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Weyl-Mahonian Statistics for Weighted Flags of Type A-D

Roland Bacher*

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Abstract¹: We relate properties of weighted flags (or multiflags) of type A-D to statistics of the corresponding Weyl groups. For type A, we recover the Mahonian statistics on symmetric groups. Finally, we sketch briefly an easy extension incorporating statistics for so-called Euler-polynomials.

1 Introduction

Combinatorial statistics (or statistics for short) are combinatorially defined \mathbb{N} -valued functions on sets of combinatorial elements. This paper deals with three interesting statistics on Weyl groups of the three infinite families A, B-C (giving rise to isomorphic Weyl groups) and D. Weyl groups of type A are finite symmetric groups. A well-known and natural statistic on such groups is defined by the number of inversions of a permutation σ acting on the totally ordered set $\{1, \dots, d\}$. It gives the length of σ with respect to the usual Coxeter generators consisting of the $d-1$ transpositions $(1, 2), (2, 3), \dots, (d-1, d)$ exchanging two consecutive integers. Another important statistic is given by summing indices of descents of a permutation. This statistic is called the Major statistic in honour of Major Mac Mahon who proved equidistribution of the length and the Major statistic for symmetric groups, see [29]. A third interesting statistic is given by Eulerian polynomials encoding numbers of permutations with a given number of descents.

All these three statistics arise naturally when counting weighted flags over finite fields. This observation allows an extension to all Weyl groups of the infinite families A,B-C,D (with a caveat for type D: numbers of descents have to be modified slightly). Techniques coming from the theory of

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linear groups, outlined only briefly, should allow to treat the cases of the exceptional Weyl groups.

The main part of this paper deals with the construction of the joint statistic for inversion numbers and Major indices. These statistics are encoded by so-called Weyl-Mahonian polynomials. Descent numbers are a cheap bonus outlined in a last chapter.

There exists several generalizations of the above statistics, mainly to Weyl groups of type BC, see for example the incomplete list [1], [2], [3], [4], [5], [6], [7], [9], [11], [12], [13], [14], [15], [16], [17], [18], [19], [22], [23], [24], [25], [28], [31], [32], [33], [34], [35], [36], [39], [40] of related works. Our paper is an addition to this list.

Weyl groups are a special case of Coxeter groups, studied for example in the monographs [8], [20], [21], [27]. Standard books of enumerative combinatorics are [30], [37], [38]. Finally, [10] and [26] are good introductions to flag-varieties.

2 Main results for flags of type A

Vector-spaces are always finite-dimensional in the sequel.

A (*partial*) *flag* of a vector-space V over a field \mathbb{F} is a sequence of subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k \subset V$$

of strictly increasing dimensions $0 = \dim(V_0) < \dim(V_1) < \cdots < \dim(V_{k-1}) < \dim(V_k) \leq \dim(V)$. We omit henceforth the trivial subspace $V_0 = \{0\}$ and use the notation $V_1 \subset \cdots \subset V_k$ for a flag of V .

A *weighted flag* is a flag $V_1 \subset \cdots \subset V_k$ together with a sequence w_1, \dots, w_k of strictly positive integers attached to the subspaces V_1, \dots, V_k . We call the sequence w_1, \dots, w_k the *weight-sequence* of the weighted flag $F = (V_1 \subset \cdots \subset V_k; w_1, \dots, w_k)$. The *weight* of such a weighted flag F is defined as $w(F) = \sum_{i=1}^k w_i \dim(V_i)$. It is the content of the partition, called the *weight-partition*, with w_i parts of size $\dim(V_i) \geq i$ for $i = 1, \dots, k$.

Remark 2.1. *The following definition, equivalent to weighted flags, avoids weights and considers instead finite sequences of weakly increasing subspaces: A partial multiframe is a weakly increasing finite sequence $\{0\} \neq V_1 \subset V_2 \subset \cdots \subset V_k$ of a vector space (finite-dimensional, as always). Partial multiflags are in bijection with weighted flags: Repeat each part V_i of a weighted flag w_i times. The weighted flag $F = (V_1 \subset \cdots \subset V_k; w_1, \dots, w_k)$ corresponds thus to the weighted multiframe*

$$\begin{aligned} W_1 = \cdots = W_{w_1} = V_1 \subset W_{w_1+1} = \cdots = W_{w_1+w_2} = V_2 \subset \\ \subset W_{w_1+w_2+1} = \cdots = W_{w_1+w_2+w_3} = V_3 \subset \cdots \subset W_l = V_k \end{aligned}$$

consisting of $l = \sum_{i=1}^k w_i$ weakly increasing subspaces (with allowed equal consecutive subspaces $W_i = W_{i+1}$). We have of course $w(F) = \sum_{i=1}^k w_i \dim(V_i) = \sum_{i=1}^l \dim(W_i)$.

We denote by $\mathcal{F}(V)$ the set of all flags and by $\mathcal{WF}(V)$ the set of all weighted flags of a vector-space V .

An *inversion* of a permutation $\sigma \in \mathcal{S}_d$ acting on $\{1, \dots, d\}$ is given by $1 \leq i < j \leq d$ such that $\sigma(i) > \sigma(j)$. We write $\text{inv}(\sigma) = |\{i, j | i < j, \sigma(i) > \sigma(j)\}|$ for the number of inversions of a permutation σ in \mathcal{S}_d . It is well-known that $\text{inv}(\sigma)$ corresponds to the length $l(\sigma)$ of σ in terms of the Coxeter generators $(1, 2), \dots, (d-1, d)$, see for example Proposition 3.1 which generalizes this result to hyperoctahedral groups (signed permutation groups).

A *descent* of a permutation σ in \mathcal{S}_d is a value $i < d$ such that $\sigma(i) > \sigma(i+1)$. The *Major index*

$$\text{maj}(\sigma) = \sum_{i, \sigma(i) > \sigma(i+1)} i \tag{1}$$

sums up the indices of all descents of a permutation σ in \mathcal{S}_d .

The following result relates Mahonian statistics over \mathcal{S}_d with statistics of weighted flags over finite fields:

Theorem 2.2. *We have*

$$\sum_{F \in \mathcal{WF}(\mathbb{F}_q^d)} t^{w(F)} = M_d \prod_{j=1}^d \frac{1}{1-t^j} \tag{2}$$

with

$$M_d = \sum_{\sigma \in \mathcal{S}_d} q^{\text{inv}(\sigma)} t^{\text{maj}(\sigma)}$$

denoting the Mahonian statistics of \mathcal{S}_d encoding multiplicities of elements of given length and Major index.

Observe that the sum over elements in $\mathcal{WF}(\mathbb{F}_q^d)$ is the sum over weighted flags in an abstract d -dimensional vector-space over \mathbb{F}_q . It does not depend on the choice of a basis for \mathbb{F}_q^d . Observe also that Theorem 2.2 involves a notational abuse: the integer q representing the number of elements of a finite field \mathbb{F}_q (which is unique, up to isomorphism) should be considered as a formal variable. Equalities are among formal power series in q and t . Easy majorations show however that we get converging series in small neighbourhoods of $(0, 0) \in \mathbb{C}^2$.

The factor $\prod_{j=1}^d (1-t^j)^{-1}$ is the generating series for partitions involving at most d parts (or, equivalently, for partitions involving only parts of length at most d).

An obvious modification of Theorem 2.2 and its generalizations holds for arbitrary fields by considering powers of q as dimensions of a cellular decomposition (corresponding to the Bruhat decomposition of simple algebraic groups of Lie type) of the set of all weighted flags into cells carrying affine structures over a field \mathbb{F} . We stick to the elementary enumerative approach over finite fields for simplicity.

We denote by $\binom{d}{k}_q$ the q -binomial coefficient defined (for example) recursively by $\binom{d}{0}_q = \binom{d}{d}_q = 1$ and $\binom{d}{i}_q = \binom{d-1}{i-1}_q + q^i \binom{d-1}{i}_q$.

Exploiting the geometric structure on the left hand side of (2) we get:

Corollary 2.3. *The polynomials $M_d = \sum_{\sigma \in \mathcal{S}_d} q^{\text{inv}(\sigma)} t^{\text{maj}(\sigma)}$ are recursively defined by $M_0 = 1$ and by*

$$M_d = \sum_{i=0}^{d-1} t^i \left(\prod_{j=i+1}^{d-1} 1 - t^j \right) \binom{d}{i}_q M_i \quad (3)$$

(using the convention $\prod_{j=d}^{d-1} (1 - t^j) = 1$).

Evaluating (3) at $t = 1$ yields the recursion $L_d = \binom{d}{d-1}_q L_{d-1} = (1 + q + q^2 + \dots + q^{d-1}) L_{d-1}$ and implies the well-known factorization

$$L_d = \prod_{j=1}^d \frac{1 - q^j}{1 - q} \quad (4)$$

for the generating polynomial $L_d = \sum_{\sigma \in \mathcal{S}_d} q^{l(\sigma)}$ enumerating elements of \mathcal{S}_d accordingly to their length with respect to the Coxeter generators $(i, i + 1)$.

Evaluating M_d , given by Formula (3) of Corollary 2.3 at $q = 1$ shows equidistribution of the length statistic with the Major statistic:

Proposition 2.4. *The evaluation at $q = 1$ of M_d factorizes as*

$$\prod_{j=1}^d \frac{1 - t^j}{1 - t} .$$

The well-known symmetry $M_d(q, t) = M_d(t, q)$ in q, t of the polynomials M_d is however not obvious from Formula (3) in Corollary 2.3.

The sequel of the paper is organized as follows:

In Sections 3 and 4 we describe analogues of Theorem 2.2 and Corollary 2.3 for Weyl groups of type B,C and D (symmetric groups are of course Weyl groups of type A).

Section 5 sketches briefly an approach for dealing with exceptional Weyl groups.

Sections 6-18.1 are devoted to complements and proofs.

Section 5 contains a brief and sketchy description perhaps useful when dealing with exceptional Weil groups (of type E,F,G).

Finally, Section 19 describes generalizations taking also into account the number $\sum_{i=1}^k w_i$ of a weighted flag $(V_1 \subset \dots \subset V_k; w_1, \dots, w_k)$, the dimension $\dim(V_k)$ of the last subspace in a flag and statistics related to signs in the case of Weyl groups of type BC and D. Statistics of these numbers in the case of type A or BC involve Euler polynomials counting descents. Details are easy to fill in and are mostly omitted.

3 Main results for flags of type B and C

We consider a finite-dimensional vector space V over a field of characteristic $\neq 2$ (this avoids technical problems in the symmetric case) endowed with a non-degenerate symmetric or antisymmetric bilinear form b . A subspace L of V is *isotropic* if the restriction of b to $L \times L$ is zero. We suppose that V has isotropic subspaces of the maximal dimension $\lfloor \dim(V)/2 \rfloor$ compatible with non-degeneracy of b . Such a space is called a *symplectic space* if b is antisymmetric. Symplectic spaces will always be denoted by (V, ω) . *Lagrangians* are maximal isotropic subspaces of symplectic spaces.

A space (V, b) is of type C if V is of even dimension and b is a non-degenerate symplectic form, of type B if (V, b) is a quadratic space of odd dimension and of type D if (V, b) is a quadratic space of even dimension. Type B and C share Weyl groups: The common Weyl group of $(\mathbb{F}^{2d}, \omega)$ and of (\mathbb{F}^{2d+1}, b) (with b a suitable symmetric bilinear form) is given by the hyperoctahedral group \mathcal{S}_d^\pm of all permutations σ of $\{\pm 1, \dots, \pm d\}$ such that $\sigma(-i) = -\sigma(i)$ for $i = 1, \dots, d$. The case of (\mathbb{F}^{2d}, b) (with b a suitable symmetric bilinear form on a vector-space of even dimension $2d$) corresponds to the subgroup \mathcal{S}_d^D of index two in \mathcal{S}_d^\pm consisting of all elements σ in \mathcal{S}_d^\pm such that $\prod_{i=1}^d \sigma(i) = d!$.

A flag of (V, b) is a flag $V_1 \subset \dots \subset V_k$ consisting only of isotropic subspaces. Weighted flags of (V, b) and weights of flags are defined in the obvious way. It is of course again possible to replace weighted flags by multiflags involving only isotropic subspaces (ending with an even isotropic subspace in the case of type D), see Remark 2.1. We denote by $\mathcal{WF}(V, b)$ the set of all weighted flags of (V, b) .

In order to state analogues of Theorem 2.2 and Corollary 2.3 we need to introduce the relevant statistics on the corresponding Weyl groups. We call these statistics *Weyl-Mahonian* since they extend Mahonian statistic on symmetric groups to other Weyl groups.

We start by introducing a somewhat exotic order-relation, denoted by $<_\pm$, on the set \mathbb{R} of real numbers. It is defined by $x <_\pm x + \epsilon <_\pm 0 <_\pm -x - \epsilon <_\pm -x$ for strictly positive x and ϵ . The induced order relation on

\mathbb{Z} is given by

$$1 <_{\pm} 2 <_{\pm} 3 <_{\pm} \dots <_{\pm} 0 <_{\pm} \dots <_{\pm} -3 <_{\pm} -2 <_{\pm} -1 \quad (5)$$

or equivalently by

$$(\mathbb{N}^*, <) <_{\pm} 0 <_{\pm} (-\mathbb{N}^*, <)$$

with $(\mathbb{N}^*, <)$ (respectively $(-\mathbb{N}^*, <)$) denoting the ordered set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ (respectively $-\mathbb{N}^* \setminus \{0\}$) endowed with the usual order. In particular, $(\mathbb{Z}, <_{\pm})$ is totally ordered with smallest element 1 and largest element -1 . Observe however that $(\mathbb{Z}, <_{\pm})$ is not well-ordered: $\{-1, -2, -3, \dots\}$ has no smallest element.

