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Analytic digraphs of uncountable Borel chromatic number under injective definable homomorphism

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Abstract. We study the analytic digraphs of uncountable Borel chromatic number on Polish spaces, and compare them with the notion of injective Borel homomorphism. We provide some minimal digraphs incomparable with $\mathbb{G}_0$. We also prove the existence of antichains of size continuum, and that there is no finite basis.

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1 Introduction

In this paper, A, B will be (binary) relations on some sets X, Y respectively. The **diagonal** of X is $\Delta(X) := \{(x,x) \mid x \in X\}$. We will say that A is **irreflexive**, or a **digraph**, if A does not meet the diagonal (some authors call these relations **simple digraphs**). We set $A^{-1} := \{(x,y) \in X^2 \mid (y,x) \in A\}$. We say that A is **symmetric** if $A = A^{-1}$, and A is **antisymmetric** if $A \cap A^{-1} \subseteq \Delta(X)$. The set $s(A) := A \cup A^{-1}$ is the **symmetrization** of A. We say that A is a **graph** if A is irreflexive and symmetric, and A is an **oriented graph** if A is irreflexive and antisymmetric. Recall that A is a **subgraph** of the digraph B if $X \subseteq Y$ and $A \subseteq B$. An A-**path** is a finite sequence $(x_i)_{i \leq n}$ of points of X such that $(x_i, x_{i+1}) \in A$ if $i < n$. We say that A is **connected** if for any $x, y \in X$ there is an A-path $(x_i)_{i \leq n}$ with $x_0 = x$ and $x_n = y$. If A is a graph, then the **connected component** of $x \in X$ is the set $C(x)$ of points $y \in X$ for which there is an A-path $(x_i)_{i \leq n}$ with $x_0 = x$ and $x_n = y$. A graph A is **acyclic** if there is no injective A-path $(x_i)_{i \leq n}$ with $n \geq 2$ and $(x_n, x_0) \in A$. If A is an acyclic graph and $y \in C(x)$, then we denote by $p^A_{x,y}$ the unique injective A-path $(x_i)_{i \leq n}$ with $x_0 = x$ and $x_n = y$. We say that A is **locally countable** if its horizontal and vertical sections are countable.

We write $(X, A) \preceq (Y, B)$ when there is $h : X \to Y$ such that $A \subseteq (h \times h)^{-1}(B)$. If this holds, then we say that h is a **homomorphism** from (X, A) into (Y, B). When h can be injective, we write $(X, A) \preceq^{\text{inj}} (Y, B)$. The notion of injective homomorphism is very natural since it corresponds to the basic notion of subgraph. Indeed, if h is an injective homomorphism from (X, A) into (Y, B), then $(h[X], (h \times h)[A])$ is a subgraph of (Y, B). Conversely, if (X, A) is a subgraph of (Y, B), then the canonical injection is an injective homomorphism from (X, A) into (Y, B). A **coloring** from (X, A) into some set Y is a map $c : X \to Y$ such that $c(x) \neq c(x')$ if $(x, x') \in A$, i.e., a homomorphism from (X, A) into $(Y, \neq)$.

The reader should see [K] for the standard descriptive set theoretic notions and notation. Let $\mathfrak{C}$ be a class of functions between Polish spaces, e.g., continuous, Borel (denoted c, B respectively). If X, Y are Polish spaces and h can be in $\mathfrak{C}$, then we will use the notation $\preceq_{\mathfrak{C}}$, $\preceq^{\text{inj}}_{\mathfrak{C}}$ respectively. The study of definable colorings of analytic graphs was initiated in [K-S-T]. The $\mathfrak{C}$-**chromatic number** of a digraph A on a Polish space X is the smallest cardinality of a Polish space Y for which there is a $\mathfrak{C}$-coloring from (X, A) into Y.

Example. Let $\psi : \omega \to 2^{<\omega}$ be a natural bijection ($\psi(0) = \emptyset$, $\psi(1) = 0$, $\psi(2) = 1$, $\psi(3) = 0^2$, $\psi(4) = 01$, $\psi(5) = 10$, $\psi(6) = 1^2$, ...). A crucial property of ψ is that $\psi^{-1}(s) < \psi^{-1}(s\varepsilon)$ if $s \in 2^{<\omega}$ and $\varepsilon \in 2$.

Note that $|\psi(n)| \leq n$, so that we can define $s_n := \psi(n)0^{n-|\psi(n)|}$. Some crucial properties of (s_n) are that it is **dense** (for each $s \in 2^{<\omega}$, there is n such that $s \subseteq s_n$), and that $|s_n| = n$. We set $\mathbb{G}_0 := \{(s_n 0\gamma, s_n 1\gamma) \mid n \in \omega \wedge \gamma \in 2^\omega\}$. The set $s(\mathbb{G}_0)$ is considered in [K-S-T], where the following is essentially proved.

Theorem 1.1 *(Kechris, Solecki, Todorčević) Let X be a Polish space and A be an analytic digraph on X. Then exactly one of the following holds:*

(a) $(X, A) \preceq_B (\omega, \neq)$ (i.e., (X, A) has countable Borel chromatic number),

(b) $(2^\omega, \mathbb{G}_0) \preceq_c (X, A)$.

Actually, the original statement in [K-S-T] is when A is a graph, with $s(\mathbb{G}_0)$ instead of $\mathbb{G}_0$. But we can get Theorem 1.1 without any change in the proof in [K-S-T].

This result had a lot of developments since. For instance, Miller developed some techniques to recover many dichotomy results of descriptive set theory, without using effective descriptive set theory (see [Mi]). He replaces it with some versions of Theorem 1.1. In [K-S-T], it is conjectured that we can replace $\preceq_c$ with $\preceq_c^{\text{inj}}$ in the version of Theorem 1.1.(b) for graphs (the authors show in Theorem 6.6 that this is the case if A is an acyclic graph or a locally countable graph, just like $s(\mathbb{G}_0)$; their proof also works for digraphs with acyclic symmetrization or locally countable, with $\mathbb{G}_0$ instead of $s(\mathbb{G}_0)$). It is proved in [L4] that this is not the case.

Theorem 1.2 *(Lecomte) There is no pair $(\mathbb{X}_0, \mathbb{A}_0)$, where $\mathbb{X}_0$ is Polish and $\mathbb{A}_0$ is an analytic graph on $\mathbb{X}_0$, such that for every pair (X, A) of the same type, exactly one of the following holds:*

(a) $(X, A) \preceq_B (\omega, \neq)$,

(b) $(\mathbb{X}_0, \mathbb{A}_0) \preceq_c^{inj} (X, A)$.

In other words, there is no one-element basis for $\preceq_c^{\text{inj}}$ among analytic graphs of uncountable Borel chromatic number (recall that if $(Q, \leq)$ is a quasi-ordered space, then a **basis** is a subfamily $\mathcal{F}$ of Q such that any element of Q is $\leq$-above an element of $\mathcal{F}$). This led Kechris and Marks to ask the following in [K-Ma] (see Problem 3.39).

Questions (1) Is there a basis of cardinality $< 2^{\aleph_0}$ for $\preceq_B^{\text{inj}}$ among analytic graphs of uncountable Borel chromatic number?

(2) If not, is there such a basis consisting of a continuum size family of reasonably simple graphs?

Of course, we can ask the same questions with $\preceq_c^{\text{inj}}$ instead of $\preceq_B^{\text{inj}}$, and for digraphs instead of graphs. We are interested in basis as small as possible with respect to the inclusion. In other words, we want our basis to be antichains for the quasi-order we consider (recall that a subfamily $\mathcal{F}$ of Q is an **antichain** if the elements of $\mathcal{F}$ are pairwise $\leq$-incomparable). This leads to the following.

Question (3) Is there a $\preceq_{\mathfrak{C}}^{\text{inj}}$-antichain basis for the class of analytic digraphs of uncountable Borel chromatic number?

In [L-Mi], it is proved that there is neither $\leq_{\mathfrak{C}}$-antichain basis, nor $\sqsubseteq_{\mathfrak{C}}$-antichain basis, for the class of analytic graphs of uncountable Borel chromatic number (these two quasi-orders are defined like $\preceq_{\mathfrak{C}}$ and $\preceq_{\mathfrak{C}}^{\text{inj}}$ respectively, except that "$A \subseteq (h \times h)^{-1}(B)$" is replaced with "$A = (h \times h)^{-1}(B)$"). In particular, (3) may have a negative answer. A first approach for the result in [L-Mi] was the existence of an $\leq_{\mathfrak{C}}$-antichain of size $2^{\aleph_0}$ made of graphs $\leq_{\mathfrak{C}}$-minimal among analytic graphs of uncountable Borel chromatic number, essentially proved in [L4]. This leads to the following.

Question (4) Is there a $\preceq_{\mathfrak{C}}^{\text{inj}}$-antichain of size $2^{\aleph_0}$ made of digraphs $\preceq_{\mathfrak{C}}^{\text{inj}}$-minimal among analytic digraphs of uncountable Borel chromatic number?

The minimal elements of $(Q, \leq)$ are of particular importance since they have to be part of any basis, up to equivalence. The discussion after Theorem 1.1 shows that $\mathbb{G}_0$ is $\preceq_c^{\text{inj}}$ and $\preceq_B^{\text{inj}}$-minimal among analytic digraphs of uncountable Borel chromatic number. The main results in this paper are steps towards a positive answer to Question (4), and our paper is mentioned in the last version of [K-Ma].

Theorem 1.3 *Let $\mathfrak{C} \in \{c, B\}$. There is a $\preceq_{\mathfrak{C}}^{inj}$-antichain $\{\mathbb{G}_0, \mathbb{G}_1, \mathbb{G}_1^{-1}\}$ made of digraphs $\preceq_{\mathfrak{C}}^{inj}$-minimal among analytic digraphs of uncountable Borel chromatic number.*

Thus Theorem 1.2 holds for digraphs. Note that some antichains were already present in [L4]. The main result in the present paper is the minimality of $\mathbb{G}_1$, and thus a dichotomy result. We now provide a construction of the digraph $\mathbb{G}_1$. Note that $\mathbb{G}_0$ can be viewed as the union of the graphs of the partial homeomorphisms $h_n : N_{s_n 0} \to N_{s_n 1}$ given by $h_n(s_n 0\gamma) := s_n 1\gamma$. In particular, the following picture holds if $m < n$ and $h_m\big(h_n(\alpha)\big)$, $h_m(\alpha)$ are defined:

$$\begin{tikzcd} \alpha \arrow[r, "h_m"] \arrow[d, "h_n"'] & h_m(\alpha) \arrow[d, phantom, "\neq"] \\ h_n(\alpha) \arrow[r, "h_m"] & h_m\big(h_n(\alpha)\big) \end{tikzcd}$$

We define $t_n \in 2^{<\omega}$ and maps $g_n : N_{t_n 0} \to N_{t_n 1}$, and $\mathbb{G}_1$ will be the union of the graphs of the g_n's. One of the crucial properties of the g_n's is that $g_m\big(g_n(\alpha)\big) = g_m(\alpha)$ if $m < n$ and $g_m\big(g_n(\alpha)\big)$, $g_m(\alpha)$ are defined. In particular, the following picture holds and violates the previous one:

$$\begin{tikzcd} \alpha \arrow[d, "g_n"'] \arrow[dr, "g_m"] & \\ g_n(\alpha) \arrow[r, "g_m"] & g_m\big(g_n(\alpha)\big) = g_m(\alpha) \end{tikzcd}$$

This provides some cycles, which have to exist in examples orthogonal to $\mathbb{G}_0$, by the discussion after Theorem 1.1. The $\leq_{\mathfrak{C}}$-antichain mentioned before Question (4) was constructed with different configurations of cycles. We believe that some other algebraic conditions of this type could lead to a positive answer to Question (4). Our second main result is a weak version of this.

Theorem 1.4 *Let $\mathfrak{C} \in \{c, B\}$.*

(a) There is a $\preceq_{\mathfrak{C}}^{inj}$-antichain of size $2^{\aleph_0}$ made of $\mathbf{\Sigma}_2^0$ digraphs of uncountable Borel chromatic number.

(b) There is a $\preceq_{\mathfrak{C}}^{inj}$-strictly increasing chain of size $\aleph_0$ made of $\mathbf{\Sigma}_2^0$ digraphs of uncountable Borel chromatic number.

These digraphs, in fact differences of two closed sets, are not all minimal. In fact, we first construct a $\preceq_B^{\text{inj}}$-antichain of size $\aleph_0$ made of digraphs in the style of $\mathbb{G}_1$, that could be minimal as in Theorem 1.3. We then consider suitable direct sums of these digraphs. This antichain of size $\aleph_0$ is in fact made of pairwise incompatible digraphs, which gives our third main result (recall that $p, q \in Q$ are **incompatible** if there is no $r \in Q$ with $r \leq p, q$).

Theorem 1.5 *Let $\mathfrak{C}\in\{c,B\}$. Any $\preceq_{\mathfrak{C}}^{inj}$-basis for the class of analytic digraphs of uncountable Borel chromatic number on Polish spaces is infinite.*

We now provide a concrete description of $\mathbb{G}_1$. We first define a sequence $(q_n)_{n\in\omega}$ of natural numbers by setting $q_0:=0$ and $q_{n+1}:=32^{q_n}$. Note that $(q_n)_{n\in\omega}$ is strictly increasing. In particular, $|\psi(n)|\leq n\leq q_n<2^{q_n}$, so that $t_n:=\psi(n)0^{2^{q_n}-|\psi(n)|}$ is well-defined and has length 2^{q_n}. We then set $S_n:=\{2^{q_n}\cdot j\mid j\geq 1\}$, define $\theta_n:\omega\to\omega$ and g_n by

$$\theta_n(k):=\begin{cases}k \text{ if } k\notin S_n,\\ 2^{q_n}\cdot(2j+1) \text{ if } k=2^{q_n}\cdot j,\end{cases}\qquad g_n(\alpha)(k):=\begin{cases}1 \text{ if } k=2^{q_n},\\ \alpha\big(\theta_n(k)\big) \text{ if } k\neq 2^{q_n}.\end{cases}$$

The next definition catches some of the crucial properties of the sequence (g_n) defining $\mathbb{G}_1$. As noted in [L4], a great variety of very different non-potentially closed relations appear at the level of differences of two closed sets. For the kind of examples we will consider, being non-potentially closed and having uncountable Borel chromatic number are equivalent properties. Let us make this more precise.

Notation. If (f_n) is a sequence of functions, then we set $A^f:=\bigcup_{n\in\omega}\operatorname{Graph}(f_n)$.

Definition 1.6 *We say that $\big(X,(f_n)\big)$ is a* **complex situation** *if*

(a) X is a nonempty Polish space,

(b) the f_n's are partial continuous and open maps whose domain and range are open in X,

(c) $\Delta(X)\subseteq\overline{A^f}\setminus A^f$.

This kind of situations play an important role in the theory of potential complexity (see, for example, Definition 2.2 in [L3], and also Definitions 13, 26 and 31 in [L4]). These properties are sufficient to ensure that A^f is a $\mathbf{\Sigma}^0_2$ digraph of uncountable Borel chromatic number (see Corollary 2.2). We prove a result giving some additional motivation for introducing this notion, which is in fact very general.

Theorem 1.7 *Let Y be a Polish space and B be an analytic digraph on Y. Then exactly one of the following holds:*

(a) (Y,B) has countable Borel chromatic number,

(b) there is a complex situation $\big(X,(f_n)\big)$ such that the inequality $(X,A^f)\preceq_c^{inj}(Y,B)$ or $(X,A^f)\preceq_c^{inj}(Y,B^{-1})$ holds.

We prove more than the minimality of $\mathbb{G}_1$.

Theorem 1.8 *Let $\big(X,(f_n)\big)$ be a complex situation satisfying the following additional property:*

(d) $f_m\big(f_n(x)\big)=f_m(x)$ if $m<n$ and $f_m\big(f_n(x)\big)$, $f_m(x)$ are defined.

Then $(2^\omega,\mathbb{G}_1)\preceq_c^{inj}(X,A^f)$.

The organization of this paper is as follows. In Section 2, we provide some basic properties of complex situations. In Section 3, we characterize when the digraph associated with a complex situation is minimal among analytic digraphs of uncountable Borel chromatic number, and prove Theorem 1.7. In Section 4, we prove a relatively general lemma ensuring the injectivity of the homomorphism h implicitly mentioned in the statement of Theorem 1.8.

In Section 5, we show that $\mathbb{G}_1$ comes from a complex situation, introduce some finitary objects used in the construction of h, and prove their important properties. In particular, in many Cantor-like constructions of homomorphisms or reductions, we construct approximations, indexed by finite binary sequences, of the desired infinitary objects. The construction is usually made by induction on the length of the finite binary sequences. So we consider the partitions into basic clopen sets $(N_x)_{x\in 2^l}$ of 2^ω, for each $l\in\omega$. Here it will be more convenient to replace 2^l with some subset X_l of $2^{<\omega}$ containing sequences of different lengths since the g_n's "forget" some coordinates. In Section 6, we construct our homomorphism and prove Theorems 1.3 and 1.8. In Section 7, we prove Theorems 1.4 and 1.5, and show that our three main results also hold for graphs.

