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RIESZ BASES OF REPRODUCING KERNELS IN SMALL FOCK SPACES

K. KELLAY & Y. OMARI

ABSTRACT. We give a complete characterization of Riesz bases of normalized reproducing kernels in the small Fock space \mathcal{F}_φ^2 , the space of entire functions f such that $fe^{-\varphi} \in L^2(\mathbb{C})$, where $\varphi(z) = (\log^+ |z|)^{\beta+1}$, $0 < \beta \leq 1$. The first results on this subject are due to Borichev-Lyubarskii who showed that φ with $\beta = 1$ is the biggest weight for which the corresponding Fock space admits Riesz bases of reproducing kernels. Later, such bases was characterized in Baranov et al. [5] for the case when $\beta = 1$. The present paper answers a question in [5] by extending their results for all parameters $\beta \in (0, 1)$. Our results are analogues to those obtained for the case $\beta = 1$ and those proved for Riesz bases of complex exponentials for the Paley-Wiener spaces. We also obtain a description of complete interpolating sequences in small Fock spaces with associated uniform norm.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

For a subharmonic function φ tending to infinity when $|z|$ is getting large, define the weighted Fock spaces \mathcal{F}_φ^p , $p = 2, \infty$, by the set of all entire functions f for which $fe^{-\varphi}$ belongs to $L^p(\mathbb{C})$. Seip and Wallstén [19, 20] characterized interpolating and sampling sequences in these spaces when $\varphi(z) = |z|^2$. They show that there are no sequence which are simultaneously interpolating and sampling, and hence there are no Riesz bases of reproducing kernels in this situation. However, the situation changes in small Fock spaces when the weight decreases slowly. In [8, Theorem 2.8, Theorem 2.10], Borichev and Lyubarskii showed that the spaces \mathcal{F}_φ^2 with $\varphi(z) = \varphi_\beta(z) = (\log^+ |z|)^{\beta+1}$, $0 < \beta \leq 1$, possess Riesz bases of normalized reproducing kernels at real points. Another proof was given by Baranov, Belov and Borichev in [3, Theorem 1.2] for the case where $\varphi(z) = O(\log^+ |z|)^2$ under some regularity conditions on φ . In this case, the Fock spaces \mathcal{F}_φ^2 can be viewed as *de Branges spaces* [3, Theorem 1.2] see also [4]. A more general approach based on the boundedness and invertibility of a discrete Hilbert transform was given by Belov, Mengestie and Seip in [6]. In a recent work, Baranov, Dumond, Hartmann and Kellay [5] gave a characterization of complete interpolating sequences (or Riesz bases) in such spaces, when $\varphi(z) = (\log^+ |z|)^2$, in the spirit of Ingham-Kadets 1/4-th theorem in Paley-Wiener space (see [17] for the case of \mathcal{F}_φ^p , $1 \leq p < \infty$). If the weight growth more rapidly than $(\log^+ |z|)^2$,

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then the associated Fock type space has no Riesz bases of (normalized) reproducing kernels, see [3, 7, 8, 18, 20] and the references therein. Also, for the case when the corresponding Riesz measure of the weight φ is doubling, Marco, Massaneda and Ortega-Cerdà in [13] proved that \mathcal{F}_φ^2 possesses no Riesz bases of normalized reproducing kernels. Finally, it follows from [10] that there is no criterion for the existence of Riesz bases in a Fock-type space \mathcal{F}_φ^2 consisting of reproducing kernels in terms of growth of the weight function φ only.

The main question we deal with in this paper is a question asked in [5]. It consists to characterize Riesz bases of normalized reproducing kernels for $\mathcal{F}_{\varphi^\beta}^2$, where $0 < \beta < 1$. Indeed, we obtain a complete description of such bases. Our obtained results are analogue to the well known Avdonin's theorem [1] proved for Riesz bases of complex exponentials for the Paley-Wiener spaces (see [22, Theorem 14, p. 178] and [2, p. 102] for more general statement) and those obtained for $\mathcal{F}_{\varphi_1}^2$ (see [5, Theorem 1.1]). We give also a characterization of complete interpolating sequences in $\mathcal{F}_{\varphi^\beta}^\infty$ for $0 < \beta < 1$, the case when $\beta = 1$ was studied in [5, Theorem 1.2].

1.1. Description of Riesz bases of \mathcal{F}_φ^2 . Let φ be an increasing function defined on $[0, \infty)$ tending to infinity. We extend φ into the whole complex plane \mathbb{C} by $\varphi(z) = \varphi(|z|)$ and we associate a *Fock type space*

$$\mathcal{F}_\varphi^2 := \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{\varphi,2}^2 := \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dm(z) < \infty \right\}.$$

Here m stands for the area Lebesgue measure. Since point evaluations are bounded linear functionals, \mathcal{F}_φ^2 , by Riesz Theorem the space \mathcal{F}_φ^2 endowed with the inner product

$$\langle f, g \rangle_\varphi := \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-2\varphi(z)} dm(z), \quad f, g \in \mathcal{F}_\varphi^2$$

is a reproducing kernel Hilbert space (RKHS). Let k_z be its reproducing kernel at $z \in \mathbb{C}$, that is

$$f(z) = \langle f, k_z \rangle_\varphi, \quad f \in \mathcal{F}_\varphi^2, \quad f \in \mathcal{F}_\varphi^2.$$

We denote by $\mathbb{k}_z = k_z / \|k_z\|_{\varphi,2}$ the normalized reproducing kernel of \mathcal{F}_φ^2 . Let Γ be a sequence of complex numbers. A system of normalized reproducing kernels $\mathcal{K}_\Gamma = \{\mathbb{k}_\gamma\}_{\gamma \in \Gamma}$ is said to be a *Riesz basis* for \mathcal{F}_φ^2 if \mathcal{K}_Γ is *complete* and if for some constant $C \geq 1$ and for each finite sequence $\{a_\gamma\}$, we have

$$\frac{1}{C} \sum_{\gamma \in \Gamma} |a_\gamma|^2 \leq \left\| \sum_{\gamma \in \Gamma} a_\gamma \mathbb{k}_\gamma \right\|_{\varphi,2}^2 \leq C \sum_{\gamma \in \Gamma} |a_\gamma|^2,$$

This means that Γ is a *complete interpolating sequence* for \mathcal{F}_φ^2 . In an equivalent way, the operator

$$T_\Gamma : \mathcal{F}_\varphi^2 \longrightarrow \ell^2(\Gamma) \\ f \longmapsto (\langle f, \mathbb{k}_\gamma \rangle)_{\gamma \in \Gamma}$$

is bounded and invertible from \mathcal{F}_φ^2 to $\ell^2(\Gamma)$.

A given sequence $\{\mu_n\}$ of real numbers is said to be *separated* whenever there exists a constant $\delta > 0$ such that

$$\inf_{n \neq m} |\mu_n - \mu_m| > \delta.$$

Our first main result in this paper is the following

Theorem 1.1. *Let $\varphi(r) = (\log^+ r)^{\beta+1}$ where $0 < \beta < 1$, and let*

$$\Gamma := \left\{ \gamma_n := \exp\left(\frac{1+n}{1+\beta}\right)^{\frac{1}{\beta}} e^{\delta_n} e^{i\theta_n}, \quad \theta_n \in \mathbb{R} : n \geq 0 \right\}, \quad (1.1)$$

where $(\delta_n)_n$ is a real sequence and such that $|\gamma_n| \leq |\gamma_{n+1}|$. Then, $\mathcal{K}_\Gamma = \{\mathbb{k}_\gamma\}_{\gamma \in \Gamma}$ is a Riesz basis for \mathcal{F}_φ^2 if and only if the following three conditions hold:

- (1) $\{\log |\gamma|, \gamma \in \Gamma\}$ is separated,
- (2) $\left(\delta_n / (1+n)^{\frac{1}{\beta}-1}\right)_{n \geq 0}$ is a bounded sequence of real numbers,
- (3) there exists $N \geq 1$ such that

$$\limsup_n \frac{1}{(1+n+N)^{\frac{1}{\beta}} - (1+n)^{\frac{1}{\beta}}} \left| \sum_{k=n+1}^{n+N} \delta_k \right| < \frac{1}{2(1+\beta)^{\frac{1}{\beta}}}. \quad (1.2)$$

1.2. Complete interpolating sequences in uniform Fock spaces $\mathcal{F}_\varphi^\infty$. In this subsection we consider the uniform Fock type space, associated with φ , given as follows

$$\mathcal{F}_\varphi^\infty := \left\{ f \in \text{Hol}(\mathbb{C}) : \|f\|_{\varphi, \infty} := \sup_{z \in \mathbb{C}} |f(z)| e^{-\varphi(z)} < \infty \right\}.$$

$\mathcal{F}_\varphi^\infty$ endowed with the norm $\|\cdot\|_{\varphi, \infty}$ is clearly a Banach space. Let $\Gamma = \{\gamma_n\}$ be a sequence of \mathbb{C} . Γ is said to be *sampling* for $\mathcal{F}_\varphi^\infty$ whenever there exists $C \geq 1$ such that

$$\|f\|_{\varphi, \infty} \leq C \|f\|_{\varphi, \infty, \Gamma},$$

for every $f \in \mathcal{F}_\varphi^\infty$, where

$$\|f\|_{\varphi, \infty, \Gamma} := \sup_{\gamma \in \Gamma} |f(\gamma)| e^{-\varphi(\gamma)}.$$

It is called an *interpolating set* for $\mathcal{F}_\varphi^\infty$ if for every sequence $v = (v_\gamma)_{\gamma \in \Gamma}$ such that $\|v\|_{\varphi, \infty, \Gamma} < \infty$ there exists $f \in \mathcal{F}_\varphi^\infty$ that satisfies $f(\gamma) = v_\gamma$, for every $\gamma \in \Gamma$. Finally, Γ is said to constitute a *complete interpolating sequence* for $\mathcal{F}_\varphi^\infty$ if Γ is simultaneously a sampling and interpolating set for $\mathcal{F}_\varphi^\infty$.

The following result describes complete interpolating sequences for $\mathcal{F}_\varphi^\infty$, the same result was obtained in [5] for the case where $\beta = 1$.

Theorem 1.2. *Let $\varphi(r) = (\log^+ r)^{\beta+1}$ where $0 < \beta < 1$. Let Γ be a sequence of complex numbers and let $\gamma^* \in \mathbb{C} \setminus \Gamma$. Then $\Gamma \cup \{\gamma^*\}$ is a complete interpolating set for $\mathcal{F}_\varphi^\infty$ if and only if \mathcal{K}_Γ is a Riesz basis for \mathcal{F}_φ^2 .*

1.3. **Remarks.** Several remarks are in order before we turn to the proofs of our theorems:

1. Complete interpolating sequences for \mathcal{F}_φ^p , $p = 2, \infty$ are characterized by comparing them to the sequence

$$\Lambda = \left\{ \lambda_n := \exp \left(\frac{1+n}{1+\beta} \right)^{\frac{1}{\beta}} e^{i\theta_n}, \quad \theta_n \in \mathbb{R} : n \geq 0 \right\}, \quad (1.3)$$

which is a complete interpolation sequence for \mathcal{F}_φ^2 and $\Lambda \cup \{\lambda^*\}$ is a complete interpolation sequence for $\mathcal{F}_\varphi^\infty$ where $\lambda^* \in \mathbb{C} \setminus \{\lambda^*\}$ (for $p = 2$ see [3, Theorem 1.2] and Lemma 2.6 and for the case $p = \infty$ see Lemma 2.7).