We recall that \mathcal{S}_d^{\pm} denotes the hyperoctahedral group of all $2^d \cdot d!$ signed permutations of $\{\pm 1, \dots, \pm d\}$. We consider it as the Weyl group of type C generated by the transpositions $(1, 2), (2, 3), \dots, (d-1, d)$ (where $(i, i+1)$ denotes the signed permutation $j \mapsto j$ if $j \notin \{\pm i, \pm(i+1)\}$ and exchanging the pair $\pm i$ with the pair $\pm(i+1)$ by preserving signs) and by the sign change $(d, -d)$ transposing the elements of the largest pair $\{d, -d\}$ of opposite integers.

The length-function of an element in \mathcal{S}_d^{\pm} with respect to these d generators is given by the following (surely well-known) result:

Proposition 3.1. *The length $l^{\pm}(\sigma)$ of an element $\sigma \in \mathcal{S}_d^{\pm}$ with respect to the generators $(1, 2), (2, 3), \dots, (d-1, d), (d, -d)$ is given by the formula*

$$l^{\pm}(\sigma) = \sum_{0 < i < j, \sigma(i) >_{\pm} \sigma(j)} 1 + \sum_{0 < i, \sigma(i) < 0} (d+1 + \sigma(i)) . \quad (6)$$

Proposition 3.1 of [9] gives a different formula (with respect to a slightly different generating set).

We call a pair $0 < i < j \leq d$ with $\sigma(i) >_{\pm} \sigma(j)$ an *inversion* of σ . In order to avoid the use of the somewhat exotic order relation $<_{\pm}$ defined by (5) one can replace the condition $\sigma(i) >_{\pm} \sigma(j)$ with the equivalent condition $\sigma(i)\sigma(j)(\sigma(i) - \sigma(j)) > 0$.

Observe that Formula (6) boils down to Formula (1) for elements in the subgroup $\mathcal{S}_d \subset \mathcal{S}_d^{\pm}$ of ordinary (unsigned) permutations.

The Weyl-Major index of an element σ in \mathcal{S}_d^{\pm} is defined by

$$Wmaj(\sigma) = \sum_{0 < i, \sigma(i) > \sigma(i+1)} i + \sum_{i > 0, \sigma(i) < 0} 1 . \quad (7)$$

We call the polynomial

$$M_d^{\pm} = M_d^{\pm}(q, t) = \sum_{\sigma \in \mathcal{S}_d^{\pm}} q^{l^{\pm}(\sigma)} t^{Wmaj(\sigma)} \quad (8)$$

the *Weyl-Mahonian statistics* of \mathcal{S}_d^{\pm} . It encodes the number of elements of the hyperoctahedral group \mathcal{S}_d^{\pm} with given length and Weyl-Major index.

We have:

Theorem 3.2. *We have*

$$\sum_{F \in \mathcal{WF}(\mathbb{F}_q^{2d}, \omega)} t^{w(F)} = M_d^\pm \prod_{j=1}^d \frac{1}{1-t^j}.$$

Theorem 3.2 generalizes Theorem 2.2 in the following sense: Suppose that the first d basis elements of \mathbb{F}^{2d} span a Lagrangian L in \mathbb{F}^{2d} . Weighted symplectic flags contained in L correspond to weighted ordinary flags of \mathbb{F}^d and are enumerated by restricting the sum in (8) to the subgroup \mathcal{S}_d of ordinary permutations.

Weyl groups of type B are also covered by Theorem 3.2 as shown by the following result:

Theorem 3.3. *We have*

$$\sum_{F \in \mathcal{WF}(\mathbb{F}_q^{2d}, \omega)} t^{w(F)} = \sum_{F \in \mathcal{WF}(\mathbb{F}_q^{2d+1}, Q)} t^{w(F)}$$

where $\mathcal{WF}(\mathbb{F}_q^{2d+1}, Q)$ denotes the set of weighted flags over a non-degenerate quadratic space (\mathbb{F}_q^{2d+1}, Q) of odd dimension $2d+1$ over a finite field \mathbb{F}_q of odd characteristic such that the quadratic form Q admits a d -dimensional isotropic subspace.

Theorem 3.2 implies the following corollary which expresses the Weyl-Mahonian statistics M_d^\pm on \mathcal{S}_d^\pm in terms of ordinary Mahonian statistic M_0, \dots, M_d (given recursively by Corollary 2.3) on symmetric groups:

Corollary 3.4. *We have*

$$M_d^\pm = \sum_{k=0}^d t^k \left(\prod_{j=0}^{k-1} \frac{1-q^{2d-2j}}{1-q^{k-j}} \right) \left(\prod_{j=k+1}^d (1-t^j) \right) M_k.$$

Corollary 3.4 is of course an analogon of Corollary 2.3 for hyperoctahedral groups.

Remark 3.5. *The function defined by (7) is called Weyl-Mahonian index in order to avoid confusion with the flag-Major index of [3] extending the Major index in a different way to \mathcal{S}_d^\pm (and more generally to wreath-products of \mathcal{S}_d with finite cyclic group).*

4 Main results for flags of type D

We denote by $\mathcal{H} = \mathcal{H}(\mathbb{F})$ the hyperbolic plane over a field \mathbb{F} realized as the quadratic space \mathbb{F}^2 endowed with a norm given by the quadratic form $(x, y) \mapsto xy$.

We denote by \mathcal{I} a fixed maximal d -dimensional isotropic subspace (also called a *metabolizer*) of \mathcal{H}^d . As always, a flag $F = (V_1 \subset \dots \subset V_k)$ is again a strictly increasing sequence of non-trivial isotropic subspaces of \mathcal{H}^d (with \mathcal{H}^d denoting the orthogonal sum of d copies of \mathcal{H}). The \mathcal{I} -*parity* (or simply the *parity*) of a flag ending with V_k is the parity of the integer $\dim(V_k/(V_k \cap \mathcal{I})) = \dim(V_k) - \dim(V_k \cap \mathcal{I})$. Flags of even parity are simply called even flags. We denote by $\mathcal{F}^e(\mathcal{H}^d)$ the set of all even flags and by $\mathcal{WF}^e(\mathcal{H}^d)$ the set of all weighted even flags.

We consider now Weyl groups of type D, given by the subgroup \mathcal{S}_d^D of the hyperoctahedral \mathcal{S}_d^\pm consisting of all signed permutations σ such that $\prod_{i=1}^d \sigma(i) = d!$. Given an element $\sigma \in \mathcal{S}_d^D$ we set

$$l^D(\sigma) = \sum_{0 < i < j, \sigma(i) > \pm \sigma(j)} 1 + \sum_{0 < i, \sigma(i) < 0} (d + \sigma(i)). \quad (9)$$

We will see in Proposition 17.1 that l^D defines the natural length function on Weyl groups of type D.

The Weyl-Major index $Wmaj(\sigma)$ of elements in \mathcal{S}_d^D coincides with the Weyl-Major index in \mathcal{S}_d^\pm and is also given by Formula (7).

The generating series of Weyl-Major statistics on \mathcal{S}_d^D has a nice factorization given by the following result:

Theorem 4.1. *We have*

$$\sum_{\sigma \in \mathcal{S}_d^D} t^{Wmaj(\sigma)} = \frac{(1-t)^d + (1+t)^d}{2} \prod_{j=1}^d \frac{1-t^j}{1-t}. \quad (10)$$

We call the polynomial

$$M_d^D = M_d^D(q, t) = \sum_{\sigma \in \mathcal{S}_d^D} q^{l^D(\sigma)} t^{Wmaj(\sigma)} \quad (11)$$

the *Weyl-Mahonian statistics* of \mathcal{S}_d^D . It encodes the number of elements of \mathcal{S}_d^D with given length and Weyl-Major index.

We have:

Theorem 4.2. *We have*

$$\sum_{F \in \mathcal{WF}^e(\mathcal{H}_q^d)} t^{w(F)} = M_d^D \prod_{j=1}^d \frac{1}{1-t^j}$$

with \mathcal{H}_q^d denoting the orthogonal sum of d hyperbolic planes over a finite field \mathbb{F}_q of odd characteristic and with $\mathcal{WF}^e(\mathcal{H}_q^d)$ denoting the set of all weighted even flags (with respect to a fixed maximal isotropic subspace \mathcal{I} of \mathcal{H}_q^d).

The analogue of Corollaries 2.3 and 3.4 for type D is given by:

Corollary 4.3. *The polynomials M_d^D for the Weyl-Mahonian statistics on Weyl groups of type D are given by the formula*

$$M_d^D = \sum_{k=0}^d t^k \left(\prod_{j=k+1}^d (1-t^j) \right) \binom{d}{k}_q \left(\sum_{l=0}^{\lfloor k/2 \rfloor} \binom{k}{2l}_q q^{l(2d+2l-2k-1)} \right) M_k \quad (12)$$

where $\binom{d}{k}_q$ are q -binomials (see for example Corollary 2.3) and where M_k , defined for example by Corollary 2.3, gives the Mahonian statistic on \mathcal{S}_k .

Remark 4.4. *The parity condition in the type D case is due to the existence of two orbits (defined by the parity condition) of maximal isotropic subspaces.*

5 Modifications for exceptional types

We denote by G a simple group of Lie type over an algebraically closed field (say over \mathbb{C} for simplicity). Borel subgroups are maximal connected solvable subgroups of G . We fix a Borel subgroup B_0 and a maximal torus T_0 contained in B_0 . The Weyl group W of G is the finite quotient group N_0/T_0 with N_0 denoting the normalizer of T_0 in G . We get thus a Bruhat decomposition $G = \bigcup_{w \in W} B_0 w B_0$ with W denoting the Weyl group of G , represented by suitable elements of N_0 . A cell $B_0 w B_0 / B_0$ can be identified with an affine space whose points index all associated Borel subgroups (given by conjugates of B_0 by elements of $B_0 w B_0$). The dimension of such a cell $B_0 w B_0 / B_0$ is given by the natural length $l^W(w)$ of w with respect to the standard generators associated to simple roots. To each Borel group B in a cell $B_0 w B_0$, we associate the parabolic subgroup P_B of G generated by B and by representants in N_0 of all standard generators $g \in W$ such that $l(wg) = l(w) + 1$. In accordance with our previous terminology we call such a parabolic group P_B standard. (Warning: This is not the usual meaning of “standard” in the theory of Lie groups.) Its structure depends only on the cell $B_0 w B_0$ under consideration. The standard parabolic group associated to B_0 is G . In the opposite direction, standard parabolics associated to $B_0 w B_0$ for w the Coxeter element are simply Borel groups of $B_0 w B_0$.

Parabolic subgroups play the role of partial flags and standard parabolic groups play the role of standard flags. The standard weight of a standard parabolic subgroup P_B is defined as the sum of dimensions over all non-trivial invariant subspaces of $\rho(P_B)$ with ρ denoting the adjoint representation. (Warning: Weights of flags and weights of representations are of course unrelated.)

An arbitrary parabolic subgroup P' contained in a standard parabolic group P_B is generated by B and by a subset $\mathcal{G}'_B \subset \mathcal{G}_B$ with B and $\mathcal{G}_B \subset N_0$

(representating a subset of standard generators of W) generating the standard parabolic subgroup P_B . A standard parabolic subgroup P_B contains thus $2^{\sharp(\mathcal{G}_B)}$ different parabolic subgroups containing B (corresponding to all flags refining the standard flag associated to the standard parabolic subgroup P_B). Weights associated to a parabolic subgroup P' are defined by sequences of strictly positive integers indexed by invariant subspaces of $\rho(P')$ where ρ denotes the adjoint representation of G . (The associated weight is then given by $\sum_i w_i \dim(V_i)$ where the sum is over all invariant subspaces V_i of $\rho(P')$ with associated weight $w_i \in \{1, 2, \dots\}$.)

Working out the technical details of this machinery turns the determination of the Weyl-Mahonian statistics on the six exceptional Weyl groups of type E, F, G (for the flags of which there exists to my knowledge no obvious easy combinatorial description) into finite computations.

6 Complements for weighted flags

The *empty flag* is reduced to the trivial (and omitted) subspace $V_0 = \{0\}$. It is the unique flag of weight 0 as a weighted flag.

A flag $V_1 \subset V_2 \subset \dots \subset V_k$ of a finite-dimensional vector space V is *complete* if $k = \dim(V)$ (and thus $\dim(V_i) = i$ for $i = 0, \dots, \dim(V)$). A flag is *maximal* if $\dim(V_k) = \dim(V)$ and *nonmaximal* otherwise. Complete flags are maximal.

Every flag $F = (V_1 \subset \dots \subset V_k)$ can be endowed with the *minimal weight* $\sum_{i=1}^k \dim(V_i)$ defined by the weight-sequence $w_1 = w_2 = \dots = w_k = 1$. Any other weight sequence yields a strictly larger weight for F .

A weighted flag $(V_1 \subset \dots \subset V_k; w_1, \dots, w_k)$ of V defines a function $\mu : V \rightarrow \mathbb{N}$ by setting $\mu(v) = \sum_{i, v \in V_i} w_i$. We call the value $\mu(v)$ the (*weighted*) *multiplicity* of v . We have of course $\mu(v) = 0$ if and only if v is in the complement of V_k .

A flag $F = (V_1 \subset V_2 \subset \dots \subset V_k)$ of a d -dimension vector space V can be encoded by a basis f_1, \dots, f_d such that $f_1, f_2, \dots, f_{\dim(V_i)}$ span V_i for all i and by the strictly increasing sequence $0 < \dim(V_1) < \dim(V_2) < \dots < \dim(V_k)$ of dimensions. If F is weighted with weight sequence w_1, \dots, w_k , the decreasing sequence $\mu(f_1) \geq \mu(f_2) \geq \dots \geq \mu(f_k)$ of multiplicities of basis vectors as above defines the conjugate partition of the weight partition and we have thus

$$\sum_{i=1}^k w_i \dim(V_i) = \sum_{i=1}^d \mu(f_i) .$$

We call the partition defined by $\mu(f_1), \dots, \mu(f_{\dim(V_k)})$ the *conjugate (weight) partition*. A weighted flag is of course also uniquely defined by a basis f_1, \dots, f_d as above and by the conjugate partition $\mu(f_1), \dots, \mu(f_d)$ involving at most d non-zero parts.

