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DECOMPOSITION, PURITY AND FIBRATIONS BY NORMAL CROSSING DIVISORS

FOUAD EL ZEIN, DŨNG TRÁNG LÊ, AND XUANMING YE

ABSTRACT. We give a simple geometric proof of the decomposition theorem in terms of Thom-Whitney stratifications by reduction to fibrations by normal crossings divisors over the strata and explain the relation with the local purity theorem an unpublished result of Deligne and Gabber.

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1. INTRODUCTION

Let $f : X \rightarrow V$ be a projective morphism of complex algebraic varieties. The decomposition theorem, as proved in [BBDG 83], describes the derived direct image complex $Rf_*j_{!*}\mathcal{L}$ on V , of an intermediate extension $j_{!*}\mathcal{L}$ of a geometric local system \mathcal{L} on a Zariski open subset of X , as a direct sum of intermediate extensions of irreducible local systems on Zariski open subsets of V .

The proof in [BBDG 83] is given first for varieties over an algebraically closed field with positive characteristic, with coefficients in a pure perverse sheaf on X , then deduced for geometric coefficients on complex algebraic varieties.

It has been preceded by an unpublished note by Deligne and Gabber [DeG 81] on the local purity theorem. The draft of the proof is dense and it is established for an algebraic variety over a field with positive characteristic.

According to the general theory of weight this result can be translated into a local purity theorem in Hodge theory in the transcendental case.

There exist a deep connection between these two results which doesn't appear in the existing proofs in the theory. The basic innovation in this article is a combined direct proof of both theorems with coefficients in the intermediate extension of a polarized variation of Hodge structures (VHS), which establish the interaction between local purity and the decomposition and leads to an overall clarification of the theory in the complex case.

Precise statements are given below in the subsection 1.1(Theorem 1.1 and corollary 1.2).

The proof is by induction on the dimension of the strata of a Thom-Whitney stratification. At the center of the inductive step we use a geometric interpretation of the local purity to deduce the decomposition (see subsections 1.4, 2.3 and 2.4). That is a simplification of the original proof in [DeG 81], when adapted to characteristic 0.

The second innovation is to introduce a class of projective morphisms which are fibrations by normal crossing divisors (NCD) over the strata. The proof for such fibrations use Hodge theory only in the basic case of a complement of a NCD.

We show in section 3 that any algebraic morphism $f : X \rightarrow V$ admits a desingularization π of X such that π and $f \circ \pi$ are in such class of fibrations, from which we deduce without further effort the result for all projective morphisms.

We rely in this paper on computations in terms of perverse sheaves and perverse cohomology as explained in [BBDG 83]. The perverse τ -filtration on a complex K in the derived category $D_c^b(X, \mathbb{Q})$ of constructible \mathbb{Q} -sheaves is defined in ([BBDG 83] proposition 1.3.3 p.29). A recent description of this filtration in [CaMi 10] shows easily that for $K = Rf_*j_{!*}\mathcal{L}$ such filtration consists of sub-MHS on the cohomology of the complement of a NCD (see subsection 5.3). This is a basic result used in our proof and extended to the local case.

In the case of constant coefficients, the proof of decomposition by de Cataldo and Migliorini [CaMi 5] shows the potential of applications in understanding the topology of algebraic maps. The proof here for coefficients in any admissible polarized VMHS covers all analytic cases and has similar applications.

Complex computations by Kashiwara intended to check the admissibility of (graded) polarized variation of mixed Hodge structures (VMHS) on the complement of a NCD in codimension one ([Ka 86], find their natural applications here to

develop Hodge theory with coefficients in the intermediate extension of an admissible polarized VMHS.

Using the decomposition theorem, the full Hodge theory can be developed from the special case of fibrations over the strata, which gives an alternative approach to the development in terms of differential modules [Sa 88]. Our proof is outlined in the notes [EL 14] and [EL 15].

It should be noted that holonomic differential Hodge modules on a smooth complex algebraic variety [Sa 88], are associated with polarized VHS on smooth open subsets of irreducible algebraic subvarieties for which the decomposition theorem is proved here.

1.0.1. *Origin of the theory.* The first statement of a decomposition theorem in the sense of this article appears in the work of Deligne [De 68] as a criteria for a complex K in the derived category of an abelian category [Ve 77] to be isomorphic to the direct sum of its shifted cohomology: $K \xrightarrow{\sim} \bigoplus_{i \in \mathbb{Z}} H^i(K)[-i]$, which means essentially the degeneration of the spectral sequence with respect to the truncation functor τ .

The criteria in Deligne's paper is applied to the complex direct image $K := Rf_* \mathbb{C}_X$ by a smooth projective morphism $f : X \rightarrow V$. The spectral sequence with respect to the perverse truncation filtration τ coincides up to indices with Leray's spectral sequence and the degeneration is deduced from Hodge and Lefschetz theorems. The decomposition of K follows.

This case apply on a big strata on V of a Thom - Whitney stratification of any projective morphism f if X is smooth. This is the starting point of our induction.

Deligne's effort to present the proofs in full generality is a prelude to the vast generalization of the theorem to any projective morphism in [BBDG 83] on schemes X of finite type on an algebraically closed field k of characteristic $p > 0$, for pure perverse sheaves in the derived category $D_c^b(X, \mathbb{Q}_l)$ of constructible \mathbb{Q}_l -sheaves for $l \neq p$ [BBDG 83].

The statements may be transposed in characteristic 0 in terms of Hodge theory according to a dictionary established by Deligne [De 71].

A general procedure to deduce results concerning geometric statements in characteristic 0 from corresponding statements in characteristic $p > 0$, is described in ([BBDG 83], 6.2). This covers geometric variations of Hodge structures.

However, the next development waited until the introduction of the Intersection complex in [GMacP 83]. The intermediate extension of a local system on a Zariski open subset is a version of such complex. The link with Kashiwara's work on differential modules and the need to extend the theory to algebraic varieties over fields of positive characteristic lead to the introduction of perverse cohomology.

1.0.2. *Hodge theory.* According to the referee, it was necessary to add a note [EY 14] to explain how to deduce Hodge theory in terms of logarithmic complexes in the NCD case with coefficients in admissible variation of Hodge structures (subsection 1.3, more in sections 4 and 5).

This allows us to state the local purity theorem 1.8 and to deduce the decomposition in section 2. By using logarithmic complexes we avoid the elaborate development of the theory of Hodge modules.

1.0.3. *The paper is organized as follows.* In the rest of this introduction we give the main statements of the article and describe the ideas based on a simultaneous proof of decomposition and local purity by induction along the strata.

Full proofs using logarithmic complexes in section 2 are based on the reduction to the case of a special fibration in section 3, while Hodge theory on logarithmic complexes is reviewed in sections 4 – 5.

1.1. Statements. *We give below the statement of the theorem 1.1 in terms of Thom-Whitney stratifications and the perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ of a complex K with constructible cohomology sheaves on an algebraic variety V . The perverse cohomology refers to a cohomology theory with value in the abelian category of perverse sheaves [BBDG 83].*

Let $f : X \rightarrow V$ be a projective morphism of complex algebraic varieties, $\tilde{\mathcal{L}}$ an admissible polarized variation of Hodge structures (VHS) on a smooth open subset Ω in X of dimension m and $j : \Omega \rightarrow X$ the embedding. We denote by $\mathcal{L} := \tilde{\mathcal{L}}[m]$ the complex of sheaves reduced to $\tilde{\mathcal{L}}$ in degree $-m$, by $j_{!*}\mathcal{L}$ the intermediate extension of \mathcal{L} ([BBDG 83], prop. 2.1.11 p. 60, 2.2.7 p. 69) and by $K := Rf_*j_{!*}\mathcal{L}$ the direct image in the derived category of cohomologically constructible sheaves of \mathbb{Q} -vector spaces $D_c^b(V, \mathbb{Q})$.

The decomposition consists of two results. First, each perverse cohomology sheaf ${}^p\mathcal{H}^i(Rf_*(j_{!*}\mathcal{L}))$ decomposes naturally as a direct sum of intermediate extensions (formula 1.5 below). This decomposition is naturally expressed here in terms of *intersection morphisms* (1.1.2 below) which appear naturally in the theory ([BBDG 83] formula 1.4.6.1), and which importance appears with a Thom-Whitney stratification of f .

The second result is the relative Hard Lefschetz theorem. Let $\eta \in R^2f_*\mathbb{Q}_X$ denote the section of cohomology classes defined by a relative hyperplane section of f . The corresponding morphism of sheaves $K \xrightarrow{\eta} K[2]$ induce morphisms on the perverse cohomology of degree 2. Then, the result states that the i -th iterated morphism

$$(1.1) \quad {}^p\mathcal{H}^{-i}(Rf_*j_{!*}\mathcal{L}) \xrightarrow{\eta^i} {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}).$$

is an isomorphism of perverse cohomology sheaves, from which the degeneration of the perverse Leray spectral sequence is classically deduced.

One consequence is the existence of a non canonical isomorphism in $D_c^b(V, \mathbb{Q})$ of the direct image complex $Rf_*j_{!*}\mathcal{L}$ with its shifted perverse cohomology sheaves (formula 1.8 below). It is deduced from Hard Lefschetz following the same pattern as in [De 68].

1.1.1. Thom-Whitney stratification \mathcal{S} of f . Let $f : X \rightarrow V$ be a projective morphism on an algebraic variety V of dimension n . The proof presented here is by induction on the dimension of the strata of the Whitney stratification of V underlying a Thom-Whitney stratification \mathcal{S} of f (section 3.0.1 below).

We say that the stratification \mathcal{S} is adapted to $j_{!*}\mathcal{L}$ when the cohomology of $j_{!*}\mathcal{L}$ is locally constant when restricted to the strata of X , and the cohomology of $Rf_*(j_{!*}\mathcal{L})$ is locally constant when restricted on the strata of V . We consider in the text only stratifications adapted to $j_{!*}\mathcal{L}$.

Let $V_j := \cup_{\dim.S \leq j} S$ denote the union of all strata of dimension $\leq j$. We start with a smooth open subset U , union of all strata of dimension n over which f is a

topological fibration with fibers stratified by the induced stratification of X , and we set $V_{n-1} := V - U$ the complement variety. Successively, let U_1 be the union of all algebraic smooth connected strata S of dimension $n-1$ over which f is a topological fibration with induced stratification adapted to $j_{!*}\mathcal{L}$ and set $U_1 := V_{n-1} - V_{n-2}$. We continue the construction by induction on the dimension of the strata S of $V_j - V_{j-1}$ over which f is a topological fibration with induced stratification adapted to $j_{!*}\mathcal{L}$.

1.1.2. *Intersection morphisms.* Let $X_S := f^{-1}(S)$ be the inverse image of a strata S of dimension $l \leq n$, $i_{X_S} : X_S \rightarrow X$, $f_S : X_S \rightarrow S$. The intersection morphism $I_S : Ri_{X_S}^! j_{!*}\mathcal{L} \rightarrow i_{X_S}^* j_{!*}\mathcal{L}$, is defined on S by the composition of the morphism $i_{X_S}^* Ri_{X_S}^! \mathcal{L} \rightarrow \mathcal{L}$ with the restriction morphism $\mathcal{L} \rightarrow i_{X_S}^* i_{X_S}^* j_{!*}\mathcal{L}$. It defines local systems \mathcal{L}_S^i as image of cohomology sheaves:

$$(1.2) \quad \mathcal{L}_S^i := \text{Im} \left\{ R^{-l+i} f_{S*} (Ri_{X_S}^! j_{!*}\mathcal{L}) \xrightarrow{I_S} R^{-l+i} f_{S*} (i_{X_S}^* j_{!*}\mathcal{L}) \right\},$$

For a Thom-Whitney stratification adapted to $j_{!*}\mathcal{L}$, the images \mathcal{L}_S^i of I_S are local systems on the various strata. These local systems are necessarily components of any eventual decomposition as it follows from the proof of proposition 2.2.

When \mathcal{L} is a variation of HS of weight a , the image $\mathcal{L}_{S_l}^i$ of I_{S_l} on a strata S_l of dimension l is a VHS of weight $a + i - l$, as image of a variation of MHS of weight $w \geq a + i - l$ into a variation of MHS of weight $w \leq a + i - l$ (corollary 5.9).

1.1.3. *The inductive step.* The proof is by induction on the dimension of the strata of the Whitney stratification \mathcal{S} of V underlying a Thom-Whitney stratification of $f : X \rightarrow V$. Let V be a complex algebraic variety of dimension n , and \mathcal{S} the stratification of f , $V_j := \cup_{\dim.S \leq j} S$ the union of all strata of dimension $\leq j$, $k_j : (V - V_j) \rightarrow V$, $i_j : V_j \rightarrow V$. The immersion of a strata of dimension l in $V_l^* := V_l - V_{l-1}$ is denoted by $i_{S_l} : S_l \rightarrow V$.

Theorem 1.1 (Inductive step). *Let $K = Rf_* j_{!*}\mathcal{L}$ where \mathcal{L} is a shifted polarized variation of Hodge structures (VHS) of weight a , and \mathcal{S} a Thom-Whitney stratification of f adapted to $j_{!*}\mathcal{L}$.*

We suppose by induction, there is a decomposition of the perverse cohomology on $(V - V_j) \xrightarrow{k_j} V$ for all degrees i

$$(1.3) \quad {}^p\mathcal{H}^i((Rf_* j_{!*}\mathcal{L})|_{V-V_j}) \xrightarrow{\sim} \bigoplus_{j < l \leq n}^{S_l \subset V_l^*} k_j^* i_{S_l}^* \mathcal{L}_{S_l}^i[l]$$

into a direct sum of intermediate extensions of polarized VHS $\mathcal{L}_{S_l}^i$ (formula 1.2) of weight $a + i - l$ over the strata S_l for all $l > j$. Let

$$\dots \rightarrow {}^p\mathcal{H}^i((i_j)_* Ri_j^! K) \xrightarrow{p\alpha_i} {}^p\mathcal{H}^i(K) \xrightarrow{p\rho_i} {}^p\mathcal{H}^i(Rk_{j*} K|_{V-V_j}) \xrightarrow{p\delta_i} \dots$$

be the long exact sequence of perverse cohomology associated to the triangle

$$(i_j)_* Ri_j^! K \xrightarrow{\alpha} K \xrightarrow{\rho} Rk_{j*} K|_{V-V_j} \xrightarrow{[1]}.$$

i) The restriction to $V - V_{j-1}$ of the short exact sequences of perverse sheaves

$$0 \rightarrow \text{Im } p\alpha_i \rightarrow {}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L}) \xrightarrow{p\rho_i} \text{Im } p\rho_i \rightarrow 0$$

is split over $V - V_{j-1}$:

$$(1.4) \quad \begin{aligned} & \mathcal{P}\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})|_{V-V_{j-1}} = k_{j-1}^* \mathrm{Im}^p \alpha_i \oplus k_{j-1}^* \mathrm{Im}^p \rho_i \\ & k_{j-1}^* \mathrm{Im}^p \alpha_i = \mathrm{Ker}^p \rho_i \xrightarrow{\sim} \bigoplus_{S_j \subset V_j^*} k_{j-1}^* i_{S_j!} \mathcal{L}_{S_j}^i \\ & k_{j-1}^* \mathrm{Im}^p \rho_i = k_{j-1}^* (k_j)_! k_j^* \mathcal{P}\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}) \xrightarrow{\sim} \bigoplus_{\substack{S_l \subset V_l^* \\ j < l \leq n}} k_{j-1}^* i_{S_l!} \mathcal{L}_{S_l}^i[l]. \end{aligned}$$

ii) *Hard Lefschetz*: If we suppose by induction Lefschetz isomorphisms on the open subset $V - V_j$, then η^i extends into an isomorphism over $V - V_{j-1}$

$$\eta^i : \mathcal{P}\mathcal{H}^{-i}(Rf_*j_{!*}\mathcal{L})|_{V-V_{j-1}} \xrightarrow{\sim} \mathcal{P}\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})|_{V-V_{j-1}}$$

Thus we obtain the following statement of the decomposition theorem proved in [BBDG 83] in the geometric case and in [Sa 88] for Hodge modules.

Corollary 1.2. *Let $K = Rf_*j_{!*}\mathcal{L}$ where \mathcal{L} is a shifted polarized variation of Hodge structures (VHS) of weight a , and \mathcal{S} a Thom-Whitney stratification of f adapted to $j_{!*}\mathcal{L}$.*

i) *There exists a decomposition of the perverse cohomology on V in each degree*

$$(1.5) \quad \mathcal{P}\mathcal{H}^i((Rf_*j_{!*}\mathcal{L})) \xrightarrow{\sim} \bigoplus_{l \leq n}^{S_l \subset V_l^*} i_{S_l!} \mathcal{L}_{S_l}^i[l]$$

into a direct sum of intermediate extensions of shifted polarized VHS $\mathcal{L}_{S_l}^i$ (formula 1.2) of weight $a + i - l$ over the strata S_l for all $l \leq n$. Moreover,

$$(1.6) \quad \mathrm{Im} \left(\mathcal{P}\mathcal{H}^i(Rk_{j!}k_j^*K) \rightarrow \mathcal{P}\mathcal{H}^i(K) \right) = k_{j!}k_j^* \mathcal{P}\mathcal{H}^i(K)$$

On a projective variety V , we have an orthogonal decomposition of a polarized HS of weight $a + i + j$

$$(1.7) \quad Gr_i^{p\tau} \mathbb{H}^{i+j}(X, j_{!*}\mathcal{L}) \simeq \mathbb{H}^j(V, \mathcal{P}\mathcal{H}^i((Rf_*j_{!*}\mathcal{L}))) \xrightarrow{\sim} \bigoplus_{l \leq n}^{S_l \subset V_l^*} \mathbb{H}^j(V, i_{S_l!} \mathcal{L}_{S_l}^i[l]).$$

ii) *Hard Lefschetz*: The iterated cup-product η^i is an isomorphism over V

$$\eta^i : \mathcal{P}\mathcal{H}^{-i}(Rf_*j_{!*}\mathcal{L}) \xrightarrow{\sim} \mathcal{P}\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})$$

Moreover, the perverse cohomology $\mathcal{P}\mathcal{H}^{-i}(K)$ is dual to $\mathcal{P}\mathcal{H}^i(K)$, and the duality is compatible with the natural decomposition: the local systems $\mathcal{L}_{S_l}^i$ are polarized variation of Hodge structure (VHS) on S_l , $\mathcal{L}_{S_l}^{-i}$ is dual to $\mathcal{L}_{S_l}^i$, and each η^i induces an isomorphism $\mathcal{L}_{S_l}^{-i} \xrightarrow{\sim} \mathcal{L}_{S_l}^i$ for each i and l .

iii) *There exists an isomorphism in the derived category of constructible sheaves*

$$(1.8) \quad K = \bigoplus_{i \in \mathbb{Z}} \mathcal{P}\mathcal{H}^i(K)[-i].$$

The duality follows from the auto-duality of $j_{!*}\mathcal{L}$ and Verdier duality for the projective morphism f . The polarization follows from the duality and Hard Lefschetz.

We use the local topological triviality property of a morphism along a strata (subsection 3.0.1), to reduce the proof to the case of a zero dimensional isolated strata (proposition 2.2).

Indeed, we cut a strata S of V_j^* by a general normal section N_v to S at a point v , so that the proof is carried on the restriction to N_v at v .

Hard Lefschetz. We check the proof of Hard Lefschetz isomorphism (1.1) in the subsection 2.2 on the terms \mathcal{L}_S^i of the explicit formula (1.2). The decomposition follows from Hard Lefschetz by Deligne's criteria in [De 68]; which completes the proof of the decomposition (formula 1.5 above).

The rest of the article is devoted to the proof.

Remark 1.3. i) The statement is local on V . However, we may suppose V projective if necessary during the proof. In fact we may suppose V affine and then embed V into a projective variety \overline{V} , extend f into $\overline{X} \xrightarrow{\overline{f}} \overline{V}$ and the intermediate extension onto \overline{X} .

ii) In the special case of a fibration by NCD over the strata (definition 1.4 below), we use logarithmic complexes to describe Hodge theory. The use of Hodge theory occurs in the proofs of the three lemma 2.1, 2.4 and 2.5, where the last two follow from the local purity theorem stating conditions on the MHS on cohomology of the link at v with value in $Rf_*j_{!*}\mathcal{L}$ (theorem 1.8 and proposition 1.9 below).

The proof for any projective morphism follows from this special fibration case.

1.2. Fibration over the strata by normal crossing divisors. Hodge theory appears in the proof via a local purity statement (theorem 1.8 below). We reduce the proof to the case where f is a special fibration, hence we only need to develop Hodge theory on the complement of a NCD (sections 4 – 5).

Let $\pi : \tilde{X} \rightarrow X$ be a desingularization such that π and $f \circ \pi$ are special fibrations. The decomposition theorem for f follows from the case of π and $f \circ \pi$ ([De 68] proposition 2.16), in particular we can suppose X smooth. In fact, the reduction to the case where $f : X \rightarrow V$ is a fibration by NCD over the strata of an adequate Thom-Whitney stratification of f in the sense below is possible ([EL 14] Définition 2.1):

Definition 1.4 (topological fibration over the strata by NCD).

i) A morphism $f : X \rightarrow V$ is a topological fibration by NCD over the strata of a stratification $\mathcal{S} = (S_\alpha)$ of V underlying a Thom-Whitney stratification of f , if X is smooth and the spaces $V_l = \cup_{\dim S_\alpha \leq l} S_\alpha$ satisfy the following properties:

- 1) The sub-spaces $X_{V_i} := f^{-1}(V_i)$ are successive sub-NCD embedded in X .
- 2) The restriction of f to $X_S := f^{-1}(S)$ over each strata S of \mathcal{S} is a topological fibration: $f|_S : X_S \rightarrow S$.

ii) The fibration is adapted to a NCD Y in X , or to a local system \mathcal{L} defined on an open algebraic set in the complement of Y in X , if in addition, for each $1 \leq i \leq n$, the union of the sub-spaces $X_{V_i} \cup Y$ are relative NCD over the strata of V , and the intermediate extension of \mathcal{L} is constructible with respect to the strata.

1.2.1. Fibration by NCD over the strata. We explain the terminology. Let X be smooth, and $v \in V_i - V_{i-1}$ (hence V_i is smooth at v); the inverse image of a normal section N_v to V_i at v in general position, is a smooth subvariety $f^{-1}(N_v)$ of X and intersects the NCD X_{V_i} transversally, then $X_{V_i} \cap f^{-1}(N_v) = f^{-1}(v)$ is a NCD in $f^{-1}(N_v)$.

We say for simplification, that $X_{V_i - V_{i-1}} := f^{-1}(V_i - V_{i-1})$ is a relative NCD (eventually empty), and that the stratification \mathcal{S} is admissible for f .

Due to the next proposition, proved in section 3, it is enough to prove the decomposition for a fibration by NCD over the strata.

Proposition 1.5. *Let $f : X \rightarrow V$ be a projective morphism and Y a closed algebraic strict sub-space containing the singularities of X . There exist a diagramm $X \xleftarrow{\pi'} X' \xrightarrow{f'} V$ where X' is a non-singular variety, and stratifications of V , X , and X' such that π' and $f' := f \circ \pi'$ are fibrations by relative NCD over the strata, adapted to $Y' := \pi'^{-1}(Y)$.*

Moreover, π' is a modification of X , and there exist an open subset $U \subset f(X)$ dense in the image $f(X)$ of f such that f is smooth over U , π' induce an isomorphism of $f'^{-1}(U) - (f'^{-1}(U) \cap Y') \xrightarrow{\sim} f^{-1}(U) - (f^{-1}(U) \cap Y)$, and $f'^{-1}(U) \cap Y'$ is a strict relative NCD in the smooth fibers of f eventually empty (called horizontal or strict in the fibers).

1.2.2. *Reduction to the case of zero dimensional strata with NCD fiber.* The notion of relative NCD is used to reduce the proof at a general point v of a strata S , to the case of a point v in a transversal section N_v to S at v , hence to a point v in a zero dimensional strata. In such situation we can use logarithmic complexes ([ICTP 14] section 8.3.3, theorem 8.3.14) since the inverse image of v is a NCD.

1.2.3. *Purity of Intersection cohomology on a singular variety (lemma 2.25).* We illustrate the results of theorem 1.1 and corollary 1.2 in the case of a polarized variation of HS \mathcal{L} on a smooth open subset U of a singular projective variety X with intermediate extension $j_{!*}\mathcal{L}$. The HS on $H^k(X, j_{!*}\mathcal{L})$ is deduced from the compatibility of the perverse filtration with the HS on the intersection cohomology of a desingularization of X .

We consider an adequate desingularization diagram $X' \xrightarrow{\pi} X$ of X and the intermediate extension $j'_{!*}\mathcal{L}$ on X' of \mathcal{L} on $U \subset X'$. The proof is based on the fact that the perverse filtration defined by π is compatible with HS defined on a projective X' .

The polarized VHS on the components \mathcal{L}_S^i of the decomposition on X are defined on the various strata S via the Intersection formula 1.2 and Hodge theory on X' .

We deduce from the compatibility with ${}^p\tau$ the HS on $H^k(X, {}^p\mathcal{H}^0(R\pi_* j'_{!*}\mathcal{L})) \simeq Gr_0^{p\tau} H^k(X', j'_{!*}\mathcal{L})$ for X projective. The decomposition $H^k(X, \bigoplus_{S_l \in \mathcal{S}} \mathcal{L}_{S_l}^0[l])$ is compatible with the HS on $Gr_0^{p\tau} H^k(X, j'_{!*}\mathcal{L})$ which follows from the formula 1.7

1.2.4. *Variation of Hodge structure on \mathcal{L}_S^i over a strata $S \subset V$ (corollary 2.10).* To illustrate the construction of the variation of HS on \mathcal{L}_S^i , we consider an adequate diagram $X' \xrightarrow{\pi} X \xrightarrow{f} V$ with a desingularization X' of X and the intermediate extension $j'_{!*}\mathcal{L}$ on X' , then $j_{!*}\mathcal{L}$ is a component of the decomposition of ${}^p\mathcal{H}^0(R\pi_* j'_{!*}\mathcal{L})$ on X .

At a point v in the zero dimensional strata of f , $\mathbb{H}_{X_v}^i(X, j_{!*}\mathcal{L})$ (resp. $\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$) is a summand of $Gr_0^{p\tau} \mathbb{H}_{X'_v}^i(X', j'_{!*}\mathcal{L})$ (resp. $Gr_0^{p\tau} \mathbb{H}^i(X'_v, j'_{!*}\mathcal{L})$) with induced sub-mixed Hodge structure of weight $\geq a+i$ (resp. $\leq a+i$), hence the image HS on \mathcal{L}_v^i is pure of weight $a+i$. The polarization is the result of Poincaré duality and Hard Lefschetz. At a point v in a strata S of V , we are reduced to the zero dimensional case for the induced stratification on a normal section to S at v .

1.3. **Logarithmic complexes.** We give in section 4 an account of Hodge theory with coefficients in an admissible variation of mixed Hodge structure (\mathcal{L}, W, F) on the complement of a normal crossing divisor Y in a smooth complex projective variety X , using logarithmic complexes.

Moreover, we enlarge the theory to logarithmic complexes associated with a NC subdivisor $Z \subset Y$, which is necessary in the inductive step (subsection ??).

Let $\tilde{\mathcal{L}}$ be a variation of mixed Hodge structure, $\mathcal{L} := \tilde{\mathcal{L}}[m]$ the associated perverse sheaf on $X - Y$ with the conventional shifted degree, $j := (X - Y) \rightarrow X$, $j_Z := (X - Z) \rightarrow X$, and $i_Z : Z \rightarrow X$.

We describe logarithmic complexes with weight and Hodge filtrations, obtained for various functors applied to the intermediate extension of the admissible variation of MHS \mathcal{L} on $X - Y$ as $j_{Z!}(j_{!*}\mathcal{L})|_{X-Z}$, $Rj_{Z*}(j_{!*}\mathcal{L})|_{X-Z}$, $i_Z^*Rj_{Z*}(j_{!*}\mathcal{L})|_{X-Z}$, $i_Z^!j_{!*}\mathcal{L}$ and $i_Z^*j_{!*}\mathcal{L}$, defining thus a MHS on the corresponding hypercohomology groups.

(\star) We refer to such complexes as *bifiltered logarithmic complexes*.

The starting point is the realization of the direct image $Rj_{Z*}(j_{!*}\mathcal{L})|_{X-Z}$ as a sub-complex $IC^*\mathcal{L}(\text{Log}Z)$ of the logarithmic complex $\Omega^*\mathcal{L} := \Omega_X^*(\text{Log}Y) \otimes \mathcal{L}_X$ (denoted as $\Omega^*(\mathcal{L}, Z)$ in [ICTP 14] definition (8.3.31)). We refer back to a local study by Kashiwara [Ka 86] to construct the weight filtration with a property of local decomposition.

We refer to the last sections 4 and 5 for the details of the constructions. Indeed, the details are not needed earlier. Only the existence of the weight and the compatibility with the perverse filtrations are needed in the earlier sections, as well the following general properties:

- 1) the statement and the proof of local purity at the end of section 2.
- 2) the condition $w \leq a + i$ on the weight of $\mathbb{H}^i(Z, j_{!*}\mathcal{L})$ for a pure \mathcal{L} of weight a .
- 3) the condition $w \geq a + i$ on the weight of $\mathbb{H}_Z^i(X, j_{!*}\mathcal{L})$ for a pure \mathcal{L} of weight a .

1.3.1. *Compatibility of Hodge structure with perverse filtration.* Let $f : X \rightarrow V$ be a morphism, and K a complex on V with constructible cohomology. The topological middle perversity truncations on K on V ([BBDG 83] section 2 and prop. 2.1.17) define an increasing *perverse filtration* ${}^p\tau$ on K , from which we deduce for each closed sub-variety W of V , an increasing filtration ${}^p\tau$ on the hypercohomology:

$$(1.9) \quad {}^p\tau_i \mathbb{H}^k(V - W, K) := \text{Im} \{ \mathbb{H}^k(V - W, {}^p\tau_i K) \rightarrow \mathbb{H}^k(V - W, K) \}.$$

Let $Z := f^{-1}W := X_W \subset X$, $j_Z : X - Z \rightarrow X$ and $K := Rf_*R(j_Z)_*j_Z^*j_{!*}\mathcal{L}$. The *perverse filtration* ${}^p\tau$ on the hypercohomology groups $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L})$ is deduced by the isomorphism $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L}) = \mathbb{H}^k(V - W, Rf_*j_{!*}\mathcal{L})$

$$(1.10) \quad {}^p\tau_i \mathbb{H}^k(X - Z, j_{!*}\mathcal{L}) := {}^p\tau_i \mathbb{H}^k(V - W, Rf_*j_{!*}\mathcal{L})$$

Similarly, we define functorially ${}^p\tau_i$ on $\mathbb{H}_Z^k(X, j_{!*}\mathcal{L})$, $\mathbb{H}_c^k(X - Z, j_{!*}\mathcal{L})$ and $\mathbb{H}^k(Z, j_{!*}\mathcal{L})$, after ${}^p\tau$ on K .

When f is projective, we check that the filtration ${}^p\tau_i$ is a filtration by sub-MHS on the hypercohomology of $X - Z$ ([EY 14], theorems 1.1 and 3.2) using a result of [CaMi 10]. We say simply: the perverse filtration ${}^p\tau$ is compatible with the MHS on the hypercohomology of $X - Z$.

Proposition 1.6. *Let $f : X \rightarrow V$ be a projective morphism on X smooth, $j : X - Y \rightarrow X$ the open embedding of the complement of a normal crossing divisor Y in X , \mathcal{L} an admissible variation of MHS on $X - Y$ shifted by $\dim X$, $W \subset V$ a closed algebraic subset of V s.t. $X_W := f^{-1}(W)$ is a sub-normal crossing divisor of Y . Then, the perverse filtration ${}^p\tau$ on the cohomology $\mathbb{H}^*(X - X_W, j_{!*}\mathcal{L})$ (resp. $\mathbb{H}_{X_W}^*(X, j_{!*}\mathcal{L})$ and by duality $\mathbb{H}_c^*(X - X_W, j_{!*}\mathcal{L})$, $\mathbb{H}^*(X_W, j_{!*}\mathcal{L})$), is a filtration by sub-MHS.*

The proof is carried in terms of logarithmic complexes in subsection (5.3.2).

1.3.2. *Mixed Hodge structure (MHS) on the Link.* The basic criteria for a complex K on V to underly a structure of mixed Hodge complex (MHC), is given locally at a point $v \in V$ in terms of the Link at v (proposition 1.7 below). The criteria is translated into a property of the tubular neighborhood of $X_v := f^{-1}(v)$, in terms

of $X_{B_v}^* := X_{B_v} - X_v$, the inverse image of a small ball B_v with center v in V minus the central fiber X_v .

Let $j_{X_v} : (X - X_v) \rightarrow X$, $i_{X_v} : X_v \rightarrow X$, $k_v : (V - v) \rightarrow V$ and $i_v : v \rightarrow V$. We apply the functor $i_v^* R(k_v)_* k_v^*$ functorially to the filtration ${}^p\tau$ on $Rf_* j_{!*} \mathcal{L}$ to define the perverse filtration ${}^p\tau_i$ on $R\Gamma(X_v, i_{X_v}^* R(j_{X_v})_* j_{X_v}^* j_{!*} \mathcal{L})$ via the isomorphism $i_{X_v}^* R(j_{X_v})_* j_{X_v}^* j_{!*} \mathcal{L} \xrightarrow{\sim} i_v^* R(k_v)_* k_v^* Rf_* j_{!*} \mathcal{L}$.

The perverse filtration is considered on the hypercohomology $\mathbb{H}^r(X_{B_v}^*, j_{!*} \mathcal{L})$, which coincide with $R\Gamma(X_v, i_{X_v}^* R(j_{X_v})_* j_{X_v}^* j_{!*} \mathcal{L})$ if the radius of B_v is small enough. Thus, it may look a less abstract object.

Proposition 1.7 (MHS on the Link). *Let v be a point in V with fibre $X_v := f^{-1}(v)$. We suppose X_v and $X_v \cup Y$ are NCD in X and \mathcal{L} defined on $X - Y$; then, for a ball B_v with center v small enough, the perverse filtration on $\mathbb{H}^r(X_{B_v}^*, j_{!*} \mathcal{L}) \xrightarrow{\sim} \mathbb{H}^r((X_v, i_{X_v}^* R(j_{X_v})_* j_{X_v}^* j_{!*} \mathcal{L}))$ is compatible with the MHS (that is a filtration sub-MHS).*

The proof in subsection 5.3.3 depends on a local version of a result in [CaMi 10] easy to check. Once the above structure is defined, we can express the notion of local purity in positive characteristic and give a meaning to the purity theorem [DeG 81].

1.4. Deligne-Gabber's local purity. This basic result is stated in positive characteristic in [DeG 81] as follows:

Let K be a pure complex of weight a on an algebraic variety V and B_v a henselization of V at a point v in the zero dimensional strata of V . The weight w of the cohomology $\mathbb{H}^i(B_v - \{v\}, K)$, satisfy:

$$w \leq a + i \text{ if } i \leq -1 \text{ and } w > a + i \text{ if } i \geq 0.$$

According to Deligne's dictionary between the purity in positive characteristic and Hodge theory, a pure complex corresponds in the transcendental case to an intermediate extension $k_{!*} \mathcal{L}$ of a polarized VHS $\tilde{\mathcal{L}}$ on a smooth algebraic open subset $k : V^* \hookrightarrow V$ embedded in a complex algebraic variety V .

To state the above result, the first task is to construct a MHS on the hypercohomology $\mathbb{H}^*(B_v - v, k_{!*} \mathcal{L})$ where B_v is a small ball with center $v \in V$, with coefficients in an intermediate extension of $\mathcal{L} := \tilde{\mathcal{L}}[n]$ where n is the dimension of V , that is the VHS shifted as a complex by n to the left, or more intrinsically on $H^*(i_v^* R(k_v)_* k_v^* k_{!*} \mathcal{L})$. Once we have such MHS we can state the conditions on the weight as follows:

Theorem 1.8 (local purity). *Let $\tilde{\mathcal{L}}$ be a polarized variation of Hodge structures of weight b on the smooth n -dimensional open subset V^* of an algebraic variety V , $k : V^* \rightarrow V$ the embedding, and let v be a point in the zero dimensional strata of V , B_v a small ball with center v and $a = b + n$.*

The weight w of the space $H^i(i_v^ R(k_v)_* k_v^* k_{!*} \mathcal{L})$, isomorphic to the Intersection cohomology $\mathbb{H}^i(B_v - \{v\}, k_{!*} \mathcal{L})$ of the Link at v , satisfy the relations:*

$$w \leq a + i \text{ if } i \leq -1 \text{ and } w > a + i \text{ if } i \geq 0$$

(the intermediate extension $k_{!} \mathcal{L}$, after the shift $\mathcal{L} := \tilde{\mathcal{L}}[n]$, is a complex of weight $a = b + n$).*

1.4.1. *Equivalent statement.* The MHS is defined in fact along the NCD X_v in X , so we prove equivalent conditions on the weight above on X as follows

Proposition 1.9 (Semi purity). *Given a (shifted) polarized variation of Hodge structures $j_{!*}\mathcal{L}$ on X of weight a , the weights of the mixed Hodge structure on the graded-cohomology spaces*

$$Gr_i^{pr} \mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L})$$

satisfy the inequalities: $w \leq a + r$ if $r - i \leq -1$, $w > a + r$ if $r - i \geq 0$.

The crucial case of the proof is treated in section 2.4. The original proof in positive characteristic may be adapted to the transcendental case, but we give a relatively simple new proof based on the inductive hypothesis which assumes Hard Lefschetz theorem on $B_v - v$. In fact, both the proposition and Hard Lefschetz theorem (1.1) are proved simultaneously by induction on the decreasing dimension of the strata of V .

