Summations over generalized ribbon Feynman diagrams and all genus Gromov-Witten invariants

Serguei Barannikov

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Summations over generalized ribbon Feynman diagrams and all genus Gromov-Witten invariants.

Skoltech

Serguei Barannikov (Paris VII, NRU HSE)

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\( \psi_i = c_1(T_{p_i}^*) \), \( \psi_i \in H^2(\overline{M}_{g,n}) \). Consider \( H^*(\overline{M}_{g,n}) \)—valued generating function for products of \( \psi_i \)

- **Theorem (S.B. [2009])**

\[
\sum_{\sum d_i = d} \psi_1^{d_1} \cdots \psi_n^{d_n} \prod_{i=1}^{n} \frac{(2d_i - 1)!!}{\lambda_i^{2d_i+1}} = \\
= \left[ \sum_{[G] \in \Gamma_{g,n}^{dec,odd}} (G, \text{or}(G)) \frac{2^{-\chi(G)}}{|\text{Aut}(G)|} \prod_{e \in \text{Edge}(G)} \frac{1}{\lambda_i(e) + \lambda_j(e)} \right]
\]

(1)

- The sum on the right is over stable ribbon graphs of genus \( g \) with \( n \) numbered punctures, with \( 2d + n \) edges, and such that the vertices of the graph have cyclically ordered subsets of arbitrary odd cardinality.

- In the simplest case, corresponding to the top degree, the cohomology \( H^{6g-6+2n}(\overline{M}_{g,n}) \) is 1-dimensional, the summation is over 3-valent usual ribbon graphs and this formula then reproduces the main identity from Kontsevich proof of Witten conjecture.
This formula is a byproduct of the construction of cohomology classes from [2006b, 2009]

- \( l \) - an odd derivation acting on cyclic associative /\( A_\infty \)-algebra \( A \), with odd scalar product, in general \( l^2 \neq 0 \) (!)

**Theorem** (S.B.[2006b, 2009]) This data →Cohomology classes in \( H^*(\overline{M}_{g,n}) \)

**Theorem** (S.B.[2018]) This construction defines Cohomological Field Theory.

**Theorem** (S.B.[2006b, 2009]) This construction applied to “odd matrix algebra” \( A = q(N) \),

\[
q(N) = \{ [X, \pi] = 0 | X \in gl(N|N) \},
\]

where \( \pi \)—odd involution, \( l = [\Xi, \cdot] \), \( \Xi \)- odd element →the formula (1) for all products of \( \psi_i \)

**Theorem** (S.B.[2006b, 2009]) This formula was the first nontrivial computation of categorical all genus Gromov-Witten invariants.
Another byproduct is a counterexample to the Theorem 1.3 from Kontsevich “Feynman diagrams in low-dimensional topology” (1993)

It turns out this sum $\sum_\Gamma Z(\Gamma)\Gamma$ is NOT closed.
Counterexample $A = \langle 1, \xi \rangle / \xi^2 = 1$

- $A = \langle 1, \xi \rangle / \xi^2 = 1, \bar{\xi} = 1, \beta(1, \xi) = 1, d(\xi) = 1, d^2 = 0, \omega = \xi \otimes \xi$.

- For this data this sum $\sum_{\Gamma} Z(\Gamma) \Gamma$ is nonzero on the boundary of the following ribbon graph =a generator of the dual complex:

Figure: $G: Z_{\tilde{\Gamma}}(\partial G) \neq 0$
Cellular decomposition of $\tilde{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A stable ribbon graph is a connected graph $G$ (recall: a graph $G$ is a triple $(\text{Flag}(G), \lambda, \sigma)$, where $\text{Flag}(G)$ is a finite set, whose elements are called flags, $\lambda$ is a partition of $\text{Flag}(G) \leftrightarrow$ vertices, $\sigma$ is an involution acting on $\text{Flag}(G) \leftrightarrow$ edges/legs) together with:

- partitions of the set of flags adjacent to every vertex into $i(v)$ nonempty subsets

$$\text{Leg}(v) = \text{Leg}(v)^{(1)} \sqcup \ldots \sqcup \text{Leg}(v)^{(i(v))}, \ v \in \text{Vert}(G)$$

- fixed cyclic order on every subset $\text{Leg}(v)^{(k)}$,

- a number $g(v) \in \mathbb{Z}_{\geq 0}$ such that $|\text{Leg}(v)| > 2(2 - i(v) - 2g(v))$.