7 q -binomials and proofs of Corollary 2.3 and Proposition 2.4

The following lore is easy and well-known (see for example Proposition 1.3.18 of [37]):

Proposition 7.1. *The number of k -dimensional subspaces of \mathbb{F}_q^d is given by the q -binomial*

$$\binom{d}{k}_q = \prod_{j=1}^k \frac{q^{d+1-j} - 1}{q^j - 1}.$$

Proof of Corollary 2.3. We start by sorting weighted flags of \mathbb{F}_q^d according to the dimension $i = \dim(V_\omega) \in \{0, 1, \dots, d-1\}$ of their last and largest subspace V_ω . By Proposition 7.1 there are $\binom{d}{i}_q$ possibilities for the choice of V_ω and we get thus the identity

$$\sum_{F \in \mathcal{WF}(\mathbb{F}_q^d)} t^{w(F)} = \sum_{i=0}^d t^i \binom{d}{i}_q \sum_{F \in \mathcal{WF}(\mathbb{F}_q^i)} t^{w(F)}$$

equivalent to

$$(1 - t^d) \sum_{F \in \mathcal{WF}(\mathbb{F}_q^d)} t^{w(F)} = \sum_{i=0}^{d-1} t^i \binom{d}{i}_q \sum_{F \in \mathcal{WF}(\mathbb{F}_q^i)} t^{w(F)}$$

which is the generating series of all weighted non-maximal flags.

Applying Theorem 2.2 on both sides we get

$$M_d \prod_{j=1}^{d-1} \frac{1}{1-t^j} = \sum_{i=0}^{d-1} t^i \binom{d}{i}_q M_i \prod_{j=1}^i \frac{1}{1-t^j}$$

which yields the result multiplying both sides with $\prod_{j=1}^{d-1} (1-t^j)$. \square

Proof of Proposition 2.4. Equality holds trivially for $d = 1$. Induction yields for the evaluation A_d of M_d at $q = 1$

$$\begin{aligned} A_d &= \sum_{i=0}^{d-1} t^i \left(\prod_{j=i+1}^{d-1} (1-t^j) \right) \binom{d}{i} \prod_{j=1}^i \frac{1-t^j}{1-t} \\ &= \prod_{j=1}^{d-1} (1-t^j) \left(\left(\sum_{i=0}^d \binom{d}{i} \frac{t^i}{(1-t)^i} \right) - \frac{t^d}{(1-t)^d} \right) \\ &= \prod_{j=1}^{d-1} (1-t^j) \left(\left(1 + \frac{t}{1-t} \right)^d - \frac{t^d}{(1-t)^d} \right) \\ &= \prod_{j=1}^d \frac{1-t^j}{1-t}. \end{aligned}$$

□

8 Canonical presentations of weighted flags

We associate to a flag $F = (V_1 \subset V_2 \subset \cdots \subset V_k)$ of \mathbb{F}^d a uniquely defined *canonical basis* f_1, \dots, f_d of \mathbb{F}^d such that V_i is spanned by $f_1, \dots, f_{\dim(V_i)}$ for all i .

The *canonical presentation* of a weighted flag is given by its canonical basis f_1, \dots, f_d together with its conjugate weight-partition $\mu(f_1), \dots, \mu(f_d)$. Canonical presentations are given algorithmically and thus uniquely determined.

The canonical basis f_1, \dots, f_k is constructed algorithmically as follows:

If V_k is a strict subspace of \mathbb{F}^d , we increase the flag by adding a last space $V_{k+1} = \mathbb{F}^d$ with degenerate multiplicity $\mu(v) = 0$ on elements in $V_{k+1} \setminus V_k$.

The *length* of a non-zero element $v = (x_1, \dots, x_j, 0, \dots, 0)$ in \mathbb{F}^d is the largest integer j corresponding to a non-zero coordinate $x_j \in \mathbb{F} \setminus \{0\}$.

Let f_1 be the unique element of smallest length and last coordinate 1 in V_1 . Set $\lambda(1) = \text{length}(f_1)$. If V_1 is of dimension larger than 1, define f_2 as the unique shortest non-zero element of V_1 with last coordinate 1 and coordinate $x_{\lambda(1)} = 0$. If V_1 is one-dimensional, define f_2 similarly using V_2 . Set $\lambda(2) = \text{length}(f_2)$.

More generally, suppose f_1, \dots, f_{i-1} with $i < d$ already constructed. Let j be the smallest index such that V_j is not spanned by f_1, \dots, f_{i-1} . The vector f_i is defined as the unique shortest element in \mathbb{F}^d of $V_j \setminus (\mathbb{F}f_1 + \cdots + \mathbb{F}f_{i-1})$ with last coordinate 1 and with coordinates $x_{\lambda(1)} = x_{\lambda(2)} = \cdots = x_{\lambda(i-1)} = 0$. We set $\lambda(i) = \text{length}(f_i)$.

This construction yields a basis f_1, \dots, f_d of \mathbb{F}^d such that V_i is spanned by $f_1, \dots, f_{\dim(V_i)}$ for all i . We call f_1, \dots, f_d the *canonical basis*.

The sequence $\lambda(f_1), \lambda(f_2), \dots$ of lengths of the canonical basis defines a permutation $i \mapsto \lambda(f_i)$ of \mathcal{S}_d . We call λ the *length-permutation*. The length-permutation is uniquely defined in terms of the canonical basis.

Observe that many flags share a common canonical basis, see Remark 9.1 for more.

We will see in Proposition 9.3 that weighted flags corresponding to a given standard basis of \mathbb{F}^d are in one-to-one correspondence with partitions having at most d parts. Each such flag contributes one monomial to the factor $\prod_{j=1}^d (1-t^j)^{-1}$ appearing in Formula(2) of Theorem 2.2. The constant term 1 corresponds to the standard flag, introduced and studied in the next Section.

Remark 8.1. *The construction of the canonical basis is equivalent to the Bruhat decomposition $BWB = \text{SL}_d(\mathbb{F})$ of special linear groups. The length-permutation λ belongs to the Weyl group \mathcal{S}_d of $\text{SL}_d(\mathbb{F})$ and indexes the*

Schubert cell $B\lambda B$ containing the matrix with rows $f_1, \dots, f_{d-1}, (-1)^{\text{sign}(\lambda)} f_d$ (with $\text{sign}(\lambda)$ denoting the signature of λ).

9 Standard flags

A flag $F = (V_1 \subset \dots \subset V_k \subset \mathbb{F}^d)$ of \mathbb{F}^d consisting of k subspaces is a *standard flag* if F is a nonmaximal flag (contained in a strict subspace V_k of \mathbb{F}^d) associated to a length-permutation with k descents. (Recall that a descent is an index $i < d$ such that $\sigma(i) > \sigma(i+1)$.) Since $\lambda(i) < \lambda(i+1)$ if $i \notin \{\dim(V_1), \dots, \dim(V_k)\}$, standard flags involve the minimal number of subspaces determining a given length-permutation. Every canonical basis defines a unique standard flag defined by $\dim(V_i) = j$ if j is the i -th descent of the corresponding length-permutation. Any abstract flag $F = U_1 \subset \dots \subset U_l$ with canonical basis f_1, \dots, f_d is a refinement of the standard flag $V_1 \subset \dots \subset V_k$ defined by f_1, \dots, f_d in the following sense: there exists a strictly increasing sequence $1 \leq j_1 < \dots < j_k \leq l$ such that $V_i = U_{j_i}$. We denote by $st(F)$ the standard flag associated to the canonical basis of a flag F . In particular, every standard flag can be refined to a unique complete flag $(U_1 \subset \dots \subset U_d)$ by defining V_i as the span of the first i elements f_1, \dots, f_i of its canonical basis. Standard flags are thus in one-to-one correspondence with complete flags. (Observe however that the definition of complete flags is independent of the choice of a basis. Standard flags are defined in \mathbb{F}^d with respect to the natural basis of \mathbb{F}^d . The map $F \mapsto st(F)$ associating to a complete flag F the standard flag $st(F)$ is thus only defined for complete flags of \mathbb{F}^d .)

Remark 9.1. A canonical basis f_1, \dots, f_d with length permutation λ having k descents is the canonical basis of exactly 2^{d-k} different abstract flags: Each quotient space V_i/V_{i-1} of the standard flag can indeed be refined in $2^{\dim(V_i) - \dim(V_{i-1}) - 1}$ different ways (corresponding to the $2^{\dim(V_i) - \dim(V_{i-1}) - 1}$ different compositions of the natural integer $\dim(V_i) - \dim(V_{i-1})$) into abstract flags having the same canonical basis.

The *standard weight* of a standard flag $F = (V_1 \subset \dots \subset V_k)$ is given by

$$w_{\min}(F) = \sum_{i=1}^k \dim(V_i) .$$

It corresponds to the smallest possible weight-sequence $w(V_1) = \dots = w(V_k) = 1$. The associated weight-multiplicities are given by $\mu(f_j) = k+1-i$ if $\dim(V_{i-1}) < j \leq \dim(V_i)$ and by $\mu(f_j) = 0$ for $j > \dim(V_k)$.

The standard weight of a standard flag has the following combinatorial characterization:

Proposition 9.2. *The standard weight $w_{\min}(F)$ of a standard flag $F = (V_1 \subset \cdots \subset V_k)$ with length permutation $\lambda \in \mathcal{S}_d$ is given by the Major index*

$$maj(\lambda) = \sum_{i, \lambda(i) > \lambda(i+1)} i$$

of λ .

We leave the obvious proof to the reader.

We illustrate the notion of a standard flag and of its minimal weight by an example. We consider a canonical basis f_1, \dots, f_9 of \mathbb{F}^9 with length permutation $(\lambda(1), \lambda(2), \dots, \lambda(9)) = (6, 3, 8, 1, 4, 9, 7, 2, 5)$ the permutation illustrating the Wikipedia entry “Permutation” at the time of writing. Dimensions $\dim(V_i) - \dim(V_{i-1})$ of quotient-spaces V_i/V_{i-1} correspond to (non-final) maximal ascending runs (maximal sets of consecutive integers on which λ is increasing). The ascending runs of λ are

run	$\lambda(\text{run})$
1	6
2, 3	3, 8
4, 5, 6	1, 4, 9
7	7
8, 9	2, 5

and correspond to the composition $1+2+3+1+2$ of $d = 9$. Since the Major index $\sum_{i, \sigma(i) > \sigma(i+1)} i$ of λ is the sum of preimages (or indices) of maximal elements in non-final ascending runs (maximal sets of consecutive integers on which the permutation is increasing), the Major index of our example equals $1 + 3 + 6 + 7 = 17$. An associated standard flag is given by

$$\begin{aligned} V_1 &= \mathbb{F}f_1, \\ V_2 &= \mathbb{F}f_1 + \mathbb{F}f_2 + \mathbb{F}f_3, \\ V_3 &= \mathbb{F}f_1 + \cdots + \mathbb{F}f_6, \\ V_4 &= \mathbb{F}f_1 + \cdots + \mathbb{F}f_7 \end{aligned}$$

for a suitable basis f_1, \dots, f_9 with weight-multiplicities $\mu(f_1) = 4, \mu(f_2) = \mu(f_3) = 3, \mu(f_4) = \mu(f_5) = \mu(f_6) = 2, \mu(f_7) = 1, \mu(f_8) = \mu(f_9) = 0$. The standard weight of this standard flag is given by $4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 1 = 17$ and is thus equal to the Major index of σ .

The observation that the standard weight of a standard flag is the smallest possible weight among all weighted flags sharing a given canonical basis can be refined into the following result:

Proposition 9.3. *Let $F = (U_1 \subset \cdots \subset U_l)$ be a weighted flag with associated standard flag $st(F) = (V_1 \subset \cdots \subset V_k)$. There exists a partition*

$\alpha_1, \dots, \alpha_d$ of $w_F(F) - w_{\min}(st(F))$ such that the weight-multiplicities $\mu_F(f_i)$ for F are given by

$$\mu_F(f_i) = \mu_{\min}(f_i) + \alpha_i$$

with $\mu_{\min}(f_i)$ defining the standard weight-multiplicities of the associated standard flag $st(F)$.

The weight of F is given by

$$w_F(F) = w_{\min}(F) + \sum_{i=1}^d \alpha_i .$$

Proposition 9.3 implies the following result:

Corollary 9.4. *We have*

$$\sum_{F \in st^{-1}(F_{st})} t^{w(F)} = t^{w(F_{st})} \prod_{i=1}^d \frac{1}{1-t^i}$$

where $w(F_{st})$ is the standard weight of a standard flag F_{st} and where $st^{-1}(F_{st})$ is the set of all weighted flags with associated standard flag F_{st} .

Proof. This follows from the easy observation that any partition $\alpha_1, \dots, \alpha_d$ defines a sequence of weight-multiplicities $w(f_i) = \alpha_i + w_{\min}(f_i)$ on the standard basis f_1, \dots, f_d . \square

Proof of Proposition 9.3. We have to show that the sequence $\alpha_1, \dots, \alpha_d$ defined by $\alpha_i = \mu_F(f_i) - \mu_{\min}(f_i)$ has decreasing non-negative values. Suppose first that we have $\alpha_i < \alpha_{i+1}$. Since $\mu_{\min}(f_i) - \mu_{\min}(f_{i+1}) \leq 1$ and since $\mu_F(f_i)$ is decreasing, the inequality $\alpha_i < \alpha_{i+1}$ is only possible if $\mu_F(f_i) = \mu_F(f_{i+1})$ and $\mu_{\min}(f_i) = 1 + \mu_{\min}(f_{i+1})$. This contradicts the fact that F contains all subspaces of the associated standard flag $st(F)$.

