



INSTITUT ÉLIE CARTAN DE LORRAINE
INRIA BIGS TEAM

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Change-point Detection Methods in the Online Context

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Plan

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- 2 Online change-point detection
 - Recursive detection statistics
 - Stopping rule
- 3 Contribution
 - Instantaneous threshold
 - Procedure reset
 - Corrected stopping rule
- 4 Simulation results
- 5 Perspectives

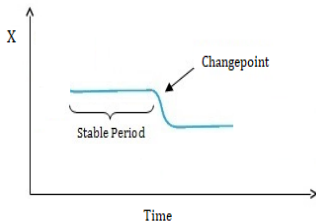


Preamble



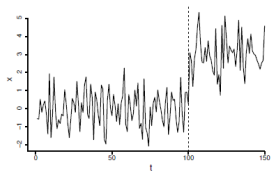
What is change-point ?

A change-point is an instance in time where the statistical properties of sequence of observations $\{X_i\}_{i=1, \dots, n}$ before and after this time point differ.

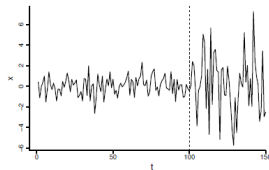




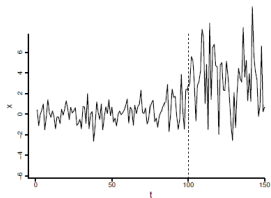
Types of change-point



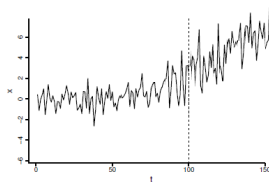
(a) Changepoint in the mean



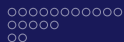
(b) Changepoint in the variance



(c) Changepoint in the mean and variance



(d) Slope changepoint



Definition

⇒ A **change-point** exists in an ordered sequence of data : $\{X_i\}_{i=1,..,n} = \{X_1, \dots, X_n\}$:

- $\exists ! v \in \{1, \dots, (n - 1)\}$;

- $\begin{cases} \{X_1, \dots, X_v\} & \text{are distributed according to a density } f(x) ; \\ \{X_{v+1}, \dots, X_n\} & \text{are distributed according to a density } g(x). \end{cases}$

where f, g are known, such as : $f(x) \neq g(x)$.

⇒ Existence of m **multiple change-points** :

- $\exists m \in \mathbb{N}, \{v_j\}_{j=1,..,m} = \{v_1, \dots, v_m\}$, such as : $\{v_j\}_{j=1,..,m} \subset \{1, \dots, (n - 1)\}$.
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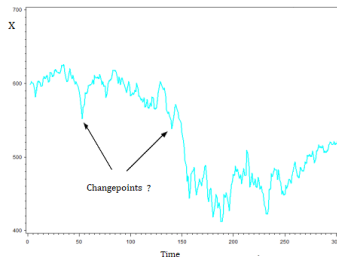
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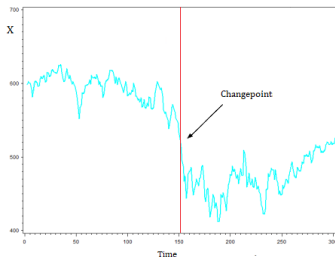
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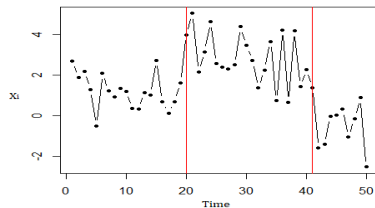




Context of analysis

Offline context :

- All data is received and processed in one go ;
- The primary aim is accurate detection of changes ;
- Inference about all change-points simultaneously.

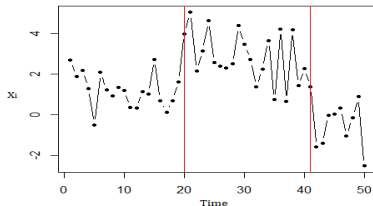




Context of analysis

Offline context :

- All data is received and processed in one go ;
- The primary aim is accurate detection of changes ;
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**Online context :**

- Data arrives either as single data-points or in batches ;
- Data must be processed quickly before new data arrives ;
- The aim is the quickest detection of a change after it has occurred ;
- Inference about most recent change only.