2. The diagonal asymptotic estimates of the reproducing kernel play an important role in the study of Riesz Bases. Let $\rho(z) := (\Delta\varphi(z))^{-1/2} = |z| (\log |z|)^{\frac{1-\beta}{2}}$. As stated in Corollary 2.3, the kernel admits the following estimate

$$\frac{e^{2\varphi(z)}}{\rho(z)^2} \lesssim \|k_z\|_{\varphi,2}^2 \lesssim \frac{e^{2\varphi(z)}}{|z|\rho(z)}, \quad |z| > 1.$$

Furthermore the kernel attains the extreme values above (see Lemma 2.2 for the precise diagonal asymptotic estimates).

3. The case where $\beta = 1$ was shown in [5, Theorem 1.1] under conditions (1), (2) and there exists $N \geq 1$ such that

$$\sup_n \frac{1}{N} \left| \sum_{k=n+1}^{n+N} \delta_k \right| < \frac{1}{4}. \quad (1.4)$$

The family $\mathcal{K}_\Lambda = \{k_\lambda\}_{\lambda \in \Lambda}$ (associated with $\beta = 1$) is a Riesz basis for $\mathcal{F}_{\varphi_1}^2$. The condition (1.2) may appear different to (1.4) in the case $\beta = 1$, however the two conditions are equivalent for the problem studied here (see Remark 5.1).

4. The condition (1.2) can be written as

$$\Delta_N := \limsup_n \frac{1}{\log |\lambda_{n+N}| - \log |\lambda_n|} \left| \sum_{k=n+1}^{n+N} \delta_k \right| < \frac{1}{2}. \quad (1.5)$$

And for $N = 1$, the condition (1.2) is equivalent to

$$\sup_{n \geq 1} \frac{|\log |\gamma_n| - \log |\lambda_n||}{n^{\frac{1}{\beta}-1}} = \sup_{n \geq 1} \frac{|\delta_n|}{n^{\frac{1}{\beta}-1}} < \frac{1}{2\beta(1+\beta)^{\frac{1}{\beta}}}, \quad (1.6)$$

(see Remark 5.1). The conditions (1.5) and (1.6) seem similar to some well-known results. Indeed, the well known 1/4 Kadets-Ingham's theorem [11] obtained for Riesz bases of exponentials for the Paley-Wiener spaces PW_α^2 , the results showed by Marzo and Seip in [14] for polynomials. For arbitrary N , (1.5) is a condition in the same spirit of Avdonin's condition [1]. It is interesting to note that contrary to the situation in the Paley-Wiener

spaces, conditions (1)-(3) describe Riesz bases of reproducing kernels for \mathcal{F}_φ^2 rather than in the Paley-Wiener spaces, they are just sufficient.

5. We notice again that our spaces possess Riesz bases of normalized reproducing kernels at real points, they can be viewed as de Branges spaces [4, 3]. Using the boundedness and invertibility problems of a discrete Hilbert transform on lacunary sequences, Belov, Mengestie and Seip gave another characterization of Riesz bases, where our summability condition (1.2) correspond to Muckenhoupt-type condition, for further details we refer to [6, Theorem 1.1]. Our approach consists of using Bari's theorem on Riesz bases in Hilbert space, as in the proof of [5, Theorem 1.1].

The plan of our paper is the following. In the next section we state diagonal asymptotic estimates of the reproducing kernel. We show Λ and $\Lambda \cup \{\lambda^*\}$ are complete interpolating sequences for \mathcal{F}_φ^2 and $\mathcal{F}_\varphi^\infty$, respectively. And we give the separation condition. Section 3 is devoted to prove Theorem 1.1. The proof of Theorem 1.2 will be presented in Section 4. We finish our paper by some remarks in the last section.

Throughout the paper, we use the following notations:

- $A \lesssim B$ means that there is an absolute constant C such that $A \leq CB$.
- $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$ hold.

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2. KEY LEMMAS AND PRELIMINARIES RESULTS

In this section we prove some preliminaries results and some key lemmas. In fact, we prove estimates of the reproducing kernel at the diagonal. We get also necessities estimates of some infinite products and also preliminary result about the separation condition. Throughout the rest of this paper we denote $\varphi(r) = \varphi_\beta(r) = (\log^+ r)^{\beta+1}$, where $0 < \beta < 1$.

2.1. Estimate of the norm of the reproducing kernel. We begin by the following lemma which was proved in [3, Lemma 3.1] and which will be used repeatedly throughout the paper.

Lemma 2.1. *For every nonnegative integer n we have*

$$\|z^n\|_{\varphi,2}^2 \asymp \left(\frac{1+n}{1+\beta}\right)^{\frac{1-\beta}{2\beta}} \exp\left(2\beta\left(\frac{1+n}{1+\beta}\right)^{\frac{1+\beta}{\beta}}\right).$$

In the following Lemma, we provide an asymptotic estimate of the reproducing kernel k_z of \mathcal{F}_φ^2 on the diagonal. Note that such estimate was obtained for the points $\lambda \in \Lambda$ by Baranov, Belov and Borichev in [3, Lemma 3.2]. They proved directly the estimate by using Hardy's convexity theorem. Here we deal with the Laplace's method to cover the rest of complex numbers $z \in \mathbb{C}$. We denote by $[x]$ the integer part of a real x .

Lemma 2.2. Let $|z| = e^s$, $n_z = [(1 + \beta)s^\beta]$ and define $g_s(t) = st - \beta\left(\frac{t}{1+\beta}\right)^{\frac{1+\beta}{\beta}}$. Let $\tilde{\varphi}(z) = \max(g_s(n_z), g_s(n_z + 1))$. We have

$$\|k_z\|_{\varphi,2}^2 \asymp \frac{e^{2\tilde{\varphi}(z)}}{|z|\rho(z)} + \frac{e^{2\varphi(z)}}{\rho^2(z)}, \quad |z| > 1$$

where $\rho(z) := (\Delta\varphi(z))^{-1/2} = |z|(\log|z|)^{\frac{1-\beta}{2}}$.

Proof. Let $|z| = e^s$, note that $g'_s(t) = 0$ for $t = (1 + \beta)s^\beta$ and $\tilde{\varphi}(z) \leq \varphi(z)$. Since $\|k_z\|_{\varphi,2}^2 = \sum_{n \geq 0} |z|^{2n} / \|z^n\|_{\varphi,2}^2$, by Lemma 2.1 we have

$$\begin{aligned} e^{2s} \|k_{e^s}\|_{\varphi,2}^2 &\asymp \sum_{n \geq 1} \frac{e^{2g_s(n)}}{(1+n)^{\frac{1-\beta}{2\beta}}} \\ &\lesssim \int_0^{n_z} \frac{e^{2g_s(t)}}{(1+t)^{\frac{1-\beta}{2\beta}}} dt + \frac{e^{2g_s(n_z)}}{(1+n_z)^{\frac{1-\beta}{2\beta}}} + \frac{e^{2g_s(n_z+1)}}{(2+n_z)^{\frac{1-\beta}{2\beta}}} + \int_{n_z+1}^\infty \frac{e^{2g_s(t)}}{(1+t)^{\frac{1-\beta}{2\beta}}} dt \\ &\lesssim \frac{e^{2\tilde{\varphi}(z)}}{s^{\frac{1-\beta}{2}}} + \int_0^\infty \frac{e^{2s^{1+\beta} \left[x - \beta \left(\frac{x}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \right]}}{s^{\frac{1-\beta}{2}} (1+x)^{\frac{1-\beta}{2\beta}}} s^\beta dx := \frac{e^{2\tilde{\varphi}(z)}}{s^{\frac{1-\beta}{2}}} + \frac{s^\beta}{s^{\frac{1-\beta}{2}}} \int_0^\infty e^{-2s^{1+\beta}\psi(x)} f(x) dx \end{aligned}$$

where $\psi(x) = \beta\left(\frac{x}{1+\beta}\right)^{\frac{1+\beta}{\beta}} - x$ and $f(x) = 1/(1+x)^{\frac{1-\beta}{2\beta}}$. We have $\psi'(x) = \left(\frac{x}{1+\beta}\right)^{\frac{1}{\beta}} - 1$. Let $x_0 = 1 + \beta$, we have $\psi'(x_0) = 0$ and $\psi(x_0) = -1$. Since $\psi''(x) = \frac{x^{\frac{1}{\beta}-1}}{\beta(1+\beta)^{\frac{1}{\beta}}} > 0$, $x \geq 1$, we get by Laplace's method [9]

$$\int_0^\infty e^{-S\psi(x)} f(x) dx \sim \sqrt{\frac{2\pi}{S\psi''(x_0)}} f(x_0) e^{-S\psi(x_0)}, \quad S \rightarrow +\infty.$$

For $S = 2s^{1+\beta} \rightarrow +\infty$, we get

$$\|k_z\|_{\varphi,2}^2 = \|k_{e^s}\|_{\varphi,2}^2 \lesssim \frac{e^{2\tilde{\varphi}(z)}}{e^{2s}s^{\frac{1-\beta}{2}}} + \frac{e^{2s^{1+\beta}}}{e^{2s}s^{1-\beta}} = \frac{e^{2\tilde{\varphi}(z)}}{|z|^2(\log|z|)^{\frac{1-\beta}{2}}} + \frac{e^{2\varphi(z)}}{|z|^2(\log|z|)^{1-\beta}}.$$

On the other hand, using Laplace's method again, we obtain

$$\begin{aligned} \mathcal{I} &:= \int_0^{1+\beta} e^{2s^{1+\beta} \left[x - \beta \left(\frac{x}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \right]} dx \sim \sqrt{\frac{2\pi}{2s^{1+\beta}\psi''(x_0)}} e^{2s^{1+\beta}} \\ &= \sqrt{\frac{\pi\beta(1+\beta)}{s^{1+\beta}}} e^{2s^{1+\beta}}. \end{aligned}$$

Also, we have

$$\begin{aligned}
\mathcal{J} &:= \int_{1+\beta-2/s^\beta}^{1+\beta} e^{2s^{1+\beta} \left[x - \beta \left(\frac{x}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \right]} dx \\
&= \int_0^{2/s^\beta} \exp \left(2s^{1+\beta} \left[(1+\beta-u) - \beta \left(1 - \frac{u}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \right] \right) du \\
&= \int_0^{2/s^\beta} \exp \left(2s^{1+\beta} \left[1 - \frac{1}{2\beta(\beta+1)} u^2 \right] \right) du \\
&\leq e^{2s^{1+\beta}} \int_0^\infty e^{-\frac{s^{1+\beta} u^2}{\beta(\beta+1)}} du \\
&= \frac{1}{2} \sqrt{\frac{\beta(\beta+1)\pi}{s^{1+\beta}}} e^{2s^{1+\beta}} \sim \frac{1}{2} \mathcal{I}.
\end{aligned}$$

