HAL open science

Decomposition, purity and fibrations by normal crossing divisors

Fouad El Zein, Dung Trang Lê, Xuanming Ye

▶ **To cite this version:**

Fouad El Zein, Dung Trang Lê, Xuanming Ye. Decomposition, purity and fibrations by normal crossing divisors. 2019. hal-01915674v2

HAL Id: hal-01915674

https://hal.science/hal-01915674v2

Preprint submitted on 19 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DECOMPOSITION, PURITY AND FIBRATIONS BY NORMAL CROSSING DIVISORS

FOUAD EL ZEIN, DŨNG TRÁNG LÊ, AND XUANMING YE

Abstract. We give a simple geometric proof of the decomposition theorem in terms of Thom-Whitney stratifications by reduction to fibrations by normal crossings divisors over the strata and explain the relation with the local purity theorem an unpublished result of Deligne and Gabber.

Contents

1. Introduction 1
1.1. Statements 3
1.2. Fibration over the strata by normal crossing divisors 7
1.3. Logarithmic complexes 8
1.4. Deligne-Gabber's local purity 10
2. Proof of the decomposition theorem 11
2.1. Proof of theorem **??** 11
2.2. Hard Lefschetz 16
2.3. Proof of the local purity theorem **??** 18
2.4. The crucial case 21
2.5. Local purity and Grauert generalized Intersection isomorphism 26
3. Fibration by normal crossing divisors 28
4. Logarithmic complexes 33
4.1. The logarithmic complex $\Omega^*\mathcal{L} := \Omega^*_X(LogY) \otimes \mathcal{L}_X$ 36
4.2. The direct image $Rj_*\mathcal{L} \simeq \Omega^*\mathcal{L}$ 36
4.3. The intermediate extension $j_{!*}\mathcal{L} \simeq IC^*\mathcal{L}$ 38
4.4. Weight filtration 41
4.5. Global definition and properties of the weight W 47
4.6. The relative logarithmic complex $\Omega^*_f\mathcal{L} := \Omega^*_{X/V}(LogY) \otimes \mathcal{L}_X$ 50
5. Logarithmic Intersection complex for an open algebraic variety 51
5.1. $IC^*\mathcal{L}(LogZ) \simeq Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})$ 51
5.2. Duality and hypercohomlogy of the link 54
5.3. Compatibility of the perverse filtration with MHS 57
References 61

Keywords: Hodge theory, algebraic geometry, decomposition, derived category, Mixed Hodge structure, Perverse sheaves, Stratifications.

Mathematics Subject Classification(2000):Primary 14D07, 32G20; Secondary 14F05

1. Introduction

Let $f : X \to V$ be a projective morphism of complex algebraic varieties. The decomposition theorem, as proved in [BBDG 83], describes the derived direct image complex $Rf_* j_{!*}\mathcal{L}$ on V, of an intermediate extension $j_{!*}\mathcal{L}$ ([BBDG 83], 2.1.7 and proposition 2.1.11) of a geometric local system $\mathcal{L}$ on a Zariski open subset of X, as a direct sum of intermediate extensions of irreducible local systems on Zariski open subsets of V.

The proof in [BBDG 83] is given first for varieties over an algebraically closed field with positive characteristic, with coefficients in a pure perverse sheaf on X, then deduced for geometric coefficients on complex algebraic varieties.

The decomposition theorem has been preceded by an unpublished note by Deligne and Gabber [DeG 81] on the local purity theorem. The draft of the proof is dense and it is established for an algebraic variety over a field with positive characteristic.

According to the general theory of weight this result can be translated into a local purity theorem in Hodge theory in the transcendental case.

There exists a deep connection between these two results which does not appear in the existing proofs of the decomposition theorem. The basic innovation of this article is a combined direct proof of both theorems with coefficients in the intermediate extension of a polarized variation of Hodge structures (VHS), which establishes the interaction between local purity and the decomposition and leads to an overall clarification of the theory in the complex case.

Precise statements are given below (subsection 1.1 Theorem 2 and corollary 1.2).

The proper morphism f can be stratified ([GMacP 88] I, 1.6) by a Thom-Whitney stratification, in opposition to just Whitney stratification of spaces. The proof is by induction on the dimension of the strata of a Thom-Whitney stratification (section 3.0.1 below). At the center of the inductive step we use a geometric interpretation of the local purity to obtain the decomposition (see subsections 1.4, 2.3 and 2.4).

The proof given here of the equivalent statement, called the semi-purity (proposition 1.9), is based on the global result of the polarization of the Intersection cohomology of X. The proof in [DeG 81], when adapted to characteristic 0, is based on the local invariant cycle theorem.

The second innovation is to introduce a class of projective morphisms which are fibrations by normal crossing divisors (NCD) over the strata. In the case of such fibrations by NCD we use Hodge theory only in the case of a complement of a NCD. We recall that NCD are essential in Deligne's construction [De 72].

We show in section 3 that any algebraic morphism $f : X \to V$ admits a desingularization π of X such that π and $f \circ \pi$ are in such class of fibrations, from which we deduce without further effort the result for all projective morphisms.

We rely in this paper on computations in terms of perverse sheaves and perverse cohomology as explained in [BBDG 83]. The perverse τ-filtration on a complex K in the derived category $D^b_c(X, \mathbb{Q})$ of constructible $\mathbb{Q}$-sheaves is defined in ([BBDG 83] proposition 1.3.3 p 29). A recent description of this filtration in [CaMi 10] shows easily that for $K = Rf_* j_{!*}\mathcal{L}$ the induced perverse filtration on the hypercohomology of the inverse image of an open algebraic subvariety U of V consists of sub-MHS of $\mathbb{H}^*(f^{-1}(U), j_{!*}\mathcal{L})$, where we suppose that $f^{-1}(U)$ the complement of a NCD in X (see subsection 5.3). This is a basic result used in our proof and extended to the case of a small ball punctured at the center.

In the case of constant coefficients, the paper of de Cataldo and Migliorini [CaMi 5] shows the potential of applications of the decomposition theorem in understanding the topology of algebraic maps. The proof here for coefficients in any admissible polarized VMHS (definition 4.2) covers all analytic coefficients and has similar applications. In particular the local purity theorem is essentially equivalent to Grauert generalized Intersection isomorphism (formula 2.15).

Complex computations by Kashiwara intended to check the admissibility of (graded) polarized variation of mixed Hodge structures (VMHS) on the complement of a NCD in codimension one ([Ka 86]), find their natural applications here to develop Hodge theory with coefficients in the intermediate extension of an admissible polarized VMHS.

Using the decomposition theorem, the full Hodge theory can be developed from the special case of fibrations by NCD over the strata, which gives an alternative approach to the development in terms of differential modules [Sa 88]. Our proof is outlined in the notes [EL 14] and [EL 15].

It should be noted that the polarized VHS on smooth open subsets of irreducible algebraic subvarieties for which the decomposition theorem is proved here correspond to the holonomic differential Hodge modules on a smooth complex algebraic variety [Sa 88].

1.0.1. *Origin of the theory.* The first statement of a decomposition theorem in the sense of this article appears in the work of Deligne [De 68] as a criteria for a complex K in the derived category of an abelian category [Ve 77] to be isomorphic to the direct sum of its shifted cohomology: $K \xrightarrow{\sim} \oplus_{i\in\mathbb{Z}} H^i(K)[-i]$, which means essentially the degeneration of the spectral sequence with respect to the truncation functor τ.

The criteria in Deligne's paper is applied to the complex direct image $K := Rf_*\mathbb{C}_X$ by a smooth projective morphism $f : X \to V$ of complex varieties. In this case the perverse truncation ${}^p\tau$ coincides with the truncation τ and the spectral sequence with respect to the perverse filtration coincides up to indices with Leray's spectral sequence where the degeneration is deduced from Hodge and Lefschetz theorems. The decomposition of K follows.