Let \mathcal{S} be a stratification of V , and $V_j := \cup_{S \in \mathcal{S}: \dim S \leq j} S$; if we assume by an inductive argument the decomposition theorem established on $V - V_j$, then we deduce first the local purity theorem at a point v of a strata $S \subset V_j$ of dimension j . Let N_v be a normal section to S at v in general position, the subvariety $f^{-1}(N_v)$ is smooth in X (we can suppose X smooth) and intersects the NCD X_S transversally. On N_v the point v is isolated in the lowest strata of dimension zero; hence, we reduce the proof of local purity to the case of an isolated point. On its turn, the semi-purity at v is used to extend the proof of the decomposition from $V - V_j$ to $(V - V_j) \cup S$, as well Hard Lefschetz theorem.

Thus, we obtain the decomposition on $V - V_{j-1}$, from which we repeat the argument of local purity at points of V_{j-1} . Finally, we obtain by induction, a simultaneous proof of the local purity and the decomposition at the same time.

2. PROOF OF THE DECOMPOSITION THEOREM

We present here a proof of theorem 1.1 and corollary 1.2 for projective morphisms, based on a reduction to a stratification by NCD over the strata (definition 1.4, proposition 1.5, section 3). Three lemma (2.1, 2.4, 2.5) below use Hodge theory for which we refer to sections 4 – 5.

2.1. Proof of theorem 1.1. Let $f : X \rightarrow V$ be a projective morphism of complex algebraic varieties, \mathcal{S} a Thom-Whitney stratification adapted to $j_{!*}\mathcal{L}$ on X and $K := Rf_*j_{!*}\mathcal{L}$. Set V_k the union of all strata of V of dimension $\leq k$.

2.1.1. *The big strata.* By construction, the restriction of f to any strata of V is a topological fibration. The union of the big strata of V form an open smooth set U of dimension $n := \dim V$ as we suppose the singularities of V contained in V_{n-1} . The restriction to U of the local cohomology sheaves $\mathcal{H}^i(K)$ are locally constant, hence they coincide with the perverse cohomology sheaves: ${}^p\mathcal{H}^i(K) = \mathcal{H}^i(K)$.

Lemma 2.1 (Initial step). *The relative Hard Lefschetz theorem apply over the big strata U of V .*

The proof follows from Hodge theory in section 4.6, where we suppose \mathcal{L} defined on the complement of Y_U an horizontal relative NCD in X_U . Then, $R^i f_* j_{!*}\mathcal{L}$ underlies a VHS such that at each point $v \in U$, the fiber $(R^i f_* j_{!*}\mathcal{L})_v \simeq \mathbb{H}^i(X_v, j_{!*}\mathcal{L})$ is isomorphic to the intersection cohomology of the restriction of \mathcal{L} to U_v the fiber

of U at v with its HS. The family $R^i f_{*j!} \mathcal{L}$ for various degrees satisfy the relative Hard Lefschetz isomorphisms. Hence, the decomposition theorem for the smooth induced morphism $f_U : X_U \rightarrow U$ follows from the results of Deligne [De 68].

2.1.2. *The inductive step.* The decomposition theorem is proved by descending induction on the dimension of the strata. We suppose the *decomposition* proved over the open subset $U_j := V - V_j$ for some $j < n$, then we extend the decomposition to U_{j-1} along $V_j - V_{j-1}$, that is across the union of smooth strata S of dimension j . Let N_v denote a general normal section to the strata S at a point $v \in S$, and $f_{N_v} : X_{N_v} \rightarrow N_v$ the restriction of f .

By construction of \mathcal{S} , f is locally trivial along S : let B_v be a small ball on S with center v , then f is locally homeomorphic to $f_{N_v} \times Id_{|B_v} : X_{N_v} \times B_v \rightarrow N_v \times B_v$ (subsection 3.0.1), and the proof may be reduced to the case of $f_{N_v} : X_{N_v} \rightarrow N_v$ with the induced stratification on N_v ; that is the case of an isolated point $v \in N_v$ with $j = 0$.

2.1.3. *The case of zero dimensional strata V_0 .* We prove now the case of the zero dimensional strata in theorem 1.1. Since the proof is local, we may suppose $V_0 = v$ reduced to one point v . Let $k_v : (V - v) \rightarrow V$ and $i_v : v \rightarrow V$, \mathcal{L} of weight a on a Zariski open subset of X :

Proposition 2.2. *Let $K = Rf_{*j!} \mathcal{L}$ on an algebraic variety V of $\dim n$, \mathcal{S} a Whitney stratification of V underlying a Thom stratification of f , $v \in V_0$ a point in the strata of dimension 0, and suppose the following conditions on $V - v$ satisfied:*

1) *There exists for each $i \in \mathbb{Z}$, a decomposition over the open subset $V - v$ into a direct sum of intermediate extensions of shifted local systems $\mathcal{L}_{S_l}^i[l]$ (formula 1.2) on the various strata $S_l \subset V - v$ of dimension $l \leq n$:*

$${}^p\mathcal{H}^i(K)|_{V-v} := k_v^* {}^p\mathcal{H}^i(K)|_{V-v} \xrightarrow{\sim} \bigoplus_{S_l \subset V-v} k_v^* i_{S_l!} \mathcal{L}_{S_l}^i[l]$$

- 2) *Hard Lefschetz: ${}^p\mathcal{H}^{-i}(K)|_{V-v} \xrightarrow{\eta^i} {}^p\mathcal{H}^i(K)|_{V-v}$ is an isomorphism on $V - v$.*
 3) *The local purity theorem apply to K at v (theorem 1.8, subsection 2.3).*

Then the decomposition as well Hard Lefschetz extend over v .

Precisely, let $\rho : K \rightarrow Rk_{v} K|_{V-v}$ and $\alpha : i_{v,*} Ri_v^! K \rightarrow K$, and consider the long exact sequence of perverse cohomology on V*

$$(2.1) \quad \begin{array}{ccccccc} & & \xrightarrow{p\delta^{i-1}} & {}^p\mathcal{H}^i(i_{v,*} Ri_v^! K) & \xrightarrow{p\alpha^i} & {}^p\mathcal{H}^i(K) & \xrightarrow{p\rho^i} & {}^p\mathcal{H}^i(Rk_{v*} K|_{V-v}) & \xrightarrow{p\delta^i} & 0 \end{array}$$

i) The perverse image of $p\rho^i$ is isomorphic to:

$$Im\ p\rho^i \xrightarrow{\sim} k_{v!} k_v^* {}^p\mathcal{H}^i(K) \xrightarrow{\sim} \bigoplus_{S_l} i_{S_l!} \mathcal{L}_{S_l}^i[l].$$

ii) We have ${}^p\mathcal{H}^i(i_{v,} Ri_v^! K) \xrightarrow{\sim} i_{v,*} H^i(i_v^! K)$. Let $\mathcal{L}_v^i := Im(H^i(i_v^! K) \rightarrow H^i(i_v^* K))$, then $Ker\ p\rho^i = Im\ p\alpha^i \simeq i_{v,*} \mathcal{L}_v^i$ is a polarized Hodge structure of weight $a + i$.*

iii) The perverse cohomology decomposes on V

$$(2.2) \quad {}^p\mathcal{H}^i(K) \xrightarrow{\sim} Ker\ p\rho^i \oplus Im\ p\rho^i.$$

In particular, we can extract from the sequence (2.1) an exact sub-sequence

$$(2.3) \quad 0 \rightarrow \bigoplus_{S_l \subset V-v, i-1-j \geq 0} R^{i-1-j} k_{v*} k_v^* (i_{S_l!} \mathcal{L}^j[l]) \xrightarrow{p\delta^{i-1}} \mathbb{H}_v^i(X, K) \xrightarrow{p\alpha^i} i_{v,*} \mathcal{L}_v^i \rightarrow 0$$

iv) Hard Lefschetz: ${}^p\mathcal{H}^{-i}(K) \xrightarrow{\eta^i} {}^p\mathcal{H}^i(K)$ is an isomorphism on V .

The proof occupy the rest of subsections 2.1 and 2.2.

In fact the third condition on local purity follows from the first condition on the decomposition on $V - v$ (see subsection 2.3); its proof is just postponed to ease the exposition. The statement is given in full generality but the proof is given first for fibrations by NCD from which case it is deduced in general.

2.1.4. *Perverse cohomology of $Rk_{v*}K|_{V-v}$ and $Rk_{v!}K|_{V-v}$.* In order to compute successively: $Im^p\rho_i$, $Im^p\alpha_i$ and to prove the splitting of ${}^p\mathcal{H}^i(K)$, we rely on the next calculus of the perverse cohomology of $Rk_{v*}(K|_{V-\{v\}})$ under the hypothesis of the decomposition on $V - v$.

Lemma 2.3. *Let $i : S \rightarrow V$ be a fixed strata of V of dim. l , $\mathcal{L}^j[l]$ a family of local systems on S shifted by l for $j \in \mathbb{Z}$, and $K' := \bigoplus_j i_{1*}(\mathcal{L}^j[l])[-j]$ the direct sum, hence ${}^p\mathcal{H}^i(K') = i_{1*}\mathcal{L}^i[l]$. Let v be a point in the closure of S and $k_v : V - v \rightarrow V$.*

i) ${}^p\mathcal{H}^i(Rk_{v*}k_v^*i_{1*}\mathcal{L}^j[l]) = R^i k_{v*}(k_v^*i_{1*}\mathcal{L}^j[l])$ for $i > 0$, vanish for $i < 0$, and we have a short exact sequence of perverse sheaves for $i = 0$

$$0 \rightarrow i_{1*}(\mathcal{L}^j[l]) \rightarrow {}^p\mathcal{H}^0(Rk_{v*}k_v^*i_{1*}(\mathcal{L}^j[l])) \xrightarrow{h} R^0 k_{v*}(k_v^*i_{1*}\mathcal{L}^j[l]) \rightarrow 0$$

in particular $\underline{H}^0({}^p\mathcal{H}^0(Rk_{v*}k_v^*i_{1*}\mathcal{L}^j[l])) \xrightarrow{\sim} R^0 k_{v*}(k_v^*i_{1*}\mathcal{L}^j[l])$, and a morphism of perverse sheaves $\varphi : \mathcal{P} \rightarrow {}^p\mathcal{H}^0(Rk_{v*}k_v^*i_{1*}\mathcal{L}^j[l])$ factors through $i_{1*}\mathcal{L}^j[l]$ if and only if $h \circ \varphi = 0$.

ii) We deduce ${}^p\mathcal{H}^i(Rk_{v*}k_v^*K') = \bigoplus_{j \leq i} {}^p\mathcal{H}^{i-j}(Rk_{v*}k_v^*i_{1*}\mathcal{L}^j[l]) = {}^p\mathcal{H}^i(Rk_{v*}k_v^{*p}\tau_{\leq i}K')$ and short exact sequences of perverse sheaves for all i

$$0 \rightarrow i_{1*}(\mathcal{L}^i[l]) \rightarrow {}^p\mathcal{H}^i(Rk_{v*}k_v^*K') \xrightarrow{h} \bigoplus_{j \leq i} R^{i-j} k_{v*}(k_v^*i_{1*}\mathcal{L}^j[l]) \rightarrow 0$$

where the last term is a sum of vector spaces supported on v in degree zero.

There are isomorphisms of sheaf cohomology concentrated on v in degree 0

$$\underline{H}^0({}^p\mathcal{H}^i(Rk_{v*}k_v^*K')) \xrightarrow{\sim} \bigoplus_{j \leq i} R^i k_{v*}(k_v^*i_{1*}\mathcal{L}^j[l-j])$$

and a morphism of perverse sheaves $\varphi : \mathcal{P} \rightarrow {}^p\mathcal{H}^i(Rk_{v*}k_v^*K')$ factors through $i_{1*}(\mathcal{L}^i[l])$ if and only if $h \circ \varphi = 0$.

iii) Dually, ${}^p\mathcal{H}^i(Rk_{v!}k_v^*K') = \bigoplus_{j \geq i} {}^p\mathcal{H}^{i-j}(Rk_{v!}k_v^*i_{1*}\mathcal{L}^j[l])$ and we have short exact sequences

$$0 \rightarrow \bigoplus_{j \geq i} i_{v*} H^{i-j-1}(i_v^*i_{1*}\mathcal{L}_{S_l}^j[l]) \xrightarrow{h'} {}^p\mathcal{H}^i(Rk_{v!}k_v^*K') \rightarrow i_{1*}\mathcal{L}^i[l] \rightarrow 0$$

where the first term is supported on v in degree zero.

A morphism φ defined on ${}^p\mathcal{H}^i(Rk_{v!}k_v^*K')$ factors through $i_{1*}(\mathcal{L}^i[l])$ if and only if $\varphi \circ h' = 0$.

Proof. In the case of a unique local system $K' = i_{1*}\mathcal{L}'[l]$, we have by definition $i_{1*}\mathcal{L}'[l] = \tau_{\leq -1}Rk_{v*}k_v^*i_{1*}\mathcal{L}'[l]$; moreover $\tau_{\geq 0}Rk_{v*}k_v^*i_{1*}\mathcal{L}'[l]$ is supported by v .

Then we have the following exact sequence and isomorphisms

$$0 \rightarrow i_{1*}\mathcal{L}'[l] \rightarrow {}^p\mathcal{H}^0(Rk_{v*}k_v^*i_{S_l!}\mathcal{L}'[l]) \rightarrow i_{v,*}R^0 k_{v*}k_v^*(i_{1*}\mathcal{L}'[l]) \rightarrow 0$$

$${}^p\mathcal{H}^i(Rk_{v*}k_v^*i_{1*}\mathcal{L}'[l]) = 0 \text{ for } i < 0$$

$${}^p\mathcal{H}^i(Rk_{v*}k_v^*i_{1*}\mathcal{L}'[l]) = R^i k_{v*}k_v^*i_{1*}(\mathcal{L}'[l]) \text{ for } i > 0$$

$$\underline{H}^0({}^p\mathcal{H}^0(Rk_{v*}k_v^*i_{1*}\mathcal{L}'[l])) \xrightarrow{\sim} R^0 k_{v*}k_v^*(i_{1*}\mathcal{L}'[l])$$

respectively, we have dual statements for $Rk_{v!}k_v^*$

$$\begin{aligned} 0 &\rightarrow i_{v*}H^{-1}(i_v^*(i_{!*}\mathcal{L}'[l]) \rightarrow {}^p\mathcal{H}^0(Rk_{v!}k_v^*i_{S_1!}\mathcal{L}'[l]) \rightarrow i_{!*}\mathcal{L}'[l] \rightarrow 0 \\ &{}^p\mathcal{H}^i(Rk_{v!}k_v^*i_{!*}\mathcal{L}'[l]) = 0 \text{ for } i > 0 \\ i_{v*}H^{i-1}(i_v^*i_{!*}\mathcal{L}'[l]) &\xrightarrow{\sim} {}^p\mathcal{H}^i(Rk_{v!}k_v^*i_{!*}\mathcal{L}'[l]) \text{ for } i < 0 \\ i_{v*}H^{-1}(i_v^*(i_{!*}\mathcal{L}'[l])) &\xrightarrow{\sim} \underline{H}^0({}^p\mathcal{H}^0(Rk_{v!}k_v^*i_{!*}\mathcal{L}'[l])) \end{aligned}$$

If we set $\mathcal{L}' = \mathcal{L}^j$ and since for any complex C , ${}^p\mathcal{H}^i(C[-j]) = {}^p\mathcal{H}^{i-j}(C)$, we deduce the contribution of the component $\mathcal{L}^j[l]$ in the sum in K' :

$$\begin{aligned} {}^p\mathcal{H}^i(Rk_{v*}k_v^*i_{!*}\mathcal{L}^j[l-j]) &= R^i k_{v*}(k_v^*i_{!*}\mathcal{L}^j[l-j]) \text{ for } i > j, \\ {}^p\mathcal{H}^i(Rk_{v*}k_v^*i_{!*}\mathcal{L}^j[l-j]) &= 0 \text{ for } i < j, \text{ and the dual statements for } Rk_{v!}k_v^*. \end{aligned}$$

Moreover, we deduce a short exact sequence

$$\begin{aligned} 0 &\rightarrow i_{!*}\mathcal{L}^j[l] \rightarrow {}^p\mathcal{H}^j(Rk_{v*}k_v^*i_{!*}\mathcal{L}^j[l-j]) \xrightarrow{h} R^j k_{v*}k_v^*i_{!*}\mathcal{L}^j[l-j] \rightarrow 0 \\ \text{and } \underline{H}^0({}^p\mathcal{H}^j(Rk_{v*}k_v^*i_{!*}\mathcal{L}^j[l-j])) &\xrightarrow{\sim} R^j k_{v*}k_v^*i_{!*}\mathcal{L}^j[l-j]. \text{ Dually} \\ 0 &\rightarrow i_{v*}H^{j-1}(i_v^*i_{!*}\mathcal{L}^j[l-j]) \xrightarrow{h'} {}^p\mathcal{H}^j(Rk_{v!}k_v^*i_{!*}\mathcal{L}^j[l-j]) \rightarrow i_{!*}\mathcal{L}^j[l] \rightarrow 0 \\ \text{and } i_{v*}H^{j-1}(i_v^*i_{!*}\mathcal{L}^j[l-j]) &\xrightarrow{\sim} \underline{H}^0({}^p\mathcal{H}^j(Rk_{v!}k_v^*i_{!*}\mathcal{L}^j[l-j])). \quad \square \end{aligned}$$

Proof of the proposition 2.2. Although the statement is for any projective f , the proof is given first in the case of fibrations by NCD. By the above lemma (2.3, ii) we have

$$k_{v!}k_v^*{}^p\mathcal{H}^i(K) = \bigoplus_{S_1 \subset V-v} i_{S_1!}\mathcal{L}_{S_1}^i[l] \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*{}^p\mathcal{H}^i(K)[-i]) \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*K)$$

i) We prove $k_{v!}k_v^*{}^p\mathcal{H}^i(K) \subset \text{Im } {}^p\rho^i \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*{}^p\mathcal{H}^i(K)[-i])$:

We deduce from the canonical morphism $\phi : Rk_{v!} \rightarrow Rk_{v*}$ and the decomposition hypothesis $k_v^*K = \bigoplus_j k_v^*{}^p\mathcal{H}^j(K)[-j]$ on $V-v$, the first equality below

$$\begin{aligned} &\text{Im} \left({}^p\mathcal{H}^i(Rk_{v!}k_v^*K) \xrightarrow{p\phi_i} {}^p\mathcal{H}^i(Rk_{v*}k_v^*K) \right) = \\ &\bigoplus_j \text{Im} \left({}^p\mathcal{H}^i(Rk_{v!}{}^p\mathcal{H}^j(k_v^*K)[-j]) \xrightarrow{p\phi_i^j} {}^p\mathcal{H}^i(Rk_{v*}{}^p\mathcal{H}^j(k_v^*K)[-j]) \right) = \\ &\text{Im} \left({}^p\mathcal{H}^0(Rk_{v!}k_v^*{}^p\mathcal{H}^i(K)) \xrightarrow{p\phi_i^i} {}^p\mathcal{H}^0(Rk_{v*}k_v^*{}^p\mathcal{H}^i(K)) \right) = k_{v!}k_v^*{}^p\mathcal{H}^i(K) \end{aligned}$$

In the direct sum over j , only the term for $j = i$ is significant since for $j < i$ ${}^p\mathcal{H}^i(Rk_{v!}k_v^*i_{S_1!}\mathcal{L}_{S_1}^j[l-j]) = 0$, and ${}^p\mathcal{H}^i(Rk_{v*}k_v^*i_{S_1!}\mathcal{L}_{S_1}^j[l-j]) = 0$ for $j > i$; hence the second equality follows. The last equality follows from the construction of $k_{v!}k_v^*{}^p\mathcal{H}^i(K)$ in ([BBDG 83], subsection 2.1.7) as image of the morphism $p\phi_i^i$.

Since $p\phi_i^i$ factors through ${}^p\mathcal{H}^i(K)$, its image $k_{v!}k_v^*{}^p\mathcal{H}^i(K)$ is contained in the image of ${}^p\mathcal{H}^i(K)$. Since $k_v^*{}^p\mathcal{H}^i(K)$ is a component of $k_v^*K[i]$, we remark that ${}^p\mathcal{H}^0(Rk_{v*}k_v^*{}^p\mathcal{H}^i(K))$ is a component of ${}^p\mathcal{H}^0(Rk_{v*}k_v^*K[i]) = {}^p\mathcal{H}^i(Rk_{v*}k_v^*K)$.

The existing morphism on $V-v$: $k_v^*{}^p\mathcal{H}^i(K) \rightarrow k_v^*K[i]$ extends by Rk_{v*} and $Rk_{v!}$ such that $p\phi_i^i$ is compatible with the factorization

$${}^p\mathcal{H}^0(Rk_{v!}k_v^*K[i]) \rightarrow {}^p\mathcal{H}^0(K[i]) \xrightarrow{{}^p\mathcal{H}^0(\rho^i)} {}^p\mathcal{H}^0(Rk_{v*}k_v^*K[i])$$

Hence $\text{Im } {}^p\rho^i$ in the formula 2.1 contains $k_{v!}k_v^*{}^p\mathcal{H}^i(K) = \bigoplus_{S_1 \subset V-v} i_{S_1!}\mathcal{L}_{S_1}^i[l]$.

To show that ${}^p\rho^i$ factors through $\bigoplus_{S_1 \subset V-v} i_{S_1!}\mathcal{L}_{S_1}^i[l] \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*K)$, we prove, in view of lemma (2.3, ii), that the induced morphism $\rho_0^i := H^0(i_v^*{}^p\rho^i)$ vanish in degree 0.

Lemma 2.4. *The morphism ρ_0^i induced by $i_v^* p\rho^i$ on the cohomology in degree zero vanish*

$$\rho_0^i : H^0(i_v^* p\mathcal{H}^i(K)) \longrightarrow H^0(i_v^* p\mathcal{H}^i(Rk_{v*} k_v^* K))$$

This is a basic argument where Hodge theory is needed in the proof. We assume that f is a fibration by NCD over the strata (section 3), in which case we refer to section 4 for Hodge theory and to (section 1.8 below), for the semi-purity theorem needed here.

First, we remark the following geometrical interpretation in terms of a small ball $B_v \subset V$ and its inverse $B_{X_v} \subset X$. By lemma (2.3 i), we have

$$H^0(i_v^* p\mathcal{H}^0(Rk_{v*} k_v^* p\mathcal{H}^i(K))) \simeq H^0(\oplus_{S_l \subset V-v} k_v^* i_{S_l!} \mathcal{L}_l^i[l]) \xrightarrow{\sim} \mathbb{H}^0(B_v - v, p\mathcal{H}^i(K))$$

Second we deduce from the triangle:

$$p_{\tau < 0}(K[i]) \rightarrow p_{\tau \leq 0}(K[i]) \rightarrow p\mathcal{H}^i(K) \xrightarrow{[1]}$$

an exact sequence:

$$\mathbb{H}^0(B_v, p_{\tau < 0}(K[i])) \rightarrow \mathbb{H}^0(B_v, p_{\tau \leq 0}(K[i])) \rightarrow \mathbb{H}^0(B_v, p\mathcal{H}^i(K)) \rightarrow \mathbb{H}^1(B_v, p_{\tau < 0}(K[i]))$$

where $\mathbb{H}^r(B_v, p_{\tau < 0}(K[i])) = 0$ for $r = 0, 1$, hence

$$\mathbb{H}^0(B_v, p_{\tau \leq i}(K)) \simeq \mathbb{H}^0(B_v, p\mathcal{H}^i(K)) \simeq H^0(i_v^* p\mathcal{H}^i(K))$$

which enables the factorization of ρ_0^i as a morphism

$$H^0(i_v^* p_{\tau \leq i} K) = \mathbb{H}^0(B_v, p_{\tau \leq i} K) \rightarrow \mathbb{H}^0(B_v - v, p_{\tau \leq i} K) \longrightarrow \mathbb{H}^0(B_v - v, p\mathcal{H}^i(K)).$$

In other terms, we have an interpretation of ρ_0^i as a composition morphism via ρ'

$$\mathbb{H}^0(B_v, p_{\tau \leq i} K) \xrightarrow{\rho'} p_{\tau \leq i} \mathbb{H}^i(B_{X_v} - X_v, j_{!*} \mathcal{L}) \longrightarrow Gr_i^{p\tau} \mathbb{H}^i(B_{X_v} - X_v, j_{!*} \mathcal{L}).$$

and through $\mathbb{H}^i(X_v, j_{!*} \mathcal{L})$ as follows:

$$\mathbb{H}^0(B_v, p_{\tau \leq i} K) \rightarrow p_{\tau \leq i} \mathbb{H}^i(X_v, j_{!*} \mathcal{L}) \rightarrow p_{\tau \leq i} \mathbb{H}^i(B_{X_v} - X_v, j_{!*} \mathcal{L})$$

where the space $\mathbb{H}^i(X_v, j_{!*} \mathcal{L})$ has weight $w \leq a + i$ (corollary 5.9). By assumption the semi purity apply to $Gr_i^{p\tau} \mathbb{H}^i(B_{X_v} - X_v, j_{!*} \mathcal{L})$ of weight $w > a + i$ (proposition 1.9), hence we deduce $\rho_0^i = 0$.

ii) *Proof of $Im p\alpha^i \xrightarrow{\sim} i_{v,*} \mathcal{L}_v^i$.* We deduce from lemma (2.3 ii) as above:

$$H^0(p\mathcal{H}^{i-1}(Rk_{v*} K|_{V-\{v\}})) \xrightarrow{\sim} R^{i-1} k_{v*} k_v^* (p_{\tau \leq i-1} K).$$

We deduce from the formula (2.1) and the computation of $p\rho^{i-1}$ above, a sub-exact sequence of perverse sheaves :

$$0 \rightarrow R^{i-1} k_{v*} k_v^* (p_{\tau \leq i-1} K) \xrightarrow{p\delta^{i-1}} \mathbb{H}_{X_v}^i(X, j_{!*} \mathcal{L}) \xrightarrow{p\alpha^i} p\mathcal{H}^i(K) \xrightarrow{p\rho^i}$$

where the perverse cohomology of $i_v^1(K)$ coincides with its cohomology $p\mathcal{H}^i(i_v^1 K) = H_{X_v}^i(X, j_{!*} \mathcal{L})$ as a complex of vector spaces, hence $Im p\alpha^i = \text{Coker } p\delta^{i-1}$.

On the other hand, by definition \mathcal{L}_v^i is the image of I^i in the exact sequence

$$(2.4) \quad H^{i-1}(i_v^* Rk_{v*} K|_{V-\{v\}}) \xrightarrow{\delta^{i-1}} \mathbb{H}_v^i(V, K) \xrightarrow{I^i} H^i(i_v^* K) \xrightarrow{\rho^i} H^i(i_v^* Rk_{v*} K|_{V-\{v\}}).$$

hence $\mathcal{L}_v^i := Im I^i = \text{Coker } \delta^{i-1}$.

We need to prove $Im \delta^{i-1} = Im p\delta^{i-1}$, to deduce the result in the form of an exact sequence

$$(2.5) \quad 0 \rightarrow i_{v,*} \mathcal{L}_v^i \rightarrow p\mathcal{H}^i(K) \rightarrow \oplus_{S_l \subset V-v} i_{S_l!} \mathcal{L}_{S_l}^j[l] \rightarrow 0.$$

We deduce from (lemma 2.3 ii) and the decomposition on $V - v$:

$$R^{i-1} k_{v*} k_v^* (p_{\tau \leq i-1} K) \xrightarrow{\sim} H^0(p\mathcal{H}^{i-1}(Rk_{v*} K|_{V-\{v\}})) = (p_{\tau \leq i-1} \mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*} \mathcal{L}))$$

and the following interpretation of the image:

$$p\delta^{i-1}(R^{i-1}k_{v*}k_v^*(p_{\mathcal{T}\leq i-1}K)) \xrightarrow{\sim} \delta^{i-1}(p_{\mathcal{T}\leq i-1}\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})).$$

Since $H^{i-1}(i_v^*Rk_{v*}K|_{V-\{v\}}) = \mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})$, we need to prove

$$\mathbf{Lemma\ 2.5.} \quad \delta^{i-1}(p_{\mathcal{T}\leq i-1}\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})) = \delta^{i-1}(\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}))$$

The proof is based again on the semi purity theorem. Indeed, the quotient space $\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})/p_{\mathcal{T}\leq i-1}$ has weight $w < a + i$ by the semi purity theorem, hence the image of $p\delta^{i-1}$ and δ^{i-1} in $\mathbb{H}_{X_v}^i(X, j_{!*}\mathcal{L})$ of weight $w \geq a + i$, are the same.

iii) *Proof of the splitting* $p\mathcal{H}^i(K) \xrightarrow{\sim} i_{v*}\mathcal{L}_v^i \oplus (\oplus_{S_l \subset V-v} i_{S_l!}\mathcal{L}_{S_l}^i[l])$. We consider the morphisms

$$(2.6) \quad p\mathcal{H}^i(Rk_{v!}k_v^*K) \xrightarrow{p\beta^i} p\mathcal{H}^i(K) \xrightarrow{p\rho^i} \text{Im } p\rho^i \subset p\mathcal{H}^i(Rk_{v*}k_v^*K)$$

where β^i is dual to ρ^{-i} as $\beta : Rk_{v!}k_v^*K \rightarrow K$ is dual to ρ . We deduce from lemma (2.3, iii) a dual argument to the proof in ii) above to assert that β^i factors through $\oplus_{S_l \subset V-v} i_{S_l!}\mathcal{L}_{S_l}^i[l]$, hence $p\rho^i$ induces an isomorphism $\text{Im } p\beta^i \rightarrow \text{Im } p\rho^i \circ p\beta^i := \oplus_{S_l \subset V-v} i_{S_l!}\mathcal{L}_{S_l}^i[l]$ and defines a splitting.

In the sequence of morphisms: $\mathbb{H}_v^i(V, K) \xrightarrow{p\alpha^i} p\mathcal{H}^i(K) \xrightarrow{p\gamma^i} H^i(i_v^*K)$ we have $\text{Im } p\alpha^i = \text{Ker } p\rho^i$ and $\text{Ker } p\gamma^i = \text{Im } p\beta^i$, hence $p\gamma^i$ induces on $\text{Im } p\alpha^i$ an isomorphism to $\text{Im } p\gamma^i \circ p\alpha^i := i_{v*}\mathcal{L}_v^i$.

Remark 2.6. i) We deduce from the above proof (formula 2.3 and 2.4)

$\text{Im } p\delta^{i-1} \simeq \oplus_{S_l \subset V-v, i-j>0} H_v^{i-j}(i_{S_l!}\mathcal{L}_{S_l}^j[l])$, $\text{Im } p\delta^{i-1} = \text{Ker } p\alpha^i = \text{Ker } I^i = \text{Im } \delta^{i-1}$ and the following isomorphism induced by $p\delta^{i-1}$:

$$\oplus_{S_l \subset V-v, i-1-j \geq 0} R^{i-1-j}k_{v*}k_v^*i_{S_l!}\mathcal{L}_{S_l}^j[l] \xrightarrow{\sim} \oplus_{S_l \subset V-v, i-j>0} H_v^{i-j}(i_{S_l!}\mathcal{L}_{S_l}^j[l])$$

ii) Hodge theory is used in the proof of the three lemmas 2.1, 2.4, 2.5. We use semi purity at X_v or equivalently local purity at v (section 2.3) to extend the decomposition property across v .

2.2. Hard Lefschetz. We check for all $i \geq 0$, $\eta^i : p\mathcal{H}^{-i}(K) \xrightarrow{\sim} p\mathcal{H}^i(K)$ is an isomorphism. By assumption the restriction of η^i to $V - v$ on $k_v^*p\mathcal{H}^{-i}(K)$ is an isomorphism, hence it remains an isomorphism on the intermediate extension

$$(k_v)_!k_v^*p\mathcal{H}^{-i}(K) \xrightarrow{\sim} \oplus_{S_l \subset V-v, j} i_{S_l!}\mathcal{L}_{S_l}^j[l] \subset p\mathcal{H}^{-i}(K)$$

across the point v in the strata V_0 of dimension 0. It remains to prove Hard Lefschetz $\mathcal{L}_v^{-i} \xrightarrow{\sim} \mathcal{L}_v^i$ where $\mathcal{L}_v^i := \text{Im}(\mathbb{H}_{Y_v}^i(X, j_{!*}\mathcal{L}) \xrightarrow{I^i} \mathbb{H}^i(Y_v, j_{!*}\mathcal{L}))$.

Lemma 2.7. i) *The cup-product with the class of an hyperplane section induces isomorphisms $\eta^i : \mathcal{L}_v^{-i} \rightarrow \mathcal{L}_v^i$ for $i > 0$.*

ii) *The HS on \mathcal{L}_v^i is Poincaré dual to \mathcal{L}_v^{-i} for $i \geq 0$.*

In the case of a big strata S , we have a local system $\mathcal{L}_S^i = R^i f_* j_{!*}\mathcal{L}$ and the lemma follows from Hard Lefschetz for intersection cohomology then the decomposition follows by Deligne's argument in [De 68] applied to the VHS on $\mathcal{L}_S^i = R^i f_* j_{!*}\mathcal{L}$ (subsection 4.6). The general statement of Verdier duality for proper morphisms apply to prove the duality between \mathcal{L}_S^i and \mathcal{L}_S^{-i} .

At a point v of a lower dimensional strata, Verdier duality between perverse cohomology sheaves in degree i and $-i$ is compatible with the decomposition established above and apply to $(k_v)_!k_v^*p\mathcal{H}^i(K)$ by induction. The duality between

\mathcal{L}_v^i and \mathcal{L}_v^{-i} in ii) is deduced from the duality of $Ri_{X_v}^!$ and $Ri_{X_v}^*$ and Verdier duality on X_v in the definition of \mathcal{L}_v^i (formula 1.2).

To prove i) at the point v we consider an hyperplane section H intersecting all NCD in X normally and proceeds by induction. Let i_H denote the closed embedding of H in X ; the cup product defines a morphism η equal to the composition of the morphisms $j_! \mathcal{L} \xrightarrow{\rho} i_{H*} i_H^* j_! \mathcal{L} \xrightarrow{\sim} i_{H*} Ri_H^!(j_! \mathcal{L})[2] \xrightarrow{G} j_! \mathcal{L}[2]$. We apply the functors $Ri_{X_v}^!$ and $i_{X_v}^*$ to the above morphisms ρ , G , and η as in the commutative diagram

$$\begin{array}{ccccc} \mathbb{H}_{X_v}^i(X, j_! \mathcal{L}) & \xrightarrow{\rho^!} & \mathbb{H}_{X_v \cap H}^i(H, j_! \mathcal{L}) & \xrightarrow{G^!} & \mathbb{H}_{X_v}^{i+2}(X, j_! \mathcal{L}) \\ I^i \downarrow & & I_H^i \downarrow & & I^{i+2} \downarrow \\ \mathbb{H}^i(X_v, j_! \mathcal{L}) & \xrightarrow{\rho^*} & \mathbb{H}^i(X_v \cap H, j_! \mathcal{L}) & \xrightarrow{G^*} & \mathbb{H}^{i+2}(X_v, j_! \mathcal{L}) \end{array}$$

where on the first line: $(\eta_v^!)_i = G_{i+2}^! \circ \rho_i^!$ and on the second line: $(\eta_v^*)_i = G_{i+2}^* \circ \rho_i^*$ are functorially induced by ρ and G , while by definition $\mathcal{L}_v^i := \text{Im } I^i$, $\mathcal{L}(H)_v^i := \text{Im } I_H^i$, $\mathcal{L}_v^{i+2} := \text{Im } I^{i+2}$ are the images of the vertical maps induced by I from the top line to the bottom line.

The morphisms ρ_i^* and the dual morphisms G_{i+2}^* induce ρ_i' and G_{i+2}' below

$$\mathcal{L}_v^i \xrightarrow{\rho_i'} \mathcal{L}(H)_v^i \xrightarrow{G_{i+2}'} \mathcal{L}_v^{i+2} \quad \text{where } \mathcal{L}(H)_v^i := \text{Im } I_H^i$$

with composition equal to η' induced by η . The proof continue by induction on H . A striking point however, there is no crucial case as in (Weil II [De 80], théorème 4.1.1) unless v is on the generic strata.

Lemma 2.8. *The induced morphism $\rho_i' : \mathcal{L}_v^i \rightarrow \mathcal{L}(H)_v^i$ is an isomorphism for $i < 0$ and by duality $G_{i+2}' : \mathcal{L}(H)_v^i \rightarrow \mathcal{L}_v^{i+2}$ is an isomorphism for $i + 2 > 0$.*

We prove first the isomorphisms for $i < 0$

$$(2.7) \quad \rho_i^! : \mathbb{H}_{X_v}^i(X, j_! \mathcal{L}) \rightarrow \mathbb{H}_{X_v \cap H}^i(H, j_! \mathcal{L})$$

We apply Artin- Lefschetz theorem ([BBDG 83], corollary 4.1.5) to the affine open subset $X_v - (H \cap X_v)$ with coefficients in a complex of sheaves $\mathcal{K} \in {}^pD^{\leq 0}(X_v)$ to prove the vanishing of $\mathbb{H}^i(X_v - (H \cap X_v), \mathcal{K})$ for $i > 0$ and its dual statement: $\mathbb{H}_c^i(X_v - (H \cap X_v), \mathcal{K}') = 0$ with coefficients in a complex of sheaves $\mathcal{K}' \in {}^pD^{\geq 0}(X_v)$ for $i < 0$.

Since $i_{X_v}^*(j_! \mathcal{L})[-1] \in {}^pD^{\leq 0}(X_v)$ and its dual $i_{X_v}^!(j_! \mathcal{L})[1] \in {}^pD^{\geq 0}(X_v)$ we deduce $\mathbb{H}_c^i(X_v - (H \cap X_v), i_{X_v}^!(j_! \mathcal{L})[1]) = 0$ for $i < 0$, hence we have for $i < 0$

$$\begin{array}{ccc} \mathbb{H}_{X_v}^i(X, j_! \mathcal{L}) & = & \mathbb{H}^{i-1}(X_v, i_{X_v}^!(j_! \mathcal{L})[1]) \\ \rho_i^! \downarrow \simeq & & \downarrow \simeq \\ \mathbb{H}_{X_v \cap H}^i(H, j_! \mathcal{L}) & = & \mathbb{H}^{i-1}(X_v \cap H, i_{X_v}^!(j_! \mathcal{L})[1]) \end{array}$$

The last equality is the main point and follows from the isomorphisms

$$i_{X_v \cap H}^* Ri_{X_v}^!(j_! \mathcal{L}) \xrightarrow{\sim} Ri_{X_v \cap H}^! i_H^*(j_! \mathcal{L}), \quad i_H^*(j_! \mathcal{L}[-1]) \xrightarrow{\sim} j_{H!} i_{H \cap (X-Y)}^* \mathcal{L}[-1]$$

due to the transversality of the intersection of H and X_v .