Online change-point detection

Hypotheses

Let $\{X_i\}_{i=1,\dots,n}$ a series of observations **sequentially observed**.

X_n is the last observed point in the dataset.

Statistically, the problem of change-point detection is to sequentially test for each new observation $n \geq 1$, the hypotheses :

$$\left\{ \begin{array}{lll} H_{0,n} : v = +\infty & x_i \sim f(x_i) & \forall i = 1, \dots, n \\ H_{1,n} : v = k, 1 \leq k < n & \begin{array}{l} x_i \sim f(x_i) \\ x_i \sim g(x_i) \end{array} & \begin{array}{l} \forall i = 1, \dots, k \\ \forall i = (k+1), \dots, n \end{array} \end{array} \right. \quad (1)$$

Where ("distribution pre-change") $f \neq g$ ("distribution post-change")

The "instantaneous" likelihood ratio is defined by : $\Lambda_i = \frac{g(X_i)}{f(X_i)}, i \geq 1$

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Cumulative Sum "CUSUM"

$$\Lambda_i = \frac{g(X_i)}{f(X_i)}, \quad L_i = \log \Lambda_i, \quad i \geq 1$$

The CUSUM statistics is introduced by [E.S.Page(1954)] :

$$V_n = \max_{1 \leq k \leq n} \prod_{j=k+1}^n \Lambda_j \quad n \geq 1 \quad (2)$$

The online CUSUM detection statistics :

$$W_n = \max\{0, \log(V_n)\}, \quad n \geq 1$$

- CUSUM detection statistics can be written recursively at instant n :

$$W_n = \max\{0, W_{n-1} + L_n\}, \quad n \geq 1, \quad W_0 = 0 \quad (3)$$



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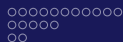
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In practice, the two distributions f and g are unknown.

⇒ [Tartakovsky, A. G. and al (2013)] suggested a non-parametric version by replacing the log likelihood ratio L_n through a score function $S_n = S_n(X_1, \dots, X_n)$.

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Stopping rule

Detection procedure

- The statistics is calculated recursively :

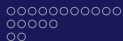
$$W_n = \max\{0, W_{n-1} + S_n\}, \quad n \geq 1, \quad W_0 = 0$$

- Online detection is based on a **Stopping Rule** :

$$C_h = \min\{n \geq 1 : W_n \geq h\}, \quad h \geq 0 : \text{threshold.}$$

When W exceeds the threshold h :

⇒ **The procedure triggers an alarm (Stopping Time) to signal that a change-point has occurred.**



Stopping rule

Stopping time

Denote T a stopping time, such as : $T = \min_{i \geq 1} \{W_i \geq h\}$

- * $T \geq v$: detection with a delay ($T - v$).
- * $T < v$: false alarm.
- * $T = +\infty$: non detection.

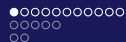


Contribution

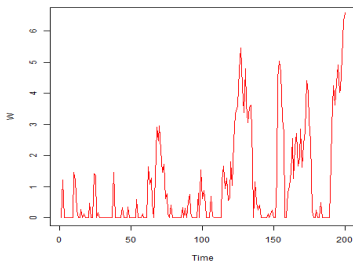
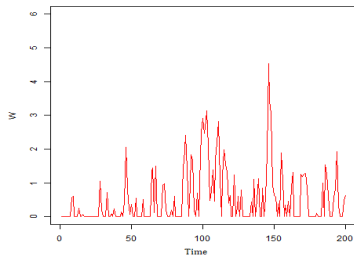
Our work

We propose an extension of non-parametric detection methods :

- 1 Replacing the constant threshold by an instantaneous threshold ;
- 2 Suggest a new configuration through resetting the detection procedure ;
- 3 Suggest a new stopping rule by modifying the classical rule.

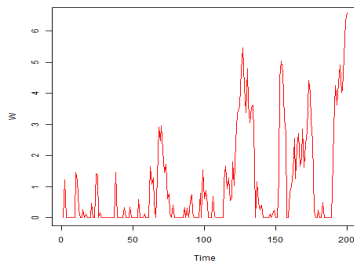
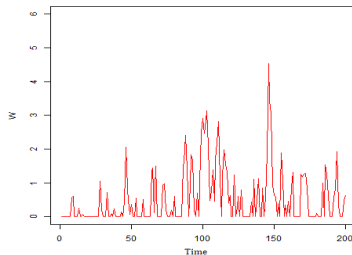


Instantaneous threshold

CUSUM statistics under H_0 



Instantaneous threshold

CUSUM statistics under H_0 

⇒ The behavior of the W -statistics under the pre-change regime H_0 is not stable.

