Optimal procedure for the identication of constitutive parameters from experimentally measured displacement fields

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Abstract

Measured displacement fields constitute a rich database for the identification of mechanical behavior. A variety of methods are available today to address this problem. A formalism is proposed showing that they all revert to the minimization of a quadratic norm (or semi-norm) between measured and computed displacement fields. However, they differ in the chosen metric.

Transporting image noise through Digital Image Correlation and identification procedures, the uncertainty in the measured displacement fields (*i.e.,* their full covariance) and the resulting uncertainty in the estimated constitutive parameters is assessed. Consequently, an optimal choice of metric (with respect to image noise) is proposed so that the resulting identification uncertainty is minimized.

Keywords: identification of constitutive law; calibration of parameters; full-field measurements; uncertainty;

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1. Introduction

Nowadays' use of materials requires high mechanical properties, complex microstructures with ever less usage of raw components (to lower energy consumption, and CO_2 imprint), in increasingly severe conditions, may potentially jeopardize the safety in domains such as energy and transportation. In these areas mechanical sciences must provide reliable and complete numerical tools to enable for decision-making [1]. The need for ever more robust and faithful constitutive models calls for advanced identification and validation strategies. The latter ones have seen numerous developments with the emergence and democratization of full-field measurement techniques [2].

Very large numbers of measured data (*i.e.,* dense fields in some cases) are accessible with rather standard equipments [3]. Their weakness is that their uncertainties are generally higher than those of point measurements. Gaussian noise (potentially correlated) is the most frequently encountered case. Its statistical properties are exhaustively captured by the full covariance affecting the measured data [4]. When using optical cameras to measure displacement fields, it is then possible to propose general frameworks that link image noise to displacement uncertainties, and the latter ones with parameter fluctuations [5, 3].

In the field of solid mechanics, various identification methods have been developed over the last decades [6]. Some of them explicitly require full-field measurements to be available [7]. Finite element model updating (FEMU) was the first proposition [8, 10, 9], which is an iterative procedure comparing measured and simulated displacement, strain or load levels [11, 12, 8]. Least squares errors were considered with no special emphasis on their weighting. Other types of gaps were introduced. In elasticity, different variational principles were considered [13, 14]. The constitutive equation error, which was initially introduced for the verification of numerical models [15], was also used for identification purposes [16, 17, 18, 19, 20]. The reciprocity gap [21, 22, 23] considers only surface measurements to determine various types of defects in the bulk of the analyzed domain. Non iterative methods such as the virtual fields method [24, 25, 26] and the equilibrium gap method [27, 28, 29, 30, 31] were introduced to calibrate elastic parameters and damage models.

In the vast majority of studies mentioned so far measurement uncertainties were not explicitly accounted for. Optimal extractors, namely, the least sensitive to measurement uncertainties were introduced for the choice of virtual fields [32], the estimation of fracture mechanics parameters [33], or the identifiability of load and contrast fields in microcantilevers [34]. Specific weighting based on global DIC uncertainties was also proposed for the identification of elastic properties [35]. Last, under the assumption of small noise level, it was shown that weighted FEMU and integrated DIC led to the same covariances of identified parameters [36].

The present paper aims at extending this optimality feature by analyzing each of the afore-mentioned identification methods in view of their sensitivity to measurement uncertainties. Once these various sensitivities have been discussed, it is proposed to formulate an optimal approach. The statement of the problem is introduced in Section 2. Section 3 reviews various identification methods in a unified framework. An optimal method is proposed in Section 4. Extensions to nonlinear problems are discussed in Section 5.

2. Statement of the problem

The identification problem consists in estimating the constitutive parameters "at best" from a measured displacement field (and possibly additional data such as applied forces, temperature fields).

It is worth noting that even the very existence of a constitutive law may be relaxed in data-driven approaches where stresses and strains are related from a cloud of data rather than an algebraic relationship [37, 38]. In the present study, the choice has been made to stick to a more traditional view where a constitutive law is formulated explicitly and parameterized.

To ease the discussion, only the case of linear elasticity (*i.e.,* the most universal constitutive postulate) will be addressed first, and extensions to more general constitutive laws will be addressed at the end of this paper. Hence, the constitutive law reduces to the Hooke's tensor, $\mathcal{C}$, parameterized by a set of parameters, $\{\boldsymbol{p}\} = \{p_1, ..., p_m\}^\top$, collected into a column vector (*e.g.,* isotropic elasticity requires only two parameters, the Lamé's moduli, or the Young's modulus and Poisson's ratio).

Let us also stress that no model error are assumed to be present. That is to say the actual constitutive law of the material under study lies within the space of the considered constitutive laws, and the difficulty is to find the best estimates of the constitutive parameters, knowing that one such set exists. In other words, an elastic law will not be sought when the behavior is elastoplastic, or elasticity will be restricted to isotropy when the behavior is anisotropic. Such cases of model error are much more demanding as they call for an appropriate metric in a space of constitutive laws, a question that is difficult to address on objective grounds. The simpler issue of asking whether the behavior that is known to belong to a large class of laws may be restricted to a subclass (that of specific symmetries or properties), thereby leading to strategies of gradual enrichments of considered models [39] is even considered beyond the scope of the following analyses. Similarly, it will be assumed that discretization errors are not relevant. Equipped with error estimates, one may consider today that the issue of numerical verification is mature enough to be under control [15, 40, 41, 42].

One issue of the problem is to define "at best." A distinctive feature of experimental measurements is that they are corrupted by noise [43, 44]. Consequently, the determination of $\{\boldsymbol{p}\}$ will inherit from the noisy input data a fluctuating part $\{\boldsymbol{\delta p}\}$, which may display a *systematic error* $\langle\{\boldsymbol{\delta p}\}\rangle$, and a *(co)variance* $\langle\{\boldsymbol{\delta p}\}\{\boldsymbol{\delta p}\}^\top\rangle$ where the angular brackets stand for the expectation over statistical realizations of the noise. The best determination will thus be defined as that 1) which has no bias, and 2) minimizes the uncertainty.

Problem. *From a given measured displacement field, $\boldsymbol{u}^m(\boldsymbol{x})$, over a region of interest Ω, the problem consists in estimating the constitutive parameters, $\{\boldsymbol{p}\}$, with no bias and the least uncertainties.*

Let us note that the first condition (*i.e.,* no bias), is not necessarily met. For instance, a method that would aim at the computation of the elastic energy from the given displacement field would involve a quadratic form on the displacement, and hence the displacement uncertainties will give rise to a contribution to this energy, which in turn will result in a systematic error. Hence, estimates of $\{\boldsymbol{p}\}$ based on nonlinear functions of the displacements are to be rejected based on the bias argument. Alternatively obtaining a bias-free estimate can be solved straightforwardly from any evaluation of $\{\boldsymbol{p}\}$ that is linear (or whose incremental correction is linear) in the measured displacement.

2.1. Simplifying assumptions

Nowadays, mechanical tests are more and more systematically equipped with imaging devices to provide surface (be it planar stemming from single-view or 3D surface displacements from multiview systems) or volume (based on tomography) displacement [45, 46]. Thus one experiment is often studied all along a loading history and long time series of images are available (this is mandatory for complex constitutive laws). It is even sometimes necessary to combine several experiments to reach a complete description. Strategies designed to address such image series, be they relative to single or multiple experiments have been proposed, but they will not be discussed hereafter. For the sake of simplicity, the most simple case is first addressed, namely a single deformed state. Dense fields are considered, thus either in 2D based on DIC measurement or 3D using Digital Volume Correlation (DVC). The case of stereocorrelation, which only informs on the surface deformations of a solid without volume kinematic measurement is not considered herein.