Non-negativity of α_i follows from $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d = \mu_F(f_d) - \mu_{\min}(f_d) = \mu_F(f_d) - 0 \geq 0$. \square

10 Canonical bases with a given length-permutation

Proposition 10.1. *The set $\mathcal{B}(\lambda)$ of all canonical bases of \mathbb{F}^d with length-permutation $\lambda \in \mathcal{S}_d$ is an affine vector-space of dimension*

$$inv(\lambda) = \sum_{i < j, \lambda(i) > \lambda(j)} 1 .$$

The crucial ingredients for proving Theorem 2.2 are Corollary 9.4 and the following result:

Corollary 10.2. Denoting by $\mathcal{F}_{st}(\mathbb{F}^d)$ the set of all standard flags of \mathbb{F}^d we have

$$\sum_{F \in \mathcal{F}_{st}(\mathbb{F}_q^d)} t^{w_{st}(F)} = \sum_{\lambda \in \mathcal{S}^d} t^{maj(\lambda)} q^{inv(\lambda)}$$

where $w_{st}(F)$ denotes the standard weight of a standard flag F and where maj and inv denote the Major index and the number of inversions of a permutation λ in \mathcal{S}_d .

Proof of Corollary 10.2. We apply Propositions 9.2 and 10.1 to the sum over all elements of \mathcal{S}_d . \square

Proof of Proposition 10.1. We construct the set $\mathcal{B}(\lambda)$ of all canonical bases of \mathbb{F}^d with length-permutation λ in \mathcal{S}_d .

The first element $f_1 = (x_{1,1}, x_{1,2}, \dots, x_{1,\lambda(1)-1}, 1, 0, \dots, 0)$ has $\lambda(1) - 1$ arbitrary coefficients followed by a coefficient $x_{1,\lambda(1)} = 1$. Coefficients with indices larger than $\lambda(1)$ are 0. More generally, the i -th basis element f_i of a canonical basis has prescribed coefficients $x_{i,\lambda(j)} = 0$ for $j = 1, \dots, i - 1$, $x_{i,\lambda(i)} = 1$, $x_{i,j} = 0$ for $j > \lambda(i)$. All remaining coefficients are free. Such coefficients are in bijection with $j > i$ such that $\lambda(j) < \lambda(i)$. Their number is thus the number of inversions $i < j$, $\lambda(i) > \lambda(j)$ involving i as their smaller argument. This shows that there are $q^{inv(\lambda)}$ canonical bases of \mathbb{F}_q^d with length-permutation a given element λ in \mathcal{S}_d . \square

The *Rothe-diagram* of λ visualizes the nature of the coefficients occurring in a canonical basis of the set $\mathcal{B}(\lambda)$ constructed while proving Proposition 10.1: Bullets \bullet represent coefficients 1, crosses \times represent free coefficients and blanks represent coefficients which are necessarily 0.

For example, the Rothe diagram of permutation $(\lambda(1), \lambda(2), \dots, \lambda(9)) = (6, 3, 8, 1, 4, 9, 7, 2, 5)$ of \mathcal{S}_9 is given by

$i \setminus \lambda(i)$	1	2	3	4	5	6	7	8	9
1	\times	\times	\times	\times	\times	\bullet			
2	\times	\times	\bullet						
3	\times	\times		\times	\times		\times	\bullet	
4	\bullet								
5		\times		\bullet					
6		\times			\times		\times		\bullet
7		\times			\times		\bullet		
8		\bullet							
9					\bullet				

cf. the section “Numbering permutations” of the Wikipedia entry “Permutation” in [41]. Its 18 crosses correspond to the 18 inversions giving the length of λ . Each cross has indeed exactly one bullet at its right and one bullet below it. The lines $i < j$ indexing these two bullets define an inversion $\lambda(i) > \lambda(j)$ of λ .

11 Type C: Symplectic flags

A symplectic space V is always of even dimension $2d$ and has a basis $e_1, \dots, e_d, f_1, \dots, f_d$ such that $\omega(e_i, f_i) = -\omega(f_i, e_i) = 1$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = \omega(e_i, f_j) = 0$ if $i \neq j$. We denote a symplectic space over a field \mathbb{F} henceforth by $(\mathbb{F}^{2d}, \omega)$ or \mathbb{F}^{2d} for short. A subspace W of V is *isotropic* if the restriction of ω to $W \times W$ is identically zero. *Lagrangians* in $(\mathbb{F}^{2d}, \omega)$ are maximally isotropic subspaces and are of dimension d .

A symplectic flag of $(\mathbb{F}^{2d}, \omega)$ is a flag $V_1 \subset V_2 \subset \dots \subset V_k$ of \mathbb{F}^{2d} which is contained in a Lagrangian of \mathbb{F}^{2d} . The symplectic form ω restricts thus to 0 on $V_k \times V_k$ and V_k is of dimension at most d . A symplectic flag is *maximal* if $\dim(V_k) = d$ and *complete* if $k = d$. Complete symplectic flags are maximal.

Weighted flags and weights of flags are defined in the obvious way. We denote by $\mathcal{WF}(\mathbb{F}^{2d}, \omega)$ the set of all weighted symplectic flags of the symplectic space $(\mathbb{F}^{2d}, \omega)$.

11.1 Proof of Corollary 3.4

The proof of Corollary 3.4 is essentially identical to the proof of Corollary 2.3. The main ingredient is the following well-known result which can be seen as an analogue of q -binomial coefficients:

Lemma 11.1. (i) *Given an odd prime power q , the symplectic space $(\mathbb{F}_q^{2d}, \omega)$ and a non-degenerate orthogonal space of dimension $2d+1$ over \mathbb{F}_q have both q^{2d} isotropic elements.*

(ii) *Given an odd prime power q , the symplectic space $(\mathbb{F}_q^{2d}, \omega)$ and a non-degenerate orthogonal space of dimension $2d+1$ over \mathbb{F}_q have both*

$$\prod_{j=0}^{k-1} \frac{q^{2d-j} - q^j}{q^k - q^j} = \prod_{j=0}^{k-1} \frac{1 - q^{2d-2j}}{1 - q^{k-j}} \quad (13)$$

isotropic subspaces of dimension k .

We leave the proof to the reader.

Proof of Corollary 3.4. Assertion (ii) of Lemma 11.1 and Theorem 2.2 imply

$$\sum_{F \in \mathcal{WF}(\mathbb{F}_q^{2d}, \omega)} t^{w(F)} = \sum_{k=0}^d t^k \left(\prod_{j=0}^{k-1} \frac{q^{2d-2j} - 1}{q^{k-j} - 1} \right) M_k \prod_{j=1}^k \frac{1}{1 - t^j}.$$

Use of Theorem 3.2 and multiplication by $\prod_{j=1}^d (1 - t^j)$ end the proof. \square

11.2 Small values and a few properties for M_d^\pm

The first few polynomials M_d^\pm are as follows: $M_1^\pm = 1 + qt$, coefficients of M_2^\pm, M_3^\pm are given by

		1	q	q^2	q^3	q^4		1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	
1	1							1										
t	1							t	1	1	1	1	1					
t^2		1	1	1				t^2	1	2	2	2	2	1	1			
t^3		1	1	1				t^3	1	1	3	2	2	3	1	1		
t^4						1			t^4	1		1	2	2	2	2	1	
t^5									t^5				1	1	1	1	1	
t^6								t^6										1

and coefficients of M_4^\pm are given by

		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	1																		
t		1	1	1	1	1	1	1											
t^2		1	2	2	3	3	3	3	2	2	1	1							
t^3		1	2	4	4	6	6	6	6	5	4	2	2						
t^4		1	2	4	6	7	8	9	9	8	7	5	3	2	1				
t^5				1	3	5	7	9	10	12	10	9	7	5	3	1			
t^6					1	2	3	5	7	8	9	9	8	7	6	4	2	1	
t^7							2	2	4	5	6	6	6	6	4	4	2	1	
t^8									1	1	2	2	3	3	3	3	2	2	1
t^9												1	1	1	1	1	1		
t^{10}																		1	

(with columns yielding the coefficients of $1, q, \dots, q^{16}$). Maximal degrees in q and t of M_d^\pm are due to the Coxeter element $c(i) = -i$, $i = 1, \dots, d$ (which is the unique non-trivial central element in \mathcal{S}_d^\pm) of maximal length $\binom{d}{2} + \binom{d+1}{2} = d^2$ and maximal flag-Major index $\binom{d}{2} + d = \binom{d+1}{2}$ contributing the monomial $q^{d^2} t^{\binom{d+1}{2}}$ of leading degree in q and t to M_d^\pm . The easy identities $l(\sigma) + l(c\sigma) = d^2$ and $Wmaj(\sigma) + Wmaj(c\sigma) = \binom{d+1}{2}$ imply the symmetry $t^{\binom{d}{2}} q^{d^2} M_d^\pm(1/q, 1/t) = M_d^\pm(q, t)$, obvious in the above examples.

The polynomial M_d^\pm evaluates to $2^d d!$ at $q = t = 1$.

Corollary 3.4 and Formula (4) yield the well-known factorization

$$M_d^\pm(q, 1) = \sum_{\sigma \in \mathcal{S}_d^\pm} q^{l(\sigma)} = \prod_{j=1}^d \frac{1 - q^{2j}}{1 - q}$$

(analogous to the factorization for \mathcal{S}_d given by (4)) for the generating polynomial of lengths in the hyperoctahedral group \mathcal{S}_d^\pm (obtained by evaluating M_d^\pm at $t = 1$).

A similar computation (again using Corollary 3.4 and Formula (4)) yields the factorization

$$M_d^\pm(1, t) = (1 + t)^d M_d(1, t) = (t + 1)^d \prod_{j=1}^d \frac{t^j - 1}{t - 1}$$

at $q = 1$.

The easy congruence

$$\binom{d}{i}_q \equiv \prod_{j=0}^{i-1} \frac{1 - q^{2d-2j}}{1 - q^{i-j}} \pmod{q^{d+1-i}}$$

implies that M_d and M_d^\pm have identical coefficients of total degree $\deg_t + \deg_q$ at most d . We have thus $\lim_{d \rightarrow \infty} M_d(q, q) = \lim_{d \rightarrow \infty} M_d^\pm(q, q) \in \mathbb{Z}[[q]]$ for coefficient-wise convergence.

12 The length-function of \mathcal{S}_d^\pm

We remind the reader that a pair $i < j$ such that $\sigma(i) >_\pm \sigma(j)$ (with $<_\pm$ defined by Formula (5)) is an inversion of an element σ in the hyperoctahedral group S_d^\pm . (Readers annoyed by the order relation $<_\pm$ can replace the condition $\sigma(i) >_\pm \sigma(j)$ by the equivalent condition $\sigma(i)\sigma(j)(\sigma(i) - \sigma(j)) > 0$.)

Inversions $i < j$ can be classified by their sign-pattern into three types: $\sigma(i) > \sigma(j) > 0$, $\sigma(i) < 0 < \sigma(j)$ and $0 > \sigma(i) > \sigma(j)$ depicted graphically by

$$\begin{array}{c} \bullet \\ \hline \bullet \end{array}, \quad \begin{array}{c} \bullet \\ \hline \bullet \end{array} \text{ and } \begin{array}{c} \hline \bullet \\ \bullet \end{array}$$

using hopefully self-explanatory notations.

Non-inversions $i < j$ are similarly classified into $0 < \sigma(i) < \sigma(j)$, $\sigma(i) > 0 > \sigma(j)$ and $\sigma(i) < \sigma(j) < 0$ represented by

$$\begin{array}{c} \bullet \\ \hline \bullet \end{array}, \quad \begin{array}{c} \bullet \\ \hline \bullet \end{array} \text{ and } \begin{array}{c} \hline \bullet \\ \bullet \end{array}$$

We call the contribution $\sum_{0 < i, \sigma(i) < 0} (d + 1 + \sigma(i))$ to $l(\sigma)$ (given by Formula (6)) the *sign-part* of the length.

We denote the ordinary transpositions $(i, i + 1)$ of M_d^\pm by s_i for $i = 1, \dots, d - 1$ and we denote by s_d the sign change of the last coordinate (exchanging d and $-d$).

Proof of Proposition 3.1. For the sake of concision, we denote by l the function of Proposition 3.1 defined by (6). The length 0 of the identity permutation $\sigma(i) = i$ for $i \in \{1, \dots, d\}$ is obviously given by $l(\sigma) = 0$.

If $i < d$, the map $\sigma \mapsto \sigma \circ s_i$ does not affect the sign-part of σ in \mathcal{S}_d^\pm . Moreover, since $i < i+1$ is an inversion of σ if and only if it is a non-inversion of $\sigma \circ s_i$ and vice-versa, the number of inversions of $\sigma \circ s_i$ and of σ differ exactly by 1 for $i = 1, \dots, d-1$. This implies $|l(\sigma) - l(\sigma \circ s_i)| = 1$ for $i < d$.