⇒ Suppose to know the behavior of W -statistic under H_0 .



Construction

Construct the instantaneous threshold according to empirical quantile of law of the statistic under H_0 "without change-point".

Construction steps :

- 1 Under H_0 : simulate B series of n observations

$$\{X_i^j\}_{i=1, \dots, n; j=1, \dots, B}$$

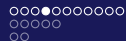


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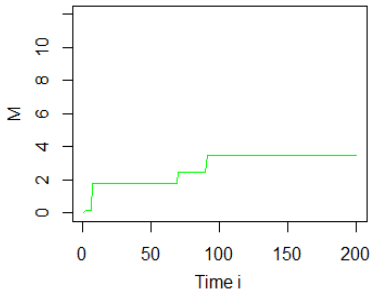
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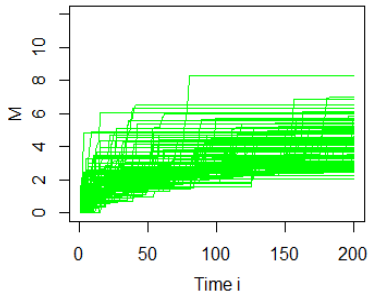
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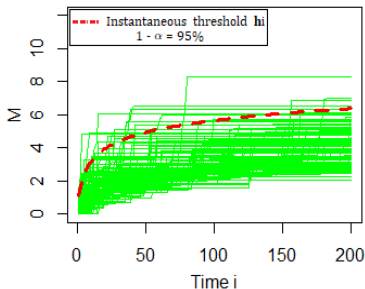
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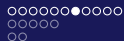
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- 3 The instantaneous thresholds would be the empirical quantile of order $(1-\alpha)$:

$$h_i = \mathbf{q}_{(1-\alpha)} \left[\left(M_i^j \right)_{j=1, \dots, B} \right], i = 1, \dots, n$$





CUSUM detection statistics : $W_n = \max\{0, W_{n-1} + S_n\}, \quad n \geq 1; \quad W_0 = 0$

- **Stopping rule :** $C_{(h_{\cdot})} = \min_{n \geq 1} \{n; W_n \geq h_n\}, \quad (h_1, \dots, h_n) \geq 0$: instantaneous threshold.

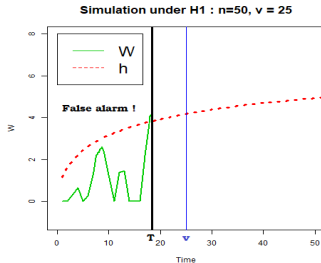


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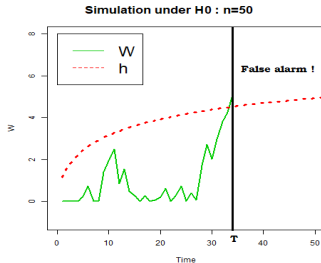
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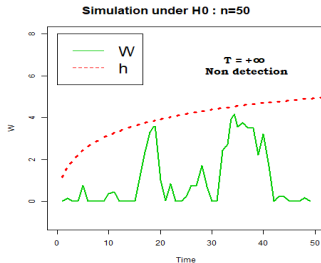
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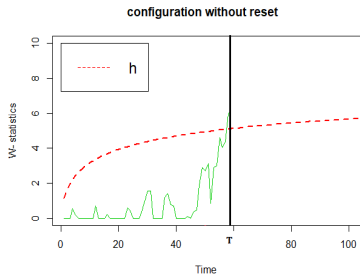
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Configuration without reset

The instantaneous threshold remains fixed throughout the detection test.