When X is smooth and f is a projective morphism, f is smooth over the big strata of a Thom - Whitney stratification of V, so we can apply Deligne's result. We shall see that this is the starting point of our inductive proof.

Deligne's effort to present the proofs in full generality is a prelude to the vast generalization of the theorem to any projective morphism in [BBDG 83] on a scheme X of finite type over an algebraically closed field k of characteristic $p > 0$, for pure perverse sheaves in the derived category $D^b_c(X, \mathbb{Q}_l)$ of constructible $\mathbb{Q}_l$-sheaves for $l \neq p$ [BBDG 83].

The statements may be transposed in characteristic 0 in terms of Hodge theory according to a dictionary established by Deligne [De 71].

A procedure to obtain results concerning *geometric* statements in characteristic 0 from corresponding statements in characteristic $p > 0$, is described in ([BBDG 83], 6.2). This procedure applies to geometric variations of Hodge structures in opposition to axiomatic VHS.

The decomposition theorem is related to the existence of a HS on Intersection cohomology [GMacP 83]. A result known also as a purity result. In the case of a

complex algebraic curve V, pure HS on the cohomology of V with degenerating coefficients, that is an extension of a local system on a Zariski open subset underlying a polarized VHS, is constructed in [Zu 79]. However the next development waited until the introduction of the Intersection complex in [GMacP 83].

The introduction of perverse sheaves extends the scope of the theory to cover algebraic varieties independently of the characteristic of the base field. In this terminology the Intersection complex is extended to the case of local systems on a Zariski open subset and it is referred to as the intermediate extension.

Also, the link with Kashiwara's work on differential modules has been established, as perverse sheaves correspond to holonomic modules.

1.0.2. *Hodge theory.* The development of Hodge theory in terms of logarithmic complexes in the NCD case with coefficients in admissible variation of Hodge structures (subsection 1.3) is not well documented, according to the referee of the notes, as he asked to add the note [EY 14] on the subject. Hence we added the sections 4 and 5 at the end to serve as reference to the proof in the first three sections.

Hodge theory allows us to state the local purity theorem 1.8 and to deduce the decomposition in section 2. By using logarithmic complexes we avoid the elaborate development of the theory of Hodge modules.

1.0.3. *The paper is organized as follows.* In the rest of this introduction we give the main statements of the article and describe the ideas based on a simultaneous proof of decomposition and local purity by induction along the strata.

Full proofs using logarithmic complexes in section 2 are based on the reduction to the case of a special fibration in section 3, while Hodge theory on logarithmic complexes is reviewed in sections $4-5$.

1.1. **Statements.** *We give below the statement of theorem* 2 in terms of Thom-Whitney stratifications and the perverse cohomology sheaves ${}^p\mathcal{H}^i(K)$ of a complex K with constructible cohomology sheaves on an algebraic variety V. The perverse cohomology refers to a cohomology theory with value in the abelian category of perverse sheaves [BBDG 83].

Let $f : X \to V$ be a projective morphism of complex algebraic varieties, $\tilde{\mathcal{L}}$ an admissible polarized variation of Hodge structures (VHS) defined over $\mathbb{Q}$ on a smooth open dense subset Ω in X of dimension m and $j : \Omega \to X$ the embedding. We denote by $\mathcal{L} := \tilde{\mathcal{L}}[m]$ the complex of sheaves reduced to $\tilde{\mathcal{L}}$ in degree $-m$ (*referred to as a shifted VHS*), by $j_{!*}\mathcal{L}$ the intermediate extension of $\mathcal{L}$ ([BBDG 83], prop. 2.1.11 p. 60, 2.2.7 p. 69) and by $K := Rf_*j_{!*}\mathcal{L}$ the direct image in the derived category of cohomologically constructible sheaves of $\mathbb{Q}$-vector spaces $D^b_c(V, \mathbb{Q})$.

The decomposition theorem consists of two results. First, each perverse cohomology sheaf ${}^p\mathcal{H}^i(Rf_*(j_{!*}\mathcal{L}))$ decomposes naturally in a direct sum of intermediate extensions (formula 1.5 below). This decomposition is naturally expressed here in terms of *intersection morphisms* (1.1.2 below) which appear naturally in the theory ([BBDG 83] formula 1.4.6.1), and which importance appears with a Thom-Whitney stratification of f.

The second result is the relative Hard Lefschetz theorem. Let $\eta \in R^2f_*\mathbb{Q}_X$ denote the section of cohomology classes defined by a relative hyperplane section of f. The corresponding morphism of sheaves $K \xrightarrow{\eta} K[2]$ induces morphisms of degree 2 on the perverse cohomology. Then, the result states that the i-th iterated

morphism

$$ {}^p\mathcal{H}^{-i}(Rf_*j_{!*}\mathcal{L}) \xrightarrow{\eta^i} {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}). \tag{1.1} $$

is an isomorphism of perverse cohomology sheaves, from which the degeneration of the perverse Leray spectral sequence is classically deduced.

One consequence is the existence of a non canonical isomorphism in $D^b_c(V,\mathbb{Q})$ of the direct image complex $Rf_*j_{!*}\mathcal{L}$ with its shifted perverse cohomology sheaves (formula 1.8 below). It is deduced from Hard Lefschetz following the same pattern as in [De 68].

1.1.1. *Thom-Whitney stratification $\mathcal{S}$ of f (section 3.0.1 below).* Let $f : X \to V$ be a projective morphism on an algebraic variety V of dimension n.

A Thom-Whitney stratification of f is adapted to $j_{!*}\mathcal{L}$, if the restriction of the cohomology of $j_{!*}\mathcal{L}$, to the various strata of the underlying Whitney stratification $\mathcal{S}$ of X, is locally constant. Since we consider only Thom-Whitney stratification of f adapted to $j_{!*}\mathcal{L}$, the cohomology of $Rf_*(j_{!*}\mathcal{L})$ is also locally constant when restricted to the strata of V. The stratification of X induces a stratification on the fibers of f adapted to $j_{!*}\mathcal{L}$.
Let $V_j := \cup_{\dim S \leq j} S$ denote the union of all strata of dimension $\leq j$.

Ideally, we start with a smooth open subset U as the big strata of dimension n over which f is a topological fibration, and set $V_{n-1} := V - U$ the complement variety. Then, we define $\mathcal{S}_1$ as the union of all algebraic smooth connected strata S of dimension $n-1$ over which f is a topological fibration, $U_1 := U \cup \mathcal{S}_1$, and $V_{n-2} := V - U_1$.

We continue this construction by induction on the dimension of the strata S of $V_j - V_{j-1}$ over which f is a topological fibration.

However, we may refine the stratification according to our need in the proof.

1.1.2. *Intersection morphisms.* Let $X_S := f^{-1}(S)$ be the inverse image of a strata S of dimension $l \leq n$, $i_{X_S} : X_S \to X$, $f_S : X_S \to S$. The intersection morphism $I_S : Ri^!_{X_S}j_{!*}\mathcal{L} \to i^*_{X_S}j_{!*}\mathcal{L}$, is defined on S by the composition of the morphism $i_{X_S*}Ri^!_{X_S}j_{!*}\mathcal{L} \to j_{!*}\mathcal{L}$ with the restriction morphism $j_{!*}\mathcal{L} \to i_{X_S*}i^*_{X_S}j_{!*}\mathcal{L}$. The image of the induced morphism on cohomology sheaves is a local system $\mathcal{L}^i_S$ on S:

$$ \mathcal{L}^i_S := \mathrm{Im}\left\{ R^{-l+i}f_{S*}(Ri^!_{X_S}j_{!*}\mathcal{L}) \xrightarrow{I_S} R^{-l+i}f_{S*}(i^*_{X_S}j_{!*}\mathcal{L}) \right\}, \tag{1.2} $$

For a Thom-Whitney stratification adapted to $j_{!*}\mathcal{L}$, the image $\mathcal{L}^i_S$ of I_S is a local system on each strata S. These local systems are necessarily components of any eventual decomposition as it follows from the proof of proposition 2.2.

