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Numerical simulation of suspensions: lubrication correction, including fluid correction

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Abstract

When simulating systems of particles embedded in a Stokes flow, it is necessary to question the treatment of close interacting particles. Indeed, when two solids come close one to another, it becomes difficult to approximate the velocity and pressure fields which become singular. However, taking the corresponding singularity is essential, both from a numerical and physical point of view. Moreover, experimentalists now need more and more precise results, taking into account the effect of these interactions on the whole flow. The method we propose is based on a decomposition of the fluid/particle problem into two subproblems: a *singular* problem (when the distance between the particles goes to zero) and a *regular* problem. The singular field is supposed to be known and the resolution of the problem comes back to solving the regular problem. A first approach have been proposed in [6], where the singular field is tabulated. Here, we propose a new method, based on an asymptotic expansion of the singular field. This method allows to catch the effect of the lubrication on the whole velocity and pressure flows and can deal with any forms of particles. We focus on a toy problem in two dimensions to present the method. Numerical results are given, using a finite element discretization.

Keywords. Suspensions. Numerical simulations. Stokes. Lubrication.

Simulation numérique de suspensions : prise en compte des forces de lubrification avec correction du champ fluide.

Résumé. Lors de simulations numériques d'écoulements de particules dans un fluide de Stokes, se pose inévitablement la question de la gestion des interactions entre particules proches. En effet, quand deux solides se rapprochent, les champs de vitesse et de pression sont singuliers et il devient difficile de les approcher numériquement. Or, la bonne prise en compte de ces interactions est primordiale, tant d'un point de vue physique que numérique. Par ailleurs, les expérimentateurs ont besoin de résultats de plus en plus précis, prenant en particulier en compte l'effet de ces interactions sur la totalité du champ fluide. La méthode que nous proposons ici consiste à résoudre le problème fluide/particules en le décomposant en deux sous problèmes : un problème *singulier* (quand les distances inter-particulaires tendent vers zéro) et un problème *régulier*. La partie singulière du champs est supposée connue et le problème initial se ramène alors à résoudre un problème régulier. Une première approche avait été proposée dans [6], dans laquelle le champs singulier était tabulé. Nous proposons ici une nouvelle méthode, basée sur un développement asymptotique du champs singulier. Cette méthode permet de prendre en compte les effets de la lubrification sur les champs de vitesse et de pression dans tout le domaine et permettent la prise en compte de particules de formes quelconques. Afin de présenter la méthode, on se concentre ici sur un problème jouet en deux dimensions d'espace. Nous présentons des résultats numériques basés sur une discrétisation éléments finis.

Mots clés. Suspensions, simulations numériques, Stokes, lubrification

1. INTRODUCTION

Dense suspensions are present in various domains coming from industry (nuclear waste reprocessing, concretes, reinforced plastics) to natural environment (silting up), biology (blood tests) or even sanitary concerns (wastewater treatment). This large area of applications gave rise to a great amount of research. However, the flow properties of these systems are still partly misunderstood. The applications we have in mind such as studying migration or understanding the interphase stress contribution to the total fluid stress involve dense systems, close to the maximal packing and for which the grains are most of the time in contact (or quasi-contact). It consists in a challenging problem, for which no result is available (neither experimental, nor theoretical, nor numerical).

From a computational point of view, it leads to suspension simulations with density between 30% and 50%, containing up to about one million of suspended particles. Experimental results show that the key points to make a breakthrough in the domain is to compute the velocity and pressure fields in the whole fluid domain for a dense suspension, to model carefully the multi-body lubrication and contact interactions and their feedback on the flow. An evaluation using numerical simulations is not possible with classical methods since they do not calculate accurately the fluid pressure and the resulting flow when the particles are close to each other. As a consequence, there is a great need of designing new mathematical models, stable and efficient numerical methods, together with HPC programming.

The objectives are the following : calculate accurately the whole pressure and velocity fields even for coarse meshes and small distances, deal with any form of particles (poly-disperse suspensions of spheres, fibres, anisotropic particles...), take advantage of the existing fast solvers for Stokes equations and obtain a parallelizable method. For the sake of clarity, we consider in this manuscript a toy problem involving 2 particles in 2 dimensions. Each of the methods presented can be easily extended to multi-particle suspensions in 3 dimensions with general relative displacements.

In section 2, we present the problem and the lubrication singularity. Then, we explain how the relative movement of close particles can be decomposed in the sum of two movements : a *regular* one and a *singular* one. We explain how this decomposition has been used since the Stokesian Dynamics was proposed in 1988 [2] to compute precisely the total force exerted on the particles. Then, we present a recent work [6] where the decomposition was used to compute precisely the whole fluid and pressure fields. In both of these methods, tabulations are used to compute accurately the singular forces or fields, which prevents the use of the methods for non mono-disperse suspensions.

