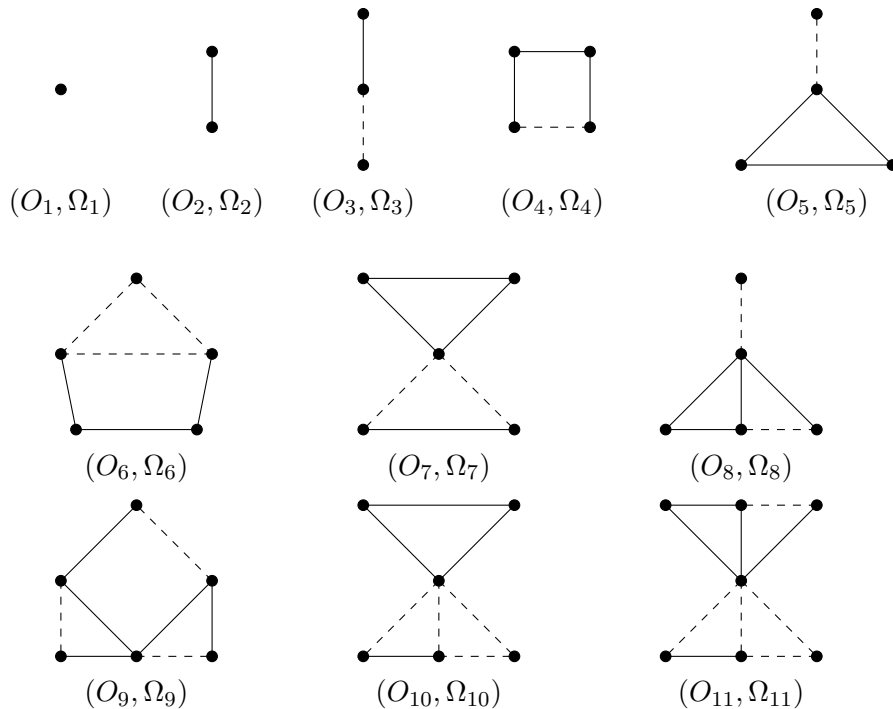


Figure 1: A list of absolute 2-edge-colored outerplanar cliques. Here “solid lines” denote edges with positive sign and “dashed lines” denote edges with negative signs. The set of the underlying simple graphs of the above signed graphs is denoted by L_{2e} .

A handy characterization of absolute signed cliques was given in the same paper [10]. A signed graph (C, Λ) is an absolute signed clique if and only if each pair of non-adjacent vertices of (C, Λ) is part of an unbalanced 4-cycle.

An undirected simple graph C is an *underlying absolute signed clique* if there exists a signature Λ such that (C, Λ) is an absolute signed clique. Despite such a characterization, the problem of recognizing underlying absolute signed cliques is an NP-complete decision problem.

Theorem 1.3 (Bensmail, Duffy and Sen 2017 [3]). *Given an undirected simple graph C , it is NP-complete to determine if it is an underlying signed clique or not.*

Turning our heads towards the family of planar graphs, we know that an absolute signed planar clique has at most 8 vertices.

Theorem 1.4 (Naserasr, Rollova and Sopena 2015 [10]). *If an absolute signed clique (C, Λ) is a signed planar graph, then $|V(C)| \leq 8$.*

A related useful tool for studying absolute signed cliques is the notion of absolute 2-edge-colored clique. A signed graph (C, Λ) is an *absolute 2-edge-colored clique* if each pair of non-adjacent vertices of (C, Λ) is connected by a 2-path having two differently signed edges. Furthermore, an undirected simple graph C_2 is an *underlying absolute 2-edge-colored clique* if there exists a signature Λ such that (C, Λ) is an absolute 2-edge-colored clique. We prove a useful result on this which will help us in the context of our work.