Proof of the lemma. We extend the diagram above by introducing the kernel of the Intersection morphisms I to get two columns of short exact sequences

$$\begin{array}{ccc} H^{i-1}(i_v^* R(k_v)_* K|_{V-\{v\}}) & \xrightarrow{\rho_{i-1}^*} & H^{i-1}(i_v^* R(k_v)_* K_{H|_{V-\{v\}}}) \\ \delta^{i-1} \downarrow & & \delta_H^{i-1} \downarrow \\ \mathbb{H}_v^i(V, K) \xrightarrow{\sim} \mathbb{H}_{X_v}^i(X, j_! \mathcal{L}) & \xrightarrow{\rho_i^!} & \mathbb{H}_{X_v \cap H}^i(H, j_! \mathcal{L}) \xrightarrow{\sim} \mathbb{H}_v^i(V, K_H) \\ I^i \downarrow & & I_H^i \downarrow \\ \mathcal{L}_v^i = \text{Im } I^i & \xrightarrow{\rho_i'} & \mathcal{L}(H)_v^i = \text{Im } I_H^i \end{array}$$

where $K_H = R(f|_H)_*(j|_H)_! j_! \mathcal{L}|_H$, $\mathcal{L}_v^i \subset \mathbb{H}^i(i_v^* K) = \mathbb{H}^i(X_v, j_! \mathcal{L})$ and $\mathcal{L}(H)_v^i \subset \mathbb{H}^i(X_v \cap H, j_! \mathcal{L}) = \mathbb{H}^i(i_v^* K_H)$.

On the right column the short exact sequence is defined by the perverse shifted restriction $\mathcal{L}|_H[-1]$ of \mathcal{L} , for which we suppose Hard Lefschetz by induction.

We prove that for $i < 0$, the morphism $\rho_i^!$, which is an isomorphism by the lemma, induces isomorphisms

$$(2.8) \quad \rho_{i|}^! : \text{Im } \delta^{i-1} = \text{Ker } I^i \xrightarrow{\sim} \text{Im } \delta_H^{i-1} = \text{Ker } I_H^i$$

hence induces isomorphisms: $\rho_i' : \mathcal{L}_v^i \xrightarrow{\sim} (\mathcal{L}|_H)_v^i$ for $i < 0$.

Proof of the isomorphism $\rho_{i|}^! : \text{Ker } I^i \xrightarrow{\sim} \text{Ker } I_H^i$. By the remark (2.6 i)

$$\text{Ker}(I^i) = \text{Im}(\delta^{i-1}) = \text{Im}(\oplus_{S_l \subset V-v, j < i} H^{i-1-j}(i_v^* R k_{v*} k_v^* i_{S_l!} \mathcal{L}_{S_l}^j[l]))$$

$$\text{Ker}(I_H^i) = \text{Im}(\delta_H^{i-1}) = \text{Im}(\oplus_{S_l \subset V-v, j < i} H^{i-1-j}(i_v^* R(k_v)_* k_v^* i_{S_l!} (\mathcal{L}|_H)_{S_l}^j[l])).$$

The proof is reduced to the comparison of $\mathcal{L}_{S_l}^j$ and $(\mathcal{L}|_H)_{S_l}^j$ on each component S_l .

We consider for each S_l , a normal section N_{v_l} to S_l at a general point $v_l \in S_l$, then:

$$(R^{-l+j} f_* i_{X_{S_l}}^! j_! \mathcal{L})_{v_l} \xrightarrow{\sim} \mathbb{H}_{X_{v_l}}^{-l+j}(X_{N_{v_l}}, j_! \mathcal{L}) \xrightarrow{\sim} \mathbb{H}_{X_{v_l}}^j(X_{N_{v_l}}, j_! \mathcal{L}[-l])$$

where $j_! \mathcal{L}[-l]$ restricts to a perverse sheaf on $X_{N_{v_l}}$ of dimension $\dim X - l$. Since $i < 0$ and $j < i$ we have $j < -1$, moreover $X_{v_l} - (H \cap X_{v_l})$ is affine, hence the restriction of $\rho_j^!$ to $X_{N_{v_l}}$ is an isomorphism by the hyperplane section theorem on the fibres of f over S_l : $\rho_{j|}^! : \mathbb{H}_{X_{v_l}}^j(X_{N_{v_l}}, j_! \mathcal{L}[-l]) \rightarrow \mathbb{H}_{X_{v_l} \cap H}^j(X_{N_{v_l}} \cap H, j_! \mathcal{L}[-l])$ where the last term is isomorphic to $(R^{-l+j} f_* i_{H \cap X_{S_l}}^! j_! \mathcal{L}|_H)_{v_l}$.

Corollary 2.9. *The iterated cup-product with the class of a relative hyperplane section induces isomorphisms $\eta^i : {}^p\mathcal{H}^{-i}(K) \rightarrow {}^p\mathcal{H}^i(K)$ for $i > 0$.*

Corollary 2.10. *For each strata S of V , the local system \mathcal{L}_S^i is a polarized variation of Hodge structure on the smooth variety S .*

We consider here the case of a fibration $f : X \rightarrow V$ by NCD over the strata with X smooth, we consider at a point $v \in S$ open neighborhoods B_v of v in V and S_v in S , and a projection $p_v : B_v \rightarrow S_v$, inducing the identity on S_v , such that $p_v \circ f : X_{B_v} \rightarrow S_v$ is smooth.

The fibers $N_a := p_v^{-1}(a)$ at points $a \in S_v$ form a family of normal sections to S_v such that $X_{N_a} := f^{-1}(N_a)$ is a smooth sub-variety of X_{B_v} . By an argument based on the relative complex $\Omega_{X_{B_v}/S_v}^* \mathcal{L}$ as in section 4.6, the families $R^k f_* (i_{X_{S_v}}^! (j_! \mathcal{L}))$ (resp. $R^k f_* (i_{X_{S_v}}^* (j_! \mathcal{L}))$) form variations of MHS on S of weight $\geq a + k + l$ (resp. $\leq a + k + l$) (corollary 5.9).

Then the image of the Intersection morphisms $\mathcal{L}_{S_v}^k$ are pure variation of HS of weight $a + k + l$ on S_v .

2.3. Proof of the local purity theorem 1.8. We did assume in the statement of the proposition 2.2 the local purity at v to simplify the exposition. We show now the local purity at v follows in fact from the decomposition on $V - v$. We assume here that f is a fibration by NCD over the strata in order to apply the construction of Hodge theory in sections 4 – 5. We refer precisely to the subsection (5.2.4) for the construction of the mixed Hodge structure used in this theorem and to the compatibility with the perverse filtration.

The proof of the equivalent semi purity at X_v (proposition 1.9) differs from [DeG 81] as we use the polarization of the Intersection cohomology of X .

Let $j_{!*}\mathcal{L}$ be a shifted polarized VHS of weight a on the smooth and compact variety X , $X_v := f^{-1}(v)$ the fiber at $v \in V$, and $B_{X_v} = f^{-1}(B_v)$ the inverse of a small neighborhood B_v of v .

2.3.1. Inductive hypothesis. We assume by induction the decomposition theorem on $V - v$, hence we have an isomorphism:

$$(2.9) \quad Gr_i^{p\tau} \mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L}) \xrightarrow{\sim} \mathbb{H}^{r-i}(B_v - \{v\}, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})).$$

This isomorphism is used to carry the MHS from the left term (subsection 5.3) to the right. Under such isomorphism, we prove the following inequalities on the weights w of the MHS

i) $w > a + r$ on ${}^{p\tau}_{\leq r} \mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L})$, and dually:

i') $w \leq a + r$ on $\mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L}) / {}^{p\tau}_{\leq r} \mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L})$.

or equivalently for $j = r - i$:

ii) $w > a + i + j$ on $H^j(B_v - v, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}))$ if $j \geq 0$, and dually:

ii') $w \leq a + i + j$ on $H^j(B_v - v, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}))$ if $j \leq -1$.

Example. The dual of $Gr_{a+l}^W \mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}))$ for all i and $l \leq i - 1$ in the assertion ii'), is $Gr_{a-l}^W \mathbb{H}^0(B_v - v, {}^p\mathcal{H}^{-i}(Rf_*j_{!*}\mathcal{L}))$ for all i and $a - l \geq a - i + 1$ in the assertion ii); we remark the change of the variable i into the variable $-i$.

2.3.2. Duality. Let $K := Rf_*j_{!*}\mathcal{L}$ and $k_v : V - \{v\} \rightarrow V$. We have:

$$K(*v) := i_v^* Rk_{v*} k_v^* K \xrightarrow{\sim} R\Gamma(B_v - v, K) \xrightarrow{\sim} R\Gamma(B_{X_v} - X_v, j_{!*}\mathcal{L}).$$

The duality isomorphism $D(K(*v))[1] \xrightarrow{\sim} K(*v)$ where D stands for Verdier dual, is deduced from the duality between Rk_{v*} and $Rk_{v!}$ (resp. i_v^* and $Ri_v^!$) as follows.

We apply to $k_v^* K$ two sequences of functors $Rk_{v!} \rightarrow Rk_{v*} \rightarrow i_{v*} i_v^* Rk_{v*}$ and $Rk_{v!} \rightarrow Rk_{v*} \rightarrow i_{v*} Ri_v^! Rk_{v!}[1]$, defining dual triangles, from which we deduce

$$D(K(*v))[1] \xrightarrow{\sim} i_{v*} Ri_v^! Rk_{v!} k_v^* K[1] \xrightarrow{\sim} K(*v).$$

which corresponds to the duality of the Intersection cohomology of the “link”: $\partial B_{X_v} = f^{-1}(\partial B_v)$ with coefficients in the restriction of $j_{!*}\mathcal{L}$.

Since the perverse filtration is compatible with the MHS over $V - v$, we deduce

$$D(Gr_{a-q}^W H^{-j}(K(*v))) \xrightarrow{\sim} Gr_{a+q}^W H^j(DK(*v)) \xrightarrow{\sim} Gr_{a+q}^W H^{j-1}(K(*v)).$$

and since the duality: $D({}^p\mathcal{H}^{-i}(K)) \xrightarrow{\sim} {}^p\mathcal{H}^i(K)$ follows from the auto-duality of $j_{!*}\mathcal{L}$ by Verdier’s direct image theorem for f proper, we deduce

$$D(Gr_{a-q}^W Gr_{-i}^{p\tau} H^{-j}(K(*v))) \xrightarrow{\sim} Gr_{a+q}^W Gr_i^{p\tau} H^{j-1}(K(*v)).$$

Hence, the proof is reduced by the above duality, to one of the two cases ii) or ii') in degree $j \geq 0$ or $j \leq -1$.

2.3.3. Proof by induction on $\dim X$. Let H be a general hyperplane section of X transversal to all strata, $i_H : H \rightarrow X$ and $j_H : (X - H) \rightarrow X$. The restriction to H of the intermediate extension $i_H^* j_{H!} \mathcal{L}$ is equal to $j_{H!}(\mathcal{L}|_H)$ by transversality.

We assume the local purity theorem for the perverse cohomology sheaves of $Rf_* j_{H!}(\mathcal{L}|_H[-1])$, and we use Artin-Lefschetz vanishing theorem to deduce the local purity for the perverse cohomology sheaves of K in degree $i \neq 0$ as follows.

Lemma 2.11. *Let $K_c = Rf_*(j_H)_! j_H^* j_{H!} \mathcal{L}$ (resp. $K(*) = Rf_*(j_H)_* j_H^* j_{H!} \mathcal{L}$), then the complex $K_c \in {}^p D_{\bar{V}}^{\geq 0}$, or equivalently ${}^p \mathcal{H}^i(K_c) = 0$ for $i < 0$ (resp. $K(*) \in {}^p D_{\bar{V}}^{\leq 0}$ or equivalently ${}^p \mathcal{H}^i(K(*)) = 0$ for $i > 0$).*

Proof. Since the morphism $f \circ j_H : (X - H) \rightarrow X$ is affine, it follows that the functor $Rf_* \circ j_{H!} = R(f \circ j_H)_!$ is left t-exact ([BBDG 83], corollary 4.1.2), which means that it transforms ${}^p D_{\bar{X}-H}^{\geq 0}$ into ${}^p D_{\bar{V}}^{\geq 0}$. It applies to $Rf_*(j_H)_! j_H^* j_{H!} \mathcal{L}$ and shows that $K_c \in {}^p D_{\bar{V}}^{\geq 0}$. The result is a version of Artin-Lefschetz vanishing theorem. The statement ${}^p \mathcal{H}^i(K(*)) = 0$ for $i > 0$ follows by duality. \square

Let $K = Rf_* j_{H!} \mathcal{L}$, and $K_H = R(f \circ i_H)_* i_H^* j_{H!} \mathcal{L}$. The restriction morphism $\rho : K \rightarrow K_H$, and dually the Gysin morphism: $G : K_H[-2] \rightarrow K$, induce morphisms compatible with the ${}^p \tau$ filtration

$$\rho_i : {}^p \mathcal{H}^i(K) \rightarrow {}^p \mathcal{H}^i(K_H), G_i : {}^p \mathcal{H}^{i-2}(K_H) \rightarrow {}^p \mathcal{H}^i(K), L_i : {}^p \mathcal{H}^i(K) \rightarrow {}^p \mathcal{H}^{i+2}(K),$$

where $L_i = G_{i+2} \circ \rho_i$ (resp. $L : K \rightarrow K[2]$), is defined by the cup product with the Chern class c_1 of a relative ample bundle.

Corollary 2.12. *The restriction $\rho_i : {}^p \mathcal{H}^i(Rf_* j_{H!} \mathcal{L}) \rightarrow {}^p \mathcal{H}^i(Rf_*(i_H)_* i_H^* j_{H!} \mathcal{L})$ is an isomorphism for each integer $i < -1$ and injective for $i = -1$.*

Dually, the Gysin morphism $G_i : {}^p \mathcal{H}^{i-2}(Rf_(i_H)_* i_H^* j_{H!} \mathcal{L}) \rightarrow {}^p \mathcal{H}^i(Rf_* j_{H!} \mathcal{L})$ is an isomorphism for $i > 1$, and it is surjective for $i = 1$.*

We deduce from the triangle $(j_H)_! j_H^* j_{H!} \mathcal{L} \rightarrow j_{H!} \mathcal{L} \rightarrow (i_H)_* i_H^* j_{H!} \mathcal{L} \xrightarrow{(1)}$, and its direct image by Rf_* , an exact sequence of perverse cohomology

$$(2.10) \quad \dots \rightarrow {}^p \mathcal{H}^i(K_c) \xrightarrow{\text{can}_i} {}^p \mathcal{H}^i(K) \xrightarrow{\rho_i} {}^p \mathcal{H}^i(K_H) \xrightarrow{\partial_i} {}^p \mathcal{H}^{i+1}(K_c) \rightarrow \dots$$

The corollary follows from lemma 2.11 as the complex $K_c \in {}^p D_{\bar{V}}^{\geq 0}$, or equivalently ${}^p \mathcal{H}^i(K_c) = 0$ for $i < 0$.

Next, we use the inductive assumptions over $V - v$ in the crucial case for $i = 0$.

Lemma 2.13. *If we suppose the relative Hard Lefschetz theorem for the morphisms f and $f \circ i_H$ restricted to $V - v$, then G_0 is injective and ρ_0 is surjective on $V - v$. Moreover, we have a decomposition*

$$(2.11) \quad {}^p \mathcal{H}^0(Rf_* j_{H!} \mathcal{L})|_{V-v} \xrightarrow{\sim} \text{Im } G_0 \oplus \ker \rho_0.$$

Respectively, ρ_{-1} is injective, G_1 is surjective and we have a decomposition

$$(2.12) \quad {}^p \mathcal{H}^{-1}(Rf_*(i_H)_* i_H^* j_{H!} \mathcal{L})|_{V-v} \xrightarrow{\sim} \text{Im } \rho_{-1} \oplus \ker G_1$$

Moreover $G_0 : {}^p \mathcal{H}^{-2}(Rf_*(i_H)_* i_H^* j_{H!} \mathcal{L})|_{V-v} \xrightarrow{\sim} \text{Im } G_0 \subset {}^p \mathcal{H}^0(Rf_* j_{H!} \mathcal{L})|_{V-v}$ is an isomorphism onto its image.

Proof. Recall that \mathcal{L} is defined on $X - Y$. Let $j' : (H - H \cap Y) \rightarrow H$. We have by transversality: $i_H^* j_{!*} \mathcal{L} \xrightarrow{\sim} j'_{!*} (i_H^* \mathcal{L})$, hence ${}^p\mathcal{H}^i(K_H) = {}^p\mathcal{H}^i(Rf_* j'_{!*} (i_H^* \mathcal{L}))$. Since by the inductive hypothesis, Hard Lefschetz theorem apply on $V - v$ to $K_H[-1]$, image of the intermediate extension $j'_{!*} i_H^* \mathcal{L}[-1]$ of the shifted restriction of \mathcal{L} on H , we deduce that the composition morphism $\smile c_1 = \rho_0 \circ G_0$ in the diagram is an isomorphism on $V - v$

$${}^p\mathcal{H}^{-2}(K_H)_{|V-v} \xrightarrow{G_0} {}^p\mathcal{H}^0(K)_{|V-v} \xrightarrow{\rho_0} {}^p\mathcal{H}^0(K_H)_{|V-v}$$

where the first term is ${}^p\mathcal{H}^{-1}(Rf_* j'_{!*} i_H^* \mathcal{L}[-1])$ and the last ${}^p\mathcal{H}^1(Rf_* j'_{!*} i_H^* \mathcal{L}[-1])$. Hence ρ_0 is surjective, G_0 is injective and the decomposition 2.11 follows.

Respectively, the composition morphism $\smile c_1 = G_1 \circ \rho_{-1}$ in the diagram

$${}^p\mathcal{H}^{-1}(K)_{|V-v} \xrightarrow{\rho_{-1}} {}^p\mathcal{H}^{-1}(K_H)_{|V-v} \xrightarrow{G_1} {}^p\mathcal{H}^1(K)_{|V-v}$$

is an isomorphism and the decomposition 2.12 follows. \square

The next result is based on Hodge theory as in (proposition 1.7, definitions 5.11 and 5.21)

Corollary 2.14. *If the local purity theorem applies for H , then it applies for X , except eventually for $(\text{Ker } \rho_0)_{|V-v} \subset {}^p\mathcal{H}^0(Rf_* j_{!*} \mathcal{L})_{|V-v}$.*

Proof. The restriction morphisms ρ_i are compatible with MHS for all i and j

$$H^j(B_v - v, {}^p\mathcal{H}^i(K)) \rightarrow H^j(B_v - v, {}^p\mathcal{H}^i(K_H))$$

and they are isomorphisms for $i < -1$ by the corollary above. Then, the conditions of local purity on X are satisfied for $i < -1$, since they are satisfied on H by induction. The dual argument for G_i apply, and we are left with the cases $i = 0, -1, 1$.

The case $i = -1$ follows from the decomposition (formula 2.12), as ρ_{-1} is an isomorphism onto a direct summand $\text{Im } \rho_{-1}$ of ${}^p\mathcal{H}^{-1}(Rf_* (i_H)_* i_H^* j_{!*} \mathcal{L})_{|V-v}$.

By Lefschetz isomorphism on $V - v$, we deduce the case $i = 1$ from the case $i = -1$. For $i = 0$, as G_0 is an isomorphism by the lemma, the inequality holds for $\text{Im } G_0$. Only the case of $\text{Ker } \rho_0$ can not be deduced by induction from H . \square

2.4. The crucial case. $w \leq a + j$ on $H^j(B_v - v, {}^p\mathcal{H}^0(Rf_* j_{!*} \mathcal{L}))$ for $j < 0$.

The result is local at v , so we can suppose V affine, then choose a projective embedding to allow the use of the polarization of the cohomology.

The proof is subdivided in many steps. We apply the next lemma for ${}^p\mathcal{H}^i(Rf_* j_{!*} \mathcal{L})$, in the case of $i = 0$, but it is equally proved next for all i .

Lemma 2.15. *Let $j_{!*} \mathcal{L}$ be of weight a , then the MHS on*

$$Gr_i^{p\tau} H^{i+j}(B_{X_v} - X_v, j_{!*} \mathcal{L}) \simeq H^j(B_v - v, {}^p\mathcal{H}^i(Rf_* j_{!*} \mathcal{L}))$$

is of weight $\omega > a + i + j$ for $j > 0$ and dually of weight $\omega \leq a + i + j$ for $j < -1$.

Proof. Let H be a general hyperplane section of V containing v , $H_v = B_v \cap H$ and $K = Rf_* j_{!*} \mathcal{L}$. We suppose H normally embedded outside v so that the perverse truncation commutes with the restriction to $H - v$ up to a shift in degrees; We have a Gysin exact sequence

$$\mathbb{H}^{j-2}(H_v - \{v\}, {}^p\mathcal{H}^i(K))(-1) \xrightarrow{G_j} \mathbb{H}^j(B_v - \{v\}, {}^p\mathcal{H}^i(K)) \rightarrow \mathbb{H}^j(B_v - H_v, {}^p\mathcal{H}^i(K))$$

Since ${}^p\mathcal{H}^i(K)$ is in the category ${}^pD_c^{\leq 0}V$, and $B_v - H_v$ is Stein, we apply Artin Lefschetz hyperplane section theorem to show that $\mathbb{H}^j(B_v - H_v, {}^p\mathcal{H}^i(K)) \simeq 0$ for $j > 0$. Then G_j is an isomorphism for $j > 1$ and it is surjective for $j = 1$.

The smooth strict transform H' of H intersects transversally in X the various subspaces Y_l inverse of the strata S_l so that Gysin morphisms are compatible with the *MHS*.

Hence, we deduce the statement for $B_v - v$ in the lemma for $j > 0$ from the inductive hypothesis on the local purity of the *MHS* structure on $H^{j-1}(H_v - v, {}^p\mathcal{H}^i(K)[-1])$ for $j - 1 \geq 0$, since $H^{j-2}(H_v - v, {}^p\mathcal{H}^i(K))(-1) \simeq H^{j-1}(H_v - v, {}^p\mathcal{H}^i(K)[-1])$. \square

2.4.1. *The large inequality: $w \leq a$ on $H^{-1}(B_v - v, {}^p\mathcal{H}^0(Rf_*j_*\mathcal{L}))$.* This case also is easily deduced by induction

Lemma 2.16. *i) $\mathbb{H}^0(B_v - \{v\}, {}^p\mathcal{H}^i(Rf_*j_*\mathcal{L}))$ is of weight $\geq a + i$.
ii) Dually: $\mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^i(K))$ is of weight $\leq a + i$.*

Proof. Let $k_v : (V - v) \rightarrow V$ and consider a general hyperplane section H_1 not containing v . We deduce from the triangle of complexes:

$Rk_{v!}k_v^* {}^p\mathcal{H}^i(K) \rightarrow Rk_{v*}k_v^* {}^p\mathcal{H}^i(K) \rightarrow i_{v*}i_v^* Rk_{v*}k_v^* {}^p\mathcal{H}^i(K)$, the exact sequence

$$\mathbb{H}^0(V - H_1, Rk_{v*}k_v^* {}^p\mathcal{H}^i(K)) \xrightarrow{\gamma_0} H^0(i_v^* Rk_{v*}k_v^* {}^p\mathcal{H}^i(K)) \rightarrow \mathbb{H}^1(V - H_1, Rk_{v!}k_v^* {}^p\mathcal{H}^i(K))$$

which shows that γ_0 is surjective, since $\mathbb{H}^1(V - H_1, Rk_{v!}k_v^* {}^p\mathcal{H}^i(K))$ vanishes as $V - H_1$ is affine. We deduce the assertion i) since the weights w of the cohomology $\mathbb{H}^0(V - H_1, Rk_{v*}k_v^* {}^p\mathcal{H}^i(K))$ of the open set $V - H_1$ satisfy $w \geq a + i$. \square

2.4.2. *The crucial step: $w \neq a$.* It remains to exclude the case $w = a$, which does not follow by induction, that is

$$(2.13) \quad Gr_a^W Gr_0^{p\tau} \mathbb{H}^{-1}(B_{X_v} - X_v, j_*\mathcal{L}) \xrightarrow{\sim} Gr_a^W \mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^0(K)) \xrightarrow{\sim} 0.$$

from which we deduce by duality $Gr_a^W Gr_0^{p\tau} \mathbb{H}^0(B_{X_v} - X_v, j_*\mathcal{L}) \xrightarrow{\sim} 0$.

By corollary 2.14, it remains to consider the case of the sub-perverse sheaf $\text{Ker}\rho_0 \subset {}^p\mathcal{H}^0(K)$ in which case we use the following lifting of cohomology classes

Lemma 2.17. *Let $B_v^* := B_v - v$ and H a general hyperplane section of X . The following map is surjective:*

$$W_a \tau_{\leq 0} \mathbb{H}^{-1}(B_v^*, Rf_*j_{H!}j_H^*j_*\mathcal{L}) \rightarrow Gr_a^W Gr_0^{p\tau} \mathbb{H}^{-1}(B_v^*, \text{Ker}\rho_0).$$

The proof relies on Hodge theory applied to K_c (formula 2.10) as follows.

2.4.3. *Cohomology with compact support of the fibers of $f|_{X-H}$.* By transversality of H with all strata of X , the restriction to H on $j_*\mathcal{L} \simeq IC^*\mathcal{L}$ is defined in terms of complexes of logarithmic type (subsection 4.5.1)

$\rho : (IC^*\mathcal{L}, F) \rightarrow i_{H*}i_H^*(IC^*\mathcal{L}, F)$, where $(IC^*(\mathcal{L}|_H[-1]), F)[1] = i_H^*(IC^*\mathcal{L}, F)$.

We introduce the cone $(C(\rho)[-1], F)$ and let $(K_c, F) := Rf_*(C(\rho)[-1], F)$. From the triangle $K_c \rightarrow K \rightarrow K_H \xrightarrow{(1)}$, we deduce an exact sequence of perverse cohomology

$${}^p\mathcal{H}^{i-1}(K) \xrightarrow{\rho^{i-1}} {}^p\mathcal{H}^{i-1}(K_H) \xrightarrow{\partial^{i-1}} {}^p\mathcal{H}^i(K_c) \xrightarrow{\text{can}_i} {}^p\mathcal{H}^i(K) \xrightarrow{\rho^i} {}^p\mathcal{H}^i(K_H)$$

Proof of lemma 2.17. In the local case, over a neighborhood of v , we consider the long exact sequence

$$\rightarrow \mathbb{H}^{i-1}(B_v^*, K_H) \rightarrow \mathbb{H}^i(B_v^*, K_c) \xrightarrow{\gamma} \mathbb{H}^i(B_v^*, K) \rightarrow \mathbb{H}^i(B_v^*, K_H) \rightarrow \dots$$

We put a MHS on the terms as follows (subsection 5.2.4). The mixed cone $C(I_X)$ (resp. $C(I_H)$) puts MHS on $\mathbb{H}^*(B_v^*, K)$ (resp. $\mathbb{H}^*(B_v^*, K_H)$) (lemma 5.12) in terms

of complexes of logarithmic type. The restriction $\rho : C(I_X) \rightarrow i_{H*}C(I_H)$ is well defined and compatible with the weight filtration W and the Hodge filtration F . We introduce the mixed cone $(C(\rho)[-1], W, F)$ satisfying

$$(Gr_i^W C(\rho)[-1], F) = Gr_i^W(C(I_X) \oplus Gr_{i+1}^W(i_{H*}C(I_H)[-1])).$$

We remark that $C(\rho)[-1] \simeq (j_H)_!j_H^*C(I_X)$, and we have a triangle

$$C(\rho)[-1] \rightarrow C(I_X) \rightarrow i_{H*}i_H^*C(I_H) \xrightarrow{(1)}$$

inducing MHS on the terms of the exact sequence.

Such MHS is compatible with the perverse filtration (proposition 1.7, sections 5.3, 5.3.3), that is the subspaces $\mathbb{H}^i(B_v^*, {}^p\tau_{\leq j}K)$, $\mathbb{H}^i(B_v^*, {}^p\tau_{\leq j}K_H)$, $\mathbb{H}^i(B_v^*, {}^p\mathcal{H}^j(K))$, and $\mathbb{H}^i(B_v^*, {}^p\mathcal{H}^j(K_H))$ underly corresponding MHS, as well $\mathbb{H}^i(B_v^*, {}^p\tau_{\leq j}K_c)$ and $\mathbb{H}^i(B_v^*, {}^p\mathcal{H}^j(K_c))$. We apply the previous results to the exact sequence

$$\dots \rightarrow \mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K_c)) \xrightarrow{\gamma_1} \mathbb{H}^{-1}(B_v^*, \text{Ker}\rho_0) \xrightarrow{\partial} \mathbb{H}^0(B_v^*, \text{Coker}\rho_{-1}) \rightarrow$$

Taking Gr_a^W , we have an exact sequence

$$Gr_a^W \mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K_c)) \xrightarrow{\gamma_1} Gr_a^W \mathbb{H}^{-1}(B_v^*, \text{Ker}\rho_0) \xrightarrow{\partial} Gr_a^W \mathbb{H}^0(B_v^*, \text{Coker}\rho_{-1}) \rightarrow$$

By the inductive hypothesis on H , $Gr_a^W \mathbb{H}^0(B_v^*, \text{Coker}\rho_{-1}) = 0$, hence γ_1 is surjective so that elements of $Gr_a^W \mathbb{H}^{-1}(B_v^*, \text{Ker}\rho_0)$ can be lifted to elements in $W_a \mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K_c))$. Moreover, $\tau_{\leq 0}K_c = {}^p\mathcal{H}^0(K_c)$ as $\tau_{< 0}K_c = 0$, hence we can lift the elements to $W_a \mathbb{H}^{-1}(B_v^*, \tau_{\leq 0}K_c)$. This ends the proof of lemma 2.17.

2.4.4. Polarization. In this step we use the polarization of Intersection cohomology. The following proof is based on the idea that the cohomology of $B_v - v$ fits in two exact sequences issued from the two triangles for $K := Rf_*j_{!*}\mathcal{L}$

$$(2.14) \quad i_v^*Rk_{v!}k_v^*K \rightarrow i_v^*K \rightarrow i_v^*Rk_{v*}k_v^*K, \quad Rk_v^!K \rightarrow i_v^*K \rightarrow i_v^*Rk_{v*}k_v^*K$$

from which we deduce the commutative diagram (see also the diagram 5.6)

$$\begin{array}{ccc} \mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}) & \xrightarrow{\partial_X} & \mathbb{H}_c^0(X - X_v, j_{!*}\mathcal{L}) \\ \downarrow \partial & \searrow \gamma & \downarrow \alpha_X \\ \mathbb{H}_{X_v}^0(X, j_{!*}\mathcal{L}) & \xrightarrow{A} & \mathbb{H}^0(X, j_{!*}\mathcal{L}) \end{array}$$

Let γ denotes the composition morphisms $\gamma = \alpha_X \circ \partial_X = A \circ \partial$.

Lemma 2.18. *Let $u \in Gr_a^W \mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})$ be in the image of the canonical morphism $Gr_a^W \mathbb{H}^{-1}(B_{X_v} - X_v, j_H^!j_H^*j_{!*}\mathcal{L}) \xrightarrow{\text{can}} Gr_a^W \mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})$, then $\gamma(u) = 0$ in $Gr_a^W \mathbb{H}^0(X, j_{!*}\mathcal{L}) = \mathbb{H}^0(X, j_{!*}\mathcal{L})$.*

Proof. 1) $\gamma(u)$ is primitive. Let $B_{X_v}^* := B_{X_v} - X_v$, $b \in Gr_a^W \mathbb{H}^{-1}(B_{X_v}^*, j_H^!j_H^*j_{!*}\mathcal{L})$ such that $u = \text{can}(b)$, and consider the diagram corresponding to an hyperplane section H

$$Gr_a^W \mathbb{H}^{-1}(B_{X_v}^*, j_H^!j_H^*j_{!*}\mathcal{L}) \xrightarrow{\partial_!} Gr_a^W \mathbb{H}_c^0(X^*, j_H^!j_H^*j_{!*}\mathcal{L}) \xrightarrow{\alpha_!} Gr_a^W \mathbb{H}^0(X, j_H^!j_H^*j_{!*}\mathcal{L})$$

Let $\gamma_! := \alpha_! \circ \partial_!$, then $\gamma(u) = \gamma(\text{can } b) = \text{can } \gamma_!(b)$, hence the restriction to H : $\rho_H(\gamma(u)) = \rho_H(\text{can } \gamma_!(b)) = 0$, and $\gamma(u)$ is a primitive element.

2) $\gamma(u) = 0$. We consider the diagram

$$\begin{array}{ccc}
& Gr_a^W \mathbb{H}^0(X, j_{!*} \mathcal{L}) & \\
A \nearrow & & \searrow A^* \\
Gr_a^W \mathbb{H}_{X_v}^0(X, j_{!*} \mathcal{L}) & \xrightarrow{I} & Gr_a^W \mathbb{H}^0(X_v, j_{!*} \mathcal{L})
\end{array}$$

where A^* is the dual of A . Let P denotes the scalar product defined by Poincaré duality on $\mathbb{H}^0(X, j_{!*} \mathcal{L})$ and C the Weil operator defined by the HS. The polarization Q on the primitive part of $\mathbb{H}^0(X, j_{!*} \mathcal{L})$ is defined by $Q(a, b) := P(Ca, \bar{b})$.

A non-degenerate pairing P_v

$$P_v : Gr_a^W \mathbb{H}_{X_v}^0(X, j_{!*} \mathcal{L}) \otimes Gr_a^W \mathbb{H}^0(X_v, j_{!*} \mathcal{L}) \rightarrow \mathbb{C}.$$

is also defined by duality. The duality between A and A^* is defined for all $b \in Gr_a^W \mathbb{H}_{X_v}^0(X, j_{!*} \mathcal{L})$ and $c \in Gr_a^W \mathbb{H}^0(X, j_{!*} \mathcal{L})$ by the formula :

$$P(Ab, c) = P_v(b, A^*c).$$

Let C be the Weil operator defined by the HS on $Gr_{a+i}^W \mathbb{H}_{X_v}^i(X, j_{!*} \mathcal{L})$ as well on $Gr_{a+i}^W \mathbb{H}^i(X, j_{!*} \mathcal{L})$, then:

$$P(C.A(\partial u), A(\overline{\partial u})) = P_v(C\partial u, A^* \circ A(\overline{\partial u})) = P_v(C.\partial(u), I(\overline{\partial u}))$$

and since $I(\partial u) = A^* \circ A \circ \text{can}(b) = 0$, we deduce $P(C.\gamma(u), \gamma(\overline{\partial u})) = P(C.A(\partial u), A(\overline{\partial u})) = 0$, hence $\gamma(u) = 0$ by polarization as $\gamma(u)$ is primitive. \square

2.4.5. The last step of the proof of the crucial case is based on two lemma.

Lemma 2.19. *The connecting morphism $\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*} \mathcal{L}) \xrightarrow{\partial_X} \mathbb{H}_c^i(X - X_v, j_{!*} \mathcal{L})$ induced by the triangles in formula (2.14) is injective on $Gr_i^W Gr_{a+i}^W Gr_i^{p\tau} \mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*} \mathcal{L}) \xrightarrow{\partial_X} Gr_{a+i}^W Gr_i^{p\tau} \mathbb{H}_c^i(X - X_v, j_{!*} \mathcal{L})$.*

We prove equivalently $Gr_{a+i}^W \mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^i(K)) \xrightarrow{\partial_X} Gr_{a+i}^W \mathbb{H}_c^0(V - \{v\}, {}^p\mathcal{H}^i(K))$ (precisely $Gr_{a+i}^W(\partial_V)$) is injective, by considering the long exact sequence:

$$\mathbb{H}^{-1}(V - \{v\}, {}^p\mathcal{H}^i(K)) \xrightarrow{\partial_X} H_c^0(V - \{v\}, {}^p\mathcal{H}^i(K)) \rightarrow \mathbb{H}^0(V - v, {}^p\mathcal{H}^i(K)).$$

It is enough to prove that $H^{-1}(V - \{v\}, {}^p\mathcal{H}^i(K))$ is pure of weight $a + i - 1$ or by duality $H_c^1(V - \{v\}, {}^p\mathcal{H}^i(K))$ is pure of weight $a + i + 1$. We consider as above the hyperplane section H_1 not containing v and the exact sequence:

$$\mathbb{H}_{H_1}^1(V, Rk_{v!} k_v^* {}^p\mathcal{H}^i(K)) \xrightarrow{\varphi} \mathbb{H}^1(V, Rjk_{v!} k_v^* {}^p\mathcal{H}^i(K)) \rightarrow \mathbb{H}^1(V - H_1, Rk_{v!} k_v^* {}^p\mathcal{H}^i(K))$$

where φ is surjective since the last term vanish; the space

$$\mathbb{H}^{-1}(H_1, Rk_{v!} k_v^* {}^p\mathcal{H}^i(K))(-1) \xrightarrow{\sim} \mathbb{H}_{H_1}^1(V, Rk_{v!} k_v^* {}^p\mathcal{H}^i(K))$$

is a pure HS of weight $a + i + 1$ (as a sub-quotient of the pure intersection cohomology of the inverse image H'_1 smooth in X).