- Define an orientation

$$\text{or}(G) \in \text{Det}(\bigotimes_{v \in \text{Vert}(G)} (k^{\text{Flag}(v)} \oplus k^{\text{Cycle}(v)}))$$

- $\bigoplus_G k(G, \text{or}(G))$ has natural “generalized contraction of edges” differential $\delta_{\text{graph}}$, $\delta^2_{\text{graph}} = 0$
Cellular decomposition of $\tilde{\mathcal{M}}_{g,n} \times \mathbb{R}^n$

A metric on the stable ribbon graph is a function
$l : Edge(G) \to \mathbb{R}_{>0}$. Given a stable ribbon graph $G$ w/out legs
and a metric on $G$ one can construct by standard procedure a
punctured Riemann surface $\Sigma(G)$, which have double points in
general.

- replace every edge by a pair of oriented strips $[0, l] \times [0 + i\infty]$\n  one for each flag and glue them side $0 \times [0 + i\infty]$ to
  $l \times [0 + i\infty]$ according to the cyclic order of the cyclically
  ordered subsets at each vertex. Then glue the two strips for\n  each edge $[0, l] \leftrightarrow [l, 0]$
- identify points corresponding to 2 subsets at vertices with\n  $g(v) = 0, i(v) = 2$, (double points); for points with\n  $2g(v) + i(v) > 2$ remplace the vertex by some Riemann
  surface of genus $g(v)$, which does not contain any marked
  point, connected to the rest via $i(v)$ double points.
- This gives an isomorphism of complexes\n  $\bigoplus_G k \left[ (G, \text{or} (G)), \delta_{\text{graph}} \right] \to \bigoplus_{g,v} C_*(\overline{\mathcal{M}}_{g,v} \times \mathbb{R}_0^v / S_v)$
Example: The class of $\psi_1$ in $H^2(\overline{M}_{g,n})$ is represented as linear combination of stable $(G, or(G))$ with $|\text{Edge}(G)| = 2 + n$. It is the coefficient in front of $\frac{1}{\lambda_1^3} \prod_{i=2}^{n} \frac{1}{\lambda_i}$ in the formula
Compactifications and $\psi_i$—classes

- The compactification $\bar{\mathcal{M}}_{g,n}$ is a quotient of Deligne-Mumford compactification $\bar{\mathcal{M}}^{DM}_{g,n}$: the natural map $\bar{\mathcal{M}}^{DM} \to \bar{\mathcal{M}}$ is the contraction which forgets complex structure on every component of singular Riemann surface which does not contain marked points.

- One can consider ribbon graph decomposition of such components. This way we get some intermediate compactification $\bar{\mathcal{M}}^{DM} \to \bar{\mathcal{M}}' \to \bar{\mathcal{M}}$. On the boundary of new stratas again some components must be contracted and so on.

- The line bundles $T_{p_i}$ on $\bar{\mathcal{M}}^{DM}_{g,n}$ are pullbacks of the line bundles $T_{p_i}$ on $\bar{\mathcal{M}}_{g,n}$.

- Hence the $\psi_i$—classes on $\bar{\mathcal{M}}^{DM}_{g,n}$ are pullbacks of the $\psi_i$—classes on $\bar{\mathcal{M}}_{g,n}$.
The Quantum Master Equation on Cyclic Cochains

The stable ribbon graph complex is intimately related with the Quantum Master Equation on Cyclic Cochains (QMCC).

Let $V = V_0 \oplus V_1$ be $\mathbb{Z}/2\mathbb{Z}$-graded vector space, $\dim_k V < \infty$, scalar product $\beta : V \otimes V \rightarrow k [p]$

$C^\lambda = (\bigoplus_{j=1}^{\infty} \text{Hom}(\Pi V \otimes j, k)^{\mathbb{Z}/j\mathbb{Z}})$, 

$S \in \text{Sym}(C^\lambda [1 - p])[\hbar]$ (symmetric products for odd $\beta$, antisymmetric for even $\beta$)

The Quantum Master Equation on Cyclic Cochains / The noncommutative Batalin-Vilkovisky equation (S.B.[2006a])

\[ \hbar \Delta_{NC} S + \frac{1}{2} \{ S, S \} = 0 \quad \text{QMCC} \]

\[ S = \sum_{g \geq 0, i > 0} \hbar^{2g-1+i} S_{g,i}, S_{g,i} \in \text{Sym}^i(C^\lambda [1 - p]). \]

\[ \{ S_{0,1}, S_{0,1} \} = 0, \]