As a further simplification in phase with the recourse to a single image, a linear elastic description is assumed to hold. A measured displacement field $\boldsymbol{u}^m$, say by DIC or DVC, is considered to be available. Similarly, Neumann boundary conditions as well as the density of body forces are assumed to be known exactly. The problem then consists in finding the elastic properties such that the displacement field computed from a numerical model, for instance using Finite Element Analyses (FEAs) matches as well as possible the measured field. Further, geometry is expected to be known exactly. Hence the specimen shape to be meshed in the model is known precisely, and the reference frame of the image is assumed to be perfectly registered onto that of the numerical twin so that the matching of points does not introduce any discrepancy per se.

Last, the discretization of the kinematic measurement in the FE modeling, which can differ from that of the DIC analysis is not addressed at this point. For global DIC/DVC techniques that can use arbitrary meshes, this question is not critical, but for local approaches this point is more relevant as boundaries are more delicate to be described [47]. It is assumed that the discrete measurement data are localized at points $\boldsymbol{x}_i$, where the modeled displacement field is to be evaluated for comparison to measurements.

The previously cited assumptions introduce some restrictions, which are initially introduced to ease the discussion. After having reached the stage where an optimal identification method has been defined, the above assumptions will be revisited in order to show that they can be relaxed within the identification framework.

2.1.1. FEA

Within the framework of a finite element discretization of the problem, the Lamé's equation takes a very classical form [41]

$$[\boldsymbol{K}]\{\boldsymbol{v}\} = \{\boldsymbol{f}\} \tag{1}$$

where $[\boldsymbol{K}]$ is the stiffness matrix, $\{\boldsymbol{v}\}$ the nodal displacement vector and $\{\boldsymbol{f}\}$ the nodal forces expressing the presence of body forces and/or surface tractions. Boundary nodes have degrees of freedom that are assumed to be subjected to either Neumann (*i.e.,* known tractions) or Dirichlet (*i.e.,* known displacements), possibly component-wise. Other types of linear boundary conditions such as Robin type can be incorporated in a similar description by moving the boundary to include the elastic coupling within the domain, and hence the restriction to either Neumann or Dirichlet is not limiting.

2.1.2. Notations

It will be convenient for the following discussion to introduce a diagonal matrix $[\boldsymbol{D}]$, valued 1 for interior and Neumann degrees of freedom, and 0 for the other (Dirichlet) degrees of freedom where displacements are known. For the Dirichlet degrees of freedom, it is natural to consider that the displacement is set by the measurement. Thus, a modeled displacement field $\{\boldsymbol{v}\}$ is introduced, such that

$$\begin{array}{rcl} ([\boldsymbol{I}]-[\boldsymbol{D}])\{\boldsymbol{v}\} & = & ([\boldsymbol{I}]-[\boldsymbol{D}])\{\boldsymbol{u}^m\} \\ [\boldsymbol{D}][\boldsymbol{K}]\{\boldsymbol{v}\} & = & [\boldsymbol{D}]\{\boldsymbol{f}\} \end{array} \tag{2}$$

where the first condition stands for the Dirichlet boundary conditions, and the second equation expresses both the constitutive law and equilibrium. It may be noted that the second equation can also be rewritten as

$$[\boldsymbol{D}][\boldsymbol{K}][\boldsymbol{D}]\{\boldsymbol{v}\} = [\boldsymbol{D}]\{\boldsymbol{f}\} - [\boldsymbol{D}][\boldsymbol{K}]([\boldsymbol{I}] - [\boldsymbol{D}])\{\boldsymbol{u}^m\} \tag{3}$$

where the stiffness matrix, restricted to inner nodes and Neumann degrees of freedom, $[\boldsymbol{K}^{\boldsymbol{I}}] \equiv [\boldsymbol{D}][\boldsymbol{K}][\boldsymbol{D}]$, is symmetric, positive and definite (because the Dirichlet boundary conditions are accounted for). The notation $[\boldsymbol{K}^{\boldsymbol{I}}]$ is introduced to simplify the following derivations. The right hand side defines effective nodal forces, $\{\boldsymbol{f}^{\text{eff}}\}$, for non-Dirichlet boundary conditions, or

$$[\boldsymbol{D}]\{\boldsymbol{f}^{\text{eff}}\} \equiv [\boldsymbol{D}]\left(\{\boldsymbol{f}\} - [\boldsymbol{K}]([\boldsymbol{I}] - [\boldsymbol{D}])\{\boldsymbol{u}^m\}\right) \tag{4}$$

With only displacement data, a rescaling of all moduli by a fixed amount does not affect the data, and hence the stress unit cannot be determined. The problem should be rephrased with dimensionless constitutive parameters where moduli are scaled by, say, the Young's modulus. Prescribing known tractions or body forces is a simple way to circumvent the previous argument, leaving open the possibility of identifying all constitutive parameters.

2.1.3. Spectral sensitivity

Because the measured displacement field is subjected to noise, it will be important to characterize the sensitivity of the following operators or norms to high frequency modulations. For this purpose, trial displacement fields $\boldsymbol{w}$ in the form of a plane wave $\boldsymbol{w} = \boldsymbol{w}_0 \exp(i\boldsymbol{k} \cdot \boldsymbol{x})$ are introduced, and the scaling of norms (or semi-norms) $\{\boldsymbol{w}\}^\top[\boldsymbol{A}]\{\boldsymbol{w}\}$ with the wave-number amplitude $k = |\boldsymbol{k}|$, is defined as

$$\langle\{\boldsymbol{w}\}^\top[\boldsymbol{A}]\{\boldsymbol{w}\}\rangle \sim k^\varsigma \tag{5}$$

where the brackets here stand for spatial averages. In practice, for the finite element method – which is beyond the scope of the usage of the present argument – the corresponding wavelength $2\pi/|\boldsymbol{k}|$ should be much larger than the element size, and much smaller than the domain size, so that the sine waves can be well resolved and well sampled. A large value of the spectral sensitivity ς will mean a high sensitivity to white noise, and without further information on the noise spectral properties, are not recommended. Matrices such as $[\boldsymbol{M}]$ (*e.g.,* mass matrix [41] or DIC matrix [47]) or $[\boldsymbol{I}]$ have a spectral sensitivity of $\varsigma = 0$. The stiffness matrix is the discretized form of the balance operator $\boldsymbol{\nabla} \cdot (\mathcal{C} : \boldsymbol{\nabla} \otimes \bullet)$ applied to the displacement field and the presence of two differential operators implies that the spectral sensitivity of $[\boldsymbol{K}]$ is $\varsigma = 2$.

In the following, some classical identification techniques will be rephrased, emphasizing their similarities and differences, in particular in terms of their spectral sensitivities.

3. Identification techniques

3.1. Finite Element Model Updating (FEMU)

The oldest and most natural method consists in matching $\{\boldsymbol{v}\}$ and $\{\boldsymbol{u}^m\}$ [8, 12, 11]. Introducing an L2 (Euclidian) norm, the FEMU method was formulated as finding the constitutive parameters (embedded in the stiffness matrix $[\boldsymbol{K}]$) that minimizes the following functional

$$\begin{aligned} \mathfrak{T}^{(\text{FEMU})} &= \|\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}\|^2 \\ &= (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top(\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \end{aligned} \tag{6}$$

Because $\{\boldsymbol{v}\}$ and $\{\boldsymbol{u}^m\}$ already coincide on the Dirichlet boundary conditions, it is equivalent to introducing the diagonal $[\boldsymbol{D}]$ matrix inside the scalar product

$$\mathfrak{T}^{(\text{FEMU})} = (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top[\boldsymbol{D}](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \tag{7}$$

It is noteworthy that if a nonuniform mesh is used, the above expression is not uniformly sampled in space. If such a uniform weighting is preferred, the above norm should be generalized by a summation over space with a uniform Lebesgue measure. Taking into account the fact that at the element scale, shape functions are known, the continuous summation can be expressed in terms of the discretized nodal values. This will automatically produce another L2 norm, which is expressed with the "mass" matrix, $[\boldsymbol{M}]$

$$\mathfrak{T}^{(\text{FEMU})} = (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top[\boldsymbol{M}](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \tag{8}$$

The latter weighting has the merit of being apparently transparent to the choice of mesh provided the region of interest remains the same, and that the mesh does not impede an accurate description of the displacement field.