When $\mathcal{L}$ is a variation of HS of weight a, the cohomology sheaves with support $R^{-l+i}f_{S*}(Ri^!_{X_S}j_{!*}\mathcal{L})$ (resp. the restrictions $R^{-l+i}f_{S*}(i^*_{X_S}j_{!*}\mathcal{L})$) are VMHS with weight $w \geq a+i-l$ (resp. $w \leq a+i-l$) (corollary 5.9). We deduce that the image $\mathcal{L}^i_{S_l}$ of I_{S_l} on each strata S_l of dimension l is a VHS of weight $a+i-l$.

1.1.3. *The inductive step.* The proof is by induction on the dimension of the strata of the Whitney stratification $\mathcal{S}$ of V underlying a Thom-Whitney stratification of $f : X \to V$. Let V be a complex algebraic variety of dimension n, and $\mathcal{S}$ the stratification of f, $V_j := \cup_{\dim S \leq j} S$ the union of all strata of dimension $\leq j$, $k_j : (V - V_j) \to V$, $i_j : V_j \to V$. The immersion of a strata of dimension l in $V^*_l := V_l - V_{l-1}$ is denoted by $i_{S_l} : S_l \to V$.

Theorem 1.1 (Inductive step)**.** *Let $K = Rf_*j_{!*}\mathcal{L}$ where $\mathcal{L}$ is a shifted polarized variation of Hodge structures (VHS) of weight a, and $\mathcal{S}$ a Thom-Whitney stratification of f adapted to $j_{!*}\mathcal{L}$.*

We suppose by induction, there is a decomposition of the perverse cohomology on $(V - V_j) \xrightarrow{k_j} V$ for all degrees i

$$ {}^p\mathcal{H}^i((K)_{|V-V_j}) \xrightarrow{\sim} \oplus_{j<l\leq n}^{S_l\subset V_l^*} k_j^* i_{S_l!*}\mathcal{L}^i_{S_l}[l] \tag{1.3}$$

into a direct sum of intermediate extensions of polarized VHS $\mathcal{L}^i_{S_l}$ (formula 1.2) of weight $a+i-l$ over the strata S_l for all $l > j$. Let

$$\cdots \to {}^p\mathcal{H}^i((i_j)_*Ri_j^!K) \xrightarrow{{}^p\alpha_i} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\rho_i} {}^p\mathcal{H}^i(Rk_{j*}K_{|V-V_j}) \xrightarrow{{}^p\delta_i} \cdots$$

be the long exact sequence of perverse cohomology associated to the triangle $(i_j)_*Ri_j^!K \xrightarrow{\alpha} K \xrightarrow{\rho} Rk_{j*}K_{|V-V_j} \xrightarrow{[1]}$.

i) The restriction to $V - V_{j-1}$ of the short exact sequences of perverse sheaves

$$0 \to Im\,{}^p\alpha_i \to {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\rho_i} Im\,{}^p\rho_i \to 0$$

is split over $V - V_{j-1}$:

$$\begin{aligned} {}^p\mathcal{H}^i(K)_{|V-V_{j-1}} &= k_{j-1}^*\mathrm{Im}\,{}^p\alpha_i \oplus k_{j-1}^* Im\,{}^p\rho_i \\ k_{j-1}^*\mathrm{Im}\,{}^p\alpha_i = \mathrm{Ker}\,{}^p\rho_i &\xrightarrow{\sim} \oplus_{S_j\subset V_j^*} k_{j-1}^* i_{S_j!*}\mathcal{L}^i_{S_j} \\ k_{j-1}^* Im\,{}^p\rho_i = k_{j-1}^*(k_j)_{!*}k_j^*\,{}^p\mathcal{H}^i(K) &\xrightarrow{\sim} \oplus_{j<l\leq n}^{S_l\subset V_l^*} k_{j-1}^* i_{S_l!*}\mathcal{L}^i_{S_l}[l]. \end{aligned} \tag{1.4}$$

ii) Hard Lefschetz: If we suppose by induction Lefschetz isomorphisms on the open subset $V - V_j$, then η^i extends into an isomorphism over $V - V_{j-1}$

$$\eta^i : {}^p\mathcal{H}^{-i}(K)_{|V-V_{j-1}} \xrightarrow{\sim} {}^p\mathcal{H}^i(K)_{|V-V_{j-1}}$$

Thus we obtain the following statement of the decomposition theorem proved in [BBDG 83] in the geometric case and in [Sa 88] for Hodge modules.

Corollary 1.2. *Let $K = Rf_*j_{!*}\mathcal{L}$ where $\mathcal{L}$ is a shifted polarized variation of Hodge structures of weight a, and $\mathcal{S}$ a Thom-Whitney stratification of f adapted to $j_{!*}\mathcal{L}$.*

i) There exists a decomposition of the perverse cohomology on V in each degree

$$ {}^p\mathcal{H}^i(K) \xrightarrow{\sim} \oplus_{l\leq n}^{S_l\subset V_l^*} i_{S_l!*}\mathcal{L}^i_{S_l}[l] \tag{1.5}$$

into a direct sum of intermediate extensions of shifted polarized VHS $\mathcal{L}^i_{S_l}$ (formula 1.2) of weight $a+i-l$ over the strata S_l for all $l \leq n$. Moreover,

$$\mathrm{Im}\left({}^p\mathcal{H}^i(Rk_{j!}k_j^*K) \to {}^p\mathcal{H}^i(K)\right) = k_{j!*}k_j^*\,{}^p\mathcal{H}^i(K) \tag{1.6}$$

On a projective variety V, we have an orthogonal decomposition of a polarized HS of weight $a+i+j$

$$Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(X, j_{!*}\mathcal{L}) \simeq \mathbb{H}^j(V, {}^p\mathcal{H}^i(K)) \xrightarrow{\sim} \oplus_{l\leq n}^{S_l\subset V_l^*} \mathbb{H}^j(V, i_{S_l!*}\mathcal{L}^i_{S_l}[l]). \tag{1.7}$$

ii) Hard Lefschetz: The iterated cup-product η^i is an isomorphism over V

$$\eta^i : {}^p\mathcal{H}^{-i}(K) \xrightarrow{\sim} {}^p\mathcal{H}^i(K)$$

Moreover, the perverse cohomology ${}^p\mathcal{H}^{-i}(K)$ is dual to ${}^p\mathcal{H}^i(K)$, and the duality is compatible with the natural decomposition: the local systems $\mathcal{L}^i_{S_l}$ are polarized

variation of Hodge structure (VHS) on S_l, $\mathcal{L}^{-i}_{S_l}$ is dual to $\mathcal{L}^{i}_{S_l}$, and each η^i induces an isomorphism $\mathcal{L}^{-i}_{S_l} \xrightarrow{\sim} \mathcal{L}^{i}_{S_l}$ for each i and l.

iii) There exists an isomorphism in the derived category of constructible sheaves on V

$$K \simeq \oplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(K)[-i]. \tag{1.8}$$

The rest of the article is devoted to the proof.
In summary, the main features of the proof are as follows.

A reduction to the zero dimensional strata (proposition 2.2). We use the local topological triviality property of a morphism along a strata (subsection 3.0.1). Indeed, we cut a strata S of V_j^* by a general normal section N_v to S at a point v, so that the proof is carried on the restriction to N_v at v.

Hard Lefschetz. We check the proof of Hard Lefschetz isomorphism (1.1) in the subsection 2.2 on the terms $\mathcal{L}^i_S$ of the explicit formula (1.2). The decomposition in the formula 1.8 above, follows from Hard Lefschetz by Deligne's criteria in [De 68].

The duality between $\mathcal{L}^{-i}_{S_l}$ and $\mathcal{L}^{i}_{S_l}$ follows from the auto-duality of $j_{!*}\mathcal{L}$ and Verdier duality for the projective morphism f.