$S_{0,1}$ - $A_\infty$ algebra with (even/odd) scalar product, so $S$— multiloop, higher genus generalization of $A_\infty$— algebra.
The noncommutative BV differential

- (QMCC) ⟷ $\Delta_{NC}(\exp \frac{1}{\hbar} S) = 0$
- The noncommutative BV differential on $F = \text{Sym}(C^\lambda [1 - p])$

$$\Delta_{NC}(a_{\rho_1} \ldots a_{\rho_r})^\lambda (a_{\tau_1} \ldots a_{\tau_t})^\lambda =$$

$$= \sum_{p, q} (-1)^{\varepsilon} \beta_{\rho_p \rho_q}^\vee (a_{\rho_1} \ldots a_{\rho_{p-1}} a_{\tau_{q+1}} \ldots a_{\tau_{q-1}} a_{\rho_{p+1}} \ldots a_{\rho_r})^\lambda +$$

$$\sum_{p \pm 1 \neq q} (-1)^{\varepsilon} \beta_{\tau_p \tau_q}^\vee (\ldots a_{\rho_{p-1}} a_{\rho_{q+1}} \ldots a_{\rho_r})^\lambda (a_{\rho_{p+1}} \ldots a_{\rho_{q-1}})^\lambda (a_{\tau_1} \ldots a_{\tau_t})^\lambda$$

$$\sum_{p \pm 1 \neq q} (-1)^{\varepsilon} \beta_{\tau_p \tau_q}^\vee (\ldots a_{\rho_{r}})^\lambda (a_{\tau_1} \ldots a_{\tau_{p-1}} a_{\tau_{q+1}} \ldots a_{\tau_t})^\lambda (a_{\tau_{p+1}} \ldots a_{\tau_{q-1}})^\lambda$$

- signs are the standard Koszul signs taking into account that

$$(a_{\rho_1} \ldots a_{\rho_r})^\lambda = 1 - p + \sum \bar{a}_{\rho_i}, a_i \in \text{Hom}(\Pi V).$$

- Theorem (S.B. [2006a]) $\Delta_{NC}^2 = 0$
\((A, m)\) is a \(d \mathbb{Z}/2\mathbb{Z}\) graded associative algebra with odd scalar product \(\beta, \dim_k A < \infty\).

\(H\) is an odd selfadjoint operator \(H : A \to \Pi A, H^\dagger = H\), such that \(Id - [d, H] = P, dP = 0, P^2 = P\). \(B\) - the image of \(P\).

Let \(\Gamma\) be a 3-valent ribbon graph with legs, then put:

- on every vertex \(v \to\) the 3-tensors of the cyclic product on \(A\)
  \[
m^v \in (\Pi A \otimes^3)^\dagger
  \]
- on every interieur edge \(e = (ff') \to\) the two tensor
  \[
  \beta^\dagger (H^\dagger u_f, v_{f'}) , \ \beta^\dagger_{H}^e \in (\Pi A) \otimes^2,
  \]
- on every leg \(l \in \text{Leg}(\Gamma) \to\) element \(a_l \in \Pi B\)
- make the contraction

\[
W_\Gamma(\bigotimes_{l \in \text{Leg}(\Gamma)} a_l) = \langle \bigotimes_{v \in \text{Vert}(\Gamma)} m^v, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^\dagger_{H}^e \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \rangle
\]
The summation over 3-valent ribbon graphs with legs (cont’d).

\[
W_\Gamma \left( \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \right) = \langle \bigotimes_{\nu \in \text{Vert}(\Gamma)} m^\nu, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^{\nu, e}_H \right) \bigotimes_{l \in \text{Leg}(\Gamma)} a_l \rangle
\]

\[
S = \sum_{\{\Gamma\}} \hbar^{1 - \chi(\Gamma)} W_\Gamma
\]

\[
S \in \text{Sym} \left( \bigoplus_{j=1}^\infty \text{Hom}((\Pi B \otimes j), k)^{\mathbb{Z}/j\mathbb{Z}} \right) \left[ \left[ \hbar \right] \right]
\]

Theorem (S.B.[2010]) The sum over ribbon graphs \( S \) satisfy the QMCC/noncommutative Batalin-Vilkovisky equation:

\[
\hbar \Delta_{NC} S + \frac{1}{2} \{ S, S \} = 0
\]
Let $S \in \text{Sym}(C^\lambda [1 - p])[[\hbar]]$ be a solution to the QMCC, with $S_{0,1,2} = 0$ ( $d_V = 0$ ), let $\Gamma$ be a stable ribbon graph, then put:

- on every vertex $v \to$ the multi-cyclic tensors
  \[ S^v_{g,i} \in \text{Sym}(\bigoplus_{j=1}^\infty \text{Hom}((\Pi V^\otimes j), k)^{\mathbb{Z}/j\mathbb{Z}} [1 - p]) \]

- on every edge $e = (ff') \to$ the two tensors
  \[ \beta^v(u_f, v_{f'}), \; \beta^{v,e} \in (\Pi V)^\otimes 2, \]

- take the contraction $W_\Gamma = \langle \bigotimes_{v \in \text{Vert}(\Gamma)} S^v_{g,i}, \left( \bigotimes_{e \in \text{Edge}(\Gamma)} \beta^{v,e} \right) \rangle$

**Theorem** (S.B.[2006a]) For any $S$- a solution to the QMCC equation the following chain is a cycle in the stable ribbon graph complex

\[ W(S) = \sum_{\{\Gamma\} \in \text{SRG}} \hbar^{1 - \chi(\Sigma_{\Gamma})} W_\Gamma \left[ \Gamma \right] \]

\[ \delta_{\text{graph}} W(S) = 0, \]

therefore $[W(S)] \in H_* \left( \overline{M}_{*,*} \right)$
More constructions of solutions to the QMCC equation

- **Conjecture** (S.B.[2006a]) Counting of holomorphic curves 
  \((\Sigma, \partial \Sigma, p_i) \rightarrow (M, \bigsqcup L_i, \bigoplus H_*(L_i \cap L_j))\), with \(\mathbb{Z}/2\mathbb{Z}\)-graded local systems, gives solution to the QMCC equation.

- **Theorem** (S.B. 2013) If \(A\) is an \(A_\infty\)—infinity algebra with the degeneration of the Hodge to de Rham spectral sequence, then the solution to the QMCC is constructed step by step starting from \(\{S_{0,1}, S_{0,1}\} = 0\)
Homotopy theory of the QMCC equation

- **Theorem** (S.B. [2006a]) Solutions to the QMCC are in one-to-one correspondence with the structure of algebra over the *Feynman transform* of $k[S_n]$

- **Theorem** (S.B. [2006a]) The Feynman transform of $\hat{k}[S_n](0, \gamma, \nu) \vee \simeq C_*(\overline{\mathcal{M}_{\gamma, \nu}}/S_{\nu})$
Associative algebra plus odd derivation $I^2 \neq 0$.

- **A-** associative algebra , with odd scalar product
- **Assume:** $I$ - an *odd derivation* acting on $A$, preserving the scalar product: , for example $I = [\Lambda, \cdot]$, $\Lambda \in A^{\text{odd}}$, in general $I^2 \neq 0 (!)$, $\exists \tilde{I}$, $[I, \tilde{I}] = 1$, $\text{str}([a, \cdot]) = 0$ for any $a \in A$.
- **Theorem** (S.B.[2006b, 2009]) This data $\rightarrow$ Cohomology classes in $H^*(\overline{M}_{g,n})$
Example $q(N)$, $q(N) = \{[X, \pi] = 0 | X \in gl(N|N)\}$, where $\pi$—odd involution, $q(N)$ has odd trace otr, $I = [\Xi, \cdot]$, $\Xi$-odd element $\Xi = (0 \mid \text{diag}(\lambda_1, \ldots, \lambda_n))$, ($I^2 \neq 0$ (!))

Theorem (S.B. [2006b, 2009]) This gives the generating function for products of tautological classes $\psi_i = c_1(T_{p_i}^*)$. 
Conjecture (S.B.[2009]) This construction, applied to $A_\infty$—algebra $A = \text{End}(C)$, where $C$ is a generating object of $D^b(\text{Coh}(Y))$, $Y$ is the mirror dual Calabi-Yau manifold to $X$ →...

→ all genus Gromov-Witten invariants of $X$
Evidences for the conjecture

- In the case of $X = \text{pt}$ the construction reproduces the all-genus Gromov-Witten descendent potential of the point.
- For projective Calabi-Yau manifolds the summation over $g = 0$ generalized ribbon graphs reproduces the summation over trees giving Frobenius manifold of the $\frac{\infty}{2} - VHS$ defined by the $HC^-$. It coincides with the $g = 0$ Gromov-Witten of the mirror dual for hypersurfaces.
- The construction defines a Cohomological Field Theory (S.B.[2018]).