In section 3, we present a new method to decompose the velocity and pressure fields using an explicit expansion of the singular fields, which is available for any form of particles. We explain how the decomposition can be implemented using finite elements and finally, we present numerical results and compare the precision of the new method with a classical direct method.

2. NUMERICAL SIMULATION OF SUSPENSIONS : A SINGULAR PROBLEM.

2.1. A toy problem. We consider (see Fig. 1) $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. We denote by $(B_i)_{i=1,2}$ two rigid spheres in Ω . $B = \cup_i B_i$ is the whole rigid domain. We suppose that $\Omega \setminus B$ is filled with a Newtonian fluid governed by the Stokes equations. The two spherical particles are aligned along the horizontal direction : if $\mathbf{X}_i \in \mathbb{R}^2$ denotes the center of particle i , we have

$$\mathbf{X}_1 = (0.5 - d/2 - r_1, 0.5) \quad \text{and} \quad \mathbf{X}_2 = (0.5 + d/2 + r_2, 0.5)$$

where r_i is the radius of particle i and d stands for the distance between the two particles.

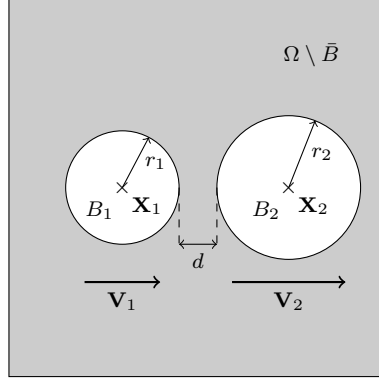


FIGURE 1. Notations.

We consider homogeneous Dirichlet boundary conditions on $\partial\Omega$. We suppose that the velocities $\mathbf{V}_1, \mathbf{V}_2$ of the particles are given and are oriented along the horizontal direction. The problem we consider consists in computing the corresponding pressure field, velocity field and forces :

$$(1) \left\{ \begin{array}{l} \text{Find } (\mathbf{u}, p) \in H^1(\Omega \setminus \bar{B}) \times L^2(\Omega \setminus \bar{B}) \text{ such that} \\ \quad -\Delta \mathbf{u} + \nabla p = 0 \quad \text{in } \Omega \setminus \bar{B} \\ \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \setminus \bar{B} \\ \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega \\ \quad \mathbf{u} = \mathbf{V}_i \quad \text{on } \partial B_i, i = 1, 2 \\ \text{and compute the corresponding forces exerted on the particles :} \\ \quad \mathbf{F}_i = \int_{\partial B_i} \sigma(\mathbf{u}, p) \cdot \mathbf{n}, i = 1, 2. \end{array} \right.$$

where σ is the stress tensor and \mathbf{n} the outward normal to the boundary of the particles.

2.2. A singular problem due to lubrication. It is well known that the main difficulty in designing numerical methods for approximating (1) arises when the configuration involves close particles. Indeed, when the two particles are almost in contact, the possible discrepancy between the velocities on both sides of the thin gap between them leads to large variations of the velocity fields. This phenomena combines with the incompressibility constraint to generate lubrication forces : large hydrodynamic forces with densities located in small areas.

More precisely, let us consider the toy problem for two spheres of radius 1 with opposite velocities $\mathbf{V}_{1,2} = \pm \mathbf{e}_x$ (see Figure 2, extracted from [6] where the force distribution was numerically computed using a decomposition in spherical harmonics). The magnitude of the force density exerted on ∂B_1 is displayed on the figure for different values of d . We can observe that the force densities concentrate on spherical caps with radii of order of \sqrt{d} .

For such large localized densities, if one wants to compute the hydrodynamic forces exerted on the particles, a large number of degrees of freedom is required in order to capture the relevant phenomenon. Therefore, simulations based on volume discretization (such as finite element or finite volume methods) will require very

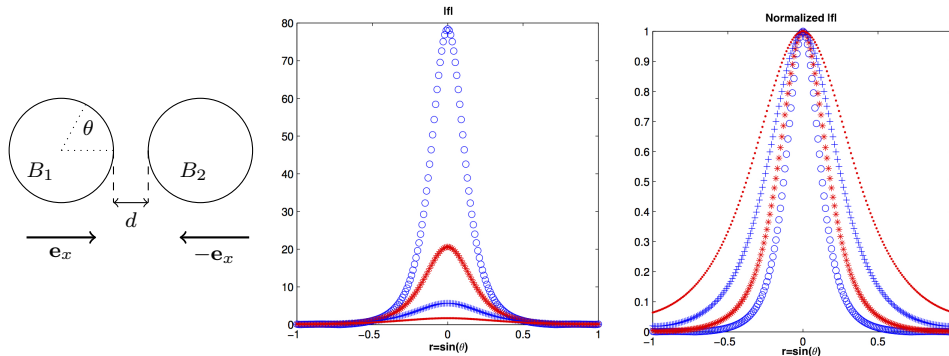


FIGURE 2. Pair of close spheres of radius 1. Notations (left). Magnitude (middle) and normalized magnitude (right) of the force density near the (almost) contact point on ∂B_1 for various distances : $d = 2.10^{-2}$ ($\circ \circ \circ$), $d = 4.10^{-2}$ ($\star \star \star$), $d = 8.10^{-2}$ ($+++$) and $d = 16.10^{-2}$ (\dots).

fine meshes to take these forces into account. For spectral discretizations (such as multipole methods) this corresponds to a truncation order L satisfying $L \gg 1/\sqrt{d}$. Such a precision cannot be achieved in simulations of 3-dimensional dense suspension and a numerical strategy has to be implemented to take the lubrication effects into account.