So, we get

$$\begin{aligned}
e^{2s} \|k_{e^s}\|_{\varphi,2}^2 &\gtrsim \int_0^{n_z-1} \frac{e^{2g_s(t)}}{(1+t)^{\frac{1-\beta}{2\beta}}} dt + \frac{e^{2g_s(n_z)}}{(1+n_z)^{\frac{1-\beta}{2\beta}}} + \frac{e^{2g_s(n_z+1)}}{(2+n_z)^{\frac{1-\beta}{2\beta}}} \\
&\gtrsim \frac{e^{2\tilde{\varphi}(z)}}{s^{\frac{1-\beta}{2}}} + \frac{s^\beta}{s^{\frac{1-\beta}{2}}} \int_0^{1+\beta-2/s^\beta} \frac{e^{2s^{1+\beta} \left[x - \beta \left(\frac{x}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \right]}}{(1+x)^{\frac{1-\beta}{2\beta}}} dx \\
&= \frac{e^{2\tilde{\varphi}(z)}}{s^{\frac{1-\beta}{2}}} + \frac{s^\beta}{s^{\frac{1-\beta}{2}}} (\mathcal{I} - \mathcal{J}) \\
&\asymp \frac{e^{2\tilde{\varphi}(z)}}{s^{\frac{1-\beta}{2}}} + \frac{s^\beta}{s^{\frac{1-\beta}{2}}} \frac{e^{2s^{1+\beta}}}{\sqrt{s^{1+\beta}}}.
\end{aligned}$$

Therefore,

$$\|k_z\|_{\varphi,2}^2 \gtrsim \frac{e^{2\tilde{\varphi}(z)}}{|z|^2 (\log |z|)^{\frac{1-\beta}{2}}} + \frac{e^{2\varphi(z)}}{|z|^2 (\log |z|)^{1-\beta}}, \quad |z| > 1.$$

The proof now is complete. □

We have the following corollary, the identity (2.2) was obtained in [3, Lemma 3.2]

Corollary 2.3. *Let $z \in \mathbb{C}$ be fixed, we have*

$$\frac{e^{2\varphi(z)}}{\rho(z)^2} \lesssim \|k_z\|_{\varphi,2}^2 \lesssim \frac{e^{2\varphi(z)}}{|z|\rho(z)}, \quad |z| > 1. \tag{2.1}$$

Furthermore, for every $\lambda_n \in \Lambda$, where Λ is the sequence given in (1.3), we have

$$\|k_{\lambda_n}\|_{\varphi,2}^2 \asymp \frac{e^{2\varphi(\lambda_n)}}{|\lambda_n|\rho(\lambda_n)} \asymp |e_n(\lambda_n)|^2, \tag{2.2}$$

where $e_n(z) = z^n / \|z^n\|_{\varphi,2}$. Also, if $\sigma_n = \exp\left(\frac{n+1/2}{1+\beta}\right)^{\frac{1}{\beta}}$, $n \geq 2$, then

$$\|k_{\sigma_n}\|_{\varphi,2}^2 \asymp \frac{e^{2\varphi(\sigma_n)}}{\rho(\sigma_n)^2}. \quad (2.3)$$

Proof. The estimate (2.1) follows directly from the last lemma. (2.2) was proved in [3, Lemma 3.2]. We will prove (2.3). Following the notations of Lemma 2.2, let $|z| = e^s$ we set $n_z := [(1+\beta)s^\beta]$ and $g_s(t) = st - \beta\left(\frac{t}{1+\beta}\right)^{\frac{\beta+1}{\beta}}$. From Lemma 2.2 we have

$$\|k_z\|_{\varphi,2}^2 \asymp \frac{1}{|z|^2(\log|z|)^{\frac{1-\beta}{2}}} \left[e^{2\max\{g_s(n_z), g_s(n_z+1)\}} + e^{2\varphi(z) - 2\frac{1-\beta}{4}\log\log|z|} \right].$$

We write $n_z = (1+\beta)s^\beta + \delta_z$, for some $\delta_z \in (-1, 0]$. Simple calculations show that for $j \in \{0, 1\}$ we have

$$\begin{aligned} g_s(n_z + j) - \varphi(z) + \frac{1-\beta}{4}\log s &= s((1+\beta)s^\beta + \delta_z + j) \\ &\quad - \beta\left(\frac{(1+\beta)s^\beta + \delta_z + j}{1+\beta}\right)^{\frac{\beta+1}{\beta}} - s^{\beta+1} + \frac{1-\beta}{4}\log s \\ &= \frac{1-\beta}{4}\log s - \frac{1+o(1)}{2\beta(1+\beta)}(\delta_z + j)^2 s^{1-\beta} \ll 0 \end{aligned}$$

if and only if

$$\frac{\beta(1-\beta^2)}{2+o(1)} \frac{\log s}{s^{1-\beta}} \ll (\delta_z + j)^2.$$

Now, if $z = \sigma_n$ we then have $\delta_z^2 = (\delta_z + 1)^2 = \frac{1}{4}$ and hence the last identity holds. Thus (2.3) follows immediately. \square

Remark. For some radial weight that satisfies $(\log r)^2 = O(\varphi(r))$, when $r \rightarrow \infty$, the reproducing kernel of \mathcal{F}_φ^2 possesses the following estimate $k_z(z) \asymp \Delta\varphi(z)e^{2\varphi(z)}$ (see [3, 7, 8]). This estimate remains valid for the kernel of \mathcal{F}_φ^2 when the associated Riesz measure of the weight φ is doubling [13, Lemma 21]. The estimate in (2.2) prove that the case $\beta \neq 1$ is quite different to the case when $\beta = 1$. Indeed, in the case $\beta = 1$ the reproducing kernel admits the estimate $\|k_z\|_{\varphi_1,2}^2 \asymp \Delta\varphi_1(z)e^{2\varphi_1(z)}$, for every $z \in \mathbb{C}$. However, in (2.2) we have for every $\lambda \in \Lambda$ that $\|k_\lambda\|_{\varphi_\beta,2}^2 \asymp e^{2\varphi_\beta(\lambda)} / \left(|\lambda|^2 \left(\log(\lambda)\right)^{\frac{1-\beta}{2}}\right) \gg \Delta\varphi_\beta(\lambda)e^{2\varphi_\beta(\lambda)}$, whenever $\beta \neq 1$. This can maybe explain the difference in the results obtained about Riesz bases of normalized reproducing kernels.

2.2. **Complete interpolating sequences in \mathcal{F}_φ^p , $p = 2, \infty$.** With a given sequence Γ of complex numbers, we associate the infinite product

$$G_\Gamma(z) = \prod_{\gamma \in \Gamma} \left(1 - \frac{z}{\gamma}\right), \quad z \in \mathbb{C},$$

whenever it converges. In what follows, we denote by $\text{dist}(z, \Gamma)$ the Euclidean distance between z and the sequence Γ .

The following lemma describes estimates of the infinite product associated with a sequence Γ given as in (1.1). Recall that

$$\Delta_N := \limsup_n \frac{1}{\log |\lambda_{n+N}| - \log |\lambda_n|} \left| \sum_{k=n+1}^{n+N} \delta_k \right|.$$

Lemma 2.4. *Let $\Gamma = \{\gamma_n\}_n$ be a sequence given by (1.1). We write $\gamma_n = \lambda_n e^{\delta_n} e^{i\theta_n}$, where $\delta_n, \theta_n \in \mathbb{R}$, and suppose that $|\gamma_n| \leq |\gamma_{n+1}|$ and $|\delta_n| = O((1+n)^{\frac{1}{\beta}-1})$. The infinite product G_Γ converges on every compact set of \mathbb{C} and for small $\varepsilon > 0$ there exists a positive constant C such that*

$$\frac{1}{C} \frac{\text{dist}(z, \Gamma)}{(1+|z|)^{\frac{3}{2} + \Delta_N + \varepsilon}} \leq |G_\Gamma(z)| e^{-\varphi(z)} \leq C \frac{\text{dist}(z, \Gamma)}{(1+|z|)^{\frac{3}{2} - \Delta_N - \varepsilon}}, \quad z \in \mathbb{C}.$$

Proof. Let $z \in \mathbb{C}$ such that $|z| = e^t = \exp\left(\frac{s}{1+\beta}\right)^{\frac{1}{\beta}}$. Let also m be the integer such that $|\gamma_{m-1}| < |z| < |\gamma_m|$ and suppose that $\text{dist}(z, \Gamma) = |z - \gamma_{m-1}|$. Therefore we have

$$\begin{aligned} \log |G_\Gamma(z)| &= \sum_{0 \leq n < m-1} \log \left|1 - \frac{z}{\gamma_n}\right| + \log \left|1 - \frac{z}{\gamma_{m-1}}\right| + O(1) \\ &= \sum_{n=0}^{m-1} \log \left|\frac{z}{\gamma_n}\right| + \log \text{dist}(z, \Gamma) - t + O(1) \\ &= mt - \sum_{n=1}^m \left(\frac{k}{1+\beta}\right)^{\frac{1}{\beta}} - \sum_{n=1}^{m-1} \delta_k + \log \text{dist}(z, \Gamma) - t + O(1), \end{aligned}$$

however,

$$\sum_{k=1}^m k^\alpha = \frac{m^{\alpha+1}}{\alpha+1} + \frac{m^\alpha}{2} (1 + o(1)). \quad (2.4)$$

Now for $N \in \mathbb{N}$ there exist p and r two positive integers such that $m = pN + r$ and $0 \leq r < N$. Let $u_k := \log |\lambda_k|$, we have

$$\begin{aligned} \left| \sum_{k=1}^m \delta_k \right| &\leq \sum_{j=1}^{p-1} \left| \sum_{k=1}^N \delta_{jN+k} \right| + \left| \sum_{k=pN+1}^{pN+r} \delta_k \right| \\ &\leq \left(\Delta_N + \frac{\varepsilon}{2} \right) \sum_{j=0}^{p-1} (u_{jN+N} - u_{jN}) + O(m^{\frac{1}{\beta}-1}) \\ &\leq \left(\Delta_N + \frac{\varepsilon}{2} \right) t + o(t). \end{aligned}$$

Consequently,

$$\begin{aligned} \log |G_\Gamma(z)| &\geq mt - \beta \left(\frac{m}{1+\beta} \right)^{\frac{1}{\beta}+1} - \left(\frac{3}{2} \mp \Delta_N \mp \frac{\varepsilon}{2} \right) \left(\frac{m}{1+\beta} \right)^{\frac{1}{\beta}} + \log \text{dist}(z, \Gamma) + o(t) \\ &\geq t^{\beta+1} - \left(\frac{3}{2} \mp \Delta_N \mp \frac{\varepsilon}{2} \right) t + \log \text{dist}(z, \Gamma) + o(t). \end{aligned}$$

Finally, there exist A and B two positive constants depend on ε such that

$$A \frac{\text{dist}(z, \Lambda)}{(1+|z|)^{\frac{3}{2}+\Delta_N+\varepsilon}} \leq |G_\Gamma(z)| e^{-\varphi(z)} \leq B \frac{\text{dist}(z, \Lambda)}{(1+|z|)^{\frac{3}{2}-\Delta_N-\varepsilon}}, \quad z \in \mathbb{C}.$$

□

We need the following classical ingredient, for which we single out as a lemma

Lemma 2.5. *Let Γ be a sequence given by (1.1). We have*

$$\int_{\mathbb{C}} \frac{\text{dist}(z, \Gamma)^2}{(1+|z|)^{3+\alpha}} dm(z) < \infty \iff \alpha > 1.$$

Proof. Note that, for every $\frac{1}{2}(|\gamma_{n-1}| + |\gamma_n|) \leq r \leq \frac{1}{2}(|\gamma_n| + |\gamma_{n+1}|)$, we have $\text{dist}(re^{i\theta}, \Gamma) \asymp |re^{i\theta} - \gamma_n|$. So, we get

$$\int_{\mathbb{C}} \frac{\text{dist}(z, \Gamma)^2}{(1+|z|)^{3+\alpha}} dm(z) \asymp \sum_{n \geq 0} \int_{\frac{1}{2}(|\gamma_{n-1}|+|\gamma_n|)}^{\frac{1}{2}(|\gamma_n|+|\gamma_{n+1}|)} \int_0^{2\pi} \frac{|re^{i\theta} - \gamma_n|^2}{(1+r)^{3+\alpha}} r d\theta dr \asymp \sum_{n \geq 0} \frac{1}{|\gamma_n|^{\alpha-1}}.$$

Therefore, the last sum is finite if and only if $\alpha > 1$.