Procedure reset

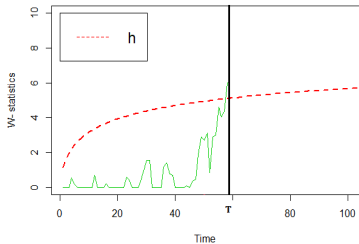
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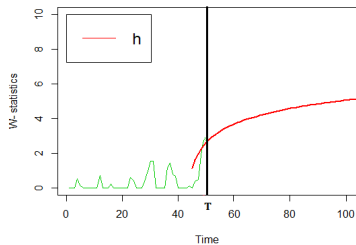
Configuration with reset

The instantaneous threshold moves whenever the statistical returns to the initial value (0 for W statistics).

configuration without reset



Configuration with reset





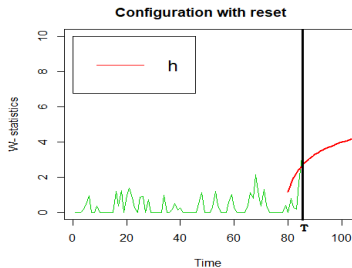
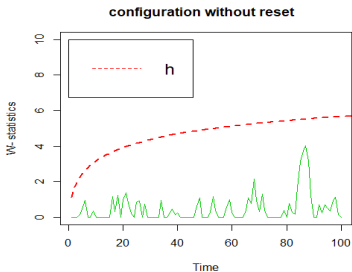
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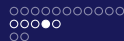
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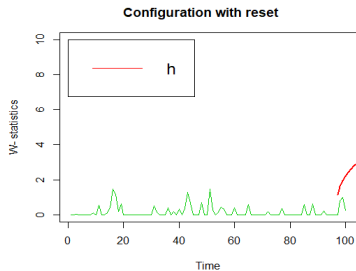
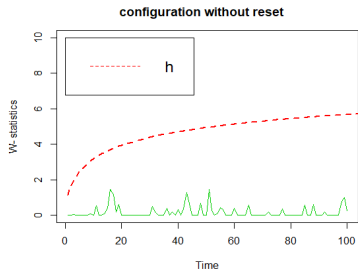
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● **Stopping rule :** $C_{(h, \cdot)} = \min_{n \geq 1} \{n; W_n \geq h_{n-k}\}$

$$k = \begin{cases} 0 & \text{: fixed threshold ;} \\ \max_{1 \leq i \leq n} \{i - 1; W_i = 0\} & \text{: moving threshold.} \end{cases}$$



Corrected stopping rule

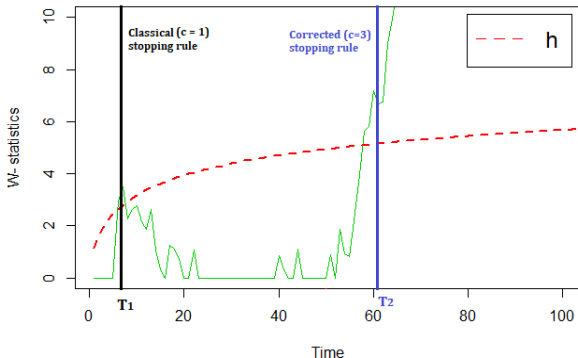
New stopping rule

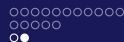
Classical stopping rule

signals the existence of a change-point when the detection statistic exceeds the instantaneous threshold.

Corrected stopping rule

signals the existence of a change-point when the detection statistic exceeds the instantaneous threshold during a time $c \geq 1$.





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- **Classical stopping rule :** $C_{(h_{\cdot})} = \min_{n \geq 1} \{n; W_n \geq h_{n-k}\}$
- **Corrected stopping rule :** $C_{(h_{\cdot})} = \min_{n \geq 1} \{n + c - 1; \bigcap_{j=n}^{n+c-1} (W_j \geq h_{j-k})\}$

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Simulation results



Empirical estimation

Perform simulations to evaluate our different proposed methods, by estimating the main evaluation parameters :

Parameters evaluated under H_0 .

False Alarm Rate (FAR)

$$\widehat{FAR}_n = \frac{1}{B} \cdot \sum_{j=1}^B \mathbf{1}_{\{T_j < +\infty\}} \quad (5)$$

Average Run length (ARL)

$$\widehat{ARL}_n = \frac{\sum_{j=1}^B \mathbf{1}_{\{T_j < +\infty\}} \cdot T_j}{\sum_{j=1}^B \mathbf{1}_{\{T_j < +\infty\}}} \quad (6)$$



Empirical estimation

Perform simulations to evaluate our different proposed methods, by estimating the main evaluation parameters :

Parameters evaluated under H_1 .