2.3. A singular/regular decomposition of the movement. All the numerical methods to include lubrication into simulations are based on two key ingredients. First, the movement of two close particles can be decomposed in the sum of two movements : a *regular* one plus a *singular* one. Then, once the decomposition is achieved, the behavior of the singular part of the solution can be investigated in a theoretical framework and the regular movement can be easily approximated using numerical solvers.

More precisely, using the linearity of Stokes equations, problem (1) can be decomposed in two similar problems with modified boundary conditions on the particles (see Fig. 3). In the first *regular* problem, the two particles undergo a global movement of translation. The solution of this problem do not contain singularity in the sense that the norm of the solution is bounded when the distance goes to zero. In the second *singular* problem, they move one towards the other with the same velocity. The key idea to estimate the singular effects due to the singular boundary conditions is to note that the asymptotic behavior of the solution is well known, provided we consider that the two particles are embedded in an unbounded fluid.

Such a strategy has been proposed by Durlofsky, Brady and Bossis in 1988 to take lubrication into effects in Stokesian Dynamics simulations [2] and has been widely used since then. Recently, their method has also been used to include lubrication effects in direct solvers [4]. Their founding remark is the following : supposing that the particles are embedded in an infinite fluid and have the same radius, the solution to the problem with opposite velocities $\mathbf{V}_{1,2}^{sing} = \pm \mathbf{e}_x$ only depends on the distance between the particles. Then, in that case, the forces exerted on the two particles can be expressed as a (vector-valued) function of d : $d \rightarrow \mathbf{F}_{i,\mathbb{R}^2}^{sing}(d)$. This function can be accurately approximated for any distance using its well-know asymptotic expansion when d goes to zero [1], together with an accurate tabulation for greater distances. As a consequence, since the asymptotic behavior when the distance goes to zero of the initial problem in Ω is the same as the one in \mathbb{R}^2 , the forces exerted

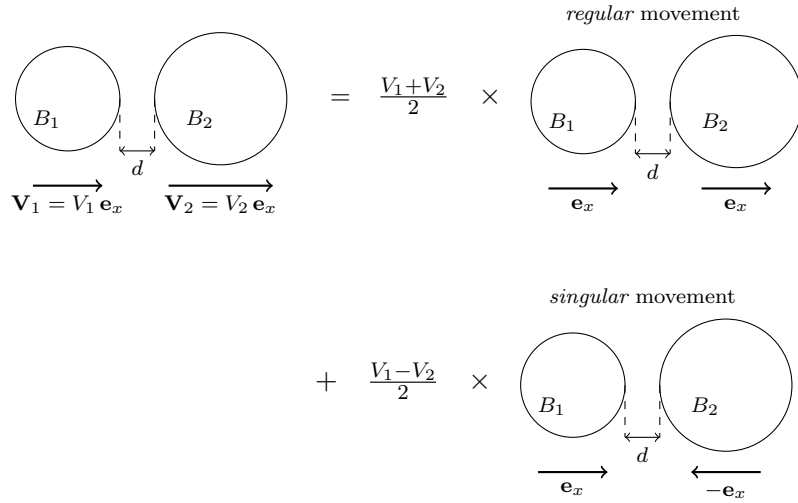


FIGURE 3. Decomposition of the movement into the sum of two movements : a *regular* movement and a *singular* movement.

on the particles can be computed as follows :

$$\mathbf{F}_i = \mathbf{F}_{i, \mathbb{R}^2}^{sing}(d) + \mathbf{F}_i^{reg}, \quad i = 1, 2$$

where the remaining part of the force \mathbf{F}_i^{reg} is *regular* (i.e. is bounded when the distance goes to zero) and can be computed using any numerical solver with a small number of degrees of freedom.

This singular/regular decomposition provides an accurate approximation of the forces exerted on the particles. However, using this method, there is no way to obtain velocity and pressure fields including lubrication effects. Here, the fluid solver is used to compute the regular part of the forces and no correction is achieved on the flow to provide an approximation of the fields (\mathbf{u}, p) solution to (1). Moreover, a precise tabulation of the forces is possible, provided the corresponding function only depends on the distance between the particles. In the case it depends on two parameters or more, this step is too demanding in CPU time, even if achieved off-line. This prevents the use of this correction method in simulations of suspensions made of poly-disperse spherical particles (the singular part of the forces depends both on the distance and the ratio between the radius of the particles) or suspensions of non spherical particles (the singular part of the forces depends on the distance, the form and the relative position of the particles).