Let us note that in all of these expressions an arbitrary scale factor can be introduced. It is usual to introduce a prefactor of (1/2), and/or a normalization by the total number of degrees of freedom in $[\boldsymbol{D}]$ or the surface / volume. At this stage, this scaling has no consequence. It will when other functionals will be considered together with $\mathfrak{T}$.

The spectral sensitivity of $\mathfrak{T}^{(\text{FEMU})}$ is $\varsigma^{(\text{FEMU})} = 0$.

3.2. Equilibrium Gap (EG)

The minimization of the equilibrium gap seeks to fulfill Equation (3) for a displacement field equal to $\boldsymbol{u}^m$. More precisely, if the measured displacement field is prescribed instead of $\{\boldsymbol{v}\}$ in Equation (3), the second member will display spurious unbalanced forces $\{\boldsymbol{\varphi}\}$ such that

$$[\boldsymbol{D}][\boldsymbol{K}][\boldsymbol{D}]\{\boldsymbol{u}^m\} - [\boldsymbol{D}]\{\boldsymbol{f}^{\text{eff}}\} = [\boldsymbol{D}]\{\boldsymbol{\varphi}\} \tag{9}$$

The euclidian norm of $[\boldsymbol{D}]\{\boldsymbol{\varphi}\}$ is the so-called "equilibrium gap" [28]. Thus, it is written as

$$\begin{aligned} \mathfrak{T}^{(\text{EG})} &= \|[\boldsymbol{D}][\boldsymbol{K}][\boldsymbol{D}]\{\boldsymbol{u}^m\} - [\boldsymbol{D}]\{\boldsymbol{f}^{\text{eff}}\}\|^2 \\ &= \|[\boldsymbol{K}^I]\{\boldsymbol{u}^m\} - [\boldsymbol{D}]\{\boldsymbol{f}^{\text{eff}}\}\|^2 \end{aligned} \tag{10}$$

Observing Equation (4), the above cost function is rewritten as

$$\begin{aligned} \mathfrak{T}^{(\text{EG})} &= \|[\boldsymbol{K}^I](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})\|^2 \\ &= (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top [\boldsymbol{K}^I][\boldsymbol{M}][\boldsymbol{K}^I](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \end{aligned} \tag{11}$$

Here again, the mass matrix reappears because the euclidian norm is considered based on a uniform measure over the domain of interest. If an equal weight is given to all degrees of freedom, the mass matrix $[\boldsymbol{M}]$ should be turned into the identity $[\boldsymbol{I}]$ matrix, and hence the metric is simply $[\boldsymbol{K}^I]^2$.

One nice feature of this formulation is that the above cost function is for some specific cases (such as scalar or modal anisotropic damage laws [29, 31]) quadratic in $\{\boldsymbol{p}\}$, thereby leading to *linear* systems for identification purposes. This is however not a universal statement, and apart from aesthetic reasons, this argument should not matter in the selection of a methodology.

For both variants, the spectral sensitivity is the double of that of $[\boldsymbol{K}^I]$, thus $\varsigma^{(\text{EG})} = 4$. This spectral sensitivity is a key property when using regularized DIC since it can be shown that the equilibrium gap functional, taken as a penalty to gray level conservation, then acts as a mechanics-based low pass filter [48, 49] of fourth order.

3.3. Constitutive Law Error (CLE)

The constitutive law error is a concept that was first developed as a tool to control and assess the quality of numerical simulations [15]. Using the same formulation to address identification from full-field measurements is an extension from its primary purpose [16, 17, 18, 19, 20]. The spirit of the method is as previously to minimize unbalanced forces $[\boldsymbol{D}]\{\boldsymbol{\varphi}\}$ with the measured displacement field. However, rather than computing the Euclidian norm of $\boldsymbol{\varphi}$, an energy-like norm is introduced, namely, the work of these unbalanced forces in a displacement field, $(\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})$ that is admissible to 0 (on Dirichlet degrees of freedom)

$$\begin{aligned} \mathfrak{T}^{(\mathrm{CLE})} &= (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})[\boldsymbol{D}]\{\boldsymbol{\varphi}\} \\ &= (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top [\boldsymbol{D}][\boldsymbol{K}][\boldsymbol{D}](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \\ &= (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top [\boldsymbol{K}^I](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \end{aligned} \tag{12}$$

This writing shows that the above form is quadratic and vanishes only if the two fields coincide (*i.e.,* $[\boldsymbol{K}^I]$ is definite). The spectral sensitivity of the CLE strategy is equal to that of $[\boldsymbol{K}^I]$, thus $\varsigma^{(\mathrm{CLE})} = 2$.

Let us note that the "Modified Constitutive Law Error," (M-CLE), was introduced as the sum of CLE and FEMU-type cost functions, $\mathfrak{T}^{(\mathrm{CLE})} + \alpha\mathfrak{T}^{(\mathrm{FEMU})}$ [19, 20, 50, 51]. The weighting parameter α gives an additional handle to optimize the identification procedure. Huang *et al.* [51] suggest using standard heuristics to determine the choice of α (such as Morozov [52] criterion or L-curve [53]). One may note that this α parameter, which relates two functionals with a different spectral sensitivity, implicitly defines a length scale, ξ, such that FEMU dominates at larger scales than ξ, and conversely, the CLE cost function rules the smaller length scales. Because the M-CLE approach is a linear combination of two "pure" cases, it will not be further distinguished.

3.4. Reconditioned Equilibrium Gap (REG)

The equilibrium gap shows a very high sensitivity to high frequency noise (since $\varsigma^{(\mathrm{EG})} = 4$), and hence is expected not to be very robust with respect to noise. In order to counteract this trend, it is desirable to "integrate twice" the unbalanced nodal forces to displacements, and this is the motivation for introducing the Reconditioned Equilibrium Gap, or REG [29]. Such an integration is provided by the solution to an elastic problem. For this purpose, a reference elastic medium is chosen, as characterized by a rigidity matrix $[\boldsymbol{K}_0]$ over the same geometry (and same mesh) as the studied sample, and as previously, the restriction of this operator to non-Dirichlet degrees of freedom is defined, $[\boldsymbol{K}_0^I] = [\boldsymbol{D}][\boldsymbol{K}_0][\boldsymbol{D}]$. The displacement field, $\{\boldsymbol{w}\}$, is computed in this reference medium with zero displacement over the Dirichlet boundary, and body forces equal to the above unbalanced forces, $\{\boldsymbol{\varphi}\}$

$$\{\boldsymbol{w}\} = [\boldsymbol{K}_0^I]^{-1}\{\boldsymbol{\varphi}\} \tag{13}$$