Theorem 1.5. *An undirected outerplanar graph is an underlying absolute 2-edge-colored clique if and only if it contains one of the graphs belonging to L_{2e} as a spanning subgraph, where L_{2e} is the set of underlying graphs of the signed graphs depicted in Fig. 1.*

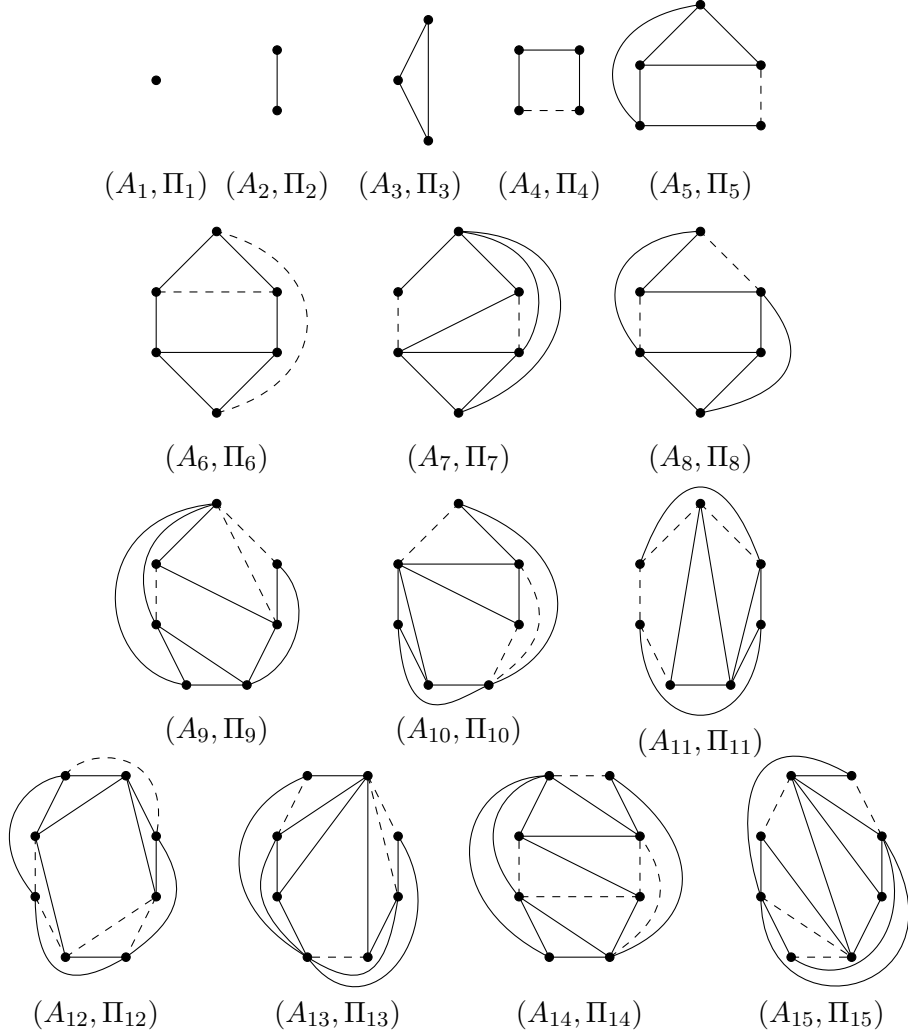


Figure 2: A list of absolute signed planar cliques. Here “solid lines” denote edges with positive sign and “dashed lines” denote edges with negative signs. The set of the underlying simple graphs of the above signed graphs is denoted by L_s .

Using the above result we provide a list L_s of all minimal (with respect to spanning subgraph inclusion) underlying absolute signed planar cliques.

Theorem 1.6. *An undirected planar graph is an underlying absolute signed clique if and only if it contains one of the graphs belonging to L_s as a spanning subgraph, where L_s is the set of underlying graphs of the signed graphs depicted in Fig 2.*

Let G_1 and G_2 be two simple graphs with $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. The 1-sum of G_1 and G_2 with respect to x_1 and x_2 is the graph obtained from G_1 and G_2 by gluing the vertex x_1 with x_2 .

Let G_1 and G_2 be two simple graphs with $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$. The 2-sum of G_1 and G_2 with respect to (x_1, y_1) and (x_2, y_2) is the graph obtained from G_1 and G_2 by gluing the vertex x_1 with x_2 , the vertex y_1 with y_2 and the edge x_1y_1 with x_2y_2 .

Let G_1 be a simple graph with a facial 3-cycle $F_1 = x_1y_1z_1x_1$ and G_2 be another simple graph with a facial 3-cycle $F_2 = x_2y_2z_2x_2$. The facial 3-sum of G_1 and G_2 with respect to F_1

and F_2 is the graph obtained from G_1 and G_2 by gluing the vertices x_1, y_1, z_1 with x_2, y_2, z_2 , respectively, and the edges x_1y_1, y_1z_1, z_1x_1 with x_2y_2, y_2z_2, z_2x_2 , respectively.