□

In the following lemma we improve a result of Borichev and Lyubarskii [8, Theorem 2.8]. They proved that $\{\mathbb{k}_{\tilde{\lambda}_n}\}_{n \geq 0}$ is a Riesz basis for \mathcal{F}_φ^2 where $\tilde{\lambda}_0 = 0$, $|\tilde{\lambda}_n| = \exp[(w_{n+1} - w_{n-1})/4]$ and

$$w_n := \log \|z^n\|^2 = c(n+1)^{1+\frac{1}{\beta}} + O(\log n).$$

Their proof was based on the Bari's theorem (see [15, section A.5.7.1]). Thanks to the estimate of the moments $\|z^n\|_{\varphi, 2}$ given in Lemma 2.1, we provide in the following result another sequence of same kind as that of [8]. The provided sequence will be viewed as a

reference set of complete interpolating sequences for \mathcal{F}_φ^2 and will play a crucial tool in the proof of the main theorems. Another proof of this lemma was given by Baranov-Belov-Borichev in [3, Theorem 1.2] by using the results of Belov-Mengestie-Seip [6]. However, for the sake of completeness, we give here the proof, we proceed here by Bari's theorem as in [8].

Lemma 2.6. *Let Λ be the sequence given by (1.3). Then $\mathcal{K}_\Lambda = \{\mathbb{k}_\lambda\}_{\lambda \in \Lambda}$ is a Riesz basis for \mathcal{F}_φ^2 .*

Proof. Let $n \geq 0$ and $e_n(z) = z^n / \|z^n\|_{\varphi,2}$ and $h_n := e^{-in\theta_n} e_n$. We have

$$\begin{aligned} \|h_n - \mathbb{k}_{\lambda_n}\|_{\varphi,2}^2 &= \left\| e^{-in\theta_n} e_n - \frac{\overline{e_n(\lambda_n)}}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}} e_n - \sum_{k \neq n} \frac{\overline{e_k(\lambda_n)}}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}} e_k \right\|_{\varphi,2}^2 \\ &= \underbrace{\left| 1 - \frac{|e_n(\lambda_n)|}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}} \right|^2}_{J_1} + \underbrace{\sum_{k \neq n} \frac{|e_k(\lambda_n)|^2}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}^2}}_{J_2}. \end{aligned}$$

However,

$$J_1 = \left| 1 - \frac{|e_n(\lambda_n)|}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}} \right|^2 \leq 1 - \frac{|e_n(\lambda_n)|^2}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}^2} = J_2.$$

Consequently,

$$\|h_n - \mathbb{k}_{\lambda_n}\|_{\varphi,2}^2 \asymp \sum_{k \neq n} \frac{|e_k(\lambda_n)|^2}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}^2} \lesssim \sum_{k \neq n} \left| \frac{e_k(\lambda_n)}{e_n(\lambda_n)} \right|^2.$$

On the other hand and for $k \neq n$ we have

$$\frac{|e_n(\lambda_n)|^2}{|e_k(\lambda_n)|^2} \asymp \left(\frac{1+k}{1+n} \right)^{\frac{1-\beta}{2\beta}} e^{2c(n,k)},$$

where

$$c(n,k) := (n-k) \left(\frac{1+n}{1+\beta} \right)^{\frac{1}{\beta}} + \beta \left[\left(\frac{1+k}{1+\beta} \right)^{\frac{1+\beta}{\beta}} - \left(\frac{1+n}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \right].$$

Therefore, by simple computations we get

$$c(n,k) \geq (1+\beta)^{-\frac{1+\beta}{\beta}} |k-n|^2 (1+n)^{\frac{1}{\beta}-1}.$$

Thus,

$$\begin{aligned} \sum_{n=0}^{\infty} \|h_n - \mathbb{k}_{\lambda_n}\|_{\varphi,2}^2 &\asymp \sum_{n=0}^{\infty} \sum_{k \neq n} \frac{|e_k(\lambda_n)|^2}{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}^2} \\ &\lesssim \sum_{n=0}^{\infty} \sum_{k \neq n} \left(\frac{1+n}{1+k} \right)^{\frac{1-\beta}{2\beta}} e^{-2(1+\beta)^{-\frac{1+\beta}{\beta}} |k-n|^2 (1+n)^{\frac{1}{\beta}-1}} < \infty. \end{aligned}$$

Note that $\Lambda = \Gamma$ with $\delta_k = 0$. So Lemmas 2.4 and 2.5 imply that Λ is a uniqueness set for \mathcal{F}_φ^2 , that is the unique function in \mathcal{F}_φ^2 that vanishes in Λ is the zero function. Also, for any $\lambda \in \Lambda$ the sequence $\Lambda \setminus \{\lambda\}$ is a zero sequence of \mathcal{F}_φ^2 and hence the system \mathcal{K}_Λ is complete and minimal in \mathcal{F}_φ^2 and $\sum_n \|h_n - \mathbb{k}_{\lambda_n}\|_{\varphi,2}^2 < \infty$. Bari's theorem [15] ensures that \mathcal{K}_Λ is a Riesz basis for \mathcal{F}_φ^2 . □

We have the following lemma corresponding to $\mathcal{F}_\varphi^\infty$

Lemma 2.7. *Let $\lambda^* \in \mathbb{C} \setminus \Lambda$. Then $\Lambda \cup \{\lambda^*\}$ is a complete interpolating sequence for $\mathcal{F}_\varphi^\infty$.*

Proof. First let us prove that $\tilde{\Lambda} = \Lambda \cup \{\lambda^*\}$ is a uniqueness set for $\mathcal{F}_\varphi^\infty$. Indeed, suppose that $\lambda^* \neq 0$ and take $f \in \mathcal{F}_\varphi^\infty$ vanishes in $\Lambda \cup \{\lambda^*\}$, so $f(z) = (1 - z/\lambda^*)h(z)G_\Lambda(z)$ where h is an entire function. By Lemma 2.4 we get

$$|h(z)| \frac{\text{dist}(z, \tilde{\Lambda})}{(1 + |z|)^{1/2+\varepsilon}} \lesssim |f(z)|e^{-\varphi(z)} \lesssim 1, \quad z \in \mathbb{C}.$$

It follows from this that h is the zero function. It remains to show that $\tilde{\Lambda}$ is an interpolating set for $\mathcal{F}_\varphi^\infty$. For this purpose let $v = (v_\lambda)_{\lambda \in \tilde{\Lambda}}$ be a sequence of complex numbers such that $\|v\|_{\varphi, \infty, \tilde{\Lambda}} < \infty$. Put $F(z) = (1 - z/\lambda^*)G_\Lambda(z)$ and consider the function

$$F_v(z) = \sum_{\lambda \in \tilde{\Lambda}} v_\lambda \frac{F(z)}{F'(\lambda)(z - \lambda)}, \quad z \in \mathbb{C}.$$

We have

$$\begin{aligned} |F_v(z)| &\leq \sum_{\lambda \in \tilde{\Lambda}} |v_\lambda| \left| \frac{F(z)}{F'(\lambda)(z - \lambda)} \right| \\ &\lesssim \|v\|_{\varphi, \infty, \tilde{\Lambda}} \sum_{\lambda \in \tilde{\Lambda}} e^{\varphi(\lambda)} \frac{|z|}{|\lambda|} \left| \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z - \lambda)} \right|. \end{aligned}$$

Let $z \in \mathbb{C}$ and $p \in \mathbb{N}$ such that $|\lambda_{p-1}| \leq |z| < |\lambda_p|$. Suppose $0 \neq |\lambda^*| < |\lambda_1|$ and write $\tilde{\Lambda} = (\lambda_n)_{n \geq -1}$ where $\lambda_{-1} = \lambda^*$. Let $|z| = e^t$ and $u_n = \log |\lambda_n|$. We have

$$\sum_{\lambda \in \tilde{\Lambda}} e^{\varphi(\lambda)} \frac{|z|}{|\lambda|} \left| \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z - \lambda)} \right| \asymp \underbrace{\sum_{n \leq p-1} \dots}_{\mathcal{I}_1} + \underbrace{\sum_{n \geq p} e^{\varphi(\lambda_n)} \frac{\text{dist}(z, \Lambda)}{|z - \lambda_n|} \frac{|z|^p}{|\lambda_n|^n} \prod_{k=0}^{p-1} \frac{1}{|\lambda_k|} \prod_{k=0}^{n-1} |\lambda_k|}_{\mathcal{I}_2}.$$

By (2.4)

$$\begin{aligned}
\mathcal{I}_1 &= \sum_{n \leq p-1} e^{\varphi(\lambda_n)} \frac{\text{dist}(z, \Lambda)}{|z - \lambda_n|} \frac{|z|^p}{|\lambda_n|^n} \prod_{k=0}^{p-1} \frac{1}{|\lambda_k|} \prod_{k=0}^{n-1} |\lambda_k| \\
&\leq |z|^p \sum_{n \leq p-1} \exp \left[u_n^{\beta+1} - (n+1)u_n + \sum_{k=0}^n u_k - \sum_{k=0}^{p-1} u_k \right] \\
&= e^{pt - \beta u_{p-1}^{\beta+1}} \sum_{n \leq p-1} \exp \left[\left(\frac{u_n}{2} + d_\beta u_n^{1-\beta} \right) - \left(\frac{u_{p-1}}{2} + d_\beta u_{p-1}^{1-\beta} \right) \right] \\
&\leq e^{pt - \beta u_{p-1}^{\beta+1}} \sum_{n \geq 0} e^{-c|u_{p-1} - u_n|} \\
&\lesssim \exp \left(\varphi(z) - \left(\frac{1}{2\beta(\beta+1)} + o(1) \right) (p-s)^2 t^{1-\beta} \right) \leq e^{\varphi(z)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_2 &\asymp |z|^{p+1} \sum_{n \geq p} \exp \left[u_n^{\beta+1} - (1+n)u_n + \sum_{k=p}^{n-1} u_k \right] \\
&\leq e^{(p+1)t - \beta u_p^{\beta+1}} \sum_{n \geq 0} \exp \left[-\frac{1}{2} |u_n - u_p| + d_\beta |u_n^{1-\beta} - u_p^{1-\beta}| \right] \\
&\lesssim e^{(p+1)t - \beta u_p^{\beta+1}} \leq e^{\varphi(z)}.
\end{aligned}$$