Lemma 2.20. *The following morphism is injective*

$$Gr_{a+i}^W \mathbb{H}_c^i(X - X_v, j_{!*} \mathcal{L}) \xrightarrow{\alpha_X} Gr_{a+i}^W \mathbb{H}^i(X, j_{!*} \mathcal{L}) \xrightarrow{\sim} \mathbb{H}^i(X, j_{!*} \mathcal{L}).$$

In the long exact sequence $H^{i-1}(X_v, j_{!*} \mathcal{L}) \rightarrow \mathbb{H}_c^i(X - X_v, j_{!*} \mathcal{L}) \xrightarrow{\alpha_X} \mathbb{H}^i(X, j_{!*} \mathcal{L})$, the weight w of $H^{i-1}(X_v, j_{!*} \mathcal{L})$ satisfy $w < a + i$ since X_v is closed, then the morphism α_X (precisely $Gr_{a+i}^W(\alpha_X)$) is injective and $\mathbb{H}^i(X, j_{!*} \mathcal{L})$ is pure of weight $a + i$.

2.4.6. The morphisms ∂_X and α_X are compatible with ${}^p\tau$ and W . We still denote by ∂_X (resp. α_X) their restriction to ${}^p\tau_*\mathbb{H}^{r-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})$ (resp. ${}^p\tau_*\mathbb{H}_c^r(X - X_v, j_{!*}\mathcal{L})$), then the composed morphism has value in $\mathbb{H}^r(X, j_{!*}\mathcal{L})$ (in fact in the subspace ${}^p\tau_*$ but we avoid the filtration ${}^p\tau$ as we have no information at the point v yet).

Corollary 2.21. $Gr_a^W Gr_0^{p\tau}\mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}) = 0$.

It is enough to apply the lemma 2.17 to $H^{-1}(B_v - v, \text{Ker } \rho_0)$, since $Gr_a^W Gr_0^{p\tau}\mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}) \simeq Gr_a^W H^{-1}(B_v - v, \text{Ker } \rho_0 \oplus \text{Im } G_0)$ splits (formula 2.11) and $Gr_a^W H^{-1}(B_v - v, \text{Im } G_0) = 0$ by induction.

By lemma 2.17, each element $\bar{u} \in Gr_0^{p\tau} Gr_a^W \mathbb{H}^{-1}(B_v^*, \text{Ker } \rho_0)$ is the class modulo ${}^p\tau_{-1}$ of an element $u \in ({}^p\tau_0 \cap W_a)\mathbb{H}^{-1}(B_{X_v}^*, j_{!*}\mathcal{L})$, where $u = \text{can}(b)$ with $b \in ({}^p\tau_0 \cap W_a)\mathbb{H}^{-1}(B_{X_v}^*, j_{H!}j_H^*j_{!*}\mathcal{L})$.

We use the commutative diagram to lift the elements as needed

$$\begin{array}{ccc} W_a\mathbb{H}^{-1}(B_v - v, \tau_{\leq 0}Rf_*j_{H!}j_H^*j_{!*}\mathcal{L}) & \xrightarrow{\text{can}} & Gr_a^W H^{-1}(B_v - v, \text{Ker } \rho_0) \\ \downarrow & & \downarrow \\ W_a\tau_{\leq 0}\mathbb{H}^{-1}(B_{X_v} - X_v, j_{H!}j_H^*j_{!*}\mathcal{L}) & \xrightarrow{\text{can}} & Gr_a^W Gr_0^{\bar{0}}\mathbb{H}^0(B_{X_v} - X_v, j_{!*}\mathcal{L}) \end{array}$$

Since $\gamma(u) = \alpha_X(\partial_X u) = 0$, we deduce $\partial_X u = 0$ in $Gr_a^W \mathbb{H}_c^0(B_{X_v}^*, j_{!*}\mathcal{L})$ by lemma (2.20), hence the class $\overline{\partial_X u} = 0$ in $Gr_0^{p\tau} Gr_a^W \mathbb{H}_c^0(B_{X_v}^*, j_{!*}\mathcal{L})$. Finally, since $\partial_X(\bar{u}) = \overline{\partial_X(u)} = 0$, we deduce $\bar{u} = 0$ by lemma (2.19). This ends the proof of the crucial case and the proposition 2.2 for f a fibration by NCD.

Remark 2.22. The following relation follows from the proof of the proposition 2.2

$$\text{Im} \left({}^p\mathcal{H}^i(Rk_{v!}k_v^*K) \rightarrow {}^p\mathcal{H}^i(K) \right) = k_{v!}k_v^*{}^p\mathcal{H}^i(K)$$

2.4.7. The inductive step theorem 1.1 and most of the corollary 1.2 follow for f a fibration by NCD. We complete now the proof of equation 1.7.

Lemma 2.23. *On a projective variety V , we have an orthogonal decomposition of a polarized HS of weight $a + i + j$*

$$Gr_i^{p\tau}\mathbb{H}^{i+j}(X, j_{!*}\mathcal{L}) \simeq \mathbb{H}^j(V, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})) \xrightarrow{\sim} \bigoplus_{l \leq n}^{S_i \subset V_i^*} \mathbb{H}^j(V, i_{S_l!}\mathcal{L}_{S_l}^i[l]).$$

The proof is by induction on $\dim.V$ and for fixed V on $\dim.X$. The case of $\dim.V = 1$ is clear and may be proved by the technique of the inductive step. We suppose the result true for $\dim.V = n - 1$. For $\dim.V = n$, by corollary 2.12, the case of ${}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})$ for $i < 0$ follows from the case ${}^p\mathcal{H}^i(Rf_*(i_H)_*i_H^*j_{!*}\mathcal{L})$ by restriction to a general ample hyperplane H , and by duality the case for $i > 0$. It remains to prove the crucial case of ${}^p\mathcal{H}^0(Rf_*j_{!*}\mathcal{L})$. First we prove that for each non generic component S_l with $l < n$, the component $\mathbb{H}^j(V, i_{S_l!}\mathcal{L}_{S_l}^0[l])$ is a sub HS.

We may suppose there exists a projection $\pi : V \rightarrow V_l$ to a projective variety V_l inducing a finite morphism on $\overline{S_l}$. By induction, the property apply for the decomposition of π_*f as $\dim.V_l < n$, which contains the component $\pi_*i_{S_l!}\mathcal{L}_{S_l}^0[l]$.

It follows that the direct sum $\bigoplus_{l < n}^{S_i \subset V_i^*} \mathbb{H}^j(V, i_{S_l!}\mathcal{L}_{S_l}^i[l]) \subset Gr_0^{p\tau}\mathbb{H}^j(X, j_{!*}\mathcal{L})$ is a sub-HS. Then, it is a standard argument to show that its orthogonal subspace, which coincides with $\mathbb{H}^j(V, i_{S_n!}\mathcal{L}_{S_n}^i[l])$ (the generic component) is a sub-HS. This is

a clarification of the proof for constant coefficients in [CaMi 5] and a generalization to coefficients.

Remark 2.24. For a quasi-projective variety, we must consider MHS. This apply in general for coefficients in a mixed Hodge complex generalizing the notion of variation of MHS. The pure case in this paper is the building bloc.

2.4.8. *Proof for any projective morphism.* The above results establish the case of fibrations by NCD. Given $j_{1*}\mathcal{L}$ on X and a diagram $X' \xrightarrow{\pi} X \xrightarrow{f} V$ such that the desingularization π and $f' := f \circ \pi$ are fibrations by NCD over the strata. We apply the above result to π and the extension $j'_{1*}\mathcal{L}$ on X' to deduce the decomposition of $K := R\pi_*j'_{1*}\mathcal{L}$ on X into a direct sum of intermediate extensions.

We indicate here, how we can extend Hodge theory and deduce the structure of polarized VHS on \mathcal{L}_S^i in all cases without reference to the the special fibration case.

Let \mathcal{L} be defined on a smooth open set Ω of X , $j : \Omega \rightarrow X$ and $j' : \Omega \rightarrow X'$ an open embedding into a desingularization of X .

Lemma 2.25. *Let $\pi : X' \rightarrow X$ be a desingularization defined by a fibration by NCD over the strata, $j'_{1*}\mathcal{L}$ the intermediate extension of \mathcal{L} on X' . If X is projective, $\mathbb{H}^i(X, j_{1*}\mathcal{L})$ is a sub-Hodge structure of $Gr_0^{p\tau}\mathbb{H}^i(X', j'_{1*}\mathcal{L})$. The induced Hodge structure on $\mathbb{H}^i(X, j_{1*}\mathcal{L})$ is independent of the choice of X' .*

Proof. We apply the above result to $\pi : X' \rightarrow X$ to deduce the decomposition of ${}^p\mathcal{H}^0(R\pi_*j'_{1*}\mathcal{L}) \simeq \bigoplus_{S \in \mathcal{S}} \mathcal{L}_S^0$ on X into a direct sum consisting of intermediate extensions of polarized VHS.

It follows from lemma 2.23, that the decomposition

$$Gr_0^{p\tau}\mathbb{H}^i(X', j'_{1*}\mathcal{L}) \simeq \mathbb{H}^i(X, {}^p\mathcal{H}^0(R\pi_*j'_{1*}\mathcal{L})) \simeq \bigoplus_{S \in \mathcal{S}} \mathbb{H}^i(X, i_{S!}\mathcal{L}_S^0)$$

is compatible with HS. Moreover, on the big strata U (as we can suppose V irreducible), we have $j_{1*}\mathcal{L} = i_{U!}\mathcal{L}_U^0$; from which we deduce a HS on $\mathbb{H}^i(X, j_{1*}\mathcal{L})$ as a sub-quotient HS of $\mathbb{H}^i(X', j'_{1*}\mathcal{L})$.

The uniqueness is deduced by the method of comparison of the two desingularizations X'_1 and X'_2 with a common desingularization X' of the fiber product $X'_1 \times_X X'_2$. \square

2.4.9. *Proof of the corollary 2.10 and lemma 2.23 for any projective morphism.* We describe the underlying structure of polarized VHS on \mathcal{L}_v^i .

Let $f : X \rightarrow V$, $\pi : X' \rightarrow X$ such that π and $f \circ \pi$ are fibrations, $v \in S$ a general point of a strata S on V , and N_v a general normal section to S at v such that the inverse image X_{N_v} is normally embedded in X (resp. X'_{N_v} in X'). By definition the fiber $\mathcal{L}_{S,v}^i$ is the image of the Intersection morphism:

$$\mathbb{H}_{X_v}^i(X_{N_v}, j_{1*}\mathcal{L}) \xrightarrow{I} \mathbb{H}^i(X_v, j_{1*}\mathcal{L}).$$

By transversality, the perverse truncation ${}^p\tau_X$ on $K := R\pi_*j'_{1*}\mathcal{L}$ on X , is compatible with the restriction of K to X_{N_v} . Hence, we can suppose v in the zero dimensional strata. As the decomposition is established for π and $f \circ \pi$, it follows for f , and $\mathbb{H}_{X_v}^i(X, {}^p\mathcal{H}^0(K)) = Gr_0^{p\tau_X}\mathbb{H}_{X'_v}^i(X', j'_{1*}\mathcal{L})$ carry an induced MHS (see subsection 5.3).

The natural decomposition on ${}^p\mathcal{H}^0(K)$ induce a decomposition of $\mathbb{H}_{X_v}^i(X, {}^p\mathcal{H}^0(K))$ into direct sum of MHS where $\mathbb{H}_{X_v}^i(X, j_{1*}\mathcal{L})$ is a summand, on which the MHS is realized as sub-quotient of the MHS on $\mathbb{H}_{X'_v}^i(X', j'_{1*}\mathcal{L})$.

A similar argument puts on $\mathbb{H}^i(X_v, j_{1*}\mathcal{L})$ a MHS as a sub-quotient of the MHS on $Gr_0^{p\tau_X}\mathbb{H}^i(X'_v, j'_{1*}\mathcal{L})$ such that the image MHS of the intersection morphism I is pure on \mathcal{L}_v^i .

Verdier duality between \mathcal{L}_S^{-k} and \mathcal{L}_S^k follows from the auto-duality of $j_{1*}\mathcal{L}$ by Verdier formula for the proper morphism f . Since Hard Lefschetz isomorphisms between \mathcal{L}_S^{-k} and \mathcal{L}_S^k are also satisfied, we deduce that the VHS \mathcal{L}_S^k are polarized.

Similarly, the decomposition for ${}^p\mathcal{H}^i(Rf_*{}^p\mathcal{H}^0(R\pi_*j'_{1*}\mathcal{L}))$ is part of the decomposition for ${}^p\mathcal{H}^i(R(f \circ \pi)_*j'_{1*}\mathcal{L})$ and contains the decomposition of ${}^p\mathcal{H}^i(Rf_*j_{1*}\mathcal{L})$. *This ends the proof of lemma 2.23 and altogether the proof of corollary 2.10.*

3. FIBRATION BY NORMAL CROSSING DIVISORS

The proof of the decomposition is reduced to the case of a fibration by NCD (definition 1.4) over the strata, in which case the proof along a strata is reduced to the zero dimensional strata by intersecting with a normal section, such that we can rely on logarithmic complexes in all arguments based on Hodge theory.

The proof of the proposition 1.5 is divided into many steps, *first we transform the morphism f* and then, simultaneously, a desingularization $\pi : X' \rightarrow X$.

3.0.1. *Thom-Whitney stratification.* Let $f : X \rightarrow V$ be an algebraic map, and $\mathcal{S} = (S_\alpha)$ a Whitney stratification by a family of strata S_α of V . The subspaces $V_l = \cup_{\dim S_\alpha \leq l} S_\alpha$ form an increasing family of closed algebraic sub-sets of V of dimension $\leq l$, with index $l \leq n$, where n is the dimension of V ,

$$V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V_n = V$$

The inverse of a sub-space $Z \subset V$, is denoted $X_Z := f^{-1}(Z)$, in particular $X_S = f^{-1}(S)$ for a strata S of \mathcal{S} .

A Thom-Whitney stratification of f has the following properties:

- (T) Over a strata S of V the morphism $f|_S : X_S \rightarrow S$ induced by f , is a locally trivial topological fibration.
- (W) The link at any point of a strata is a locally constant topological invariant of the strata [Mat 12], [LeT 83].

Lemma 3.1. *Let $f : X \rightarrow V$ be a projective morphism, and Y a strict closed algebraic subset containing the singularities of X . There exists a commutative diagram:*

$$\begin{array}{ccccccc} X & \xleftarrow{\pi_1} & X_1 & \dots & \xleftarrow{\pi_i} & X_i & \dots & X_k & \xleftarrow{\pi_{k+1}} & X_{k+1} \\ f \downarrow & & f_1 \downarrow & & & f_i \downarrow & & f_k \downarrow & & f_{k+1} \downarrow \\ V & \xleftarrow{id} & V & \dots & \xleftarrow{id} & V & \dots & V & \xleftarrow{id} & V \end{array}$$

such that $f_{i+1} := f_i \circ \pi_{i+1}$ and a decreasing sequence V^i for $0 < i \leq k$ of closed algebraic subspaces of V of dimension $d_i > 0$ such that for $j < i$, the inverse image $f_i^{-1}(V^j)$ are NCD in the non-singular variety X_i , and $f_i^{-1}(V^j) - f_i^{-1}(V^{j+1})$ is a relative NCD over $V^j - V^{j+1}$.

Moreover, there exists a Whitney stratification \mathcal{S}^i of V adapted to V^j for $j \leq i$, satisfying the following relation: the strata of \mathcal{S}^i and of \mathcal{S}^{i-1} coincide outside V^i .

The morphism π_{i+1} , obtained by blowing-ups over $f_i^{-1}(V^i)$ in X_i , transforms $f_i^{-1}(V^i)$ into a NCD and π_{i+1} induces an isomorphism:

$$(\pi_{i+1})|_i : (X_{i+1} - f_{i+1}^{-1}(V^i)) \xrightarrow{\sim} X_i - f_i^{-1}(V^i).$$

Moreover, the morphisms π_s are modifications over $f_i^{-1}(V^i)$ for all $s \geq i + 1$.

Let $\rho_i := \pi_1 \circ \dots \circ \pi_i$ for all i . The open subset $\Omega := f^{-1}(V - V^1) \subset X$ is dense in X , the restriction of ρ_i to $\rho_i^{-1}(\Omega) \subset X_i$ is an isomorphism, and the restriction of the morphism f_i to $\rho_i^{-1}(Y \cap \Omega)$ is a fibration by relative NCD over $V - V^1$.

Moreover, we can suppose the family of subspaces V^i maximal: the dimension of V^i is $n - i$ for $i > 0$ and $k + 1 = n$.

We refer to the morphism $f_{k+1} : X_{k+1} \rightarrow V$ (resp. diagram) as an admissible fibration with respect to the family V_i .

We can always suppose $V = f(X)$. Let $\pi_1 : X_1 \rightarrow X$ be a desingularisation morphism of X such that $\pi_1^{-1}(Y)$, as well the inverse image of the irreducible components of Y , are NCD in X_1 . Let $f_1 := f \circ \pi_1$ with X_1 smooth; there exists an open subset $U \subset V$ such that the restriction of f_1 over U is smooth.

If the dimension of $f(Y)$ is strictly smaller than $n := \dim V$, let $U \subset V - f(Y)$ over which the restriction of f_1 is smooth and let $V^1 := V - U$.

In the case $f(Y) = V$, we suppose moreover the restriction of f_1 to $\pi_1^{-1}(Y) \cap f_1^{-1}(U)$ is a fibration by relative NCD over U and let $V^1 := V - U$. We remark that $d_1 := \dim V^1$ is strictly smaller than $\dim V = n$ and we can always choose U , hence V^1 , such that $d_1 = n - 1$.

We consider a Thom-Whitney stratification of the morphism $f_1 : X_1 \rightarrow V$; in particular the image by f_1 of a strata of X_1 is a strata of V . We suppose also that the stratification is compatible with the divisor $\pi_1^{-1}(Y)$ and let \mathcal{S}^0 denotes such stratification of V .

We construct now the algebraic sub-spaces V^i of V , the morphisms $f_i = X_i \rightarrow V$, and $\pi_i : X_i \rightarrow X_{i-1}$ in the lemma.

Let \mathcal{S}^1 be a stratification compatible with V^1 and a refinement of \mathcal{S}^0 . Let $D^1 := f_1^{-1}(V^1)$; we introduce the horizontal divisor: $Y_h^1 := (\pi_1^{-1}(Y) \cap f_1^{-1}(V - V^1))$.

We construct $\pi_2 : X_2 \rightarrow X_1$ by blowings-up over $D^1 \cup Y_h^1$, without modification of $X_1 - D^1$, such that the inverse image $D^2 := \pi_2^{-1}(D^1)$ of D^1 and the union $D^2 \cup Y_h^2$ with $Y_h^2 := \pi_2^{-1}(Y_h^1)$ are NCD in the smooth variety X_2 .

Let $f_2 := f_1 \circ \pi_2$, hence $D^2 := f_2^{-1}(V^1)$, the next argument allow us to construct a sub-space V^2 such that $D^2 \cap f_2^{-1}(V^1 - V^2)$ is a relative NCD over $V^1 - V^2$.

Lemma 3.2. *Let $f : X \rightarrow Z$ be a projective morphism on a non-singular space X , T an algebraic sub-space of Z such that $D := f^{-1}(T)$ is a NCD in X as well the inverse image of each irreducible component of T .*

Then, there exists a non singular algebraic subset S_0 in T , such that the dimension of $T - S_0$ is strictly smaller than $\dim T$, over which f is a relative NCD.

Proof. i) Let D_{β_i} denote the components of the divisor D and $D_{\beta_1, \dots, \beta_j}$ the intersection of $D_{\beta_1}, \dots, D_{\beta_j}$. Let $d := \dim T$ be the dimension of T ; there exists an open subset $S_{\beta_1, \dots, \beta_j}$ complementary of an algebraic sub-space of T of dimension strictly smaller than d , such that $f^{-1}(S_{\beta_1, \dots, \beta_j}) \cap D_{\beta_1, \dots, \beta_j}$ is either empty, or a topological fibration over $S_{\beta_1, \dots, \beta_j}$. Over the open subset $S'_0 := \cap_{\beta_1, \dots, \beta_j} S_{\beta_1, \dots, \beta_j}$, the divisor $D \cap f^{-1}(S'_0)$ is a topological fibration, which is needed to satisfy the assertion (2) of the definition 1.4.

ii) Still, we need to check the assertion (3) of the definition 1.4. Let \overline{S}_1 be an irreducible component of T of dimension d . There exists a dense open subset S'_1 in \overline{S}_1 with inverse image a sub-NCD of D , by the above hypothesis. As the argument is local, we consider an open affine subset \mathcal{U} of Z with non empty intersection $S_1 := \overline{S}_1 \cap \mathcal{U} \subset S'_1$ and a projection $q : \mathcal{U} \rightarrow \mathbb{C}^d$ whose restriction to S_1 is a finite

projection. There exists an open affine dense subset U_1 of \mathbb{C}^d such that $q \circ f$ induces a smooth morphism over U_1 .

Considering S'_0 in i) above, we notice that f induces over $q^{-1}(U_1) \cap S'_0$ a fibration by NCD: indeed, let x be a point in $q^{-1}(U_1) \cap S'_0 \cap S_1$. There exists an open neighborhood \mathcal{U}_x of x in $(q^{-1}(U_1) \cap S'_0) \subset \mathcal{U}$, small enough such that the restriction of q to $\mathcal{U}_x \cap S_1$ is non ramified on its image.

On the other hand, for all $y \in \mathcal{U}_x$, we have: $f^{-1}(q^{-1}(q(y)) \cap \mathcal{U}_x)$ is smooth, since $q(y) \in U_1$. The dimension of $f^{-1}(q^{-1}(q(y)) \cap \mathcal{U}_x)$ is $\dim X - d$. As x is in S'_0 the dimension of $f^{-1}(y) \cap D$ is $\dim X - 1 - d_1$ and it is a divisor with normal crossings in $f^{-1}(q^{-1}(q(y)) \cap \mathcal{U}_x)$ of dimension: $\dim X - d$.

We remark that $q^{-1}(q(x)) \cap \mathcal{U}_x$ is a normal section of $U_1 \cap S'_0$ at x in V . Hence f induces on $q^{-1}(U_1) \cap S'_0$ a fibration by NCD in the normal sections.

The open subset S_0 is the union of open subsets $q^{-1}(U_1) \cap S'_0$ for the various irreducible components \overline{S}_1 of T of maximal dimension d , which proves the assertion (3) of the definition and ends the proof of the lemma 3.2.

End of the proof. Let d_1 be the dimension of V^1 . After the lemma 3.2 there exists an open algebraic subset S_0 of V^1 , over which the restriction of f_2 induces a relative NCD over S_0 , moreover, the dimension d_2 of the algebraic set $V^2 := V^1 - S_0$ is $d_2 < d_1$ strictly less than d_1 . If $d_1 = n - 1$, we can always choose S_0 such that $d_2 = n - 2$.

We define a new Whitney stratification \mathcal{S}^2 of V , compatible with V^2 which coincide with \mathcal{S}^1 away of V^1 , by keeping the same strata away of V^1 and adding Thom-Whitney strata including the connected components of S_0 just defined and strata in their complement V^2 in V^1 .

We complete the proof by repeating this argument for the closed algebraic sub-space V^2 of V^1 . This process is the beginning of an inductive argument as follows.

Hypothesis of the inductive argument: Given projective morphisms $f_j : X_j \rightarrow V$ for $1 \leq j \leq i$ as in the diagram above, and $\pi_j : X_j \rightarrow X_{j-1}$, $f_j := f_{j-1} \circ \pi_j$ where X_j is non-singular, a Thom-Whitney stratification \mathcal{S}^i of V compatible with a family of algebraic sub-spaces $V^j \subset \dots \subset V^1 \subset V$ for $j \leq i$ such that there exists a stratification associated to X_i , stratifying f_i , and such that $f_{i-1}^{-1}(V^j)$ are NCD in X_i , as well the inverse image of each irreducible component of V^j for $1 \leq j < i$, and moreover f_i induces an admissible morphism over $V - V^i$. Also, let Y_h^i be a divisor of X_i such that $f_i^{-1}(V^1) \cup Y_h^i$ is a NCD and $Y_h^i \cap (X_i - f_i^{-1}(V^1))$ is a relative NCD.

A sequence of blowings-up centered over $f_i^{-1}(V^i)$ leads to the construction of $\pi_{i+1} : X_{i+1} \rightarrow X_i$ such that X_{i+1} is non-singular and the inverse images of V^j by $f_i \circ \pi_{i+1}$, as well its irreducible components for $1 \leq j \leq i$, and their union with $Y_h^{i+1} := \pi_{i+1}^{-1}(Y_h^i)$ are NCD in X_{i+1} .

Let $f_{i+1} := f_i \circ \pi_{i+1}$. The stratification \mathcal{S}^i of V underlies a stratification of f_{i+1} . It follows from lemma 3.2 that in each maximal strata S of V^i in \mathcal{S}^i , there exists an open dense subset $S_0^i(S)$ over which π_{i+1} is a relative NCD. Let V_0^i be the union of all S_0^i . The complement $V^i - V_0^i$ of V_0^i is a closed algebraic strict sub-space V^{i+1} of V^i , and f_{i+1} is admissible over $V - V^{i+1}$. We construct a refinement of the stratification \mathcal{S}^i and then a Thom-Whitney stratification of \mathcal{S}^{i+1} compatible with V^{i+1} , keeping the same strata away from V^i and introducing as new strata

the connected components of the open subset V_0^i of V^i , then completeing by a Thom-Whitney stratification of the complement in V^i .

The inductive argument ends when $V^{k+1} = \emptyset$, which occurs after a finite number of steps since the family (V^i) is decreasing.

Remark 3.3. Let $d_i := \dim.V^i$, we can suppose $V_{d_i} = V^i$ of dimension d_i for $0 < i \leq k$.

Corollary 3.4. *i) In the lemma 3.1, the morphism f_{k+1} is a fibration by NCD over the strata, moreover we can suppose $k = n$.*

ii) For any X , there exists a modification $\pi : X' \rightarrow X$ which is a fibration by NCD over the the strata, and transform an algebraic sub-space Y containing the singularities of X into a divisor with normal crossings $Y' := \pi^{-1}(Y)$.

iii) We can suppose in the preceding diagram that each morphism π_i is a fibration by NCD over the strata.

The assertion i) is clear. In the case where f is the identity of X in the lemma 3.1, we construct a modification X' of X compatible with Y , which is a fibration by relative NCD over X , which prove ii). By the same argument, we can suppose all π_i in the lemma admissible which prove iii).

Proof of the proposition 1.5. Going back to the inductive argument for f in the lemma, we apply at each step of the induction the assertion (ii) of the corollary, in particular we can start with the desingularization by an admissible modification.

Hypothesis of the induction. We suppose there exists:

1) A diagram of morphisms D_i :

$$X \xleftarrow{\pi'_i} X_i \xrightarrow{f_i} V, \quad f_i = f \circ \pi'_i$$

where π'_i is admissible.

2) A decreasing family of algebraic sub-spaces $V^i \subset V^j$ for $j < i$ with inverse image $f_i^{-1}(V^j)$ consisting of NCD in X_i for $j < i$, and a stratification \mathcal{S}^i of V compatible with the family V^j , such that the restriction of f_i to $X_i - f_i^{-1}(V^i)$ over $V - V^i$ is a fibration by relative NCD over the strata.

Inductive step. Let d_i (resp. $n - i$) be the dimension of V^i . We want to define a sub-space $V^{i+1} \subset V^i$ of dimension strictly smaller $d_{i+1} < d_i$ (resp. $n - i - 1$) and to extend the diagram over the open subset $V^i - V^{i+1}$, that is, to construct a diagram of morphisms D_{i+1} :

$$X \xleftarrow{\pi'_{i+1}} X_{i+1} \xrightarrow{f_{i+1}} V, \quad f_{i+1} = f \circ \pi'_{i+1}$$

such that:

1) $\pi'_{i+1} : X_{i+1} \xrightarrow{\pi_{i+1}} X_i \xrightarrow{\pi'_i} X$ is admissible and defined as a composition of $\pi'_i : X_i \rightarrow X$ with a modification π_{i+1} inducing an isomorphism: $X_{i+1} - f_{i+1}^{-1}(V^i) \xrightarrow{\sim} X_i - f_i^{-1}(V^i)$.

2) $f_{i+1}^{-1}(V^i)$ is a relative NCD over the open subset $V^i - V^{i+1}$.

To achieve this step, we apply a slightly modified version of the lemma 3.1.

Lemma 3.5 (Relative case). *Let $f : X \rightarrow V$ be a projective morphism and Z a strict algebraic sub-space of V of dimension ℓ .*

With the notations of the definition 1.4, there exists an admissible diagram in the sense of the lemma 3.1 and an index i such that $Z \subset V^i$, $\dim V^i = \dim Z$, such that $Z' := f_i^{-1}(Z)$ is a NCD in X_i relative over $V^i - V^{i+1}$.

Then, we may construct a stratification of f , such that Z is a sub-space of V_ℓ , union of the strata of dimension $\leq \ell$, and $Z' := f_\ell^{-1}(Z)$ is a NCD in X_ℓ relative over $V_\ell - V_{\ell-1}$. Hence, there exists a diagram

$$X \xleftarrow{\pi'} X' \xrightarrow{f'} V, \quad f' := f \circ \pi'$$

such that $f'^{-1}(Z) = \pi'^{-1}(f^{-1}(Z))$ is a relative NCD over $Z - (Z \cap V_{\ell-1})$.

Proof. Indeed, in the preceding construction, we can suppose that all sub-spaces V^i of dimension $d_i \geq \ell$ contain Z . Let V^k and V^{k+1} such that $d_k > \ell \geq d_{k+1}$, then we choose stratifications \mathcal{S}^i of V compatible with Z and replace V^{k+1} by $V^{k+1} \cup Z$, hence we are obliged to replace all V^i for $i \leq k+1$ by the sub-spaces $V^i \cup Z \supset V^{k+1} \cup Z$, then we construct $\pi_\ell : X^\ell \rightarrow X^k$ such that $\pi_\ell^{-1}(f_k^{-1}(V^{k+1} \cup Z)) \subset X^\ell$ is a NCD. \square

We remark that the variety $f_i^{-1}(V^i)$ in X_i is over $f^{-1}(V^i)$ in X . Then, we apply the lemma (3.5) for $\pi'_i : X_i \rightarrow X$ instead of $f : X \rightarrow V$ in the lemma and $f^{-1}(V^i) \subset X$ instead of $Z \subset V$, to construct the admissible morphism π'_{i+1}

$$\pi'_i : X_i \rightarrow X, \quad f^{-1}(V^i) \subset X, \quad X_i \xleftarrow{\pi'_{i+1}} X_{i+1} \xrightarrow{\pi'_{i+1}} X, \quad \pi'_{i+1} := \pi'_i \circ \pi_{i+1}$$

That is we develop the constructions of the lemma 3.1 to construct π'_{i+1} over X by modification only of sub-spaces over $f_i^{-1}(V^i)$ to transform $f_i^{-1}(V^i)$ into a NCD, hence X_{i+1} in the diagram D_{i+1} differs from X_i in the diagram D_i only over $f_i^{-1}(V^i)$.

At this stage, we go back to the construction in the lemma to extend f_i over an open subset Ω of V^i such that $V^{i+1} := V^i - \Omega$ is of dimension smaller than d_i and we define $f_{i+1} : f_i \circ \pi_{i+1}$, then we can complete the diagram \mathcal{D}_{i+1} and the inductive step. At the end we define π'_m and f_m both admissible for some index m . \square

Corollary 3.6. *The decomposition theorem for f can be deduced from both cases π' and f' in the proposition 1.5.*

Let \mathcal{L} be a local system on $X - Y$, there exists an open algebraic set $\Omega \subset X - Y$ dense in X such that $\Omega' := \pi'^{-1}(\Omega) \subset X'$ is isomorphic to Ω , which carry the local system \mathcal{L} . Let $j' : \Omega' \rightarrow X'$, then the decomposition theorem for $j'_*\mathcal{L}$ with respect to the original proper algebraic morphism f follows from both cases of f' and π' ([De 68] proposition 2.16).

Remark 3.7. In the construction of a relative NCD X_{V_i} over $V_i - V_{i-1}$, for more clarity we ask for both conditions:

i) The restriction of f to $X_S := f^{-1}(S)$ over each strata S of $V_i - V_{i-1}$ is a topological fibration: $f|_S : X_S \rightarrow S$.

ii) For each point $v \in V_i - V_{i-1}$ smooth in V_i , let N_v be a normal section to V_i at v in general position, then $f^{-1}(N_v)$ is smooth in X and its intersection with the NCD X_{V_i} is transversal.

These two conditions may be equivalent.

4. LOGARITHMIC COMPLEXES

We develop the construction of Hodge theory by logarithmic complexes with coefficients in an admissible graded polarized variation of mixed Hodge structure (VMHS): (\mathcal{L}, W, F) with singularities along a normal crossing divisor Y . We refer to [Ka 86] and [EY 14] for basic computations and to [Ka 86, ICTP 14, StZ 85] for admissibility.

The admissibility on $X - Y$ refers to asymptotic properties of (\mathcal{L}, W, F) along the NCD. Such asymptotic properties are expressed on Deligne's extension \mathcal{L}_X defined in terms of the "multivalued" horizontal sections of ∇ on $X - Y$, on which the connection ∇ extends on \mathcal{L}_X with logarithmic singularities along Y .

The extension \mathcal{L}_X is a locally free analytic sheaf of modules, hence algebraic if X is projective. It is uniquely characterized by the residues of the logarithmic singularities of the connexion ∇ , and defined in terms of a choice of the logarithm of the eigenvalues of the monodromy ([De 70] Théorme d'existence Proposition 5.2).

The fibre of the vector bundle $\mathcal{L}(x) := \mathcal{L}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ is viewed as the space of the "multivalued" horizontal sections of \mathcal{L} at x (sections of a universal covering of the complementary of Y in a ball B_x at x).

The extension over Y of the Hodge filtration of a polarizable variation of HS by Schmid [Sc 73, GrSc 73, CaKSc 86] is a fundamental asymptotic property, that is required by assumption for a graded polarizable variation of MHS as a condition to admissibility.

The local monodromy T_i around a component Y_i of Y , defines a nilpotent endomorphism $N_i := \text{Log } T_i^u$ logarithm of the unipotent part of T preserving the extension of the filtration W of \mathcal{L} by sub-bundles $\mathcal{W}_X \subset \mathcal{L}_X$.

Deligne pointed out the problem of the existence of the relative monodromy filtration $M(\sum_i N_i, W)$ in ([De 80], I.8.15). The required properties are proved in the case of geometric variation of MHS over a punctured disc in [E 83] and studied axiomatically as conditions of admissibility in [StZ 85]. The definition of admissibility in [Ka 86] along a NCD is by reduction to the case of a punctured disc.

We construct below the weight filtration directly on the logarithmic complex (4.1), generalizing the case of constant coefficients in [De 72].

We remark that the construction of the Intersection complex (4.3.1), the intermediate extension of a local system on $X - Y$, as well the development of mixed Hodge theory, involve the behaviour at "infinity", along the NCD Y .

4.0.1. Notations. Let $Y := \cup_{i \in I} Y_i$ be a NCD, union of smooth irreducible components with index I , and for $J \subset I$, set $Y_J := \cap_{i \in J} Y_i$, $Y_J^* := Y_J - \cup_{i \notin J} (Y_i \cap Y_J)$ ($Y_\emptyset^* := X - Y$). We denote uniformly the various embeddings by $j : Y_J^* \rightarrow X$.

The local system \mathcal{L} on $X^* := X - Y$ is defined by a connection ∇ on the fibre bundle $\mathcal{L}_{X^*} = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_{X^*}$ with horizontal sections given by \mathcal{L} . The extension of \mathcal{L}_{X^*} with a regular singular connection is a couple consisting of a fibre bundle \mathcal{L}_X and a connection $\nabla : \mathcal{L}_X \rightarrow \Omega_X^1(\text{Log } Y) \otimes \mathcal{L}_X$ ([De 70], [Ma 87] definition 3.1). The residue of ∇ is defined along a component Y_i of the NCD Y as an endomorphism of the restriction $\text{Res}_{Y_i} \nabla : \mathcal{L}_{Y_i} \rightarrow \mathcal{L}_{Y_i}$, of \mathcal{L}_X to Y_i .

The eigenvalues of the residue are constant along a connected component of Y_j and related to the local monodromy T_j of \mathcal{L} at a general point of Y_j by the formula: $\text{Log } T_j = -2i\pi \text{Res}_{Y_j} \nabla$ ([De 70], theorem 1.17, proposition 3.11).

The construction of \mathcal{L}_X is local. Deligne's idea is to fix the choice of the residues of the connection by the condition that the eigenvalues of the residue belong to the image of a section of the projection $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$, determined by fixing the real part of $z : n \leq \mathcal{R}(z) < n + 1$, hence fixing the determination of the logarithm $\text{Log} : \mathbb{C}^* \rightarrow \mathbb{C}$, and forcing the uniqueness of the construction. Hence the local constructions glue into a global bundle. In Deligne's extension case $n = 0$.

Let $X(x) \xrightarrow{\sim} D^{n+l}$ be a neighborhood of a point x in Y isomorphic to a product of complex discs, such that $X(x)^* = X(x) \cap (X - Y) \xrightarrow{\sim} (D^*)^n \times D^l$ where D^* is the disc with the origin deleted. For simplification, we often set $l = 0$, then $X := D^n$ is a ball in \mathbb{C}^n with Y defined by $y_1 \cdots y_n = 0$. The fundamental group $\pi_1(X(x)^*)$ is a free abelian group generated by n elements representing classes of closed paths around the hypersurface Y_i , defined locally by the equation $y_i = 0$, at a general point of Y_i , one for each index i .

The restriction of the local system \mathcal{L} to $X(x)^*$ corresponds to a representation of $\pi_1(X(x)^*)$ in a vector space L , hence to the action of commuting automorphisms T_i of L for $i \in [1, n]$ indexed by the local components Y_i of Y and called local monodromy action around Y_i .