The quadratic norm of $\{\boldsymbol{w}\}$ is the reconditioned equilibrium gap

$$\begin{aligned} \mathfrak{T}^{(\mathrm{REG})} &= \{\boldsymbol{w}\}^\top [\boldsymbol{M}]\{\boldsymbol{w}\} \\ &= \{\boldsymbol{\varphi}\}^\top [\boldsymbol{K}_0^I]^{-1}[\boldsymbol{M}][\boldsymbol{K}_0^I]^{-1}\{\boldsymbol{\varphi}\} \\ &= (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top [\boldsymbol{K}^I][\boldsymbol{K}_0^I]^{-1}[\boldsymbol{M}][\boldsymbol{K}_0^I]^{-1}[\boldsymbol{K}^I](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \end{aligned} \tag{14}$$

It is to be noted that $[\boldsymbol{K}_0^I]^{-1}$ is never to be computed explicitly, but only the trial displacements $\{\boldsymbol{w}\}$ whose cost is that of solving a linear elastic problem. A reference medium is introduced, and among all possible choices, one natural candidate is to choose precisely that of the studied sample. In this case, $[\boldsymbol{K}_0^I] = [\boldsymbol{K}^I]$, $\mathfrak{T}^{(\mathrm{REG})}$ *coincides* with the FEMU criterion. However, if no link is established between $[\boldsymbol{K}]$ and $[\boldsymbol{K}_0]$ the above mentioned linearity property of the EG criterion when solving for damage fields is preserved, in contrast to FEMU that would be nonlinear. However, here again, computationally, the advantage of such linearity is marginal.

As expected from its very construction, the reconditioned equilibrium gap method displays a zero spectral sensitivity, $\varsigma^{(\mathrm{REG})} = 0$, and hence is much less sensitive to noise, and thus becomes comparable to FEMU.

3.5. Virtual Fields Method (VFM)

The virtual fields method, which was proposed long ago [24], and which has been matured to a large number of successful cases [26], can be cast into a similar formalism. Because unbalanced forces $\{\boldsymbol{\varphi}\}$ should be ideally vanishing, for any kinematically admissible displacement field, referred to as *virtual field*, $\{\boldsymbol{\psi}\}$, the virtual work $\{\boldsymbol{\psi}\}^\top\{\boldsymbol{\varphi}\}$ should vanish. This condition defines a scalar equation with which at most one constitutive parameter can be determined. When n such parameters are needed (depending on the complexity of the constitutive law, and hence, in the present case of linear elasticity, on its class of symmetry), one may introduce as many virtual fields $\{\boldsymbol{\psi}_i\}$, $i = 1, ..., n$, to obtain as many equations as unknowns. The equations to solve read

$$\{\boldsymbol{\psi}_i\}^\top [\boldsymbol{K}^I](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) = 0 \tag{15}$$

This set of equation is identical to the one that would result from the minimization of

$$\mathfrak{T}^{(\mathrm{VF})} = (\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\})^\top [\boldsymbol{K}^I][\boldsymbol{\Psi}][\boldsymbol{K}^I](\{\boldsymbol{v}\} - \{\boldsymbol{u}^m\}) \tag{16}$$

where $[\boldsymbol{\Psi}]$ is the projector along the subspace generated by the virtual fields

$$[\boldsymbol{\Psi}] = \sum_i \{\boldsymbol{\psi}_i\}\{\boldsymbol{\psi}_i\}^\top \tag{17}$$

Since the rank of this matrix is n, the above cost function $\mathfrak{T}^{(\mathrm{VF})}$ is a semi-norm, with a large kernel (subspace orthogonal to $[\mathbf{\Psi}]$). It is thus essential to choose the virtual fields wisely. In the review paper [54], the selection of the virtual fields is discussed, and different solutions are proposed (although the problem of selecting the best set of virtual fields is declared unsettled by the authors). The different options that were proposed are:

- a heuristic approach where the different fields $\boldsymbol{\psi}_i$ are chosen to be orthogonal to one another. In fact it could be argued that orthogonality is not relevant computationally inasmuch as only the subspace generated by the $\boldsymbol{\psi}_i$ matters.

- a second mentioned criterion is (equivalent) to achieve the best conditioning of the linear system.

- let us mention that it was recently proposed to use the sensitivity fields as computed virtual fields for using the virtual field method in the context of nonlinear constitutive laws [55]. It is interesting to note how close this choice actually brings the VFM to the original FEMU technique that extensively uses sensitivity fields [4, 56].

- another possible choice is to minimize the effect of noise, leading to so-called "special virtual fields." In ref. [32], an operational way of optimizing the virtual fields was based on the assumption that the measured *strains* are affected by a Gaussian white noise. Since a displacement field is generally measured, the strains cannot be affected by a *white* noise as it fulfills a compatibility condition, and the strain noise displays a strong anticorrelation character resulting from the partial differentiation that relates strain to displacement. However, noting that the measurement noise is the limiting feature of the identification is a very relevant observation, this calls for taking into account the actual covariance of the displacement field noise, $\mathbf{Cov}^{\boldsymbol{u}}$ (from which that of strains can be derived [57]).

It is noteworthy that in the case of linear elasticity or viscoelasticity, such an approach (for any choice of the virtual fields) may lead to linear systems provided the constitutive parameters are chosen so that the rigidity matrix is affine in $\{\boldsymbol{p}\}$ [26], akin to what was observed for the EG and REG methods [27, 29].

Because typically n is a small number, the spectral sensitivity is not meaningful. Virtual fields are usually structural (*i.e.,* not related to a particular mesh) exactly for the purpose of not being too sensitive to noise so that the k-dependence is not meaningful. Let us note that if a larger number of virtual fields is considered the minimization of $\mathfrak{T}_{VF}$ could still be considered although it would generally not be possible to reach 0 for all Equations (15). In this case, the advantage of formulating the problem as a minimization is to naturally deal with more equations. Considering all possible fields as generated by the finite element mesh (*i.e.*, Ψ would then reduce to the identity or mass operator and it would generate the entire space of discretized displacements) leads to the EG method (*i.e.*, $\varsigma^{(VFM)} = 4$). In contrast, keeping the number of fields small corresponds to ignoring all information coming from the complementary subspace (*i.e.*, $\varsigma^{(VFM)} \approx 0$).

3.6. Summary

All methods to be discussed below tend to evaluate the elastic properties, *e.g.*, Hooke's tensor $\mathcal{C}(\{\boldsymbol{p}\})$, which is parameterized by $\{\boldsymbol{p}\}$ so that $\{\boldsymbol{v}\}$ and $\{\boldsymbol{u}^m\}$ become as close as possible. A general framework that has been shown as suitable for all the above cited methods is to estimate $\mathcal{C}$, or rather $\{\boldsymbol{p}\}$ as the argument minimizing a functional $\mathfrak{T}(\{\boldsymbol{p}\})$

$$\mathcal{C}^{\text{identif}} = \text{Argmin}_{\{\boldsymbol{p}\}} \; \mathfrak{T}(\{\boldsymbol{p}\}) \tag{18}$$

where

$$\mathfrak{T}(\{\boldsymbol{p}\}) = (\{\boldsymbol{v}(\{\boldsymbol{p}\})\} - \{\boldsymbol{u}^m\})^\top [\boldsymbol{N}](\{\boldsymbol{v}(\{\boldsymbol{p}\})\} - \{\boldsymbol{u}^m\}) \tag{19}$$

is a norm (or a semi-norm) defining a metric to evaluate the distance between computed and measured displacement fields. However, all these methods differ in the expression of the symmetric matrix $[\boldsymbol{N}]$ as summarized in Table 1.

Table 1: Different proposed metrics.