2 Some basic properties of complex situations

The next lemma is essentially Lemma 3.5 in [L1], and the crucial point of its proof.

Lemma 2.1 *Let X be a nonempty Polish space and, for $n\in\omega$, D_n, R_n be dense G_δ subsets of some open subsets of X, and $f_n : D_n \to R_n$ be a continuous, open and onto map.*

(a) Let G be a dense G_δ subset of X. Then Graph$(f_n)\subseteq\overline{\text{Graph}(f_n)\cap G^2}$.

(b) We assume that $\Delta(X)\subseteq\overline{A^f}\setminus A^f$. Then (X, A^f) has uncountable Borel chromatic number.

Proof. (a) See Lemma 3.5 in [L1].

(b) As $\Delta(X)\subseteq\neg A^f$, (X, A^f) is a digraph and thus its Borel chromatic number is defined. We argue by contradiction to see that it is uncountable, which gives a countable partition (B_n) of X into Borel sets with $A^f\cap B_n^2=\emptyset$. By 13.5 in [K], there is a finer Polish topology τ on X such that the B_n's are clopen in (X,τ). By 15.2 in [K], the identity of X equipped with its initial topology into (X,τ) is Borel. By 11.5 in [K] it is Baire measurable. By 8.38, in [K], there is a dense G_δ subset G of X on which the B_n's are clopen. We pick $x\in G$, which exists since X is Polish and nonempty. We choose n with $x\in B_n$. By (a), $\overline{A^f}\cap G^2=\overline{A^f\cap G^2}\cap G^2$, so that $(x,x)\in\overline{A^f\cap G^2}\cap(B_n\cap G)^2$. As $B_n\cap G$ is clopen in G, $A^f\cap(B_n\cap G)^2$ is nonempty, which is absurd. $\square$

Corollary 2.2 *Let $\big(X,(f_n)\big)$ be a complex situation. Then A^f is a $\mathbf{\Sigma}^0_2$ digraph of uncountable Borel chromatic number.*

Proof. Note that $A^f=\bigcup_{n\in\omega}$ Graph(f_n) is $\mathbf{\Sigma}^0_2$ since the f_n's are continuous with open domain. Moreover, Lemma 2.1 ensures that (X, A^f) has uncountable Borel chromatic number. $\square$

Condition (d) in Theorem 1.8 is motivated by the following result, which is essentially Claim 1 in the proof of Theorem 10 in [L4]. This condition ensures that A^f is $\preceq^{\text{inj}}_B$-incomparable with $\mathbb{G}_0$.

Notation. In the sequel, D_n will be the domain of the function f_n.

Lemma 2.3 *Let X be a Polish space, and $g_0, g_1, \ldots : X\to X$ be fixed point free Borel partial functions such that $g_m\big(g_n(x)\big)=g_m(x)$ if $m<n$ and $g_m\big(g_n(x)\big), g_m(x)$ are defined. Then every locally countable analytic subset of A^g has countable Borel chromatic number.*

Proof. Suppose that H is a locally countable analytic subset of A^g. By 35.13 in [K] and Lemma 2.4.(a) in [L2], there are Borel partial injections f_n on X such that $H \subseteq A^f \subseteq A^g$. By replacing each f_n with its restrictions to the sets $\{x \in D_n \mid f_n(x) = g_m(x)\}$, for $m \in \omega$, we can assume that, for all $n \in \omega$, there is $k_n \in \omega$ such that $f_n = g_{k_n}|_{D_n}$. It is easily seen that the graph of a fixed point free Borel function has countable Borel chromatic number (see Proposition 4.5 of [K-S-T]). So, by replacing f_n with its restriction to countably many Borel sets, we can also assume that $D_n^2 \cap \bigcup_{k \leq k_n} \text{Graph}(g_k) = \emptyset$ for all $n \in \omega$. It only remains to note that $D_n^2 \cap \bigcup_{k > k_n} \text{Graph}(g_k) = \emptyset$. In order to see this, simply observe that if $k > k_n$ and $x, g_k(x) \in D_n$, then $f_n(x) = g_{k_n}(x) = g_{k_n} \circ g_k(x) = f_n \circ g_k(x)$, which contradicts the fact that f_n is a partial injection. $\square$

Corollary 2.4 *Let $\big(X, (f_n)\big)$ be a complex situation satisfying Condition (d) in Theorem 1.8. Then $\{\mathbb{G}_0, A^f, (A^f)^{-1}\}$ is a $\preceq_B^{\text{inj}}$-antichain.*

Proof. Assume that $\mathbb{G}_0$ is $\preceq_B^{\text{inj}}$-below A^f, with witness π. Then $(\pi \times \pi)[\mathbb{G}_0]$ is a locally countable Borel subset of A^f with uncountable Borel chromatic number, which contradicts Lemma 2.3. A^f is not $\preceq_B^{\text{inj}}$-below $\mathbb{G}_0$ since $\mathbb{G}_0$ is locally countable and A^f is not, by Corollary 2.2 and Lemma 2.3. The discussion after Theorem 1.1 shows that $(2^\omega, \mathbb{G}_0) \preceq_c^{\text{inj}} (2^\omega, \mathbb{G}_0^{-1})$. Thus $\mathbb{G}_0$ and $(A^f)^{-1}$ are $\preceq_B^{\text{inj}}$-incomparable. If $(2^\omega, (A^f)^{-1}) \preceq_B^{\text{inj}} (2^\omega, A^f)$ with witness h, then $(h \times h)[(A^f)^{-1}]$ is a locally countable subset of A^f since A^f has countable vertical sections, which contradicts Lemma 2.3. $\square$

The next two lemmas will be used in the proof of Theorem 1.8.

Lemma 2.5 *Let $\big(X, (f_n)\big)$ be a complex situation satisfying Condition (d) in Theorem 1.8, V_0, V_1 be subsets of X, and $m < n$ be natural numbers such that $V_0 \subseteq D_m \cap D_n$ and $V_1 \subseteq f_n[V_0] \cap D_m$. Then $f_m[V_1] \subseteq f_m[V_0]$.*

Proof. Pick $y \in V_1$, and $x \in V_0$ with $y = f_n(x)$. Note that $f_m\big(f_n(x)\big)$ is defined, as well as $f_m(x)$. By Condition (d), $f_m\big(f_n(x)\big) = f_m(x)$. This implies that $f_m(y) = f_m(x) \in f_m[V_0]$. $\square$

Lemma 2.6 *Let $\big(X, (f_n)\big)$ be a complex situation such that X is zero-dimensional and the D_n's are clopen, V be a nonempty open subset of X, and m be a natural number. Then we can find $n > m$ and nonempty clopen subsets V_0, V_1 of X such that $V_0 \subseteq V \cap D_n$ and $V_1 \subseteq V \cap f_n[V_0]$.*

Proof. The assumption on $\big(X, (f_n)\big)$ implies that $\Delta(X) \subseteq \overline{\bigcup_{n>m} \text{Graph}(f_n)}$ since the Graph(f_n)'s are closed. This gives $n > m$ such that $V^2 \cap \text{Graph}(f_n) \neq \emptyset$, and (x, y) in this intersection. In particular, $x \in D_n$ and $y = f_n(x)$. We choose a clopen subset V_0 of X with $x \in V_0 \subseteq V \cap D_n$, and a clopen subset V_1 of X with $y \in V_1 \subseteq V \cap f_n[V_0]$. $\square$

3 The characterization of the minimality

We will characterize when the set A^f associated with a complex situation $\big(X, (f_n)\big)$ is $\preceq_c^{\text{inj}}$-minimal among analytic digraphs of uncountable Borel chromatic number. We will need a strengthening of the notion of a complex situation.

Definition 3.1 *We say that a complex situation* $\big(X,(f_n)\big)$ *is a* **strongly complex situation** *if*

(a) X *is a nonempty zero-dimensional perfect Polish space,*

(b) the f_n*'s are partial continuous and open maps whose domain and range are clopen in* X*,*

(c) the restriction of any f_n *to any nonempty open subset of its domain is not countable-to-one.*

The reader should see [M] for the basic notions of effective descriptive set theory. Let X be a recursively presented Polish space. The topology Δ_X on X is generated by $\Delta^1_1(X)$. This topology is Polish (see the proof of Theorem 3.4 in [Lo]). The Gandy-Harrington topology Σ_X on X is generated by $\Sigma^1_1(X)$. Recall that $\Omega_X := \{x \in X \mid \omega_1^x = \omega_1^{\mathrm{CK}}\}$ is Borel and Σ^1_1, and (Ω_X, Σ_X) is a zero-dimensional Polish space (in fact, the intersection of Ω_X with any nonempty Σ^1_1 set is a nonempty clopen subset of (Ω_X, Σ_X)-see [L1]).

Lemma 3.2 *Let* $\big(X,(f_n)\big)$ *be a complex situation such that* $(2^\omega, \mathbb{G}_0) \not\leq_c^{inj} (X, A^f)$*,* P *be a Borel subset of* X*, and* (S_n) *be a sequence of analytic subsets of* X *such that*

$$A := \bigcup_{n\in\omega} \mathit{Graph}(f_{n|S_n}) \subseteq P^2$$

has uncountable Borel chromatic number. Then we can find a Borel subset S *of* P*, a finer topology* τ *on* S*, and a sequence* (C_n) *of clopen subsets of* $Y := (S,\tau)$ *such that* $\big(Y,(f_{n|C_n})\big)$ *is a strongly complex situation and* $C_n \subseteq S \cap f_n^{-1}(S) \cap S_n$ *for each* $n \in \omega$*.*

Proof. In order to simplify the notation, we will assume as usual that X is recursively presented, (f_n) is Δ^1_1 (so that A^f is Δ^1_1 too), P is Δ^1_1, and (S_n) is Σ^1_1 (so that A is Σ^1_1). We set, for each $n \in \omega$ and each $W \subseteq X$,

$$\Phi_n(W) \Leftrightarrow f_{n|D_n\cap W} \text{ is countable-to-one.}$$

Note that Φ_n is Π^1_1 on Σ^1_1. Indeed, let Z be a recursively presented Polish space, and $\mathfrak{S}$ be in $\Sigma^1_1(Z\times X)$. Then $\Phi_n(\mathfrak{S}_z) \Leftrightarrow \forall x \in X \;\; f_n^{-1}(\{x\}) \cap \mathfrak{S}_z$ is countable. Note that $f_n^{-1}(\{x\}) \cap \mathfrak{S}_z$ is $\Sigma^1_1(z,x)$. By 4F.1 in [M], $f_n^{-1}(\{x\}) \cap \mathfrak{S}_z$ is countable if and only if it is contained in $\Delta^1_1(z,x) \cap X$, which is a Π^1_1 condition (in (z,x)).

This argument shows that if $X_n := \bigcup\{\Delta \in \Delta^1_1(X) \mid \Phi_n(\Delta)\}$, then (X_n) is Π^1_1, and also that we can apply the effective version of the first reflection theorem (see 35.10 in [K]). Let us prove that if $C \in \Sigma^1_1(X)$ and $C\backslash X_n$ is not empty, then $f_{n|D_n\cap C}$ is not countable-to-one. We argue by contradiction. As $C\backslash X_n \in \Sigma^1_1$ and $\Phi_n(C\backslash X_n)$ holds, the effective version of the first reflection theorem gives $\Delta \in \Delta^1_1$ such that $C\backslash X_n \subseteq \Delta$ and $\Phi_n(\Delta)$ holds. Thus $\Delta \subseteq X_n$ and $C\backslash X_n \subseteq X_n\backslash X_n$ is empty, which is absurd.

Note that $A = \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|S_n\cap X_n}) \cup \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|S_n\backslash X_n})$. As $\bigcup_{n\in\omega} \mathrm{Graph}(f_{n|S_n\cap X_n})$ is locally countable analytic, it has countable Borel chromatic number since $(2^\omega, \mathbb{G}_0) \not\leq_c^{\mathrm{inj}} (X, A^f)$, by the discussion after Theorem 1.1. Thus $A' := \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|S_n\backslash X_n})$ is a Σ^1_1 relation on X with uncountable Borel chromatic number.

By 4D.2 and 4D.14 in [M], $D_X := \{x \in X \mid x \in \Delta^1_1\}$ is countable and Π^1_1. We set

$$S := \{x \in P \mid (x,x) \in \overline{A'}^{\Delta^2_X}\} \cap \Omega_X \backslash D_X,$$

$\tau := \Sigma_{X|S}$ and, for each $n \in \omega$, $C_n := S \cap f_n^{-1}(S) \cap S_n \backslash X_n$.

Note that S is a Borel and Σ^1_1 subset of X. Thus the C_n's are Σ^1_1 and clopen subsets of Y. As $S \in \Sigma^1_1$, Y is a zero-dimensional perfect Polish space. We set $C := \{x \in P \mid (x,x) \in \overline{A'}^{\Delta^2_X}\}$.

Let us check that $C \setminus D_X$ is nonempty. We argue by contradiction. Then C is Σ^1_1 and contained in D_X. The effective separation result gives $\Delta \in \Delta^1_1(X)$ with $C \subseteq \Delta \subseteq D_X$. If $x \in P \setminus \Delta$, then there is $U_x \in \Delta^1_1(X)$ containing x with $A' \cap U_x^2 = \emptyset$. This also holds if $x \in \Delta$ with $U_x := \{x\}$. But this contradicts the fact that A' has uncountable Borel chromatic number.

Thus $C \setminus D_X$ is a nonempty Σ^1_1 subset of X, which therefore meets Ω_X. This shows that S is not empty. Note also that $(x,x) \in \overline{A'}^{\Delta^2_X} \cap S^2 = \overline{A'}^{\Sigma^2_X} \cap S^2 = \overline{A' \cap S^2}^{Y^2}$ if $x \in S$. We proved that $\big(Y, (f_{n|C_n})\big)$ is a strongly complex situation. $\square$

Corollary 3.3 *Let $\big(X, (f_n)\big)$ be a complex situation such that $(2^\omega, \mathbb{G}_0) \not\preceq_c^{inj} (X, A^f)$. The following are equivalent:*

(a) A^f is $\preceq_c^{inj}$-minimal among analytic digraphs of uncountable Borel chromatic number,

(b) for any Borel subset S of X, any finer topology τ on S, and any sequence (C_n) of clopen subsets of $Y := (S, \tau)$, $(X, A^f) \preceq_c^{inj} \big(Y, \bigcup_{n\in\omega} \mathit{Graph}(f_{n|C_n})\big)$ if $\big(Y, (f_{n|C_n})\big)$ is a strongly complex situation.

Proof. (a) $\Rightarrow$ (b) By Corollary 2.2, $\bigcup_{n\in\omega} \mathrm{Graph}(f_{n|C_n})$ is an analytic digraph of uncountable Borel chromatic number. As $\big(Y, \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|C_n})\big) \preceq_c^{\mathrm{inj}} (X, A^f)$, we are done.

(b) $\Rightarrow$ (a) Let Z be a Polish space, and A be an analytic digraph on Z of uncountable Borel chromatic number. We assume that $(Z, A) \preceq_B^{\mathrm{inj}} (X, A^f)$, with witness $u : Z \to X$. We set $P := u[Z]$, so that P is a Borel subset of X. Note that $A' := (u \times u)[A] \subseteq A^f$ is an analytic relation on P with uncountable Borel chromatic number, which gives a sequence $(S_n)_{n\in\omega}$ of analytic subsets of X with $A' = \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|S_n})$. Lemma 3.2 gives a Borel subset T of P, a finer topology σ on T, and a sequence (O_n) of clopen subsets of $W := (T, \sigma)$ such that $\big(W, (f_{n|O_n})\big)$ is a strongly complex situation and $O_n \subseteq T \cap f_n^{-1}(T) \cap S_n$. We put $A'' := \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|O_n})$.

Note that $b := {u^{-1}}_{|W} : W \to u^{-1}(W)$ is Borel and one-to-one. Let G be a dense G_δ subset of W such that $b|_G$ is continuous. This function is a witness for the fact that $(G, A'' \cap G^2) \preceq_c^{\mathrm{inj}} (Z, A)$. By Lemma 2.1, we get $\mathrm{Graph}(f_{n|O_n}) \subseteq \overline{\mathrm{Graph}(f_{n|O_n}) \cap G^2}$. Thus $\overline{A''} = \overline{A'' \cap G^2}$, and $\Delta(G) \subseteq \overline{A'' \cap G^2}$. Let us prove that $A'' \cap G^2$ has uncountable Borel chromatic number. We argue by contradiction, which gives a countable partition (B_q) of G into Borel sets. We can find $q \in \omega$, a nonempty open subset O of W, and a dense G_δ subset H of W with $O \cap H \subseteq B_q$. The previous argument shows that $\Delta(G \cap H) \subseteq \overline{A'' \cap (G \cap H)^2}$. Therefore $\Delta(O \cap G \cap H) \subseteq \overline{A'' \cap (O \cap G \cap H)^2} \subseteq \overline{A'' \cap G^2 \cap B_q^2} = \emptyset$, which is absurd.