The polarization follows from the duality and Hard Lefschetz.

Remark 1.3. i) The statement is local on V, hence we may suppose V affine. However, we may suppose V projective during the proof as follows. We embed X into a projective variety $\overline{X}$, then the intermediate extension $j_{!*}\mathcal{L}$ on X extends over $\overline{X}$. On $\overline{X}$ the global properties such as the polarization on the Intersection cohomology apply.

ii) In the special case of a fibration by NCD over the strata (definition 1.4 below), we use logarithmic complexes to describe mixed Hodge theory. The use of mixed Hodge theory occurs in the proofs of the three lemma 2.1, 2.4 and 2.5, where the last two follow from the local purity theorem stating conditions on the MHS on cohomology of the link at v with value in $Rf_*j_{!*}\mathcal{L}$ (theorem 1.8 and proposition 1.9 below). Then, the proof for any projective morphism f is reduced to the case of such special fibration.

1.2. **Fibration over the strata by normal crossing divisors.** Hodge theory appears in the proof via a local purity statement (theorem 1.8 below). We reduce the proof to the case where f is a special fibration, hence we only need to develop Hodge theory on the complement of a NCD (sections $4-5$).

Let $\pi : \tilde{X} \to X$ be a desingularization. The decomposition theorem for f follows from the case of π and $f\circ\pi$ ([De 68] proposition 2.16), in particular we can suppose X smooth. In fact, we will suppose π and $f\circ\pi$ fibrations by NCD over the strata of adequate Thom-Whitney stratifications in the sense below.

Definition 1.4 (topological fibration over the strata by NCD)**.**

i) A morphism $f : X \to V$ is a topological fibration by NCD over the strata of a stratification $\mathcal{S} = (S_\alpha)$ of V underlying a Thom-Whitney stratification of f, if X is smooth and the spaces $V_l = \cup_{\dim S_\alpha \leq l} S_\alpha$ satisfy the following properties:

1) The sub-spaces $X_{V_i} := f^{-1}(V_i)$ are successive sub-NCD embedded in X.

2) The restriction of f to $X_S := f^{-1}(S)$ over each strata S of $\mathcal{S}$ is a topological fibration: $f_| : X_S \to S$.

3) For each point $v \in V_i - V_{i+1}$ (hence V_i is smooth at v) and a normal section $\mathcal{N}_v$ to V_i at v in general position, the subvariety $f^-(\mathcal{N}_v)$ is smooth in X and intersects the NCD X_{V_i} transversally.

ii) The fibration is adapted to a NCD Y in X, or to a local system $\mathcal{L}$ defined on a Zariski open algebraic set in the complement of Y in X, if in addition, for each $1 \leq i \leq n$, the union of the sub-spaces $X_{V_i} \cup Y$ are relative NCD over the strata of V, and the intermediate extension of $\mathcal{L}$ is constructible with respect to the strata.

iii) The fibration is adapted to a subspace Z of V, if Z is a union of strata of the underlying whitney stratification $\mathcal{S}$ of V.

1.2.1. *Fibration by NCD over the strata.* Let X be smooth, and $v \in V_i - V_{i-1} \neq \emptyset$ (hence V_i is smooth at v); the inverse image of a normal section N_v to V_i at v in general position, is a smooth subvariety $f^{-1}(N_v)$ of X intersecting the NCD X_{V_i} transversally, then $X_{V_i} \cap f^{-1}(N_v) = f^{-1}(v)$ is a NCD in $f^{-1}(N_v)$.

We say for simplification, that $X_{V_i - V_{i-1}} := f^{-1}(V_i - V_{i-1})$ is a relative NCD, and that the stratification $\mathcal{S}$ is admissible for f.

The next proposition, proved in section 3, allows to deduce the decomposition theorem for any morphism f from the case where f is a fibration by NCD.

Proposition 1.5. *Let $f : X \to V$ be a projective morphism and Y a closed algebraic strict subspace containing the singularities of X.*

There exists a diagram $X \overset{\pi'}{\leftarrow} X' \overset{f'}{\to} V$, where X' is a non-singular variety, and Thom-Whitney stratifications of V, X, and X' such that π' and $f' := f \circ \pi'$ are fibrations by relative NCD over the strata, adapted to $Y' := \pi'^{-1}(Y)$.

Moreover, π' is a modification of X, and there exists a Zariski open subset $U \subset f(X)$ dense in the image $f(X)$ of f such that f is smooth over U, π' induces an isomorphism of $f'^{-1}(U) - (f'^{-1}(U) \cap Y') \xrightarrow{\sim} f^{-1}(U) - (f^{-1}(U) \cap Y)$, and $f'^{-1}(U) \cap Y'$ is a strict relative NCD in the smooth fibers of f eventually empty (called horizontal or strict in the fibers).

1.2.2. *Reduction to the case of zero dimensional strata with NCD fiber.* The notion of relative NCD is used to reduce the proof at a general point v of a strata S, to the case of a point v in a transversal section N_v to S at v, hence to a point v in a zero dimensional strata. In such situation we can use logarithmic complexes ([ICTP 14] section 8.3.3, theorem 8.3.14) since the inverse image of v is a NCD.

1.2.3. *Purity of Intersection cohomology on a singular variety (lemma 2.25).* As a consequence of theorem 2 and corollary 1.2, in the case of a shifted polarized variation of HS $\mathcal{L}$ on a smooth Zariski open subset U of a singlular projective variety X with intermediate extension $j_{!*}\mathcal{L}$, we deduce a pure HS on $\mathbb{H}^k(X, j_{!*}\mathcal{L})$ as follows.

Let $\pi : X' \to X$ be a desingularization of X, $j' : U \to X'$ and $j'_{!*}\mathcal{L}$ the intermediate extension on X' of $\mathcal{L}$ on U. The polarized VHS components $\mathcal{L}^i_S$ of the decomposition of $R\pi_* j'_{!*}\mathcal{L}$ on X are defined on the various strata S of a stratification $\mathcal{S}$ via the Intersection formula 1.2.

The right term of the isomorphism $H^k(X, {}^p\mathcal{H}^0(R\pi_* j'_{!*}\mathcal{L})) \simeq Gr_0^{{}^p\tau} H^k(X', j'_{!*}\mathcal{L})$ carry a HS by the compatibility of the perverse filtration ${}^p\tau$ with the HS on $H^k(X', j'_{!*}\mathcal{L})$. The compatibility of the decomposition $H^k(X, \oplus_{S_l \in \mathcal{S}} \mathcal{L}^0_{S_l}[l])$ of the left term with the HS on $Gr_0^{{}^p\tau} H^k(X, j'_{!*}\mathcal{L})$ follows from the formula 1.7. The HS on $\mathbb{H}^k(X, j_{!*}\mathcal{L})$ figures as the component corresponding to the big strata.

1.2.4. *Variation of Hodge structure on $\mathcal{L}^i_S$ over a strata $S \subset V$ (corollary 2.10).* Let $f : X \to V$, to illustrate the construction of the variation of HS on a component $\mathcal{L}^i_S$ over a strata S in the decomposition of $Rf_* j_{!*}\mathcal{L}$, we consider an adequate diagram $X' \xrightarrow{\pi} X \xrightarrow{f} V$ with a desingularization X' of X and the intermediate extension $j'_{!*}\mathcal{L}$ on X', then $j_{!*}\mathcal{L}$ is a component of the decomposition of ${}^p\mathcal{H}^0(R\pi_* j'_{!*}\mathcal{L})$ on X.