Power to Detect (PD)

$$\widehat{PD}_{n,v} = \frac{\sum_{j=1}^B \mathbf{1}_{\{T_j \geq v\}}}{B - \sum_{j=1}^B \mathbf{1}_{\{T_j < v\}}} \quad (7)$$

Average Detection Delay (ADD)

$$\widehat{ADD}_{n,v} = \frac{\sum_{j=1}^B \mathbf{1}_{\{T_j \geq v\}} \cdot (T_j - v)}{\sum_{j=1}^B \mathbf{1}_{\{T_j \geq v\}}} \quad (8)$$



Simulation

Simulation model 01 :

- **Several trajectories** with possibility of having a **single change-point by trajectory** ;
- Change-point detection only on the mean (the variance is homogeneous for any series).

Results : under H_0 TABLE – FAR and ARL estimation under $H_0 : B = 10000, n = 100$

Parameter	False Alarm Rate			Average Run Length			
	-	-	-	-	-	-	
Stopping rule	c = 1	c = 2	c = 3	c = 1	c = 2	c = 3	
Threshold	Fixed	0.25	0.12	0.07	17.96	22.56	25.52
	Moving	0.92	0.62	0.4	30.44	42.40	47.5

Results : under H_0 TABLE – FAR and ARL estimation under $H_0 : B = 10000, n = 100$

Parameter		False Alarm Rate			Average Run Length		
		-	-	-	-	-	-
Stopping rule		c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
Threshold	Fixed	0.25	0.12	0.07	17.96	22.56	25.52
	Moving	0.92	0.62	0.4	30.44	42.40	47.5

Results : under H_0 TABLE – FAR and ARL estimation under $H_0 : B = 10000, n = 100$

Parameter		False Alarm Rate			Average Run Length		
		-			-		
Stopping rule		c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
Threshold	Fixed	0.25	0.12	0.07	17.96	22.56	25.52
	Moving	0.92	0.62	0.4	30.44	42.40	47.5

Results : under H_0 TABLE – FAR and ARL estimation under $H_0 : B = 10000, n = 200$

Parameter	False Alarm Rate			Average Run Length			
	-	-	-	-	-	-	
Stopping rule	c = 1	c = 2	c = 3	c = 1	c = 2	c = 3	
Threshold	Fixed	0.28	0.14	0.08	28.76	40.16	43.68
	Moving	0.99	0.86	0.63	36.80	70.46	84.81

Results : under H_1 TABLE – PD and ADD estimation under $H_1 : B = 10000, n = 100, v = 25$

Parameter		Power to Detect			Average Detection Delay		
		c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
Threshold	Fixed	1	1	1	8.68	10.34	11.57
	Moving	1	1	1	4.40	6.35	8.03

Results : under H_1 TABLE – PD and ADD estimation under $H_1 : B = 10000, n = 100, v = 25$

Parameter		Power to Detect			Average Detection Delay		
		c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
Threshold	Fixed	1	1	1	8.68	10.34	11.57
	Moving	1	1	1	4.40	6.35	8.03

Results : under H_1 TABLE – PD and ADD estimation under $H_1 : B = 10000, n = 100, v = 50$

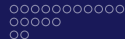
Parameter		Power to Detect			Average Detection Delay		
		-	-	-	-	-	-
Stopping rule		c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
Threshold	Fixed	0.99	0.99	0.99	9.92	11.54	12.84
	Moving	1	0.99	0.99	4.47	6.41	8.10



Simulation

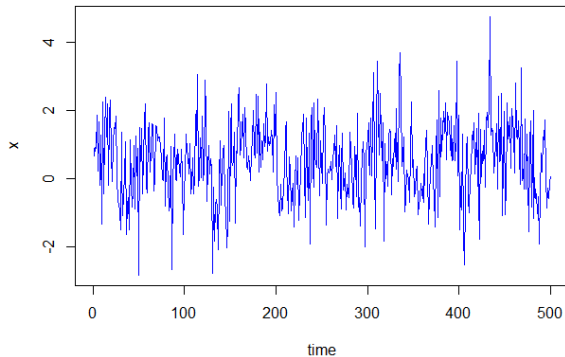
Simulation model 02 :

- **One trajectory** over a long period of time ;
- Possibility of **several change-points** $\{v_k\}_{k=1, \dots, m}$;
- The mean of the observations can switch between two states (0 , 1).