Classically L is viewed as the fibre of \mathcal{L} at the base point of the fundamental group $\pi_1(X^*)$, however to represent the fibre of Deligne's extended bundle at x , we view L as the vector space of multivalued horizontal sections of \mathcal{L} , that is the sections of the inverse of \mathcal{L} on a universal cover $\pi : \tilde{D}^n \rightarrow D^n := X(x)^*$ defined for $z = (z_1, \dots, z_n) \in \tilde{D}^n \subset \mathbb{C}^n$ by $y = \pi(z)$ with components $y_j := e^{2i\pi z_j}$, then $L := H^0(\tilde{D}^n, \pi^{-1}\mathcal{L})$.

The automorphisms T_i are defined over \mathbb{Z} and decompose as a product of semi-simple T_i^s and unipotent T_i^u commuting automorphisms $T_i = T_i^s T_i^u$. On the complex vector space L , T_i^s is diagonalizable and represented over \mathbb{C} by the diagonal matrix of its eigenvalues. The logarithm of T_i is defined as the sum $\text{Log} T_i = \text{Log} T_i^s + \text{Log} T_i^u = D_i + N_i$, where $D_i = \text{Log} T_i^s$ is diagonalizable over \mathbb{C} with entries $\text{Log} a_i$ on the diagonal for all eigenvalues a_i of T_i^s and for a fixed determination of Log on \mathbb{C}^* , while $N_i := \text{Log} T_i^u$ is a nilpotent endomorphism, defined by $N_i = -\sum_{k \geq 1} (1/k)(I - T_i^u)^k$ as a polynomial function of the nilpotent morphisms $(I - T_i^u)$, where the sum is finite. At a point x on $\cup_{i \in J \subset I} Y_i$ a product of closed paths corresponds to a sum of various N_i .

Locally, at a point $x \in Y$ on the intersection of n -components $Y_i, i \in [1, n]$, a spectral decomposition into a finite direct sum is defined on $L := \mathcal{L}(x)$ with index sequences of eigenvalues $(a.)$, with one component a_i for each T_i

$$L = \oplus_{(a.)} L^{a.}, \quad L^{a.} = \cap_{i \in [1, n]} (\cup_{j > 0} \ker (T_i - a_i I)^j).$$

4.0.2. For a detailed description of \mathcal{L}_X at $x \in Y$, let $\alpha_j \in [0, 1[$ for $j \in [1, n]$ such that $e^{-2i\pi\alpha_j} = a_j$ is an eigenvalue for T_j , and L the vector space of multivalued sections of \mathcal{L} on $X(x)^*$, then the fiber $\mathcal{L}_{X,x}$ is generated as an $\mathcal{O}_{X,x}$ -submodule of $(j_* \mathcal{L}_{X^*})_x$ by the image of the embedding of the space L of multivalued horizontal sections into $\mathcal{L}_{X,x}$ by the correspondence $v \rightarrow \tilde{v}$, defined for y near x as

$$\tilde{v}(y) = (\exp(\sum_{j \in J} (\log y_j)(\alpha_j - \frac{1}{2i\pi} N_j))).v = \prod_{j \in J} y_j^{\alpha_j} \exp(\sum_{j \in J} (\log y_j) N_j).v,$$

in ([De 70], 5.2.1- 5.2.3), then $\tilde{v}(y)$ is a uniform analytic section on $X(x)^*$. It is important to stress that a basis v_a of L is sent onto a basis \tilde{v}_a of $\mathcal{L}_{X,x}$, and if X is projective \mathcal{L}_X is an algebraic bundle by Serre's general correspondence. In the

text, we omit in general the analytic notations with X^{an} as our applications to the proof of the decomposition are on projective varieties. The action of N_i on L determines the connection ([De 70], theorem 1.17, proposition 3.11) as

$$\nabla \tilde{v} = \sum_{j \in J} [\widetilde{(\alpha_j v)} - \frac{1}{2i\pi} \widetilde{(N_j v)}] \otimes \frac{dy_j}{y_j}.$$

4.1. The logarithmic complex $\Omega^* \mathcal{L} := \Omega_X^*(\text{Log} Y) \otimes \mathcal{L}_X$.

Let \mathcal{L} be a local system (shifted by n) on the complement U of the NCD Y in a smooth complex algebraic variety X , and (\mathcal{L}_X, ∇) Deligne's extension of $\mathcal{L} \otimes \mathcal{O}_U$ with logarithmic singularities. The connection $\nabla : \mathcal{L}_X \rightarrow \Omega_X^1(\text{Log} Y) \otimes \mathcal{L}_X$ extends naturally into a complex, called the (shifted by n) logarithmic complex $\Omega_X^*(\text{Log} Y) \otimes \mathcal{L}_X$.

When \mathcal{L} underlies a variation of MHS (\mathcal{L}, W, F) , the filtration by sub-local systems W of \mathcal{L} extends as a filtration by canonical sub-analytic bundles $\mathcal{W}_X \subset \mathcal{L}_X$. By the condition of admissibility the filtration \mathcal{F}_U extends by sub-bundles $\mathcal{F}_X \subset \mathcal{L}_X$. Both \mathcal{W}_X and \mathcal{F}_X are combined here to define the structure of mixed Hodge complex.

Theorem 4.1. *Let \mathcal{L} be a shifted admissible graded polarized variation of MHS on $X - Y$. There exists a weight filtration W on the logarithmic complex with coefficients \mathcal{L}_X by perverse sheaves, and a Hodge filtration F by complexes of analytic sub-sheaves such that the bi-filtered complex*

$$(4.1) \quad \Omega^* \mathcal{L} := (\Omega_X^*(\text{Log} Y) \otimes \mathcal{L}_X, W, F)$$

underly a structure of mixed Hodge complex and induces a canonical MHS on the cohomology groups $\mathbb{H}^i(X - Y, \mathcal{L})$.

The weight is defined by constructible sub-complexes, although it consists in each degree, of analytic sub-sheaves of $\Omega_X^i(\text{Log} Y) \otimes \mathcal{L}_X$.

The filtration F is classically deduced on the logarithmic complex from the sub-bundles \mathcal{F}_X^p in \mathcal{L}_X satisfying Griffith's transversality:

$$F^p = 0 \rightarrow \mathcal{F}^p \mathcal{L}_X \cdots \rightarrow \Omega_X^i(\text{Log} Y) \otimes \mathcal{F}^{p-i} \mathcal{L}_X \rightarrow \cdots$$

In the rest of this section the direct definition of the weight filtration W as well its properties in the case of a NCD is based on the local study in [Ka 86] and [EY 14].

4.2. The direct image $Rj_* \mathcal{L} \simeq \Omega^* \mathcal{L}$. To represent the complex $Rj_* \mathcal{L} \otimes \mathbb{C}$ in the derived category, we use its de Rham realization $\Omega^* \mathcal{L} := \Omega_X^*(\text{Log} Y) \otimes \mathcal{L}_X$. Indeed, the quasi-isomorphism $Rj_* \mathcal{L} \xrightarrow{\sim} \Omega^* \mathcal{L}$ follows from Grothendieck's algebraic de Rham cohomology [Gro 66] and its generalization to local systems by Deligne ([De 70], definition 3.1).

We also describe a sub-complex $IC^* \mathcal{L}$ representing the intermediate extension [GMacP 83, Br 82, BBDG 83]. Various related definitions given here in terms of local coordinates are independent of the choice of coordinates. This approach is fit for calculus.

4.2.1. The (higher) direct image $Rj_* \mathcal{L}$. The residue of the connection ∇ on the analytic restriction \mathcal{L}_{Y_i} of \mathcal{L}_X decomposes into Jordan sum $D_i - \frac{1}{2i\pi} N_i$ where N_i is nilpotent and D_i diagonal with eigenvalues $\alpha_i \in [0, 1[$ such that the eigenvalues

of the monodromy T_j are $a_j = \mathbf{e}(\alpha_j) := e^{-2i\pi\alpha_j}$. The de Rham complex with coefficients \mathcal{L}_X , is quasi-isomorphic to $Rj_*\mathcal{L}$ ([De 70], section II.3):

$$(4.2) \quad Rj_*\mathcal{L} \xrightarrow{\sim} \Omega^*\mathcal{L} := \Omega_X^*(\text{Log}Y) \otimes \mathcal{L}_X$$

In the local situation (4.0.1) with Y defined near a point x at the origin in D^{n+l} by $y_1 \cdots y_n = 0$, the fiber of the complex $Rj_*\mathcal{L}$ is quasi-isomorphic to a Koszul complex. We associate to the component of the spectral decomposition, a strict simplicial vector space $(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i, i \in [1, n])$ such that for all sequences $(i.) = (i_1 < \cdots < i_p)$: $L(\alpha., i.) = L^{\mathbf{e}(\alpha.)}, \alpha_{i_j} Id - \frac{1}{2i\pi}N_{i_j} : L(\alpha., i. - i_j) \rightarrow L(\alpha., i.)$.

The Koszul complex is the sum of this simplicial vector space; it is denoted by $s(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i)_{i \in [1, n]}$.

Definition 4.2. The direct sum of the complexes $s(L^{\mathbf{e}(\alpha.)}, \alpha_j Id - \frac{1}{2i\pi}N_j)_{j \in [1, n]}$ over all sequences $(\alpha.)$ is denoted by $s(L, \alpha. Id - \frac{1}{2i\pi}N.)$.

It is also denoted as an exterior algebra

$$(4.3) \quad \Omega^*L := \Omega(L, \alpha. Id - \frac{1}{2i\pi}N.) = \bigoplus_{\alpha.} \Omega(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i), i \in [1, n].$$

where $\mathbf{e}(\alpha_j) = e^{-2i\pi\alpha_j} = a_j$ is an eigenvalue for T_j .

4.2.2. *The tilda embedding.* For $M \subset I$ of length $n := |M|$ and $x \in Y_M^*$, the above correspondence $v \mapsto \tilde{v}$, from L to $\mathcal{L}_{X,x}$, extends to:

$$L(i_1, \dots, i_j) \rightarrow (\Omega_X^*(\text{Log}Y) \otimes \mathcal{L}_X)_x \text{ by } v \mapsto \tilde{v} \frac{dy_{i_1}}{y_{i_1}} \wedge \dots \wedge \frac{dy_{i_j}}{y_{i_j}}.$$

It induces quasi-isomorphisms

$$(4.4) \quad (Rj_*\mathcal{L})_x \cong (\Omega_X^*(\text{Log}Y) \otimes \mathcal{L}_X)_x \cong \Omega^*L \cong s(L, \alpha. - \frac{1}{2i\pi}N.).$$

The endomorphisms $\alpha_j Id$ and N_j correspond to endomorphisms denoted by the symbols $\alpha_j Id$ and \mathcal{N}_j on the image sections \tilde{v} in \mathcal{L}_X . We recall below a proof of the following result: the subcomplex $\Omega(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i), i \in [1, n]$ is acyclic if there exists an index i such that $\alpha_i \neq 0$.

Proposition 4.3. *Let $x \in Y_M^*$ be a general smooth point of $\bigcap_{i \in M} Y_i$ for $M \subset I$, $L := \mathcal{L}_X(x)$, and $L^u \subset L$ the subspace on which the action of each monodromy T_i is unipotent.*

The complex of $\mathcal{O}_{X,x}$ -modules $(\Omega^\mathcal{L})_x$ is quasi-isomorphic to the complex of exterior algebra Ω^*L^u defined by (L^u, T_i) :*

$$\Omega^*L^u := \left(0 \rightarrow L^u \rightarrow \cdots \rightarrow \bigoplus_{\{i_1 < \dots < i_{k-1}\}} L^u \rightarrow \bigoplus_{\{i_1 < \dots < i_k\}} L^u \cdots \rightarrow 0 \right) \simeq (\Omega^*\mathcal{L})_x$$

with differential in degree $k-1$: $(d_{k-1}v_{\bullet})_{\{i_1 < \dots < i_k\}} = \sum_j (-1)^j \frac{N_{i_j}}{2i\pi} v_{\{i_1 < \dots < i_j \dots < i_k\}}$.

Proof. We show that for each power of the maximal ideal \mathfrak{m}^r at x with $r > 0$, the subcomplex $\mathfrak{m}^r(\Omega^*\mathcal{L})_x$ is acyclic, and for $r = 0$ only the component Ω^*L^u is not acyclic. As x is at the intersection of n components of Y , we associate to each monomial $y^{m.} := y^{m_1} \cdots y^{m_n}$ of degree r a complex vector subspace $sL \subset \mathcal{L}_x$ by the correspondence $v \rightarrow y^{m.}\tilde{v}$ (4.2.2) to get a complex of vector subspaces $\Omega^*Ly^{m.}$ of $\mathfrak{m}^r(\Omega^*\mathcal{L})_x$

$$(4.5) \quad \Omega^*Ly^{m.} := \left(0 \rightarrow sL \rightarrow \cdots \rightarrow \bigoplus_{\{i_1 < \dots < i_k\}} y^{m.} L \frac{dy_{i_1}}{y_{i_1}} \wedge \cdots \wedge \frac{dy_{i_k}}{y_{i_k}} \cdots \rightarrow 0 \right)$$

with differential defined by: $y^{m \cdot} \tilde{v} \rightarrow y^{m \cdot} \left((m_i + \alpha_i) Id - \frac{1}{2i\pi} N_i \right) \cdot \tilde{v} \otimes \frac{dy_i}{y_i}$. The complex $\mathfrak{m}^r(\Omega^* \mathcal{L})_x$ is an inductive limit of the sub-complexes $\Omega^* Ly^{m \cdot}$ for various sections $y^{m \cdot}$ such that $\sum_i m_i \leq r$. We show in particular that each complex $\Omega^* Ly^{m \cdot}$ is acyclic for $r > 0$.

Lemma 4.4. *For each monomial $y^{m \cdot} := y^{m_1} \cdots y^{m_n}$, the sub-complex $\Omega^* Ly^{m \cdot}$ of $(\Omega^* \mathcal{L})_x$ (4.5) is acyclic if the degree $r := \sum_{i \leq n} m_i$ of $y^{m \cdot}$ is strictly positive, hence it is quasi-isomorphic to the complex*

$$\Omega^* L := \left(0 \rightarrow L \rightarrow \bigoplus_{i \in [1, n]} L \rightarrow \cdots \rightarrow \bigoplus_{\{i_1 < \dots < i_k\}} L \cdots \rightarrow 0 \right)$$

with differential in degree $k - 1$:

$$\forall v_{\bullet}, d_{k-1}(v_{\bullet})_{\{i_1 < \dots < i_k\}} = \sum_j (-1)^j \left((m_{i_j} + \alpha_{i_j}) Id - \frac{1}{2i\pi} N_{i_j} \right) v_{\{i_1 < \dots < i_j \dots < i_k\}}.$$

Since $\nabla y^{m \cdot} \tilde{v} = \sum_i m_i y^{m \cdot} \tilde{v} \otimes \frac{dy_i}{y_i} + y^{m \cdot} \Sigma_i [\alpha_i \tilde{v} - \frac{1}{2i\pi} \widetilde{N_i v}] \otimes \frac{dy_i}{y_i} = y^{m \cdot} \left[(m_i + \alpha_i) Id - \frac{1}{2i\pi} N_i \right] \tilde{v} \otimes \frac{dy_i}{y_i}$

The complex $\Omega^* Ly^{m \cdot}$ is isomorphic to the complex $\Omega^* L$ in the lemma where the differentials appear as given by morphisms $(m_i + \alpha_i) Id - \frac{1}{2i\pi} N_i : L \rightarrow L$, hence it is acyclic if one of such morphisms is an isomorphism, that is at least one $m_j + \alpha_j \neq 0$ in which case $(m_j + \alpha_j) Id - \frac{1}{2i\pi} N_j$ is an isomorphism of L as N_j is nilpotent; indeed, the complex may be written as a cone over such morphism. We deduce the proposition as $\alpha_j \notin \mathbb{Z}$ unless $\alpha_j = 0$, then $m_j + \alpha_j = 0$ iff $m_j = 0$ and $\alpha_j = 0$. \square

Remark 4.5 (Reduction to the locally unipotent case). It follows from the proposition that the cohomology of the restriction to Y is determined locally by the unipotent subspace under the monodromy actions. This is a good reason to reduce the study of the weight filtration to local systems with locally unipotent monodromy.

In particular, if $Z \subset Y$ is a component with non locally unipotent monodromy (the monodromy T_Z around Z has no eigenvalue equal to 1), then $Rj_{Z*}(j_{!*} \mathcal{L}|_{X-Z}) = j_{Z!}(j_{!*} \mathcal{L}|_{X-Z}) = j_{!*} \mathcal{L}$, where $j_Z : X - Z \rightarrow X$ denotes the open embedding. In general, only the locally unipotent summand of \mathcal{L} is interesting.

The above description of $(Rj_ \mathcal{L})_x$ is the model for the next description of various perverse sheaves in the rest of the section.*

4.3. The intermediate extension $j_{!*} \mathcal{L} \simeq IC^* \mathcal{L}$. We describe the intermediate extension $j_{!*} \mathcal{L}$ at a point at “infinity” along the NCD Y , by a sub-complex $IC^* \mathcal{L} \subset \Omega^* \mathcal{L}$ containing the submodule $\Omega_X^* \otimes \mathcal{L}_X$. Let \mathcal{I}_Y denote the ideal product of the ideals of the components Y_i . The complex of $IC^* \mathcal{L}$ contains the product $\mathcal{I}_Y \Omega^* \mathcal{L}$ as an acyclic sub-complex by lemma 4.4. The quotient complex $\Omega^* \mathcal{L} / IC^* \mathcal{L}$ is supported by Y .

Due to (remark 4.5), we state the results for locally unipotent local system to simplify the notations and we mention as a remark the case of a general local system.

4.3.1. Definition of $IC^* \mathcal{L}$ for a locally unipotent local system. In the local situation (4.0.1) at a general point $x \in Y_M^*$ of the intersection of Y_i for $i \in M$, set for all $K = \{i_1, \dots, i_k\} \subset M = [1, n]$, $A := \mathcal{O}_x \xrightarrow{\sim} \mathbb{C}\{\{x\}\}$, $y_K := y_{i_1} \cdots y_{i_k}$ and $dy_K := dy_{i_1} \wedge \cdots \wedge dy_{i_k}$. Let \tilde{L} denote the vector space defined by sections \tilde{v} for all flat vectors $v \in L$, generating the fiber $\mathcal{L}_{X,x} := A\tilde{L}$ of the sheaf \mathcal{L}_X at x . The fiber $(\Omega^* \mathcal{L})_x$ of the sheaf $\Omega^* \mathcal{L}$ at x , is generated as an $(\Omega_X^*)_x$ -module by $\sum_{K \subset M} A\tilde{L} \frac{dy_K}{y_K}$.

Let $N_J = \prod_{j \in J} N_j$ denotes a composition of endomorphisms of L , and consider the strict simplicial sub-complex of the de Rham logarithmic complex (4.1) defined by $\text{Im} N_J := N_J L$ in $L(J) = L$.

More generally, each subspace $V \subset L$, defines a subspace $\tilde{V} \subset \tilde{L}$ generating an A -submodule of the fiber at x ; for $K, B \subset M$ we are interested in the generating subspaces $V = N_K L$ image of N_K and $V = \text{Ker } N_B$ the kernel N_B .

Definition 4.6 (Stalk of the Intersection complex $(IC^* \mathcal{L})_x$).

Let $x \in Y_M^*$ be a general point in Y_M and $A := \mathcal{O}_x$. The sub-complex $(IC^* \mathcal{L})_x \subset (\Omega^* \mathcal{L})_x$ is generated as an $\Omega_{X,x}^*$ -algebra by the \mathcal{O}_x -sub-modules:

$$\sum_{B \subset K} y_B A(\widetilde{N_{K-B} L}) \frac{dy_K}{y_K}$$

sum for all $K \subset M$ and for each K for all $B \subset K$.

As an $\mathcal{O}_{X,x}$ -module, it is generated by $\sum_{B \subset K} y_B A(\widetilde{N_{K-B} L}) \frac{dy_K}{y_K} \wedge \Omega_{X,x}^*$. In the proof, by reduction to a normal section to Y_M at x , we erase $\Omega_{X,x}^*$ in the formula.

Lemma 4.7 (Independence of the coordinates). *The definition of $(IC^* \mathcal{L})_x$ is independent of the choice of the coordinates defining Y .*

Proof. We check the independence, after restriction to a normal section, for a one variable change at a time: $z_j = f y_j$ for $j \in K$ with f invertible at x . We can suppose $j = 1$, and we write $IC^* L(z_1)$ (resp. $IC^* L(y_1)$) for the stalk at x of the complex when defined with the coordinate z_1 and y_j for $j > 1$ (resp. with the coordinate y_1 instead of z_1). We prove $IC^* L(z_1) \subset IC^* L(y_1)$.

For $K \subset M = [1, n]$ of length $|K| = k$, and $K_i \subset K$ of length $|K_i| = k - i$, since $\frac{dz_1}{z_1} = \frac{dy_1}{y_1} + \frac{df}{f}$, where $\frac{df}{f}$ is regular at x , a section $s \in z_{K_i} A(\widetilde{N_{K-K_i} L}) \frac{dz_K}{z_K} \subset IC^* L(z_1)$ is transformed into a sum

$$s = w_1 + w_2, \quad w_1 \in IC^* L(y_1), \quad w_2 \in y_{K_i} A(\widetilde{N_{K-K_i} L}) \frac{df}{f} \wedge_{j \in K - \{1\}} \frac{dy_j}{y_j}$$

to show that w_2 is also in $IC^* L(y_1)$, it is enough to check that $N_{K-K_i} L \subset N_{K-1-K_i} L$ if $1 \notin K_i$, as $\frac{df}{f}$ is regular. The proof of $IC^* L(y_1) \subset IC^* L(z_1)$ is similar. \square

Proposition 4.8 ($IC^* L$). *The intersection complex $(IC^* \mathcal{L})_x$ at a point $x \in Y_M^*$, for a locally unipotent local system \mathcal{L} , is quasi-isomorphic to the complex:*

(4.6)

$$IC^* L := \left(0 \rightarrow L \rightarrow \bigoplus_{i \in [1, n]} N_i L \cdots \rightarrow \bigoplus_{\{i_1 < \dots < i_k \subset [1, n]\}} N_{\{i_1 < \dots < i_k\}} L \cdots \rightarrow 0 \right)$$

with differentials induced by the embedding into $\Omega^* L$.

We may reduce the proof to a normal section to Y_M^* at x , that is we may suppose $(IC^* \mathcal{L})_x$ defined by the complex

$$\cdots \rightarrow \bigoplus_{K \subset M} \left(\sum_{K_i \subset K} y_{K_i} A(\widetilde{N_{K-K_i} L}) \frac{dy_K}{y_K} \right) \rightarrow \cdots$$

(instead of being generated as an $\Omega_{X,x}^*$ -module). In the definition above, $N_\emptyset = Id$, hence for $K_i = K$, $N_{K-K_i}L = L$. The complex IC^*L embeds into $(IC^*\mathcal{L})_x$ as

$$0 \rightarrow \tilde{L} \rightarrow \bigoplus_{j \in [1, n]} \widetilde{N_j L} \frac{dy_j}{y_j} \cdots \rightarrow \bigoplus_{K \subset [1, n]} \widetilde{N_K L} \frac{dy_K}{y_K} \cdots \rightarrow 0$$

We show that the quotient complex is acyclic. The proof is similar to the proof of proposition (4.3) and is carried along Y modulo $\mathcal{I}_Y \Omega^* \mathcal{L}$ which is acyclic. Let $p \in [1, n]$ be an integer, and for each monomial $y^{m^\cdot} := y_1^{m_{i_1}} \cdots y_p^{m_{i_p}}$ with all $m_{i_j} > 0$, let $K_{y^{m^\cdot}} = \{i_1, \dots, i_p\} \cap K$. We define the sub-complex $IC^*Ly^{m^\cdot}$ of $\Omega^*Ly^{m^\cdot}$ (formula 4.5) as

$$IC^*Ly^{m^\cdot} := (y^{m^\cdot}L \cdots \rightarrow \bigoplus_{K, |K|=k} s(N_{K-K_{y^{m^\cdot}}}L) \frac{dy_K}{y_K} \rightarrow \cdots \rightarrow 0).$$

Lemma 4.9. *For each section $y^{m^\cdot} := y_1^{m_1} \cdots y_p^{m_p}$ with at least one $m_i \neq 0$ for $i \in [1, p]$, the complex $IC^*Ly^{m^\cdot}$ is acyclic: $IC^*Ly^{m^\cdot} \xrightarrow{\sim} 0$.*

For $p = n$, $IC^*Ly^{m^\cdot} = \Omega^*Ly^{m^\cdot}$ and for any $v \in L$, $y^{m^\cdot}v \in (\mathcal{I}_Y \Omega^* \mathcal{L})_x$.

For $p < n$, we introduce the complex of vector spaces $IC(p)$:

$$IC(\check{p}) := \left(0 \rightarrow L \rightarrow \bigoplus_{j \in [p+1, n]} N_j L \frac{dy_j}{y_j} \cdots \rightarrow \bigoplus_{K \subset [p+1, n]} N_K L \frac{dy_K}{y_K} \cdots \rightarrow 0 \right)$$

where the differential of a vector $v \in \bigoplus_{\{i_1 < \dots < i_k\} \subset [p+1, n]} N_{\{i_1 < \dots < i_k\}} L$ of degree k is: $(d_k v)_{\{i_1 < \dots < i_{k+1}\}} = -\frac{1}{2i\pi} \sum_{j \in [1, k+1]} (-1)^j N_j v_{\{i_1 < \dots < \hat{i}_j \dots < i_{k+1}\}}$.

We associate to each subset $B \subset [1, p]$, the complex $IC(p) \wedge \frac{dy_B}{y_B}$ and to each index $j \in [1, p] - B$, a morphism

$$(m_j Id - \frac{1}{2i\pi} N_j) \frac{dy_j}{y_j} : IC(p) \wedge \frac{dy_B}{y_B} \rightarrow IC(p) \wedge \frac{dy_j}{y_j} \wedge \frac{dy_B}{y_B}.$$

We check that $IC^*Ly^{m^\cdot}$ may be written as sum of a double complex

$$IC^*Ly^{m^\cdot} := s \left(IC(\check{p}) \wedge \frac{dy_B}{y_B}, (m_j Id - \frac{1}{2i\pi} N_j) \frac{dy_j}{y_j} \right)_{B \subset [1, p]}.$$

Then, we conclude as in the case of $\Omega^*Ly^{m^\cdot}$.

The quotient complex $(IC^*\mathcal{L})_x / IC^*L$ is an inductive limit of direct sum of complexes $IC^*Ly^{m^\cdot}$ for y^{m^\cdot} is in the ideal of A generated by y_i for all $i \in [1, n]$, in which case $IC^*Ly^{m^\cdot}$ is acyclic by the lemma; then the proposition follows.

4.3.2. Local definition of $(IC^*\mathcal{L})_x$ for a general local system. We introduce in terms of the spectral decomposition of L , for each set α , the composition of endomorphisms of $L^{e(\alpha)}$: $(\alpha.Id - \frac{1}{2i\pi} N)_J = \prod_{j \in J} (\alpha_j Id - \frac{1}{2i\pi} N_j)$.

The strict simplicial sub-complex of the de Rham logarithmic complex (4.1) is defined by $Im(\alpha.Id - \frac{1}{2i\pi} N)_J$ in $L^{e(\alpha)}(J) = L^{e(\alpha)}$.

Definition 4.10 (IC^*L). The simple complex defined by the above simplicial sub-vector spaces is the intersection complex $IC^*L := \bigoplus_{\alpha} IC^*L^{e(\alpha)}$ where

$$(4.7) \quad IC^*L^{e(\alpha)} := s \left((\alpha.Id - \frac{1}{2i\pi} N)_J L^{e(\alpha)}, \alpha.Id - \frac{1}{2i\pi} N \right)_{J \subset [1, n]}.$$

We introduce for each set α the subset $M(\alpha) \subset [1, n]$ such that $j \in M(\alpha)$ if and only if $\alpha_j = 0$. Let $N_{J \cap M(\alpha)} = \prod_{j \in J \cap M(\alpha)} N_j$ (it is the identity if $J \cap M(\alpha) = \emptyset$). For each $J \subset [1, n]$, we have the equality of the image subspaces:

$(\alpha \cdot Id - \frac{1}{2i\pi} N.)_J L^{\mathbf{e}(\alpha.)} = N_{J \cap M(\alpha.)} L^{\mathbf{e}(\alpha.)}$ since the endomorphism $(\alpha_j Id - \frac{1}{2i\pi} N_j)$ is an isomorphism on $L^{\mathbf{e}(\alpha.)}$ whenever $\alpha_j \neq 0$, hence

$$(4.8) \quad IC^* L \simeq \bigoplus_{\alpha} s(N_{J \cap M(\alpha.)} L^{\mathbf{e}(\alpha.)})_{J \subset M}$$

Lemma 4.11. *Let L^u denote the subspace of L such that all local monodromies are unipotent, then we have quasi-isomorphisms $IC^* L^u \simeq IC^* L \simeq (IC^* \mathcal{L})_x$.*

Indeed, if there exists an index k such that $\alpha_k \neq 0$, then $IC^* L^{\mathbf{e}(\alpha.)}$ may be written as a cone over the quasi-isomorphism

$$\alpha_k Id - \frac{1}{2i\pi} N_k : s(N_{J \cap M(\alpha.)} L^{\mathbf{e}(\alpha.)})_{J \subset M-k} \rightarrow s(N_{J \cap M(\alpha.)} L^{\mathbf{e}(\alpha.)})_{J \subset M-k}.$$

4.3.3. *Global definition of the Intersection complex.* The local definition of $IC^* L$ is well adapted to computations. As it is independent of the local coordinates defining Y (4.7), we deduce a global definition of the Intersection complex as follows.

A filtration W of \mathcal{L} by sub-local systems, extends to a filtration of \mathcal{L}_X by sub-bundles \mathcal{W}_X . For all subsets M of I , the decomposition of the restriction to Y_M is global: $\mathcal{L}_{Y_M} = \bigoplus_{(\alpha.)} \mathcal{L}_{Y_M}^{\mathbf{e}(\alpha.)}$. The endomorphisms \mathcal{N}_i for i in M are defined on \mathcal{L}_{Y_M} , and $\mathcal{N}_M = \prod_{i \in M} \mathcal{N}_i$ is compatible with the decomposition. The image $\mathcal{N}_M \mathcal{L}_{Y_M}$ is an analytic sub-bundle of \mathcal{L}_{Y_M} . The residue of the connection ∇ along each Y_j defines an endomorphism $\alpha_j Id - \frac{1}{2i\pi} \mathcal{N}_j$ on the component $\mathcal{L}_{Y_j}^{\mathbf{e}(\alpha.)}$ of \mathcal{L}_{Y_j} compatible with the filtration by sub-analytic bundles \mathcal{W}_{Y_j} . For subsets $J \subset I$ of the set I of indices of the components of Y , let:

$$(\alpha \cdot Id - \frac{1}{2i\pi} \mathcal{N}.)_J := (\alpha_{i_1} Id - \frac{1}{2i\pi} \mathcal{N}_{i_1}) \cdots (\alpha_{i_j} Id - \frac{1}{2i\pi} \mathcal{N}_{i_j}) : \mathcal{L}_{Y_J}^{\mathbf{e}(\alpha.)} \rightarrow \mathcal{L}_{Y_J}^{\mathbf{e}(\alpha.)}$$

Definition 4.12. The Intersection complex $IC^* \mathcal{L}$ is the sub-analytic complex of $\Omega_X^*(Log Y) \otimes \mathcal{L}_X \supset \Omega_X^* \otimes \mathcal{L}_X$ whose fibre at a point $x \in Y_M^*$ is defined, in terms of a set of coordinates y_i defining equations of Y_M for $i \in M$, as an $\Omega_{X,x}^*$ sub-module generated by the sections $\tilde{v} \wedge_{j \in J} \frac{dy_j}{y_j}$ for all $v \in \text{Im}(\alpha \cdot Id - \frac{1}{2i\pi} \mathcal{N}.)_J \subset \bigoplus_{\alpha} L^{\mathbf{e}(\alpha.)}$ and $J \subset M$ where $\tilde{v} \in \mathcal{L}_{X,x}$ and $L = \mathcal{L}_X(x)$.

This definition is independent of the choice of coordinates (lemma 4.7) and a section of $IC^* \mathcal{L}$ restricts to a section in $IC^* L$ near x , since $(\alpha \cdot Id - \frac{1}{2i\pi} \mathcal{N}.)_J L \subset (\alpha \cdot Id - \frac{1}{2i\pi} N.)_{J-i} L$ for all $i \in J$.

For example in the unipotent case, for $M = \{1, 2\}$ and $x \in Y_M^*$, the sections in $(IC^* \mathcal{L})_x \subset (\Omega_X^2(Log Y) \otimes \mathcal{L}_X)_x$ are generated by $\tilde{v} \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2}$ for $v \in N_M L$, $\tilde{v} \frac{dy_1}{y_1} \wedge dy_2$ for $v \in N_1 L$, $\tilde{v} dy_1 \wedge \frac{dy_2}{y_2}$ for $v \in N_2 L$ and $\tilde{v} dy_1 \wedge dy_2$ for $v \in L$.

We remark that the notation $IC(X, \mathcal{L})$ instead of $IC^* \mathcal{L}$ is used in ([ICTP 14], chapter 8). The intermediate extension $j_{!*} \mathcal{L}$ of \mathcal{L} is topologically defined by an explicit formula in terms of the stratification ([Br 82], §3). Locally its fiber $(j_{!*} \mathcal{L})_x$ at a point $x \in Y_M^*$ is quasi-isomorphic to the above complex $IC^* L$.

4.4. **Weight filtration.** The description of the weight filtration on the logarithmic complex is based on the work of Kashiwara [Ka 86] and [EY 14]. Although the declared purpose of the paper [Ka 86] is a criteria of admissibility in codimension one, various local statements and proofs in [Ka 86] are useful to define the weight filtration ([ICTP 14], ch.8).

We start with a graded polarized variation of MHS on the underlying shifted local system \mathcal{L} on the complement of a NCD Y , satisfying certain asymptotic

properties at points of Y , summarized under the condition of the admissibility property assuming as axioms the properties of the asymptotic behavior of variation of MHS of geometric origin. In particular we ask for the filtration F on \mathcal{L}_{X^*} to extend into a filtration of \mathcal{L}_X by sub-bundles F , and the existence of the monodromy filtration relative to the natural extension of the filtration W along Y .

Let $x \in Y$ at the intersection of n locally irreducible components with local monodromy action T_i , and let $L := \mathcal{L}_X(x) = (\mathcal{L}_X)_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ denote the space of "multiform" sections of \mathcal{L} (to emphasize this point the notation $L := \psi_{x_1} \cdots \psi_{x_n} \mathcal{L}$ is used in the literature). The conditions of admissibility, stated in terms of the extension of F and the nilpotent endomorphisms $N_i := \text{Log} T_i^u$ logarithm of the unipotent monodromy T_i^u , are defined by the structure of Infinitesimal mixed Hodge structure (IMHS) on L , recalled below.

4.4.1. Mixed nilpotent orbit and Infinitesimal mixed Hodge structure (IMHS). We deduce from a variation of MHS on the complement of a NCD Y , locally at a point $x \in Y$, the data: $(L, W, F, N_1, \dots, N_n)$ with an increasing filtration W s.t. $N_j W_k \subset W_k$, a Hodge filtration F and its conjugate \overline{F} with respect to the rational structure inherited from \mathcal{L} . It is called here mixed nilpotent orbit (pre-infinitesimal mixed Hodge module in ([Ka 86], 4.2).

The filtration W is an extension of the weight W on \mathcal{L} . We may call it the finite filtration in opposition to the relative monodromy filtration to be introduced later called the limit filtration.

Nilpotent orbit. We consider an integer w and a data $(L, F, S; N_1, \dots, N_n)$ where L is a finite dimensional complex vector space with a \mathbb{Q} -structure, F is a decreasing \mathbb{C} -filtration, such that: $N_j F^p \subset F^{p-1}$, S is a non-degenerate rational bilinear form satisfying

$$S(x, y) = (-1)^w S(y, x) \quad \text{for } x, y \in L \quad \text{and} \quad S(F^p, F^q) = 0 \quad \text{for } p + q > w.$$

and N_i are \mathbb{Q} -nilpotent endomorphisms for all i , mutually commuting, such that $S(N_i x, y) = S(x, N_i y)$ and $N_i F^p \subset F^{p-1}$.

Definition 4.13 (Nilpotent orbit). ([Ka 86], 4.1) The above data is called a (polarized) nilpotent orbit of weight w if :

- i) The monodromy filtration M of the nilpotent endomorphism $N = \sum_j t_j N_j$ does not depend on the various t_j for $t_j > 0$ and all j .
- ii) The data (L, M, F) is a MHS on L of weight w (a MHS (L, M, F) is of weight w , or shifted by w , if (Gr_k^M, F) is a HS of weight $w + k$). The bilinear form S_k such that $S_k(x, y) = S(x, N^k y)$ polarizes the primitive subspace $P_k = \text{Ker}(N^{k+1} : Gr_k^M \rightarrow Gr_{-k-2}^M)$ with its induced HS of weight $w + k$.

Henceforth, all nilpotent orbits are polarized.

Remark 4.14 (Nilpotent orbit). i) As N_i acts on F , the exponential morphism $e^{i \sum t_j N_j}$ acts on F for $t_i \in \mathbb{R}$. The nilpotent orbit theorem proved by Schmid states that the above definition is equivalent to the statement:

There exists $c > 0$ such that $(L, e^{i \sum t_j N_j} F, e^{-i \sum t_j N_j} \overline{F})$ is a Hodge structure of weight w polarized by S for $t > c$.

The HS is viewed as variation of HS for variable t_i and M is called the limit MHS.

ii) On this formula, it is clear that for each subset of indices $K \subset [1, n]$, the filtrations $F_K := e^{i \sum_{j \in K} t_j N_j} F, \overline{F}_K := e^{-i \sum_{j \in K} t_j N_j} \overline{F}$ define a variation of HS on

L such that $N_i F_K^p \subset F_K^{p-1}$ with $i \notin K$. Then $(L, W(\sum_{i \notin K} N_i), F_K)$ is a MHS for fixed $t_j, j \in K$.