We apply Lemma 3.2 to $\big(W, (f_{n|O_n})\big)$, G and $\big(G \cap f_n^{-1}(G) \cap O_n\big)$, which gives a Borel subset S of G, a topology τ on S finer than σ, and a sequence (C_n) of clopen subsets of $Y := (S, \tau)$ such that $\big(Y, (f_{n|C_n})\big)$ is a strongly complex situation and $C_n \subseteq S \cap f_n^{-1}(S) \cap O_n$. It remains to note that $(X, A^f) \preceq_c^{\mathrm{inj}} \big(Y, \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|C_n})\big) \preceq_c^{\mathrm{inj}} (G, A'' \cap G^2) \preceq_c^{\mathrm{inj}} (Z, A)$, by (b). $\square$

Remark. This proof also shows that (b) implies that A^f is $\preceq_B^{\mathrm{inj}}$-minimal among analytic digraphs of uncountable Borel chromatic number.

Proof of Theorem 1.7. By Corollary 2.2, (a) and (b) cannot hold simultaneously. So assume that (a) does not hold. Theorem 1.1 provides $h:2^\omega\to Y$ continuous with $(h\times h)[\mathbb{G}_0]\subseteq B$. Note that $\mathbb{G}_0$ comes from a complex situation. By Corollary 2.2, $\mathbb{G}_0$ is a $\mathbf{\Sigma}^0_2$ relation on the compact space 2^ω. Thus $\mathbb{G}_0$ and $(h\times h)[\mathbb{G}_0]$ are K_σ digraphs of uncountable Borel chromatic number. The canonical injection is a witness for the fact that $\big(Y,(h\times h)[\mathbb{G}_0]\big)\preceq_c^{\mathrm{inj}}(Y,B)$. So we may assume that B is K_σ. Let (C_n) be a sequence of closed digraphs on Y whose union is B.

We now argue essentially as in the proof of Theorem 2.3 in [L2]. In order to simplify the notation, we will assume as usual that Y is recursively presented and (C_n) is Δ^1_1. As in the proof of Theorem 1.1, we set $X_1:=Y\setminus\big(\bigcup\{S\in\Sigma^1_1(Y)\mid B\cap S^2=\emptyset\}\big)$, so that X_1 is a nonempty Σ^1_1 subset of Y, disjoint from $\{y\in Y\mid y\in\Delta^1_1\}$, and satisfies the following property:

$$\forall S\in\Sigma^1_1(Y)\ \ (\emptyset\neq S\subseteq X_1\Rightarrow B\cap S^2\neq\emptyset).$$

We set $Z:=(X_1\cap\Omega_Y,\Sigma_Y)$, so that Z is a nonempty zero-dimensional perfect Polish space. We also set $\mathfrak{S}:=\{S\in\Sigma^1_1(Z)\mid\emptyset\neq S\subseteq Z\}$. If $S\in\mathfrak{S}$, then we can find n with $C_n\cap S^2\neq\emptyset$. Note that $C_n\cap S^2$ is a closed relation on Z. Moreover, if U is an open relation on Z, then the projections of $U\cap C_n\cap S^2$ are open subsets of Z.

Theorem 1.13 in [L2] provides, for each $S\in\mathfrak{S}$, dense G_δ subsets F_S, G_S of some nonempty open subsets of Z and $g_S:F_S\to G_S$ onto, continuous and open such that $\mathrm{Graph}(g_S)\subseteq C_n\cap S^2$ or $\mathrm{Graph}(g_S)\subseteq(C_n\cap S^2)^{-1}$. We set

$$G(g_S):=\begin{cases}\mathrm{Graph}(g_S)\text{ if }\mathrm{Graph}(g_S)\subseteq C_n\cap S^2,\\ \big(\mathrm{Graph}(g_S)\big)^{-1}\text{ if }\mathrm{Graph}(g_S)\subseteq(C_n\cap S^2)^{-1},\end{cases}$$

so that $G(g_S)\subseteq B\cap S^2$.

Let $\alpha\in2^\omega$ such that Z is recursively in α presented and the sequence $\big(G(g_S)\big)_{S\in\mathfrak{S}}$ is $\Delta^1_1(\alpha)$. Let G be a dense G_δ and $\Sigma^1_1(\alpha)$ subset of Z on which Σ_Y and $\Delta^\alpha_Z:=<\Delta^1_1(\alpha)(Z)>$ coincide, which exists by Lemma 2.1 in [L2]. Let $\Sigma^\alpha_Z:=<\Sigma^1_1(\alpha)(Z)>$, which gives Ω^α_Z like before Lemma 3.2. Moreover, the proof of Theorem 2.3 in [L2] shows that Ω^α_Z is comeager in Z, as well as $G\cap\Omega^\alpha_Z$. We set $W:=(G\cap\Omega^\alpha_Z,\Sigma^\alpha_Z)$ and define, for $S\in\mathfrak{S}$, a partial function f_S by $G(f_S):=G(g_S)\cap W^2$, so that W is a nonempty zero-dimensional Polish space and $\big(W,\bigcup_{S\in\mathfrak{S}}G(f_S)\big)\preceq_c^{\mathrm{inj}}(Y,B)$. Moreover, for each $S\in\mathfrak{S}$, f_S is a partial continuous and open map with clopen domain and range in W. We then note that

$$\Delta(W)\subseteq\overline{\bigcup_{S\in\mathfrak{S}}G(g_S)}^{Z^2}\cap W^2=\overline{\bigcup_{S\in\mathfrak{S}}G(g_S)}^{(\Delta^\alpha_Z)^2}\cap W^2=\overline{\bigcup_{S\in\mathfrak{S}}G(g_S)}^{(\Sigma^\alpha_Z)^2}\cap W^2\subseteq\overline{\bigcup_{S\in\mathfrak{S}}G(f_S)}^{W^2}.$$

We set $\mathfrak{S}_0:=\{S\in\mathfrak{S}\mid\mathrm{Graph}(g_S)\subseteq B\}$, $\mathfrak{S}_1:=\{S\in\mathfrak{S}\mid\mathrm{Graph}(g_S)\subseteq B^{-1}\}$ and, for $\varepsilon\in2$,

$$W_\varepsilon:=\{w\in W\mid(w,w)\in\overline{\bigcup_{S\in\mathfrak{S}_\varepsilon}G(f_S)}\}.$$

Note that (W_0, W_1) is a covering of W into closed sets, which gives a nonempty clopen subset X of W and $\varepsilon \in 2$ with $X \subseteq W_\varepsilon$. Note that X is a nonempty Polish space, $G(f_S) \cap X^2$ defines a partial continuous and open map with open domain and range in X if $S \in \mathfrak{S}_\varepsilon$, and

$$\Delta(X) \subseteq \overline{\bigcup_{S \in \mathfrak{S}_\varepsilon} G(f_S) \cap X^2}^{X^2} \setminus \big(\bigcup_{S \in \mathfrak{S}_\varepsilon} G(f_S) \cap X^2 \big).$$

We enumerate $(f_n) := \big((f_S)_{|X \cap f_S^{-1}(X)}\big)_{S \in \mathfrak{S}_\varepsilon}$. If $\varepsilon = 0$, then $\big(X, (f_n)\big)$ is a complex situation with $(X, A^f) \preceq_c^{\text{inj}} (Y, B)$. If $\varepsilon = 1$, then $\big(X, (f_n)\big)$ is a complex situation with $(X, A^f) \preceq_c^{\text{inj}} (Y, B^{-1})$. $\square$

4 The lemma ensuring the injectivity

Recall the set X_l mentioned at the end of the introduction. An oriented graph A_l on X_l will contain finite approximations of a subset of $\mathbb{G}_1$ with acyclic symmetrization. We now isolate some important properties of A_l leading to a relatively general lemma ensuring the injectivity of the homomorphism in Theorem 1.8.

Notation. Let X be a set, and A be an oriented graph on X. We set, for $x \in X$,

$$\begin{cases} \text{Succ}(x) := \{y \in X \mid (x, y) \in A\} \text{ and } \max_X := \{x \in X \mid \text{Succ}(x) = \emptyset\}, \\ \\ \text{Pred}(x) := \{y \in X \mid (y, x) \in A\} \text{ and } \min_X := \{x \in X \mid \text{Pred}(x) = \emptyset\}. \end{cases}$$

The following oriented graphs will be of particular importance in the sequel.

Definition 4.1 *An oriented graph A on a set X is* **unambiguously oriented** *if $|Succ(x)| \leq 1$ for each $x \in X$. If moreover A has acyclic symmetrization, then we say that A is an* **uogas**.

Lemma 4.2 *Let A be an uogas on a finite set X.*

(a) Let l be a natural number, $(y_i)_{i \leq l} \in X^{l+1}$ such that $y_{i+1} \in Succ(y_i)$ if $i < l$. Then $(y_i)_{i \leq l}$ is injective. In particular, $(y_i)_{i \leq l} = p^{s(A)}_{y_0, y_l}$.

(b) Let $y \in X$, $x \in C(y) \cap max_X$, and $p := p^{s(A)}_{y,x}$. Then $\big(p(i), p(i+1)\big) \in A$ if $i < |p| - 1$.

(c) The intersection of max_X with each $s(A)$-connected component C is a singleton $\{x_C\}$.

(d) Let $(y, x) \in A$ and $p := p^{s(A)}_{y, x_{C(y)}}$. Then $|p| \geq 2$ and $p(1) = x$.

Proof. (a) It is enough to see that $(y_i)_{i \leq l}$ is injective. This is clear if $l = 0$. As A is irreflexive, $y_i \neq y_{i+1}$ if $i < l$. As A is an oriented graph, $y_i \neq y_{i+2}$ if $i < l - 1$. As $s(A)$ is acyclic, $(y_i)_{i \leq l}$ is injective, by induction on l.

(b) We argue by induction on $m := |p| - i$. For $m = 2$, we argue by contradiction, so that $\big(x, p(i)\big) \in A$. Thus $p(i) \in \text{Succ}(x)$, which contradicts the fact that $x \in \max_X$. So assume that

$$\big(p(|p| - m), p(|p| - m + 1)\big) \in A.$$

We argue by contradiction, so that $\big(p(|p| - m), p(|p| - m - 1)\big) \in A$. Thus $p(|p| - m + 1)$ and $p(|p| - m - 1)$ are different elements of $\text{Succ}\big(p(|p| - m)\big)$, which contradicts the fact that A is unambiguously oriented.

(c) Let C be a $s(A)$-connected component, and $\{x_m \mid m<l\}$ be an injective enumeration of C. We argue by contradiction to prove that $\max_X$ meets C. This means that $\mathrm{Succ}(x_m)\neq\emptyset$ if $m<l$. We define, inductively on i, a sequence $(m_i)_{i\leq l}$ as follows. We first set $m_0:=0$. Assume that $i<l$ and m_i has been defined. As $\mathrm{Succ}(x_{m_i})\neq\emptyset$, there is $m_{i+1}<l$ such that $x_{m_{i+1}}\in\mathrm{Succ}(x_{m_i})$. By (a), $(m_i)_{i\leq l}\in l^{l+1}$ is injective, which is absurd.

Assume now that, for example, $x_0,x_1\in C\cap\max_X$. Let $p_0:=p^{s(A)}_{x_0,x_1}$ and $p_1:=p^{s(A)}_{x_1,x_0}$, so that $p_1(i)=p_0(n-i-1)$ if $i<n:=|p_1|=|p_0|$. By (b), $\big(p_0(n-2),x_1\big),\big(x_1,p_1(1)\big)\in A$, which contradicts the fact that A is an oriented graph.

(d) We set $y_0:=y$, $y_1:=x$, and choose $y_{i+1}\in\mathrm{Succ}(y_i)$ if this last set is not empty. After finitely many steps, this construction stops and provides $(y_i)_{i\leq l}$ with $l\geq 1$, by (a) and since X is finite. Note that $y_l=x_{C(y)}$ and $(y_i)_{i\leq l}=p$, by construction and by (a) and (c). $\square$

The homomorphism h in Theorem 1.8 will be obtained thanks to a Cantor-like construction. In the inductive step of this construction, we will consider an uogas on a finite set X. We will associate open sets to the elements of X. As we want h to be injective, we will have to ensure the disjunction of these open sets. This will be achieved in Lemma 4.7 to come. Its proof will use the following objects.

Notation. Let A be an uogas on a finite set X. If $y\in X$, then Lemma 4.2 allows us to set $p_y:=p^{s(A)}_{y,x_{C(y)}}$. We set $L:=|X|$, and enumerate $X:=\{x_m|m<L\}$ injectively in such a way that $(|p_{x_m}|)_{m<L}$ is increasing. Let $L_0\leq L$ be maximal such that $x_m\in\max_X$ if $m<L_0$, so that $\{x_m|m<L_0\}$ enumerates injectively $\max_X$. The idea is to make, inductively on m, L copies of x_m if $m\geq L_0$, keeping the A-relations. In order to do this, we will give labels to the elements of X. We set

$$\mathcal{X}_m:=\big\{\big(x_k,(0)\big)\mid k<L\big\}$$

and $\mathcal{A}_m:=\Big\{\Big(\big(x_k,(0)\big),\big(x_l,(0)\big)\Big)\mid (x_k,x_l)\in A\Big\}$ if $m<L_0$. Assume that $L_0\leq m+1<L$. We put the elements of $\mathcal{X}_m$ whose X-coordinate is not x_{m+1} or one of its (iterated) predecessors in

$$\mathcal{R}_{m+1}:=\big\{(x_k,\sigma)\in\mathcal{X}_m\mid x_{m+1}\notin p_{x_k}\big\}.$$

We also set, for $\sigma\in L^{\leq m+1}$ and $j<L$, $\mathcal{X}^{\sigma}_{m+1}:=\big\{(x_k,\sigma)\in\mathcal{X}_m\mid x_{m+1}\in p_{x_k}\big\}$,

$$\mathcal{X}^{\sigma j}_{m+1}:=\big\{(x_k,\sigma j)\in X\times L^{<\omega}\mid (x_k,\sigma)\in\mathcal{X}_m\wedge x_{m+1}\in p_{x_k}\big\}$$

(some of these sets can be empty). Then we set $\mathcal{X}_{m+1}:=\mathcal{R}_{m+1}\cup\bigcup_{\sigma\in L^{\leq m+1},j<L}\mathcal{X}^{\sigma j}_{m+1}$ and

$$\begin{aligned}\mathcal{A}_{m+1}:=\big\{\big((x_k,\sigma),(x_l,\tau)\big)\in\mathcal{A}_m\mid k,l\leq m\big\}\cup&\\ \big\{\big((x_{m+1},\sigma j),(x_i,\sigma)\big)\in(\mathcal{X}_{m+1})^2\mid \big((x_{m+1},\sigma),(x_i,\sigma)\big)\in\mathcal{A}_m\wedge j<L\big\}\cup&\\ \big\{\big((x_k,\sigma),(x_l,\sigma)\big)\in\mathcal{A}_m\mid k>m+1\wedge x_{m+1}\notin p_{x_l}\big\}\cup&\\ \big\{\big((x_k,\sigma j),(x_l,\sigma j)\big)\in(\mathcal{X}_{m+1})^2\mid \big((x_k,\sigma),(x_l,\sigma)\big)\in\mathcal{A}_m\wedge j<L\wedge x_{m+1}\in p_{x_l}\big\}.&\end{aligned}$$

We enumerate injectively $\{\sigma\in L^{<\omega}\mid (x_{m+1},\sigma)\in\mathcal{X}_m\}$ by $\{\sigma_n\mid n<N\}$. We set, for $p\leq N$, $\mathcal{X}^p_{m+1}:=\mathcal{R}_{m+1}\cup\bigcup_{n<p,j<L}\mathcal{X}^{\sigma_n j}_{m+1}\cup\bigcup_{p\leq n<N}\mathcal{X}^{\sigma_n}_{m+1}$, so that $\mathcal{X}^0_{m+1}=\mathcal{X}_m$ and $\mathcal{X}^N_{m+1}=\mathcal{X}_{m+1}$. We also define the corresponding intermediate versions of $\mathcal{A}_{m+1}$ as follows.