We prove now that this holds also for $i = d$, i.e. we have $|l(\sigma) - l(\sigma \circ s_d)| = 1$: Up to replacing σ with $\sigma \circ s_d$ we can suppose $\sigma(d) = a > 0$. We depict σ schematically by the following representation

$\sigma(j)$	i_1	i_2	i_3	i_4	d
a				\bullet_4	\bullet
$-a$		\bullet_2	\bullet_3		
	\bullet_1				\circ

(the horizontal line represents 0, the vertical line separates the last index d from previous ones, the last value of σ , respectively of $\tilde{\sigma} = \sigma \circ s_d$, is represented by \bullet , respectively \circ). We denote by i_j indices taking values depicted by \bullet_j , i.e.

$$-d \leq \sigma(i_1) < -a < \sigma(i_2) < 0 < \sigma(i_3) < a < \sigma(i_4) \leq d .$$

The following Table depicts the status with respect to inversions (Yes for inversions, No for non-inversions) of σ and $\tilde{\sigma} = \sigma \circ s_d$ for $i_j < d$:

j	$\sigma(i_j) >_{\pm} \sigma(d)$	$\tilde{\sigma}(i_j) >_{\pm} \tilde{\sigma}(d)$
1	Yes	No
2	Yes	Yes
3	No	No
4	Yes	No

Setting

$$\nu_j = \#\{i < d \mid i \text{ is of type } i_j\}$$

(where "type" means represented by \bullet_j) we get now

$$\begin{aligned} l(\sigma \circ s_d) - l(\sigma) &= d + 1 - a - \nu_1 - \nu_4 \\ &= d + 1 - a - (d - a) \\ &= 1 \end{aligned}$$

using the trivial identity $\nu_1 + \nu_4 = d - a$. This shows the equality

$$|l(\sigma) - l(\sigma \circ s_d)| = 1. \tag{14}$$

So far we have proven that $l(\sigma)$, given by Formula (6) of Proposition 3.1, is at least equal to the length of σ in terms of the generators s_1, \dots, s_d .

Indeed, since composition with a generator changes the value of l exactly by 1 and since $l(\sigma) = 0$ if σ is the identity, at least $l(\sigma)$ generators are necessary for writing an arbitrary element σ of \mathcal{S}_d^\pm .

Equality holds if we show that for each σ with strictly positive $l(\sigma)$ there exists a generator s_i such that $l(\sigma \circ s_i) = l(\sigma) - 1$. We consider thus an arbitrary signed permutation σ of \mathcal{S}_d^\pm .

If $\sigma(d) < 0$, then $l(\sigma \circ s_d) = l(\sigma) - 1$ as can be seen after interchanging $\sigma, \sigma \circ s_d$ and applying the identity (14).

If $\sigma(d) > 0$ then σ is either the identity or there exists a largest integer $i < d$ such that either $\sigma(i) < 0 < \sigma(i+1)$ or $\sigma(i) > \sigma(i+1) > 0$. In both cases, $i < i+1$ defines an inversion of σ but not of $\sigma \circ s_i$ and we have thus $l(\sigma \circ s_i) = l(\sigma) - 1$. \square

Remark 12.1. *The proof of Proposition 3.1 is algorithmic. Given an element $\sigma \neq id$ of \mathcal{S}_d^\pm , we choose an index i_1 (for example as large as possible) such that $\sigma \circ s_{i_1}$ is of shorter length than σ and we iterate until arriving at the identity $s \circ s_{i_1} \circ s_{i_2} \circ \dots \circ s_{i_l}$ (with $l = l(\sigma)$ denoting the length of σ given by Formula (6) in Proposition 3.1). This leads to a shortest expression*

$$\sigma = s_{i_l} \circ s_{i_{l-1}} \circ \dots \circ s_{i_1}$$

of σ using $l = l(\sigma)$ (not necessarily different) generators s_{i_j} belonging to the set $\{s_1 = (1, 2), s_2 = (2, 3), \dots, s_{d-1} = (d-1, d), s_d = (d, -d)\}$. Every shortest expression with respect to the Coxeter generators s_1, \dots, s_d is obtained in this way.

Denoting a signed partition σ by $(\sigma(1), \dots, \sigma(d))$, we get for example for $\sigma = (-2, -3, 1)$ (of length 6, it has 3 inversions and a sign-part of $(4-2) + (4-3) = 3$):

$$\begin{array}{r|l} \sigma & (-2, -3, 1) & 6 \\ \sigma \circ s_2 & (-2, 1, -3) & 5 \\ \sigma \circ s_2 \circ s_3 & (-2, 1, 3) & 4 \\ \sigma \circ s_2 \circ s_3 \circ s_1 & (1, -2, 3) & 3 \\ \sigma \circ s_2 \circ s_3 \circ s_1 \circ s_2 & (1, 3, -2) & 2 \\ \sigma \circ s_2 \circ s_3 \circ s_1 \circ s_2 \circ s_3 & (1, 3, 2) & 1 \\ \sigma \circ s_2 \circ s_3 \circ s_1 \circ s_2 \circ s_3 \circ s_2 & (1, 2, 3) & 0 \end{array}$$

(with the last column indicating lengths) when choosing always largest possible indices. This gives for $\sigma = (-2, -3, 1)$ the expression

$$\sigma = (s_2 \circ s_3 \circ s_1 \circ s_2 \circ s_3 \circ s_2)^{-1} = s_2 \circ s_3 \circ s_2 \circ s_1 \circ s_3 \circ s_2$$

in terms of the generators $s_1 = (2, 1, 3), s_2 = (1, 3, 2), s_3 = (1, 2, -3)$.

13 The canonical half-basis of a symplectic flag

We embed a flag $F = (V_1 \subset \dots \subset V_k)$ of $\mathbb{F}^{2d} = (\mathbb{F}^{2d}, \omega)$ canonically in a Lagrangian (maximal isotropic subspace) $L(V_k)$ depending only on V_k and we associate to such a flag a length-permutation $\sigma \in \mathcal{S}_d^\pm$ and a canonical basis f_1, \dots, f_d of $L(V_k)$ such that $f_1, \dots, f_{\dim(V_i)}$ span V_i for $i = 1, \dots, k$.

We identify the symplectic space \mathbb{F}^{2d} with $\bigoplus_{i=1}^d (\mathbb{F}b_i + \mathbb{F}b_{-i})$, $i \in \{1, \dots, d\}$, endowed with the symplectic form $\omega(b_i, b_j) = 0$ if $i + j \neq 0$, $\omega(b_i, b_{-i}) = -\omega(b_{-i}, b_i) = 1$ for $i = 1, \dots, d$.

We order indices of coordinates x_i of an element $\sum_{i=1}^d (x_i b_i + x_{-i} b_{-i})$ in \mathbb{F}^{2d} always increasingly with respect to the order relation $<_\pm$ giving the order relation

$$1 <_\pm 2 <_\pm 3 <_\pm \dots <_\pm -3 <_\pm -2 <_\pm -1$$

of (5). We write thus elements of \mathbb{F}^{2d} in the form

$$(x_1, x_2, \dots, x_{d-1}, x_d, x_{-d}, x_{-d+1}, \dots, x_{-2}, x_{-1}) . \quad (15)$$

We use this order also on indices of the standard basis of $(\mathbb{F}^{2d}, \omega)$ writing it in the form

$$b_1, b_2, b_3, \dots, b_{d-1}, b_d, b_{-d}, b_{1-d}, b_{2-d}, \dots, b_{-2}, b_{-1} .$$

We call the position of the index i with respect to the order $<_\pm$ the *length* of a basis element b_i . More precisely, the length $\lambda(b_i)$ of $b_i \in \mathbb{F}^{2d}$ is given by

$$\lambda(b_i) = \begin{cases} i & \text{if } i > 0, \\ 2d + 1 - i & \text{if } i < 0. \end{cases}$$

The length $\lambda(\mathbf{x})$ of $\mathbf{x} = (x_1, \dots, x_d, x_{-d}, \dots, x_{-1}) \in \mathbb{F}^{2d}$ is the maximal length of a basis-vector involved with non-zero coefficient. The length of an element depends crucially on the dimension $2d$ of the surrounding symplectic space as shown by the following example: The element $3b_2 + b_3 - 5b_{-3}$ of (\mathbb{F}^6, ω) with coordinate-vector $(0, 3, 1, -5, 0, 0)$ has length 4. The same element $3b_2 + b_3 - 5b_{-3}$ has length 6 when considered as an element of (\mathbb{F}^8, ω) since its coordinate-vector is then given by $(0, 3, 1, 0, 0 - 5, 0, 0)$.

Let $F = (V_1 \subset V_2 \subset \dots \subset V_k)$ be a symplectic flag of \mathbb{F}^{2d} . We associate to F a basis f_1, \dots, f_d of a canonically defined maximal Lagrangian L containing V_k and a length-permutation $\sigma \in \mathcal{S}_d^\pm$ as follows: f_1 is the element of minimal length in $V_1 \setminus \{0\}$ with last non-zero coordinate 1. We define $\sigma(1)$ as the unique element of $\{\pm 1, \dots, \pm d\}$ such that f_1 and $b_{\sigma(1)}$ have the same length. More generally, given f_1, \dots, f_i (with $i < d$) we define f_{i+1} as follows: If $\dim(V_k) > i$, let $j \leq k$ be the smallest index such that $\dim(V_j) > i$ and let g_{i+1} be the shortest non-zero element all whose non-zero coordinates have indices in $\{\pm 1, \dots, \pm d\} \setminus \{\pm \sigma(1), \dots, \pm \sigma(i)\}$, which has a

last non-zero coordinate with coefficient 1 and which is such that there exist constants $\lambda_1, \dots, \lambda_i$ in \mathbb{F} with $g_{i+1} + \sum_{k=1}^i \lambda_k b_{-\sigma(k)} \in V_j \setminus \{\oplus_{k=1}^i \mathbb{F} f_k\}$. Set $f_{i+1} = g_{i+1} + \sum_{k=1}^i \lambda_k b_{-\sigma(k)}$ and define $\sigma(i+1)$ as the unique integer of $\{\pm 1, \dots, \pm d\} \setminus \{\pm \sigma(1), \dots, \pm \sigma(i)\}$ such that g_{i+1} and $b_{\sigma(i+1)}$ have the same length. If $\dim(V_k) \leq i < d$ proceed similarly by considering g_{i+1} in $\mathbb{F}^{2d} \setminus \left(\oplus_{j=1}^i \mathbb{F} f_j\right)$ and define the constants $\lambda_1, \dots, \lambda_i$ in order to get orthogonality (with respect to ω) of $f_{i+1} = g_{i+1} + \sum_{k=1}^i \lambda_k b_{-\sigma(k)}$ with f_1, \dots, f_i .

In particular $\sigma(d) < 0$ implies $\dim(V_k) = d$ (and the flag F is maximal).

An important point of this construction is the observation that the generally non-zero coordinates $x_{-\sigma(1)}, x_{-\sigma(2)}, \dots, x_{-\sigma(i)}$ of f_{i+1} are not involved in the “length” of g_{i+1} defining $\sigma(i+1)$. These coordinates are only used for forcing the orthogonality relations $\omega(f_{i+1}, f_1) = \dots = \omega(f_{i+1}, f_i) = 0$.

Observe that the length-permutation σ belongs to the subgroup \mathcal{S}_d of ordinary permutations in \mathcal{S}_d^\pm if and only if the flag F is contained in the maximal Lagrangian generated by b_1, \dots, b_d . The construction of the canonical basis f_1, \dots, f_d coincides then with the construction of the canonical basis of F considered as an ordinary (non-symplectic) flag of $\mathbb{F}^d = \mathbb{F}b_1 + \dots + \mathbb{F}b_d$.

Observe also that the lengths of $b_{\sigma(i)}$ and $b_{\sigma(i+1)}$ are increasing if $f_i, f_{i+1} \in V_j \setminus V_{j-1}$ for some j . Two such consecutive integers contribute however a descent to the flag-Major index of σ if $\sigma(i) > 0 > \sigma(i+1)$.

We call a maximal flag $F = (V_1, \dots, V_k)$ of \mathbb{F}^{2d} *standard* if $\sigma(d) < 0$ and if $b_{\sigma(\dim(V_i))}$ is longer than $b_{\sigma(\dim(V_i)+1)}$ whenever $i < k$. A non-maximal flag $F = (V_1, \dots, V_k)$ is *standard* if $b_{\sigma(\dim(V_i))}$ is longer than $b_{\dim(V_i)+1}$ for $i \leq k$ (recall that a flag is maximal if and only if V_k is d -dimensional).

Every symplectic flag $V_1 \subset \dots \subset V_k$ contains a unique standard subflag $V_{i_1} \subset \dots \subset V_{i_j}$ (for some integer $j \leq k$ and for $i_1 < i_2 < \dots < i_j$ a subset of $\{1, \dots, j\}$) with the same length-permutation.

The *standard weight* of a standard flag $F = (V_1 \subset \dots \subset V_k)$ is given in the usual way by $\sum_{i=1}^k \dim(V_i)$ and is the minimal weight for F considered as a weighted flag.

Theorem 3.2 will be an easy consequence of the following generalization of Theorem 10.2:

Theorem 13.1. *We have*

$$\sum_{F \in \mathcal{F}_{st}(\mathbb{F}_q^{2d}, \omega)} t^{w_{st}(F)} = \sum_{\sigma \in \mathcal{S}_d^\pm} q^{l^\pm(\sigma)} t^{Wmaj(\sigma)}$$

where $\mathcal{F}_{st}(\mathbb{F}_q^{2d}, \omega)$ denotes the set of all standard flags of $(\mathbb{F}_q^{2d}, \omega)$ and where $w_{st}(F)$ is the standard weight of a standard flag F .

Theorem 13.1 will be proven in the next section by showing that the set of all standard flags of $(\mathbb{F}_q^{2d}, \omega)$ with length-permutation $\sigma \in \mathcal{S}_d^\pm$ contains $q^{l^\pm(\sigma)}$ elements and that all these standard flags have the same standard weight $Wmaj(\sigma)$.