Let \bar{L}_s denote the set of all planar graphs having one of the elements of L_s as a spanning subgraph. Let L_{ext} be the set of all signed graphs whose underlying graphs can be obtained by performing (repeated, but finitely many) 1-sums, 2-sums and facial 3-sums recursively on the elements of \bar{L}_s . Observe that the elements of L_{ext} are planar graphs. Thus we present the following as an evidence that (P_9^+, Γ^+) might actually be the unique optimized universal target of \mathcal{P}_3 .

Theorem 1.7. *For each $C \in L_{ext}$ and for each signature Λ of C , (C, Λ) admits a homomorphism to (P_9^+, Γ^+) .*

In this article, we present the proofs of the above three theorems in three separate sections and conclude after that.

2 Proof of Theorem 1.5

First of all, note that an underlying absolute 2-edge-colored clique is a connected graph having diameter at most two. Also observe that each of the eleven signed graphs depicted in Fig. 1 is an absolute 2-edge-colored outerplanar clique.

Bensmail, Duffy and Sen [3] defined and studied (m, n) -cliques (we are omitting the definition). An absolute 2-edge-colored clique is the same as a $(0, 2)$ -clique. As they have shown that an outerplanar (m, n) -clique can have at most $3(2m + n) + 1$ vertices whenever $2m + n \geq 2$, an underlying absolute 2-edge-colored outerplanar clique has at most seven vertices.

Another useful definition is that of $(0, 2)$ -colored mixed coloring and chromatic number. It is better known as 2-edge-colored coloring and chromatic number [1]. A *2-edge-colored k -coloring* of a signed graph (G, Σ) is k -vertex-coloring of G such that all the edges of any bichromatic subgraph of (G, Σ) have the same sign. The *2-edge-colored chromatic number* $\chi_2((G, \Sigma))$ of (G, Σ) is the minimum k such that (G, Σ) admits a 2-edge-colored k -coloring.

Observation 2.1 (Bensmail, Duffy and Sen 2017 [3]). *A signed graph (C, Λ) is an absolute 2-edge-colored clique if and only if $\chi_2((C, \Lambda)) = |V(C)|$. Therefore, if (C, Λ) is an absolute 2-edge-colored clique, then $|V(C)| \leq \chi_2((C, \Lambda))$.*

After that we prove a useful result.

Lemma 2.2. *Let G^+ be a graph obtained by adding a new vertex ∞ to G and making it adjacent to each vertex of G . If $\chi_2((G, \Sigma)) \leq k$, then $\chi_2((G^+, \Sigma^*)) \leq 2k + 1$ where the signature Σ^* restricted to G is the same as Σ .*

Proof. Let $f : V(G) \rightarrow \{1, 2, \dots, k\}$ be a 2-edge-colored k -coloring of (G, Σ) . Then

$$f^*(v) = \begin{cases} 0 & \text{if } v = \infty, \\ (f(v), +) & \text{if } v \in N^+(\infty) \subseteq V(G), \\ (f(v), -) & \text{if } v \in N^-(\infty) \subseteq V(G). \end{cases}$$

is a 2-edge-colored $(2k + 1)$ -coloring of (G^+, Σ^*) . □

Given a signed graph (G, Σ) , two vertices u and v see each other if they are adjacent or are endpoints of a 2-path with edges having different signs. Also if u and v are endpoints of a 2-path with edges having different signs and the third vertex of the 2-path is w , then we say that u and v sees each other *through* w .

Lemma 2.3. *If O is an underlying absolute 2-edge-colored outerplanar clique having $|V(O)| = 7$, then O contains O_{11} as a spanning subgraph.*

Proof. Suppose that $|V(O)| = 7$ with (O, Ω) being an absolute 2-edge-colored outerplanar clique. Thus $\delta(O) \in \{1, 2\}$.

Let $\delta(O) = 1$ with a vertex u having degree one and let v be its only neighbor. Furthermore, let v be an α -neighbor of u for some $\{\alpha, \bar{\alpha}\} = \{+, -\}$. Therefore we must have $N^{\bar{\alpha}}(v) = V(O) \setminus \{u, v\}$ in order for a vertex $x \in V(O) \setminus \{u, v\}$ to see u . Observe that a pair of vertices from $V(O) \setminus \{u, v\}$ sees each other without using u or v . Thus the graph induced by $N^{\bar{\alpha}}(v)$ is an underlying absolute 2-edge-colored clique. Moreover it is a path as O is an outerplanar graph. As we know that [1] a signed forest (F, Φ) has $\chi_2((F, \Phi)) \leq 3$, by Observation 2.1 we have $|V(O)| \leq 5$. Therefore, $\delta(O) = 2$.