By intersecting with a general normal section to S at a general point v on S, we reduce the proof to the case where v is a point in the zero dimensional strata of f. Then, $\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L})$ (resp. $\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$) is a summand of $Gr_0^{{}^p\tau}\mathbb{H}^i_{X'_v}(X', j'_{!*}\mathcal{L})$ (resp. $Gr_0^{{}^p\tau}\mathbb{H}^i(X'_v, j'_{!*}\mathcal{L})$) with induced sub-mixed Hodge structure of weight $\geq a+i$ (resp. $\leq a+i$), hence the image HS on $\mathcal{L}^i_v$ is pure of weight $a+i$. The polarization of $\mathcal{L}^i_v$ is the result of Poincaré duality with $\mathcal{L}^{-i}_v$ and Hard Lefschetz.

1.3. **Logarithmic complexes.** We give in section 4 an account of mixed Hodge theory with coeficients in an admissible variation of mixed Hodge structure $(\mathcal{L}, W, F)$ (definition 4.2) on the complement of a normal crossing divisor Y in a smooth complex projective variety X, using logarithmic complexes.

Moreover, we enlarge the theory to logarithmic complexes asssociated with a NC subdivisor $Z \subset Y$, which is necessary in the inductive step (subsection 5.1).

Let $\widetilde{\mathcal{L}}$ be an admissible variation of mixed Hodge structure on $X-Y$ of dimension m, $\mathcal{L} := \widetilde{\mathcal{L}}[m]$ the associated perverse sheaf with the conventional shifted degree, $j := (X-Y) \to X$, $j_Z := (X-Z) \to X$, and $i_Z : Z \to X$.

We describe logarithmic complexes with weight and Hodge filtrations, obtained for various functors applied to the intermediate extension of the admissible variation of MHS $\mathcal{L}$ on $X-Y$ as $j_{Z!}(j_{!*}\mathcal{L})_{|X-Z}, Rj_{Z*}(j_{!*}\mathcal{L})_{|X-Z}$, $i^*_Z Rj_{Z*}(j_{!*}\mathcal{L})_{|X-Z}, i^!_Z j_{!*}\mathcal{L}$ and $i^*_Z j_{!*}\mathcal{L}$, defining thus a MHS on the corresponding hypercohomology groups.
($\star$) *We refer to such complexes as bifiltered logarithmic complexes.*
The starting point is the realization of the direct image $Rj_{Z*}(j_{!*}\mathcal{L})_{|X-Z}$ as a subcomplex $IC^*\mathcal{L}(LogZ)$ of the logarithmic complex $\Omega^*\mathcal{L} := \Omega^*_X(LogY)\otimes\mathcal{L}_X$ (denoted as $\Omega^*(\mathcal{L}, Z)$ in [ICTP 14] definition (8.3.31)).

We refer to a local study by Kashiwara [Ka 86] to construct the weight filtration with a property of local decomposition.

The details of the constructions are given in the last sections 4 and 5 as they are not needed earlier. In the first two sections we admit the existence of the weight filtration and the following general properties of the weight w (corollary 5.9) :
1) the condition $w \leq a+i$ on the weight of $\mathbb{H}^i(Z, j_{!*}\mathcal{L})$ for a pure $\mathcal{L}$ of weight a.
2) the condition $w \geq a+i$ on the weight of $\mathbb{H}^i_Z(X, j_{!*}\mathcal{L})$ for a pure $\mathcal{L}$ of weight a.

We need also the existence of the MHS in the statement of the semi-purity theorem (definition 5.11) in order to prove the local purity conditions on the weight at the end of section 2.

1.3.1. *Compatibility of Hodge structure with perverse filtration.* Let $f : X \to V$ be a projective morphism, and K a complex on V with constructible cohomology. The topological middle perversity truncations on K on V ([BBDG 83] section 2 and prop. 2.1.17) define an increasing *perverse filtration* ${}^p\tau$ on K, from which we deduce for each closed sub-variety W of V, an increasing filtration ${}^p\tau$ on the hypercohomology:

$$(1.9) \qquad {}^p\tau_i\mathbb{H}^k(V-W, K) := \operatorname{Im}\big\{\mathbb{H}^k(V-W, {}^p\tau_i K) \to \mathbb{H}^k(V-W, K)\big\}.$$

Let $Z := f^{-1}W := X_W \subset X$, $j_Z : X - Z \to X$ and $K := Rf_*R(j_Z)_*j_Z^*j_{!*}\mathcal{L}$. The *perverse filtration* ${}^p\tau$ on the hypercohomology groups $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L})$ is deduced by the isomorphism $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L}) = \mathbb{H}^k(V - W, Rf_*j_{!*}\mathcal{L})$

$$ {}^p\tau_i\mathbb{H}^k(X - Z, j_{!*}\mathcal{L}) := {}^p\tau_i\mathbb{H}^k(V - W, Rf_*j_{!*}\mathcal{L}) \tag{1.10}$$

Similarly, we define functorially ${}^p\tau$ on $\mathbb{H}^k_Z(X, j_{!*}\mathcal{L}), \mathbb{H}^k_c(X-Z, j_{!*}\mathcal{L})$ and $\mathbb{H}^k(Z, j_{!*}\mathcal{L})$, from ${}^p\tau$ on K.

We check that the filtration ${}^p\tau$ is a filtration by sub-MHS on the hypercohomology of $X - Z$ ([EY 14], theorems 1.1 and 3.2) using a result of [CaMi 10]. We say simply: the perverse filtration ${}^p\tau$ is compatible with the MHS on the hypercohomology of $X - Z$.

Proposition 1.6. *Let $f : X \to V$ be a projective morphism where X is smooth, $j : X-Y \to X$ the open embedding of the complement of a normal crossing divisor Y in X, $\mathcal{L}$ an admissible variation of MHS on $X-Y$ shifted by* $\dim X$, *$W \subset V$ a closed algebraic subset of V such that $X_W := f^{-1}(W)$ is a sub-normal crossing divisor of Y. Then, the perverse filtration ${}^p\tau$ on the cohomology $\mathbb{H}^*(X - X_W, j_{!*}\mathcal{L})$ (resp. $\mathbb{H}^*_{X_W}(X, j_{!*}\mathcal{L})$ and by duality $\mathbb{H}^*_c(X - X_W, j_{!*}\mathcal{L})$, $\mathbb{H}^*(X_W, j_{!*}\mathcal{L})$), is a filtration by sub-MHS.*

The proof is carried in terms of logarithmic complexes in subsection (5.3.1).

1.3.2. *Mixed Hodge structure (MHS) on the Link.* The basic criteria for a complex K on V to underly a structure of mixed Hodge complex (MHC), is given locally at a point $v \in V$ in terms of the Link at v (proposition 1.7 below). The criteria is translated into a property of the tubular neighborhood of $X_v := f^{-1}(v)$, in terms of $X^*_{B_v} := X_{B_v} - X_v$, the inverse image of a small ball B_v with center v in V minus the central fiber X_v.

Let $j_{X_v} : (X - X_v) \to X$, $i_{X_v} : X_v \to X$, $k_v : (V - v) \to V$ and $i_v : v \to V$. We apply the functor $i_v^*R(k_v)_*k_v^*$ functorially to the filtration ${}^p\tau$ on $Rf_*j_{!*}\mathcal{L}$ to define the perverse filtration ${}^p\tau_i$ on $R\Gamma(X_v, i^*_{X_v}R(j_{X_v})_*j^*_{X_v}j_{!*}\mathcal{L})$ via the isomorphism $i^*_{X_v}R(j_{X_v})_*j^*_{X_v}j_{!*}\mathcal{L} \xrightarrow{\sim} i_v^*R(k_v)_*k_v^*Rf_*j_{!*}\mathcal{L}$.

The perverse filtration is considered on the hypercohomology $\mathbb{H}^r(X^*_{B_v}, j_{!*}\mathcal{L})$, which coincide with $R\Gamma(X_v, i^*_{X_v}R(j_{X_v})_*j^*_{X_v}j_{!*}\mathcal{L})$ if the radius of B_v is small enough. Thus, it may look a less abstract object.