So F_v the interpolating function belongs to $\mathcal{F}_\varphi^\infty$. This ends the proof. \square

2.3. log-separated sequences. We next set

$$d(z, w) = \frac{|z - w|}{1 + \min(|z|, |w|)}, \quad z, w \in \mathbb{C}.$$

The sequence $\Sigma \subset \mathbb{C}$ is said to be *d-separated* if there is $d_\Sigma > 0$ such that

$$\inf \{d(\sigma, \sigma^*) : \sigma, \sigma^* \in \Sigma, \sigma \neq \sigma^*\} \geq d_\Sigma.$$

We denote by $\mathbb{D}(z, r)$ the euclidean disc of radius r centered at z . For $\delta < 1$ and $0 \neq z \in \mathbb{C}$ the ball corresponding to the distance d is given by

$$D_d(z, \delta) := \{w \in \mathbb{C} : d(z, w) < \delta\}.$$

Hence $D_d(z, \delta)$ is comparable to $\mathbb{D}(z, c_\delta |z|)$ with a suitable constant c_δ depending on δ . It follows from this that Σ is *d-separated* if and only if there exists $c > 0$ such that the euclidean disks $\mathbb{D}(\sigma, c|\sigma|)$, $\sigma \in \Sigma$, are disjoint and this is equivalent to the fact that the sequence $\{\log |\sigma| : \sigma \in \Sigma\}$ is separated.

The following lemma was proved in [5] in the case where $\beta = 1$ by using Bernstein type inequality. The proof in the case $0 < \beta < 1$ is different from [5, Lemma 2.6].

Lemma 2.8. *Let Γ be a sequence given by (1.1). Then*

$$\sum_{\gamma \in \Gamma} |\langle f, \mathbb{k}_\gamma \rangle|^2 \leq C(\Gamma) \|f\|_{\varphi,2}^2, \quad (2.5)$$

for every $f \in \mathcal{F}_\varphi^2$ if and only if $\{\log |\gamma| : \gamma \in \Gamma\}$ is a finite union of separated sequences.

Proof. Let f be a function of \mathcal{F}_φ^2 . According to Lemma 2.6, the system \mathcal{K}_Λ is a Riesz basis for \mathcal{F}_φ^2 and hence

$$f(z) = \sum_{\lambda \in \Lambda} \langle f, \mathbb{k}_\lambda \rangle g_\lambda(z), \quad z \in \mathbb{C},$$

where $\mathcal{G}_\Lambda := \{g_\lambda : \lambda \in \Lambda\}$ is the unique biorthogonal system of \mathcal{K}_Λ and it is given by

$$g_\lambda(z) = \frac{\|\mathbb{k}_\lambda\|}{G'_\Lambda(\lambda)} \frac{G_\Lambda(z)}{z - \lambda}, \quad z \in \mathbb{C}, \quad (2.6)$$

where G_Λ is the infinite product associated with Λ . We then get

$$\begin{aligned} \sum_{m \geq 0} \left| \langle f, \mathbb{k}_{\gamma_m} \rangle \right|^2 &= \sum_{m \geq 0} \left| \left\langle \sum_{n \geq 0} \langle f, \mathbb{k}_{\lambda_n} \rangle g_{\lambda_n}, \mathbb{k}_{\gamma_m} \right\rangle \right|^2 \\ &= \sum_{m \geq 0} \left| \sum_{n \geq 0} \langle f, \mathbb{k}_{\lambda_n} \rangle \langle g_{\lambda_n}, \mathbb{k}_{\gamma_m} \rangle \right|^2. \end{aligned}$$

Since the operator $f \mapsto (\langle f, \mathbb{k}_{\lambda_n} \rangle)$ is an isomorphism between \mathcal{F}_φ^2 and ℓ^2 , then (2.5) is equivalent to the fact that the matrix $\mathcal{C} = (C_{n,m})_{n,m}$ defines a bounded operator in ℓ^2 , where

$$|C_{n,m}| = |\langle g_{\lambda_n}, \mathbb{k}_{\gamma_m} \rangle| = \frac{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}}{\|\mathbb{k}_{\gamma_m}\|_{\varphi,2}} \left| \frac{G_\Lambda(\gamma_m)}{G'_\Lambda(\lambda_n)(\lambda_n - \gamma_m)} \right|$$

Denote first by $p_m = [(1 + \beta)(\log |\gamma_m|)^\beta] - 1$. We have

$$|G_\Lambda(\gamma_m)| \asymp \frac{\text{dist}(\gamma_m, \Lambda)}{\gamma_m} \prod_{k=0}^{p_m} \left| \frac{\gamma_m}{\lambda_k} \right|$$

and

$$|G'_\Lambda(\lambda_n)| \asymp \frac{1}{|\lambda_n|} \prod_{k=0}^n \left| \frac{\lambda_n}{\lambda_k} \right|$$

Hence

$$|C_{n,m}| \asymp \frac{\|\mathbb{k}_{\lambda_n}\|_{\varphi,2}}{\|\mathbb{k}_{\gamma_m}\|_{\varphi,2}} \frac{\text{dist}(\gamma_m, \Lambda)}{|\lambda_n - \gamma_m|} \frac{|\gamma_m|^{p_m}}{|\lambda_n|^n} \prod_{k=0}^{p_m} \frac{1}{|\lambda_k|} \prod_{k=0}^n |\lambda_k|. \quad (2.7)$$

By (2.2), we have

$$\|\mathbb{k}_{\lambda_n}\|_{\varphi,2} \asymp |e_n(\lambda_n)| \quad \text{and} \quad \|\mathbb{k}_{\gamma_m}\|_{\varphi,2} \geq |e_{p_m}(\gamma_m)|$$

and we get

$$\begin{aligned} |C_{n,m}| &\lesssim \frac{\|z^{p_m}\|_{\varphi,2} \operatorname{dist}(\gamma_m, \Lambda)}{\|z^n\|_{\varphi,2} |\lambda_n - \gamma_m|} \prod_{k=0}^{p_m} \frac{1}{|\lambda_k|} \prod_{k=0}^n |\lambda_k| \\ &\asymp \left(\frac{1+p_m}{1+n} \right)^{\frac{1-\beta}{4\beta}} \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_n - \gamma_m|} e^{-\alpha(n,m)} \end{aligned}$$

where

$$\begin{aligned} \alpha(n,m) &= \beta \left(\frac{1+n}{1+\beta} \right)^{\frac{1+\beta}{\beta}} - \beta \left(\frac{1+p_m}{1+\beta} \right)^{\frac{1+\beta}{\beta}} + \sum_{k=0}^{p_m} \left(\frac{1+k}{1+\beta} \right)^{\frac{1}{\beta}} - \sum_{k=0}^n \left(\frac{1+k}{1+\beta} \right)^{\frac{1}{\beta}} \\ &= \frac{1}{2} \left[\left(\frac{1+p_m}{1+\beta} \right)^{\frac{1}{\beta}} - \left(\frac{1+n}{1+\beta} \right)^{\frac{1}{\beta}} \right] + \frac{1+o(1)}{12\beta(1+\beta)} \left[\left(\frac{1+p_m}{1+\beta} \right)^{\frac{1-\beta}{\beta}} - \left(\frac{1+n}{1+\beta} \right)^{\frac{1-\beta}{\beta}} \right]. \end{aligned}$$

Now, if $|\gamma_m| \geq |\lambda_n|$, then we have $\operatorname{dist}(\gamma_m, \Lambda) \leq |\lambda_n - \gamma_m|$, and hence

$$|C_{n,m}| \lesssim e^{-c_\beta \left| (1+p_m)^{\frac{1}{\beta}} - (1+n)^{\frac{1}{\beta}} \right|}. \quad (2.8)$$

If $|\gamma_m| < |\lambda_n|$, then $\operatorname{dist}(\gamma_m, \Lambda) \leq |\lambda_{p_m}|$ (or $|\gamma_m|$) and $|\lambda_n - \gamma_m| \asymp |\lambda_n|$. Again we have (2.8). So, we get in any case that we have (2.8).

On another hand, by Corollary 2.3 and by (2.7), we get

$$\begin{aligned} |C_{n,m}| &\gtrsim \left(\frac{\log |\gamma_m|}{\log |\lambda_n|} \right)^{\frac{1-\beta}{4}} e^{\varphi(\lambda_n) - \varphi(\gamma_m)} \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_n - \gamma_m|} \frac{|\gamma_m|^{p_m+1}}{|\lambda_n|^{n+1}} \prod_{k=0}^{p_m} \frac{1}{|\lambda_k|} \prod_{k=0}^n |\lambda_k| \\ &\asymp \left(\frac{1+p_m}{1+n} \right)^{\frac{1-\beta}{4\beta}} \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_n - \gamma_m|} e^{-\tilde{\alpha}(n,m)}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\alpha}(n,m) &= \left(\frac{1+p_m}{1+\beta} \right)^{\frac{1+\beta}{\beta}} - \left(\frac{1+n}{1+\beta} \right)^{\frac{1+\beta}{\beta}} \\ &\quad + (1+n) \left(\frac{1+n}{1+\beta} \right)^{\frac{1}{\beta}} - (1+p_m) \left(\frac{1+p_m}{1+\beta} \right)^{\frac{1+\beta}{\beta}} + \sum_{k=0}^{p_m} \left(\frac{1+k}{1+\beta} \right)^{\frac{1}{\beta}} - \sum_{k=0}^n \left(\frac{1+k}{1+\beta} \right)^{\frac{1}{\beta}} \\ &= \frac{1}{2} \left[\left(\frac{1+p_m}{1+\beta} \right)^{\frac{1}{\beta}} - \left(\frac{1+n}{1+\beta} \right)^{\frac{1}{\beta}} \right] + \frac{1+o(1)}{12\beta(1+\beta)} \left[\left(\frac{1+p_m}{1+\beta} \right)^{\frac{1-\beta}{\beta}} - \left(\frac{1+n}{1+\beta} \right)^{\frac{1-\beta}{\beta}} \right]. \end{aligned}$$

Again and by the same way as before we get

$$|C_{n,m}| \gtrsim e^{-\tilde{c}_\beta \left| (1+p_m)^{\frac{1}{\beta}} - (1+n)^{\frac{1}{\beta}} \right|}.$$

Finally,

$$e^{-\tilde{c}_\beta \left| (1+p_m)^{\frac{1}{\beta}} - (1+n)^{\frac{1}{\beta}} \right|} \lesssim |C_{n,m}| \lesssim e^{-c_\beta \left| (1+p_m)^{\frac{1}{\beta}} - (1+n)^{\frac{1}{\beta}} \right|}.$$

If $\{\log |\gamma| : \gamma \in \Gamma\}$ is a finite union of separated sequences, then $C = (C_{n,m})$ is bounded in ℓ^2 .