Example. In the case of a polarized variation of HS of pure weight w on the product n -times of a small punctured disc D^* , the limit Hodge filtration exists on the finite dimensional vector space $L := \psi_{x_1} \cdots \psi_{x_n} \mathcal{L}$. It defines with the monodromy filtration M a nilpotent orbit approximating the variation of HS ([CaK 82], [CaKSc 86]).

Mixed nilpotent orbit. We consider a data $(L, W, F, N_1, \dots, N_n)$, where L is a finite dimensional complex vector space with a \mathbb{Q} -structure, N_i are \mathbb{Q} -nilpotent endomorphisms for all i , W (resp. F) is an increasing \mathbb{Q} -filtration (resp. decreasing \mathbb{C} -filtration), such that: $N_j F^p \subset F^{p-1}$ and $N_j W_k \subset W_k$.

Definition 4.15 (Mixed nilpotent orbit). The above data is called a mixed nilpotent orbit (graded polarized) if the data with restricted structures

$$(Gr_i^W L, F|_i, (N_1)|_i, \dots, (N_n)|_i)$$

is a nilpotent orbit for each integer i , of weight i with some polarization S_i (it is called pre-infinitesimal mixed Hodge module in ([Ka 86], 4.2).

Definition 4.16 (IMHS). ([Ka 86], 4.3)

A mixed nilpotent orbit $(L, W, F, N_1, \dots, N_n)$ is called an infinitesimal mixed Hodge structure (IMHS) if the following conditions are satisfied:

- i) For each $J \subset I = \{1, \dots, n\}$, the monodromy filtration $M(J)$ of $\sum_{j \in J} N_j$ relative to W exists and satisfy $N_j M_i(J) \subset M_{i-2}(J)$ for all $j \in J$ and $i \in \mathbb{Z}$.
- ii) The filtrations $M(I), F$ define a graded polarized MHS. The filtrations W and $M(J)$ are compatible with the MHS and the morphisms N_i are of type $(-1, -1)$.

IMHS are called IMHM in [Ka 86]; Deligne remarked, the fact that if the relative monodromy filtration $M(\sum_{i \in I} N_i, W)$ exists in the case of a mixed nilpotent orbit, then it is necessarily the weight filtration of a MHS.

Kashiwara proves in ([Ka 86], theorem 4.4.1) the main theorem: it is enough to check the existence of $W(N_i)$ (in cxdimension 1). Moreover, IMHS form an abelian category for which the filtrations W, F , and M are strict ([Ka 86], prop. 5.2.6).

The morphism N_i is not an auto-morphism of IMHS as it shifts F but not W . However, when combined with the descent lemma it leads to an important application below (subsection 4.4.5).

4.4.2. Definition of the filtration $N * W$. ([Ka 86], 3.4)

Let (L, W, N) denote an increasing filtration W on a vector space L with a nilpotent endomorphism N compatible with W s.t. the relative monodromy filtration $M(N, W)$ exists. Then a new filtration $N * W$ of L is defined by the formula ([Ka 86], 3.4)

$$(4.9) \quad (N * W)_k := NW_{k+1} + M_k(N, W) \cap W_k = NW_{k+1} + M_k(N, W) \cap W_{k+1}$$

where the last equality follows from ([Ka 86], Prop 3.4.1).

The endomorphism $N : L \rightarrow L$ and the identity I on L induce morphisms

$$N : W_k \rightarrow (N * W)_{k-1}, \quad I : (N * W)_{k-1} \rightarrow W_k.$$

We mention two important properties of $N * W$ ([Ka 86], lemma 3.4.2):

The monodromy filtration relative to $N * W$ exists and $M(N, N * W) = M(N, W)$.

Moreover, the following decomposition property is satisfied

$$(4.10) \quad \begin{aligned} Gr_k^{N*W} L &\simeq \text{Im} (N : Gr_{k+1}^W L \rightarrow Gr_k^{N*W} L) \oplus \ker (I : Gr_k^{N*W} L \rightarrow Gr_{k+1}^W L) \\ &\text{Im} (N : Gr_{k+1}^W L \rightarrow Gr_k^{N*W} L) \xrightarrow{\simeq} \text{Im} (N : Gr_{k+1}^W L \rightarrow Gr_{k+1}^W L). \end{aligned}$$

To refer to this decomposition, the couple $(L, N * W)$ is said to form a (graded) distinguished pair.

4.4.3. The filtration W^J associated to an IMHS. Let $(L, W, F, N_i, i \in M)$ denote the IMHS at $x \in Y_M^*$ and $J \subset M$. The relative weight filtration

$$(4.11) \quad M(J, W) := M(N, W) \text{ for } N \in C(J) := \{\sum_{j \in J} t_j N_j, t_j > 0\},$$

is well defined by assumption. A basic lemma ([Ka 86], cor. 5.5.4) asserts that:

Lemma 4.17. $(L, N_1 * W, F, N_i, i \in M)$ and $(L, M(N_1, W), F, N_i, i \neq 1)$ are IMHS.

Remark 4.18. Geometrically, $N_1 * W$ and $M(N_1, W)$ may be viewed on \mathcal{L}_{Y_1} near x . In term of the equation z_1 of Y_1 , the filtration $N_1 * W$ viewed on $\psi_{z_1} \mathcal{L}$ is not a variation of MHS.

In ([Ka 86], lemma 3.4.2 ii and proposition 5.2.5) the following equality is proved:

$$M(\sum_{i \in M} N_i, N_1 * W) = M(\sum_{i \in M - \{1\}} N_i, M(N_1, W)) = M(\sum_{i \in M} N_i, W).$$

We deduce a recursive definition of an increasing filtration W^J of L by the star operation

$$(4.12) \quad W^J := N_{i_1} * (\dots (N_{i_j} * W) \dots) \text{ for } J = \{i_1, \dots, i_j\}$$

It is denoted by $\Psi_J * W$ in ([Ka 86], 5.8.2) as it is interpreted as defined repeatedly on Ψ_{x_i} for $x_i \in J$ in the theory of nearby cycles.

It follows by induction that $(L, W^J, F, N_i, i \in M)$ is an IMHS. The filtration W^J does not depend on the order of composition of the respective transformations $N_{i_k} *$ since $N_{i_p} * (N_{i_q} * W) = N_{i_q} * (N_{i_p} * W)$ for all $i_p, i_q \in J$ according to a basic result in ([Ka 86], proposition 5.5.5). The star operation has the following properties ([Ka 86], formula 5.8.5 and 5.8.6):

$$\forall J_1, J_2 \subset M : M(J_1, W^{J_2}) = M(J_1, W)^{J_2}, \forall J \subset K \subset M : M(K, W^J) = M(K, W).$$

The filtrations W^J fit together to define the weight filtration on $\Omega^* L$:

Definition 4.19. The filtrations W and F on the de Rham complex associated to an IMHS $(L, W, F, N_i, i \in M)$ are defined as simplicial complexes

$$W_k \Omega^* L := s(W_{k-|J|}^J, N.)_{J \subset M}, \quad F^p \Omega^* L := s(F^{k-|J|}, N.)_{J \subset M}$$

Remark 4.20. The induced morphism $N_i : W_k^J \rightarrow (N_i * W^J)_{k-1}$ drops the degree of the filtration W^J . For further use, it is important to add to the above data, the canonical inclusion $I : (N_i * W^J)_{k-1} \rightarrow W_k^J$.

Beware that the weight filtrations W on \mathcal{L} underlying the variation of MHS, extends as a constant filtration W' on $\Omega^* L$. The weight W on $\Omega^* L$ is the local definition of a global weight on the de Rham complex. The relation between W and W' on the terms of $\Omega^* L$ is described by ([Ka 86] lemma 3.4.2 iii) and corollary 3.4.3).

$$\begin{aligned} Gr_i^{W^1} L &\simeq N_1 Gr_{i+1}^W L \oplus P_i^1 L, \text{ and } Gr_i^{W^2} L \simeq N_2 Gr_{i+1}^W L \oplus P_i^2 L \\ Gr_{i-1}^{W^{12}} L &\simeq N'_2 Gr_i^{W^1} L \oplus P_{i-1}^{12,1} L \simeq N'_2 N_1 Gr_{i+1}^W L \oplus N'_2 P_i^1 L \oplus P_{i-1}^{12,1} L, \text{ and} \\ Gr_{i-1}^{W^{12}} L &\simeq N'_1 N_2 Gr_{i+1}^W L \oplus N'_1 P_i^2 L \oplus P_{i-1}^{12,2} L, \end{aligned}$$

Let $P_{i-1}^{12} L := P_{i-1}^{12,2} L \cap P_{i-1}^{12,1} L$. It remains to prove

$$P_{i-1}^{12,1} L = N'_2 P_i^1 L \oplus P_{i-1}^{12} L \text{ and } P_{i-1}^{12,2} L = N'_1 P_i^2 L \oplus P_{i-1}^{12} L.$$

which follows from the linear algebra relations:

$$I'_2 \circ N'_1 = N_1 \circ I_2, \text{ and } I_2 \circ I'_1 = I_1 \circ I'_2.$$

Remark 4.22. In the global decomposition of the logarithmic complex, the HS $P_k^J(L)$ is viewed as a fibre of a VHS on Y_J (corollary 4.23 below, and theorem 4.30).

The following corollary is a basic step to check the structure of MHC on the logarithmic de Rham complex

Corollary 4.23 (local decomposition). *The graded vector space of the filtration W on $\Omega^* L$ (definition 4.19) satisfy the decomposition property into a direct sum of Intersection complexes*

$$(4.13) \quad Gr_k^W(\Omega^* L) \xrightarrow{\sim} \oplus IC^*(P_{k-|K|}^K(L)[-|K|])_{K \subset M}$$

The corollary is the local statement of the structure of MHC on the logarithmic de Rham complex.

It follows from the lemma by definition of IC^* (proposition 4.8) by a general description of perverse sheaves in the normally crossing divisor case ([Ka 86], section 2). The description is in terms of a family of vector spaces and a diagram of morphisms. The minimal extensions are characterized by relations of distinguished pairs as in the formula 4.10.

The family of vector spaces is deduced from a perverse sheaf by repeated application of the functors nearby and vanishing cycles with respect to the coordinates.

4.4.5. *Decomposition of the Intersection complex $IC^* L$.* We need to prove a similar decomposition of the Intersection complex $IC^* L$ associated to $(L, W, F, N_i, i \in M)$ an IMHS. For each $j \in M$, we consider the data $(N_j L, N_j * W, F, N_i, i \in M)$ with induced filtrations from the IMHS $(L, N_i * W, F, N_i, i \in M)$ (we should write $(N_j * W)|_{N_j L}, F|_{N_j L}, (N_i)|_{N_j L}$ but the symbol of restriction is erased). We have a graded exact sequence defined by the induced filtrations on $N_j L$:

$$0 \longrightarrow Gr_k^{W^j} N_j L \longrightarrow Gr_k^{W^j} L \longrightarrow Gr_k^{W^j} L / N_j L \longrightarrow 0$$

Lemma 4.24 (descent lemma for IMHS). *i) Let $(L, W, F, N_i, i \in M)$ be an IMHS, then $(N_j L, W^j := ((N_j * W)|_{N_j L}, F, N_i, i \in M)$ is an IMHS for all $j \in M$.*

ii) The couple $(L, W, F, N_i) \xrightarrow{N_j} (N_j L, W^j, F, N_i)$ form a graded distinguished pair: $Gr_k^{W^j} N_j L$ splits into a direct sum

$$Im(N_j : Gr_{k+1}^W L \rightarrow Gr_k^{W^j} N_j L) \oplus Ker(I_j : Gr_k^{W^j} N_j L \rightarrow Gr_{k+1}^W L)$$

Proof. The filtration W^j is first constructed on L , then induced on $N_j L$. The proof is simultaneous for i) and ii). We consider the diagram

$$\begin{array}{ccccc} 0 & \rightarrow & Gr_{k+1}^W L & = & Gr_{k+1}^W L \\ & & \downarrow N'_j \uparrow I'_j & & \downarrow N_j \uparrow I_j \\ 0 & \rightarrow & Gr_k^{W^j} N_j L & \xrightarrow{s} & Gr_k^{W^j} L \end{array}$$

where s is injective, N'_j and I'_j are other symbols for the restriction of N_j and I_j . Set $P_k := \ker I_j$, $\text{Im } N_j$ for the image of N_j , $\text{Im } N'_j$ for the image of N'_j and $\widetilde{\text{Im}} N_j := \text{Im}(N_j : Gr_{k+1}^W L \rightarrow Gr_{k+1}^W L)$.

The terms on the right column form a distinguished pair, that is we have a splitting $Gr_k^{W^j} L \simeq \text{Im } N_j \oplus \ker I_j$ and an isomorphism $\text{Im } N_j \simeq \widetilde{\text{Im}} N_j$.

Set $P'_k := (Gr_k^{W^j} N_j L) \cap P_k$. As $\text{Im } N'_j \simeq s(\text{Im } N'_j) = \text{Im } N_j$, we deduce the splitting

$$Gr_k^{W^j} N_j L = \text{Im } N'_j \oplus P'_k$$

where $P'_k = \ker(I'_j : Gr_k^{W^j} N_j L \rightarrow Gr_{k+1}^{\widetilde{W}} N_K L)$. We prove that both components of this splitting are nilpotent orbits of weight k .

1) We apply the descent lemma ([CaKSc 87] descent lemma 1.16) to $\text{Im } N'_j$ since it is isomorphic to the image $\widetilde{\text{Im}} N_j$ (the image of N_j acting on the nilpotent orbit $Gr_{k+1}^W L$). We deduce that the data $(\text{Im } N'_j, F, N_i, i \in M)$ is a nilpotent orbit of weight k .

2) The kernel $P_k = \text{Ker } I_j \subset Gr_{k+1}^W N_j L \subset Gr_{k+1}^W L$ is compatible with the action of N_i for all $i \in M$, and since $N_j = N_j \circ I_j$ vanish on P_k we deduce that $P_k \simeq Gr_k^{W(N_j)}(\text{Ker } N_j)$ has a pure polarized HS induced by the MHS of $W(N_j) \subset Gr_{k+1}^{W^j} L$. In particular $W(N_j)_{k-1} \text{Ker } N_j$ is in $\text{Im } N_j \simeq \widetilde{\text{Im}} N_j$ which is consistent with the descent lemma.

Moreover the orthogonal subspace to $P'_k = \text{Ker } I'_j \subset P_k$ in the primitive part $Gr_k^{W(N_j)} \text{Ker } N_j$ (resp. in $Gr_k^{W(N_j)} \text{Ker } N_j$) defines a splitting compatible with the action of N_i for all $i \in M$ as the N_i are infinitesimal isometry of the bilinear product $S : S(N_i a, b) + S(a, N_i b) = 0$, hence P'_k is a sub-nilpotent orbit of the nilpotent orbit structure on $Gr_k^{W(N_j)} Gr_{k+1}^{W^j} L$. \square

Corollary 4.25. *i) Let $(L, W, F, N_i, i \in M)$ be an IMHS, then $(N_K L, W^K, F, N_i, i \in M)$ is an IMHS for all $K \subset M$, and the embedding $(N_K L, W^K, F) \subset (L, W^K, F)$ in the category of IMHS is strict for the filtrations.*

ii) The couple $(N_K L, W^K, F, N_i) \xrightarrow{N_j} (N_J L, W^j, F, N_i)$ form a graded distinguished pair: $Gr_k^{W^j} N_J L$ splits into a direct sum

$$\text{Im}(N_j : Gr_{k+1}^{W^K} N_K L \rightarrow Gr_k^{W^j} N_J L) \oplus \text{Ker}(I_j : Gr_k^{W^j} N_J L \rightarrow Gr_{k+1}^{W^K} N_K L)$$

and the image $\text{Im } N_j$ is isomorphic to $\text{Im}(N_j : Gr_{k+1}^{W^K} N_K L \rightarrow Gr_{k+1}^{W^K} N_K L)$.

We prove the corollary by induction on the length of K . By the lemma, the embedding $(N_J L, W^j, F, N_i) \subset (L, W^j, F, N_i)$ is an injective morphism in the abelian category of IMHS.

We assume by induction that $(N_K L, W^K, F, N_i, i \in M)$ is an IMHS, and let $J := K \cup i$ for $i \notin K$. First we consider the induced morphism $N_j : W^K \rightarrow W^K$, the filtration $N_j * W_{|N_K L}^K$ defined by the star operation on $N_K L$ and its induced filtration on $N_J L \subset N_K L$. There is another filtration obtained from W^K on L : $N_j * W^K$ on L and its induced filtration on $N_K L$ such that we have a diagram

$$(N_j * W^K)_{|N_J L} \rightarrow N_j * W_{|N_K L}^K \rightarrow W^J := N_j * W^K$$

By the lemma the first morphism is in the category of IMHS and by the inductive hypothesis the second morphism is compatible with the IMHS, hence the morphisms are strict and the filtration W^J in L induces $N_j * W_{|N_J L}^K$ on $N_J L$. Hence

$(N_J L, W^J, F, N_i, i \in M)$ is an IMHS, and the embedding into $(L, W^J, F, N_i, i \in M)$ is compatible with the IMHS.

4.4.6. Decomposition of $IC^* L$. As in the case of $\Omega^* L$ (subsection 4.4.4) we introduce the data $DR(IC^*(L)) :=$

$$\{N_J L, W^J, F^J, I_{J,K} : N_J L \rightarrow N_K L, N_{J-K} : N_K L \rightarrow N_J L\}_{K \subset J \subset M}$$

Set for each $J \subset M$

$$P_k^J(N_J L) := \cap_{K \subset J, K \neq J} \text{Ker}(Gr I_{J,K} : Gr_k^{W^J} N_J L \rightarrow Gr_{k+|J-K|}^{W^K} N_K L) \subset Gr_k^{W^J} L$$

then $P_k^J(N_J L) = P_k^J(L) \cap Gr_k^{W^J} N_J L$ has pure weight k with respect to the weight filtration $M(\sum_{j \in J} N_j, L)$ and it is polarized.

Corollary 4.26. *i)* $Gr_k^{W^J} N_J L \xrightarrow{\sim} \oplus_{K \subset J} N_{J-K} P_{k+|J-K|}^K(N_K L)$.

$$ii) \quad Gr_k^W(IC^* L) \xrightarrow{\sim} \oplus_{K \subset M} IC(P_{k-|K|}^K(N_K L)[-|K|]).$$

4.5. Global definition and properties of the weight W . The local study ended with the local decomposition into Intersection complexes. We develop now the corresponding global result. The local cohomology of $\Omega^* \mathcal{L}$ is reduced to \mathcal{L} on $X - Y$ and to the various analytic restrictions \mathcal{L}_{Y_J} on Y (proposition 4.3).

In the case of a pure VHS, the weight starts with the subcomplex $IC^* \mathcal{L}$.

Notations in the global case. We suppose \mathcal{L} locally unipotent by the remark 4.5 as other components of the spectral decomposition have acyclic de Rham complexes. We recall that $\mathcal{L} := \tilde{\mathcal{L}}[n]$ with $n := \dim X$ is a shifted variation of MHS, hence all complexes and bundles below are already shifted by n to the left. For all $J \subset I$, let \mathcal{L}_{Y_J} denote the restriction of the analytic module \mathcal{L}_X to Y_J . The weight filtration W on \mathcal{L} defines a filtration by sub-bundles \mathcal{W}_X of \mathcal{L}_X with restriction \mathcal{W}_{Y_J} to Y_J , such that the relative monodromy filtrations of the endomorphism $\sum_{i \in J} \mathcal{N}_i$ of $(\mathcal{L}_{Y_J}, \mathcal{W}_{Y_J})$ exist for all $J \subset I$. We write $\mathcal{M}(J, \mathcal{W}_{Y_J}) := \mathcal{M}(\sum_{i \in J} \mathcal{N}_i, \mathcal{W}_{Y_J})$, it is a filtration by analytic sub-bundles, so that we can define

$$(\mathcal{N}_i * \mathcal{W}_{Y_i})_k := \mathcal{N}_i \mathcal{W}_{Y_i, k+1} + \mathcal{M}_k(\mathcal{N}_i, \mathcal{W}_{Y_i}) \cap \mathcal{W}_k = \mathcal{N}_i \mathcal{W}_{Y_i, k+1} + \mathcal{M}_k(\mathcal{N}_i, \mathcal{W}_{Y_i}) \cap \mathcal{W}_{k+1}$$

and an increasing filtration \mathcal{W}^J of \mathcal{L}_{Y_J} is recursively defined by the star operation

$$\mathcal{W}^J := \mathcal{N}_{i_1} * (\dots (\mathcal{N}_{i_j} * \mathcal{W}) \dots) \text{ for } J = \{i_1, \dots, i_j\}.$$

Definition 4.27 (weight W on $\Omega^* \mathcal{L}$). Let (\mathcal{L}, W, F) be an admissible variation of MHS on $U := X - Y$. The weight filtration by sub-analytic complexes, denoted also by W on $\Omega^* \mathcal{L}$, is defined locally in terms of the various restrictions \mathcal{W}_{Y_J} to the strata Y_J^* .

At a point $y \in Y_M^*$ for $M \subset I$, $W_r(\Omega_X^*(\text{Log} Y) \otimes \mathcal{L}_X)_y$ is defined locally in a neighborhood of $y \in Y_M^*$, in terms of the IMHS (L, W, F, N) at y and a set of coordinates y_i for $i \in M$ (including local equations of Y at y), as follows:

The term W_r of the filtration is generated as an $\Omega_{X,y}^*$ -sub-module by the germs of the sections $\tilde{v} \otimes \wedge_{j \in J} \frac{dy_j}{y_j}$ for $J \subset M$ and $v \in W_{r-|J|}^J L$.

In particular, for $M = \emptyset$, at a point $y \in U$, W_r is generated as an $\Omega_{X,y}^*$ -sub-module by the germs of the sections $v \in W_r \mathcal{L}_y$.

The definition of W above is independent of the choice of coordinates on a neighborhood $U(y)$, since if we change the coordinate into $y'_i = f y_i$ with f invertible holomorphic at y , we check first that the submodule $W_{r-|J|}^J(\mathcal{L}_X)_y$ of $\mathcal{L}_{X,y}$ defined

by the image of $W_{r-|J|}^J L$ is independent of the coordinates as in the construction of the canonical extension (see also lemma 4.7). For a fixed $\alpha \in W_{r-|J|}^J \mathcal{L}_{X,y}$, as the difference $\frac{dy'_i}{y'_i} - \frac{dy_i}{y_i} = \frac{df}{f}$ is holomorphic at y , the difference of the sections $\alpha \otimes \wedge_{j \in J} \frac{dy'_j}{y'_j} - \alpha \otimes \wedge_{j \in J} \frac{dy_j}{y_j}$ is still a section of the $\Omega_{X,y}^*$ -sub-module generated by the germs of the sections $W_{r-|J|}^J (\mathcal{L}_X)_y \otimes \wedge_{j \in (J-i)} \frac{dy_j}{y_j}$. Finally, we remark that the sections defined at y restrict to sections defined on $(U(y) - Y_M^* \cap U(y))$.

4.5.1. *Global statements.* The Hodge filtration on the Intersection complex defined algebraically on $IC^* \mathcal{L}$ induces a HS on the Intersection cohomology with coefficients in a polarized VHS.

Theorem 4.28. *Let (\mathcal{L}, F) be a shifted polarized VHS of weight a , then the sub-complex $(IC^* \mathcal{L}, F)$ of the logarithmic complex with induced filtration F is a Hodge complex which defines a pure HS of weight $a+i$ on its hypercohomology $\mathbb{H}(X, IC^* \mathcal{L})$, equal to the Intersection cohomology $IH^i(X, \mathcal{L})$.*

We rely on the proof by Kashiwara ([KaK 86], Theorem 1 and Proposition 3).

Remark 4.29. i) The proof of the local purity theorem here relies on the polarization of the intersection cohomology in the case of the complement of a NCD in X .

The original proof of the purity theorem is local at v [DeG 81]. As noted by the referee of the notes, the MHS on the boundary of the exceptional divisor at a point $v \in V$ (definition 5.11, lemma 5.12) is independent of the global variety X but depends only on the components of the NCD X_v and their embedding in X .

It is interesting and possible to adapt such local proof, then the algebraic-geometric constructions of W and F leads to an alternative construction of Hodge theory as follows.

The case of curves is studied in [Zu 79]. We proceed by induction on $\dim X$, and assume the purity of $IC^* \mathcal{L}$ in dimension $n-1$ in the inductive step. We construct a morphism $f : X \rightarrow V$ to a smooth curve V for which the decomposition theorem apply with just a local proof of purity, from which we deduce that the intersection cohomology $\mathbb{H}^i(X, \mathcal{L})$ carries a pure HS induced by the Hodge filtration defined in this paper.

ii) The subtle proof of the comparison in [KaK 86] is based on the auto-duality of the Intersection cohomology.

We are not aware of a direct comparison of the filtration F on $IC^* \mathcal{L}$ defined here in an algebro-geometric way, with the filtration F in terms of L^2 -cohomology in [KaK 87] and [CaKSc 86].

4.5.2. *The bundles $\mathcal{P}_k^J(\mathcal{L}_{Y_J})$.* Given a subset $J \subset I$, the filtration \mathcal{W}^J induces a filtration by sub-analytic bundles of \mathcal{L}_{Y_J} . We introduce the following bundles

$$\mathcal{P}_k^J(\mathcal{L}_{Y_J}) := \cap_{K \subset J, K \neq J} \text{Ker}(Gr I_{J,K} : Gr_k^{\mathcal{W}^J} \mathcal{L}_{Y_J} \rightarrow Gr_{k+|J-K|}^{\mathcal{W}^K} \mathcal{L}_{Y_J}) \subset Gr_k^{\mathcal{W}^J} \mathcal{L}_{Y_J}$$

where $I_{J,K}$ is defined as in the local case. In particular $\mathcal{P}_k^\emptyset(\mathcal{L}_X) = Gr_k^{\mathcal{W}} \mathcal{L}_X$ and $\mathcal{P}_k^J(\mathcal{L}_{Y_J}) = 0$ if $Y_J^* = \emptyset$.

Theorem 4.30. *i) The weight W is a filtration by sub-complexes of $Rj_* \mathcal{L}$ consisting of perverse sheaves defined over \mathbb{Q} .*

ii) The bundles $\mathcal{P}_k^J(\mathcal{L}_{Y_J})$ are Deligne's extensions of polarized VHS $\mathcal{P}_k^J(\mathcal{L})$ on Y_J^* of weight k with respect to the weight induced by $\mathcal{M}(\sum_{j \in J} \mathcal{N}_j, \mathcal{L}_{Y_J^*})$.

iii) The graded perverse sheaves for the weight filtration, satisfy the decomposition property into intermediate extensions for all k

$$Gr_k^W(\Omega^* \mathcal{L}) := Gr_k^W(\Omega_X^*(\text{Log}Y) \otimes \mathcal{L}_X) \xrightarrow{\sim} \bigoplus_{J \subset I} (i_{Y_J})_* j_{!*} \mathcal{P}_{k-|J|}^J(\mathcal{L})[-|J|].$$

where j denotes uniformly the inclusion of Y_J^* into Y_J for each $J \subset I$.

The proof is essentially based on the local study above. The various graded complexes $Gr_k^W(\Omega_X^*(\text{Log}Y) \otimes \mathcal{L}_X)$ are direct sum of Intersections complexes defined over \mathbb{Q} . In particular, we deduce that the whole filtration W_k on the de Rham complex is defined over \mathbb{Q} , and W_k is perverse with respect to the stratification defined by Y .

The local rational structure of the complexes W_k glue into a global rational structure, since perverse sheaves may be glued as the usual sheaves, although they are not concentrated in a unique degree ([BBDG 83] corollary 2.1.23 and 2.2.19).

Another proof of the existence of the rational structure is based on Verdier's specialization [ELM 10].

The next result on weights satisfy a property cited in ([De 80], cor 3.3.5).

Corollary 4.31. *The de Rham logarithmic mixed Hodge complex $(\Omega^* \mathcal{L}, W, F)$ of an admissible variation of MHS of weight $\omega \geq a$ induces on the hypercohomology $\mathbb{H}^i(X - Y, \mathcal{L})$ a MHS of weight $\omega \geq a + i$.*

Indeed, $W_k = 0$ on the logarithmic complex for $k \leq a$.

Proposition 4.32. *The Intersection complex $(IC(X, \mathcal{L}), W, F)$ of an admissible variation of MHS, as an embedded sub-complex of $(\Omega^* \mathcal{L}, W, F)$ with induced filtrations, is a mixed Hodge complex satisfying for all k :*

$$(IC^*(Gr_k^W \mathcal{L}), F) \simeq (Gr_k^W IC^* \mathcal{L}, F)$$

The proposition follows from corollary 5.2. In general the intersection complex of an extension of two local systems, is not the extension of their intersection complex.

4.6. The relative logarithmic complex $\Omega_f^* \mathcal{L} := \Omega_{X/V}^*(\text{Log}Y) \otimes \mathcal{L}_X$.

Let $f : X \rightarrow V$ be a smooth proper morphism of smooth complex algebraic varieties, and let Y be an ‘‘horizontal’’ NCD in X , that is a relative NCD with smooth components over V . For each point $v \in V$, the fiber Y_v is a NCD in the smooth fiber X_v . In this case, the various intersections Y_{i_1, \dots, i_p} of p -components are smooth over V and $Y \rightarrow V$ is a topological fiber bundle.

Let $U := X - Y$, $j : U \rightarrow X$. The sheaf of modules $i_{X_v}^* \mathcal{L}_X$ induced on each fiber X_v , by the canonical extension \mathcal{L}_X , is isomorphic to the canonical extension $(i_{U_v}^* \mathcal{L})_{X_v}$ of the induced local system $i_{U_v}^* \mathcal{L}$. The family of cohomology spaces $H^i(U_v, \mathcal{L})$ (resp. $\mathbb{H}^i(X_v, j_{!*} \mathcal{L})$) for $v \in V$, form a variation of MHS. The logarithmic complex $\Omega_f^* \mathcal{L} := \Omega_{X/V}^*(\text{Log}Y) \otimes \mathcal{L}_X$ ([De 70], section 2.22) satisfy, in the case of an horizontal NCD: $i_{X_v}^* \Omega_{X/V}^*(\text{Log}Y) \otimes \mathcal{L}_X \simeq \Omega_{X_v}^*(\text{Log}Y_v) \otimes (i_{U_v}^* \mathcal{L})_{X_v}$.

When \mathcal{L} underlies an admissible graded polarized variation of MHS (\mathcal{L}, W, F) , its restriction to the open subset U_v in X_v is also admissible.

4.6.1. *The relative logarithmic complex with coefficients $\Omega_f^* \mathcal{L}$ for a smooth f .* The image of the filtrations W and F , by the map $\Omega^* \mathcal{L} \rightarrow \Omega_f^* \mathcal{L} := \Omega_{X/V}^*(\text{Log} Y) \otimes \mathcal{L}_X$

$$F := \text{Im}(R^i f_* F_X \rightarrow R^i f_*(\Omega_{X/V}^*(\text{Log} Y) \otimes \mathcal{L}_X)),$$

$$W := \text{Im}(R^i f_* W_X \rightarrow R^i f_*(\Omega_{X/V}^*(\text{Log} Y) \otimes \mathcal{L}_X))$$

define a variation of MHS on $R^i(f \circ j)_* \mathcal{L}$ inducing at each point $v \in V$ the corresponding weight W and Hodge F filtrations of the MHS on $(\mathbb{H}^i(U_v, \mathcal{L}), W, F)$.

Proposition 4.33. *i) The direct image $R^i(f \circ j)_* \mathcal{L}$ is a local system on V and*

$$R^i(f \circ j)_* \mathcal{L} \otimes \mathcal{O}_V \simeq R^i f_*(\Omega_{X/V}^*(\text{Log} Y) \otimes \mathcal{L}_X)$$

ii) Moreover, the connecting morphism in Katz-Oda's construction [KaOd 68] coincides with the connection on V defined by the local system $R^i(f \circ j)_ \mathcal{L}$*

$$R^i f_*(\Omega_{X/V}^*(\text{Log} Y) \otimes \mathcal{L}_X) \xrightarrow{\nabla_V} \Omega_V^1 \otimes R^i f_*(\Omega_{X/V}^*(\text{Log} Y) \otimes \mathcal{L}_X)$$

iii) The filtration F , is horizontal with respect to ∇_V , while W is locally constant, and $(R^i(f \circ j)_ \mathcal{L}, W, F)$ is a graded polarized variation of MHS on V .*

Deligne's proof of ([De 70], proposition 2.28) extends in (i), as well the connecting morphism [KaOd 68] in (ii).

Remark 4.34. We refer to the remark 4.29 for the case of a "vertical" NCD, that is when f is smooth over the complement of a NCD $W \subset V$ and $Y := f^{-1}W$.

4.6.2. *The relative Intersection complex $IC_f^* \mathcal{L}$.*

We deduce from the bi-filtered complex $(IC^* \mathcal{L}, W, F)$ the i -th direct image

$$(\mathcal{L}^i, F, W) := (R^i f_* j_{!*} \mathcal{L}, \text{Im}(R^i f_* F \rightarrow R^i f_* IC^* \mathcal{L}), \text{Im}(R^i f_* W \rightarrow R^i f_* IC^* \mathcal{L}))$$

To prove that *the filtrations W and F defined on $\mathcal{L}^i := R^i f_* j_{!*} \mathcal{L}$, form a structure of a graded polarized variation of MHS on V* , we consider the image complex of $IC^* \mathcal{L}$,

$IC_f^* \mathcal{L} := \text{Im}\left(IC^* \mathcal{L} \rightarrow \Omega_{X/V}^*(\text{Log} Y) \otimes \mathcal{L}_X\right)$ with image filtrations W and F , then:

$W_V := \text{Im}(R^i f_* W \rightarrow R^i f_*(IC_f^* \mathcal{L}))$ is locally constant on V , and

$F_V := \text{Im}(R^i f_* F \rightarrow R^i f_*(IC_f^* \mathcal{L}))$ on $R^i f_* j_{!*} \mathcal{L} \otimes \mathcal{O}_V \simeq R^i f_*(IC_f^* \mathcal{L})$ is horizontal with respect to ∇_V .

Proposition 4.35. *i) The direct image $R^i f_* j_{!*} \mathcal{L}$ of the intersection complex is a local system on V and*

$$R^i f_* j_{!*} \mathcal{L} \otimes \mathcal{O}_V \simeq R^i f_*(IC_f^* \mathcal{L})$$

ii) The connecting morphism in Katz-Oda's construction coincides with the connection on V defined by the local system $R^i f_ j_{!*} \mathcal{L}$*

$$R^i f_*(IC_f^* j_{!*} \mathcal{L}) \xrightarrow{\nabla_V} \Omega_V^1 \otimes R^i f_*(IC_f^* \mathcal{L})$$

ii) The de Rham complex defined by the connection on $(R^i f_ IC_f^* \mathcal{L}, W_V, F_V)$ underlies a structure of mixed Hodge complex on V defined by the variation of MHS induced on $R^i f_* j_{!*} \mathcal{L}$.*

iii) The truncation filtration τ induces a filtration on $\mathbb{H}^{i+j}(X, j_{!} \mathcal{L})$ compatible with the MHS and $Gr_i^\tau \mathbb{H}^{i+j}(X, j_{!*} \mathcal{L}) \simeq \mathbb{H}^j(V, R^i f_* IC_f^* \mathcal{L})$.*

5. LOGARITHMIC INTERSECTION COMPLEX FOR AN OPEN ALGEBRAIC VARIETY

Let $Z = \cup_{i \in I_Z \subset I} Y_i \subset Y$ be a sub-divisor of Y , union of components of Y with index in a subset I_Z of I , $j_Z := (X - Z) \rightarrow X$, and $i_Z : Z \rightarrow X$.

We construct a logarithmic sub-complex $IC^* \mathcal{L}(\text{Log} Z) \subset \Omega^* \mathcal{L}$ which is a realization of the direct image $Rj_{Z*}((j_! \mathcal{L})_{|_{X-Z}})$ (denoted $\Omega^*(\mathcal{L}, Z)$ in [ICTP 14] definition 8.3.31); from which we deduce by duality various logarithmic complexes with weight and Hodge filtrations, realizing cohomological constructions associated to Z in Y such as the functors:

$$j_{Z!}((j_! \mathcal{L})_{|_{X-Z}}), Rj_{Z*}((j_! \mathcal{L})_{|_{X-Z}}), i_Z^* Rj_{Z*}((j_! \mathcal{L})_{|_{X-Z}}), Ri_Z^! j_! \mathcal{L} \text{ and } i_Z^* j_! \mathcal{L}.$$

We describe a structure of mixed Hodge complexes (MHC) on $IC^* \mathcal{L}(\text{Log} Z)$ that is $Rj_{Z*}((j_! \mathcal{L})_{|_{X-Z}})$, and $i_{Z*} i_Z^* Rj_{Z*}((j_! \mathcal{L})_{|_{X-Z}}) := i_{Z*} i_Z^* IC^* \mathcal{L}(\text{Log} Z)$ (def . 5.11 and lemma 5.12 on cohomology of the boundary of a tubular neighborhood of Z).

5.1. $IC^* \mathcal{L}(\text{Log} Z) \simeq Rj_{Z*}((j_! \mathcal{L})_{|_{X-Z}})$.

We suppose \mathcal{L} locally unipotent as we refer to the remark 4.5 in general. The subcomplex of analytic sheaves $IC^* \mathcal{L}(\text{Log} Z)$ of the logarithmic de Rham complex $\Omega_X^*(\text{Log} Y) \otimes \mathcal{L}_X$ is defined below locally at a point $x \in Y_M^*$ in terms of a set of coordinates y_i for $i \in M$ defining a set of equations of Y_M .

5.1.1. *The Logarithmic Intersection complex: $IC^* L(\text{Log} Z)$.*

Given an IMHS $(L, W, F, N_i, i \in M)$ and a subset $M_Z \subset M$, we consider for each $J \subset M$ the subsets $J_Z := J \cap M_Z$ and $J'_Z = J - J_Z$ such that $J = J_Z \cup J'_Z$, in particular $M'_Z := M - M_Z$.