Let $\delta(O) = 2$ with a vertex u having degree two and let v, w be its neighbors. Furthermore, let v be an α -neighbor and w be a β -neighbor of u for some $\{\alpha, \bar{\alpha}\} = \{\beta, \bar{\beta}\} = \{+, -\}$. Therefore we must have $N^{\bar{\alpha}}(v) \cup N^{\bar{\beta}}(w) = V(O) \setminus \{u, v, w\}$ in order for a vertex $x \in V(O) \setminus \{u, v, w\}$ to see u . Note that $|N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w)| \leq 1$, as otherwise it will create a $K_{2,3}$ contradicting the fact that O is an outerplanar graph.

If $|N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w)| = 0$, then a pair of vertices from $V(O) \setminus \{u, v, w\}$ sees each other without using u, v or w . Moreover, the vertices of $V(O) \setminus \{u, v, w\}$ induce a forest. Thus the vertices of $V(O) \setminus \{u, v, w\}$ must induce an underlying absolute 2-edge-colored clique of order four which is a forest. This is not possible. Hence we must have $|N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w)| = 1$.

Finally, let $|N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w)| = 1$. Moreover, let $N^{\bar{\alpha}}(v) \cap N^{\bar{\beta}}(w) = z$. Assume that $A = N^{\bar{\alpha}}(v) \setminus \{z\}$ and $B = N^{\bar{\beta}}(w) \setminus \{z\}$. Note that any edge between A and B will force a $K_{2,3}$ -minor contradicting the fact that O is an outerplanar graph.

Without loss of generality assume that $|A| \geq |B|$. If $|B| > 0$, then the only way for the vertices of A to see the vertices of B is through z . This creates a $K_{2,3}$ -minor, contradicting the outerplanarity of O . Hence $|B| = 0$.

If $|B| = 0$, then $|A| = 3$. The only two options for a vertex $x \in A$ to see w are through v or z . If all three vertices of A see w through z , then a $K_{2,3}$ -minor is created, a contradiction. Thus at least one vertex from A sees w through v forcing the edge vw in O .

Observe that the vertices of A induce a forest. Furthermore, the vertices of A cannot see each other only by going through u, v, w . Also it is not possible that all three vertices of A see each other through z . Therefore, at least two vertices of A see through the third vertex of A forcing a 2-path in the graph induced by A .

However, this forces the graph O_{11} inside O as a subgraph. □

Finally, we prove the remaining part of the result.

Lemma 2.4. *Let O be an underlying absolute 2-edge-colored outerplanar clique. If $|V(O)| \leq 6$, then O contains one of O_1, O_2, \dots, O_{10} from L_{2e} as a spanning subgraph.*

Proof. Suppose (O, Ω) is an absolute 2-edge-colored clique.

As any absolute 2-edge-colored clique is connected, if $|V(O)| = 1, 2$ or 3 , then O contains O_1, O_2 or O_3 as a spanning subgraph, respectively.

Next suppose that $|V(O)| \in \{4, 5, 6\}$. Thus O is either hamiltonian or has a cut vertex v .

Let us first suppose that O has a cut vertex v . Assume that $O \setminus \{v\}$ has components C_1, C_2, \dots, C_k . Then the vertices of C_i must see the vertices of C_j , for all $i \neq j$, through v in (O, Ω) . Therefore, $k = 2$ and $V(C_1) \subseteq N^{\alpha}(u)$ and $V(C_2) \subseteq N^{\bar{\alpha}}(u)$ for some $\{\alpha, \bar{\alpha}\} = \{+, -\}$. Thus the graph induced by $V(C_i)$ from (O, Ω) is an absolute 2-edge-colored clique for each $i \in \{1, 2\}$. Moreover, as O is outerplanar, C_i 's are paths. As $\chi_2((F, \Phi)) \leq 3$ when F is a forest,

$|V(C_i)| \leq 3$. Thus if O has a cut vertex, then O must contain O_5, O_7, O_8 or O_{10} as a spanning subgraph.

If $|V(O)| = 4$ and O is hamiltonian, then O must contain O_4 as a spanning subgraph.