13.1 Half-bases with given length-permutation

We denote by $\mathcal{B}(\sigma)$ the set of all half-bases of \mathbb{F}^{2d} with associated length-permutation σ . The set $\mathcal{B}(id)$ for example corresponds to the half-basis b_1, b_2, \dots, b_d associated to the empty flag. The half-basis $b_{-1}, b_{-2}, \dots, b_{-d}$ belongs to $\mathcal{B}(c)$ where $c(i) = -i$ is the Coxeter element. The standard flag associated to $b_{-1}, b_{-2}, \dots, b_{-d}$ is the complete flag $V_1, \dots, V_i = \bigoplus_{j=1}^i \mathbb{F}b_{-j}, \dots, V_d = \bigoplus_{j=1}^d \mathbb{F}b_{-j}$. The set $\mathcal{B}(c)$ corresponds to complete generic flags and is in some sense as large as possible.

Proposition 13.2. *The set $\mathcal{B}(\sigma)$ of all half-bases of \mathbb{F}^{2d} with associated length-permutation σ is in one-to-one correspondence with the vector space $\mathbb{F}^{l(\sigma)}$ where $l(\sigma)$ is given by Formula (6) of Proposition 3.1.*

Proof. (We suggest to contemplate the example given in Section 13.2 while reading the proof.) Elements f_1, \dots, f_d of $\mathcal{B}(\sigma)$ are of the following form: f_i has a coefficient 1 (represented by a bullet \bullet in the example of Section 13.2) at position $\sigma(i)$, has coefficients 0 (represented by empty spaces in the example of Section 13.2) at positions $\sigma(1), \dots, \sigma(i-1)$ and at positions not of the form $-\sigma(1), \dots, -\sigma(i-1)$ which follow $\sigma(1)$ (with indices in the order $1, 2, 3, \dots, d, -d, -d+1, -d+2, \dots, -2, -1$ given by $<_{\pm}$) and has arbitrary coefficients (represented by \times or \otimes_* in Section 13.2, see below for explanations) at positions not of the form $\pm\sigma(1), \dots, \pm\sigma(i-1)$ which precede $\sigma(i)$. The coefficients at positions $-\sigma(1), \dots, -\sigma(i-1)$ (represented by \perp in Section 13.2) are uniquely determined by orthogonality of f_i with f_1, \dots, f_{i-1} .

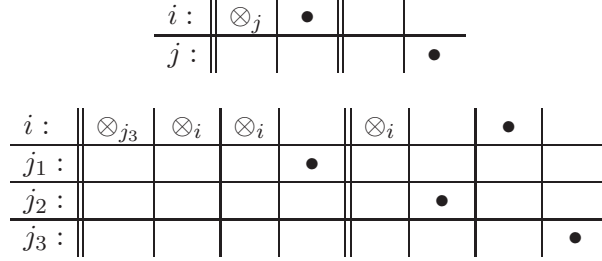
Proposition 13.2 will follow from the fact that the number of arbitrary free coefficients of such a half-basis f_1, \dots, f_d in $\mathcal{B}(\sigma)$ equals the length $l(\sigma)$ of σ .

Free coefficients of f_i are of two types: We call such a coefficient involved in f_i of *inversion-type* (represented by a cross \times in the example of Section 13.2) if it corresponds to an index k before $\sigma(i)$ with $j = \sigma^{-1}(k) > i$ (ie. if there exists $j > i$ such that the index $\sigma(j) = k$ comes before $\sigma(i)$ with respect to the order $1, 2, 3, \dots, d, -d, -d+1, \dots, -2, -1$ given by $<_{\pm}$).

The total number of such coefficients (which are all free) is given by the inversion contribution $\sum_{0 < i < j, \sigma(i) > \pm\sigma(j)} 1$ to $l(\sigma)$ in Formula (6).

The remaining free coefficients (represented by tensor products \otimes in Section 13.2) of f_i correspond to indices $k <_{\pm} \sigma(i)$ such that $\sigma(-k) \geq i$. We call such indices of *sign-type*. If $\sigma(i) > 0$, free coefficients of sign-type involved in f_i correspond to integers $j > i$ such that $0 < -\sigma(j) < \sigma(i)$. If $\sigma(i) < 0$, free coefficients of sign-type have indices $-\sigma(j)$ for $j > i$ such that $|\sigma(j)| > -\sigma(i) > 0$ and $-\sigma(j)$ for $j \geq i$ such that $\sigma(j) < 0$. (In particular, if $\sigma(i) < 0$, the coefficient of $-\sigma(i)$ is always free of sign-type). The following simplified Rothe diagrams (with the second double vertical bar separating positive from negative indices) resume the different cases for free coefficients

of sign-type (the meaning of the indices will be explained later):



We will show that the number of coefficients of sign-type is given by the last summand $\sum_{0 < i, \sigma(i) < 0} (d + 1 + \sigma(i))$ of Formula (6) defining $l(\sigma)$.

We associate an integer (written as an index of \otimes in Section 13.2) in the set $\{1 \leq i \leq d \mid \sigma(i) < 0\}$ to each free coefficient of sign-type in the following way: A free coefficient of sign-type is associated to i (with $\sigma(i) < 0$) if it corresponds either to a free coefficient of f_i with $\sigma(i) < 0$ whose index belongs to the set

$$\{-\sigma(i), \pm(1 - \sigma(i)), \dots, \pm d\}$$

or if it corresponds to a coefficient indexed by $-\sigma(i)$ of f_j with $j < i$ such that $|\sigma(j)| > -\sigma(i)$.

We claim that every free coefficient of sign-type is associated to an integer i with $\sigma(i) < 0$ and that there are exactly $d + 1 + \sigma(i)$ such free coefficients of sign-type associated to i . This ends the proof since it implies the existence of exactly $\sum_{i > 0, \sigma(i) < 0} d + 1 + \sigma(i)$ free coefficients of sign-type.

Let us first consider a free coefficient of sign-type, say a coefficient of index k involved in f_i . Since it is a free coefficient, we have $k <_{\pm} \sigma(i)$. Suppose first that $\sigma(i) > 0$. This implies $1 < k < \sigma(i)$. Since it is a free coefficient of sign-type, there exists $j > i$ such that $k = -\sigma(j) < \sigma(i)$ and the coefficient is associated to j . Suppose now $\sigma(i) < 0$. There exists again j such that $k = -\sigma(j)$. If $k < 0$, then $|k| = |\sigma(j)| > |\sigma(i)|$ and the coefficient is associated to i . If $k > 0$ then the free coefficient is associated to i if $k \geq -\sigma(i)$ and to j otherwise.

At last we have to show that there are $d + 1 + \sigma(i)$ free coefficients of sign-type associated to an integer i such that $\sigma(i) < 0$. This is realized by a bijection between such coefficients and the set $\{-\sigma(i), -\sigma(i) + 1, \dots, d - 1, d\}$ of all $d + 1 + \sigma(i)$ integers between $-\sigma(i)$ and d . Indeed, let k be such an integer. There exists j such that $|\sigma(j)| = k$. If $j > i$ then the coefficient of b_{-k} in f_i is of sign-type. If $j \leq i$ then the coefficient of $b_{-\sigma(i)}$ in f_j is of sign-type. \square

13.2 Rothe diagrams

The proof of Proposition 13.2 can be visualized by generalizing the notion of a Rothe diagram to the symplectic setting. The Rothe diagram of the

signed permutation $(\sigma(1), \dots, \sigma(6)) = (-5, 3, -1, 6, 4, -2)$ is then given by

$i \setminus \sigma(i)$	1	2	3	4	5	6	-6	-5	-4	-3	-2	-1
1	\otimes_3	\otimes_6	\times	\times	\otimes_1	\times	\otimes_1	\bullet				
2	\otimes_3	\otimes_6	\bullet		\perp							
3	\otimes_3	\otimes_3		\times	\perp	\times	\otimes_3		\otimes_3	\perp	\times	\bullet
4	\perp	\otimes_6		\times	\perp	\bullet				\perp		
5	\perp	\otimes_6		\bullet	\perp		\perp			\perp		
6	\perp	\otimes_6			\perp		\perp		\perp	\perp	\bullet	

Crosses \times (corresponding to the seven inversions

$$(1, 3), (1, 4), (1, 5), (3, 4), (3, 5), (3, 6), (4, 5)$$

defining intersections of rows and columns delimited with bullets) and tensor products \otimes_i (with indices $i = 1, 3, 6$ corresponding to the 3 negative values $\sigma(1) = -5, \sigma(3) = -1, \sigma(6) = -1$) are arbitrary elements, bullets \bullet represent 1, symbols for perpendicularity \perp are coefficients determined uniquely by orthogonality relations and empty cases represent coefficients which are 0. The length $l(\sigma)$ of σ is given by Formula (6) and equals the number of crosses \times (corresponding to the seven signed inversions

$$1 < 2, 1 < 4, 1 < 5, 3 < 4, 3 < 5, 3 < 6, 4 < 5$$

of σ whose number is counted by the first summand of (6)) added to the number of tensor products \otimes_i for $i \in \{1, 3, 6\}$ (corresponding to the three negative values $\sigma(1) = -5, \sigma(3) = -1, \sigma(6) = -2$). More precisely, the number of symbols \otimes_i is given by $6 + 1 + \sigma(i)$ for $i \in \{1, 3, 6\}$.

14 Standard weights of standard flags associated to elements of $\mathcal{B}(\sigma)$

Proposition 14.1. *For every element σ of \mathcal{S}_d^\pm , the standard weight of a standard flag associated to a half-basis in $\mathcal{B}(\sigma)$ is given by the Weyl-Major index*

$$Wmaj(\sigma) = \sum_{i>0, \sigma(i+1)<\sigma(i)} i + \sum_{i>0, \sigma(i)<0} 1$$

(cf. Formula (7)) of σ .

We illustrate Proposition 14.1 by the example of Section 13.2. Standard flags associated to $(\sigma(1), \dots, \sigma(6)) = (-5, 3, -1, 6, 4, -2)$ are given by (V_1, V_2, V_3, V_4) where

$$\begin{aligned} V_1 &= \mathbb{F}f_1, \\ V_2 &= \mathbb{F}f_1 + \mathbb{F}f_2 + \mathbb{F}f_3, \\ V_3 &= \mathbb{F}f_1 + \mathbb{F}f_2 + \mathbb{F}f_3 + \mathbb{F}f_4, \\ V_4 &= \mathbb{F}f_1 + \mathbb{F}f_2 + \mathbb{F}f_3 + \mathbb{F}f_4 + \mathbb{F}f_5 + \mathbb{F}f_6 \end{aligned}$$

with standard weight $\sum_{i=1}^4 \dim(V_i) = 1 + 3 + 4 + 6 = 14$.

The ascending runs of σ are

run	$\sigma(\text{run})$
1, 2	-5, 3
3, 4	-1, 6
5	4
6	-1

and correspond to the composition $2 + 2 + 1 + 1$ of $d = 6$. The Weyl-Major index of σ equals thus also

$$(2 + 4 + 5) + 3 = 14$$

(with the summand 3 corresponding to the three elements i such that $\sigma(i) < 0$).

Proof of Proposition 14.1. We consider a symplectic standard flag $F = (V_1 \subset \dots \subset V_{k-1} \subset V_k)$ with associated length-permutation $\sigma \in \mathcal{S}_d^\pm$. We have to show that F has standard weight $Wmaj(\sigma)$.

We prove Proposition 14.1 by induction on d . If $d = 1$, the unique element of $\mathcal{B}(\text{id})$ corresponds to the empty standard flag of weight $Wmaj(\text{id}) = 0$ and elements of $\mathcal{B}(c)$ associated to the Coxeter element $c(1) = -1$ define complete flags $\mathbb{F}(x, 1)$ in (\mathbb{F}^2, ω) . They are standard flags of standard weight $1 = Wmaj(c)$.

Given an integer $d \geq 2$, we denote by $\hat{\sigma} \in \mathcal{S}_{d-1}^\pm$ the signed permutation obtained by “erasing” $\sigma(d)$. More precisely, $\hat{\sigma}(i)$ for $i \in \{1, \dots, d-1\}$ is defined by

$$\hat{\sigma}(i) = \begin{cases} \sigma(i) & \text{if } |\sigma(i)| < |\sigma(d)|, \\ \sigma(i) - 1 & \text{if } \sigma(i) > |\sigma(d)|, \\ \sigma(i) + 1 & \text{if } \sigma(i) < -|\sigma(d)|. \end{cases}$$







We transform the canonical half-basis f_1, \dots, f_d associated to F into a canonical half-basis $\hat{f}_1, \dots, \hat{f}_{d-1}$ of $\mathbb{F}_q^{2(d-1)}$ with associated length-permutation $\hat{\sigma}$ as follows: We remove first the coefficients corresponding to indices $\pm\sigma(d)$ from all vectors f_1, \dots, f_{d-1} . The destroyed orthogonality (with respect to the symplectic form) is now restored by correcting the coefficients of f_i with indices $-\sigma(1), -\sigma(2), \dots, -\sigma(i-1)$ (the correction is defined uniquely). Finally, we rename indices of absolute value larger than $|\sigma(d)|$ by keeping their signs and decreasing their absolute value by 1. We call the resulting elements $\hat{f}_1, \dots, \hat{f}_{d-1}$. We denote the standard flag associated to $\hat{f}_1, \dots, \hat{f}_{d-1}$ by \hat{F} . It is obviously associated with the length-permutation $\hat{\sigma}$.

The induction step follows now from the equality

$$Wmaj(\sigma) - Wmaj(\hat{\sigma}) = w_{st}(F) - w_{st}(\hat{F})$$

(with w_{st} denoting the standard weight of a standard flag) which we are going to establish.

