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► **To cite this version:**

Serguei Barannikov. Summations over generalized ribbon Feynman diagrams and all genus Gromov-Witten invariants. 2018. hal-01915676

HAL Id: hal-01915676

<https://hal.science/hal-01915676>

Preprint submitted on 7 Nov 2018

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Summations over generalized ribbon Feynman diagrams and all genus Gromov-Witten invariants.

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24/09/2018

$\psi_i = c_1(T_{p_i}^*)$, $\psi_i \in H^2(\bar{\mathcal{M}}_{g,n})$. Consider $H^*(\bar{\mathcal{M}}_{g,n})$ -valued generating function for products of ψ_i

- ▶ Theorem (S.B.[2009])

$$\sum_{\sum d_i=d} \psi_1^{d_1} \cdots \psi_n^{d_n} \prod_{i=1}^n \frac{(2d_i - 1)!!}{\lambda_i^{2d_i+1}} = \left[\sum_{[G] \in \Gamma_{g,n}^{dec, odd}} (G, or(G)) \frac{2^{-\chi(G)}}{|\text{Aut}(G)|} \prod_{e \in \text{Edge}(G)} \frac{1}{\lambda_{i(e)} + \lambda_{j(e)}} \right] \quad (1)$$

- ▶ the sum on the right is over *stable ribbon* graphs of genus g with n numbered punctures, with $2d + n$ edges, and such that the vertices of the graph have cyclically ordered subsets of *arbitrary odd* cardinality
- ▶ In the simplest case, corresponding to the top degree, the cohomology $H^{6g-6+2n}(\bar{\mathcal{M}}_{g,n})$ is 1-dimensional, the summation is over 3-valent usual ribbon graphs and this formula then reproduces the main identity from Kontsevich proof of Witten conjecture.

This formula is a byproduct of the construction of cohomology classes from [2006b, 2009]

- ▶ l - an odd derivation acting on cyclic associative A_∞ -algebra A , with odd scalar product, in general $l^2 \neq 0$ (!)
- ▶ **Theorem** (S.B.[2006b, 2009]) This data \rightarrow Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$
- ▶ **Theorem** (S.B.[2018]) This construction defines Cohomological Field Theory.
- ▶ **Theorem** (S.B.[2006b, 2009]) This construction applied to “odd matrix algebra” $A = q(N)$,
 $q(N) = \{[X, \pi] = 0 \mid X \in gl(N|N)\}$, where π -odd involution, $l = [\Xi, \cdot]$, Ξ - odd element \rightarrow the formula (1) for all products of ψ_i
- ▶ This formula was the first nontrivial computation of categorical all genus Gromov-Witten invariants.

Counterexample to a Theorem of Kontsevich

Another byproduct is a counterexample to the Theorem 1.3 from Kontsevich “Feynman diagrams in low-dimensional topology” (1993)

Dual construction

We describe here a way to produce *homology* classes on $\mathcal{M}_{g,n}$. The starting ingredient is a finite-dimensional differential associative algebra V with a non-degenerate *odd* scalar product and trivial cohomology.

The right inverse to the scalar product can be considered as an odd element δ of $V \otimes V$. It follows from the compatibility of the scalar product with the differential that δ is closed. From the trivality of $H(V)$ it follows that there exists $\omega \in V \otimes V$ such that $d\omega = \delta$. We can use ω as a “propagator” and tensors $T_{(k)} : V^{\otimes k} \rightarrow \mathbf{C}$

$$T(v_1 \otimes \dots \otimes v_k) = (v_1 v_2 \dots v_{k-1}, v_k)$$

as “interactions”. Again, we obtain a state model on ribbon graphs. Now we will consider the complex which is *dual* to the cochain complex from the previous section.

Theorem 1.3 $\sum_{\Gamma} Z(\Gamma) \Gamma$ is a well-defined chain and it is closed. Its homology class does not depend on the choice of ω .

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It turns out this sum $\sum_{\Gamma} Z(\Gamma) \Gamma$ is NOT closed.