If $|V(O)| = 5$ and O is hamiltonian, then O contains a 5-cycle. However, any signature on a 5-cycle forces two incident edges of the same sign. The endpoints of the 2-path induced by those edges cannot see each other. Thus O must have at least one chord in this case forcing O_6 as a spanning subgraph.

If $|V(O)| = 6$ and O is hamiltonian, then O contains a 6-cycle. However, a 6-cycle does not have diameter two. If we add some chords to a 6-cycle in order to construct an outerplanar graph having diameter two, then we are forced to have O_9 as a spanning subgraph. \square

Thus the proof of Theorem 1.5 follows directly from the above two lemmas and the fact that an absolute 2-edge-colored outerplanar clique can have at most seven vertices due to Bensmail, Duffy and Sen [3].

3 Proof of Theorem 1.6

First observe that each of the fifteen signed graphs depicted in Fig 2 is an absolute signed planar clique.

For the rest of this section we will assume that (H, Π) is an absolute signed planar clique.

Let u, v be a pair of vertices of an undirected simple graph G . We say that u reaches v if either u and v are adjacent or u and v have at least two common neighbors. Furthermore, when writing that u reaches v through w , we will mean that w is a common neighbor of u and v . Also if each pair of vertices of G reaches each other, then we say that G is reach-complete. This motivates our first observation.

Observation 3.1. *An underlying absolute signed clique is reach-complete.*

We are going to list a series of observations and lemmas to aid our proof.

Observation 3.2. *A reach-complete graph G cannot have a cut vertex. Moreover, if $|V(G)| \geq 3$, then $\delta(G) \geq 2$.*

Proof. If a reach-complete graph G has a cut vertex v , then the vertices from different components of $G \setminus \{v\}$ are neither adjacent nor have at least two common neighbors in G .

If $|V(G)| \geq 3$ and $\delta(G) = 1$, then the neighbor of a vertex having degree one is a cut vertex. \square

Note that a reach-complete graph is either a complete graph or contains a 4-cycle. Thus the following is immediate.

Lemma 3.3. *If (H, Π) is an absolute signed planar clique having $|V(H)| \leq 4$, then H contains one of O_1, O_2, O_3, O_4 from L_s as a spanning subgraph.*

Now let us consider the case of graphs having five vertices.

Lemma 3.4. *If (H, Π) is an absolute signed planar clique having $|V(H)| = 5$, then H contains A_5 from L_s as a spanning subgraph.*

Proof. As H does not have a cut vertex, H contains either a 5-cycle or a $K_{2,3}$. We know that a 5-cycle is not an underlying absolute 2-edge-colored clique, and thus obviously not an underlying absolute signed clique. Therefore, we need to add at least one chord to make the graph an underlying absolute signed clique. In that case, the subgraph A_5 is created.

On the other hand, a $K_{2,3}$ is not an underlying absolute signed clique as the vertices from the partite set of cardinality three cannot reach each other. Thus we need to add at least one edge in the concerning partite set. This creates the subgraph A_5 . \square

We already know that if (H, Π) is an absolute signed planar clique having $|V(H)| \geq 6$, then $\delta(H) \geq 2$. First we analyse the case when $\delta(H) = 2$.

Lemma 3.5. *If (H, Π) is an absolute signed planar clique having $|V(H)| \geq 6$ and $\delta(H) = 2$, then H contains A_6 from L_s as a spanning subgraph.*

Proof. Let u be a vertex of H having exactly two neighbors v and w . Then each non-neighbor of u must be adjacent to both v and w , resulting in a $K_{2,|V(G)|-2}$. For being an absolute signed clique, v (or w) must be adjacent to each non-neighbor of u by edges of the same sign. Now note that the only way for (H, Π) to be an absolute signed planar clique is to have the non-neighbors of u induce a 2-path with edges having different signs. In that case, $|V(H)| - 2 = 4$ and thus H contains A_6 as a subgraph. \square

We provided a list of edge-minimal outerplanar underlying absolute 2-edge-colored cliques (see Fig. 1). If an absolute signed planar clique (H, Π) has a dominating vertex v , then we can resign the $-$ -neighbors of v to obtain a new signature Π^* of H . As absolute signed cliques are invariant under resigning operations, (H, Π^*) is also an absolute signed planar clique. Observe that the signed graph obtained by deleting v from (H, Π^*) is an absolute 2-edge-colored outerplanar clique. Hence:

Observation 3.6. *A signed graph (H, Π) is an absolute 2-edge-colored outerplanar clique if and only if (H^+, Π^+) is an absolute signed planar clique.*

Recall that (O_k^+, Ω_k^+) denotes the k^{th} graph in Figure 1. Then:

Observation 3.7. *If \sqsubseteq denotes spanning subgraph inclusion, then $A_2 \sqsubseteq O_1^+$, $A_3 \sqsubseteq O_2^+$, $A_4 \sqsubseteq O_3^+$, $A_5 \sqsubseteq O_4^+$, $A_5 \sqsubseteq O_5^+$, $A_6 \sqsubseteq O_6^+$, $A_8 \sqsubseteq O_7^+$, $A_7 \sqsubseteq O_8^+$, $A_9 \sqsubseteq O_9^+$, $A_{10} \sqsubseteq O_{10}^+$, $A_{13} \sqsubseteq O_{11}^+$,*

Due to the above results, we need to only focus on the edge-minimal underlying absolute signed planar cliques having at least six vertices with minimum degree three and domination number at least two.

Now let us introduce a few graphs. Let P_2 and P_4 denote the paths on two and four vertices, respectively. Let A be the graph obtained by taking two vertices ∞ and $-\infty$ along with the disjoint union of P_2 and P_4 , and making each of $\infty, -\infty$ adjacent to each vertex of P_2 and P_4 . Also let B be the graph obtained from A by adding an edge between the vertices ∞ and $-\infty$. Furthermore, let C denote the graph obtained by taking a 5-cycle and a vertex ∞ and adding edges between ∞ and each of the vertices of the 5-cycle.

Lemma 3.8. *The graphs A, B and C are not underlying absolute signed cliques.*

Proof. Observe that A is a subgraph of B . Thus if we prove that B is not an underlying absolute signed clique, then it will imply that A is also not an underlying absolute signed clique.

Assume that B is an underlying absolute signed clique and Σ is a signature of B such that (B, Σ) is an absolute signed clique. Now resign vertices of (B, Σ) to obtain the signed graph

(B, Σ^*) such that the edges incident to ∞ are all positive and one edge between $-\infty$ and a vertex x of P_2 is also positive. We know that (B, Σ^*) is also an absolute signed clique. Therefore, in order for x to reach the vertices of P_4 , the edges between $-\infty$ and the vertices of P_4 must be negative. In this case, the endpoints of P_4 cannot reach each other.

Observe that C is a planar graph. If C is an underlying absolute signed planar clique, then by Observation 3.6 a 5-cycle is an underlying absolute 2-edge-colored outerplanar clique. This is a contradiction due to Theorem 1.5. \square

As A, B and C are not underlying absolute signed planar cliques, one may wonder what happens if we add some edges to them. If we add any edge to B to obtain a graph B^* , then B^* contains A_{13} as a spanning subgraph. Similarly, if we add any edge to C to obtain a graph C^* , then C^* contains A_6 as a spanning subgraph. Therefore:

Lemma 3.9. *If an underlying absolute signed planar clique H contains A, B or C as a spanning subgraph, then H must contain either A_6 or A_{13} as a spanning subgraph.*

Finally, we conclude the proof of Theorem 1.6 using the following result which is implicitly proved by Bensmail, Nandi and Sen [4] (please note the erratum [5] as well).

Lemma 3.10. *If H is a planar edge-minimal reach-complete graph having at least six vertices, minimum degree at least three and domination number at least two, then H contains one of $A_6, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A, B, C$ as a spanning subgraph.*

Proof. This proof is implicitly present in Section 5 of [4]. Let H be as in the statement.

If $|V(H)| = 6$, then Lemma 5.6 proves that H contains one of A_6, A_7, A_8, C (which are the graphs H_7, H_6, H_9, H_8 in [4], respectively).

If $|V(H)| = 7$, then Lemma 5.9 proves that H contains one of A_9, A_{10}, A_{11} (which are the graphs H_{10}, H_{11}, H_{12} in [4], respectively).