In the opposite direction, since \mathcal{K}_Λ is a Riesz basis, let g_λ be the function given in (2.6), $\|g_\lambda\|_{\varphi,2}^2 \asymp 1$ and as before we have

$$\begin{aligned} 1 &\gtrsim \sum_{\gamma \in \Gamma} \frac{|g_\lambda(\gamma)|^2}{k_\gamma(\gamma)} \\ &\gtrsim \sum_{\gamma \in \Gamma \cap D_d(\lambda, \delta)} \frac{|g_\lambda(\gamma)|^2}{k_\gamma(\gamma)} \\ &\gtrsim \sum_{\gamma \in \Gamma \cap D_d(\lambda, \delta)} e^{-c_\beta |\log |\gamma| - \log |\lambda||} \\ &\asymp \text{Card}(\Gamma \cap D_d(\lambda, \delta)). \end{aligned}$$

Since $\text{dist}(\gamma, \Lambda) \asymp |\gamma - \lambda|$, we get

$$\sup_{\lambda \in \Lambda} \text{Card}(\Gamma \cap D_d(\lambda, \delta)) < \infty.$$

Hence Γ is a finite union of d -separated sequence. □

3. PROOF OF THEOREM 1.1

Proof. " \Leftarrow "

First of all, since the sequence $\{\log |\gamma_n|\}$ is separated, then by Lemma 2.8 the operator T_Γ is bounded from \mathcal{F}_φ^2 to ℓ^2 . On the other hand, the sequence Γ is a uniqueness set for \mathcal{F}_φ^2 , and for any $\gamma \in \Gamma$, the sequence $\Gamma \setminus \{\gamma\}$ is a zero set of \mathcal{F}_φ^2 . Indeed, let $f \in \mathcal{F}_\varphi^2$ such that $f|_\Gamma = 0$, then by the Hadamard's factorization theorem [12] we can write $f = hG_\Gamma$, for an entire function h . Since $f \in \mathcal{F}_\varphi^2$, by Lemma 2.3 we have

$$|f(z)| \leq \|f\|_{\varphi,2} \|k_z\|_{\varphi,2} \lesssim \frac{e^{\varphi(z)}}{|z|(\log |z|)^{\frac{1-\beta}{4}}}, \quad |z| > 1.$$

Lemma 2.4 implies that

$$|h(z)| \frac{\text{dist}(z, \Gamma)}{(1 + |z|)^{3/2+c_N}} \lesssim |f(z)| e^{-\varphi(z)} \lesssim \frac{1}{|z|(\log |z|)^{\frac{1-\beta}{4}}}, \quad |z| > 1$$

where $c_N = \Delta_N + \varepsilon$, for some ε small enough so that $c_N < 1/2$. Then h must be a polynomial. Combining this fact with Lemma 2.5, we conclude that h must be identically zero, and hence Γ is a set of uniqueness for \mathcal{F}_φ^2 , therefore T_Γ is injective. Also since $T_\Gamma g_n = e_{n+1}$, the $(n+1)$ -th element of the canonical basis of ℓ^2 , for every $n \geq 0$, where

$$g_n(z) = \frac{\|k_{\gamma_n}\|_{\varphi,2} G_\Gamma(z)}{G'_\Gamma(\gamma_n) z - \gamma_n}, \quad z \in \mathbb{C}$$

this implies that the range of T_Γ is dense in ℓ^2 . In order to prove that the operator T_Γ is onto, we associate with each $a \in \ell^2$ the function as follows

$$H_a(z) = \sum_{\gamma \in \Gamma} a_\gamma \frac{\|k_\gamma\|_{\varphi,2} G_\Gamma(z)}{G'_\Gamma(\gamma) z - \gamma}, \quad z \in \mathbb{C}.$$

The estimate of the reproducing kernel given in Corollary 2.3 and the estimate of G_Γ proved in Lemma 2.4 ensure that the above series converges uniformly on every compact set of \mathbb{C} . It remains now to show that $H_a \in \mathcal{F}_\varphi^2$. Indeed, since \mathcal{K}_Λ is a Riesz basis for \mathcal{F}_φ^2 (see Lemma 2.6), we have

$$\begin{aligned} \|H_a\|_{\varphi,2}^2 &\asymp \sum_{m \geq 0} |\langle H_a, k_{\lambda_m} \rangle|^2 \\ &= \sum_{m \geq 0} \left| \sum_{n \geq 0} a_n \frac{G_\Gamma(\lambda_m)}{G'_\Gamma(\gamma_n)(\lambda_m - \gamma_n)} \frac{\|k_{\gamma_n}\|_{\varphi,2}}{\|k_{\lambda_m}\|_{\varphi,2}} \right|^2. \end{aligned}$$

Hence $\{H_a\}_{a \in \ell^2} \subset \mathcal{F}_\varphi^2$ if and only if the matrix $A = (A_{n,m})_{n,m \geq 0}$ defines a bounded operator in ℓ^2 , where

$$|A_{n,m}| = \left| \frac{G_\Gamma(\lambda_m)}{G'_\Gamma(\gamma_n)(\lambda_m - \gamma_n)} \frac{\|k_{\gamma_n}\|_{\varphi,2}}{\|k_{\lambda_m}\|_{\varphi,2}} \right|$$

Recall that for every $z \in \mathbb{C}$ there exists $p \in \mathbb{N}$ such that $\gamma_{p-1} \leq |z| < |\gamma_p|$, we have

$$|G_\Gamma(z)| \asymp \frac{\text{dist}(z, \Gamma)}{|z|} \prod_{k=0}^{p-1} |z/\gamma_k|$$

and also

$$|G'_\Gamma(\gamma_n)| \asymp \frac{1}{|\gamma_n|} \prod_{k=0}^{n-1} \left| \frac{\gamma_n}{\gamma_k} \right|.$$

Since $\delta_n = O((1+n)^{\frac{1}{\beta}-1})$, there exists $M > 0$ such that for every $m \geq 0$ there exists $|i| \leq M$ such that the correspond p to λ_m is $m+i$. Consequently,

$$|A_{n,m}| \asymp \frac{\|k_{\gamma_n}\|_{\varphi,2}}{\|k_{\lambda_m}\|_{\varphi,2}} \frac{\text{dist}(\lambda_m, \Gamma)}{|\lambda_m - \gamma_n|} \frac{|\gamma_n|}{\lambda_m} \prod_{k=0}^{m+i-1} \left| \frac{\lambda_m}{\gamma_k} \right| \prod_{k=0}^{n-1} \left| \frac{\gamma_k}{\gamma_n} \right|.$$

By Lemma 2.2 we have

$$\|k_z\|_{\varphi,2}^2 \asymp |e_{n_z-1}(z)|^2 + |e_{n_z}(z)|^2 + \frac{e^{2\varphi(z)}}{\rho(z)^2},$$

where $n_z = [(1+\beta) \log^\beta |z|]$. Again, since $\delta_n = O((1+n)^{\frac{1}{\beta}-1})$, for every $n \geq 0$ there exists $|j| \leq M$ satisfying

$$\|k_{\gamma_n}\|_{\varphi,2}^2 \asymp |e_{n+j}(\gamma_n)|^2 + \frac{e^{2\varphi(\gamma_n)}}{|\gamma_n|^2 (\log |\gamma_n|)^{1-\beta}}.$$

Also for every $m \geq 0$ we have

$$\|k_{\lambda_m}\|_{\varphi,2} \asymp |e_m(\lambda_m)| \asymp \frac{e^{\varphi(\lambda_m)}}{|\lambda_m| (\log |\lambda_m|)^{\frac{1-\beta}{4}}}.$$

From these estimates we can write $|A_{n,m}| \asymp I_{n,m} + J_{n,m}$, where

$$\begin{aligned} I_{n,m} &\asymp \frac{|e_{n+j}(\gamma_n)|}{|e_m(\lambda_m)|} \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} \frac{|\gamma_n|}{|\lambda_m|} \prod_{k=0}^{m+i-1} \left| \frac{\lambda_m}{\gamma_k} \right| \prod_{k=0}^{n-1} \left| \frac{\gamma_k}{\gamma_n} \right| \\ &=: \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} e^{\Theta(n,m)} \end{aligned}$$

and

$$\begin{aligned} J_{n,m} &\asymp \frac{(\log |\lambda_m|)^{\frac{1-\beta}{4}}}{(\log |\gamma_n|)^{\frac{1-\beta}{2}}} e^{\varphi(\gamma_n) - \varphi(\lambda_m)} \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} \prod_{k=0}^{m+i-1} \left| \frac{\lambda_m}{\gamma_k} \right| \prod_{k=0}^{n-1} \left| \frac{\gamma_k}{\gamma_n} \right| \\ &=: \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} e^{\theta(n,m)}. \end{aligned}$$

To estimate the coefficients $I_{n,m}$ and $J_{n,m}$, we first write

$$K_{n,m} = \prod_{k=0}^{m+i-1} \left| \frac{\lambda_m}{\gamma_k} \right| \prod_{k=0}^{n-1} \left| \frac{\gamma_k}{\gamma_n} \right|.$$

We next put $u_k := \log |\lambda_k|$ for every $k \geq 0$. We have

$$\begin{aligned} \log K_{n,m} &= (m+i)u_m - n(u_n + \delta_n) - \sum_{k=0}^{m+i-1} u_k + \sum_{k=0}^{n-1} u_k - \sum_{k=0}^{m+i-1} \delta_k + \sum_{k=0}^{n-1} \delta_k \\ &= \frac{u_{m+i} - u_n}{2} + [\beta u_n^{\beta+1} - n(u_n + \delta_n)] - [\beta u_{m+i}^{\beta+1} - (m+i)u_m] \\ &\quad + d_\beta (u_n^{1-\beta} - u_{m+i}^{1-\beta}) - \sum_{k=0}^{m+i-1} \delta_k + \sum_{k=0}^{n-1} \delta_k \end{aligned}$$

Hence we have

$$\begin{aligned} \Theta(n,m) &= \log K_{n,m} + \log |e_{n+j}(\gamma_n)| - \log e_m(\lambda_m) + (u_n + \delta_n) - u_m \\ &= \frac{u_n - u_{m+i}}{2} + \left(\frac{1}{2\beta(\beta+1)} + o(1) \right) (j^2 u_n^{1-\beta} - i^2 u_{m+i}^{1-\beta}) + d_\beta (u_n^{1-\beta} - u_{m+i}^{1-\beta}) \\ &\quad - \sum_{k=0}^{m+i-1} \delta_k + \sum_{k=0}^n \delta_k + \frac{1-\beta}{4} \log \frac{1+m}{1+n}. \end{aligned}$$

Now if $n \leq m + i$, then

$$I_{n,m} \asymp \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} e^{\Theta(n,m)} \asymp e^{\Theta(n,m)}$$

and

$$\Theta(n,m) = - \left(\frac{1}{2} + o(1) \right) (u_{m+i} - u_n) - \sum_{k=n+1}^{m+i-1} \delta_k.$$

If $n \geq m + i$, then

$$I_{n,m} \asymp \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} e^{\Theta(n,m)} \asymp e^{\Theta(n,m) + u_m - u_n - \delta_n}$$

and

$$\Theta(n,m) + u_m - u_n - \delta_n = - \left(\frac{1}{2} + o(1) \right) (u_n - u_{m+i}) + \sum_{k=m+i}^{n-1} \delta_k.$$