Method	Metric $[\boldsymbol{N}]$	Spectral sensitivity
FEMU (dof)	$[\boldsymbol{I}]$	0
FEMU (uniform)	$[\boldsymbol{M}]$	0
EG (dof)	$[\boldsymbol{K}^I]^2$	4
EG (uniform)	$[\boldsymbol{K}^I][\boldsymbol{M}][\boldsymbol{K}^I]$	4
CEE	$[\boldsymbol{K}^I]$	2
REG	$[\boldsymbol{K}^I][\boldsymbol{K}_0^I]^{-1}[\boldsymbol{M}][\boldsymbol{K}_0^I]^{-1}[\boldsymbol{K}^I]$	0
VFM	$[\boldsymbol{K}^I][\boldsymbol{\Psi}][\boldsymbol{K}^I]$	(not relevant)

4. Optimal methods

In the previous section, a set of different criteria have been shown to differ only by their "metric." Thus the next question to answer is to point out the "best" one. The first difficulty to address is to define properly what is understood as "best." The present choice for the definition of best is to not only evaluate the constitutive parameters, but also their uncertainty. This will provide an objective criterion to decide on the best method, namely, the one that will lead to the minimum uncertainty.

4.1. Identification algorithm

With the above common framework, one may proceed to the solution through generic algorithms. Because of the generally nonlinearity of the minimization, a very common one is a Newton-Raphson minimization of the norm [4]. The determination of the elastic parameters is iteratively corrected. To this aim, the variation of the displacement field with respect to each of the constitutive parameters, p_i, namely, the so-called sensitivity fields [4, 56], $\{\boldsymbol{s}_i\}$,

$$\{\boldsymbol{s}_i\} = \frac{\partial\{\boldsymbol{v}\}}{\partial p_i} \tag{20}$$

are needed. Hence, varying the constitutive parameters by a small amount leads to a change in the computed displacement field

$$\{\boldsymbol{\delta v}\} = \sum_i \{\boldsymbol{s}_i\}\delta p_i \tag{21}$$

and hence, at step t of the algorithm, the minimization of $\mathfrak{T}(\{\boldsymbol{p}^{(t)}\})$ with respect to the variations δp_i leads to

$$\{\boldsymbol{s}_i^{(t)}\}^\top[\boldsymbol{N}]\left(\{\boldsymbol{v}^{(t)}\} - \{\boldsymbol{u}^m\} + \sum_j \{\boldsymbol{s}_j^{(t)}\}\delta p_j\right) = 0 \tag{22}$$

where the superscript (t) has been added to $\boldsymbol{v}^{(t)}$ and $\boldsymbol{s}_i^{(t)}$ to recall that they are computed with the current determination of the constitutive parameters, here the Hooke's tensor $\mathcal{C}(\{\boldsymbol{p}^{(t)}\})$.

Introducing the Hessian $[\boldsymbol{H}^{(t)}]$,

$$H_{ij}^{(t)} = \{\boldsymbol{s}_i^{(t)}\}^\top[\boldsymbol{N}]\{\boldsymbol{s}_j^{(t)}\} \tag{23}$$

and second member, $\{\boldsymbol{J}^{(t)}\}$,

$$J_i^{(t)} = \{\boldsymbol{s}_i^{(t)}\}^\top[\boldsymbol{N}]\left(\{\boldsymbol{u}^m\} - \{\boldsymbol{v}^{(t)}\}\right) \tag{24}$$

the incremental correction of the constitutive parameters $\{\boldsymbol{p}^{(t)}\}$, is obtained from

$$\{\boldsymbol{p}^{(t+1)}\} = \{\boldsymbol{p}^{(t)}\} + [\boldsymbol{H}^{(t)}]^{-1}\{\boldsymbol{J}^{(t)}\} \tag{25}$$

Such a scheme is quite general, and not unique. In particular, because updating the sensitivity fields should ideally be performed at each iteration is a costly operation, different variants can be considered keeping a non updated Hessian for some iterations, or updating the sensitivities only along the direction of the last increment $\{\boldsymbol{p}^{(t+1)}\} - \{\boldsymbol{p}^{(t)}\}$. Numerical algorithms optimized for an efficient solution is an interesting topic far from being exhausted here. However, a plain Newton-Raphson algorithm is applicable and provides a solution, irrespective of chosen method.

4.2. Uncertainty quantification

In the following, the structure of the tangent problem about the ideal solution is considered for perfect input data, $\boldsymbol{u}_0^m$. The problem is assumed to be well-posed, and hence its solution provides the constitutive parameter $\{\boldsymbol{p}_0\}$, as well as the set of sensitivities $\boldsymbol{s}_i^0$. However, in real life, the measurement data are corrupted by noise. Hence $\{\boldsymbol{u}^m\} = \{\boldsymbol{u}_0^m\} + \{\boldsymbol{\eta}^u\}$, where $\{\boldsymbol{\eta}^u\}$ is the noise in the input measurement. It is assumed that the DIC analysis has been performed correctly, thus producing an *unbiased result*, or $\langle\{\boldsymbol{\eta}^u\}\rangle = \boldsymbol{0}$ [57]. Moreover, the noise is assumed to be Gaussian. One argument for this assumption is the stability of Gaussian noise through linear combinations, so that even if initially non-Gaussian, linear combinations of local noise will converge asymptotically to a Gaussian distribution (*i.e.,* convergence in law) when the number of independent perturbations tends to infinity (*i.e.,* central limit theorem). One such example would be white noise affecting the gray level value of each pixel in images to be processed by DIC. When the number of pixels used for all kinematic degrees of freedom is large, the noise affecting the discrete displacement variables is Gaussian.

The displacement noise is then completely characterized by its covariance matrix

$$\mathrm{Cov}_{ij}^{\boldsymbol{u}} = \langle \eta_i^u \eta_j^u \rangle \tag{26}$$

Assuming that the tangent identification problem about the ideal solution remains valid for the amplitude of input noise, the displacement uncertainties can be propagated down to the noise affecting the identified parameters, $\{\boldsymbol{\eta}^p\}$

$$\eta_i^p = H_{ij}^{-1}\{\boldsymbol{s}_j^0\}^\top[\boldsymbol{N}]\{\boldsymbol{\eta}^u\} \tag{27}$$

As a result, for all identification methods (*i.e.,* all $[\boldsymbol{N}]$ metrics), the identified parameters are *unbiased*, and as linear combination of Gaussian noises, $\boldsymbol{\eta}^p$ is also Gaussian. Thus it is completely characterized by its covariance

$$\begin{aligned} \mathrm{Cov}_{ij}^{\boldsymbol{p}} &= \langle \eta_i^p \eta_j^p \rangle \\ &= H_{ik}^{-1}\{\boldsymbol{s}_k^0\}^\top[\boldsymbol{N}][\mathbf{Cov}^{\boldsymbol{u}}][\boldsymbol{N}]\{\boldsymbol{s}_l^0\}H_{lj}^{-1} \end{aligned} \tag{28}$$

Equation (27) has a simple geometric interpretation. The operator $[\boldsymbol{P}]$ defined as

$$[\boldsymbol{P}] \equiv \{\boldsymbol{s}_i^0\}H_{ij}^{-1}\{\boldsymbol{s}_j^0\}^\top[\boldsymbol{N}] \tag{29}$$

is the projector from the space of displacements $\mathcal{U}$ onto the subspace $\mathcal{S}$ generated by the sensitivity vectors, and this projection is orthogonal with respect to the scalar product defined by $[\boldsymbol{N}]$. It can easily be checked that

$$\begin{aligned} [\boldsymbol{P}]\{\boldsymbol{s}_k^0\} &= \{\boldsymbol{s}_i^0\}H_{ij}^{-1}\{\boldsymbol{s}_j^0\}^\top[\boldsymbol{N}]\{\boldsymbol{s}_k^0\} \\ &= \{\boldsymbol{s}_i^0\}H_{ij}^{-1}H_{jk} \\ &= \{\boldsymbol{s}_i^0\}\delta_{ik} \\ &= \{\boldsymbol{s}_k^0\} \end{aligned} \tag{30}$$

and for any vector $\{\boldsymbol{v}\}$ orthogonal to all sensitivities $\{\boldsymbol{s}_k^0\}^\top[\boldsymbol{N}]\{\boldsymbol{v}\} = \{\mathbf{0}\}$, then $[\boldsymbol{P}]\{\boldsymbol{v}\} = 0$. Thus Equation (27) expresses that the noise on the constitutive parameters translated into displacements from their sensitivity fields is simply the normal projection of the displacement noise onto the subspace $\mathcal{S}$, or

$$\sum_i \eta_i^p\{\boldsymbol{s}_i^0\} = [\boldsymbol{P}]\{\boldsymbol{\eta}^u\} \tag{31}$$