If $|V(H)| = 8$, then Lemma 5.9 proves that H contains one of $A_{12}, A_{13}, A_{14}, A_{15}, A$ (which are the graphs $H_{13}, H_{14}, H_{15}, H_{16}, A$ in [4], respectively). \square

4 Proof of Theorem 1.7

A signed graph (G_1, Σ_1) is *edge-transitively homomorphic* to (G_2, Σ_2) if for each $x_1y_1 \in E(G_1)$ and each $x_2y_2 \in E(G_2)$ there exists a homomorphism $f : (G_1, \Sigma_1) \rightarrow (G_2, \Sigma_2)$ such that $f(x_1) = x_2$ and $f(y_1) = y_2$.

A signed graph (G_1, Σ_1) is *triangle-transitively homomorphic* to (G_2, Σ_2) if for each unbalanced (respectively, balanced) 3-cycle $x_1y_1z_1x_1$ in (G_1, Σ_1) and each unbalanced (respectively, balanced) 3-cycle $x_2y_2z_2x_2$ in (G_2, Σ_2) , there exists a homomorphism $f : (G_1, \Sigma_1) \rightarrow (G_2, \Sigma_2)$ such that $f(x_1) = x_2$, $f(y_1) = y_2$ and $f(z_1) = z_2$.

An *isomorphism* is a bijective homomorphism whose inverse is also a homomorphism. An isomorphism of a signed graph (G, Σ) to itself is an *automorphism*.

We know that (P_9^+, Γ^+) is edge- and triangle-transitive [12], or, in other words, that (P_9^+, Γ^+) is edge- and triangle-transitively automorphic to itself. Thus if a signed (E, Λ) admits a homomorphism to (P_9^+, Γ^+) , then (E, Λ) is also edge- and triangle-transitively homomorphic to (P_9^+, Γ^+) .

Recall that \bar{L}_s is the set of all planar graphs having one of the elements of L_s as a spanning subgraph. Let L_{tri} denote the set of all triangulations of the elements of L_s . Observe that every element of \bar{L}_s is a subgraph of an element of L_{tri} . If we can show that $(E, \Lambda) \rightarrow (P_9^+, \Gamma^+)$ for any $E \in L_{tri}$ and any signature Λ , then we are done as (P_9^+, Γ^+) is edge- and triangle-transitive.

Lemma 4.1. *For each $C \in L_{tri}$ and for each signature Λ of C , (C, Λ) admits a homomorphism to (P_9^+, Γ^+) .*

The proof of the above lemma is contained in some observations and lemmas presented in the following. We will use the following computer-assisted observation from [12] as a tool to our proof.

Observation 4.2 (Ochem, Pinlou and Sen 2015 [12]). *Any 4-connected planar signed graph with at most fifteen vertices admits a homomorphism to (P_9^+, Γ^+) .*

As any triangulation of A_1, A_2, A_3 and A_4 is 4-connected, then:

Lemma 4.3. *Any signed planar graph whose underlying graph is a triangulation of A_1, A_2, A_3 or A_4 admits a homomorphism to (P_9^+, Γ^+) .*

Let G be a planar graph. Let E_1 and E_2 be two embeddings of G on a sphere. If it is possible to continuously deform the sphere along with the embedding E_1 to obtain a sphere along with the embedding E_2 , then the two embeddings of G are *congruent*. Furthermore, let f be an automorphism of G . Now in the embedding E_2 of G , rename each vertex $f(v)$ to v for all $v \in V(G)$. If the embedding E_2 , after renaming, is congruent to E_1 , then E_1 is *equivalent* to E_2 .

Lemma 4.4. *Each of the planar graphs A_5, A_6, \dots, A_{15} has a unique embedding on a sphere upto equivalence.*

Proof. Notice that A_5, A_6, \dots, A_{15} are hamiltonian planar graphs as each of them has the hamiltonian cycle $abc \dots a$. This hamiltonian cycle, under any planar embedding, becomes a closed curve.

Now the other edges, called *extras*, of these graphs are either inside the cycle (closed curve) or outside. Let *in-extras* be the extras drawn inside the hamiltonian cycle(s) in Fig. 2 and let the other extras be *out-extras*.

Observe that we may select any such edge of the graph and assume without loss of generality that the edge will be inside the cycle. For any $X \in \{A_5, A_6, A_7, A_9, A_{11}, A_{12}, A_{14}, A_{15}\}$ and any in-extra e of X , if we draw e inside the hamiltonian cycle of X , then we are forced to draw each in-extra of X inside the hamiltonian cycle and each out-extra of X outside the hamiltonian cycle.

Thus $A_5, A_6, A_7, A_9, A_{11}, A_{12}, A_{14}, A_{15}$ have unique embeddings on a sphere upto equivalence.