We set, for $p \leq N$,

$$\begin{aligned}\mathcal{A}^p_{m+1} := \big\{ \big((x_k,\sigma),(x_l,\tau)\big) \in \mathcal{A}_m \mid k,l \leq m \big\} \cup \\ \bigcup_{n<p} \ \big\{ \big((x_{m+1},\sigma_n j),(x_i,\sigma_n)\big) \in (\mathcal{X}^p_{m+1})^2 \mid \big((x_{m+1},\sigma_n),(x_i,\sigma_n)\big) \in \mathcal{A}_m \wedge j<L \big\} \cup \\ \bigcup_{p\leq n<N} \ \big\{ \big((x_{m+1},\sigma_n),(x_i,\sigma_n)\big) \in (\mathcal{X}^p_{m+1})^2 \mid \big((x_{m+1},\sigma_n),(x_i,\sigma_n)\big) \in \mathcal{A}_m \big\} \cup \\ \big\{ \big((x_k,\sigma),(x_l,\sigma)\big) \in \mathcal{A}_m \mid k>m+1 \wedge x_{m+1} \notin p_{x_l} \big\} \cup \\ \bigcup_{n<p} \ \big\{ \big((x_k,\sigma_n j),(x_l,\sigma_n j)\big) \in (\mathcal{X}^p_{m+1})^2 \mid \big((x_k,\sigma_n),(x_l,\sigma_n)\big) \in \mathcal{A}_m \wedge j<L \wedge x_{m+1} \in p_{x_l} \big\} \cup \\ \bigcup_{p\leq n<N} \ \big\{ \big((x_k,\sigma_n),(x_l,\sigma_n)\big) \in \mathcal{A}_m \mid x_{m+1} \in p_{x_l} \big\}.\end{aligned}$$

Lemma 4.3 *Let A be an uogas on a finite set X. Then $\mathcal{A}_m$ (respectively, $\mathcal{A}^p_{m+1}$) is also an uogas on the finite set $\mathcal{X}_m$ (respectively, $\mathcal{X}^p_{m+1}$) if $m<L$ (respectively, $L_0 \leq m+1<L$ and $p \leq N$).*

Proof. Note first that $(\mathcal{X}_m, \mathcal{A}_m)$ is a copy of (X, A) if $m < L_0$. In particular, $\mathcal{X}_m$ is finite, $\mathcal{A}_m$ is an oriented graph on $\mathcal{X}_m$ with acyclic symmetrization, and $|\mathrm{Succ}(q)| \leq 1$ for each $q \in \mathcal{X}_m$ (with respect to $\mathcal{A}_m$). Then we assume that $L_0 \leq m+1 < L$. Note that $\mathcal{X}_{m+1}$ is finite, $\mathcal{A}_{m+1}$ is an oriented graph on $\mathcal{X}_{m+1}$, and $|\mathrm{Succ}(q)| \leq 1$ for each $q \in \mathcal{X}_{m+1}$ (with respect to $\mathcal{A}_{m+1}$). Let us check that $s(\mathcal{A}_{m+1})$ is acyclic. The restriction of $\mathcal{A}_{m+1}$ to a fixed $\mathcal{X}^{\sigma j}_{m+1}$ is isomorphic to a subgraph of $\mathcal{A}_m$, and has therefore acyclic symmetrization. Note that the $\mathcal{X}^{\sigma j}_{m+1}$'s are pairwise disjoint and not $s(\mathcal{A}_{m+1})$-related, and that $\mathcal{X}^{\sigma j}_{m+1}$ contains $(x_{m+1}, \sigma j)$ if it is not empty (i.e., if (x_{m+1}, σ) is in $\mathcal{X}_m$). The restriction of the oriented graph $\mathcal{A}_{m+1}$ to $\mathcal{R}_{m+1}$ is also isomorphic to a subgraph of $\mathcal{A}_m$. Moreover, the only possible $s(\mathcal{A}_{m+1})$-edge between an element of $\mathcal{X}^{\sigma j}_{m+1}$ and element of $\mathcal{R}_{m+1}$ is between $(x_{m+1}, \sigma j)$ and (x_i, σ), where $x_i \in \mathrm{Succ}(x_{m+1})$. This shows the acyclicity of $s(\mathcal{A}_{m+1})$. We argue similarly for $(\mathcal{X}^p_{m+1}, \mathcal{A}^p_{m+1})$. □

Definition 4.4 *We say that the tuple $\mathcal{T} := \big(X, A, Z, (f_n)\big)$ is a* **mapping tuple** *if A is an uogas on the finite set X and $\big(Z, (f_n)\big)$ is a strongly complex situation.*

Notation. Let $\mathcal{T} := \big(X, A, Z, (f_n)\big)$ be a mapping tuple. We set

$$\begin{cases} E_{\mathcal{T}} := \big\{ \ \big(u, (V_x)_{x\in X}\big) \in \omega^X \times (\mathbf{\Sigma}^0_1(Z) \setminus \{\emptyset\})^X \mid \forall (x,y) \in A \ \ V_x \subseteq D_{u(x)} \wedge V_y = f_{u(x)}[V_x] \ \big\}, \\ U_{\mathcal{T}} := \big\{ \ \big(u, (V_x)_{x\in X}\big) \in \omega^X \times (\mathbf{\Sigma}^0_1(Z) \setminus \{\emptyset\})^X \mid \forall (x,y) \in A \ \ V_x \subseteq D_{u(x)} \wedge V_y \subseteq f_{u(x)}[V_x] \ \big\}. \end{cases}$$

Lemma 4.5 *Let $\mathcal{T} := \big(X, A, Z, (f_n)\big)$ be a mapping tuple, and $\big(u, (V_x)_{x\in X}\big) \in U_{\mathcal{T}}$. Then we can find a family $(W_x)_{x\in X}$ of subsets of Z such that*

(a) $\big(u, (W_x)_{x\in X}\big) \in E_{\mathcal{T}}$,

(b) $W_x \subseteq V_x$ if $x \in X$, and $W_x = V_x$ if $x \in max_X$.

Proof. We define W_x by induction on $|p_x|$. If $|p_x| = 1$, then we set $W_x := V_x$. Assume that $|p_x| \geq 2$, so that W_y has been defined if $y := p_x(1)$. We set $W_x := V_x \cap f^{-1}_{u(x)}(W_y)$. We are done, by Lemma 4.2. □

Lemma 4.6 *Let $\mathcal{T} := \big(X, A, Z, (f_n)\big)$ be a mapping tuple, $\big(u, (V_x)_{x\in X}\big) \in E_{\mathcal{T}}$, $x_0 \in X$, and W_{x_0} be a nonempty open subset of V_{x_0}. Then we can find a family $(W_x)_{x\in X\setminus\{x_0\}}$ of subsets of Z such that*

(a) $\big(u, (W_x)_{x\in X}\big) \in E_{\mathcal{T}}$,

(b) $W_x \subseteq V_x$ if $x \in X$, and $W_x = V_x$ if $x \notin C(x_0)$.

Proof. We set, for $x \in C(x_0)$, $q_x := p^{s(A)}_{x,x_0}$. We define W_x by induction on $|q_x|$, the case $|q_x| = 1$ (i.e., $x = x_0$) being done. So assume that $|q_x| \geq 2$, so that W_y has been defined if $y := q_x(1)$. We set

$$W_x := \begin{cases} f_{u(y)}[W_y] \text{ if } (y,x) \in A, \\ \\ V_x \cap f^{-1}_{u(x)}(W_y) \text{ if } (x,y) \in A. \end{cases}$$

If $x \in X \setminus C(x_0)$, then we set $W_x := V_x$. $\square$

Notation. Let A be an uogas on X finite, and $x \in X$. We set $M_x := \max\{|p^{s(A)}_{y,x}| \mid y \in \min_X \wedge x \in p_y\}$.

Lemma 4.7 *Let $\mathcal{T} := \big(X, A, Z, (f_n)\big)$ be a mapping tuple, $d \in \omega$, and $\big(u, (V_x)_{x \in X}\big) \in U_{\mathcal{T}}$. Then we can find a family $(W_x)_{x \in X}$ of subsets of Z such that*

(a) $\big(u, (W_x)_{x \in X}\big) \in E_{\mathcal{T}}$,

(b) $W_x \subseteq V_x$, $W_x \in \mathbf{\Delta}^0_1(Z)$ and diam$(W_x) \leq 2^{-d}$ if $x \in X$,

(c) $W_x \cap W_y = \emptyset$ if $x \neq y \in X$.

Proof. We consider the oriented graphs $\mathcal{A}_m$ (respectively, $\mathcal{A}^p_{m+1}$) on $\mathcal{X}_m$ (respectively, $\mathcal{X}^p_{m+1}$) defined before Lemma 4.3. We set, for $m < L$, $\mathcal{T}_m := \big(\mathcal{X}_m, \mathcal{A}_m, Z, (f_n)\big)$. Similarly, we set, when $L_0 \leq m+1 < L$ and $p \leq N$, $\mathcal{T}^p_{m+1} := \big(\mathcal{X}^p_{m+1}, \mathcal{A}^p_{m+1}, Z, (f_n)\big)$. By Lemma 4.3, all these tuples are mapping tuples. We define, for $m < L$, $u_m : \mathcal{X}_m \to \omega$ by $u_m(x_k, \sigma) := u(x_k)$. We also define, when $L_0 \leq m+1 < L$ and $p \leq N$, $u^p_{m+1} : \mathcal{X}^p_{m+1} \to \omega$ by $u^p_{m+1}(x_k, \sigma) := u(x_k)$.

We first construct, by induction on $m < L$, a family $(W^m_q)_{q \in \mathcal{X}_m}$ of nonempty open subsets of Z satisfying

(1) $W^{m+1}_{x_k,\sigma'} \subseteq W^m_{x_k,\sigma} \subseteq V_{x_k}$
(2) $\big(u_m, (W^m_q)_{q \in \mathcal{X}_m}\big) \in E_{\mathcal{T}_m}$
(3) diam$(W^m_{x_m,\sigma}) \leq 2^{-d}$
(4) $\forall j \neq j' < L \;\; W^m_{x_m,\sigma j} \cap W^m_{x_m,\sigma j'} = \emptyset$

We will apply Lemmas 4.5 and 4.6 to perform this construction. For $m = 0$, we will apply Lemma 4.5 to $\mathcal{T}_0$ and u_0. We choose a nonempty open subset $V_{x_0,(0)}$ of V_{x_0} with diameter at most 2^{-d}, and we set $V_{x_k,(0)} := V_{x_k}$ if $0 < k < L$. Note that $\big(u_0, (V_q)_{q \in \mathcal{X}_0}\big) \in U_{\mathcal{T}_0}$. Lemma 4.5 provides a family $(W^0_q)_{q \in \mathcal{X}_0}$ of subsets of Z such that

(a) $\big(u_0, (W^0_q)_{q \in \mathcal{X}_0}\big) \in E_{\mathcal{T}_0}$,
(b) $W^0_{x_k,(0)} \subseteq V_{x_k,(0)} \subseteq V_{x_k}$.

This completes the construction for $m = 0$. If $m+1 < L_0$, then we proceed similarly: we choose a nonempty open subset $V_{x_{m+1},(0)}$ of $V_{x_{m+1}}$ (which is equal to $W^m_{x_{m+1},(0)}$ since we only applied Lemma 4.5 to perform this construction up to this point) with diameter at most 2^{-d}, and we set $V_{x_k,(0)} := W^m_{x_k,(0)}$ if $m+1 \neq k < L$. Note that $\big(u_{m+1}, (V_q)_{q \in \mathcal{X}_{m+1}}\big) \in U_{\mathcal{T}_{m+1}}$. Lemma 4.5 provides a family $(W^{m+1}_q)_{q \in \mathcal{X}_{m+1}}$ of subsets of Z such that

(a') $\big(u_{m+1}, (W^{m+1}_q)_{q \in \mathcal{X}_{m+1}}\big) \in E_{\mathcal{T}_{m+1}}$,
(b') $W^{m+1}_q \subseteq W^m_q$.

So we may assume that $L_0 \leq m+1 < L$. We will, starting with $(Z^0_q)_{q\in\mathcal{X}_m} := (W^m_q)_{q\in\mathcal{X}_m}$ and inductively on $p \leq N$, construct families $(Z^p_q)_{q\in\mathcal{X}^p_{m+1}}$ of subsets of Z such that

(a") $\big(u^p_{m+1}, (Z^p_q)_{q\in\mathcal{X}^p_{m+1}}\big) \in E_{\mathcal{T}^p_{m+1}}$,
(b") $Z^{p+1}_{x_k,\sigma'} \subseteq Z^p_{x_k,\sigma}$.

At the end we will set $W^{m+1}_q := Z^N_q$. Assume that $p < N$ and $(Z^p_q)_{q\in\mathcal{X}^p_{m+1}}$ has been constructed, which is the case for $p=0$. We will apply Lemma 4.5 to $\mathcal{T}^{p+1}_{m+1}$ and u^{p+1}_{m+1}. Let $u' := u^p_{m+1}(x_{m+1}, \sigma_p)$, so that $Z^p_{x_{m+1},\sigma_p} \subseteq D_{u'}$ and $f_{u'}[Z^p_{x_{m+1},\sigma_p}] = Z^p_{x_i,\sigma_p}$, by (a"). As $f_{u'|Z^p_{x_{m+1},\sigma_p}}$ is not countable-to-one, we can find $z \in Z^p_{x_i,\sigma_p}$ and $(z_j)_{j<L} \in (Z^p_{x_{m+1},\sigma_p})^L$ injective such that $f_{u'}(z_j) = z$ for each $j < L$. We choose a sequence $(V_{x_{m+1},\sigma_p j})_{j<L}$ of pairwise disjoint clopen subsets of Z with diameter at most 2^{-d} such that $z_j \in V_{x_{m+1},\sigma_p j} \subseteq Z^p_{x_{m+1},\sigma_p}$ for each $j < L$, and put $V_{x_i,\sigma_p} := Z^p_{x_i,\sigma_p} \cap \bigcap_{j<L} f_{u'}[V_{x_{m+1},\sigma_p j}]$, which is an open neighborhood of z. We apply Lemma 4.6 to $\mathcal{T}^p_{m+1}$, $\big(u^p_{m+1}, (Z^p_q)_{q\in\mathcal{X}^p_{m+1}}\big)$, (x_i, σ_p) and V_{x_i,σ_p}, which gives $\big(u^p_{m+1}, (Y_q)_{q\in\mathcal{X}^p_{m+1}}\big)$.

We then set $V_{x_k,\tau} := Y_{x_k,\tau}$ if $x_k \in p_{x_i}$ and $\tau \subseteq \sigma_p$. We also set $V_{x_k,\tau} := Z^p_{x_k,\tau}$ if $x_k \notin p_{x_{m+1}}$, or ($x_k \in p_{x_i}$ and $\tau \nsubseteq \sigma_p$), or ($k = m+1$ and τ is not of the form $\sigma_p j$ for some $j < L$). This defines $(V_q)_{q\in\mathcal{X}^{p+1}_{m+1}}$, and $\big(u^{p+1}_{m+1}, (V_q)_{q\in\mathcal{X}^{p+1}_{m+1}}\big) \in U_{\mathcal{T}^{p+1}_{m+1}}$. Lemma 4.5 provides a family $(Z^{p+1}_q)_{q\in\mathcal{X}^{p+1}_{m+1}}$ of subsets of Z such that

(a"') $\big(u^{p+1}_{m+1}, (Z^{p+1}_q)_{q\in\mathcal{X}^{p+1}_{m+1}}\big) \in E_{\mathcal{T}^{p+1}_{m+1}}$,
(b"') $Z^{p+1}_q \subseteq V_q$ if $q \in \mathcal{X}^{p+1}_{m+1}$.

This finishes the construction of the family $(W^{m+1}_q)_{q\in\mathcal{X}_{m+1}}$ as desired. It remains to get clopen sets and ensure (c). We first ensure the disjointness. In order to do this, we construct, inductively on m, a family $(z_m)_{m<L}$ of points of Z, and a family $(\sigma_m)_{m<L}$ of finite sequences of elements of L such that

(α) $z_{m+1} \notin \{z_l \mid l \leq m\}$
(β) $(x_m, \sigma_m) \in \mathcal{X}_{L-1}$
(γ) $z_m \in W^{L-1}_{x_m,\sigma_m}$
(δ) $f_{u(x_{m+1})}(z_{m+1}) = z_i$ if $x_i \in \mathrm{Succ}(x_{m+1})$

We first set $\sigma_0 := (0)$, so that $(x_0, \sigma_0) \in \mathcal{X}_{L-1}$, and choose $z_0 \in W^{L-1}_{x_0,\sigma_0}$. If $m+1 < L_0$, then we set $\sigma_{m+1} := (0)$, so that $(x_{m+1}, \sigma_{m+1}) \in \mathcal{X}_{L-1}$, and choose $z_{m+1} \in W^{L-1}_{x_{m+1},\sigma_{m+1}} \setminus \{z_l \mid l \leq m\}$, which is possible since Z is perfect.