3. A SINGULAR/REGULAR DECOMPOSITION OF THE FLOW

3.1. Description of the method. We proposed in [6] a new method based on a singular/regular decomposition of the whole flow. We write

$$(2) \quad \begin{cases} \mathbf{u} &= \mathbf{u}^{reg} + \mathbf{u}^{sing}, \\ p &= p^{reg} + p^{sing}. \end{cases}$$

The *singular* field is supposed to catch the singularity of the solution when the distance goes to zero in such a way that the remaining *regular* field is bounded independently of the distance in $H^1(\Omega \setminus \bar{B}) \times L^2(\Omega \setminus \bar{B})$. Let us for the moment suppose that such a singular field can be computed (this point will be discussed

later). Then, to solve the initial problem (1) for (\mathbf{u}, \mathbf{p}) , it is enough to compute $(\mathbf{u}^{reg}, \mathbf{p}^{reg})$, solution to the following regular problem :

$$(3) \quad \left\{ \begin{array}{l} \text{Find } (\mathbf{u}^{reg}, \mathbf{p}^{reg}) \in H^1(\Omega \setminus \bar{B}) \times L^2(\Omega \setminus \bar{B}) \text{ such that} \\ -\Delta \mathbf{u}^{reg} + \nabla \mathbf{p}^{reg} = \Delta \mathbf{u}^{sing} - \nabla \mathbf{p}^{sing} \quad \text{in } \Omega \setminus \bar{B} \\ \nabla \cdot \mathbf{u}^{reg} = -\nabla \cdot \mathbf{u}^{sing} \quad \text{in } \Omega \setminus \bar{B} \\ \mathbf{u}^{reg} = -\mathbf{u}^{sing} \quad \text{on } \partial\Omega \\ \mathbf{u}^{reg} = \mathbf{V}_i - \mathbf{u}^{sing} \quad \text{on } \partial B_i, i = 1, 2 \\ \text{and compute the corresponding forces exerted on the particles :} \\ \tilde{\mathbf{F}}_i^{reg} = \int_{\partial B_i} \sigma(\mathbf{u}^{reg}, \mathbf{p}^{reg}) \cdot \mathbf{n}, i = 1, 2. \end{array} \right.$$

3.2. Choice for the singular field. If $(\mathbf{u}^{reg}, \mathbf{p}^{reg})$ is computed as a solution to problem (3), the field (\mathbf{u}, \mathbf{p}) given by (2) is solution the initial problem (1), whatever the singular field is. The remaining question is to manage to design $(\mathbf{u}^{sing}, \mathbf{p}^{sing})$ such that the *regular* problem (3) can be numerically approximated with a number of degrees of freedom not depending on the distance between the particles.

This can be achieved using, as for the Stokesian Dynamics correction, the singular part of the movement described in Fig. 3. Note that the decomposition has to be achieved as soon as the relative movement of the two particles is non zero i.e. as soon as V_1 is different from V_2 . In [6], the singular field $(\mathbf{u}^{sing}, \mathbf{p}^{sing})$ is defined as

$$(\mathbf{u}^{sing}, \mathbf{p}^{sing}) = \frac{V_2 - V_1}{2} (\mathbf{u}_{\mathbb{R}^2}^{sing}, \mathbf{p}_{\mathbb{R}^2}^{sing})$$

where $(\mathbf{u}_{\mathbb{R}^2}^{sing}, \mathbf{p}_{\mathbb{R}^2}^{sing})$ is the solution to the fluid-particle problem in \mathbb{R}^2 with the boundary conditions $\mathbf{V}_{1,2}^{sing} = \pm \mathbf{e}_x$. Again, for particles with the same radius, this singular field only depends on the distance between the particles and in [6] we propose to tabulate its coefficients in a well-chosen basis. This leads to a precise method to compute the solution to problem (1). Here, the lubrication is taken into account in the computed flow (\mathbf{u}, \mathbf{p}) . However, since the singularity is computed using tabulations, the method is limited to mono-disperse suspensions of spherical particles. Moreover, the singular field is not truncated and therefore, the boundary conditions for the regular problem has to be modified in the whole boundary of $\Omega \setminus \bar{B}$, which will induce heavy computations when extending the method to dense suspensions.