Let S be two disjoint vertices and let T be two disjoint paths. Add all the edges between the vertices of S and T . Observe that A_5, A_8, A_{10} and A_{13} are planar graphs of the above types. They have unique embeddings on a sphere upto equivalence. \square

First we handle the signed planar graphs whose underlying graphs are triangulations of A_5, A_8, A_{10} and A_{13} .

Lemma 4.5. *Any signed planar graph (A, Π) whose underlying graph is a triangulation of A_5, A_7, A_8, A_{10} or A_{13} admits a homomorphism to (P_9^+, Γ^+) .*

Proof. Any triangulation of A_5, A_7, A_8, A_{10} or A_{13} is either a 4-connected graph or a planar graph with domination number one.

If A (as in the lemma) is 4-connected, then we are done by Observation 4.2.

Otherwise A has a dominating vertex v . We may resign some of the neighbors of v to obtain a signed graph (A, Π^*) where all the edges incident to v are positive. Observe that the graph

obtained by deleting the vertex v from (A, Π^*) is a signed outerplanar graph. We know that every signed outerplanar graph admits a vertex-mapping to (P_9, Γ) preserving edge signs [8]. Thus we can extend the mapping by mapping the vertex v to the vertex ∞ of (P_9^+, Γ^+) to obtain $(A, \Pi^*) \rightarrow (P_9^+, \Gamma^+)$. \square

A couple of necessary observations follow.

Observation 4.6. *Any triangulation of A_6 or A_9 is a subgraph of A_{15} and any triangulation of A_{11} is a subgraph of A_{14} .*

Observation 4.7 (Ochem, Pinlou and Sen 2017 [12]). *Let v be a vertex of a signed graph (A, Π) having degree at most three such that the neighbors of v induce a clique in A . Then we can extend any homomorphism of $(A \setminus \{v\}, \Pi) \rightarrow (P_9^+, \Gamma^+)$ to a homomorphism of $(A, \Pi) \rightarrow (P_9^+, \Gamma^+)$.*

Finally we handle the signed planar graphs whose underlying graphs are triangulations of A_{12} , A_{14} and A_{15} .

Lemma 4.8. *Any signed planar graph (A, Π) whose underlying graph is a triangulation of A_{12} , A_{14} or A_{15} admits a homomorphism to (P_9^+, Γ^+) .*

Proof. Any triangulation of A_{12} is 4-connected and thus we are done by Observation 4.2.

There are exactly two triangulations of A_{14} , one by adding the edge ae and the other by adding the edge bf . The graph $A_{14} \cup \{bf\}$ is 4-connected, thus we are done by Observation 4.2. On the other hand, the graph $A_{14} \cup \{ae\}$ is not 4-connected. Observe that the graph $A_{14} \cup \{ae\}$ has exactly two vertices of b and f having degree three each. The neighbors of b and f are cliques of order three. Also if we delete the vertices b and f from $A_{14} \cup \{ae\}$, then the graph we obtain is 4-connected. Therefore, we are done by Observations 4.2 and 4.7.

There are exactly four triangulations of A_{15} , obtained by adding two edges from $\{ag, bh\} \times \{ce, df\}$. The graph $A_{14} \cup \{bh, df\}$ is 4-connected, thus we are done by Observation 4.2. On the other hand, the other triangulations have at most four vertices having degree three. The neighbors of each of these vertices are cliques of order three. Also if we delete the vertices having degree three from the triangulation(s), then the graph(s) we obtain are 4-connected. Therefore, we are done by Observations 4.2 and 4.7. \square

5 Conclusions

We would like to report that there are, in total, 47 non-isomorphic planar underlying push cliques (1 on 1 vertex, 1 on 2 vertices, 1 on 3 vertices, 3 on 4 vertices, 4 on 5 vertices, 10 on 6 vertices, 14 on 7 vertices and 13 on 8 vertices). See the lists in the webpage: <http://jbensmai.fr/code/signed/> for details.

Moreover, we managed to show that an important infinite family of signed planar graphs all admit homomorphism to (P_9^+, Γ^+) . We feel confident enough to make the following conjecture:

Conjecture 5.1. *Any signed planar graph (E, Σ) admits a homomorphism to (P_9^+, Γ^+) .*

In particular, the conjecture, if true, would imply that $\chi_s(\mathcal{P}_3) = 10$.

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