The correspondence which attach to each index $J \subset M$ the subset $N_{J'_Z} L$ of L , define a sub-complex $IC^* L(\text{Log} Z)$ of $\Omega^* L$ as a sum over $J \subset M$

$$(5.1) \quad IC^* L(\text{Log} Z) := s(N_{J'_Z} L)_{J \subset M} \subset \Omega^* L$$

Example. On the 3-dimensional disc $D^3 \subset \mathbb{C}^3$, $Y = Y_1 \cup Y_2 \cup Y_3$ the NCD defined by the coordinates $y_1 y_2 y_3 = 0$; The local system \mathcal{L} is defined by a vector space L with the action of 3 nilpotent endomorphisms N_i . Let $Z = Y_1 \cup Y_2$ be defined by $y_1 y_2 = 0$, we consider $(L, N_i, i \in [1, 3])$ and $M_Z = \{1, 2\}$, then $IC^*(\text{Log} Z)$ is defined by the diagram:

$$\begin{array}{ccccc} L & \xrightarrow{N_1, N_2} & L \oplus L & \xrightarrow{N_1, N_2} & L \\ \downarrow N_3 & & \downarrow N_3 & & \downarrow N_3 \\ N_3 L & \xrightarrow{N_1, N_2} & N_3 L \oplus N_3 L & \xrightarrow{N_1, N_2} & N_3 L \end{array}$$

with differentials defined by N_i with + or - sign. Then, we have a quasi-isomorphism $\mathbb{R}\Gamma(D^3 - (D^3 \cap Z), j_! \mathcal{L}) \simeq IC^*(\text{Log} Z)$.

5.1.2. *Weight filtration.*

Definition 5.1. The filtration W^J on the term $N_{J'_Z} L$ of index J in $IC^* L(\text{Log} Z) \subset \Omega^* L$ is induced by the embedding $(N_{J'_Z} L, W^J) \subset (L, W^J)$

$$IC^* L(\text{Log} Z) := s(N_{J'_Z} L)_{J \subset M}, \quad W_k IC^* L(\text{Log} Z) := s(W_{k-|J|}^J N_{J'_Z} L)_{J \subset M}.$$

As in the case of $\Omega^* L$ and $IC^* L$, to compute $Gr_k^W IC^* L(\text{Log} Z)$ we introduce the data $DR(IC^* L(\text{Log} Z)) :=$

$$\{N_{J'_Z} L, W^J, F^J, I_{J,K} : N_{J'_Z} L \rightarrow N_{K'_Z} L, N_{J-K} : N_{K'_Z} L \rightarrow N_{J'_Z} L\}_{K \subset J \subset M}$$

satisfying various properties as in the subsection 4.4.4, including the properties of distinguished pairs for consecutive terms $N_i : N_{K'_Z} L \rightarrow N_{J'_Z} L$ for $J = K \cup i$ (corollary 4.25). Set for each $J \subset M$

$$P_k^J(N_{J'_Z} L) := \cap_{K \subset J, K \neq J} \text{Ker}(Gr_{I_{J,K}} : Gr_k^{W^J} N_{J'_Z} L \rightarrow Gr_{k+|J-K|}^{W^K} N_{K'_Z} L)$$

then $P_k^J(N_{J'_Z} L) \subset Gr_k^{W^J} N_{J'_Z} L$ has pure weight k with respect to the weight filtration $M(\sum_{j \in J} N_j, L)$ and it is polarized.

Corollary 5.2. *We have: $Gr_k^{W^J} N_{J'_Z} L \xrightarrow{\sim} \oplus_{K \subset J} N_{J-K} P_{k+|J-K|}^K(N_{K'_Z} L)$, and $Gr_k^W(IC^*L(\text{Log}Z)) \xrightarrow{\sim} \oplus_{K \subset M} IC(P_{k-|K|}^K(N_{K'_Z} L)[-|K|])$.*

5.1.3. *Global statements.* In terms of the diagram $(X - Y) \xrightarrow{j'} (X - Z) \xrightarrow{j_Z} X$, $j = j_Z \circ j'$ such that $Rj_{Z*}((j'_*\mathcal{L})|_{X-Z}) \simeq Rj_{Z*}j'_{!*}\mathcal{L}$.

Definition 5.3 ($IC^*\mathcal{L}(\text{Log}Z)$). For $J \subset M$, let $J_Z = J \cap I_Z$, $J'_Z = J - J_Z$. The fiber of $IC^*\mathcal{L}(\text{Log}Z)$ is generated at $x \in Y_M^*$ as an $\Omega_{X,x}^*$ -submodule, by the sections $\tilde{v} \wedge_{j \in J} \frac{dy_j}{y_j}$ for $J \subset M$ for all $v \in N_{J'_Z} L$.

Here $N_\emptyset = Id$. In other terms, in degree k , for each subset $K \subset M$ of length $|K| = k$, let $K' = (I - I_Z) \cap K$, and $K_i \subset K'$ denote a subset of length $|K_i| = k - i$ and $A := \mathcal{O}_{X,x}$, the subcomplex $IC^*\mathcal{L}(\text{Log}Z)_x \subset \Omega^*\mathcal{L}_x$ is defined with the notations of (4.3.1), as the $\Omega_{X,x}^*$ -submodule:

$$\oplus_{|K|=k} \left(y_{K'} A \tilde{L} \frac{dy_K}{y_K} + \cdots + \sum_{K_i \subset K'} y_{K_i} A (\widetilde{N_{K'-K_i} L}) \frac{dy_K}{y_K} + \cdots \right)$$

Next we check the isomorphism $(Rj_{Z*}j'_{!*}\mathcal{L})_x \simeq IC^*L(\text{Log}Z)$ of the fiber at a point $x \in Y_M^*$.

Lemma 5.4. *We have: $(Rj_{Z*}(j'_*\mathcal{L})|_{X-Z})_x \xrightarrow{\sim} (IC^*\mathcal{L}(\text{Log}Z))_x$*

We check the proof on the terms of the spectral sequence with respect to the truncation filtration on $j'_{!*}\mathcal{L}$, that is $E_2^{p,q} = R^p j'_{Z*} H^q((j'_{!*}\mathcal{L})_x)$ (after décalage in Deligne's notations), converging to $H^{p+q}(IC^*L(\text{Log}Z))$.

Indeed, the complex $IC^*L(\text{Log}Z)$ may be described as the exterior algebra defined by the action of $N_i, i \in M_Z$ on the Koszul complex $s(N_{J'_Z} L)_{(J'_Z) \subset (M'_Z)}$. That is the simple complex associated to the double complex $s(s(N_{J'_Z} L)_{(J'_Z) \subset (M'_Z)}, N_i)_{i \in M_Z}$. $IC^*L(\text{Log}Z) := s(\Omega(N_{J'_Z} L, N_i, i \in M_Z)_{J'_Z \subset M'_Z} = \Omega(IC^*(L, N_i, i \in M'_Z), N_i, i \in M_Z))$. then the spectral sequence is associated to the double complex.

We define as well the modules $P_k^J(N_{J'_Z} L)$, the corresponding global sheaves $\mathcal{P}_k^J(N_{J'_Z} \mathcal{L})$ on Y_M^* and deduce from the above corollary

Proposition 5.5. *The induced filtrations W and F on $IC^*\mathcal{L}(\text{Log}Z)$ form a structure of MHC such that the graded perverse sheaves for the weight filtration, satisfy the decomposition property into intermediate extensions for all k (corollary 5.2)*

$$Gr_k^W IC^*\mathcal{L}(\text{Log}Z) \xrightarrow{\sim} \oplus_{J \subset I} (i_{Y_{J_Z}})_* j'_{!*} \mathcal{P}_{k-|J|}^J(N_{J'_Z} \mathcal{L})[-|J|].$$

where j denotes uniformly the inclusion of Y_J^* into Y_J for each $J \subset I, Z := \cup_{i \in I_Z} Y_i, J_Z := J \cap I_Z$ and $Y_{J_Z} := \cap_{i \in J_Z} Y_i$. In particular, for $J = \emptyset$ we have $j'_{!*} Gr_k^W \mathcal{L}$, otherwise the summands are supported by Z .

A MHS is deduced on the hypercohomology $\mathbb{H}^*(X - Z, j'_{!*}\mathcal{L})$.

Remark 5.6. For any NCD Z such that $Z \cup Y$ is still a NCD, we may always suppose that \mathcal{L} is a variation of MHS on $X - (Y \cup Z)$ (by enlarging Y) and consider Z equal to a union of components of $Y \cup Z$.

5.1.4. *Thom-Gysin isomorphism.* Let H be a smooth hypersurface intersecting transversally Y such that $H \cup Y$ is a NCD, $i_H : H \rightarrow X$, $j_H : H - (Y \cap H) \rightarrow H$, then $i_H^* j_{H!} \mathcal{L}$ is isomorphic to the (shifted) intermediate extension $(j_H)_! (i_H^* \mathcal{L})$ of the restriction of \mathcal{L} to H , with abuse of notations as we should write $((j_H)_! ((i_H^* \mathcal{L})[-1]))[1]$. The residue with respect to H , $R_H : i_H^*(IC^* \mathcal{L}(\text{Log}H)/IC^* \mathcal{L}) \xrightarrow{\sim} i_H^* IC^* \mathcal{L}[-1]$ induce an isomorphism, inverse to Thom-Gysin isomorphism

$$i_H^* j_{H!} \mathcal{L}[-1] \xrightarrow{\sim} i_H^! j_{H!} \mathcal{L}[1] \xrightarrow{\sim} i_H^*(IC^* \mathcal{L}(\text{Log}H)/IC^* \mathcal{L}).$$

Moreover, if H intersects transversally $Y \cup Z$ such that $H \cup Y \cup Z$ is a NCD, we have a triangle

$$(i_H)_* Ri_H^! IC^* \mathcal{L}(\text{Log}Z) \rightarrow IC^* \mathcal{L}(\text{Log}Z) \rightarrow IC^* \mathcal{L}(\text{Log}Z \cup H) \xrightarrow{[1]}$$

then we deduce an isomorphism of the quotient complex with the cohomology with support: $(IC^* \mathcal{L}(\text{Log}Z \cup H)/IC^* \mathcal{L}(\text{Log}Z)) \xrightarrow{\sim} Ri_H^! IC^* \mathcal{L}(\text{Log}Z)[1]$, induced by the connecting isomorphism.

We have also an isomorphism of the restriction $i_H^* IC^* \mathcal{L}(\text{Log}Z)$ with the complex $IC^*(i_H^* \mathcal{L})(\text{Log}Z \cap H)$ constructed directly on H .

The residue with respect to H : $IC^* \mathcal{L}(\text{Log}Z \cup H)[1] \rightarrow i_{H*} IC^* i_H^* \mathcal{L}(\text{Log}Z \cap H)$ vanish on $IC^* \mathcal{L}(\text{Log}Z)$ and induces an inverse to the Thom-Gysin isomorphism $i_H^* IC^* \mathcal{L}(\text{Log}Z) \xrightarrow{\sim} Ri_H^! IC^* \mathcal{L}(\text{Log}Z)[2]$. The above constructions are compatible with the filtrations up to a shift in degrees.

5.2. **Duality and hypercohomology of the link.** The link at a point $v \in V$ refers to the boundary of a ball with center v intersecting transversally the strata in a small neighborhood of v . It is a topological invariant, hence its hypercohomology is well defined. In the case of a morphism $f : X \rightarrow V$, it refers to the hypercohomology of the boundary of a tubular neighborhood of $Z := f^{-1}(v)$, that is the inverse image of the link at v .

The duality functor D in the derived category of sheaves of vector spaces over \mathbb{Q} (resp. \mathbb{C}) (see [BBDG 83], the references there and subsection 5.4), and the cone construction, are used here to deduce various logarithmic complexes from the structure of mixed Hodge complex on $IC^* \mathcal{L}(\text{Log}Z)$.

5.2.1. *Duality.* To develop a comprehensive theory of weights, we need to fix some conventions. The fields \mathbb{Q} and \mathbb{C} form a HS of weight 0. If H is a HS of weight a , we write $H(r)$ for the HS of weight $a - 2r$.

i) A Hodge complex K of \mathbb{Q} -vector spaces is of weight a if its cohomology $H^i(K)$ is a HS of weight $a + i$. We set $\mathbb{Q}[r]$ for the complex with \mathbb{Q} in degree $-r$ and zero otherwise, then $H^{-r}(\mathbb{Q}[r]) = \mathbb{Q}$ is a HS of weight 0, hence $\mathbb{Q}[r]$ must be a HC of weight r .

On a smooth proper variety X , the dual of $H^i(X, \mathbb{Q})$ has weight $-i$, while $H^{2n-i}(X, \mathbb{Q}) = \mathbb{H}^{-i}(X, \mathbb{Q}[2n])$ has weight $2n - i$. We can write Poincaré duality with value in $\mathbb{Q}[2n](n)$ of weight 0, such that $H^{2n-i}(X, \mathbb{Q}(n)) = \mathbb{H}^{-i}(X, \mathbb{Q}[2n](n))$ with weight $-i$ corresponds to the dual HS on $H^i(X, \mathbb{Q})$.

Hence, it is convenient to set $\omega_X := \mathbb{Q}_X[2n](n)$ for the dualizing complex in the category of sheaves of \mathbb{Q} - HS.

ii) A *pure variation* of HS $\tilde{\mathcal{L}}$ on a Zariski open subset $U \subset X$ of weight $w(\tilde{\mathcal{L}}) = b$, has a dual variation $\tilde{\mathcal{L}}^*$ of HS of weight $w(\tilde{\mathcal{L}}^*) = -b$. A polarization on $\tilde{\mathcal{L}}$ induces an isomorphism defined by the underlying non degenerate bilinear product $S : \tilde{\mathcal{L}} \xrightarrow{\sim} \tilde{\mathcal{L}}^*(-b)$.

As a complex of sheaves, $\mathcal{L} := \tilde{\mathcal{L}}[n]$ has weight $w = a = b + n$. Its dual in the derived category $D\mathcal{L} := \mathcal{H}om(\mathcal{L}, \mathbb{Q}_X[2n](n)) \xrightarrow{\sim} \tilde{\mathcal{L}}^*(n)[n]$, has weight $w = -b + n - 2n = -a$. Then, the polarization induces an isomorphism $S : \mathcal{L} \xrightarrow{\sim} D\mathcal{L}(-a)$. This isomorphism extends in the category of topological constructible sheaves, into:

$$j_{1*}\mathcal{L} \simeq j_{1*}(D\mathcal{L}(-a)) \xrightarrow{\sim} D(j_{1*}\mathcal{L})(-a) := R\mathcal{H}om(j_{1*}\mathcal{L}, \mathbb{Q}_X[2n])(-a).$$

Lemma 5.7. *For X compact and a shifted polarized variation of HS \mathcal{L} of weight a , we have the auto - duality isomorphism with equal weights*

$$(5.2) \quad \mathbb{H}^i(X, j_{1*}\mathcal{L})^* \xrightarrow{\sim} H^{-i}(X, j_{1*}\mathcal{L})(a)$$

where $w(\mathbb{H}^i(X, j_{1*}\mathcal{L})^*) = -(a + i)$ and $w(H^{-i}(X, j_{1*}\mathcal{L})(a)) = a - i - 2a$.

The lemma follows from the polarization $S : j_{1*}\mathcal{L} \xrightarrow{\sim} D(j_{1*}\mathcal{L})(-a)$ and the duality $\mathbb{H}^i(X, j_{1*}\mathcal{L})^* = H^{-i}(\mathcal{H}om(\mathbb{R}\Gamma(X, j_{1*}\mathcal{L}), \mathbb{Q})) \xrightarrow{\sim} H^{-i}(X, D(j_{1*}\mathcal{L})) \xrightarrow{\sim} H^{-i}(X, j_{1*}\mathcal{L})(a)$

iii) *Dual filtrations.* The dual of a variation of MHS $(\tilde{\mathcal{L}}, W, F)$ on a smooth Zariski open subset U , is a variation of MHS $(\tilde{\mathcal{L}}^*, W, F)$, with the filtrations dual to W and F on \mathcal{L} , such that $Gr_{-i}^W(\tilde{\mathcal{L}}^*) \simeq (Gr_i^W \tilde{\mathcal{L}})^*$.

In the bifiltered derived category of an abelian category with a dualizing functor D , the dual of a bifiltered complex (K, W, F) , is denoted by (DK, W, F) with dual filtrations defined by:

$$W_{-i}DK := D(K/W_{i-1}), F^{-i}DK := D(K/F^{i+1})$$

such that: $DGr_i^W K \xrightarrow{\sim} Gr_{-i}^W DK$ and $DGr_F^i K \xrightarrow{\sim} Gr_F^{-i} DK$.

As a complex of sheaves, $(\mathcal{L} := \tilde{\mathcal{L}}[n], W, F)$ has its weight increased by n , such that $W^i(\mathcal{L}) := W^{i-n}(\tilde{\mathcal{L}})[n]$, and $F_i(\mathcal{L}) := F_i(\tilde{\mathcal{L}})[n]$. Thus the filtration on its cohomology $H^{-n}(\mathcal{L})$, decreased by n , satisfy $(H^{-n}(\mathcal{L}), W) = (\tilde{\mathcal{L}}, W)$.

The dual in the bifiltered derived category

$$D(\mathcal{L}, W, F) := \mathbb{R}\mathcal{H}om((\mathcal{L}, W, F), \mathbb{Q}_U[2n](n)),$$

is a shifted variation of MHS $(D\mathcal{L}, W, F)$ such that: $Gr_{-i}^W(D\mathcal{L}, F) \simeq D(Gr_i^W \mathcal{L}, F)$.

Thus, the dual of a mixed Hodge complex on a variety X is a MHC.

A graded polarization on a variation of MHS is defined by a family of polarizations S_i on $(Gr_i^W \mathcal{L}, F)$. It induces isomorphisms:

$$(Gr_i^W \mathcal{L}, F) \xrightarrow{\sim} D(Gr_i^W \mathcal{L}, F)(-i) \xrightarrow{\sim} (Gr_{-i}^W D\mathcal{L}, F)(-i)$$

5.2.2. *Structure of mixed Hodge complex on $Ri_Z^! j_{1*}\mathcal{L}$, $i_Z^* j_{1*}\mathcal{L}$ and $j_{Z!}(j_{1*}\mathcal{L}|_{X-Z})$.* We use the duality $DRi_Z^! j_{1*}\mathcal{L} \simeq i_Z^* D j_{1*}\mathcal{L} \simeq i_Z^* j_{1*}\mathcal{L}(a)$ to deduce the weight filtrations on $i_Z^* j_{1*}\mathcal{L}$.

Definition 5.8 (Dual filtrations). Let \mathcal{L} be a shifted polarized VHS of weight a on a Zariski open subset $U := X - Y$ where Y is a NCD in X smooth projective, and Z a closed sub-NCD of Y .

i) The filtrations on $Ri_Z^! j_{1*}\mathcal{L}$ are defined by the isomorphism

$$(Ri_Z^! j_{1*}\mathcal{L}, W, F) \simeq i_Z^*(IC^* \mathcal{L}(\text{Log}Z)/IC^* \mathcal{L}, W, F)[-1].$$

such that $W_i(i_{Z*}Ri_Z^!j_{!*}\mathcal{L}) = W_{i+1}(IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L})[-1]$.

ii) The filtrations on the complex $i_Z^*j_{!*}\mathcal{L}$ are deduced by duality

$$(i_Z^*j_{!*}\mathcal{L}, W, F) \simeq D(i_Z^!j_{!*}\mathcal{L}(a), W, F)$$

iii) The filtrations on $j_{Z!}(j_{!*}\mathcal{L}|_{X-Z})$ are deduced by duality:

$$(j_{Z!}((j_{!*}\mathcal{L})|_{X-Z}), W, F) \simeq D(IC^*\mathcal{L}(a)(\text{Log}Z), W, F)$$

Corollary 5.9. *The weights w satisfy the following inequalities:*

$$\begin{aligned} w\mathbb{H}^i(X-Z, j_{!*}\mathcal{L}) &\geq a+i, w\mathbb{H}_Z^i(X, j_{!*}\mathcal{L}) \geq a+i \\ w\mathbb{H}^i(Z, j_{!*}\mathcal{L}) &\leq a+i, w\mathbb{H}_c^i(X-Z, j_{!*}\mathcal{L}) \leq a+i \end{aligned}$$

The weights of $IC^*\mathcal{L}(\text{Log}Z)$ as a MHC, are $\geq a$ by construction. The weights of $Ri_Z^!\mathcal{L}$ as a MHC, are $\geq a$ since the weights of $IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L}$ are $\geq a+1$.

We use duality computations for the weights of $\mathbb{H}^i(Z, j_{!*}\mathcal{L})$:

$$\mathbb{H}^i(Z, Gr_j^W i_Z^*j_{!*}\mathcal{L}) \simeq \mathbb{H}^i(Z, Gr_j^W Di_Z^!j_{!*}\mathcal{L}(a)) \simeq \mathbb{H}^i(Z, D(Gr_{-j}^W i_Z^!j_{!*}\mathcal{L}(a))(-j)) \simeq \mathbb{H}^{-i}(Z, (Gr_{-j}^W i_Z^!j_{!*}\mathcal{L}(a))(-j))^*$$

where $w(Gr_{-j}^W i_Z^!j_{!*}\mathcal{L}(a))(-j) = -a-j+2j$, hence $w\mathbb{H}^{-i}(Z, (Gr_{-j}^W i_Z^!j_{!*}\mathcal{L}(a))(-j))^* = a+i-j \leq a+i$ since it vanishes for $j < 0$.

Lemma 5.10. *Let \mathcal{L} be a polarized VHS of weight a on a Zariski open subset U and Z a closed subvariety of X projective.*

i) *There exists a long exact sequence of MHS*

$$(5.3) \quad \cdots \rightarrow \mathbb{H}_Z^i(X, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^i(X, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^i(X-Z, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}_Z^{i+1}(X, j_{!*}\mathcal{L}) \rightarrow \cdots$$

with weights

$$w(\mathbb{H}_Z^i(X, j_{!*}\mathcal{L})) \geq a+i, w(\mathbb{H}^i(X, j_{!*}\mathcal{L})) = a+i, w(\mathbb{H}^i(X-Z, j_{!*}\mathcal{L})) \geq a+i.$$

ii) *We have a dual exact sequence of MHS*

$$(5.4) \quad \rightarrow \mathbb{H}_c^i(X-Z, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^i(X, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^i(Z, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}_c^{i+1}(X-Z, j_{!*}\mathcal{L}) \rightarrow \cdots$$

with weights

$$w(\mathbb{H}_c^i(X-Z, j_{!*}\mathcal{L})) \leq a+i, w(\mathbb{H}^i(X, j_{!*}\mathcal{L})) = a+i, w(\mathbb{H}^i(Z, j_{!*}\mathcal{L})) \leq a+i.$$

The lemma is proved first for X smooth projective and $U := X - Y$ is the complement of a NCD with Z a sub-NCD of Y .

For example, the exact sequence is associated to the triangle $Ri_Z^!j_{!*}\mathcal{L} \rightarrow j_{!*}\mathcal{L} \rightarrow Rj_{Z*}(\mathcal{L}|_{X-Z})$ and the inequalities on the weights follow from the corollary above.

In the general case, the Hodge theory is deduced from the case of NCD by application of the decomposition theorem as in the subsection 2.4.8.

Example. i) For a pure \mathcal{L} of weight a , $i_{Z*}Ri_Z^!j_{!*}\mathcal{L}$, is supported by Z , of weights $w_i \geq a$, and $i_{Z*}i_Z^*j_{!*}\mathcal{L}$ of weights $w_i \leq a$. The morphism I induces the morphism: $Gr_i^W i_{Z*}Ri_Z^!j_{!*}\mathcal{L} \rightarrow Gr_i^W j_{!*}\mathcal{L} \rightarrow Gr_i^W i_{Z*}i_Z^*j_{!*}\mathcal{L}$ for $i = a$ and 0 for $i \neq a$.

ii) A polarized VHS on \mathbb{C}^* corresponds to the action of a nilpotent endomorphism N on L . The duality in the assertion ii) at 0 between $Gr_{-r+1}^W i_Z^*(j_{!*}\mathcal{L})[-1]$ and $Gr_r^W i_Z^*(IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L})$ corresponds to the duality between $Gr_{-r+1}^W \text{Ker}N$ and $Gr_{r-1}^W L/NL$. For higher dimension, one needs to introduce strict simplicial coverings of a NCD to have such interpretation.

5.2.3. *Structure on the Mixed cone.* We set $W_{j+1}(K[1]) := (W_j K)[1]$. The mixed cone over a morphism $f : (K, W, F) \rightarrow (K', W', F')$ is the cone $(C(f), W, F)$ with the corresponding filtrations on $K[1] \oplus K'$.

We construct a MHC as the mixed cone over a morphism of MHC. Due to the definition of morphisms of complexes up to homotopy, a special attention is needed for the compatibility of the constructions over \mathbb{Q} and \mathbb{C} . This can be achieved by using simplicial coverings as in [De 75] (see section 5.4 in our case).

5.2.4. *Structure of MHC on $i_{Z*}i_Z^*Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})$ and the cohomology of the boundary of a tubular neighborhood of Z .*

Let $\mathcal{K} := (j_{!*}\mathcal{L})_{|X-Z}$, we have the following triangles

$$(5.5) \quad j_{Z!}\mathcal{K} \xrightarrow{can} Rj_{Z*}\mathcal{K} \rightarrow i_{Z*}i_Z^*Rj_{Z*}\mathcal{K}, \quad Ri_Z^!j_{!*}\mathcal{L} \xrightarrow{I} i_Z^*j_{!*}\mathcal{L} \rightarrow i_Z^*Rj_{Z*}\mathcal{K}$$

where can is the composition of $j_{Z!}\mathcal{K} \xrightarrow{p} j_{!*}\mathcal{L} \xrightarrow{i} Rj_{Z*}\mathcal{K}$ and I is the restriction of the composition of $i_{Z*}Ri_Z^!j_{!*}\mathcal{L} \xrightarrow{q} j_{!*}\mathcal{L} \xrightarrow{r} i_{Z*}i_Z^*Rj_{Z*}\mathcal{K}$. From which we deduce two descriptions of the complex $i_Z^*Rj_{Z*}\mathcal{K}$, as the restriction of the cone complex over the morphism can or as the cone over I .

We remark that the morphism p is a morphism of perverse sheaves so that we have a triangle $\text{Ker } p \rightarrow j_{Z!}\mathcal{K} \xrightarrow{p} j_{!*}\mathcal{L} \xrightarrow{[1]}$, and $\text{Ker } p \simeq i_{Z*}i_Z^*Rj_{Z*}\mathcal{K}[-1]$. A description of $\text{Ker } p$ is given by lifting to a strict simplicial covering of Z in the spirit of Deligne's simplicial construction, which is dual to the logarithmic construction of $IC^*\mathcal{L}(\text{Log } Z)/IC^*\mathcal{L}$. In particular we put on $\text{Ker } p$ the dual filtrations W and F (subsection 5.4.5 and proposition 5.17).

Definition 5.11 ($i_{Z*}i_Z^*Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})$). Let can be defined as a bifiltered morphism of complexes $can : j_{Z!}(j_{!*}\mathcal{L}_{|X-Z}) \rightarrow Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})$ with their filtrations W and F . The mixed cone $C(can)$ over can is isomorphic to $i_{Z*}i_Z^*Rj_{Z*}(j_{!*}\mathcal{L})_{|X-Z}$.

The cone depends only on the neighborhood of Z and not on X , and we have a duality isomorphism

$$i_Z^*Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})[-1] \xrightarrow{\sim} D(i_Z^*Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})).$$

To realize the morphism I , we introduce the shifted cone $C(i)[-1]$ over the embedding $i : IC^*\mathcal{L} \rightarrow IC^*\mathcal{L}(\text{Log } Z)$. We have an isomorphism $C(i)[-1] \simeq i_{X_v*}Ri_Z^!j_{!*}\mathcal{L}$ such that the projection $C(i)[-1] \rightarrow IC^*\mathcal{L}$ is well defined and compatible with the filtrations; then I is the composition with the restriction map $IC^*\mathcal{L} \rightarrow i_{Z*}i_Z^*j_{!*}\mathcal{L}$ (see also definition 5.21).

Let B_Z denote a small neighborhood of Z in X , then

$$\mathbb{H}^i(Z, i_{Z*}i_Z^*Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})) \simeq \mathbb{H}^i(B_Z - Z, j_{!*}\mathcal{L})$$

and we have an exact sequence of MHS

$$\cdots \rightarrow \mathbb{H}^i(X - Z, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^i(B_Z - Z, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^{i+1}(X, j_{!*}(j_{!*}\mathcal{L}_{|X-Z})) \rightarrow \cdots$$

The MHS on the hypercohomology of $B_Z^* := B_Z - Z$ is defined equivalently by the mixed cone over the morphism can or the Intersection morphism I

Lemma 5.12. *There exist an isomorphism of mixed cones $C(I) \xrightarrow{\sim} i_Z^*C(can)$ compatible with both filtrations W and F .*

The isomorphism is constructed by comparison with the cone over the morphism $j_{Z!}(j_{!*}\mathcal{L}|_{X-Z}) \oplus i_{Z*}Ri_Z^!j_{!*}\mathcal{L} \rightarrow j_{!*}\mathcal{L}$, such that we have a diagram with coefficients in $j_{!*}\mathcal{L}$ where $X^* = X - Z$

$$(5.6) \quad \begin{array}{ccccccc} \mathbb{H}_Z^r(X) & & \rightarrow & & \mathbb{H}^r(Z) & & \mathbb{H}_Z^{r+1}(X) \\ & \searrow & & \nearrow & & \searrow & \nearrow \\ & & \mathbb{H}^r(X) & & & & \mathbb{H}^r(B_Z^*) \\ & \nearrow & & \searrow & \nearrow & & \searrow \\ \mathbb{H}_c^r(X^*) & & \rightarrow & & \mathbb{H}^r(X^*) & & \mathbb{H}_c^{r+1}(X^*) \end{array}$$

5.3. Compatibility of the perverse filtration with MHS. Let V be a quasi-projective variety and $K \in D_c^b(V, \mathbb{Q})$ a complex with constructible cohomology sheaves. The perverse filtration ${}^p\tau$ on K induces a perverse filtration ${}^p\tau$ on the hypercohomology groups $\mathbb{H}^k(V, K)$. This topological filtration has an interesting construction described by algebraic-geometric techniques in [CaMi 10].

We consider two families of hyperplanes $\Lambda_* := \Lambda_i$ and $\Lambda'_* := \Lambda'_i$ for $1 \leq i \leq n$ in \mathbb{P}^N , defining by intersection with V two families H_* and W_* of increasing closed sub-varieties on V , where $H_{-r} := \cap_{1 \leq i \leq r} \Lambda_i \cap V$ and $W_{-r} := \cap_{1 \leq i \leq r} \Lambda'_i \cap V$:

$$H_* : V = H_0 \supset H_{-1} \supset \dots \supset H_{-n}, \quad W_* : V = W_0 \supset W_{-1} \supset \dots \supset W_{-n}$$

and $H_{-n-1} = \emptyset = W_{-n-1}$. Let $h_i : (V - H_{i-1}) \rightarrow V$ with indices in $[-n, 0]$ denote the open embeddings. The following filtration

$$(5.7) \quad \delta_p \mathbb{H}^*(V, K) := \text{Im} \left(\bigoplus_{i+j=p} \mathbb{H}_{W_{-j}}^*(V, (h_i)_! h_i^* K) \rightarrow \mathbb{H}^*(V, K) \right)$$

is defined in ([CaMi 10], remark 3.6.6), where the main result in ([CaMi 10], Thm 4.2.1) states that for an affine embedding of V into a projective space and for a generic choice of both families depending on K and the embedding of V , the filtration δ is equal to the perverse filtration ${}^p\tau$ up to a shift in indices.

5.3.1. Relative case. Let $f : X \rightarrow V$ be a projective morphism of complex varieties where V is projective. Let $Y \subset X$ be a closed subvariety, \mathcal{L} a local system on $X - Y$, $j : (X - Y) \rightarrow X$, W a closed subspace of V , $Z := f^{-1}W$ the inverse image of W and $j_Z : (X - Z) \rightarrow X$. We apply the above result in the following cases: $K_1 := Rf_* R(j_Z)_*(j_Z)_* j_{!*}\mathcal{L}$ (including the case $Z = Y$ or $Z = \emptyset$), and $K_2 := Rf_* Ri_Z^! j_{!*}\mathcal{L}$ (the cases $Rf_*(j_Z)_!(j_Z)_* j_{!*}\mathcal{L}$ and $Rf_*(i_Z)^* j_{!*}\mathcal{L}$ are dual).

The perverse filtration ${}^p\tau$ is defined on $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L})$ via the isomorphism $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L}) \simeq \mathbb{H}^k(V, K_1)$ (resp. on $\mathbb{H}_Z^k(X, j_{!*}\mathcal{L})$ via $\mathbb{H}_Z^k(X, j_{!*}\mathcal{L}) \simeq \mathbb{H}_W^k(V, K_2)$).

5.3.2. Proof of the proposition 1.6. We suppose f a fibration by NCD over the strata, Y a NCD in X and \mathcal{L} an admissible variation of MHS on $X - Y$, such that $Z := f^{-1}W$ and $Z \cup Y$ are NCD and W union of strata. We consider the bifiltered logarithmic complex $K_1 = Rf_* IC^* \mathcal{L}(\text{Log} Z)$ (resp. K_2 as in definition 5.8 i).

We check that the induced filtration ${}^p\tau$ on $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L})$ (resp. on $\mathbb{H}_Z^k(X, j_{!*}\mathcal{L})$) is compatible with the MHS, that is a filtration by sub-MHS as stated in the propositions 1.6.

Starting with the complex $(IC^* \mathcal{L}(\text{Log} Z), W, F)$ (4.3.1), the system of truncations maps $\dots \rightarrow {}^p\tau_{\leq -i} Rf_* IC^* \mathcal{L}(\text{Log} Z) \rightarrow {}^p\tau_{\leq -i+1} Rf_* IC^* \mathcal{L}(\text{Log} Z) \dots$ is isomorphic in $D_c^b(V, \mathbb{Q})$ to a system of inclusions maps $\dots \rightarrow P^i K \rightarrow P^{i-1} K \dots$ where we suppose K , all $P^i K$ and $Gr_P^i K$ are injective complexes ([BBDG 83], 3.1.2.7), then

the filtration P on K is defined up to unique isomorphism in the category of filtered complexes $(D_c^b(V, \mathbb{Q}), F)$.

We need to consider not only a fine Dolbeault resolution of $(IC^*\mathcal{L}(\text{Log}Z), W, F)$ to deal with Hodge theory, but an acyclic resolution for the filtrations P, W and F on the same complex on V .

Let (K_1, P, W, F) be a representative complex of $Rf_*IC^*\mathcal{L}(\text{Log}Z), W, F)$ with acyclic filtrations on V and where P represents the perverse filtration. To prove that $R\Gamma(V, Gr_i^P K_1, W, F)$ is a MHC, or $\mathbb{H}^r(V, Gr_i^P K_1, W, F)$ is a MHS, we need to prove that the filtration δ defined by the formula 5.7 above is a filtration by sub-MHS.

Lemma 5.13. *i) Let $f : X \rightarrow V$ be a fibration by NCD over the strata, and W a closed subvariety of V such that $Z := f^{-1}(W)$ is a NCD in X . Then the MHS on $\mathbb{H}^j(X, IC^*\mathcal{L}(\text{Log}Z))$ (resp. $\mathbb{H}^j(X, Ri_Z^! j_{!*}\mathcal{L})$) is well defined, as well on the terms of the filtration δ (5.7) for $Rf_*IC^*\mathcal{L}(\text{Log}Z)$ (resp. $Rf_*Ri_Z^! j_{!*}\mathcal{L}$). Consequently the perverse filtration ${}^p\tau$ on $\mathbb{H}^j(X - Z, j_{!*}\mathcal{L})$ (resp. $\mathbb{H}_Z^j(X, j_{!*}\mathcal{L})$) is compatible with the MHS.*

ii) The dual statement and the corresponding filtration δ show that the perverse filtration ${}^p\tau$ on $\mathbb{H}_c^j(X - Z, j_{!}\mathcal{L})$ (resp. $\mathbb{H}^j(Z, j_{!*}\mathcal{L})$) is compatible with the MHS.*

Proof. We reduce the proof to the case of an affine embedding of $V - W$. Let $\pi : \tilde{V} \rightarrow V$ be a blowing up of W such that the embedding of $U := V - W$ in \tilde{V} is affine. Since $f^{-1}(U)$ is a NCD in X , the morphism $f : X \rightarrow V$ factors as $\pi \circ g$ for $g : X \rightarrow \tilde{V}$. We apply the lemma to U in \tilde{V} , to construct the families H_* and W_* as intersection of two general families of hypersurfaces Λ_* and Λ'_* in \tilde{V} such that the union of their inverse image $g^{-1}\Lambda_*$ and $g^{-1}\Lambda'_*$, the existing NCD Y and $f^{-1}(U)$, form together a large NCD in X . In particular, the intersections of the various inverse image $f^{-1}(H_i \cap W_j)$ are *transversal and intersect transversally* the various NCD in X .

Then, we consider this large NCD to construct $IC^*\mathcal{L}(\text{Log}Z) \subset \Omega^*\mathcal{L}$ with the weight and Hodge filtrations on it.