The proof splits into six cases given by the relative positions of the three integers $0, \sigma(d-1), \sigma(d)$. The following table resumes the six cases:

		$\Delta Wmaj$	\hat{V}_ω	Δw_{st}
$0 < \sigma(d-1) < \sigma(d)$		0	\hat{V}_k	0
$\sigma(d-1) < 0 < \sigma(d)$		0	\hat{V}_k	0
$\sigma(d) < 0 < \sigma(d-1)$		d	\hat{V}_{k-1}	d
$\sigma(d) < \sigma(d-1) < 0$		d	\hat{V}_{k-1}	d
$0 < \sigma(d) < \sigma(d-1)$		$d-1$	\hat{V}_{k-1}	$d-1$
$\sigma(d-1) < \sigma(d) < 0$		1	\hat{V}_k	1

The first column describes all six possible relative positions of $0, \sigma(d-1), \sigma(d)$. The second column depicts them graphically using the conventions for Rothe diagrams (positive indices are separated from negative indices by a vertical bar). The third and fifth columns contain the differences $Wmaj(\sigma) - Wmaj(\hat{\sigma})$, respectively $w_{st}(F) - w_{st}(\hat{F})$. The fourth column gives the index ω of the largest space in the standard flag \hat{F} .

The cases $0 < \sigma(d-1) < \sigma(d)$ and $\sigma(d-1) < 0 < \sigma(d)$ are similar: V_k does not involve f_k and corresponds to \hat{V}_k . The standard flags F and \hat{F} have identical weights (with respect to the standard weight). An easy computation shows that σ and $\hat{\sigma}$ have identical Weyl-Major indices.

The cases $\sigma(d) < 0 < \sigma(d-1)$ and $\sigma(d) < \sigma(d-1) < 0$ are also similar: The standard flag \hat{F} ends in both cases with the “projection” \hat{V}_{k-1} spanned by $\hat{f}_1, \dots, \hat{f}_{\dim(V_{k-1})}$ of V_{k-1} while F ends with a Lagrangian V_k of dimension d . This implies that weights of F and \hat{F} differ by d and an easy computation shows that this holds also for the Weyl-Major indices.

In the case $0 < \sigma(d) < \sigma(d-1)$, the space V_k is spanned by f_1, \dots, f_{d-1} while the largest space of \hat{F} is \hat{V}_{k-1} spanned by $\hat{f}_1, \dots, \hat{f}_{\dim(V_{k-1})}$. This implies a difference of $d-1$ for the weights of the standard flags F and \hat{F} . The same difference is realized by the flag-Major indices $Wmaj(\sigma)$ and $Wmaj(\hat{\sigma})$ of the corresponding length-permutations.

Finally, in the last case $\sigma(d-1) < \sigma(d) < 0$, the space \hat{V}_k is a maximal Lagrangian of dimension $d-1$ while V_k is a maximal Lagrangian of dimension d . This implies a difference of 1 between weights and the same difference of 1 is also realized by Weyl-Major indices. \square

14.1 Proof of Theorem 3.2

Proof. Analogues of Proposition 9.3 and Corollary 9.4 hold obviously in the symplectic setting. \square

15 Type B: Orthogonal groups of odd dimension

For simplicity, we work again only over finite fields of odd characteristic.

Sketch of Proof of Theorem 3.3. We consider again the order $<_{\pm}$, see (5). Coordinate-vectors of an element in \mathbb{F}^{2d+1} are always given in the form

$$(x_1, x_2, \dots, x_d, x_0, x_{-d}, x_{1-d}, \dots, x_{-2}, x_{-1}) .$$

We endow the vector space \mathbb{F}^{2d+1} generated by $b_{\pm 1}, \dots, b_{\pm d}, b_0$ over a field of odd characteristic with the quadratic form

$$Q(x_1, \dots, x_d, x_0, x_{-d}, \dots, x_{-1}) = x_0^2 + \sum_{i=1}^d x_i x_{-i} .$$

We consider the following bijection between elements of the set $\mathcal{B}_C(\lambda)$ of canonical bases for the symplectic space and the set $\mathcal{B}_B(\lambda)$ of canonical bases (defined in the obvious way) for the quadratic space (\mathbb{F}_q^{2d+1}, Q) .

The coordinates $(x_1, \dots, x_d, x_0, x_{-d}, \dots, x_{-1})$ of the i -th vector f_i in a basis in $\mathcal{B}_B(\lambda)$ are as follows: Coordinates with indices $<_{\pm} \lambda(i)$ in $\{0, \pm 1, \dots, \pm d\} \setminus \{\pm \lambda(1), \dots, \pm \lambda(i)\}$ are free. The coordinate $x_{\lambda(i)}$ equals 1. The coordinates $x_{-\lambda(1)}, \dots, x_{-\lambda(i-1)}$ are determined by orthogonality of f_i to f_1, \dots, f_{i-1} . The coordinate $x_{-\lambda(i)}$ of f_i is determined by isotropy of f_i (it is in fact always equal to 0 if $\lambda(i) > 0$). All coordinates $>_{\pm} \lambda(i)$ with indices in $\{\lambda(1), \dots, \lambda(i-1)\}$ or with indices in $\{0, \pm 1, \dots, \pm d\} \setminus \{\pm \lambda(1), \dots, \pm \lambda(i)\}$ are zero. Observe that $x_{-\lambda(i)}$ is never free. However x_0 is free if $\lambda(i) < 0$ and it is equal to 0 otherwise. This shows that elements of $\mathcal{B}_B(\lambda)$ behave exactly in the same way as elements of $\mathcal{B}_C(\lambda)$. We leave the easy details to the reader. \square

We illustrate this Section by the Rothe diagram “of type B” associated to the signed permutation $(\sigma(1), \dots, \sigma(6)) = (-5, 3, -1, 6, 4, -2)$. Using conventions analogous to those of Section 13 it is given by

$i \setminus \sigma(i)$	1	2	3	4	5	6	0	-6	-5	-4	-3	-2	-1
1	\otimes_3	\otimes_6	\times	\times	\perp	\times	\otimes_1	\otimes_1	\bullet				
2	\otimes_3	\otimes_6	\bullet		\perp						\perp		
3	\perp	\otimes_3		\times	\perp	\times	\otimes_3	\otimes_3		\otimes_3	\perp	\times	\bullet
4	\perp	\otimes_6		\times	\perp	\bullet		\perp			\perp		
5	\perp	\otimes_6		\bullet	\perp			\perp		\perp	\perp		
6	\perp	\perp			\perp		\otimes_6	\perp		\perp	\perp	\bullet	

16 Type D: Orthogonal groups in even dimensions

As in Section 15, all finite fields are of odd characteristic.

We denote by $\mathcal{H} = \mathcal{H}(\mathbb{F})$ the hyperbolic plane over a field \mathbb{F} realized as the quadratic space \mathbb{F}^2 endowed with the quadratic form $(x, y) \mapsto xy$, called *norm* in the sequel.

We denote by \mathcal{I} a fixed maximal d -dimensional isotropic subspace of \mathcal{H}^d (denoting d orthogonal copies of \mathcal{H}). As always, a flag $F = (V_1 \subset \cdots \subset V_k)$ is a strictly increasing sequence of non-trivial isotropic subspaces of \mathcal{H}^d . The \mathcal{I} -*parity* of a flag ending with V_k is the parity of the integer $\dim(V_k/(V_k \cap \mathcal{I})) = \dim(V_k) - \dim(V_k \cap \mathcal{I})$. Flags of even parity are simply called even flags. We denote by $\mathcal{F}^e(\mathcal{H}^d)$ the set of all even flags and by $\mathcal{WF}^e(\mathcal{H}^d)$ the set of all weighted even flags.

Remark 16.1. *The parity condition of type D flags has the following explanation: Two complete flags of type D are related by an isometry of determinant 1 if and only if they are both even or both odd. Otherwise an isometry of determinant -1 is needed. The set of all complete flags of type D decays thus into two orbits under the action of the simple linear group of Lie type D.*

In contrast, complete flags of type A, B, C are always related by an isomorphism (of the corresponding structure) of determinant 1 and the action of the corresponding simple linear group is thus transitive.

16.1 Proof of Corollary 4.3

Proposition 16.2. *The space \mathcal{H}^d over the finite field \mathbb{F}_q contains exactly*

$$q^{l(2d+l-2k-1)/2} \binom{d}{k}_q \binom{k}{l}_q$$

isotropic subspaces V of dimension k such that $\dim(V/(V \cap \mathcal{I})) = l$.

Proof. There are $\binom{d}{l}_q$ different l -dimensional subspaces of the d -dimensional quotient space $\mathcal{H}^d/\mathcal{I}$. Such a subspace $A \subset \mathcal{H}^d/\mathcal{I}$ can be lifted in $q^{dl - \binom{l+1}{2}}$ ways into an l -dimensional isotropic subspace \tilde{A} of \mathcal{H}^d . We add to \tilde{A} a $k-l$ dimensional subspace B of $\mathcal{I} \cap \tilde{A}^\perp$. This can be done in $\binom{d-l}{k-l}_q$ different ways. The subspace $V = \tilde{A} + B$ has the required properties. Such subspaces arise however with multiplicity $q^{l(k-l)}$. Indeed, for $B, B' \subset \tilde{A}^\perp \cap \mathcal{I}$, the equality $\tilde{A} + B = \tilde{A}' + B'$ holds if and only if $B = B'$ and $\tilde{A}' = \tilde{A} \pmod{B}$. The total number of such spaces is thus given by

$$\binom{d}{l}_q \binom{d-l}{k-l}_q q^{dl - \binom{l+1}{2} - l(k-l)}$$

which amounts to the formula given by Proposition 16.2. \square

16.2 First values and some properties of the polynomials M_d^D for type D

The first few polynomials M_d^D are as follows: $M_1^D = 1$, coefficients of M_2^D, M_3^D are given by

	1	q	q^2		1	q	q^2	q^3	q^4	q^5	q^6
1	1			1	1						
t		1		t	1	1					
t^2		1		t^2	1	1	1	1	1		
t^3			1	t^3	1	1	2	1	1	1	
				t^4		1	2	2	1		
				t^5			1	1	1		

and coefficients of M_4^\pm are given by

	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}
1	1												
t	1	1	1										
t^2	1	2	1	1	1	1	2	1	1				
t^3	1	1	3	3	4	4	3	3	1	1			
t^4	1	2	3	4	5	6	6	5	3	1			
t^5		1	3	6	7	8	7	6	3	1			
t^6		1	3	5	6	6	5	4	3	2	1		
t^7		1	1	3	3	4	4	3	3	1	1		
t^8			1	1	2	1	1	1	1	2	1		
t^9									1	1	1		
t^{10}													1

Evaluation of M_d^D at $t = 1$ yields

$$\frac{1 - q^d}{1 - q} \prod_{j=1}^{d-1} \frac{1 - q^{2j}}{1 - q}.$$

(Proof amounts to equality

$$\sum_{l=0}^{\lfloor d/2 \rfloor} \binom{d}{2l}_q q^{l(2l-1)} = \prod_{j=1}^{d-1} (1 + q^j) \quad .)$$

Evaluating the obvious identity

$$M_d^\pm = \sum_{k=0}^d t^k \left(\prod_{j=k+1}^d (1 - t^j) \right) \binom{d}{k}_q \left(\sum_{l=0}^k \binom{k}{l}_q q^{l(2d+l-2k-1)/2} \right) M_k$$

at $t = 1$ and comparing with ??? we get

$$\sum_{j=0}^d \binom{d}{j}_q q^{\binom{j}{2}} = \prod_{j=0}^{d-1} (1 + q^j) .$$

The following result generalizes the classical binomial theorem (corresponding to the case $q = 1$):

Theorem 16.3. *We have*

$$\sum_{j=0}^d \binom{d}{j}_q q^{\binom{j+a}{2}} t^j = q^{\binom{a}{2}} \prod_{j=0}^{d-1} (1 + tq^{j+a}) .$$

for all $d \in \mathbb{N}$.

Proof. The identity holds trivially for $d = 0$. The recursive definition $\binom{d+1}{j}_q = \binom{d}{j-1}_q + q^j \binom{d}{j}_q$ implies

$$\begin{aligned} & \sum_{j=0}^{d+1} \binom{d+1}{j}_q q^{\binom{j+a}{2}} t^j \\ &= \sum_{j=1}^{d+1} \binom{d}{j-1}_q q^{\binom{(j-1)+a+1}{2}} t^{(j-1)+1} + q^{-a} \sum_{j=0}^d \binom{d}{j}_q q^{j+a+\binom{j+a}{2}} t^j \\ &= (t + q^{-a}) q^{\binom{a+1}{2}} \prod_{j=0}^{d-1} (1 + q^{a+1+j}) \\ &= q^{\binom{a}{2}} \prod_{j=0}^d (1 + tq^{a+j}) \end{aligned}$$

which ends the proof by induction. \square

Proof of Theorem 10. Evaluating M_d^D , given by Corollary 4.3, at $q = 1$ we get

$$\begin{aligned} & \prod_{j=1}^d (1 - t^j) + \sum_{k=1}^d t^k \prod_{j=k+1}^d (1 - t^j) \binom{d}{k} 2^{k-1} \prod_{j=1}^k \frac{1 - t^j}{1 - t} \\ &= \prod_{j=1}^d (1 - t^j) \left(1 + \frac{1}{2} \sum_{k=1}^d \left(\frac{2t}{1-t} \right)^k \binom{d}{k} \right) \\ &= \prod_{j=1}^d (1 - t^j) \frac{1}{2} \left(1 + \left(1 + \frac{2t}{1-t} \right)^d \right) . \end{aligned}$$

Simplification yields

$$\frac{(1-t)^d + (1+t)^d}{2} \prod_{j=1}^d \frac{1-t^j}{1-t}.$$

which ends the proof. \square

17 The length function for type D

We consider the usual generators of the Weyl group of type D given by $s_i = (i, i+1)$, $i < d$ and $s_d(d-1) = -d$, $s_d(d) = 1-d$, $s_d(i) = i$ for $i \notin \{\pm(d-1), \pm d\}$.