And hence

$$I_{n,m} \asymp \exp \left[- \left(\frac{1}{2} + o(1) \right) |u_n - u_{m+i}| \pm \sum_{k=m+i}^{n-1} \delta_k \right].$$

Similarly, we have

$$\begin{aligned} \theta(n,m) &= \log K_{n,m} + \varphi(\gamma_n) - \varphi(\lambda_m) + \frac{1-\beta}{4} \log \frac{1+m}{(1+n)^2} \\ &= \frac{u_n - u_{m+i}}{2} + \left(\frac{1}{2\beta(\beta+1)} + o(1) \right) \left(j^2 u_n^{1-\beta} - i^2 u_{m+i}^{1-\beta} \right) + \frac{1-\beta}{4} \log \frac{1+m}{(1+n)^2} \\ &\quad + d_\beta \left(u_n^{1-\beta} - u_{m+i}^{1-\beta} \right) - \sum_{k=0}^{m+i-1} \delta_k + \sum_{k=0}^n \delta_k. \end{aligned}$$

Now if $n \leq m + i$, we have

$$J_{n,m} \asymp \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} e^{\theta(n,m)} \asymp e^{\theta(n,m)},$$

and

$$\theta(n,m) = \exp \left[- \left(\frac{1}{2} + o(1) \right) (u_{m+i} - u_n) - \sum_{k=n+1}^{m+i-1} \delta_k \right].$$

If $n \geq m + i$, then

$$J_{n,m} \asymp \frac{|\lambda_m|}{|\lambda_m - \gamma_n|} e^{\theta(n,m)} \asymp e^{\theta(n,m) + u_m - u_n - \delta_n}$$

and

$$\theta(n,m) + u_m - u_n - \delta_n = - \left(\frac{1}{2} + o(1) \right) (u_n - u_{m+i}) + \sum_{k=m+i}^{n-1} \delta_k.$$

Hence,

$$J_{n,m} \asymp \exp \left[- \left(\frac{1}{2} + o(1) \right) |u_n - u_{m+i}| \pm \sum_{k=m+i}^{n-1} \delta_k \right].$$

Consequently,

$$|A_{n,m}| \asymp I_{n,m} + J_{n,m} \asymp \exp \left[- \left(\frac{1}{2} + o(1) \right) |u_n - u_{m+i}| \pm \sum_{k=m+i}^{n-1} \delta_k \right].$$

Recall that

$$\Delta_N := \limsup_n \frac{1}{u_{n+N} - u_n} \left| \sum_{k=n+1}^{n+M} \delta_k \right| < \frac{1}{2}.$$

Which implies that for a very small ε (chosen in such a way that $\Delta_N + \varepsilon < 1/2$) and for every $n \geq 0$ we have

$$\left| \sum_{k=n+1}^{n+M} \delta_k \right| \leq \left(\Delta_N + \frac{\varepsilon}{2} \right) (u_{n+N} - u_n).$$

Thus,

$$|A_{n,m}| \lesssim \exp \left[- \left(\frac{1}{2} - \left(\Delta_N + \frac{\varepsilon}{2} \right) + o(1) \right) |u_n - u_{m+i}| \right].$$

Since $\Delta_N < \frac{1}{2}$, the matrix $A = (A_{n,m})$ defines a bounded operator in ℓ^2 and hence \mathcal{K}_Γ is a Riesz basis for \mathcal{F}_φ^2 .

" \implies "

Proof of (1). Let $\varphi(r) = \varphi_\beta(r) = (\log^+ r)^{1+\beta}$, $0 < \beta \leq 1$. If \mathcal{K}_Γ is a Riesz basis for $\mathcal{F}_{\varphi_\beta}^2$ then Γ is an interpolation sequence for $\mathcal{F}_{\varphi_\beta}^2$ and hence for $\mathcal{F}_{\varphi_1}^2$. By [5, Corollary 2.3] $\{\log \gamma_n\}_{n \geq 0}$ is separated.

Proof of (2). Let $\Gamma = \{\gamma_n\}$ be a sequence of complex numbers such that \mathcal{K}_Γ is a Riesz basis for \mathcal{F}_φ^2 . Then for every $\gamma \in \Gamma$ there exists a unique function $f_\gamma \in \mathcal{F}_\varphi^2$ that satisfies the interpolation problem :

$$\langle f_\gamma, \mathbb{k}_\gamma \rangle = 1 \quad \text{and} \quad \langle f_\gamma, \mathbb{k}_{\gamma'} \rangle = 0, \quad \gamma' \in \Gamma \setminus \{\gamma\}.$$

Consequently, $\Gamma \setminus \{\gamma\}$ is a subset of the zero set of the function f_γ . Since Γ (is complete) is a uniqueness set, then $\Gamma \setminus \{\gamma\}$ is exactly the zero set of f_γ . By the Hadamard's factorization theorem we have $f_\gamma(z) = c \frac{G_\Gamma(z)}{G'_\Gamma(\gamma)(z-\gamma)}$, for some constant $c \in \mathbb{C}$. Since, $\langle f_\gamma, \mathbb{k}_\gamma \rangle = 1$, we then get $c = \|k_\gamma\|_{\varphi,2}$. Therefore,

$$f_\gamma(z) = \|k_\gamma\|_{\varphi,2} \frac{G_\Gamma(z)}{G'_\Gamma(\gamma)(z-\gamma)}, \quad z \in \mathbb{C}.$$

Also $\|f_\gamma\|_{\varphi,2} \asymp 1$. Hence,

$$|f_\gamma(\lambda_m)| \leq \|f_\gamma\|_{\varphi,2} \|k_{\lambda_m}\| \asymp |e_m(\lambda_m)|,$$

for every $\lambda_m \in \Lambda$.

Suppose now that $(\delta_n/(1+n)^{\frac{1}{\beta}-1})$ is unbounded. Without loss of generality we assume the existence of a subsequence $(\delta_{n_k}/(1+n_k)^{\frac{1}{\beta}-1})$ which tends to $+\infty$ (the case convergence to $-\infty$ is similar). Then, for every k there exists m_k such that $|n_k - m_k| \rightarrow \infty$ and γ_{n_k} close to λ_{m_k} . we then obtain

$$|f_{\gamma_{n_k}}(\lambda_{m_k})| \lesssim \|k_{\lambda_{m_k}}\|_{\varphi,2} \asymp |e_{m_k}(\lambda_{m_k})|. \quad (3.1)$$

On the other hand, we have $\|k_{\gamma_{n_k}}\|_{\varphi,2} \geq |e_{m_k}(\gamma_{n_k})|$ and

$$|G_{\Gamma}(\lambda_{m_k})| \asymp \frac{|\lambda_{m_k} - \gamma_{n_k}|}{|\lambda_{m_k}|} \prod_{j=0}^{n_k-1} \left| \frac{\lambda_{m_k}}{\gamma_j} \right|, \quad |G'_{\Gamma}(\gamma_{n_k})| \asymp \frac{1}{|\gamma_{n_k}|} \prod_{j=0}^{n_k-1} \left| \frac{\gamma_{n_k}}{\gamma_j} \right|.$$

We next denote $n := n_k$ and $m := m_k$, if no confusion is possible. The identity (3.1) becomes

$$|e_m(\lambda_m)| \gtrsim |f_{\gamma_n}(\lambda_m)| \gtrsim \frac{|\gamma_n|}{|\lambda_m|} |e_m(\gamma_n)| \prod_{j=0}^{n-1} \left| \frac{\lambda_m}{\gamma_j} \right| \left| \frac{\gamma_j}{\gamma_n} \right|.$$

Hence,

$$\left| \frac{e_m(\lambda_m)}{e_m(\gamma_n)} \right| \gtrsim \left| \frac{\lambda_m}{\gamma_n} \right|^{n-1} \iff \left| \frac{\lambda_m}{\gamma_n} \right|^{n-m-1} \lesssim 1. \quad (3.2)$$

Note that we can suppose that $|\lambda_m| \leq |\gamma_n|$ and hence $m = n + \delta'_n + \delta_{n,m}$, where (δ'_n) is a sequence which tends to infinity $(\delta'_n = \text{Const}(\beta)\delta_n/(1+n)^{\frac{1}{\beta}-1})$ and $(\delta_{n,m})$ bounded negative sequence such that $\delta_{n,m} \leq -1$ (otherwise we replace m by $m - m'$ for a suitable integer m'). Thus

$$\begin{aligned} \log \left| \frac{\lambda_m}{\gamma_n} \right| &= u_m - (u_n + \delta_n) \\ &= b_{\beta} \left((1 + \beta)(u_n + \delta_n)^{\beta} + \delta_{n,m} \right)^{\frac{1}{\beta}} - (u_n + \delta_n) \\ &= \left(\frac{1}{\beta(\beta+1)} + o(1) \right) \delta_{n,m} (u_n + \delta_n)^{1-\beta}. \end{aligned}$$

Since $n - m - 1 = n + \delta'_n - m - \delta'_n - 1 = -\delta_{n,m} - \delta_n - 1$, (3.2) becomes

$$1 \gtrsim -(\delta_n + \delta_{n,m} + 1) \left(\frac{1}{\beta(\beta+1)} + o(1) \right) \delta_{n,m} (u_n + \delta_n)^{1-\beta} \geq c_{\beta} \delta_n u_n^{1-\beta}$$

which is impossible.

Proof of (3). Recall first that the matrix of coefficients

$$|A_{n,m}| = \left| \frac{G_{\Gamma}(\lambda_m)}{G'_{\Gamma}(\gamma_n)(\lambda_m - \gamma_n)} \frac{\|k_{\gamma_n}\|_{\varphi,2}}{\|k_{\lambda_m}\|_{\varphi,2}} \right|$$

defines a bounded operator in ℓ^2 . From the proof of the first part we have

$$|A_{n,m}| \gtrsim \exp \left[-\frac{|u_{m+i} - u_n|}{2} + c_\beta \left(u_n^{1-\beta} - u_{m+i}^{1-\beta} \right) \mp \sum_{k=n+1}^{m+i-1} \delta_k \right].$$

Assume that for every $N \geq 1$, we have

$$\Delta_N := \limsup_n \frac{1}{u_{n+N} - u_n} \left| \sum_{k=n+1}^{n+N} \delta_k \right| = \frac{1}{2} + \varepsilon_N.$$

For every $N \geq 1$ there exists an integer n_N (sufficiently large) such that

$$\left| \sum_{k=n_N+1}^{n_N+N} \delta_k \right| \geq \frac{u_{n_N+N} - u_{n_N}}{2} + \varepsilon_N \frac{u_{n_N+N} - u_{n_N}}{2}.$$

- Assume that there exists a subsequence (N_l) such that $\sum_{k=n_l+1}^{n_l+N_l} \delta_k > 0$ (n_l is the integer n_{N_l}), we then get

$$\begin{aligned} |A_{n_l+N_l, n_l}| &\gtrsim \exp \left[-\frac{|u_{n_l} - u_{n_l+N_l}|}{2} + c_\beta \left(u_{n_l+N_l}^{1-\beta} - u_{n_l}^{1-\beta} \right) + \sum_{k=n_l+1}^{n_l+N_l} \delta_k \right] \\ &\gtrsim \exp \left[c_\beta \left(u_{n_l+N_l}^{1-\beta} - u_{n_l}^{1-\beta} \right) \right]. \end{aligned}$$

Which implies that $(A_{n_l+N_l, n_l})$ is unbounded ($0 < \beta < 1$) and hence the matrix $(A_{n,m})$ cannot represent a bounded operator in ℓ^2 .