Counterexample $A = \langle 1, \xi \rangle / \xi^2 = 1$

- ▶ $A = \langle 1, \xi \rangle / \xi^2 = 1, \bar{\xi} = 1, \beta(1, \xi) = 1, d(\xi) = 1, d^2 = 0, w = \xi \otimes \bar{\xi}$.
- ▶ For this data this sum $\sum_{\Gamma} Z(\Gamma)\Gamma$ is nonzero on the boundary of the following ribbon graph = a generator of the dual complex:

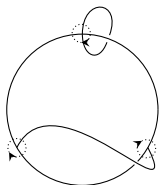


Figure: $G: Z_{\Gamma}(\partial G) \neq 0$

Cellular decomposition of $\bar{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A *stable ribbon graph* is a connected graph G (recall: a graph G is a triple $(Flag(G), \lambda, \sigma)$, where $Flag(G)$ is a finite set, whose elements are called flags, λ is a partition of $Flag(G) \leftrightarrow$ vertices, σ is an involution acting on $Flag(G) \leftrightarrow$ edges/legs) together with:

- ▶ partitions of the set of flags adjacent to every vertex into $i(v)$ nonempty subsets

$$Leg(v) = Leg(v)^{(1)} \sqcup \dots \sqcup Leg(v)^{(i(v))}, v \in Vert(G)$$

- ▶ fixed cyclic order on every subset $Leg(v)^{(k)}$,
- ▶ a number $g(v) \in \mathbb{Z}_{\geq 0}$ such that $|Leg(v)| > 2(2 - i(v) - 2g(v))$.

- ▶ Define an orientation

$$or(G) \in Det(\otimes_{v \in Vert(G)} (k^{Flag(v)} \oplus k^{Cycle(v)}))$$

- ▶ $\oplus_{\{G\}} k(G, or(G))$ has natural “generalized contraction of edges” differential δ_{graph} , $\delta_{graph}^2 = 0$

Cellular decomposition of $\overline{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A metric on the stable ribbon graph is a function $l : \text{Edge}(G) \rightarrow \mathbb{R}_{>0}$. Given a stable ribbon graph G w/out legs and a metric on G one can construct by standard procedure a punctured Riemann surface $\Sigma(G)$, which have double points in general.

- ▶ replace every edge by a pair of oriented strips $[0, l] \times [0 + i\infty[$ one for each flag and glue them side $0 \times [0 + i\infty[$ to $l \times [0 + i\infty[$ according to the cyclic order of the cyclically ordered subsets at each vertex. Then glue the two strips for each edge $[0, l] \leftrightarrow [l, 0]$
- ▶ identify points corresponding to 2 subsets at vertices with $g(v) = 0, i(v) = 2$, (double points); for points with $2g(v) + i(v) > 2$ remplace the vertex by some Riemann surface of genus $g(v)$, which does not contain any marked point, connected to the rest via $i(v)$ double points.
- ▶ This gives an isomorphism of complexes
$$\left(\bigoplus_{\{G\}} k[(G, \text{or}(G)), \delta_{\text{graph}}]\right) \rightarrow \bigoplus_{g,v} C_*(\overline{\mathcal{M}}_{g,v} \times \mathbb{R}_{>0}^v / \mathbb{S}_v)$$

Example of the formula for products of ψ_i

Example: The class of ψ_1 in $H^2(\bar{\mathcal{M}}_{g,n})$ is represented as linear combination of stable $(G, \text{or}(G))$ with $|\text{Edge}(G)| = 2 + n$. It is the coefficient in front of $\frac{1}{\lambda_1^3} \prod_{i=2}^n \frac{1}{\lambda_i}$ in the formula

Compactifications and ψ_i -classes

- ▶ The compactification $\bar{\mathcal{M}}_{g,n}$ is a quotient of Deligne-Mumford compactification $\bar{\mathcal{M}}_{g,n}^{DM}$: the natural map $\bar{\mathcal{M}}^{DM} \rightarrow \bar{\mathcal{M}}$ is the contraction which forgets complex structure on every component of singular Riemann surface which does not contain marked points.
- ▶ One can consider ribbon graph decomposition of such components. This way we get some intermediate compactification $\bar{\mathcal{M}}^{DM} \rightarrow \bar{\mathcal{M}}' \rightarrow \bar{\mathcal{M}}$. On the boundary of new strata again some components must be contracted and so on.
- ▶ The line bundles T_{p_i} on $\bar{\mathcal{M}}_{g,n}^{DM}$ are pullbacks of the line bundles T_{p_i} on $\bar{\mathcal{M}}_{g,n}$
- ▶ hence the ψ_i -classes on $\bar{\mathcal{M}}_{g,n}^{DM}$ are pullbacks of the ψ_i -classes on $\bar{\mathcal{M}}_{g,n}$

The Quantum Master Equation on Cyclic Cochains

- ▶ The stable ribbon graph complex is intimately related with the Quantum Master Equation on Cyclic Cochains (QMCC).
- ▶ Let $V = V_0 \oplus V_1$ be $\mathbb{Z}/2\mathbb{Z}$ -graded vector space, $\dim_k V < \infty$, scalar product $\beta : V^{\otimes 2} \rightarrow k[p]$
- ▶ $C^\lambda = (\bigoplus_{j=1}^{\infty} \text{Hom}((\Pi V^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})$,
 $S \in \text{Sym}(C^\lambda [1-p])[[\hbar]]$ (symmetric products for odd β , antisymmetric for even β)
- ▶ The Quantum Master Equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.[2006a])