Here, we propose a new choice for the singular field, based on an asymptotic expansion of the flow when the distance goes to zero. The computations to obtain such an expansion are similar to the ones achieved in [1] to obtain the asymptotic behavior of the total lubrication force. More recently, in [5], the authors provide explicit formulas for the expansion and prove that the remaining part of the flow is regular in the sense that its norm is bounded in $H_0^1(\Omega \setminus \bar{B}) \times L^2(\Omega \setminus \bar{B})$ independently of the distance between the particles. We propose to use

$$(4) \quad (\mathbf{u}^{sing}, \mathbf{p}^{sing}) = \frac{V_2 - V_1}{2} (\mathbf{u}_{exp}^{sing}, \mathbf{p}_{exp}^{sing})$$

where $(\mathbf{u}_{exp}^{sing}, \mathbf{p}_{exp}^{sing})$ is a truncation of the explicit asymptotic expansion for the fluid-particle problem in \mathbb{R}^2 with the boundary conditions $\mathbf{V}_{1,2}^{sing} = \pm \mathbf{e}_x$. An example of such a field is given in Fig 4.

Then, solving the corresponding regular problem (3) and computing the solution as the sum of the regular and singular fields(2), the lubrication is taken into account in the computed flow (\mathbf{u}, \mathbf{p}) . The new singular field is explicit and only depends (explicitly) on the local geometry of the particles near the (almost) contact point

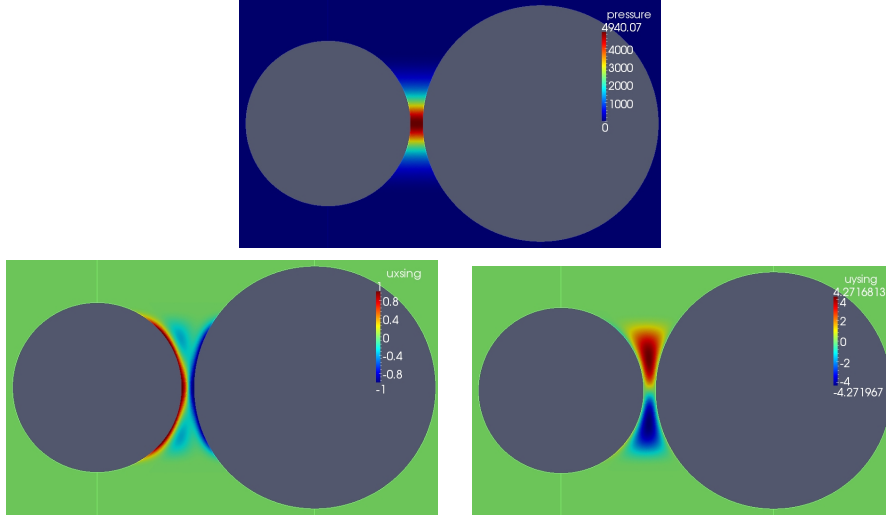


FIGURE 4. Truncated asymptotic singular field. $r_1 = 0.07$, $r_2 = 0.1$, $d = r_2/10$. p_{exp}^{sing} (top) and \mathbf{u}_{exp}^{sing} , horizontal component (bottom left) and vertical component (bottom right).

between them. Then, the method can be implemented for poly-disperse suspensions of spherical particles and for various forms of particles. Moreover, the singular field is localized in space, which avoids too many computations when modifying the boundary conditions to solve the regular problem (3).

3.3. Numerical implementation. We propose to solve (3) using a finite element method. Let $V_h \subset H^1(\Omega \setminus \bar{B})$ and $M_h \subset L^2(\Omega \setminus \bar{B})$ be a stable pair of finite element spaces for the Stokes problem. The singular field is given by the truncated asymptotic expansion (4). We want to compute the approximation of the regular field $(\mathbf{u}_h^{reg}, p_h^{reg})$ solution to

$$(5) \quad \left\{ \begin{array}{l} \text{Find } (\mathbf{u}_h^{reg}, p_h^{reg}) \in V_h \times M_h \text{ such that} \\ -\Delta \mathbf{u}_h^{reg} + \nabla p_h^{reg} = \Delta \mathbf{u}^{sing} - \nabla p^{sing} \quad \text{in } \Omega \setminus \bar{B} \\ \nabla \cdot \mathbf{u}_h^{reg} = -\nabla \cdot \mathbf{u}^{sing} \quad \text{in } \Omega \setminus \bar{B} \\ \mathbf{u}_h^{reg} = 0 \quad \text{on } \partial\Omega \\ \mathbf{u}_h^{reg} = \mathbf{V}_i - \mathbf{u}^{sing} \quad \text{on } \partial B_i, i = 1, 2, \end{array} \right.$$

where the right-hand side is computed as precisely as possible. Then, from (2), the discrete field approximating (\mathbf{u}, p) solution to problem (1) is

$$\begin{aligned} \mathbf{u}_h &= \mathbf{u}^{sing} + \mathbf{u}_h^{reg} \in \mathbf{u}^{sing} + V_h \\ p_h &= p^{sing} + p_h^{reg} \in p^{sing} + M_h \end{aligned}$$

Non-homogeneous Stokes problems as (3) or (5) have a unique solution if and only if a compatibility condition is satisfied by the right-hand side of the problem, expressing the fact that one may have

$$\int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}^{reg} = \int_{\partial\Omega \setminus \bar{B}} \mathbf{u}^{reg} \cdot \mathbf{n} \quad \text{and} \quad \int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}_h^{reg} = \int_{\partial\Omega \setminus \bar{B}} \mathbf{u}_h^{reg} \cdot \mathbf{n}.$$