The inverse image of $H'_* := g^{-1}(H_*)$ and $W'_* := g^{-1}(W_*)$ are transversal in X and the restrictions $i_{H'_i}^*\mathcal{L}$ are shifted VHS on $H'_i \cap Y$ for various indices i , whenever \mathcal{L} is a shifted VHS. Let $h'_i : (X - H'_i) \rightarrow X$, we construct by the techniques of sections 3, 4, and Thom-Gysin isomorphisms (subsection 5.1.4), bifiltered logarithmic mixed Hodge complexes $Ri_{W'_j}^!(h'_i)_! h'^i_* j_{!*}\mathcal{L}$, such that the embedding in the formula 5.7 is compatible with MHS. The proof for all other hypercohomology functors is similar. \square

5.3.3. *The proof of the proposition 1.7 is similar but does not follow from [CaMi 10]. The hypercohomology of a tubular neighborhood with the central fibre deleted $B_{X_v}^* = B_{X_v} - X_v$ with coefficients in $j_{!*}\mathcal{L}$, is defined (definition 5.11, lemma 5.12, see also 5.21) by the cone over the morphism $can : j_{X_v}!(j_{!*}\mathcal{L}|_{X - X_v}) \rightarrow Rj_{X_v,*}(j_{!*}\mathcal{L}|_{X - X_v})$ which is a realization of $i_{X_v,*}i_{X_v}^*Rj_{X_v,*}((j_{!*}\mathcal{L})|_{X - X_v})$.*

The mixed cone $(C(can), W, F)$ is endowed with the filtrations W and F deduced from the filtrations on its components.

We remark that the preceding characterization of the perverse filtration, as a filtration δ (5.7), apply to a punctured ball $B_v - \{v\}$ as B_v is Stein. In fact the preceding lemma apply to two hyperplane sections in general position containing

v . As a consequence the filtration δ on $\mathbb{H}^*(B_{X_v}^*, j_{1*}\mathcal{L})$ coincide up to indices with the perverse filtration and moreover it consists of sub-MHS.

5.4. Complements on Duality. We deduced from $\Omega^*\mathcal{L}$, by duality, various logarithmic complexes. In this section, we develop what we may call duality calculus to describe the filtered realizations of the complexes $i_Y^*j_{1*}\mathcal{L}$, $Ri_Y^!j_{1*}\mathcal{L}$, and $j_!\mathcal{L}$ obtained from the filtered structure of $\Omega^*\mathcal{L}$.

An explicit *filtered version of the auto-duality isomorphism of the Intersection complex* : $j_{1*}\mathcal{L} \xrightarrow{\sim} R\mathcal{H}om(j_{1*}\mathcal{L}, \mathbb{Q}_X[2n])$ is also described (proposition 5.17).

For a local system \mathcal{L} with singularities along a normal crossing divisor Y , the dual of $\Omega^*\mathcal{L}$ with value in Grothendieck's residual complex is described, in terms of residues and Grothendieck's symbols, using the simplicial covering space of Y (formula 5.13).

In the case of a NCD sub-divisor $Z \subset Y$, we deduce from $i_Z^*\Omega^*\mathcal{L}(\text{Log}Z)$ various logarithmic complexes needed in the proof of the decomposition, in particular a cone construction in the case of the tubular neighborhood. The description presents similarity with Deligne's simplicial techniques in [De 72].

In order to deduce a MHC by the cone construction over a morphism of MHC, the compatibility between \mathbb{Q} and \mathbb{C} coefficients must be well defined, not only up to homotopy, which is achieved by the simplicial constructions.

5.4.1. Serre duality. On a smooth compact complex algebraic variety X of dimension n , Serre duality for locally free modules, has value in the canonical sheaf of modules Ω_X^n [Se 55]. Since the logarithmic complex $\Omega^*\mathcal{L} := \Omega_X^*(\text{Log}Y) \otimes_{\mathcal{O}_X} \mathcal{L}_X$ consists of locally free sheaves, its dual can be defined in the following way.

Let y_1, \dots, y_p denote a subset of a coordinate set of parameters defining an equation of Y locally at a point x . The ideal \mathcal{I}_Y of Y is generated at x by the product $(y_1 \dots y_p) := \mathcal{I}_{Y,x}$. The wedge product defines a map

$$\mathcal{I}_Y \Omega_X^j(\text{Log}Y) \otimes \Omega_X^{n-j}(\text{Log}Y) \rightarrow \Omega_X^n$$

In each degree the sub-module $\mathcal{I}_Y \Omega^j \mathcal{L} := \mathcal{I}_Y(\Omega_X^j(\text{Log}Y) \otimes_{\mathcal{O}_X} \mathcal{L}_X)$ is contained in $\Omega_X^j \otimes_{\mathcal{O}_X} \mathcal{L}_X$; it is acyclic at points of Y and it is a sub-complex of the Intersection sub-complex $IC^*\mathcal{L}$; We use the product $S : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathbb{C}$, to define a morphism $\phi : \mathcal{I}_Y \Omega^j \mathcal{L} \xrightarrow{\phi} \mathcal{H}om_{\mathcal{O}_X}(\Omega^{n-j}\mathcal{L}, \Omega_X^n)$ as follows: for $\omega \in \mathcal{I}_Y \Omega^j \mathcal{L}$ and $\alpha \in \Omega^{n-j}\mathcal{L}$

$$(5.8) \quad \omega \otimes \tilde{v} \mapsto \{\alpha \otimes \tilde{v}' \mapsto S(v, v') \omega \wedge \alpha.\}$$

as $\omega \wedge \alpha \in \Omega_X^n$ for $\omega \in \mathcal{I}_Y \Omega^j \mathcal{L}$. We check locally term by term, that ϕ is an isomorphism (with \mathbb{C} -linear differentials and not \mathcal{O}_X -linear). As the modules of the logarithmic complex are locally free, the duality isomorphism apply

$$(5.9) \quad \mathcal{I}_Y \Omega^* \mathcal{L} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\Omega^* \mathcal{L}, \Omega_X^n) \xrightarrow{\sim} DRj_* \mathcal{L} \xrightarrow{\sim} j_! \mathcal{L}.$$

Let $i_Y : Y \rightarrow X$, we remark the following quasi-isomorphisms:

$$i_{Y*} i_Y^* Rj_* \mathcal{L} \xrightarrow{\sim} \Omega^* \mathcal{L} / \mathcal{I}_Y \Omega^* \mathcal{L}, \quad i_{Y*} i_Y^* j_{1*} \mathcal{L} \xrightarrow{\sim} IC^* \mathcal{L} / \mathcal{I}_Y \Omega^* \mathcal{L}.$$

5.4.2. Grothendieck-Verdier duality and the Dualizing complex.

Grothendieck introduced the concept of residual complex K_X^\bullet on complex algebraic varieties [Gro 58, Ha 66] to extend duality theory to the category of coherent sheaves; the complex K_X^\bullet consists of injective sheaves. In the case of a smooth X , it is an injective resolution of $\Omega_X^n[n]$.

The complex $\mathbb{C}[2n]$ is dualizing in the category of sheaves of \mathbb{C} -vector spaces on X^{an} (the terminology is related to Poincaré duality). Both duality theory, in

the category of sheaves of modules and in the category of \mathbb{C} -vector spaces, are related. To explain the relation between Serre duality and Poincaré duality, the category of complexes of \mathcal{O}_X -modules with differential operators of order ≤ 1 is used in ([HeL 71], §2) where a graded module over the algebra of differential forms Ω_X^* (resp. $\Omega_{X^{an}}^*$ in the analytic case) is associated to such complexes. A basic formula ([HeL 71], proposition (2.9), 3) establishes for each Ω_X^* -graded module \mathcal{F}^\bullet an isomorphism

$$(5.10) \quad \mathcal{H}om_{\Omega_X^*}^k(\mathcal{F}^\bullet, \Omega_X^*) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^{n-k}, \Omega_X^n)$$

Then Poincaré duality on X^{an} between the hypercohomology $\mathbb{H}^*(X^{an}, \Omega_{X^{an}}^*)$ and $\text{Ext}_{\Omega_{X^{an}}^*}^*(\Omega_{X^{an}}^*, \Omega_{X^{an}}^*[2n]) \simeq \text{Ext}_{\mathcal{O}_{X^{an}}}^*(\Omega_{X^{an}}^*, \Omega_{X^{an}}^*[n])$ follows from Serre duality at the level of associated spectral sequences ([HeL 71], Introduction).

As a consequence of Grothendieck's algebraic de Rham theory [Gro 66], we may introduce the algebraic complex $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^*, \Omega_X^n[n])$ in duality theory [Ha 72, E 78], although it is not a resolution of $\mathbb{C}[2n]$ in the algebraic case.

The dual of a complex of \mathcal{O}_X -modules \mathcal{F}^\bullet with differential operators of order ≤ 1 , is the complex $\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{F}^\bullet, K_X^\bullet)$ with differential operators of order ≤ 1 , with value in the injective resolution K_X^\bullet of $\Omega_X^n[n]$.

Definition 5.14 (Dual complex). The dual complex of $\Omega^*\mathcal{L}$ is the complex with differential operators of order ≤ 1

$$\mathcal{H}om_{\mathcal{O}_X}^\bullet(\Omega^*\mathcal{L}, K_X^\bullet) \simeq R\mathcal{H}om_{\mathcal{O}_X}^\bullet(\Omega^*\mathcal{L}, \Omega_X^n[n]).$$

The dual filtrations of W and F are defined with value in K_X^\bullet .

Remark 5.15. i) On the analytic variety X^{an} , the complex $\mathcal{O}_{X^{an}} \otimes K_X^\bullet$ is dualizing for complexes of analytic sheaves associated to algebraic sheaves. However, we continue to use the notation K_X^\bullet in both algebraic and analytic cases. A dualizing complex exists for analytic coherent sheaves [RaR 70], but it is not needed here.

ii) The relation between the dualizing complex and currents is discussed in ([RaR 74], §5). In fact K_X^\bullet embeds into currents on X^{an} (see also [E 74]).

5.4.3. *The simplicial covering of a NCD and the duality map.* To construct a duality map for MHS, we use a simplicial covering of Y .

We attach to Y with normally crossing components Y_i indexed by $i \in I$, the strict simplicial covering Y_\bullet (resp. Y_\bullet^+) defined as follows: $\pi : Y_\bullet \rightarrow X$ is indexed by \mathbb{N}^* such that $Y_n := \coprod_{|A|=n} Y_A$, $A \subset I$, $|A| = n$, $n \neq 0$ (hence $A \neq \emptyset$) where $Y_A := \bigcap_{i \in A} Y_i$, with connecting morphisms defined by the natural embeddings for each inclusion $A \subset B \subset I$; resp. we add $Y_0 := Y_\emptyset := X$ with index 0 and write $\pi_+ : Y_\bullet^+ \rightarrow X$. For example, in the case of two components: $Y_i \cup Y_j$, $Y_\bullet^+ = (Y_2 := Y_{i,j} \rightarrow Y_1 := Y_i \amalg Y_j \rightarrow Y_0 := X)$.

A complex $K \in D(X, \mathbb{Z})$ on X , lifts to a simplicial complex π^*K on Y_\bullet (resp. π_+^*K on Y_\bullet^+). We write $\pi_*\pi^*K$ or $s(\pi^*K)$ (resp. $\pi_{+*}\pi_+^*K$ or $s(\pi_+^*K)$) for the simple associated complex, then we have a quasi-isomorphism $i_Y^*K \xrightarrow{\sim} s(\pi^*K)$ (resp. $j_!(K|_{X-Y}) \xrightarrow{\sim} s(\pi_+^*K)$ where $j : (X - Y) \rightarrow X$). In some sense, simplicial coverings correspond in topological terms, to simplicial resolutions of complexes.

Definition 5.16. The simplicial topological inverse image by π_+^* (resp. π^*) of $\Omega^*\mathcal{L}$ (resp. $IC^*\mathcal{L}$) is denoted by $\Omega^*\mathcal{L}|_{Y_\bullet^+}$ on Y_\bullet^+ (resp. $IC^*\mathcal{L}|_{Y_\bullet^+}$ on Y_\bullet^+ , and $\Omega^*\mathcal{L}|_{Y_\bullet}$,

$IC^*\mathcal{L}|_{Y_\bullet}$ on Y_\bullet). The direct images by π_{+*} on X (resp. π_* on Y) are:

$$\begin{aligned}\Omega_+\mathcal{L}(\bullet) &:= s(\Omega^*\mathcal{L}|_{Y_\bullet^+}) \simeq j_!\mathcal{L}, & IC_+\mathcal{L}(\bullet) &:= s(IC^*\mathcal{L}|_{Y_\bullet^+}) \simeq j_!\mathcal{L} \\ \Omega\mathcal{L}(\bullet) &:= s(\Omega^*\mathcal{L}|_{Y_\bullet}) \simeq \Omega^*\mathcal{L}|_Y, & IC\mathcal{L}(\bullet) &:= s(IC^*\mathcal{L}|_{Y_\bullet}) \simeq IC^*\mathcal{L}|_Y.\end{aligned}$$

For example, in dimension 2, $\Omega\mathcal{L}(\bullet)$ is defined by the double complex $\Omega^*\mathcal{L}|_{Y_1} \oplus \Omega^*\mathcal{L}|_{Y_2} \rightarrow \Omega^*\mathcal{L}|_{Y_{1,2}}$, while $\Omega_+\mathcal{L}(\bullet)$ is defined by $(\Omega^*\mathcal{L} \rightarrow \Omega\mathcal{L}(\bullet)[-1])$.

We have an exact sequence: $0 \rightarrow IC\mathcal{L}(\bullet)|_Y[-1] \rightarrow IC_+\mathcal{L}(\bullet) \rightarrow IC^*\mathcal{L} \rightarrow 0$.

5.4.4. *Duality map and Grothendieck's symbols.* We construct a dualizing map with value in the residual complex by local computations with Grothendieck's symbols

$$(5.11) \quad \Omega_+\mathcal{L}(\bullet) := s(\Omega^*\mathcal{L}|_{Y_\bullet^+}) \xrightarrow{\phi} \mathcal{H}om_{\mathcal{O}_X}^\bullet(\Omega^*\mathcal{L}, K_X^\bullet).$$

Let z denotes a generic point of an irreducible sub-variety Z of codimension i in X . The module of local cohomology $H_z^i(\Omega_X^n)$ defines a constant sheaf on Z denoted by $i_z H_z^i(\Omega_X^n)$, then $K_X^i = \sum_{\text{codim. } Z=i} i_z H_z^i(\Omega_X^n)$ in degree i .

A set of local equations y_j for $j \in [1, i]$ of Z at z generates the ideal of Z at z , so that the local cohomology group is computed by an inductive limit of Koszul resolutions defined by various m -th powers of y_j , hence an element $\bar{w} \in H_z^i(\Omega_X^n)$

is written as a Grothendieck symbol $\left[\begin{array}{c} \omega \\ y_1^m, \dots, y_i^m \end{array} \right]$.

Since $IC^*\mathcal{L}$ consists of modules which are not locally free, the dualizing map has value in the injective resolution K_X^\bullet of $\Omega_X^n[n]$. In the following computation, we need only the terms with value in the subcomplex $\underline{\Gamma}_Y K_X^\bullet$ with support in the algebraic divisor Y . A duality calculus based on Grothendieck's symbols [Gro 58, Ha 66, E 78] leads to the construction of the dual weight filtrations and the duality map ϕ (formula 5.13 below).

Let $x_0 \in X$ be the generic point of X , the wedge product defines a map : $\Omega_X^i(\text{Log}Y) \otimes \Omega_X^j(\text{Log}Y) \rightarrow i_{x_0} \Omega_{X,x_0}^{i+j}$ for $i+j=n$ with value in the constant sheaf of rational n -forms. We define the duality map as:

$$(5.12) \quad \phi : \Omega^*\mathcal{L} \otimes_{\mathcal{O}_X} \Omega^*\mathcal{L} \rightarrow i_{x_0} \Omega_{X,x_0}^*[n] : (\omega \otimes \tilde{v}) \otimes (\alpha \otimes \tilde{v}') \mapsto S(v, v')\omega \wedge \alpha$$

where the product $S : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathbb{C}$ is defined by the polarization, and ω and α have logarithmic singularities. This map is compatible with the differentials of forms, as : $d(S(v, v')\omega \wedge \alpha) = S(v, v')((d\omega) \wedge \alpha + (-1)^i \omega \wedge (d\alpha))$ since $S(v, v')$ is constant as v and v' are flat. In the analytic setting, we consider the complex of analytic sheaves $K_X^\bullet \otimes \mathcal{O}_{X^{an}}$ as a dualizing complex on X^{an} . As the NCD Y is algebraic, the denominator terms are algebraic and $\mathcal{O}_{X^{an}}$ is flat over \mathcal{O}_X . For example on X^{an} , we use the sheaf $\mathcal{O}_{X^{an}} \otimes i_{x_0} \Omega_{X,x_0}^*$, instead of $i_{x_0} \Omega_{X,x_0}^*$. But we omit the corresponding notation for X^{an} .

However, to construct a map: $\Omega^*\mathcal{L} \otimes_{\mathcal{O}_X} \Omega^*\mathcal{L} \rightarrow K_X^\bullet$, we view Ω_{X,x_0}^n as the term $H_{x_0}^0(\Omega_X^n)$ of the complex K_X^\bullet . This map is no more compatible with the total differential since we introduce the differential $d_1 : H_{x_0}^0(\Omega_X^n) \rightarrow \bigoplus_{y_i} H_{y_i}^1(\Omega_X^n)$ where the codimension of the closure Y_i of y_i is 1.

For example, in the case of $\Omega^i\mathcal{L} \xrightarrow{\phi} \mathcal{H}om_{\mathcal{O}_X}(\Omega^{n-i}\mathcal{L}, \Omega_{X,x}^n)$

$$d_1(\omega \wedge \frac{dy_{i_1}}{y_{i_1}} \wedge \dots \wedge \frac{dy_{i_k}}{y_{i_k}}) = \bigoplus_s \left[\begin{array}{c} (-1)^s \omega \wedge \frac{dy_{i_1}}{y_{i_1}} \wedge \dots \wedge dy_{i_s} \wedge \dots \wedge \frac{dy_{i_k}}{y_{i_k}} \\ y_{i_s} \end{array} \right]$$

(In comparison with the case of currents instead of K_X^\bullet ([Dol 71]), the defect of compatibility is corrected there by the residue in the formula $\phi(D\omega) - D\phi(\omega) = \phi(\text{Res}\omega)$ ([Dol 71] formula 9)).

We extend the map ϕ to the terms $\Omega^* \mathcal{L}_{|Y_\bullet^+} := \pi_+^*(\Omega^*(\text{Log}Y) \otimes_{\mathcal{O}_X} \mathcal{L}_X)$ on the semi-simplicial covering Y_\bullet^+ . For sections $\alpha \in \Omega^i \mathcal{L}_{|Y_{i_1, \dots, i_k}}$ and $\beta \in \Omega^{n-i} \mathcal{L}$, we use in the next formula, the symbols in $\text{Ext}_{\mathcal{O}_X}^k(\mathcal{O}_{Y_{i_1, \dots, i_k}}, \Omega^* \mathcal{L})_{y_{i_1, \dots, i_k}}$ considered as a subset of $H_{y_{i_1, \dots, i_k}}^k(\Omega_X^*)$ where y_{i_1, \dots, i_k} is a generic point of a component of $\cap_j Y_{i_j}$ and $y_{i_1} \cdots y_{i_k} \alpha \wedge \beta$ is regular at y_{i_1, \dots, i_k} :

$$(5.13) \quad \begin{aligned} & (\Omega^i \mathcal{L})_{i_1, \dots, i_k} \otimes_{\mathcal{O}_X} \Omega^{n-i} \mathcal{L} \xrightarrow{\phi_{i_1, \dots, i_k}} H_{y_{i_1, \dots, i_k}}^k(\Omega_X^n) \\ & \alpha \otimes \beta \mapsto \epsilon(i_1, \dots, i_k) \begin{bmatrix} y_{i_1} \cdots y_{i_k} \alpha \wedge \beta \\ y_{i_1}, \dots, y_{i_k} \end{bmatrix} \in H_{y_{i_1, \dots, i_k}}^k(\Omega_X^n). \end{aligned}$$

that is the class of $y_{i_1} \cdots y_{i_k} \alpha \wedge \beta \in \Omega_{X, y_{i_1, \dots, i_k}}^n$ with value in the module of n -forms at the generic point y_{i_1, \dots, i_k} modulo the sub-module generated by y_{i_1}, \dots, y_{i_k} . This class is also called $\text{Res}_{Y_{i_1, \dots, i_k}}(\alpha \wedge \beta)$.

We deduce from the above, an extension of ϕ (formula 5.13) to $\Omega_+ \mathcal{L}(\bullet)$ on X (definition 5.16), defining the duality isomorphism of complexes

$$(5.14) \quad \begin{array}{ccc} \Omega_+ \mathcal{L}(\bullet) & \xrightarrow{\phi} & \mathcal{H}om_{\mathcal{O}_X}^\bullet(\Omega^* \mathcal{L}, K_X^\bullet) \\ \downarrow \simeq & & \downarrow \simeq \\ j_! \mathcal{L} & \xrightarrow{\sim} & D_X Rj_* \mathcal{L} \end{array}$$

5.4.5. *The complex $\text{Ker } \phi(\bullet)$ kernel of ϕ and the auto-duality of $j_! \mathcal{L}$.* We construct a filtered version of the triangles of perverse sheaves

$$i_{Y^*} i_Y^* j_! \mathcal{L}[-1] \rightarrow j_! \mathcal{L} \rightarrow D j_! \mathcal{L} \text{ dual to } IC^* \mathcal{L} \rightarrow \Omega^* \mathcal{L} \rightarrow \Omega^* \mathcal{L} / IC^* \mathcal{L}$$

We deduce from the duality map ϕ an induced map

$$(5.15) \quad IC_+ \mathcal{L}(\bullet) := s(IC^* \mathcal{L}_{|Y_\bullet^+}) \xrightarrow{\phi} \mathcal{H}om_{\mathcal{O}_X}^\bullet(IC^* \mathcal{L}, K_X^\bullet).$$

The map ϕ does not vanish on the subcomplex $s(IC^* \mathcal{L}_{|Y_\bullet^+})[-1] \simeq i_Y^* IC^* \mathcal{L}[-1]$. The kernel of ϕ , denoted $\text{Ker } \phi(\bullet)[-1]$ is a sub-complex of $s(IC^* \mathcal{L}_{|Y_\bullet^+})[-1]$.

Proposition 5.17. *[$i_Y^* IC^* \mathcal{L} \simeq \text{Ker } \phi(\bullet)$ and auto-duality]*

i) *The kernel of ϕ (formula 5.14, 5.15, 5.13) satisfy*

$$\text{Ker } \phi(\bullet) \xrightarrow{\sim} s(IC^* \mathcal{L}_{|Y_\bullet^+}) \xrightarrow{\sim} i_Y^* IC^* \mathcal{L}.$$

and $\text{Ker } \phi(\bullet)[-1]$ is perverse.

ii) *We have a filtered auto-duality isomorphism induced by ϕ :*

$$(5.16) \quad \begin{aligned} IC^* \mathcal{L} & \xrightarrow{\sim} IC_+ \mathcal{L}(\bullet) / \text{Ker } \phi(\bullet)[-1] \xrightarrow{\phi} \mathcal{H}om_{\mathcal{O}_X}^\bullet(IC^* \mathcal{L}, K_X^\bullet) \\ \text{Ker } \phi(\bullet)[-1] & \xrightarrow{\phi} \mathcal{H}om_{\mathcal{O}_Y}^\bullet(\Omega^* \mathcal{L} / IC^* \mathcal{L}, K_Y^\bullet) \end{aligned}$$

where $K_Y^\bullet := \mathcal{H}om^\bullet(\mathcal{O}_Y, K_X^\bullet)$ is the dualizing complex on Y .

The complex $\text{Ker } \phi(\bullet)[-1]$ is denoted by $\text{Ker } p$ in subsection 5.2.4.

With the notations of subsection 4.3.1, let Y_J denote a component space of the simplicial space Y_\bullet^+ , $x \in Y_M^*$ and L the fiber of the local system. To each subset $K_i \subset K \subset M$ and to each subspace $V := (\text{Ker } N_{J-J \cap K}) \cap (N_{K-K_i} L)$, we associate the submodule $A\tilde{V} := A((\text{Ker } N_{J-J \cap K}) \cap (N_{K-K_i} L)) \subset \mathcal{L}_x$.

Definition 5.18 (The Kernel complex $\text{Ker } \phi(\bullet)$).

The simplicial sub-complex of modules $(\text{Ker } \phi)_J^* \subset IC^* \mathcal{L}_{J,x}$ consist of fibers of analytic sheaves. It is defined as the subcomplex generated as an $(\Omega_{Y_J}^*)_x$ -algebra by the sub-modules

$$\bigoplus_{|K|=k} \left(\sum_{K_i \subset K} y_{K_i} A((\text{Ker } N_{J-(J \cap (K-K_i))}) \cap N_{K-K_i} L) \frac{dy_K}{y_K} \right)$$

$\text{Ker } \phi(\bullet) := s((\text{Ker } \phi)_J^*)_{\emptyset \neq J \subset M}$ is the sum.

We explain, how $\text{Ker } N_{J-(J \cap (K-K_i))} \cap N_{K-K_i} L$ appears in the description of the kernel of ϕ . Let $K := \{i_1, \dots, i_k\}$, $Y_K := \bigcap_{i \in K} Y_i$ with generic point y_K , and $dy_K/y_K := dy_{i_1}/y_{i_1} \wedge \dots \wedge dy_{i_k}/y_{i_k}$. For $\alpha \in i_{Y_K}^* \Omega^p \mathcal{L}$ and $\beta \in \Omega^q \mathcal{L}$, the pairing at a point $x \in Y_K$:

$(\alpha, \beta) := \phi_{i_1, \dots, i_k}(\alpha \wedge \beta) \in H_{y_K}^k(\Omega_X^{p+q}) \subset K_X^\bullet(\Omega_X^*)_x$ vanish unless dy_K/y_K divides $\alpha \wedge \beta$ (as its value is considered modulo the ideal generated by $\{y_{i_1}, \dots, y_{i_k}\}$).

For $\beta = \tilde{v} dy_Q/y_Q$ for $Q \subset K$, let $P := K - Q$ and $\alpha = \tilde{v}' dy_P/y_P$, the pairing $(\alpha, \beta) = 0$ vanish if and only if $S(v', v) = 0$.

In particular $(\alpha, \beta) = 0$ for all $v \in N_Q L$ if and only if $v' \in \text{Ker } N_Q$.

Respectively, to describe the dual filtrations we remark that for $v \in W_{-r-1} L$, $S(v', v) = 0$ vanish if and only if $v' \in (W_r)^* L$.

5.4.6. *The filtration $N!W$ dual to $N * W$.* The dual filtrations may be constructed directly. Let N be a nilpotent endomorphism respecting (L, W) a filtered space with relative monodromy filtration M . We introduce the filtration ([Ka 86, 3.4.2])

$$(N!W)_k := W_{k-1} + M_k(N, W) \cap N^{-1} W_{k-1},$$

with induced morphisms $I : W_{k-1} \rightarrow (N!W)_k$ and $N : (N!W)_k \rightarrow W_{k-1}$ satisfying the relations $N \circ I = N$ and $I \circ N = N$.

Then $N!W$ is dual to the filtration $N * W$ on L (formula 4.9), with induced morphisms $W_{k+1} \xrightarrow{N} (N * W)_k \xrightarrow{I} W_{k+1}$ satisfying $I \circ N = N$ and $I \circ N = N$.

Let (L^*, N^*) denote the dual of (L, N) and W^* the dual filtration to W defined by: $W_k^* := \text{Hom}(L/W_{-k-1}, \mathbb{Q})$ such that $Gr_k^{W^*} L^* \xrightarrow{\sim} (Gr_{-k}^W L)^*$. There is a duality relation between $(N^*!W^*)$ and $(N * W)^*$, in the following sense:

Lemma 5.19. *Let W^* denote the filtration on the vector space $L^* := \text{Hom}(L, \mathbb{Q})$ dual to a filtration W on L ; then for all a ,*

$$(N^*!W^*)_a = (N * W)_a^* \subset L^*.$$

Remark 5.20. We may develop this construction, in parallel to the constructions corresponding to $N * W$, and define the dual weight filtration on $\text{Ker } \phi(\bullet)$.

5.4.7. *Simplicial coverings of $Z \subset Y$.* We realize the cone construction of the boundary of the tubular neighborhood of Z .

To dualize the filtration W on $IC^* \mathcal{L}(\text{Log } Z)$, we use the strict simplicial covering $\pi_+ : Z_\bullet^+ \rightarrow X$ (resp. $\pi : Z_\bullet \rightarrow Z$) of X , defined by the NCD Z , with index subsets $A \subset I_Z$ including $Z_\emptyset := X$ (resp. excluding Z_\emptyset). It is naturally embedded in Y_\bullet^+ (resp. Y_\bullet). For any complex \mathcal{K} on X , the sum $\pi_{+*} \pi_+^* \mathcal{K} := s(\pi_+^* \mathcal{K}) \xrightarrow{\sim} j_{Z!}(\mathcal{K}|_{X-Z})$. If $Z_\emptyset := X$ is excluded, then $i_Z^* \mathcal{K} \xrightarrow{\sim} \pi_* \pi^* \mathcal{K}$.

5.4.8. *Simplicial complex.* The Intersection complex $IC^*\mathcal{L}$ lifts to a simplicial complex $IC^*\mathcal{L}|_{Z_\bullet^+} := (IC^*\mathcal{L}|_{Z_J})_{J \subset I_Z}$ on Z_\bullet^+ . The simple complex $IC_+\mathcal{L}(Z, \bullet)$ is defined by summing $IC^*\mathcal{L}|_{Z_J}$ over Z_J for $J \subset I_Z$ including $J = \emptyset$. We write $IC\mathcal{L}(Z, \bullet)$ when $Z_\emptyset := X$ is excluded from the sum:

$$\begin{aligned} IC_+\mathcal{L}(Z, \bullet) &:= s(IC^*\mathcal{L}|_{Z_J})_{J \subset I_Z} \xrightarrow{\sim} j_{Z!}((j_{!*}\mathcal{L})|_{X-Z}), \\ IC\mathcal{L}(Z, \bullet) &:= s(IC^*\mathcal{L}|_{Z_J})_{J \neq \emptyset \subset I_Z} \xrightarrow{\sim} i_Z^*(j_{!*}\mathcal{L}). \end{aligned}$$

With the notations of section 5.4.5 the restriction of ϕ (formula 5.13) on components of Z_\bullet

$$IC_+\mathcal{L}(Z, \bullet) \xrightarrow{\phi} \mathcal{H}om_{\mathcal{O}_X}^\bullet(IC^*\mathcal{L}(\text{Log}Z), K_X^\bullet)$$

induce the duality isomorphism $j_{Z!}((j_{!*}\mathcal{L})|_{X-Z}) \xrightarrow{\sim} DRj_{Z*}((j_{!*}\mathcal{L})|_{X-Z})$.

The kernel of ϕ (prop. 5.17), is a simplicial sub-complex $(\text{Ker } \phi(\bullet)_J^*)_{J \subset I_Z, J \neq \emptyset}$. Its sum $\text{Ker } \phi(\bullet)(Z, \bullet) := s(\text{Ker } \phi(\bullet)_J^*)_{J \subset I_Z, J \neq \emptyset}$ is a sub-complex of $IC\mathcal{L}(Z, \bullet)$.

It is embedded with a shift in the degree, as a subcomplex $\text{Ker } \phi(\bullet)(Z, \bullet)[-1] \subset IC_+\mathcal{L}(Z, \bullet)$ equal to $\text{Ker } \phi$ on Z_\bullet^+ . There exist quasi-isomorphisms

$$\text{Ker } \phi(\bullet)(Z, \bullet) \xrightarrow{\sim} IC\mathcal{L}(Z, \bullet) \xrightarrow{\sim} i_Z^*j_{!*}\mathcal{L}.$$

The morphism ϕ induces a duality isomorphism

$$IC_+\mathcal{L}(Z, \bullet)[-1] \xrightarrow{\sim} D_X(IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L})$$

5.4.9. $i_Z^*IC^*\mathcal{L}$ and $j_!\mathcal{L}$. The filtration W on $\text{Ker } \phi(\bullet)(Z, \bullet)$ is defined by duality.

The complex $IC_+\mathcal{L}(Z, \bullet) \simeq j_{Z!}j_Z^*j_{!*}\mathcal{L}$ has a structure of mixed Hodge complex with the following weight filtration:

$$(5.17) \quad W_r IC_+\mathcal{L}(Z, \bullet) := W_r(IC^*\mathcal{L}) \oplus (W_{r+1} \text{Ker } \phi(\bullet)(Z, \bullet))[-1]$$

It defines a MHS on $\mathbb{H}^*(X, j_{Z!}j_Z^*j_{!*}\mathcal{L})$ dual to the MHS on $\mathbb{H}^*(X, Rj_{Z*}j_Z^*j_{!*}\mathcal{L})$.

There exist filtered duality isomorphisms

$$(i_Z^*j_{!*}\mathcal{L}, W) \xrightarrow{\sim} D_X(Ri_Z^!j_{!*}\mathcal{L}, W) \xrightarrow{\sim} D_X(IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L})[-1, W),$$

induced by the morphism $\phi : \text{Ker } \phi(\bullet)(Z, \bullet)[-1] \xrightarrow{\sim} D_X((IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L}))$, and a filtered duality isomorphisms $(j_{Z!}j_Z^*j_{!*}\mathcal{L}, W) \xrightarrow{\sim} D_X(IC^*\mathcal{L}(\text{Log}Z), W)$.

5.4.10. *Cohomology with support* $Ri_Z^!j_{!*}\mathcal{L}[1]$. The complex $IC^*\mathcal{L}(\text{Log}Z)$ lifts to a simplicial complex on Z_\bullet

$IC^*\mathcal{L}(\text{Log}Z)|_{Z_J} := i_{Z_J}^*IC^*\mathcal{L}(\text{Log}Z)$. The simple associated complex is the sum

$$IC^*\mathcal{L}(\text{Log}Z)(\bullet) := s(IC^*\mathcal{L}|_{Z_J})_{J \subset I_Z, J \neq \emptyset} \xrightarrow{\sim} i_Z^*Rj_{Z*}(j_Z^*j_{!*}\mathcal{L}).$$

We have an embedding $IC^*\mathcal{L}(Z, \bullet) \subset IC^*\mathcal{L}(\text{Log}Z)(\bullet)$ with quotient complex:

$$IC^*\mathcal{L}(\text{Log}Z)(\bullet)/IC^*\mathcal{L}(Z, \bullet) \simeq i_Z^*(IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L}) \simeq Ri_Z^!j_{!*}\mathcal{L}[1].$$

5.4.11. *The complex* $i_{Z*}i_Z^*Rj_{Z*}((j_{!*}\mathcal{L})|_{X-Z})$ *and the cohomology of the boundary of a tubular neighborhood of* Z . Let B_Z denotes a small neighborhood of Z in X

$$\mathbb{H}^i(Z, i_{Z*}i_Z^*Rj_{Z*}((j_{!*}\mathcal{L})|_{X-Z})) \simeq \mathbb{H}^i(B_Z - Z, j_{!*}\mathcal{L})$$

and we have an exact sequence

$$\cdots \rightarrow \mathbb{H}^i(X - Z, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^i(B_Z - Z, j_{!*}\mathcal{L}) \rightarrow \mathbb{H}^{i+1}(X, j_!(j_{!*}\mathcal{L}|_{X-Z})) \rightarrow \cdots$$

Definition 5.21 ($i_{Z*}i_Z^*Rj_{Z*}((j_{!*}\mathcal{L})|_{X-Z})$). Let $p : j_{Z!}(j_{!*}\mathcal{L})|_{X-Z} = IC_+\mathcal{L}(Z, \bullet) \rightarrow IC_0^*\mathcal{L} = IC^*\mathcal{L}$ denote the projection on the term with index \emptyset , and $i : IC^*\mathcal{L} \rightarrow IC^*\mathcal{L}(\text{Log}Z)$ the natural embedding, the composition morphism $\tilde{I} := i \circ p$ defines a morphism

$$\tilde{I} : IC_+\mathcal{L}(Z, \bullet) \rightarrow IC^*\mathcal{L}(\text{Log}Z), \quad (\tilde{I} : j_{Z!}j_Z^*j_{!*}\mathcal{L} \rightarrow Rj_{Z*}j_Z^*j_{!*}\mathcal{L})$$

The cone $C(\tilde{I})$ over \tilde{I} is isomorphic to $i_{Z*}i_Z^*Rj_{Z*}(j_{!*}\mathcal{L})|_{X-Z}$.

The weight filtration W is deduced from the weights on $IC^*\mathcal{L}(\text{Log}Z)$ and $IC_+\mathcal{L}(Z, \bullet)$: $W_r C(\tilde{I}) := W_{r-1}(IC_+\mathcal{L}(Z, \bullet))[1] \oplus W_r(IC^*\mathcal{L}(\text{Log}Z))$.

Remark 5.22. i) The cone depends only on the neighborhood of Z and not on X as we have a triangle $i_{Z*}\text{Ker } \phi(\bullet)(Z, \bullet) \rightarrow C(\tilde{I}) \rightarrow (IC^*\mathcal{L}(\text{Log}Z)/IC^*\mathcal{L})$.

ii) We have a duality isomorphism :

$D(Gr_r^W C(\tilde{I}, \mathcal{L}) \simeq Gr_{-r}^W DC(\tilde{I}, \mathcal{L}) \simeq (Gr_{-r+1}^W C(\tilde{I}, D\mathcal{L}))[-1]$, since $Di_{Z*}i_Z^*Rj_{Z*}((j_{!*}\mathcal{L})|_{X-Z}) \simeq i_{Z*}Ri_Z^!Rj_{Z!}((j_{!*}D\mathcal{L})|_{X-Z}) \simeq i_{Z*}i_Z^*Rj_{Z*}((j_{!*}D\mathcal{L})|_{X-Z})[1]$, hence $\mathbb{H}^j(Z, C(\tilde{I}))^* \simeq \mathbb{H}^{-j-1}(Z, C(\tilde{I}))$.

iii) If \mathcal{L} is a polarized VHS of weight a , we have

$Gr_{a+i}^W C(\tilde{I}) \simeq Gr_{a+i}^W IC^*\mathcal{L}(\text{Log}Z)$ for $i > 0$, and $Gr_{a+i}^W C(\tilde{I}) \simeq Gr_{a+i}^W \text{Ker } \phi(\bullet)(Z, \bullet)$ for $i \leq 0$.

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