If $L_0 \leq m+1 < L$, then let $i \leq m$ such that x_i is the unique element of $\mathrm{Succ}(x_{m+1})$, so that $(x_{m+1}, \sigma_i j) \in \mathcal{X}_{L-1}$ for each $j < L$. We choose $j_{m+1} < L$ such that $\{z_l \mid l \leq m\} \cap W^{L-1}_{x_{m+1},\sigma_i j_{m+1}} = \emptyset$, which is possible by (4). We then set $\sigma_{m+1} := \sigma_i j_{m+1}$, so that $(x_{m+1}, \sigma_{m+1}) \in \mathcal{X}_{L-1}$, and choose $z_{m+1} \in W^{L-1}_{x_{m+1},\sigma_{m+1}}$ with $f_{u(x_{m+1})}(z_{m+1}) = z_i$.

Note that $\big(u, (W^{L-1}_{x_m,\sigma_m})_{m<L}\big) \in E_{\mathcal{T}}$. As $(z_m)_{m<L}$ is injective, we can find a family $(O_x)_{x\in X}$ of pairwise disjoint clopen subsets of Z such that $z_m \in O_{x_m} \subseteq W^{L-1}_{x_m,\sigma_m}$.

Recall the definition of M_x just before Lemma 4.7. We then define, for $y\in X$, and inductively on M_y, $U_y := O_y \cap \bigcap_{x\in \mathrm{Pred}(y)} f_{u(x)}[U_x]$, so that $z_m \in U_{x_m} \subseteq V_{x_m}$, the U_x's are pairwise disjoint, and $\big(u, (U_x)_{x\in X}\big) \in U_{\mathcal{T}}$.

Lemma 4.5 provides a family $(W_x)_{x\in X}$ of subsets of Z such that

(a"") $\big(u, (W_x)_{x\in X}\big) \in E_{\mathcal{T}}$,
(b"") $W_x \subseteq U_x$.

It remains to get clopen sets. This can be done if we apply the proof of Lemma 4.6, inductively on M_y as above, using the fact that this proof uses pre-images to go towards the elements of $\min_X$. $\square$

5 The important properties of $\mathbb{G}_1$

We first introduce the other elements of the countable antichain mentioned after Theorem 1.4.

Notation. Let (g_n) be a sequence of partial functions. We set, for $s\in\omega^{<\omega}\setminus\{\emptyset\}$,

$$g_s := g_{s(0)} \circ \cdots \circ g_{s(|s|-1)}$$

and $s^* :=< s(1), \cdots, s(|s|-1) >$. If $L=1$ and $j\geq 1$, then we set $\theta(j) := 2j+1$. Fix $L\geq 2$. We set $P_L := \{2^p \cdot 3^l \mid p\in\omega \wedge l < L-2\}$ and $M_L := \{2^{31}\cdot 3\cdot k \mid k\geq 1 \wedge k\notin P_L\}$. We define a map $\theta : \{j\in\omega \mid j\geq 1\} \to \{3k \mid k\geq 1\}$ as follows:

$$\theta(j) := \begin{cases} 3j \text{ if } j\notin \big(M_L \cup (M_L+1)\big), \\ 3j+3 \text{ if } j\in M_L, \\ 3j-3 \text{ if } j\in M_L+1. \end{cases}$$

Recall the definition of (S_n) after Theorem 1.5. We define $\theta_n : \omega\to\omega$ by

$$\theta_n(k) := \begin{cases} k \text{ if } k\notin S_n, \\ 2^{q_n}\cdot\theta(j) \text{ if } k = 2^{q_n}\cdot j. \end{cases}$$

Lemma 5.1 *Fix $L\geq 1$. Then (S_n) and (θ_n) satisfy the following properties:*

(1) $\{2^j \mid j\geq q_n\} \subseteq \omega\setminus\theta_n[\omega]$
(2) *θ_n is injective*
(3) $\omega\setminus S_n = \{k\in\omega \mid \theta_n(k) = k\}$
(4) $\forall s\in\omega^{<\omega}$ *strictly decreasing with* $2\leq |s|\leq L$ $\theta_{s^*}[\omega] \cap S_{s(0)}$ *is infinite*
(5) $\forall s\in\omega^{<\omega}$ *strictly decreasing with* $|s| = L+1$ $\theta_{s^*}[\omega] \cap S_{s(0)} = \emptyset$
(6) $\theta_n\big(2^{q_n}\cdot(j+1)\big) - \theta_n(2^{q_n}\cdot j) \leq 2^{q_n+1}\cdot 3$

Proof. (1) If $k\in S_n$, then $\theta_n(k)$ is a multiple of an odd number ≥ 3. It remains to note that $\{2^j \mid j\geq q_n\} \subseteq S_n$ and θ_n is the identity on $\omega\setminus S_n$.

(2) If $L=1$, then θ_n is a bijection between $\omega\setminus S_n$ onto itself on one side, and from S_n onto

$$\{2^{q_n}\cdot(2\cdot k+1) \mid k\geq 1\} \subseteq S_n$$

on the other side. So we may assume that $L\geq 2$.

The map $\theta' : \{j \in \omega \mid j \geq 1\} \to \{3k \mid k \geq 1\}$ defined by $\theta'(j) := 3j$ is a bijection. The difference between θ and θ' is that θ exchanges $\theta'(2^{31} \cdot 3 \cdot k)$ and $\theta'(2^{31} \cdot 3 \cdot k+1)$ for each $k \geq 1$ with $k \notin P_L$. Thus θ is a bijection too. This implies that θ_n is a bijection between $\omega \setminus S_n$ onto itself on one side, and from S_n onto $\{2^{q_n} \cdot 3 \cdot k \mid k \geq 1\} \subseteq S_n$ on the other side.

(3) It is enough to check that θ is fixed point free. We may assume that $L \geq 2$. In the first two cases of the definition of θ, $\theta(j) \geq 3j > j$. In the last case, if $\theta(j) = j$, then $2j = 3$, which is absurd.

(4) Fix $r \in \omega$. Note that

$$\theta_{s^*}(2^{q_{s(0)}+r}) = \theta_{<s(1),\cdots,s(|s|-2)>}(2^{q_{s(0)}+r} \cdot 3) = \cdots = \theta_{s(1)}(2^{q_{s(0)}+r} \cdot 3^{|s|-2}) = 2^{q_{s(0)}+r} \cdot 3^{|s|-1} \in S_{s(0)}.$$

(5) We may assume that $L \geq 2$. Note first that if $\theta(3j) = 2^{31} \cdot 3 \cdot m$, then $j \in P_L$. Indeed, we argue by contradiction. If $3j \notin \big(M_L \cup (M_L+1)\big)$, then $\theta(3j) = 3^2 \cdot j$, so that $3j = 2^{31} \cdot m$. In particular, m is a multiple of 3, which implies that m is of the form $2^p \cdot 3^{l+1}$ with $l < L-2$ and $j = 2^{31+p} \cdot 3^l$. If $3j = 2^{31} \cdot 3 \cdot k$ and $k \notin P_L$, then $\theta(3j) = 3^2 \cdot j + 3$, so that $3j$ is a multiple of 2^{31} and $3j+1 = 2^{31} \cdot m$, which is absurd. Finally, $3j$ cannot be of the form $2^{31} \cdot 3 \cdot k+1$.

If $\theta_{s^*}(J) \in S_{s(0)}$, then $\theta_{s(|s|-1)}(J) \in S_{s(|s|-2)}$ is a multiple of 3 and is of the form $2^{q_{s(|s|-2)}} \cdot 3 \cdot j$. Thus $\theta_{<s(|s|-2),s(|s|-1)>}(J) = \theta_{s(|s|-2)}(2^{q_{s(|s|-2)}} \cdot 3 \cdot j) = 2^{q_{s(|s|-2)}} \cdot \theta(3j) \in S_{s(|s|-3)}$ is of the form $2^{q_{s(|s|-3)}} \cdot k$. Thus $\theta(3j) = 2^{q_{s(|s|-3)} - q_{s(|s|-2)}} \cdot k$ is of the form $2^{31} \cdot 3 \cdot m$ since

$$q_{s(|s|-3)} - q_{s(|s|-2)} \geq q_{s(|s|-2)+1} - q_{s(|s|-2)} \geq 32^{q_1} - q_1 = q_2 - q_1 = 31.$$

The previous point implies that $j \in P_L$. Thus

$$\theta_{s^*}(J) = \theta_{<s(1),\cdots,s(|s|-2)>}(2^{q_{s(|s|-2)}+p} \cdot 3^{l+1}) = \theta_{<s(1),\cdots,s(N)>}(2^{q_{s(|s|-2)}+p} \cdot 3^{|s|+l-N-1})$$

as in (4), as long as $|s|+l-N-1 \leq L-2$, i.e., $l+2 \leq N$. In other words,

$$\theta_{s^*}(J) = \theta_{<s(1),\cdots,s(l+2)>}(2^{q_{s(|s|-2)}+p} \cdot 3^{L-2}) = \theta_{<s(1),\cdots,s(l+1)>}(2^{q_{s(|s|-2)}+p} \cdot 3^{L-1}).$$

As $\theta_{s^*}(J) \in S_{s(0)}$, $q_{s(|s|-2)} + p \geq q_{s(0)}$. Now $\theta_{s(l+1)}(2^{q_{s(|s|-2)}+p} \cdot 3^{L-1}) = 2^{q_{s(|s|-2)}+p} \cdot 3^L + 3$ is not in $S_{s(l)}$, which is the desired contradiction.

(6) We may assume that $L \geq 2$. Note that $\theta(j+1) - \theta(j) \in \{3, 6, -3\}$, so that

$$\theta_n\big(2^{q_n} \cdot (j+1)\big) - \theta_n(2^{q_n} \cdot j) \leq 2^{q_n+1} \cdot 3.$$

This finishes the proof. $\square$

Notation. We define, for $s \in \omega^{<\omega} \setminus \{\emptyset\}$, $s^- := < s(0), \cdots, s(|s|-2) >$ and

$$s^{-1} := < s(|s|-1), \cdots, s(0) > .$$

We then set, for $n \in \omega$, $\mathbb{D}_n^1 := N_{t_n 0}$. We extend the definition of $g_n : N_{t_n 0} \to N_{t_n 1}$ after Theorem 1.5 to any $L \geq 2$, writing g_n^L instead of g_n when a confusion is possible. Fix $L \geq 2$. We set, for $n \in \omega$,

$$\mathbb{D}_n^L := \big\{\alpha \in N_{t_n 0} \mid \forall m \leq n \;\; \alpha(2^{q_n} \cdot 3^m) \neq \alpha(2^{q_n} \cdot 3^{m+1})\big\}$$

and $\mathbb{G}_L := \bigcup_{n \in \omega} \operatorname{Graph}(g^L_{n\,|\mathbb{D}_n^L})$, so that $\mathbb{D}_n^L$ is a nonempty clopen subset of 2^ω and $\mathbb{G}_L$ is a $\mathbf{\Sigma}_2^0$ digraph on 2^ω.

Lemma 5.2 *Fix $L \geq 1$.*

(a) $\big(2^\omega, (g^L_{n\,|\mathbb{D}^L_n})\big)$ is a strongly complex situation.

(b) $g_s(\alpha) \neq g_{s^-}(\alpha)$ if $s \in \omega^{<\omega}$ is strictly increasing, $2 \leq |s| \leq L$, $\alpha \in \mathbb{D}^L_{s(|s|-1)}$ and $g_s(\alpha), g_{s^-}(\alpha)$ are defined.

(c) $g_s(\alpha) = g_{s^-}(\alpha)$ if $s \in \omega^{<\omega}$ is strictly increasing, $|s| = L+1$ and $g_s(\alpha), g_{s^-}(\alpha)$ are defined.

Proof. (a) It is known that 2^ω is a nonempty zero-dimensional perfect Polish space. Note that g_n is defined, partial, and continuous since $g_n(\alpha)(k)$ depends only on $\alpha|\big(\theta_n(k)+1\big)$. If $x \in 2^{<\omega}$, then by Lemma 5.1.(2) $g_n[N_{t_n 0x}] = \big\{\beta \in N_{t_n 1} \mid \forall 2^{q_n} < i < 2^{q_n}+1+|x| \ \ \big(i = \theta_n(k) \Rightarrow \beta(k) = (t_n 0x)(i)\big)\big\}$ is clopen, so that g_n is open and onto with clopen domain and range, and $g^L_{n\,|\mathbb{D}^L_n}$ is partial continuous open with clopen domain and range. If $\alpha \in 2^\omega$ and $l \in \omega$, then there is n with $\alpha|l = \psi(n)$. Pick $\alpha_0 \in \mathbb{D}^L_n$. Then $\big(\alpha_0, g^L_n(\alpha_0)\big) \in \mathbb{G}_L \cap N^2_{\alpha|l}$. Finally, assume that U is an open subset of 2^ω and $\alpha \in U \cap \mathbb{D}^L_n$. Then we can find $x \in 2^{<\omega}$ such that $\alpha \in N_{t_n 0x} \subseteq U \cap \mathbb{D}^L_n$. If $\beta(k) = \alpha(k)$ for $k \in \theta_n[\omega] \cup 2^{q_n+1+|x|}$, then $\beta \in N_{t_n 0x} \subseteq U \cap \mathbb{D}^L_n$ and $g_n(\beta) = g_n(\alpha)$. As $\omega \setminus \theta_n[\omega]$ is infinite by Lemma 5.1.(1), $\omega \setminus (\theta_n[\omega] \cup 2^{q_n+1+|x|})$ is also infinite, and the set of such β's is not countable.

(b) We argue by contradiction. We set $k := 2^{q_{s(|s|-1)}}$. As (q_n) and s are strictly increasing and $|s| \geq 2$, $k \neq 2^{q_{s(0)}}$, and $g_{s(0)}(\beta)(k) = \beta\big(\theta_{s(0)}(k)\big)$ if $\beta \in N_{t_{s(0)}0}$. Thus

$$g_{s(0)}(\beta)(k) = \beta\big(\theta_{s(0)}(2^{q_{s(0)}} \cdot 2^{q_{s(|s|-1)} - q_{s(0)}})\big) = \beta\big(2^{q_{s(0)}} \cdot \theta(2^{q_{s(|s|-1)} - q_{s(0)}})\big) = \beta(2^{q_{s(|s|-1)}} \cdot 3)$$

if $\beta \in N_{t_{s(0)}0}$. As $2^{q_{s(|s|-1)}} \cdot 3 \neq 2^{q_{s(1)}}$,

$$\begin{aligned} g_{<s(0),s(1)>}(\beta)(k) &= g_{s(1)}(\beta)(2^{q_{s(|s|-1)}} \cdot 3) = \beta\big(\theta_{s(1)}(2^{q_{s(|s|-1)}} \cdot 3)\big) \\ &= \beta\big(2^{q_{s(1)}} \cdot \theta(2^{q_{s(|s|-1)} - q_{s(1)}} \cdot 3)\big) = \beta(2^{q_{s(|s|-1)}} \cdot 3^2) \end{aligned}$$

if $\beta \in N_{t_{s(1)}0}$ and $g_{s(1)}(\beta) \in N_{t_{s(0)}0}$. Similarly, $g_s(\alpha)(k) = \alpha(2^{q_{s(|s|-1)}} \cdot 3^{|s|})$ since $2 \leq |s| \leq L$, which contradicts the fact that $\alpha \in \mathbb{D}^L_{s(|s|-1)}$ since $|s|-1 \leq s(|s|-1)$ cause s is stricly increasing.

(c) As above, $g_s(\alpha)(k) = \alpha\big(\theta_{s^{-1}}(k)\big)$ if $k \neq 2^{q_{s(0)}}$. As $\theta_{(s^-)^{-1}}(k)$ is in $\theta_{(s^-)^{-1}}[\omega]$, it is not in $S_{s(|s|-1)}$ by Lemma 5.1.(5). Thus $\theta_{s^{-1}}(k) = \theta_{(s^-)^{-1}}(k)$ by Lemma 5.1.(3) and we are done. $\square$

We introduce some finitary objects used in the construction of our homomorphism. Fix $L \geq 1$.

Notation. We inductively define, for each $l \in \omega$,

- a subset X_l of $2^{\leq l}$ such that 2^ω is the disjoint union of $(N_x)_{x \in X_l}$,
- an oriented graph B_l on X_l, containing finite approximations of $\mathbb{G}_L$ and providing some control on the cycles,
- a map $\varphi_l : B_{l+1} \to l$ such that $t_{\varphi_l(y,x)}0 \subseteq y$ and $t_{\varphi_l(y,x)}1 \subseteq x$ if $(y,x) \in B_{l+1}$, giving the number of the function approximated by an element of B_{l+1},
- an uogas A_l on X_l contained in B_l, containing finite approximations of a subset of $\mathbb{G}_L$ with acyclic symmetrization,
- a subset E_l of X_l, determining the future elements of X_{l+1}.

We first perform the construction of these objects for $l=0$, sometimes more.