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0 \quad \text{QMCC}$$

$$S = \sum_{g \geq 0, i > 0} \hbar^{2g-1+i} S_{g,i}, \quad S_{g,i} \in \text{Sym}^i(C^\lambda [1-p]).$$



$$\{S_{0,1}, S_{0,1}\} = 0,$$

$S_{0,1}$ - A_∞ - algebra with (even/odd) scalar product, so S -*multiloop*, *higher genus* generalization of A_∞ - algebra.

The noncommutative BV differential

- ▶ (QMCC) $\iff \Delta_{NC}(\exp \frac{1}{\hbar} S) = 0$
- ▶ The noncommutative BV differential on $F = \text{Sym}(C^\lambda [1 - \rho])$

$$\begin{aligned} \Delta_{NC}(a_{\rho_1} \dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda &= \\ &= \sum_{p,q} (-1)^\varepsilon \beta_{\rho_p \tau_q}^\vee (a_{\rho_1} \dots a_{\rho_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_{q-1}} a_{\rho_{p+1}} \dots a_{\rho_r})^\lambda + \end{aligned}$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\rho_p \rho_q}^\vee (\dots a_{\rho_{p-1}} a_{\rho_{q+1}} \dots a_{\rho_r})^\lambda (a_{\rho_{p+1}} \dots a_{\rho_{q-1}})^\lambda (a_{\tau_1} \dots a_{\tau_t})^\lambda$$

$$\sum_{p \pm 1 \neq q} (-1)^{\tilde{\varepsilon}} \beta_{\tau_p \tau_q}^\vee (\dots a_{\rho_r})^\lambda (a_{\tau_1} \dots a_{\tau_{p-1}} a_{\tau_{q+1}} \dots a_{\tau_t})^\lambda (a_{\tau_{p+1}} \dots a_{\tau_{q-1}})^\lambda$$

- ▶ signs are the standard Koszul signs taking into account that $(a_{\rho_1} \dots a_{\rho_r})^\lambda = 1 - \rho + \sum \bar{a}_{\rho_i}$, $a_i \in \text{Hom}(\Pi V)$.
- ▶ **Theorem** (S.B.[2006a]) $\Delta_{NC}^2 = 0$

(A, m) is a $d\mathbb{Z}/2\mathbb{Z}$ graded associative algebra with odd scalar product β , $\dim_k A < \infty$.

H is an odd selfadjoint operator $H : A \rightarrow \Pi A$, $H^\vee = H$, such that $Id - [d, H] = P$, $dP = 0$, $P^2 = P$. B - the image of P .

Let Γ be a 3-valent ribbon graph with legs, then put:

- ▶ on every vertex $v \rightarrow$ the 3-tensors of the cyclic product on A

$$m^v \in ((\Pi A)^{\otimes 3})^\vee$$

- ▶ on every interior edge $e = (ff') \rightarrow$ the two tensor

$$\beta^\vee(H^\vee u_f, v_{f'}), \beta_H^{\vee, e} \in (\Pi A)^{\otimes 2},$$

- ▶ on every leg $l \in \text{Leg}(\Gamma) \rightarrow$ element $a_l \in \Pi B$
- ▶ make the contraction

$$W_\Gamma \left(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left(\bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{\vee, e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle$$

The summation over 3-valent ribbon graphs with legs (cont'd).



$$W_{\Gamma} \left(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left(\bigotimes_{e \in \text{Edge}(\Gamma)} \beta_H^{v,e} \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right\rangle$$

$$S = \sum_{\{\Gamma\}} \hbar^{1-\chi(\Gamma)} W_{\Gamma}$$

- ▶ $S \in \text{Sym}(\oplus_{j=1}^{\infty} \text{Hom}((\Pi B^{\otimes j}), k)^{\mathbb{Z}/j\mathbb{Z}})[[\hbar]]$
- ▶ **Theorem** (S.B.[2010]) The sum over ribbon graphs S satisfy the QMCC/noncommutative Batalin-Vilkovisky equation:

$$\hbar \Delta_{NC} S + \frac{1}{2} \{S, S\} = 0$$

Let $S \in \text{Sym}(C^\lambda [1 - \rho])[[\hbar]]$ be a solution to the QMCC, with $S_{0,1,2} = 0$ ($d_V = 0$), let Γ be a *stable* ribbon graph, then put:

- ▶ on every vertex $v \rightarrow$ the multi-cyclic tensors

$$S_{g,i}^v \in \text{Sym}(\bigoplus_{j=1}^i \text{Hom}((\Pi V^{\otimes j}), k))^{\mathbb{Z}/j\mathbb{Z}} [1 - \rho]$$

- ▶ on every edge $e = (ff') \rightarrow$ the two tensor

$$\beta^v(u_f, v_{f'}), \beta^{v,e} \in (\Pi V)^{\otimes 2},$$

- ▶ take the contraction $W_\Gamma = \left\langle \bigotimes_{v \in \text{Vert}(\Gamma)} S_{g,i}^v, \left(\bigotimes_{e \in \text{Edge}(\Gamma)} \beta^{v,e} \right) \right\rangle$

- ▶ **Theorem** (S.B.[2006a]) For any S - a solution to the QMCC equation the following chain is a cycle in the *stable ribbon* graph complex

$$W(S) = \sum_{\{\Gamma\} \in \text{SRG}} \hbar^{1-\chi(\Sigma_\Gamma)} W_\Gamma [S]$$

$$\delta_{\text{graph}} W(S) = 0,$$

therefore $[W(S)] \in H_*(\overline{\mathcal{M}}_{*,*})$

More constructions of solutions to the QMCC equation

- ▶ Conjecture (S.B.[2006a]) Counting of holomorphic curves $(\Sigma, \partial\Sigma, p_i) \rightarrow (M, \coprod L_i, \oplus H_*(L_i \cap L_j))$, with $\mathbb{Z}/2\mathbb{Z}$ -graded local systems, gives solution to the QMCC equation.
- ▶ **Theorem** (S.B. 2013) If A is an A_∞ -infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC is constructed step by step starting from $\{S_{0,1}, S_{0,1}\} = 0$

Homotopy theory of the QMCC equation

- ▶ **Theorem** (S.B.[2006a]) Solutions to the QMCC are in one-to-one correspondence with the structure of algebra over the *Feynman transform* of $\widehat{k[\mathbb{S}_n]}$
- ▶ **Theorem** (S.B.[2006a]) (The Feynman transform of $\widehat{k[\mathbb{S}_n]}(0, \gamma, \nu)^\vee \simeq C_*(\overline{\mathcal{M}}_{\gamma, \nu} / \mathbb{S}_\nu)$)

Associative algebra plus odd derivation $I^2 \neq 0$.

- ▶ A - associative algebra , with odd scalar product
- ▶ Assume: I - an *odd derivation* acting on A , preserving the scalar product: ,for example $I = [\Lambda, \cdot]$, $\Lambda \in A^{\text{odd}}$, in general $I^2 \neq 0$ (!), $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, $str([a, \cdot]) = 0$ for any $a \in A$.
- ▶ **Theorem**(S.B.[2006b, 2009]) This data \rightarrow Cohomology classes in $H^*(\overline{\mathcal{M}}_{g,n})$

“Odd matrix algebra” $Q(N)$

- ▶ Example $q(N)$, $q(N) = \{[X, \pi] = 0 \mid X \in \mathfrak{gl}(N|N)\}$, where π -odd involution, $q(N)$ has *odd trace* otr , $I = [\Xi, \cdot]$, Ξ -odd element $\Xi = \begin{pmatrix} 0 & \\ & \text{diag}(\lambda_1, \dots, \lambda_n) \end{pmatrix}$, ($I^2 \neq 0$ (!))
- ▶ **Theorem** (S.B.[2006b, 2009]) This gives the generating function for products of tautological classes $\psi_i = c_1(T_{\rho_i}^*)$.

- ▶ **Conjecture** (S.B.[2009]) This construction, applied to A_∞ -algebra $A = \text{End}(C)$, where C is a generating object of $D^b(\text{Coh}(Y))$, Y is the mirror dual Calabi-Yau manifold to X
→...
- ▶ →all genus Gromov-Witten invariants of X

Evidences for the conjecture

- ▶ In the case of $X = \text{pt}$ the construction reproduces the all-genus Gromov-Witten descendent potential of the point.
- ▶ For projective Calabi-Yau manifolds the summation over $g = 0$ generalized ribbon graphs reproduces the summation over trees giving Frobenius manifold of the $\frac{\infty}{2} - VHS$ defined by the HC^- . It coincides with the $g = 0$ Gromov-Witten of the mirror dual for hypersurfaces.
- ▶ The construction defines a Cohomological Field Theory (S.B.[2018]).

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