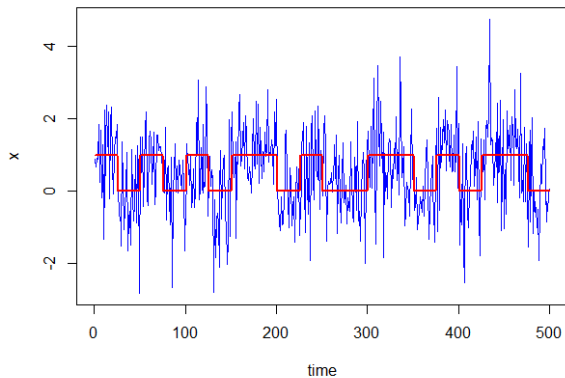
Simulation

$n = 500, m = 20$



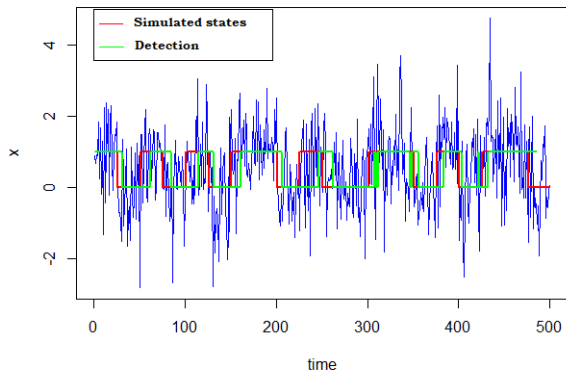


Simulation

 $n = 500, m = 20$ 

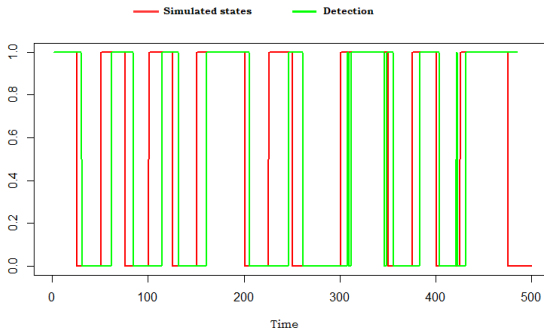


Simulation

 $n = 500, m = 20$ 



Simulation



Results : under H_1 TABLE – PD and ADD estimation under $H_1 : n = 100000, m = 19486$

Change-point occurs at least every 25 points.

Parameter		Power to Detect			Average Detection Delay		
		-	-	-	-	-	-
Stopping rule		c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
Threshold	Fixed	0.97	0.96	0.95	8.64	10.21	11.21
	Moving	0.93	0.96	0.96	6.44	6.75	7.58

Results : under H_1 TABLE – PD and ADD estimation under $H_1 : n = 100000, m = 9445$

Change-point occurs at least every 50 points.

Parameter		Power to Detect			Average Detection Delay		
		c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
Threshold	Fixed	0.99	0.99	0.99	10.21	11.90	13.00
	Moving	0.96	0.96	0.97	7.76	8.01	8.26



Perspectives



Perspectives

- Estimation of the data distribution and its parameters under pre-change regime ;
- Use the methods in the multivariate case ;
- Thesis framework : prediction of a dreaded event during online monitoring of lung transplant patients.



Thank you !



Annex



The SR statistics is defined by :

$$R_n = \sum_{k=1}^n \prod_{j=k+1}^n \Lambda_j \quad n \geq 1 \quad (9)$$

● **SR recursive detection statistics :**

$$R_n = (1 + R_{n-1}) \cdot e^{L_n} , \quad n \geq 1; \quad R_0 = 0 \quad (10)$$



The S_n score is defined by a quadratic linear form for a mean and variance changepoint detection :

$$S_n = C_1 \cdot Y_n + C_2 \cdot Y_n^2 - C_3 \quad (11)$$

$Y_n = (X_n - \mu_\infty)/\sigma_\infty$: the centered and standardized data under H_0 .

$$C_1 = \delta \cdot q^2, \quad C_2 = \frac{1-q^2}{2}, \quad C_3 = \frac{\delta^2 \cdot q^2}{2} - \log(q)$$

$$\delta = (\mu_v - \mu_\infty)/\sigma_\infty \quad q = \sigma_\infty/\sigma_v$$

- δ : minimum level of change in the mean that is required to be detected.