Proposition 1.7 (MHS on the Link)**.** *Let v be a point in V with fibre $X_v := f^{-1}(v)$. We suppose X_v and $X_v \cup Y$ are NCD in X and $\mathcal{L}$ be a shifted polarized VHS defined on $X - Y$; then, for a ball B_v with center v small enough, the perverse filtration on $\mathbb{H}^r(X^*_{B_v}, j_{!*}\mathcal{L}) \xrightarrow{\sim} \mathbb{H}^r((X_v, i^*_{X_v}R(j_{X_v})_*j^*_{X_v}j_{!*}\mathcal{L})$ is compatible with the MHS (that is a filtration by sub-MHS).*

The proof in subsection 5.3.2 depends on a local version of a result in [CaMi 10] easy to check. Once the above structure is defined, we can express the notion of local purity in positive characteristic and give a meaning to the purity theorem [DeG 81].

1.4. **Deligne-Gabber's local purity.** This basic result is stated in positive characteristic in [DeG 81] as follows:
Let K be a pure complex of weight a on an algebraic variety V and B_v a henselization of V at a point v in the zero dimensional strata of V. The weight w of the

cohomology $\mathbb{H}^i(B_v - \{v\}, K)$*, satisfy:*

$$w \leq a + i \text{ if } i \leq -1 \text{ and } w > a + i \text{ if } i \geq 0.$$

The henselization of a complex algebraic variety at a point v corresponds to a small neighbourhood of v for the transcendental topology. According to Deligne's dictionary between the purity in positive caracteristic and Hodge theory, a pure complex corresponds in the transcendental case to an intermediate extension $k_{!*}\mathcal{L}$ of a shifted (as a complex by $\dim V$ to the left) polarized VHS on a smooth algebraic open subset $k : V^* \hookrightarrow V$ embedded in a complex algebraic variety V.

To state the above result, the first task is to construct a MHS on the hypercohomology $\mathbb{H}^*(B_v - v, k_{!*}\mathcal{L})$ where B_v is a small ball with center $v \in V$, with coefficients in an intermediate extension of $\mathcal{L}$ of a shifted polarized VHS on V, or more intrinsically on $H^*(i_v^* R(k_v)_* k_v^* k_{!*}\mathcal{L})$. Once we have such MHS we can state the conditions on the weight as follows:

Theorem 1.8 (local purity)**.** *Let $\tilde{\mathcal{L}}$ be a polarized variation of Hodge structures of weight b on the smooth n-dimensional open subset V^* of an algebraic variety V, $k : V^* \to V$ the embedding, and let v be a point in the zero dimensional strata of V, B_v a small ball with center v and $a = b + n$.*

The weight w of the space $H^i(i_v^* R(k_v)_* k_v^* k_{!*}\mathcal{L})$, isomorphic to the Intersection cohomology $\mathbb{H}^i(B_v - \{v\}, k_{!*}\mathcal{L})$ of the Link at v, satisfies the relations:

$$w \leq a + i \text{ if } i \leq -1 \text{ and } w > a + i \text{ if } i \geq 0$$

(the intermediate extension $k_{!*}\mathcal{L}$, after the shift $\mathcal{L} := \tilde{\mathcal{L}}[n]$, is a complex of weight $a = b + n$).

1.4.1. *Equivalent statement.* The MHS is defined in fact along the NCD X_v in X, so we prove equivalent conditions on the weight above on X as follows

Proposition 1.9 (Semi purity)**.** *Given a (shifted) polarized variation of Hodge structures $j_{!*}\mathcal{L}$ on X of weight a, the weights of the mixed Hodge structure on the graded-cohomology spaces*

$$Gr_i^{p_\tau}\mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L})$$

satisfy the inequalities: $w \leq a + r$ if $r - i \leq -1$, $w > a + r$ if $r - i \geq 0$.

The crucial case of the proof is treated in section 2.4. The original proof in positive characteristic may be adapted to the transcendental case, but we give a relatively simple new proof based on the inductive hypothesis which assumes Hard Lefschetz theorem on $B_v - v$. In fact, both the proposition and Hard Lefschetz theorem (1.1) will be proved simultaneously in section 2, by induction on the decreasing dimension of the strata of V.

2. Proof of the decomposition theorem

We present here a proof of theorem 2 and corollary 1.2 for projective morphisms, based on a reduction to a stratification by NCD over the strata (definition 1.4, proposition 1.5, section 3). Three lemma (2.1, 2.4, 2.5) use Hodge theory for which we refer to sections $4 - 5$.

We start the proof with f smooth over the big strata. In this case, Deligne's classical proof [De 68] extends to the case of Intersection cohomology on the fibers.

Next, the proposition 2.2 states the detailed statements of the results in the case of a zero dimensional strata.
In the paragraph 2.1.4, we give an example of useful calculus with perverse sheaves.
The proof of the local purity theorem is given in subsection 2.3 by reduction to an hyperplane section via Artin Lefschetz theorem except the crucial case of middle perverse cohomology.
At the end we deduce the decomposition theorem for a projective morphism from the case of fibrations by NCD over strata (lemma 2.23 and subsection 5.17).

2.1. **Proof of theorem 2.** Let $f : X \to V$ be a projective morphism of complex algebraic varieties, $\mathcal{S}$ a Thom-Whitney stratification adapted to $j_{!*}\mathcal{L}$ on X and $K := Rf_*j_{!*}\mathcal{L}$. Set V_k the union of all strata of V of dimension $\leq k$.

2.1.1. *The big strata.* By construction, the restriction of f to any strata of V is a topological fibration and the big strata of V is an open smooth set U of dimension $n := dim.V$ as we suppose the singularities of V contained in V_{n-1}. The restrictions to U of the local cohomology sheaves $\mathcal{H}^i(K)$ are locally constant, hence they coincide with the perverse cohomology sheaves: ${}^p\mathcal{H}^i(K) = \mathcal{H}^i(K)$.

Lemma 2.1 (Initial step)**.** *The relative Hard Lefschetz theorem apply to Intersection cohomology of the fibers of the restriction of the morphism f smooth over the big strata U of V.*

The proof follows from Hodge theory in section 4.6, where we suppose $\mathcal{L}$ defined on the complement of Y_U an horizontal relative NCD in X_U. Then, $R^if_*j_{!*}\mathcal{L}$ underlies a VHS such that at each point $v \in U$, the fiber $(R^if_*j_{!*}\mathcal{L})_v \simeq \mathbb{H}^i(X_v, j_{!*}\mathcal{L})$ is isomorphic to the intersection cohomology of the restriction of $\mathcal{L}$ to U_v the fiber of U at v with its HS. The family $R^if_*j_{!*}\mathcal{L}$ in various degrees i satisfy the relative Hard Lefschetz isomorphisms. Hence, the decomposition theorem for the smooth induced morphism $f_U : X_U \to U$ follows from the results of Deligne [De 68].

2.1.2. *The inductive step.* The decomposition theorem is proved by descending induction on the dimension of the strata. We suppose the *decomposition* proved over the open subset $U_j := V - V_j$ for some $j < n$, then we extend the decomposition to U_{j-1} along $V_j - V_{j-1}$, that is across the union of smooth strata S of dimension j. Let N_v denote a general normally embedded section to the strata S at a point $v \in S$, and $f_{N_v} : X_{N_v} \to N_v$ the restriction of f.

By construction of $\mathcal{S}$, f is locally trivial along S: let B_v be a small ball on the strata S with center v, then f is locally homeomorphic to $f_{N_v} \times Id_{|B_v} : X_{N_v} \times B_v \to N_v \times B_v$ (subsection 3.0.1) , and the proof may be reduced to the case of $f_{N_v} : X_{N_v} \to N_v$ with the induced stratification on N_v; that is the case of an isolated point $v \in N_v$ with $j = 0$.