Henceforth, all norms $[\boldsymbol{N}]$ produce the *same* unbiased result when averaging out noise. However, they differ in their "noisiness," *i.e.,* in the uncertainties of the noise that will affect the identified parameters, and hence it is attractive to look for the "optimal norm," which is here *defined* as the one that minimizes the noise amplitude on $\{\boldsymbol{p}\}$. Let us underline that because $\{\boldsymbol{p}\}$ is multidimensional, the noise amplitude has to be characterized by the full covariance matrix $[\mathbf{Cov}^{\boldsymbol{p}}]$. Thus, it is not obvious that such an optimal norm even exists, since only a partial order is available for such symmetric matrices. However, there is a simple shortcut to the result that allows one to understand that such an optimum exists and provides its expression.

If displacement noise were uncorrelated and uniform, $[\mathbf{Cov}^{\boldsymbol{u}}] = \sigma^2[\boldsymbol{I}]$, then the optimal norm is the Euclidian one, $[\boldsymbol{N}^{\text{opt}}] = [\boldsymbol{I}]$. Noise could be seen as blurring the displacement field to a hypersphere of radius σ in $\mathcal{U}$ centered onto the actual displacement. Any norm would project this hypersphere onto the hyperplane $\mathcal{S}$ as a hyper-ellipse. The smallest domain that can be reached is the ordinary (Euclidian) orthogonal projection.

The difficulty is to extend this simple result to a more complex case where the covariance matrix in $\mathcal{U}$ is arbitrary. Let us observe that if a linear transformation of the displacement space $\{\boldsymbol{v}\} = [\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2}\{\boldsymbol{u}\}$ is introduced, then noise affecting $\{\boldsymbol{v}\}$ reads

$$\{\boldsymbol{\eta}^v\} = [\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2}\{\boldsymbol{\eta}^u\} \tag{32}$$

then

$$\begin{aligned} \langle\{\boldsymbol{\eta}^v\}\{\boldsymbol{\eta}^v\}^\top\rangle &= [\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2}\langle\{\boldsymbol{\eta}^u\}\{\boldsymbol{\eta}^u\}^\top\rangle[\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2} \\ &= [\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2}[\mathbf{Cov}^{\boldsymbol{u}}][\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2} \\ &= [\boldsymbol{I}] \end{aligned} \tag{33}$$

Therefore, in the new coordinates $\{\boldsymbol{v}\}$, the displacement is only affected by a white and uniform noise for which the optimal solution is known. Thus it suffices to see how such a linear transformation affects the scalar product

$$\begin{aligned} \{\boldsymbol{u}_1\}^\top[\boldsymbol{N}]\{\boldsymbol{u}_2\} &= \{\boldsymbol{v}_1\}^\top[\mathbf{Cov}^{\boldsymbol{u}}]^{1/2}[\boldsymbol{N}][\mathbf{Cov}^{\boldsymbol{u}}]^{1/2}\{\boldsymbol{v}_1\} \\ &\equiv \{\boldsymbol{v}_1\}^\top[\boldsymbol{N}']\{\boldsymbol{v}_2\} \end{aligned} \tag{34}$$

or $[\boldsymbol{N}'] = [\mathbf{Cov}^{\boldsymbol{u}}]^{1/2}[\boldsymbol{N}][\mathbf{Cov}^{\boldsymbol{u}}]^{1/2}$. Therefore the optimal norm $[\boldsymbol{N}'_{\text{opt}}] = [\boldsymbol{I}]$ is transported from $\{\boldsymbol{v}\}$ to $\{\boldsymbol{u}\}$

$$\begin{aligned} [\boldsymbol{N}_{\text{opt}}] &= [\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2}[\boldsymbol{N}'_{\text{opt}}][\mathbf{Cov}^{\boldsymbol{u}}]^{-1/2} \\ &= [\mathbf{Cov}^{\boldsymbol{u}}]^{-1} \end{aligned} \tag{35}$$

This last result is both simple and powerful. There are many ways to evaluate norms or to measure distances with noisy data, but among those one and only one (up to a multiplicative factor) is optimal in the sense that it minimizes the effect of noise. This "best" norm has to account precisely for the character of noise, which is summed up in the covariance matrix for a Gaussian noise, and the optimal metric has to unravel the correlations that are present there. This optimal norm has long been known in statistics and is due to P.C. Mahalanobis [58], who has given his name to it, as the Mahalanobis distance. For two vectors $\{\boldsymbol{u}_1\}$ and $\{\boldsymbol{u}_2\}$, their Mahalanobis distance $d(\{\boldsymbol{u}_1\},\{\boldsymbol{u}_2\})$ is expressed as

$$d(\{\boldsymbol{u}_1\},\{\boldsymbol{u}_2\}) = (\{\boldsymbol{u}_1\} - \{\boldsymbol{u}_2\})^\top[\mathbf{Cov}^{\boldsymbol{u}}]^{-1}(\{\boldsymbol{u}_1\} - \{\boldsymbol{u}_2\}) \tag{36}$$

It is also noteworthy that this norm can be encountered from a different reasoning. Assuming that the difference between two vectors, $\{\boldsymbol{u}_1\}$ and $\{\boldsymbol{u}_2\}$, is only due to noise, and that the latter is Gaussian with a known covariance, the exact probability that such a noise vector $\{\boldsymbol{\eta}\} = (\{\boldsymbol{u}_1\} - \{\boldsymbol{u}_2\})$ be observed reads

$$\mathcal{P}(\{\boldsymbol{\eta}\}) = \frac{1}{(2\pi)^{n/2}\det([\mathbf{Cov}^{\boldsymbol{u}}])^{1/2}} \exp\left(-\frac{1}{2}\{\boldsymbol{\eta}\}^\top[\mathbf{Cov}^{\boldsymbol{u}}]^{-1}\{\boldsymbol{\eta}\}\right) \tag{37}$$

The maximum of this probability or likelihood is reached when the Mahalanobis norm is precisely minimum. Actually the cologarithm of the likelihood coincides with this norm. Let us turn back to the initial problem of calibrating elastic properties. It is striking that none of the metrics listed in Table 1 actually depend on the nature of the noise that affects the measured data. Thus, none may claim to be optimal with respect to their (least) sensitivity to noise. The optimal identification method is simply the one that minimizes the Mahalanobis distance between the measured and the computed displacement fields, namely, $[\boldsymbol{N}_{\text{opt}}] = [\mathbf{Cov}^{\boldsymbol{u}}]^{-1}$. The name "weighted FEMU" was proposed [35], as a result of the fact that if the noise affecting the displacement data were white, then $[\mathbf{Cov}^{\boldsymbol{u}}]$ being the identity matrix, the optimal identification norm would be that of the FEMU method.