Proposition 17.1. *The length $l^D(\sigma)$ of an element $\sigma \in \mathcal{S}_d^D$ with respect to the generators s_1, \dots, s_d is given by the formula*

$$l^D(\sigma) = \sum_{0 < i < j, \sigma(i) > \pm \sigma(j)} 1 + \sum_{0 < i, \sigma(i) < 0} (d + \sigma(i)) \quad (16)$$

with $>_{\pm}$ denoting the order-relation of \mathbb{Z} defined by (5).

Observe that the length $l^D(\sigma)$ of an element $\sigma \in \mathcal{S}_d^D$ with respect to the generators $s_1, \dots, s_d \in \mathcal{S}_d^D$ is always bounded above by its length $l(\sigma)$ with respect to the natural generators s_1, \dots, s_d of \mathcal{S}_d^{\pm} . Equality holds if and only if $\sigma \in \mathcal{S}_d$. More precisely, the difference is exactly the even number of elements in $-\mathbb{N} \cap \{\sigma(1), \dots, \sigma(d)\}$, as can be seen by comparing Formula (16) with Formula (6).

Proof of Proposition 17.1. The proof is by induction on the length and completely analogous to the proof of Proposition 3.1.

The result holds of course if σ is the identity.

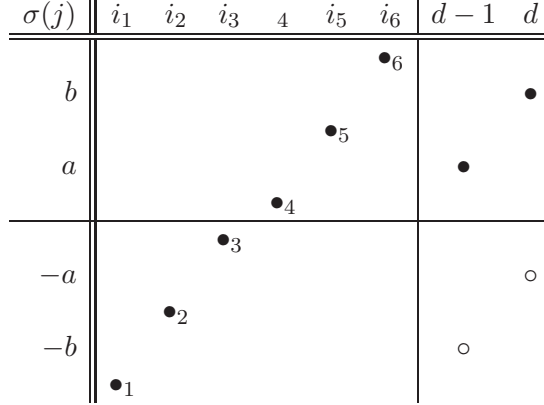
The crucial point for induction is again the equality $|l^D(\sigma) - l^D(\sigma \circ s_d)| = 1$. (The behaviour with respect to the $d-1$ first generators $\sigma_1, \dots, \sigma_{d-1}$ is as in the proof of Proposition 3.1.) We denote l^D simply by l until the end of the proof.

We write $a = \sigma(d-1)$ and $b = \sigma(d)$.

We consider two cases, depending on the sign of ab .

We discuss first the case $ab > 0$. If $a > b$, we replace σ by $\tilde{\sigma} = \sigma \circ s_{d-1}$. We have then $l(\sigma) = l(\tilde{\sigma}) + 1$ and $l(\sigma \circ s_d) = l(\tilde{\sigma} \circ s_d) + 1$. We can thus assume that $a < b$. Up to replacing σ with $\sigma \circ s_d$, we can furthermore assume that $0 < a < b$. The following representation, similar to the representation

used in the proof of Proposition 3.1, depicts all possible subcases:



(the horizontal line represents 0, the vertical line separates the two last indices $d-1$ and d from previous ones, the last two values of σ , respectively of $\tilde{\sigma} = \sigma \circ s_d$, are represented by •, respectively ○). We denote by i_j indices taking values depicted by • _{j} , i.e.

$$\sigma(i_1) < -b < \sigma(i_2) < -a < \sigma(i_3) < 0 < \sigma(i_4) < a < \sigma(i_5) < b < \sigma(i_6).$$

The following Table represents the status with respect to inversions (Yes for inversions, No for non-inversions) of σ and $\tilde{\sigma} = \sigma \circ s_d$ for $i_j < (d-1)$ and for $i_j < d$:

j	$\sigma(i_j) >_{\pm} \sigma(d-1)$	$\sigma(i_j) >_{\pm} \sigma(d)$	$\tilde{\sigma}(i_j) >_{\pm} \tilde{\sigma}(d-1)$	$\tilde{\sigma}(i_j) >_{\pm} \tilde{\sigma}(d)$
1	Yes	Yes	No	No
2	Yes	Yes	Yes	No
3	Yes	Yes	Yes	Yes
4	No	No	No	No
5	Yes	No	No	No
6	Yes	Yes	No	No

Setting

$$\nu_j = \#\{i < d-1 \mid i \text{ is of type } i_j\}$$

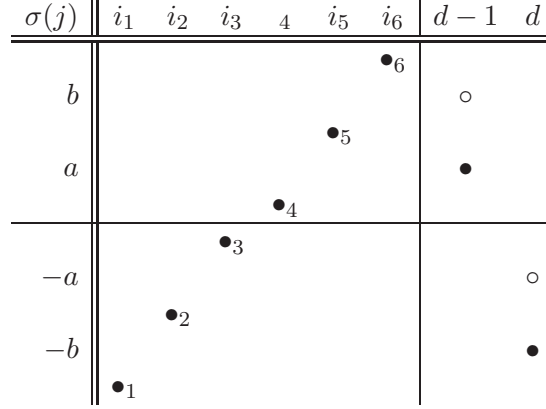
we get now

$$\begin{aligned}
l(\sigma \circ s_d) - l(\sigma) &= d - a + d - b - 2\nu_1 - \nu_2 - \nu_5 - 2\nu_6 \\
&= 2d - a - b - (2(\nu_1 + \nu_6) + (\nu_2 + \nu_5)) \\
&= 2d - a - b - (2(d-b) + (b-1-a)) \\
&= 1
\end{aligned}$$

where we have used the trivial identities $\nu_1 + \nu_6 = d-b$ and $\nu_2 + \nu_5 = b-1-a$. This settles the case $ab > 0$.

We consider now the case of $ab < 0$ with $a = \sigma(d-1), b = \sigma(d)$. If $a < 0 < b$, we set $\tilde{\sigma} = \sigma \circ s_{d-1}$. Since $l(\sigma) - l(\tilde{\sigma}) = 1$ and $l(\sigma \circ s_d) - l(\tilde{\sigma} \circ s_d) = 1$

we can replace σ with $\tilde{\sigma}$ without loss of generality. We can thus assume $a = \sigma(d-1) > 0 > -b = \sigma(d)$. Up to replacing σ with $\sigma \circ s_d$, we can moreover assume that $a < b$. The situation is now represented by



The table describing inversions involving $d-1$ or d is

j	$\sigma(i_j) >_{\pm} \sigma(d-1)$	$\sigma(i_j) >_{\pm} \sigma(d)$	$\tilde{\sigma}(i_j) >_{\pm} \tilde{\sigma}(d-1)$	$\tilde{\sigma}(i_j) >_{\pm} \tilde{\sigma}(d)$
1	Yes	No	Yes	No
2	Yes	Yes	Yes	No
3	Yes	Yes	Yes	Yes
4	No	No	No	No
5	Yes	No	No	No
6	Yes	No	Yes	No

(with $\tilde{\sigma} = \sigma \circ s_d$, as before).

Defining the numbers ν_i as above, we get

$$\begin{aligned}
 l(\sigma \circ s_d) - l(\sigma) &= (d-a) - (d-b) - (\nu_2 + \nu_5) \\
 &= b - a - (b-1-a) \\
 &= 1
 \end{aligned}$$

which settles the case $ab < 0$.

Since $l(\sigma)$ and $l(\sigma \circ s_i)$ differ always exactly by 1, the length of an element σ in \mathcal{S}_d^D is at least $l(\sigma)$. Let now σ be a non-trivial element of \mathcal{S}_d^D (we have of course $l(id) = 0$ for the identity permutation id of \mathcal{S}_d^D). If σ has an inversion then it has an inversion involving two consecutive indices $i, j = i+1$ and replacing σ with $\sigma \circ s_i$ decreases its length l by one. If $\sigma \neq id$ is without inversions then it ends with $\sigma(d-1) = -2, \sigma(d) = -1$ and applying s_d decreases its length by 1. \square

The proof of Proposition 17.1 is again algorithmic. We illustrate it by considering the permutation $(\sigma(1), \dots, \sigma(4)) = (-2, 4, -3, 1)$ of \mathcal{S}_4^D . It has length 8 (the pairs $(1, 2), (1, 3), (1, 4), (2, 4), (3, 4)$ define inversions and we

get a two sign-contributions $4 + \sigma(1) = 4 - 2 = 2$ and $4 + \sigma(3) = 4 - 3 = 1$. We denote a permutation τ of S_4^D always by $(\tau(1), \dots, \tau(4))$. We have

σ	$(-2, 4, -3, 1)$	8
$\sigma \circ s_3$	$(-2, 4, 1, -3)$	7
$\sigma \circ s_3 \circ s_2$	$(-2, 1, 4, -3)$	6
$\sigma \circ s_3 \circ s_2 \circ s_1$	$(1, -2, 4, -3)$	5
$\sigma \circ s_3 \circ s_2 \circ s_1 \circ s_2$	$(1, 4, -2, -3)$	4
$\sigma \circ s_3 \circ s_2 \circ s_1 \circ s_2 \circ s_4$	$(1, 4, 3, 2)$	3
$\sigma \circ s_3 \circ s_2 \circ s_1 \circ s_2 \circ s_4 \circ s_3$	$(1, 4, 2, 3)$	2
$\sigma \circ s_3 \circ s_2 \circ s_1 \circ s_2 \circ s_4 \circ s_3 \circ s_2$	$(1, 2, 4, 3)$	1
$\sigma \circ s_3 \circ s_2 \circ s_1 \circ s_2 \circ s_4 \circ s_3 \circ s_2 \circ s_3$	$(1, 2, 3, 4)$	0

yielding the minimal expression

$$\sigma = s_3 \circ s_2 \circ s_3 \circ s_4 \circ s_2 \circ s_1 \circ s_2 \circ s_3$$

of σ in terms of the generators

$$s_1 = (2, 1, 3, 4), s_2 = (1, 3, 2, 4), s_3 = (1, 2, 4, 3), s_4 = (1, 2, -4, -3) .$$

18 Halfbases for type D

Half-bases for type D are similar to half-bases in the symplectic case. The only difference is the fact that the coefficient of index $-\lambda(i)$ in f_i is always determined by isotropy. (It is free in the symplectic case if $\lambda(i) < 0$. This difference in behaviour translates to a difference of 1 in the summands of the second summation occurring in Formulae (6) and (16).) We illustrate this by the Rothe diagram of $(\sigma(1), \dots, \sigma(6)) = (-5, 3, -1, -6, 4, -2)$ which is given by

$i \setminus \sigma(i)$	1	2	3	4	5	6	-6	-5	-4	-3	-2	-1
1	\otimes_3	\otimes_6	\times	\times	\perp	\times	\otimes_1	\bullet				
2	\otimes_3	\otimes_6	\bullet		\perp					\perp		
3	\perp	\otimes_3		\times	\perp	\times	\otimes_3		\otimes_3	\perp	\times	\bullet
4	\perp	\otimes_6		\times	\perp	\perp	\bullet			\perp		
5	\perp	\otimes_6		\bullet	\perp	\perp			\perp	\perp		
6	\perp	\perp			\perp	\perp			\perp	\perp	\bullet	

18.1 Proof of Theorem 4.1

Up to obvious modifications, the proof is as for the type C .

19 Incorporating statistics for Eulerian polynomials of type A and BC

According to [9], a *descent* of an element w in a Weyl group W is a canonical generator s_i such that $w \circ s_i$ is shorter than w . Using our conventions, it is easy to check that the number of descents of σ in \mathcal{S}_d or in \mathcal{S}_d^\pm is given by

$$\beta(\sigma) = \delta(\sigma(d) < 0) + \sum_{1 \leq i < d, \sigma(i) >_{\pm} \sigma(i+1)} 1 \quad (17)$$

where $\delta(\text{true}) = 1$ and $\delta(\text{false}) = 0$.

Formula (17) does not coincide with the number of descents in Weyl groups of type D (the difference is however always bounded by 1).

We consider now the extended Weyl-Mahonian statistics defined by

$$\tilde{M}_d^* = \sum_{\sigma \in \mathcal{S}_d^*} q^{l^*(\sigma)} s^{\beta(\sigma)} t^{Wmaj(\sigma)}$$

of type A, BC and D.

Given a weighted flag $F = (V_1 \subset \dots \subset V_k; w_1, \dots, w_k)$, we set $\alpha(F) = \sum_{i=1}^k w_i$. We have obviously $\alpha(F) \leq w(F) = \sum_{i=1}^k w_i \dim(V_i)$.

Straightforward modifications of the proofs of Theorems 2.2, 3.2 and 4.1 show easily the following result:

Theorem 19.1. *We have*

$$\sum_{F \in \mathcal{WF}(*, q)_d} s^{\alpha(F)} t^{w(F)} = \tilde{M}_d^* \prod_{j=1}^d \frac{1}{1 - st^j}$$

with $\mathcal{WF}(*, q)_d$ denoting the obvious set of weighted flags of type $*$ with $*$ standing for A, BC or D.

Formulae for \tilde{M}_d^* are given by the following result:

Theorem 19.2. *For type A we get*

$$\tilde{M}_d = \left(\prod_{j=1}^{d-1} 1 - st^j \right) + s \sum_{k=1}^{d-1} t^k \binom{d}{k}_q \left(\prod_{j=k+1}^{d-1} 1 - st^j \right) \tilde{M}_k. \quad (18)$$

For type BC we get

$$\tilde{M}_d^\pm = \left(\prod_{j=1}^d 1 - st^j \right) + s \sum_{k=1}^d t^k \left(\prod_{j=0}^{k-1} \frac{1 - q^{2d-2j}}{1 - q^{k-j}} \right) \left(\prod_{j=k+1}^d 1 - st^j \right) \tilde{M}_k. \quad (19)$$

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