No changepoint detection on the mean :

$$\mu_\infty = \mu_v \Rightarrow \delta = 0, \text{ therefore } C_1 = 0.$$

- q : minimum level of change in the variance that is required to be detected.

No changepoint detection on the variance :

$$\sigma_\infty = \sigma_v \Rightarrow q = 1, \text{ therefore } C_2 = 0.$$



To define the score function, let first :

$\mu_\infty = \mathbb{E}_\infty(X_n)$, $\sigma_\infty^2 = \mathbb{V}_\infty(X_n)$: average and variance pre change (under H_0), $n \geq 1$.

$\mu_v = \mathbb{E}_v(X_n)$, $\sigma_v^2 = \mathbb{V}_v(X_n)$: average and variance post change (under H_1), $n \geq v$.

Note

The score function can only be used to the knowledge of the parameters **average and variance of pre change data** $\mu_\infty, \sigma_\infty$.

- Use a portion of observed data on the normal state without breaking
 \Rightarrow **Estimate** μ_∞ et σ_∞ .
- Depending on the objective (mean and/or variance) and level of change-point detection
 \Rightarrow **Determine** μ_v et σ_v that one would like to detect with some risk.

To define the score function, let first :

$\mu_\infty = \mathbb{E}_\infty(X_n)$, $\sigma_\infty^2 = \mathbb{V}_\infty(X_n)$: average and variance pre change (under H_0), $n \geq 1$.

$\mu_v = \mathbb{E}_v(X_n)$, $\sigma_v^2 = \mathbb{V}_v(X_n)$: average and variance post change (under H_1), $n \geq v$.

Note

The score function can only be used to the knowledge of the parameters **average and variance of pre change data** $\mu_\infty, \sigma_\infty$.

- Use a portion of observed data on the normal state without breaking
⇒ **Estimate** μ_∞ **et** σ_∞ .
- Depending on the objective (mean and/or variance) and level of change-point detection
⇒ **Determine** μ_v **et** σ_v **that one would like to detect with some risk.**



Let $\mathbb{P}_\infty[\cdot]$, $\mathbb{E}_\infty[\cdot]$: respectively the probability and the average under H_0 .

Evaluation criteria evaluated under H_0 .

False Alarm Rate (FAR)

is the probability of signaling a changepoint, while it did not occur. It is given by :

$$FAR(T) = \mathbb{P}_\infty (T < +\infty) \quad (12)$$

Average Run Length (ARL)

the average number of observations that the procedure samples before it triggers a false alarm.

$$ARL(T) = \mathbb{E}_\infty [T] \quad (13)$$



Let $\mathbb{P}_v[\cdot]$, $\mathbb{E}_v[\cdot]$: respectively the probability and the average under H_1 .

Evaluation criteria evaluated under H_1 .

Power to Detect (PD)

is the ability of a procedure to detect the existence of a changepoint at a given moment.

$$PD_v(T) = \mathbb{P}_v(T \geq v), \quad 1 \leq v < +\infty \quad (14)$$

Average Delay to Detect (ADD)

$$ADD_v(T) = \mathbb{E}_v[T - v | T \geq v], \quad 1 \leq v < +\infty \quad (15)$$



Evaluation criteria

Results : under H_1

Under H_1	Parameter	False Alarm Rate			Power to Detect			Average Detection Delay		
	n=100000	Stopping rule			Stopping rule			Stopping rule		
	Threshold	c = 1	c = 2	c = 3	c = 1	c = 2	c = 3	c = 1	c = 2	c = 3
np=10	Fixed	0.01	0.004	0.004	0.76	0.70	0.64	6.02	7.23	7.87
	Moving	0.02	0.01	0.007	0.86	0.82	0.80	4.40	5.54	6.24
np=25	Fixed	0.01	0.003	0.001	0.97	0.96	0.95	8.64	10.21	11.21
	Moving	0.03	0.01	0.008	0.93	0.96	0.96	6.44	6.75	7.58
np=50	Fixed	0.006	0.002	0.001	0.99	0.99	0.99	10.21	11.90	13.00
	Moving	0.03	0.01	0.01	0.96	0.96	0.97	9.76	8.01	8.26