Concerning the continuous problem (3), the compatibility condition is verified since

$$\int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}^{sing} = \int_{\partial\Omega \setminus \bar{B}} \mathbf{u}^{sing} \cdot \mathbf{n}.$$

It is not true anymore for the discrete problem (5). Indeed, inside Ω , the divergence equation gives

$$\int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}_h^{reg} = \int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}^{sing}.$$

Concerning the boundary we have

$$\mathbf{u}_h^{reg}|_{\partial\Omega \setminus \bar{B}} = \mathbf{V}_i - \text{Proj}(\mathbf{u}^{sing}) \quad \text{on } \partial B_i$$

where Proj is the projection onto the space made of the traces on $\partial\Omega \setminus \bar{B}$ of the functions of V_h . Then,

$$\int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}_h^{reg} = \int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}^{sing} \neq \int_{\partial\Omega \setminus \bar{B}} \text{Proj}(\mathbf{u}^{sing}) \cdot \mathbf{n} = \int_{\partial\Omega \setminus \bar{B}} \mathbf{u}_h^{reg} \cdot \mathbf{n}.$$

As a consequence, problem (5) may not have solutions.

To circumvent this problem, we chose $\varepsilon > 0$ and we introduce a modified discrete regular problem :

$$(6) \quad \left\{ \begin{array}{l} \text{Find } (\mathbf{u}_{h,\varepsilon}^{reg}, p_{h,\varepsilon}^{reg}) \in V_h \times M_h \text{ such that} \\ -\Delta \mathbf{u}_{h,\varepsilon}^{reg} + \nabla p_{h,\varepsilon}^{reg} = \Delta \mathbf{u}^{sing} - \nabla p^{sing} \quad \text{in } \Omega \setminus \bar{B} \\ \nabla \cdot \mathbf{u}_{h,\varepsilon}^{reg} + \varepsilon p_{h,\varepsilon}^{reg} = -\nabla \cdot \mathbf{u}^{sing} \quad \text{in } \Omega \setminus \bar{B} \\ \mathbf{u}_{h,\varepsilon}^{reg} = 0 \quad \text{on } \partial\Omega \\ \mathbf{u}_{h,\varepsilon}^{reg} = \mathbf{V}_i - \mathbf{u}^{sing} \quad \text{on } \partial B_i, i = 1, 2. \end{array} \right.$$

This problem has a unique solution. Testing the divergence equation with $v_h \equiv 1 \in V_h$ we see that

$$\varepsilon \left(\int_{\Omega \setminus \bar{B}} p \right) = - \int_{\Omega \setminus \bar{B}} \nabla \cdot \mathbf{u}^{sing} + \int_{\partial\Omega \setminus \bar{B}} \text{Proj}(\mathbf{u}^{sing}) \cdot \mathbf{n} \neq 0.$$

The classical unknown constant in the pressure field will be chosen by the solver so as to compensate the fact that the compatibility condition is not satisfied from the numerical point of view.

The constant ε has to be chosen sufficiently small in order that the error introduced in the divergence equation is smaller than the one due to the finite element discretization. On the other hand, the finite element discretization being fixed, letting ε go to zero in the numerical tests shall induce numerical errors.

4. NUMERICAL RESULTS

The numerical tests have been implemented with **freefem++** [3]. We chose $V_h = P_b^1$ and $M_h = P^1$. The configuration is the one given in Fig. 1 with $r_1 = 0.07$, $r_2 = 0.1$, $\mathbf{V}_1 = \mathbf{e}_x$, $\mathbf{V}_2 = -\mathbf{e}_x$. Three distances have been tested : $d = r_2/2$, $d = r_2/5$ and $d = r_2/10$.

We plot in Fig. 5 the numerical solutions obtained for $d = r_2/10$. The solution is given by the singular/regular decomposition with $\varepsilon = 10^{-4}$ and using a quadrature formula of order 10 to compute the right-hand side of the regular problem. The direct solver Crout is used to solve the linear system. The reference is computed using the direct finite element method for (1), the same direct linear system solver, and a very fine mesh (until convergence). The constant in the pressure field is computed such that $\int p = 0$. We can observe that, even for a very coarse mesh, the solution is well approximated.