- There exists $N_0 \geq 1$ such that for every $N \geq N_0$ we have $\sum_{k=n_l+1}^{n_l+N_l} \delta_k < 0$. For every n the following holds

$$|A_{n, n+N}| \gtrsim \exp \left[-\frac{u_{n+N} - u_n}{2} - c_\beta \left(u_{n+N}^{1-\beta} - u_n^{1-\beta} \right) - \sum_{k=n+1}^{n+N} \delta_k \right].$$

If (ε_N) contains a subsequence ε_N which is bounded below by some $\varepsilon > 0$ then for N in this subsequence we have

$$|A_{n_N, n_N+N}| \gtrsim \exp \left[-c_\beta \left(u_{n_N+N}^{1-\beta} - u_{n_N}^{1-\beta} \right) + \varepsilon \frac{u_{n_N+N} - u_{n_N}}{2} \right].$$

This ensures that (A_{n_N, n_N+N}) tends to infinity. Therefore, the matrix A cannot define a bounded operator in ℓ^2 . Suppose now that (ε_N) converges to zero. Simple computations

yield

$$\begin{aligned}
|A_{n_N, n_N+N}| &\gtrsim \exp \left[-c_\beta \left(u_{n_N+N}^{1-\beta} - u_{n_N}^{1-\beta} \right) + \varepsilon_N \frac{u_{n_N+N} - u_{n_N}}{2} \right] \\
&\gtrsim \exp \left[-c_\beta \left(u_{n_N+N}^{1-\beta} - u_{n_N}^{1-\beta} \right) + \frac{\varepsilon_N \beta(1+n_N)}{2(1-\beta^2)} \left(u_{n_N+N}^{1-\beta} - u_{n_N}^{1-\beta} \right) \right] \\
&= \exp \left[\left(u_{n_N+N}^{1-\beta} - u_{n_N}^{1-\beta} \right) \left(\frac{\varepsilon_N \beta(1+n_N)}{2(1-\beta^2)} - c_\beta \right) \right].
\end{aligned}$$

Note that n_N can be chosen so large, we then choose the integer n_N in such a way that $\varepsilon_N(1+n_N) > \frac{4c_\beta}{\beta}(1-\beta^2)$, ($0 < \beta < 1$). It follows from this that

$$\begin{aligned}
|A_{n_N, n_N+N}| &\gtrsim \exp \left[\left(u_{n_N+N}^{1-\beta} - u_{n_N}^{1-\beta} \right) (2c_\beta - c_\beta) \right] \\
&= \exp \left[c_\beta \left(u_{n_N+N}^{1-\beta} - u_{n_N}^{1-\beta} \right) \right].
\end{aligned}$$

Thus $(A_{n_N, n_N+N})_N$ is unbounded and hence the matrix $A = (A_{n,m})$ cannot define a bounded operator in ℓ^2 . This ends the proof. \square

4. PROOF OF THEOREM 1.2

” \Leftarrow ”

Suppose that \mathcal{K}_Γ is a Riesz basis for \mathcal{F}_φ^2 . Then Γ satisfies conditions (1), (2) and (3) of Theorem 1.1.

First, the sequence $\tilde{\Gamma} = \Gamma \cup \{\gamma^*\}$ is a uniqueness set for $\mathcal{F}_\varphi^\infty$. Indeed, if f is a function from $\mathcal{F}_\varphi^\infty$ that vanishes on $\tilde{\Gamma}$, then $f = (1 - z/\gamma^*)G_\Gamma h$, for an entire function h . Again, the estimates of G_Γ imply

$$|h(z)| \frac{\text{dist}(z, \Gamma)}{(1+|z|)^{1/2+\Delta_N+\varepsilon}} \lesssim |f(z)| e^{-\varphi(z)} \lesssim 1, \quad z \in \mathbb{C}.$$

Hence,

$$|h(z)| \lesssim \frac{(1+|z|)^{1/2+\Delta_N+\varepsilon}}{\text{dist}(z, \Gamma)}, \quad z \in \mathbb{C} \setminus \Gamma.$$

Since $\Delta_N < 1/2$, the function h must be identically zero.

Let us prove that $\tilde{\Gamma} = \{\gamma_n\}_{n \geq -1}$ is an interpolating sequence for $\mathcal{F}_\varphi^\infty$, where $\gamma_{-1} = \gamma^*$. For this, let $v = (v_n)$ be a sequence such that $\|v\|_{\varphi, \infty, \tilde{\Gamma}} < \infty$ and consider the following entire function

$$L_v(z) = \sum_{n \geq -1} v_n \frac{F_\Gamma(z)}{F'_\Gamma(\gamma_n)(z - \gamma_n)}, \quad z \in \mathbb{C}.$$

where $F_\Gamma = (1 - z/\gamma^*)G_\Gamma$. Prove that $L_v \in \mathcal{F}_\varphi^\infty$. According to Lemma 2.7, $\Lambda \cup \{\lambda^*\}$ is a complete interpolating set for $\mathcal{F}_\varphi^\infty$, consequently

$$\|L_v\|_{\varphi, \infty} \asymp \sup_{m \geq -1} e^{-\varphi(\lambda_m)} \left| \sum_{n \geq 0} v_n \frac{F_\Gamma(\lambda_m)}{F'_\Gamma(\gamma_n)(\lambda_m - \gamma_n)} \right| = \sup_{m \geq -1} \left| \sum_{n \geq 0} v_n e^{-\varphi(\gamma_n)} B_{n,m} \right|$$

where

$$\begin{aligned} |B_{n,m}| &= e^{\varphi(\gamma_n) - \varphi(\lambda_m)} \left| \frac{F_\Gamma(\lambda_m)}{F'_\Gamma(\gamma_n)(\lambda_m - \gamma_n)} \right| \\ &\asymp e^{\varphi(\gamma_n) - \varphi(\lambda_m)} \frac{|\lambda_m|}{|\gamma_n|} \left| \frac{G_\Gamma(\lambda_m)}{G'_\Gamma(\gamma_n)(\lambda_m - \gamma_n)} \right| \\ &\asymp |A_{n,m}| e^{o(1)|u_m - u_n|}, \end{aligned}$$

where $u_n = \log |\lambda_n|$. The estimates of the matrix $A = (A_{n,m})$ imply that L_v belongs to $\mathcal{F}_\varphi^\infty$.

” \implies ”

Suppose now that $\Gamma \cup \{\gamma^*\}$ is a complete interpolating set for $\mathcal{F}_\varphi^\infty$ and prove that \mathcal{K}_Γ is a Riesz basis for \mathcal{F}_φ^2 . That is, Γ satisfies conditions (1)-(3) of Theorem 1.1. For that purpose, firstly Γ is d -separated because every interpolating set for $\mathcal{F}_{\varphi_\beta}^\infty$ is also an interpolating sequence for $\mathcal{F}_{\varphi_1}^\infty$. Otherwise, remark that

$$|B_{n,m}| \asymp e^{\varphi(\gamma_n) - \varphi(\lambda_m)} \frac{|\lambda_m|}{|\gamma_n|} \left| \frac{G_\Gamma(\lambda_m)}{G'_\Gamma(\gamma_n)(\lambda_m - \gamma_n)} \right| \gtrsim \frac{(\log \gamma_n)^{\frac{1-\beta}{2}}}{(\log \lambda_n)^{\frac{1-\beta}{4}}} |A_{n,m}|.$$

From the proof of Theorem 1.1. If (δ_n) contains a subsequence noted (δ_{n_k}) such that $(\delta_{n_k}/(1+n_k)^{\frac{1}{\beta}-1})$ is unbounded, then $(A_{n,m})$ is unbounded and hence for $(B_{n,m})$ and this ensures (2). Suppose now that $\Delta_N = \frac{1}{2} + \varepsilon_N$, for a nonnegative sequence (ε_N) . Again as in the proof of Theorem 1.1 the sequence $|A_{n_k+N_k, n_k}| + |A_{n_k, n_k+N_k}|$ is unbounded and hence for the matrix $(B_{n,m})$. This proves (3) and ends the proof.

5. FINAL REMARKS

The following remark shows that the limit superior in condition (3) in Theorem 1.1 can be replaced by a supreme. This result implies that for the case $\beta = 1$ Theorem 1.1 and [5, Theorem 1.1] are equivalent.

Remark 5.1. *Let (δ_n) be a sequence of real numbers such that $(\delta_n/(1+n)^{\frac{1}{\beta}-1}) \in \ell^\infty$. Let N be a positive integer and $\delta = \frac{1}{2\beta(1+\beta)^{\frac{1}{\beta}}}$. The two following conditions are equivalent for the study of Riesz bases of normalized reproducing kernels for \mathcal{F}_φ^2*

$$(1) \sup_n \frac{1}{N(1+n)^{\frac{1}{\beta}-1}} \left| \sum_{k=n+1}^{n+N} \delta_k \right| < \delta,$$

$$(2) \limsup_n \frac{1}{N(1+n)^{\frac{1}{\beta}-1}} \left| \sum_{k=n+1}^{n+N} \delta_k \right| < \delta.$$

Proof. First, it is obvious that

$$\limsup_n \frac{1}{N(1+n)^{\frac{1}{\beta}-1}} \left| \sum_{k=n+1}^{n+N} \delta_k \right| \leq \sup_n \frac{1}{N(1+n)^{\frac{1}{\beta}-1}} \left| \sum_{k=n+1}^{n+N} \delta_k \right|.$$

Conversely, let $\Gamma = \{\gamma_n\}$ be a d -separated sequence of complex numbers ordered in such a way that $|\gamma_n| < |\gamma_{n+1}|$ and write $\gamma_n = \lambda_n e^{\delta_n} e^{i\theta_n}$. Suppose that

$$\Delta_N := \limsup_n \frac{1}{N(1+n)^{\frac{1}{\beta}-1}} \left| \sum_{k=n+1}^{n+N} \delta_k \right| < \delta.$$

Let $\varepsilon \in (0, \delta - \Delta_N)$, there exists $m \geq 0$ such that

$$\sup_{n \geq m} \frac{1}{N(1+n)^{\frac{1}{\beta}-1}} \left| \sum_{k=n+1}^{n+N} \delta_k \right| \leq \Delta_N + \varepsilon < \frac{1}{4}.$$

Thus, $\mathcal{K}_{\tilde{\Gamma}}$ is a Riesz basis for \mathcal{F}_{φ}^2 , where

$$\tilde{\Gamma} := \left\{ e^{\left(\frac{n+1}{1+\beta}\right)^{\frac{1}{\beta}}} : 0 \leq n \leq m-1 \right\} \cup \{\gamma_n \in \Gamma : n \geq m\}.$$

Consequently, \mathcal{K}_{Γ} is also a Riesz basis for \mathcal{F}_{φ}^2 . \square

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