4.3. DIC measurements

Given the fact that the computed displacement fields are assumed to be the result of FEAs, it was assumed that the displacement measurements were expressed on the same mesh. Under such conditions, global DIC [59, 60] is a natural choice to measure displacement fields. However, different discretizations may be considered between measured and computed displacement fields [61], or even local DIC [2] for displacement measurements. In all these cases, the covariance matrices of the measured displacements can be assessed [57]. Other measurement techniques may also be considered [45, 7, 46] and will then require the covariance matrices to be estimated.

The spirit of DIC consists in minimizing the norm of the registration residual between a reference image $f(\boldsymbol{x})$ and a deformed image, $g(\boldsymbol{x})$, corrected by the measured displacement field $\boldsymbol{u}(\boldsymbol{x})$, $\hat{g}_{\boldsymbol{u}}(\boldsymbol{x}) = g(\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}))$. The registration residual is $\rho_{\boldsymbol{u}}(\boldsymbol{x}) = \hat{g}_{\boldsymbol{u}}(\boldsymbol{x}) - f(\boldsymbol{x})$. Because images are prone to noise, (both f and g), the displacement fields that are measured are corrupted by a fluctuation $\boldsymbol{\delta u}$. Usually, images are assumed to be affected by Gaussian white noise of small amplitude. This assumption of a white and uniform Gaussian image noise implies that images are to be compared with an optimal Mahalanobis distance that is simply the L2 Euclidian norm. Such is the case for some DIC strategies (local or global). The interesting point here is that if noise deviates from a such simple case, the above derived result allows to optimally measure the registration between images.

Local and global DIC approaches can be distinguished [57, 2]. The first ones partition the domain into elementary windows (or “subsets”), the center of which is characterized by a mean displacement that is evaluated independently from window to window. If no overlap is considered between subsets, then the noise affecting the displacement is white (*i.e.,* uncorrelated), and thus the optimal identification method is standard FEMU. However, often subsets overlap, and interpolation or smoothing operations are performed thereby creating correlations. The mere fact of expressing the measurement within the language of the computed ones involving a projection induces non-trivial spatial correlations, and hence one should be extremely careful in the post-processing of such data where reference with the raw data (*i.e.,* images) is lost.

The global approach proceeds through a unique and simultaneous determination of all degrees of freedom simultaneously, allowing them to be coupled. The advantage of this methodology is that it can naturally be interfaced with numerical models as it can share the very same discretization (say a finite element mesh). Yet one consequence of these couplings is that global DIC has always a non trivial covariance matrix [57]. It will be shown however that the latter is naturally available from the DIC code, as it is needed to measure the displacements.

The interested reader is referred to Ref. [47] for a general presentation of global DIC. Let us mention one key result in relation with the above uncertainty analysis. The displacement field is written in a chosen discretization that can take many a form. A basis of vector fields $\boldsymbol{\phi}_i(\boldsymbol{x})$ being chosen, the displacement field is sought under the discrete form $\boldsymbol{u}(\boldsymbol{x}) = \sum_i u_i \boldsymbol{\phi}_i(\boldsymbol{x})$. Usually an iterative algorithm is chosen to perform the registration whereby the displacement field is progressively corrected until a satisfactory match was found. Close to convergence, displacement correction δu_i is found through a linearization of the effect of a further displacement. Introducing the Hessian

$$H_{ij}^{\mathrm{DIC}} = \sum_{\boldsymbol{x}} (\boldsymbol{\phi}_i(\boldsymbol{x}) \cdot \boldsymbol{\nabla} f(\boldsymbol{x}))(\boldsymbol{\phi}_j(\boldsymbol{x}) \cdot \boldsymbol{\nabla} f(\boldsymbol{x})) \tag{38}$$

and second member

$$r_i = \sum_{\boldsymbol{x}} (\boldsymbol{\phi}_i(\boldsymbol{x}) \cdot \boldsymbol{\nabla} f(\boldsymbol{x}))(f(\boldsymbol{x}) - \widetilde{g}(\boldsymbol{x})) \tag{39}$$

where $\widetilde{g}$ has been corrected by the current determination of the displacement field, the incremental correction obeys

$$H_{ij}^{\mathrm{DIC}} \delta u_j = r_i \tag{40}$$

The effect of noise in the f and g images produces a fluctuation in the displacements η^u that can be compared to the last iterations of the displacement corrections, in the sense that the tangent problem is similar, upon the substitution of image noise $(\eta^g - \eta^f)$ instead of mere residuals $(f(\boldsymbol{x}) - \widetilde{g}(\boldsymbol{x}))$ in Equation (39). Thus introducing

$$\rho_i = \sum_{\boldsymbol{x}} (\boldsymbol{\phi}_i(\boldsymbol{x}) \cdot \boldsymbol{\nabla} f(\boldsymbol{x}))(\eta^f(\boldsymbol{x}) - \eta^g(\boldsymbol{x})) \tag{41}$$

the noise on $\{\boldsymbol{u}_i\}$, $\{\boldsymbol{\eta}_i^u\}$, reads

$$\{\boldsymbol{\eta}^u\} = [\boldsymbol{H}^{\mathrm{DIC}}]^{-1}\{\boldsymbol{\rho}\} \tag{42}$$

from which the covariance matrix becomes [57]

$$\mathbf{Cov}^{\boldsymbol{u}} = 2\sigma^2[\boldsymbol{H}^{\mathrm{DIC}}]^{-1} \tag{43}$$

where σ is the standard deviation of the white noise affecting the images (assumed to be statistically similar but independent in the two images, which provides the factor 2 in the r.h.s. term). Thus, it is observed that the Mahalanobis distance for the identification that involves $[\mathbf{Cov}^{\boldsymbol{u}}]^{-1}$ reduces to simply using the Hessian of DIC, $[\boldsymbol{H}^{\mathrm{DIC}}]$, up to the noise variance, and hence it does not involve any additional cost to deliver the full covariance matrix of the displacement field noise.

Let us finally underline a final result of interest. "Integrated DIC", (or I-DIC [62, 36, 63]), was introduced in the context of identification as being a special form of global DIC where the kinematic basis $\boldsymbol{\phi}_i(\boldsymbol{x})$ is chosen to be directly the sensitivity fields $\boldsymbol{s}_i(\boldsymbol{x})$ previously defined in Section 4.1. It is easy to show that using global DIC and further the optimal identification procedure, or weighted FEMU, is mathematically equivalent to using I-DIC to perform the identification directly [36]. This remarkable property however relies on the assumption that no model error was introduced, such as for instance using too coarse a mesh as may be required to keep a low uncertainty on $\boldsymbol{u}^m$. In contrast, I-DIC may use a very fine spatial discretization devoid of spatial discretization error while still keeping the number of unknowns to a fixed (small) value [62, 64], namely, equal to the number of constitutive parameters (plus rigid body motions when needed).

5. Extensions

Within the above framework it is possible to include additional unknowns or measurements.

5.1. Noisy boundary conditions

The above discussion considered boundary conditions to be exactly known. It may be true in particular for Neumann boundary conditions when traction-free surfaces are present. Otherwise, these boundary conditions are generally corrupted by noise, and moreover noise is known to be amplified at boundaries. After identification, in fact, boundary conditions may even constitute the major source of uncertainty. The above presented case thus only covers partly the conditions for optimality with respect to uncertainty (*i.e.,* only when boundary condition uncertainties can be neglected).