To be more precise, we plot on Fig. 6 the relative errors in norm $L^2(\Omega \setminus \bar{B})$ for the pressure and $H^1(\Omega \setminus \bar{B})$ for the velocity. The convergence of the direct method and the decomposition method are compared for different distances. We can observe that the classical order of convergence for the $P_b^1 \times P^1$ approximation of

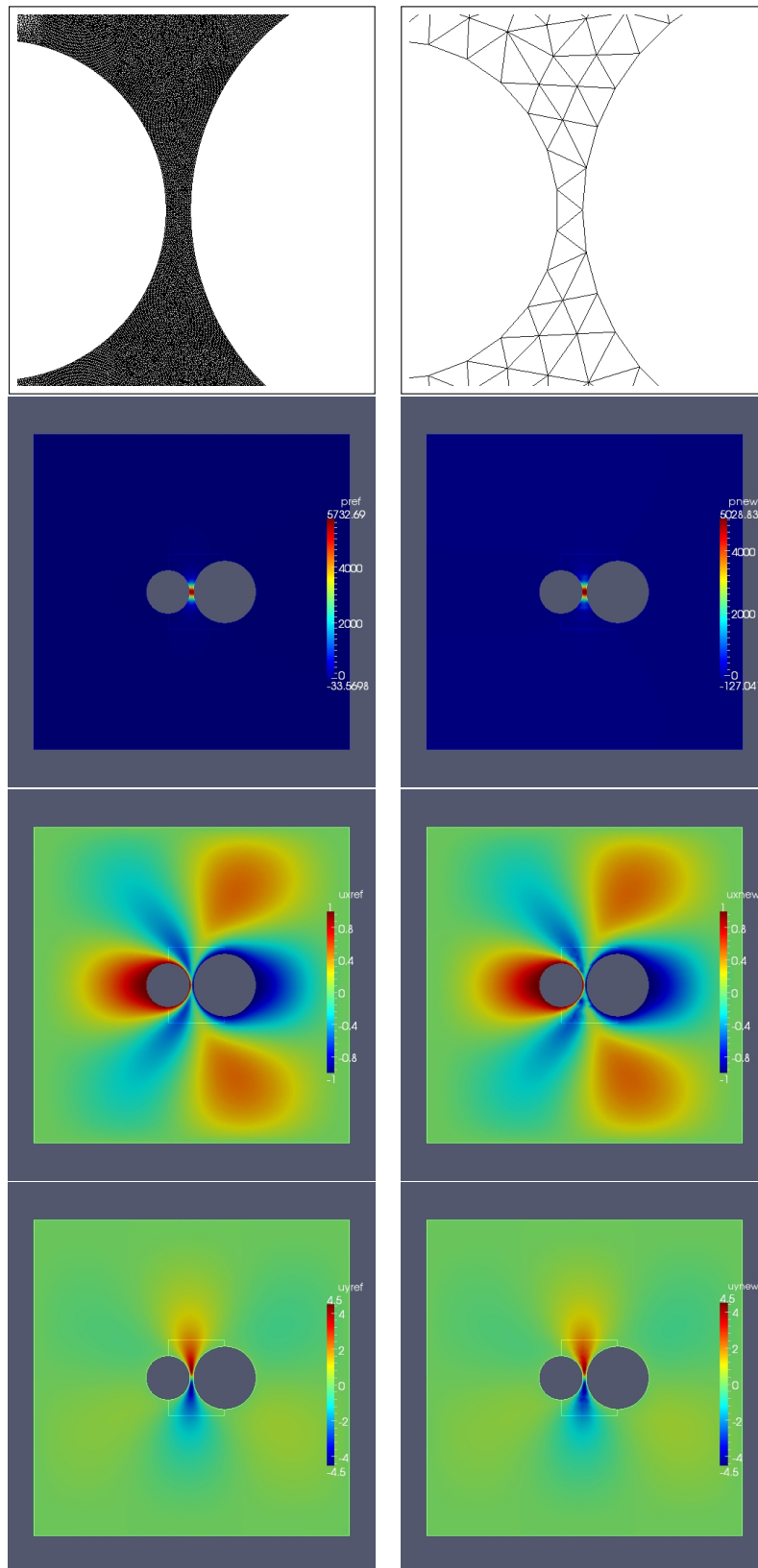


FIGURE 5. Numerical solution. Reference (left) and numerical solution (right). Zoom of the mesh in the gap between the particles (top), pressure (line 2), horizontal velocity (line 3) and vertical velocity (line 4)

Stokes equations is recovered. The decomposition method is equivalent to the direct method for $d = r_2/2$ but becomes more precise than the direct method when the distance between the particles decreases. These numerical tests also confirm that the precision is less dependent on the distance when the decomposition method is used.

5. CONCLUSION

We presented in this manuscript a new method to include lubrication in numerical simulations of particles. The lubrication effects are transmitted to the whole velocity and pressure flows. The singularity is taken into account by solving a numerical *regular* problem which is a Stokes problem with a modified right-hand side. This allows the use of existing fast solvers to compute its solution. It can deal with poly-disperse suspensions and with any forms of particles since the explicit asymptotic expansion only depends on the local geometry of the particles.

This method can straightforwardly be extended to the three dimensional case with any number of particles. In this manuscript we tested the singularity induced by normal relative translation of the two particles. The general relative movement of two particles can be decomposed as a *regular* movement (a global rigid movement) plus a *singular* movement including normal and tangential relative translations as well as relative rotations. All these singular relative movements can be taken into account in the simulations since the asymptotic expansion of the corresponding singular fields are available.