- We set $X_0:=\{\emptyset\}$, so that $X_0\subseteq 2^{\leq 0}$ and 2^ω is the disjoint union of $(N_x)_{x\in X_0}$.
- Generally speaking, we set

$$B_l:=\big\{(y,x)\in X_l^2 \mid (N_y\times N_x)\cap \mathbb{G}_L\neq\emptyset \wedge \exists i<|y|,|x|\ \ y(i)\neq x(i)\big\},$$

so that $B_0=\emptyset$. Note that B_l is an oriented graph on X_l since $B_l\subseteq <_{\text{lex}}$.

- We set $A_0:=\emptyset$.
- If $x\in X_l$, then the decision of putting x in E_l or not is made by induction on M_x, related to A_l, defined before Lemma 4.7. If $M_x=1$, then we put x in E_l. We set, when the decision of putting y in E_l is made,

$$l_y:=\begin{cases}|y| \text{ if } y\notin E_l,\\ |y|+1 \text{ if } y\in E_l.\end{cases}$$

Assume that the decision of putting x in E_l is made if $1\leq M_x\leq m$, which is the case for $m=1$, and fix $x\in X_l$ with $M_x=m+1$, so that $x\notin \min_{X_l}$ and $\text{Pred}(x)\neq\emptyset$. Fix $y\in\text{Pred}(x)$, so that $(y,x)\in A_l$. We put $n:=\varphi_{l-1}(y,x)$. We say that x is y**-expandable** if $\theta_n(|x|)<l_y$. Generally speaking, we put x in E_l exactly when x is y-expandable for each $y\in\text{Pred}(x)$ and, when there is $q\in\omega$ with $t_q\in E_l$, there is no $1\leq i<|p_{t_q}|$ with $x=p_{t_q}(i)$. In particular, $E_0=\{\emptyset\}$.

Assume that our objects have been constructed for $p\leq l$, which is the case for $l=0$.

- We set $X_{l+1}:=\big\{s\varepsilon \mid s\in E_l\wedge\varepsilon\in 2\big\}\cup(X_l\setminus E_l)$, so that $X_{l+1}\subseteq 2^{\leq l+1}$ and 2^ω is the disjoint union of $(N_x)_{x\in X_{l+1}}$.
- If $(y,x)\in B_{l+1}$, then there is a unique $n\in\omega$ with $(N_y\times N_x)\cap\text{Graph}(g_n^L)\neq\emptyset$, and $2^{q_n}=|y\wedge x|$. Note that $n\leq q_n<2^{q_n}<|y|\leq l+1$. We set $\varphi_l(y,x):=n$, which defines $\varphi_l:B_{l+1}\to l$.
- We now define A_{l+1}. If $t_q\in\max_{X_l}\cap E_l$, then we put (t_q0,t_q1) in A_{l+1}. If $y\in X_l\setminus\max_{X_l}$, then let x be the unique element of $\text{Succ}(y)$, so that $(y,x)\in A_l$. As $A_l\subseteq B_l$, we can define $n:=\varphi_{l-1}(y,x)$. If $x\notin E_l$, then we put

$$\begin{cases}(y,x) \text{ if } y\notin E_l,\\ (y\eta,x) \text{ if } y\in E_l^-:=E_l\setminus\{t_m\mid m\in\omega\}\wedge\eta\in 2,\\ (y1,x),(y0,y1) \text{ if } y=t_q\in E_l\end{cases}$$

in A_{l+1}. If $x\in E_l$, then $y\notin\{t_m\mid m\in\omega\}$, x is y-expandable and $\theta_n(|x|)<l_y$. We put $(y\eta,x\varepsilon)$ in A_{l+1} if $\eta\in 2^{\leq 1}$, $y\eta\in X_{l+1}$ and $\varepsilon=(y\eta)\big(\theta_n(|x|)\big)$. We first check the following announced facts.

Lemma 5.3 *Let $l\in\omega$. Then A_l is an uogas on X_l contained in B_l.*

Proof. We argue by induction on l, and the case $l=0$ is clear. Note that $A_{l+1}\subseteq B_{l+1}$, by definition of A_{l+1}, so that A_{l+1} is an oriented graph on X_{l+1}. By definition, A_{l+1} is unambiguously oriented. We argue by contradiction to see that $s(A_{l+1})$ is acyclic. Let $(u_j)_{j\leq N}$ be a $s(A_{l+1})$-cycle. We choose $j\leq N$ such that u_j is $<_{\text{lex}}$-minimal, and we may assume that $0<j<N$. Then $(u_j,u_{j-1}),(u_j,u_{j+1})\in A_{l+1}$, so that $u_{j-1}=u_{j+1}$ since A_{l+1} is unambiguously oriented, which is absurd. □

Lemma 5.4 *Let $l\in\omega$, and $x\in X_{l+1}$. Then $|p_x|\leq l+1$.*

Proof. We argue by induction on l. For $l=0$, $X_1=\{(0),(1)\}$, $p_{(1)}=\big((1)\big)$ and $p_{(0)}=\big((0)\big)$, so we are done. Let $x\in X_{l+2}$. Note first that if $i<L:=|p_x|$, then we can find a unique couple (x_i,ε_i) in $X_{l+1}\times 2^{\leq 1}$ with $p_x(i)=x_i\varepsilon_i$. Let $i<L-1$. By definition of A_{l+2}, either $(x_i,x_{i+1})\in A_{l+1}$, or there is $q\in\omega$ such that $t_q\in E_{l+1}$ and $\big(p_x(i),p_x(i+1)\big)=(t_q0,t_q1)$. In particular, $(x_i)_{i<L}$ is $\leq_{\text{lex}}$-increasing. By definition of E_{l+1}, there is at most one $i<L-1$ for which there is $q\in\omega$ such that $t_q\in E_{l+1}$ and $\big(p_x(i),p_x(i+1)\big)=(t_q0,t_q1)$. If such a q does not exist, then $(x_i)_{i<L}=p^{s(A_{l+1})}_{x_0,x_{L-1}}$ and $L\leq l+1$, by induction assumption. If now there is such a q, then $(x_0,\cdots,x_i,x_{i+2},\cdots,x_{L-1})=p^{s(A_{l+1})}_{x_0,x_{L-1}}$ and $L-1\leq l+1$, by induction assumption, so that $L\leq l+2$. $\square$

Lemma 5.5 *Fix $L\geq 1$. Then there is $(L_n)_{n\in\omega}\in(\omega\setminus\{0\})^\omega$ strictly increasing satisfying the following properties:*

(a) $t_n\in E_{L_n}\setminus(\bigcup_{k<L_n}\ E_k)$,
(b) $X_{L_n}\cap X_{L_{n+1}}=\emptyset$,
(c) $2^{q_n}\leq L_n<L_n+(6L_n)^{L_n-1}\leq 2^{q_{n+1}}$.

Proof. We construct L_n inductively on n. We set $L_0:=1$, which is correct since $t_0=(0)$ is in $E_1\setminus E_0$, $p_0=0$ and $p_1=1$. As $X_1=\{(0),(1)\}$, $|x|>0$ if $x\in X_{L_0}$. Assume that L_n has been constructed.

Claim. *Let $l\geq L_n+(3\cdot 2^{q_n+1})^{L_n-1}$. Then X_l does not meet X_{L_n}. In fact, we can add $m\geq 1$ coordinates at least to $x\in X_{L_n}$ in X_l if $l\geq L_n+m\cdot(3\cdot 2^{q_n+1})^{L_n-1}$ and no t_q appears in some E_k with $L_n<k<l$.*

Indeed, fix $x\in X_{L_n}$. We want to extend properly x in some X_l with $l>L_n$, and to give an estimate on l. We proceed by induction on M_x, which was defined just before Lemma 4.7. If $M_x=1$, then $l:=L_n+1\leq L_n+(3\cdot 2^{q_n+1})^{L_n-1}$ is suitable. Assume now that $M_x=2$, which gives $y\in\min_{X_{L_n}}$ with $x\in p_y$, and in fact $x=p_y(1)$. In particular, $(y,x)\in A_{L_n}\subseteq B_{L_n}$ and $N:=\varphi_{L_n-1}(y,x)<L_n-1$ is defined. The definition of g^L_N and Lemma 5.1.(6) show that if $\alpha\in N_y$, then we can find $l\leq L_n+3\cdot 2^{q_N+1}$ and $z\subseteq\alpha$ with $z\in X_l$ and $z(|z|-1)=g^L_N(\alpha)(|x|)$. Note that $q_N\leq q_n$ since (q_n) is increasing, $L_n\geq 2$ since $B_0=B_1=\emptyset$, and $L_n<L_n+3\cdot 2^{q_n+1}\leq L_n+6L_n\leq L_n+(6L_n)^{L_n-1}\leq 2^{q_{n+1}}$. Thus x is properly extended in $X_{L_n+3\cdot 2^{q_n+1}}$, and also in $X_{L_n+(3\cdot 2^{q_n+1})^{L_n-1}}$. This argument also shows that x can be extended by m coordinates in $X_{L_n+m\cdot 3\cdot 2^{q_n+1}}$ if $L_n<L_n+m\cdot 3\cdot 2^{q_n+1}\leq 2^{q_{n+1}}$. More generally, this argument shows that x can be properly extended in $X_{L_n+(3\cdot 2^{q_n+1})^{L_n-1}}$, by Lemma 5.4, which implies that $M_x\leq L_n$. $\diamond$

Now note that $|\psi(n+1)|\leq n+1\leq 2^n\leq 2^{q_n}\leq L_n<L_n+(3\cdot 2^{q_n+1})^{L_n-1}\leq 2^{q_{n+1}}$. We will extend $\psi(n+1)$ until we reach $t_{n+1}=\psi(n+1)0^{2^{q_{n+1}}-|\psi(n+1)|}$ in some X_q with $q\geq 2^{q_{n+1}}$. Note first that there is $x\in X_{L_n}$ with $x\subseteq\psi(n+1)0^\infty$, and $\psi(n+1)\subseteq x$ since $|x|>n$.

In order to do this, we add $m:=2^{p_{n+1}}-|x|$ coordinates to x. By the claim, this will be possible in X_q for some $2^{q_{n+1}}\leq q\leq L_n+m\cdot(3\cdot 2^{q_n+1})^{L_n-1}$ if no t_r appears in some E_k with $L_n<k<q$. This is the case since $q<L_n+(m+1)\cdot(3\cdot 2^{q_n+1})^{L_n-1}\leq 2^{q_{n+1}}+2^{q_{n+1}}\cdot(6L_n)^{L_n-1}\leq(2^{q_{n+1}})^2<2^{q_{n+2}}$ because $q_{n+2}=32^{q_{n+1}}>2q_{n+1}$. It remains to check that $L_{n+1}+(6L_{n+1})^{L_{n+1}-1}\leq 2^{q_{n+2}}$. As $L_{n+1}\leq(2^{q_{n+1}})^2$, $L_{n+1}+(6L_{n+1})^{L_{n+1}-1}\leq(6L_{n+1})^{L_{n+1}}\leq 2^{(2q_{n+1}+3)\cdot 2^{2q_{n+1}}}$. It remains to note that $(2q_{n+1}+3)\cdot 2^{2q_{n+1}}\leq 2^{5q_{n+1}}=32^{q_{n+1}}=q_{n+2}$. $\square$

Corollary 5.6 *For each $x \in 2^{<\omega}$ there is $l \in \omega$ such that $x \in X_l$.*

Proof. We argue by induction on $|x|$. For $x = \emptyset$, we can take $l := 0$. Assume that $x \in X_l$ and $\varepsilon \in 2$. If there is $l' > l$ with $x\varepsilon \in X_{l'}$, then we are done. Otherwise, $x \in X_{l'}$ for each $l' \geq l$. Using Lemma 5.5, we choose $n \in \omega$ such that $L_n \geq l$. Then $x \in X_{L_n} \cap X_{L_{n+1}}$, which is absurd. $\square$

Lemma 5.7 *We work with $L = 1$. Let $l \in \omega$, $(y, x) \in B_{l+1}$ and $p := p^{s(A_{l+1})}_{y, x_{C(y)}}$. Then*

(a) $x = p(j)$ for some $1 \leq j < |p|$,

(b) $\varphi_l\big(p(j-1), p(j)\big) = \varphi_l(y, x) = \min_{i<j} \varphi_l\big(p(i), p(i+1)\big)$,

(c) $\Big(\varphi_l\big(p(i), p(i+1)\big)\Big)_{i<j}$ is injective.

Proof. We argue by induction on l. We are done if $l = 0$ since $B_1 = \emptyset$. So assume that $(y, x) \in B_{l+2}$, which gives $n := \varphi_{l+1}(y, x) \leq l$ with $t_n 0 \subseteq y$ and $t_n 1 \subseteq x$. In particular, $l + 1 \geq L_n$. If $l + 1 = L_n$, then $(y, x) = (t_n 0, t_n 1)$ and $p = (t_n 0) p^{s(A_{l+2})}_{t_n 1, x_{C(t_n 0)}}$ by Lemma 4.2 and since $A_{l+2} \subseteq <_{\text{lex}}$. Thus $j = 1$ is convenient. So we may assume that $l \geq L_n$. Note that we can find $(y', x') \in B_{l+1}$ with $t_n 0 \subseteq y' \subseteq y$ and $t_n 1 \subseteq x' \subseteq x$. By the induction assumption, $p' := p^{s(A_{l+1})}_{y', x_{C(y')}}$ is defined and there is $1 \leq j' < |p'|$ with $x' = p'(j')$.

Case 1. We cannot find $q \in \omega$ with $t_q \in p^{s(A_{l+1})}_{y', x'} \cap E_{l+1}$.

Let $\varepsilon_0 \in 2^{\leq 1}$ such that $y = y'\varepsilon_0$. Note that there is a unique $\varepsilon_1 \in 2^{\leq 1}$ such that $\big(y, p'(1)\varepsilon_1\big)$ is in A_{l+2}, by definition of A_{l+2}. Similarly, if $1 \leq i < j'$, then there is a unique $\varepsilon_{i+1} \in 2^{\leq 1}$ such that $\big(p'(i)\varepsilon_i, p'(i+1)\varepsilon_{i+1}\big) \in A_{l+2}$. Note that $p^{s(A_{l+2})}_{y, p'(j')\varepsilon_{j'}}$ is defined and equal to $\big(p'(i)\varepsilon_i\big)_{i \leq j'}$. We will be able to set $j := j'$ if we prove that $x = x'\varepsilon_{j'}$, by Lemma 4.2 and since $A_{l+2} \subseteq <_{\text{lex}}$ again. This is the case if $\varepsilon_{j'} = \emptyset$, so we may assume that $\varepsilon_{j'} \in 2$, which implies that $x' \in E_{l+1}$. Let $\varepsilon := x(|x'|)$, so that $x = x'\varepsilon$ and we have to see that $\varepsilon_{j'} = \varepsilon$. Note that $p'(i+1)\varepsilon_{i+1}$ is $\big(p'(i)\varepsilon_i\big)$-expandable if $\varepsilon_{i+1} \in 2$, for each $i < j'$. More precisely, if $n_i := \varphi_l\big(p'(i), p'(i+1)\big)$, then $\varepsilon_{i+1} = \big(p'(i)\varepsilon_i\big)\Big(\theta_{n_i}\big(|p'(i+1)|\big)\Big)$. This implies that $\varepsilon_{j'} = y\big(\theta_n(|x'|)\big)$ since $n_{j'-1} = n = \min_{i<j'} n_i$ and $(n_i)_{i<j'}$ is injective, by induction assumption. In particular, y is long enough to ensure that $\varepsilon_{j'} = \varepsilon$ since $(y, x) \in B_{l+2}$ with witness n.

Case 2. $t_q \in p^{s(A_{l+1})}_{y', x'} \cap E_{l+1}$ for some $q \in \omega$, which implies that $l + 1 = L_q$.

Fix $i_0 \leq j'$ with $p'(i_0) = t_q$. The definition of ε_i is as in Case 1 if $i \leq i_0$. If $\varepsilon_{i_0} = 0$, then we set $\varepsilon_i := \emptyset$ if $i_0 < i \leq j'$ and we note that $p^{s(A_{l+2})}_{y,x} = \big(p'(0)\varepsilon_0, \cdots, p'(i_0)0, p'(i_0)1, p'(i_0+1), \cdots, x'\big)$ if $i_0 < j'$, and $p^{s(A_{l+2})}_{y,x} = \big(p'(0)\varepsilon_0, \cdots, p'(i_0)0\big)$ if $i_0 = j'$. Apart from that, we argue as in Case 1. If $\varepsilon_{i_0} = 1$, then we set $\varepsilon_i := \emptyset$ if $i_0 < i \leq j'$ and we note that

$$p^{s(A_{l+2})}_{y,x} = \big(p'(0)\varepsilon_0, \cdots, p'(i_0)1, p'(i_0+1), \cdots, x'\big)$$

if $i_0 < j'$, and $p^{s(A_{l+2})}_{y,x} = \big(p'(0)\varepsilon_0, \cdots, p'(i_0)1\big)$ if $i_0 = j'$. Apart from that, we argue as in Case 1.