2.1.3. *The case of zero dimensional strata V_0.* We prove now the case of the zero dimensional strata in theorem 2. Since the proof is local, we may suppose $V_0 = v$ reduced to one point v.
Let $k_v : (V - v) \to V$, $i_v : v \to V$, and $\mathcal{L}$ of weight a on a Zariski open subset of X:

Proposition 2.2. *Let $K = Rf_*j_{!*}\mathcal{L}$ denote the higher direct image on the algebraic variety V of dim.n, $\mathcal{S}$ a Whitney stratification of V underlying a Thom stratification of f, $v \in V_0$ a point in the strata of dimension 0, and suppose the following conditions on $V - v$ satisfied:*

1) There exists for each $i \in \mathbb{Z}$, a decomposition over the open subset $V - v$ into a direct sum of intermediate extensions of shifted local systems $\mathcal{L}^i_{S_l}[l]$ (formula 1.2) on the various strata $S_l \subset V - v$ of dimension $l \leq n$:

$$ {}^p\mathcal{H}^i(K)_{|V-v} := k_v^* \, {}^p\mathcal{H}^i(K)_{|V-v} \xrightarrow{\sim} \oplus_{S_l \subset V-v} \; k_v^* i_{S_l !*} \mathcal{L}^i_{S_l}[l] $$

2) Hard Lefschetz: ${}^p\mathcal{H}^{-i}(K)_{|V-v} \xrightarrow{\eta^i} {}^p\mathcal{H}^i(K)_{|V-v}$ is an isomorphism on $V - v$.

3) The local purity theorem apply to K at v (theorem 1.8, subsection 2.3).

Then the decomposition as well Hard Lefschetz extend over v.

Precisely, let $\rho : K \to Rk_{v*}K_{|V-v}$ and $\alpha : i_{v,*}Ri_v^! K \to K$, and consider the long exact sequence of perverse cohomology on V

$$ \xrightarrow{{}^p\delta^{i-1}} {}^p\mathcal{H}^i(i_{v,*}Ri_v^! K) \xrightarrow{{}^p\alpha^i} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\rho^i} {}^p\mathcal{H}^i(Rk_{v*}K_{|V-v}) \xrightarrow{{}^p\delta^i} \tag{2.1} $$

i) The perverse image of ${}^p\rho^i$ is isomorphic to:

$$ Im\, {}^p\rho^i \xrightarrow{\sim} k_{v!*}k_v^* \, {}^p\mathcal{H}^i(K) \xrightarrow{\sim} \oplus_{S_l} i_{S_l !*} \mathcal{L}^i_{S_l}[l]. $$

ii) We have ${}^p\mathcal{H}^i(i_{v,*}Ri_v^! K) \xrightarrow{\sim} i_{v,*}H^i(i_v^! K)$. Let $\mathcal{L}^i_v := \mathrm{Im}(H^i(i_v^! K) \to H^i(i_v^* K))$, then $\mathrm{Ker}\, {}^p\rho^i = \mathrm{Im}\, {}^p\alpha^i \simeq i_{v,*}\mathcal{L}^i_v$ is a polarized Hodge structure of weight $a + i$.

iii) The perverse cohomology decomposes on V

$$ {}^p\mathcal{H}^i(K) \xrightarrow{\sim} \mathrm{Ker}\, {}^p\rho^i \oplus Im\, {}^p\rho^i. \tag{2.2} $$

In particular, we can extract from the sequence (2.1) an exact sub-sequence

$$ 0 \to \oplus_{S_l \subset V-v, i-1-j \geq 0} R^{i-1-j} k_{v*}k_v^*(i_{S_l !*}\mathcal{L}^j[l] \xrightarrow{{}^p\delta^{i-1}} \mathbb{H}^i_v(X, K) \xrightarrow{{}^p\alpha^i} i_{v,*}\mathcal{L}^i_v \to 0 \tag{2.3} $$

iv) Hard Lefschetz: ${}^p\mathcal{H}^{-i}(K) \xrightarrow{\eta^i} {}^p\mathcal{H}^i(K)$ is an isomorphism on V.

The proof occupy the rest of subsections 2.1 and 2.2.
In fact the third condition on local purity follows from the first condition on the decomposition on $V - v$ (see subsection 2.3); its proof is just postponed to ease the exposition. The statement is given in full generality but the proof is given first for fibrations by NCD from which case it is then deduced in general.

2.1.4. *Perverse cohomology of $Rk_{v*}K_{|V-v}$ and $Rk_{v!}K_{|V-v}$.* In order to compute successively: $Im\, {}^p\rho_i$, $Im\, {}^p\alpha_i$ and to prove the splitting of ${}^p\mathcal{H}^i(K)$, we rely on the next calculus of the perverse cohomology of $Rk_{v*}(K_{|V-\{v\}})$ under the hypothesis of the decomposition on $V - v$.

Lemma 2.3. *Let $i : S \to V$ be a fixed strata of V of dim.l, $\mathcal{L}^j[l]$ a family of local systems on S shifted by l for $j \in \mathbb{Z}$, and $K' := \oplus_j i_{!*}(\mathcal{L}^j[l])[-j]$ the direct sum, hence ${}^p\mathcal{H}^i(K') = i_{!*}\mathcal{L}^i[l]$. Let v be a point in the closure of S and $k_v : V - v \to V$.*

i) ${}^p\mathcal{H}^i(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l]) = R^i k_{v*}(k_v^* i_{!*}\mathcal{L}^j[l])$ for $i > 0$, vanish for $i < 0$, and we have a short exact sequence of perverse sheaves for $i = 0$

$$ 0 \to i_{!*}(\mathcal{L}^j[l]) \to {}^p\mathcal{H}^0(Rk_{v*}k_v^* i_{!*}(\mathcal{L}^j[l])) \xrightarrow{h} R^0 k_{v*}(k_v^* i_{!*}\mathcal{L}^j[l]) \to 0 $$

in particular $\underline{H}^0({}^p\mathcal{H}^0(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l]))) \xrightarrow{\sim} R^0 k_{v*}(k_v^* i_{!*}\mathcal{L}^j[l])$, and a morphism of perverse sheaves $\varphi : \mathcal{P} \to {}^p\mathcal{H}^0(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l])$ factors through $i_{!*}\mathcal{L}^j[l]$ if and only if $h \circ \varphi = 0$.

ii) We deduce ${}^p\mathcal{H}^i(Rk_{v*}k_v^* K') = \oplus_{j \leq i} {}^p\mathcal{H}^{i-j}(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l]) = {}^p\mathcal{H}^i(Rk_{v*}k_v^* {}^p\tau_{\leq i} K')$

and short exact sequences of perverse sheaves for all i

$$0 \to i_{!*}(\mathcal{L}^i[l]) \to {}^p\mathcal{H}^i(Rk_{v*}k_v^*K') \xrightarrow{h} \oplus_{j\leq i} R^{i-j}k_{v*}(k_v^* i_{!*}\mathcal{L}^j[l]) \to 0$$

where the last term is a sum of vector spaces supported on v in degee zero.
There are isomorphisms of sheaf cohomology concentrated on v in degree 0

$$\underline{H}^0({}^p\mathcal{H}^i(Rk_{v*}k_v^*K')) \xrightarrow{\sim} \oplus_{j\leq i} R^i k_{v*}(k_v^* i_{!*}\mathcal{L}^j[l-j])$$

and a morphism of perverse sheaves $\varphi : \mathcal{P} \to {}^p\mathcal{H}^i(Rk_{v*}k_v^*K')$ factors through $i_{!*}(\mathcal{L}^i[l])$ if and only if $h\circ\varphi = 0$.

iii) Dually, ${}^p\mathcal{H}^i(Rk_{v!}k_v^*K') = \oplus_{j\geq i}\, {}^p\mathcal{H}^{i-j}(Rk_{v!}k_v^* i_{!*}\mathcal{L}^j[l])$ and we have short exact sequences

$$0 \to \oplus_{j\geq i} i_{v*}H^{i-j-1}(i_v^* i_{!*}\mathcal{L}^j_{S_l}[l]) \xrightarrow{h'} {}^p\mathcal{H}^i(Rk_{v!}k_v^*K') \to i_{!*}\mathcal{L}^i[l] \to 0$$

where the first term is supported on v in degee zero.
A morphism φ defined on ${}^p\mathcal{H}^i(Rk_{v!}k_v^*K')$ factors through $i_{!*}(\mathcal{L}^i[l])$ if and only if $\varphi\circ h' = 0$.