We believe this method can be useful for intensive simulations of dense three-dimensional suspensions. Indeed, in that case, the lubrication effects are predominant but only coarse meshes can be used. Moreover, the size of the problem imposes to use iterative solvers. Using the decomposition method, we end with a regular numerical problem and then, less iterations of the solvers shall be needed to obtain a given precision. Moreover, the computation of the right-hand side can be parallelized since the contributions of each pairs of particles are independent.

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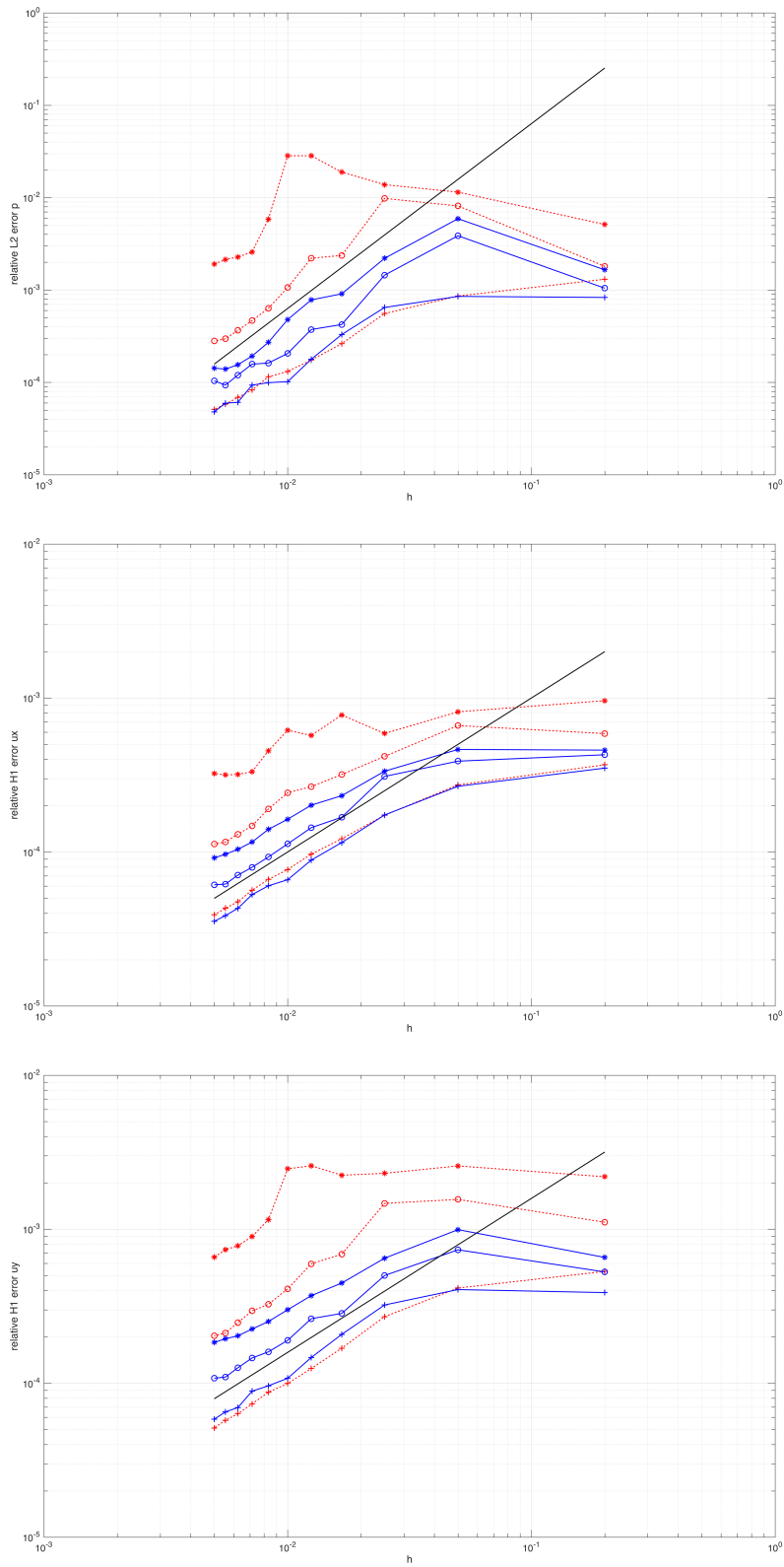


FIGURE 6. Relative errors versus h for the direct method (\cdots) and the decomposition method ($-$). The tests are run for different values of the distance : $d = r_2/2$ ($+$), $d = r_2/5$ (o) and $d = r_2/10$ ($*$). Slope 2 is given on the top plot (black line). Slope 1 is given on the middle and bottom plot (black line).