When noisy Dirichlet boundary conditions are considered, one should distinguish between actual and estimated displacement fields along those boundaries. Hence, actual displacements are to be considered as unknown. However, estimates are available together with the statistical properties of their uncertainties [63]. Thus, one may proceed exactly as for constitutive parameters. They are needed for a direct problem, yet only an approximate version may be sufficient. In the course of the identification procedure, boundary conditions as well as constitutive parameters, $\{\boldsymbol{p}\}$, are to be identified, using the very same procedure, namely, with sensitivity fields that characterize the changes in displacement field induced by a Dirichlet boundary degree of freedom [65].

5.2. Unknown geometry

In some cases, the geometry of the observed medium is not entirely known. In particular, it is a usual problem for propagating cracks say in fatigue problems where the tip position may not be known precisely. In this case, parameters describing the geometry may be added to the list of constitutive parameters, $\{\boldsymbol{p}\}$, and treated similarly. Namely, sensitivity fields accounting for the change in displacement field due to a small variation of the geometrical parameters can be computed to complement the library of fields over which the displacement is to be projected [66].

5.3. Fusion of information

More than often, load information is measured and this measurement comes with some uncertainty, in contrast to the previous section where applied forces where assumed to be known exactly. In such a case, not only should one match the displacement field in the identification process but also the applied forces should be compared to computed ones. It has been shown above that the appropriate distance between displacement field was the Mahalanobis distance to be used in $\mathfrak{T}_u$, (see Equation (19)). Similarly, the distance between the temporal series of external forces and the computed ones should be the Mahalanobis distance. In most cases, white noise may be fully justified and hence this will reduce to the Euclidian norm of force differences $\mathfrak{T}_F = \|\boldsymbol{F}^{comput} - \boldsymbol{F}^{meas}\|^2$.

These two functionals are to be minimized together, and hence, they are to be alloyed into a single one. The appropriate weight to be given to each of these two terms may not be obvious at first glance. In fact, the colog-likelihood argument shows that these two functional are to be summed with the *same* weight if the covariance matrix of each measurement is used in both $\mathfrak{T}$ functionals [36]. The same result is straightforwardly obtained by gathering all measurements (be they kinematic or static) into a global vector, and writing the covariance matrix of this full vector. Hence, no choice is to be exerted here if one aims at the optimal solution. Let us stress however that, here as before when using any norm $[\boldsymbol{N}]$, it is *not wrong* to give other weights than unity, but it results in a larger uncertainty for the result as compared to the optimal one.

5.4. Nonlinear constitutive laws

The above discussion about identification was restricted to linear elastic problems. Let us however note that heterogeneous elastic media are naturally included in the previous analysis. Identification of the elastic parameters can be restricted to specific sub-domains, and potentially, the geometry of sub-domains can also be included in the list of unknowns as previously mentioned [28, 35].

For nonlinear constitutive laws, it is to be emphasized that the local tangent stress/strain relationship can be written as

$$\boldsymbol{\sigma} = \mathcal{C} : \boldsymbol{\varepsilon} + \boldsymbol{\tau} \tag{44}$$

where the difference with an elastic law is gathered in the so-called "stress polarization tensor" $\boldsymbol{\tau}$ (equivalently a stress-free strain could have been introduced). However, the polarization stress as well as the tangent tensor $\mathcal{C}$ depend on the current strain $\boldsymbol{\epsilon}$ tensor and possibly on one or several internal variables. When accounted for in the finite element formalism, the polarization will be included in the nodal force vector, $\{\boldsymbol{F}\}$. The previous remark is also relevant herein. The very existence of a constitutive law implies that the nodal force vector is constrained (in fact determined) by the constitutive law, and hence iterative methods should drive the system to a fixed point where constitutive law and balance equations are both satisfied. In the final steps of this iterative procedure, the tangent problem is expected to remain essentially stationary and hence the mathematical complexity is not much different than the previously discussed elastic problem.

The specific issue to get a rough description of the constitutive law, sufficiently robust to provide a reliable tangent law is not discussed herein. When this is achieved, the identification problem takes precisely the same form as previously. Although a nonlinear constitutive law is an additional difficulty, the availability of robust numerical methods and softwares renders sensitivity fields accessible from mere finite differences. Usually, several loading levels (say N) are considered, so that the displacement fields are actually a stack of N fields indexed by the load at which images were captured. Similarly, sensitivity fields will also be a stack of N elementary sensitivity fields, one per load level. However once computed these sensitivities are transformed into a very long vector, and hence the multiplicity of load levels does not change the mathematical nature of the problem as compared to a single level. The very nature of the problem remains actually quite close. The current determination of the constitutive parameters, at iteration n, $\{\boldsymbol{p}^n\}$, and the known boundary conditions are sufficient to compute the displacement field at all nodes and measurement times, $\{\boldsymbol{v}^n\}$. Similarly sensitivities may be computed from $\boldsymbol{s}_i^n = \partial \boldsymbol{v}^n / \partial p_i$, so that within a small neighborhood of $\{\boldsymbol{p}^n\}$, the displacement field may be written as $\{\boldsymbol{v}\} = \{\boldsymbol{v}^n\} + \sum_i \{\boldsymbol{s}_i^n\}(p_i - p_i^n)$. This computed displacement field is now to be compared with the measured one $\boldsymbol{u}^m$ and the squared Mahalanobis distance

$$\mathfrak{T}_u = \|\{\boldsymbol{v}^n\} + \sum_i \{\boldsymbol{s}_i^n\}(p_i - p_i^n) - \{\boldsymbol{u}^m\}\|^2 \tag{45}$$

minimized with respect to $(p_i - p_i^n)$. Each update from $\{\boldsymbol{p}^n\}$ to $\{\boldsymbol{p}^{n+1}\}$ theoretically implies that all displacements and sensitivities should be recomputed at each iteration.

One limitation to be mentioned, is the possible existence of a high sensitivity to some parameters (*e.g.,* close to instabilities) that may induce a deviation from the Gaussian statistics for the uncertainty. For those cases, a closer inspection may lead to the disqualification of some nodes at some time steps, but this may not hamper identification.

As a final comment, let us note that this optimal procedure was already followed quite a few times in the past although without necessarily claiming to be optimal [32, 33, 35].

6. Conclusions

A unified framework was proposed to recast many of the identification procedures used nowadays. The main difference is the metric used to measure the distance between measured and computed displacement fields. Let us stress that, for some methods, no reference is usually made to a computed field. In the proposed unified framework, the computed displacement field may belong to the kernel of the operator $[\boldsymbol{N}]$ restricted to non-Dirichlet degrees of freedom making the underlying method strictly equivalent mathematically in spite of an unusual presentation. Moreover, the spectral sensitivity of all these identification methods was assessed. It allows the sensitivity of all these methods to measurement uncertainties to be compared.

The optimality of the identification being defined as its least sensitivity to measurement noise, it was shown that the metric based on the inverse covariance matrix of the measured displacements minimizes the Mahalanobis distance. It corresponds to weighted finite element model updating (or equivalently to integrated DIC when the signal to noise ratio remains high).

For the sake of simplicity, the discussion was restricted to a single loading step. The present results may be generalized to time series of images (and loading steps). Similarly, the boundary conditions were assumed to be noiseless. This assumption can also be relaxed using similar approaches.

Let us also emphasize that in the entire paper, it was assumed that no model error was present. Using the same formalism as above presented, it is sometimes argued that the constitutive law satisfaction could be relaxed to better account for the measured data [37, 38]. In this way, identification appears as a compromise between a constitutive law describing a material and noise corrupting the measurements. When phrased in this way, it appears clearly that now the metric on measured and computed displacement field, and in particular the optimal one based on Mahalanobis distance, has to be balanced by a metric in the constitutive law space. The latter is however very rarely addressed, but even if it were, how to gauge it with the measurement metric appears as a difficult, not to say impossible, epistemological question.

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