Note that (b), (c) follow from the previous discussion since $\Big(\varphi_l\big(p'(i),p'(i+1)\big)\Big)_{i<j'}$ is equal to $\Big(\varphi_{l+1}\big(p(i),p(i+1)\big)\Big)_{i<j}$, except when $i_0<j'$ in the first subcase of the Case 2 where one number bigger than the others has been added strictly before the last position. $\square$

Lemma 5.8 *Fix $L\geq 1$. If $n\in\omega$, $y_{|t_n|+1}:=t_n1$ and $k>|t_n|+1$, then for each $\alpha\in N_{t_n0}$ we can find $l_k\in\omega$ and $y_k\in X_{l_k}$ such that $(\alpha|k,y_k)\in B_{l_k}$ and $y_{k-1}\subseteq y_k$; moreover, $\{g_n(\alpha)\}=\bigcap_{k>|t_n|}\ N_{y_k}$.*

Proof. By Corollary 5.6, there is $l_k\in\omega$ such that $\alpha|k\in X_{l_k}$. Note that $l_k>L_n+1$ since $k>|t_n|+1$. There is $y_k\in X_{l_k}$ such that $g_n(\alpha)\in N_{y_k}$ since 2^ω is the disjoint union of $(N_x)_{x\in X_{l_k}}$. Note that $y_k\supseteq t_n1$ since $t_n1\in X_{L_n+1}$ and $l_k>L_n+1$, so that $(\alpha|k,y_k)\in B_{l_k}$. $\square$

6 The main construction

We now come to the construction of our homomorphism.

Theorem 6.1 *Let $\big(Z,(f_n)\big)$ be a strongly complex situation satisfying Condition (d) in Theorem 1.8. Then $(2^\omega,\mathbb{G}_1)\preceq_c^{inj}(Z,A^f)$.*

Proof. We construct, inductively on l,

$\diamond$ a sequence $(U_x^l)_{x\in X_l}$ of nonempty clopen subsets of Z,
$\diamond$ a natural number $\phi(n)$ if $L_n<l$.

We want these objects to satisfy the following conditions, using Lemma 5.7.

(1) $U_{x\varepsilon}^{l+1}\subseteq U_x^l$ if $\varepsilon\in 2^{\leq 1}\wedge x\varepsilon\in X_{l+1}$
(2) $\mathrm{diam}(U_x^l)\leq 2^{-l}$
(3) $U_{x0}^{l+1}\cap U_{x1}^{l+1}=\emptyset$ if $x0,x1\in X_{l+1}$
(4) $U_y^{l+1}\subseteq D_{\phi(\varphi_l(y,x))}\wedge U_x^{l+1}\subseteq f_{\phi(\varphi_l(y,x))}[U_y^{l+1}]$ if $(y,x)\in A_{l+1}$
(5) $U_{p(m)}^{l+1}\subseteq D_{\phi(\varphi_l(y,x))}$ if $(y,x)\in B_{l+1}\wedge m<j\wedge$
$\varphi_l\big(p(j-1),p(j)\big)=\varphi_l(y,x)=\min_{i<j}\varphi_l\big(p(i),p(i+1)\big)$
(6) $\phi(r)>\sup_{n<r}\phi(n)$

Assume that this is done. Fix $\alpha\in 2^\omega$ and $l\in\omega$. As 2^ω is the disjoint union of the N_x's for $x\in X_l$, there is a unique $k_l\in\omega$ for which $\alpha|k_l\in X_l$. The sequence $(U_{\alpha|k_l}^l)_{l\in\omega}$ defines $h(\alpha)\in Z$, using (1) and (2). Note that $h:2^\omega\to Z$ is continuous, and injective by (3) and Corollary 5.6 (which implies that k_l tends to infinity as l tends to infinity).

Let us prove that $U_y^{l+1}\subseteq D_{\phi(\varphi_l(y,x))}$ and $U_x^{l+1}\subseteq f_{\phi(\varphi_l(y,x))}[U_y^{l+1}]$ if $(y,x)\in B_{l+1}$. By Lemma 5.7, $\varphi_l\big(p(j-1),p(j)\big)=\varphi_l(y,x)=\min_{i<j}\varphi_l\big(p(i),p(i+1)\big)$ and $\varphi_l(y,x)<\varphi_l\big(p(i),p(i+1)\big)$ if $i<j-1$. By Lemmas 5.3, 4.2.(b) and (4), $U_{p(i)}^{l+1}\subseteq D_{\phi(\varphi_l(p(i),p(i+1)))}$ and $U_{p(i+1)}^{l+1}\subseteq f_{\phi(\varphi_l(p(i),p(i+1)))}[U_{p(i)}^{l+1}]$ if $i<j$. (5) implies that $U_{p(m)}^{l+1}\subseteq D_{\phi(\varphi_l(y,x))}$ if $m<j$.

We apply Lemma 2.5 to $V_0 := U^{l+1}_{p(i)}$, $V_1 := U^{l+1}_{p(i+1)}$, $m := \phi\big(\varphi_l(y,x)\big)$ and

$$n := \phi\Big(\varphi_l\big(p(i), p(i+1)\big)\Big)$$

when $i < j-1$. We get $f_{\phi(\varphi_l(y,x))}[U^{l+1}_{p(i+1)}] \subseteq f_{\phi(\varphi_l(y,x))}[U^{l+1}_{p(i)}]$, so that

$$U^{l+1}_x \subseteq f_{\phi(\varphi_l(y,x))}[U^{l+1}_{p(j-1)}] \subseteq \cdots \subseteq f_{\phi(\varphi_l(y,x))}[U^{l+1}_y].$$

Thus $U^{l+1}_x \subseteq f_{\phi(\varphi_l(y,x))}[U^{l+1}_y]$, as desired.

Let $(\alpha,\beta) \in \mathbb{G}_1$, and n with $\alpha \in N_{t_n 0}$ and $\beta = g_n(\alpha)$. By Lemma 5.8, $(\alpha|k, \beta||y_k|) \in B_{l_k}$ if $k > |t_n|+1$. By the previous point, $U^{l_k}_{\alpha|k} \subseteq D_{\phi(n)}$ and $U^{l_k}_{\beta||y_k|} \subseteq f_{\phi(n)}[U^{l_k}_{\alpha|k}]$ if $k > |t_n|+1$. In particular, $h(\alpha) \in D_{\phi(n)}$. As $f_{\phi(n)}$ is continuous at $h(\alpha)$, $\mathrm{diam}(f_{\phi(n)}[U^{l_k}_{\alpha|k}])$ converges to zero as k converges to infinity. Thus $d\big(f_{\phi(n)}\big(h(\alpha)\big), h(\beta)\big)$ is zero and $f_{\phi(n)}\big(h(\alpha)\big) = h(\beta)$.

So it is enough to prove that the construction is possible. We fix a compatible metric with $\mathrm{diam}(Z) \leq 1$. We first set $U_\emptyset := Z$. Assume that $\big((U_x)_{x\in X_p}\big)_{p\leq l}$ and $\big(\phi(n)\big)_{L_n<l}$ satisfying (1)-(6) have been constructed, which is the case for $l=0$.

By Lemmas 5.3 and 4.2.(c), we can set, for each $y \in X_l$, $p^l_y := p^{s(A_l)}_{y,x_{C(y)}}$. We choose $u_l \in \omega^{X_l}$ such that $u_l(y) := \phi\Big(\varphi_{l-1}\big(y, p^l_y(1)\big)\Big)$ if $y \in X_l \setminus \max_{X_l}$ (this can be done, by the induction assumption). We define, for $y \in X_l$, and inductively on $|p^l_y|$,

$$W_y := \begin{cases} U^l_y \text{ if } y \in \max_{X_l}, \\ U^l_y \cap f^{-1}_{u_l(y)}(W_x) \text{ if } x \in \mathrm{Succ}(y), \end{cases}$$

which defines nonempty clopen subsets of Z, by (4) of the induction assumption. Note that

$$f_{u_l(y)}[W_y] = W_x$$

if $x \in \mathrm{Succ}(y)$ since $x = p^l_y(1)$ and $W_x \subseteq U^l_x \subseteq f_{\phi(\varphi_{l-1}(y,x))}[U^l_y]$ by the induction assumption.

If l is of the form L_r, then Lemma 2.6 applied to $V := W_{t_r}$ and $m := \sup_{L_n<l} \phi(n) = \sup_{n<r} \phi(n)$ gives $\phi(r) > \sup_{n<r} \phi(n)$ and nonempty clopen subsets $O_{t_r 0}, O_{t_r 1}$ of Z such that $O_{t_r 0} \subseteq W_{t_r} \cap D_{\phi(r)}$ and $O_{t_r 1} \subseteq W_{t_r} \cap f_{\phi(r)}[O_{t_r 0}]$.

We will apply, thanks to Lemma 5.3, Lemma 4.7 to $\mathcal{T} := \big(X_{l+1}, A_{l+1}, Z, (f_n)\big)$,

$$d := \max_{x\in X_{l+1}} |x|,$$

$u \in \omega^{X_{l+1}}$ such that $u(y) := \phi\Big(\varphi_l\big(y, p^{l+1}_y(1)\big)\Big)$ if $y \in X_{l+1} \setminus \max_{X_{l+1}}$, and $(V_x)_{x\in X_{l+1}}$ defined as follows. If $x \in X_{l+1}$, we denote by x^- the unique element X_l for which there is $\varepsilon \in 2^{\leq 1}$ with $(x^-)\varepsilon = x$. We also set $q_y := p^{l+1}_{t_r 1, x_{C(y)}}$ if $y \in C(t_r 1)$. If $x \notin C(t_r 1)$, or if $x \notin q_x$ and $x^- \neq t_r$, then we set $V_x := U^l_{x^-}$. We also set $V_{t_r 1} := O_{t_r 1}$ and $V_{t_r 0} := O_{t_r 0}$. If now $x \in C(t_r 1)$, $x \in q_x \setminus \{t_r 1\}$ and $x = q_x(i)$ with $i \geq 1$, then we define V_x by induction on i. We set $V_x := f_{u_l((q_x(i-1))^-)}[V_{q_x(i-1)}]$.

This is possible since, inductively on $k < |q_x|$, $V_{q_x(k)}$ is defined, nonempty and contained in $W_{(q_x(k))^-}$, since $\Big(\big(q_x(k)\big)^-, \big(q_x(k+1)\big)^-\Big) \in A_l$, so that $f_{u_l((q_x(k))^-)}[W_{(q_x(k))^-}] = W_{(q_x(k+1))^-}$.

Note that $\big(u, (V_x)_{x\in X_{l+1}}\big) \in U_{\mathcal{T}}$. Indeed, let $(y,x) \in A_{l+1}$. If $y \notin C(t_r1)$, then $y^- \neq t_r$, so that $(y^-, x^-) \in A_l$. Moreover, $x \notin C(t_r1)$ and $V_y = U^l_{y^-}$, $V_x = U^l_{x^-}$. Thus $V_y \subseteq D_{\phi(\varphi_{l-1}(y^-,x^-))}$ and $V_x \subseteq f_{\phi(\varphi_{l-1}(y^-,x^-))}[V_y]$, by the induction assumption. It remains to note that

$$u(y) = \phi\big(\varphi_l(y,x)\big) = \phi\big(\varphi_{l-1}(y^-,x^-)\big)$$

to see that $V_y \subseteq D_{u(y)}$ and $V_x \subseteq f_{u(y)}[V_y]$. If $y \in C(t_r1)$ and $y \in q_y$, then $x \in q_y \setminus \{t_r1\}$. By definition, $V_x = f_{u_l(y^-)}[V_y]$. Note that $u(y) = \phi\big(\varphi_l(y,x)\big) = \phi\big(\varphi_{l-1}(y^-,x^-)\big) = u_l(y^-)$. If $y = t_r0$, then $x = t_r1$, $V_{t_r0} = O_{t_r0} \subseteq D_{\phi(r)} = D_{u(t_r0)}$ and $V_{t_r1} = O_{t_r1} \subseteq f_{u(t_r0)}[V_{t_r0}]$. Otherwise, $V_y = U^l_{y^-}$ and $x \in C(t_r1)$. If $x \notin q_y$ and $x^- \neq t_r$, then we argue as in the case $y \notin C(t_r1)$. If $x \in q_y$ or $x^- = t_r$, then $V_x \subseteq W_{x^-} = f_{u_l(y^-)}[W_{y^-}] \subseteq f_{u_l(y^-)}[U^l_{y^-}] = f_{u(y)}[V_y]$.

Lemma 4.7 provides a sequence $(U^{l+1}_x)_{x\in X_{l+1}}$ of pairwise disjoint nonempty clopen subsets of Z with diameter at most 2^{-l-1} such that $U^{l+1}_y \subseteq D_{u(y)}$ if $y \notin \max_{X_{l+1}}$ and

$$U^{l+1}_x \subseteq V_x \cap \bigcap_{y\in \mathrm{Pred}(x)} f_{u(y)}[U^{l+1}_y]$$

if $x \in X_{l+1}$. Note that $U^{l+1}_x \subseteq V_x \subseteq U^l_{x^-}$, by definition of the W_x's, which shows that (1) is satisfied. The construction of the U^{l+1}_x's shows that (2)-(4) are satisfied. For (5), let $(y,x) \in B_{l+1}$, and $m < j$ with $\varphi_l\big(p(j-1), p(j)\big) = \varphi_l(y,x) = \min_{i<j} \varphi_l\big(p(i), p(i+1)\big)$. If $p(q) \neq t_r1$ for each $0 < q \leq j$, then we are done, by induction assumption. If $p(q) = t_r1$ for some $0 < q \leq j$, then $q < j$ or $j = 1$, and we may assume that $q < j$. It remains to note that $U^{l+1}_{p(q-\varepsilon)} \subseteq U^l_{(p(q))^-} \subseteq D_{\phi(\varphi_l(y,x))}$ for each $\varepsilon \in 2$, by the induction assumption. □

Proof of Theorem 1.8. We apply Lemma 3.2 to $\big(X, (f_n)\big)$, $P := X$ and $S_n := D_n$. This is possible by Corollary 2.2. Lemma 3.2 provides a Borel subset S of X, a finer topology τ on S, and a sequence (C_n) of clopen subsets of $Y := (S,\tau)$ such that $\big(Y, (f_{n|C_n})\big)$ is a strongly complex situation and $C_n \subseteq S \cap g_n^{-1}(S)$. By Theorem 6.1, $(2^\omega, \mathbb{G}_1) \preceq_c^{\mathrm{inj}} \big(Y, \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|C_n})\big)$. As $\big(Y, \bigcup_{n\in\omega} \mathrm{Graph}(f_{n|C_n})\big) \preceq_c^{\mathrm{inj}} (X, A^f)$, we are done. □

Proof of Theorem 1.3. By Corollary 2.2, $\mathbb{G}_1$ is a $\mathbf{\Sigma}^0_2$ digraph of uncountable Borel chromatic number, and is in particular analytic. Theorem 1.8 shows that if S is a Borel subset of 2^ω, τ is a finer topology on S, and (C_n) is a sequence of clopen subsets of $Y := (S,\tau)$ such that $\big(Y, (g_{n|C_n})\big)$ is a strongly complex situation, then $(2^\omega, \mathbb{G}_1) \preceq_c^{\mathrm{inj}} \big(Y, \bigcup_{n\in\omega} \mathrm{Graph}(g_{n|C_n})\big)$. We apply Corollary 3.3 and the remark after it to get the minimality of $\mathbb{G}_1$. This implies the minimality of $\mathbb{G}_1^{-1}$. We saw in the introduction that $\mathbb{G}_0$ is also minimal. It remains to apply Lemma 5.2 and Corollary 2.4. □

7 The structure of $\preceq_{\mathfrak{C}}^{\mathbf{inj}}$

Theorem 7.1 *$(2^\omega, \mathbb{G}_L)_{L\in\omega}$ is a sequence made of $\mathbf{\Sigma}^0_2$ digraphs pairwise $\preceq_B^{inj}$-incompatible among analytic digraphs of uncountable Borel chromatic number on Polish spaces. In particular, the sequence $(2^\omega, \mathbb{G}_L)_{L\in\omega}$ is a $\preceq_B^{inj}$-antichain. Moreover, if $L\neq M$ are natural numbers, $s\in 2^{<\omega}$ and G is a dense G_δ subset of 2^ω, then $\big(N_s\cap G, \mathbb{G}_L\cap (N_s\cap G)^2\big) \not\preceq_c^{inj} (2^\omega, \mathbb{G}_M)$.*

Proof. By Lemma 5.2.(a), $\big(2^\omega, (g^L_{n\,|\mathbb{D}^L_n})\big)$ is a complex situation if $L\geq 1$. By Theorem 1.1 and Corollary 2.2, $\mathbb{G}_L$ is a $\mathbf{\Sigma}^0_2$ digraph on 2^ω of uncountable Borel chromatic number.

Claim. *Let $L<M$ be natural numbers, and $\big(X, (f_n)\big)$ be a complex situation satisfying the Conditions (a), (b) of a strongly complex situation. Then $(X, A^f) \preceq_c^{inj} (2^\omega, \mathbb{G}_L), (2^\omega, \mathbb{G}_M)$ cannot hold simultaneously.*

Indeed, we argue by contradiction, which gives witnesses u, u'. We set

$$R := \{(n,r,r',x)\in\omega^3\times X \mid x\in D_n \wedge u(x)\in\mathbb{D}^L_r \wedge u'(x)\in\mathbb{D}^M_{r'} \wedge u\big(f_n(x)\big)=g^L_r\big(u(x)\big) \wedge u'\big(f_n(x)\big)=g^M_{r'}\big(u'(x)\big)\}.$$

Note that R is closed, by continuity. If $n\in\omega$, U is a nonempty clopen subset of D_n and $x\in U$, then there are $r, r'\in\omega$ such that $(n,r,r',x)\in R$. By Baire's theorem, we can find a nonempty clopen subset C of U and $r, r'\in\omega$ such that $(n,r,r',x)\in R$ if $x\in C$.

We inductively construct a strictly increasing sequence $(n_k)_{k\in\omega}\in\omega^\omega$ as follows. We first apply the previous point to $n_0:=0$ and $U:=D_0$, which provides $C_0\subseteq D_{n_0}$ and $r_0, r'_0\in\omega$ such that $(n_0, r_0, r'_0, x)\in R$ if $x\in C_0$. As $\big(X, (f_n)\big)$ is a strongly complex situation, we can find $n_{k+1}>n_k$ such that Graph$(f_{n_{k+1}})\cap C_k^2\neq\emptyset$. We apply the previous point to n_{k+1} and $U\subseteq C_k\cap f^{-1}_{n_{k+1}}(C_k)$ with diameter at most 2^{-k}, which provides $C_{k+1}\subseteq U$ and $r_{k+1}, r'_{k+1}\in\omega$ such that $(n_{k+1}, r_{k+1}, r'_{k+1}, x)$ is in R if $x\in C_{k+1}$.

As (C_k) is a decreasing sequence of nonempty closed subsets of X whose diameters tend to zero, there is $x\in\bigcap_{k\in\omega} C_k$. Note that $(n_k, r_k, r'_k, x)\in R$ for each k. Let us prove that (r_k) is unbounded. We argue by contradiction, which gives $r\in\omega$ and $I\subseteq\omega$ infinite such that $r_k=r$ if $k\in I$. By continuity, we get $u(x)=g^L_r\big(u(x)\big)$ since $\lim_{k\to\infty, k\in I} f_{n_k}(x)=x$, which contradicts the fact that g^L_r is fixed point free. So, extracting a subsequence if necessary, we may assume that (r_k) and (r'_k) are strictly increasing.

As $x\in C_1$ and $f_{n_1}(x)\in C_0$, $u\Big(f_{n_0}\big(f_{n_1}(x)\big)\Big) = g^L_{r_0}\Big(u\big(f_{n_1}(x)\big)\Big) = g^L_{r_0}\Big(g^L_{r_1}\big(u(x)\big)\Big)$. More generally, $u\big(f_{n_s}(x)\big)=g^L_{r_s}\big(u(x)\big)$ and $u'\big(f_{n_s}(x)\big)=g^M_{r'_s}\big(u'(x)\big)$ if $s\in\omega^{<\omega}$ is strictly increasing.

Assume first that $L\geq 1$. We choose $s\in\omega^{L+1}$ strictly increasing. By Lemma 5.2.(c), we get $g^L_{r_s}\big(u(x)\big) = g^L_{r_{s^-}}\big(u(x)\big)$, so that $g^M_{r'_s}\big(u'(x)\big) = g^M_{r'_{s^-}}\big(u'(x)\big)$ by injectivity of u, which contradicts Lemma 5.2.(b) since $u'(x)\in\mathbb{D}^M_{r_{s(|s|-1)}}$. If $L=0$, we argue simililarly, using the fact that the g^0_r's are injective and fixed point free. $\diamond$

Let $L<M$. We argue by contradiction, which gives a Polish space Y and an analytic digraph B on Y of uncountable Borel chromatic number such that $(Y,B)\preceq_B^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M)$. Theorem 1.7 gives a complex situation $\big(X,(f_n)\big)$ such that $(X,A^f)\preceq_c^{\text{inj}}(Y,B)$ or $(X,A^f)\preceq_c^{\text{inj}}(Y,B^{-1})$.

Let us prove that $(X,A^f)\preceq_c^{\text{inj}}(Y,B)$. We argue by contradiction, so that

$$(X,A^f)\preceq_c^{\text{inj}}(Y,B^{-1}).$$

As $\mathbb{G}_L$ has countable vertical sections, B too, so that B^{-1} and A^f have countable horizontal sections. Thus A^f is locally countable. By Corollary 2.2, A^f is a $\mathbf{\Sigma}^0_2$ digraph of uncountable Borel chromatic number. The discussion after Theorem 1.1 shows that $(2^\omega,\mathbb{G}_0)\preceq_c^{\text{inj}}(X,A^f)$, and also

$$(2^\omega,\mathbb{G}_0)\preceq_c^{\text{inj}}(2^\omega,\mathbb{G}_0^{-1})\preceq_c^{\text{inj}}(Y,B)\preceq_B^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M).$$

Let H be a dense G_δ subset of 2^ω such that $(H,\mathbb{G}_0\cap H^2)\preceq_c^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M)$. The proof of Lemma 2.1.(b) shows that $\mathbb{G}_0\cap H^2$ has uncountable Borel chromatic number. The discussion after Theorem 1.1 shows that $(2^\omega,\mathbb{G}_0)\preceq_c^{\text{inj}}(H,\mathbb{G}_0\cap H^2)$, and thus $(2^\omega,\mathbb{G}_0)\preceq_c^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M)$, which contradicts the claim.

This shows that $(X,A^f)\preceq_B^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M)$. So we may assume that $Y=X$ and $B=A^f$. Let P be a dense G_δ subset of X such that $(P,A^f\cap P^2)\preceq_c^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M)$. The proof of Lemma 2.1.(b) shows that $A^f\cap P^2$ has uncountable Borel chromatic number. The previous point shows that $(2^\omega,\mathbb{G}_0)\not\preceq_c^{\text{inj}}(X,A^f)$. By Lemma 3.2, we may assume that $\big(X,(f_n)\big)$ is a strongly complex situation and $(X,A^f)\preceq_c^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M)$, which contradicts the claim.

The last assertion comes from the fact that $\mathbb{G}_L\cap(N_s\cap G)^2$ has uncountable Borel chromatic number, by the proof of Lemma 2.1.(b). $\square$

Proof of Theorem 1.5. We argue by contradiction, which gives a natural number N and a basis $(X_i,A_i)_{i<N}$. By Theorem 7.1, we can find $i<N$ and $L\neq M$ with $(X_i,A_i)\preceq_B^{\text{inj}}(2^\omega,\mathbb{G}_L),(2^\omega,\mathbb{G}_M)$. This contradicts Theorem 7.1. $\square$

We now prove Theorem 1.4.

Notation. We set, for each $n\in\omega$, $\mathbb{Q}_n:=\oplus_{L\leq n}2^\omega$ and

$$\mathbb{H}_n:=\big\{\big((L,\gamma),(M,\delta)\big)\in\mathbb{Q}_n^2\mid L=M\wedge(\gamma,\delta)\in\mathbb{G}_L\big\}.$$

Theorem 7.2 *Let $\mathfrak{C}\in\{c,B\}$. Then $(\mathbb{Q}_n,\mathbb{H}_n)_{n\in\omega}$ is a $\preceq_{\mathfrak{C}}^{inj}$-strictly increasing chain made of $\mathbf{\Sigma}^0_2$ digraphs of uncountable Borel chromatic number.*

Proof. By Theorem 7.1, $\mathbb{H}_n$ is a $\mathbf{\Sigma}^0_2$ digraph of uncountable Borel chromatic number on $\mathbb{Q}_n$. The identity map shows that our sequence is increasing. We argue by contradiction, which gives $n\in\omega$ such that $(\mathbb{Q}_{n+1},\mathbb{H}_{n+1})\preceq_B^{\text{inj}}(\mathbb{Q}_n,\mathbb{G}_n)$ with witness u. The map associating $u_0(n,\gamma)$ to $\gamma\in2^\omega$ is Borel, which gives $L<n$, $s\in2^{<\omega}$ and a dense G_δ subset G of 2^ω such that $u_0(L,\gamma)=L$ if γ is in $N_s\cap G$.

The map v associating $u_1(n,\gamma)$ to $\gamma\in N_s\cap G$ is injective and Borel. We can restrict G, so that we may assume that v is continuous. Thus $\big(N_s\cap G,\mathbb{G}_n\cap(N_s\cap G)^2\big)\preceq_c^{\mathrm{inj}}(2^\omega,\mathbb{G}_L)$ with witness v, which contradicts Theorem 7.1 since $n\neq L$. $\square$

Notation. Let $(p_n)_{n\in\omega}$ be the sequence of prime numbers. We define, for each $\alpha\in 2^\omega$, $E_\alpha\subseteq\omega$ by $E_\alpha:=\{p_0^{\alpha(0)+1}\cdots p_n^{\alpha(n)+1}\mid n\in\omega\}$. Then we set $\mathbb{P}_\alpha:=\oplus_{L\in E_\alpha}2^\omega$ and

$$\mathbb{G}_\alpha:=\big\{\big((L,\gamma),(M,\delta)\big)\in\mathbb{P}_\alpha^2\mid L=M\wedge(\gamma,\delta)\in\mathbb{G}_L\big\}.$$

Theorem 7.3 *$(\mathbb{P}_\alpha,\mathbb{G}_\alpha)_{\alpha\in 2^\omega}$ is a $\preceq_B^{\mathrm{inj}}$-antichain made of $\mathbf{\Sigma}^0_2$ digraphs of uncountable Borel chromatic number.*

Proof. By Theorem 7.1, $\mathbb{G}_\alpha$ is a $\mathbf{\Sigma}^0_2$ digraph of uncountable Borel chromatic number on $\mathbb{P}_\alpha$. We argue by contradiction, which gives $\alpha\neq\beta$ such that $(\mathbb{P}_\alpha,\mathbb{G}_\alpha)\preceq_B^{\mathrm{inj}}(\mathbb{P}_\beta,\mathbb{G}_\beta)$ with witness u. Note that $E_\alpha\cap E_\beta$ is finite, which gives $L\in E_\alpha\setminus E_\beta$. The map associating $u_0(L,\gamma)$ to $\gamma\in 2^\omega$ is Borel, which gives $M\in E_\beta$, $s\in 2^{<\omega}$ and a dense G_δ subset G of 2^ω such that $u_0(L,\gamma)=M$ if γ is in $N_s\cap G$. The map v associating $u_1(L,\gamma)$ to $\gamma\in N_s\cap G$ is injective and Borel. We can moreover restrict G, so that we may and will assume that v is continuous. Thus v is a witness for the fact that $\big(N_s\cap G,\mathbb{G}_L\cap(N_s\cap G)^2\big)\preceq_c^{\mathrm{inj}}(2^\omega,\mathbb{G}_M)$, which contradicts Theorem 7.1 since $L\neq M$. $\square$

Our main results also hold for graphs.

Theorem 7.4 *Let $\mathfrak{C}\in\{c,B\}$.*

(a) There is a $\preceq_{\mathfrak{C}}^{\mathrm{inj}}$-antichain $\{s(\mathbb{G}_0),s(\mathbb{G}_1)\}$ made of graphs $\preceq_{\mathfrak{C}}^{\mathrm{inj}}$-minimal among analytic graphs of uncountable Borel chromatic number.

(b) There is a $\preceq_{\mathfrak{C}}^{\mathrm{inj}}$-antichain of size $2^{\aleph_0}$ made of $\mathbf{\Sigma}^0_2$ graphs of uncountable Borel chromatic number.

(c) Any $\preceq_{\mathfrak{C}}^{\mathrm{inj}}$-basis for the class of analytic graphs of uncountable Borel chromatic number on Polish spaces is infinite.

Proof. We first prove the following.

Claim. *$\big(2^\omega,s(\mathbb{G}_L)\big)_{L\in\omega}$ is a sequence made of $\mathbf{\Sigma}^0_2$ graphs pairwise $\preceq_B^{\mathrm{inj}}$-incompatible among analytic graphs of uncountable Borel chromatic number on Polish spaces.*

Indeed, we argue by contradiction, which gives $L\neq M$ and an analytic graph G of uncountable Borel chromatic number on a Polish space X such that $(X,G)\preceq_B^{\mathrm{inj}}\big(2^\omega,s(\mathbb{G}_L)\big),\big(2^\omega,s(\mathbb{G}_M)\big)$, with witnesses u, u' respectively. Note that

$$\begin{aligned}G=&\big(G\cap(u\times u)^{-1}(\mathbb{G}_L)\cap(u'\times u')^{-1}(\mathbb{G}_M)\big)\cup\big(G\cap(u\times u)^{-1}(\mathbb{G}_L)\cap(u'\times u')^{-1}(\mathbb{G}_M^{-1})\big)\cup\\&\big(G\cap(u\times u)^{-1}(\mathbb{G}_L^{-1})\cap(u'\times u')^{-1}(\mathbb{G}_M)\big)\cup\big(G\cap(u\times u)^{-1}(\mathbb{G}_L^{-1})\cap(u'\times u')^{-1}(\mathbb{G}_M^{-1})\big).\end{aligned}$$

The second and the third of these subgraphs are locally countable. If one of them has uncountable Borel chromatic number, then it is above $\mathbb{G}_0$ by the discussion after Theorem 1.1. By Theorem 7.1, we must have $L=0=M$, which is absurd. Thus the first or the fourth of these subgraphs has uncountable Borel chromatic number, which contradicts the incompatibility of $\mathbb{G}_L$ and $\mathbb{G}_M$. $\diamond$

(a) By the claim, $s(\mathbb{G}_0)$ and $s(\mathbb{G}_1)$ are incompatible, and thus incomparable. Assume that we can find $L\in 2$ and an analytic graph G of uncountable Borel chromatic number on a Polish space X such that $(X,G)\preceq^{\mathrm{inj}}_{\mathfrak{c}}\big(2^\omega, s(\mathbb{G}_L)\big)$, with witness u. Note that

$$G=\big(G\cap(u\times u)^{-1}(\mathbb{G}_L)\big)\cup\big(G\cap(u\times u)^{-1}(\mathbb{G}_L^{-1})\big).$$

One of these subgraphs has uncountable Borel chromatic number. Assume for example that it is the first one. Then the minimality of $\mathbb{G}_L$ shows that $(2^\omega,\mathbb{G}_L)\preceq^{\mathrm{inj}}_{\mathfrak{c}}\Big(2^\omega,\big(G\cap(u\times u)^{-1}(\mathbb{G}_L)\big)\Big)$, and thus $\big(2^\omega, s(\mathbb{G}_L)\big)\preceq^{\mathrm{inj}}_{\mathfrak{c}}(X,G)$.

(b) We essentially argue as in the proof of Theorem 7.3. By the claim, $s(\mathbb{G}_\alpha)$ is a $\mathbf{\Sigma}^0_2$ graph of uncountable Borel chromatic number on $\mathbb{P}_\alpha$. We argue by contradiction, which gives $\alpha\neq\beta$ such that $\big(\mathbb{P}_\alpha, s(\mathbb{G}_\alpha)\big)\preceq^{\mathrm{inj}}_{B}\Big(\mathbb{P}_\beta, s(\mathbb{G}_\beta)\Big)$ with witness u. Note that $E_\alpha\cap E_\beta$ is finite, which gives $L\in E_\alpha\setminus E_\beta$. The map associating $u_0(L,\gamma)$ to $\gamma\in 2^\omega$ is Borel, which gives $M\in E_\beta$, $s\in 2^{<\omega}$ and a dense G_δ subset G of 2^ω such that $u_0(L,\gamma)=M$ if γ is in $N_s\cap G$. The map v associating $u_1(L,\gamma)$ to $\gamma\in N_s\cap G$ is injective and Borel. We can moreover restrict G, so that we may and will assume that v is continuous. Thus v is a witness for the fact that $\big(N_s\cap G, s(\mathbb{G}_L)\cap(N_s\cap G)^2\big)\preceq^{\mathrm{inj}}_{c}\big(2^\omega, s(\mathbb{G}_M)\big)$, which contradicts the claim 7.1 since $L\neq M$.

(c) We argue as in the proof of Theorem 1.5, using the claim. □

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