Proof. In the case of a unique local system $K' = i_{!*}\mathcal{L}'[l]$, we have by definition $i_{!*}\mathcal{L}'[l] = \tau_{\leq -1}Rk_{v*}k_v^* i_{!*}\mathcal{L}'[l]$; moreover $\tau_{\geq 0}Rk_{v*}k_v^* i_{!*}\mathcal{L}'[l]$ is supported by v.
Then we have the following exact sequence and isomorphisms

$$0 \to i_{!*}\mathcal{L}'[l] \to {}^p\mathcal{H}^0(Rk_{v*}k_v^* i_{S_l!*}\mathcal{L}'[l]) \to i_{v,*}R^0k_{v*}k_v^*(i_{!*}\mathcal{L}'[l] \to 0$$
$${}^p\mathcal{H}^i(Rk_{v*}k_v^* i_{!*}\mathcal{L}'[l]) = 0 \text{ for } i<0$$
$${}^p\mathcal{H}^i(Rk_{v*}k_v^* i_{!*}\mathcal{L}'[l]) = R^ik_{v*}k_v^* i_{!*}(\mathcal{L}'[l]) \text{ for } i>0$$
$$\underline{H}^0({}^p\mathcal{H}^0(Rk_{v*}k_v^* i_{!*}\mathcal{L}'[l])) \xrightarrow{\sim} R^0k_{v*}k_v^*(i_{!*}\mathcal{L}'[l]$$

respectively, we have dual statements for $Rk_{v!}k_v^*$

$$0 \to i_{v*}H^{-1}(i_v^*(i_{!*}\mathcal{L}'[l]) \to {}^p\mathcal{H}^0(Rk_{v!}k_v^* i_{S_l!*}\mathcal{L}'[l]) \to i_{!*}\mathcal{L}'[l] \to 0$$
$${}^p\mathcal{H}^i(Rk_{v!}k_v^* i_{!*}\mathcal{L}'[l]) = 0 \text{ for } i>0$$
$$i_{v*}H^{i-1}(i_v^* i_{!*}\mathcal{L}'[l]) \xrightarrow{\sim} {}^p\mathcal{H}^i(Rk_{v!}k_v^* i_{!*}\mathcal{L}'[l]) \text{ for } i<0$$
$$i_{v*}H^{-1}(i_v^*(i_{!*}\mathcal{L}'[l]) \xrightarrow{\sim} \underline{H}^0({}^p\mathcal{H}^0(Rk_{v!}k_v^* i_{!*}\mathcal{L}'[l]))$$

If we set $\mathcal{L}' = \mathcal{L}^j$ and since for any complex C, ${}^p\mathcal{H}^i(C[-j]) = {}^p\mathcal{H}^{i-j}(C)$, we deduce the contribution of the component $\mathcal{L}^j[l]$ in the sum in K':
${}^p\mathcal{H}^i(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l-j]) = R^ik_{v*}(k_v^* i_{!*}\mathcal{L}^j[l-j])$ for $i>j$,
${}^p\mathcal{H}^i(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l-j]) = 0$ for $i<j$, and the dual statements for $Rk_{v!}k_v^*$.

Moreover, we deduce a short exact sequence
$0 \to i_{!*}\mathcal{L}^j[l] \to {}^p\mathcal{H}^j(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l-j]) \xrightarrow{h} R^jk_{v*}k_v^* i_{!*}\mathcal{L}^j[l-j] \to 0$
and $\underline{H}^0({}^p\mathcal{H}^j(Rk_{v*}k_v^* i_{!*}\mathcal{L}^j[l-j]) \xrightarrow{\sim} R^jk_{v*}k_v^* i_{!*}\mathcal{L}^j[l-j]$. Dually
$0 \to i_{v*}H^{j-1}(i_v^* i_{!*}\mathcal{L}^j[l-j]) \xrightarrow{h'} {}^p\mathcal{H}^j(Rk_{v!}k_v^* i_{!*}\mathcal{L}^j[l-j]) \to i_{!*}\mathcal{L}^j[l] \to 0$
and $i_v^*H^{j-1}(i_v^* i_{!*}\mathcal{L}^j[l-j]) \xrightarrow{\sim} \underline{H}^0({}^p\mathcal{H}^j(Rk_{v!}k_v^* i_{!*}\mathcal{L}^j[l-j]))$. □

Proof of the proposition 2.2. Although the statement is for any projective f, the proof is given first in the case of fibrations by NCD. By the above lemma (2.3, ii) we have

$$k_{v!*}k_v^*\, {}^p\mathcal{H}^i(K) = \oplus_{S_l\subset V-v} i_{S_l!*}\mathcal{L}^i_{S_l}[l] \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*\, {}^p\mathcal{H}^i(K)[-i]) \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*K)$$

i) *We prove $k_{v!*}k_v^*\, {}^p\mathcal{H}^i(K) \subset \operatorname{Im} {}^p\rho^i \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*\, {}^p\mathcal{H}^i(K)[-i])$:*

We deduce from the canonical morphism $\phi : Rk_{v!} \to Rk_{v*}$ and the decomposition hypothesis $k_v^*K = \oplus_j k_v^* {}^p\mathcal{H}^j(K)[-j]$ on $V - v$, the first equality below

$$\mathrm{Im}\left({}^p\mathcal{H}^i(Rk_{v!}k_v^*K) \xrightarrow{{}^p\phi_i} {}^p\mathcal{H}^i(Rk_{v*}k_v^*K)\right) =$$

$$\oplus_j \mathrm{Im}\left({}^p\mathcal{H}^i(Rk_{v!}\, {}^p\mathcal{H}^j(k_v^*K)[-j]) \xrightarrow{{}^p\phi_i^j} {}^p\mathcal{H}^i(Rk_{v*}\, {}^p\mathcal{H}^j(k_v^*K)[-j])\right) =$$

$$\mathrm{Im}\left({}^p\mathcal{H}^0(Rk_{v!}k_v^*\, {}^p\mathcal{H}^i(K)) \xrightarrow{{}^p\phi_i^i} {}^p\mathcal{H}^0(Rk_{v*}k_v^*\, {}^p\mathcal{H}^i(K))\right) = k_{v!*}k_v^*\, {}^p\mathcal{H}^i(K)$$

In the direct sum over j, only the term for $j = i$ is significant since for $j < i$ ${}^p\mathcal{H}^i(Rk_{v!}k_v^* i_{S_l!*}\mathcal{L}^j_{S_l}[l-j]) = 0$, and ${}^p\mathcal{H}^i(Rk_{v*}k_v^* i_{S_l!*}\mathcal{L}^j_{S_l}[l-j]) = 0$ for $j > i$; hence the second equality follows. The last equality follows from the construction of $k_{v!*}k_v^*\, {}^p\mathcal{H}^i(K)$ in ([BBDG 83], subsection 2.1.7) as image of the morphism ${}^p\phi_i^i$.

Since $k_v^*\, {}^p\mathcal{H}^i(K)$ is a component of $k_v^*K[i]$, we remark that: ${}^p\mathcal{H}^0(Rk_{v*}k_v^*\, {}^p\mathcal{H}^i(K))$ is a component of ${}^p\mathcal{H}^0(Rk_{v*}k_v^*K[i]) = {}^p\mathcal{H}^i(Rk_{v*}k_v^*K)$. The existing morphism on $V - v$: $k_v^*\, {}^p\mathcal{H}^i(K) \to k_v^*K[i]$ extends by Rk_{v*} and $Rk_{v!}$ such that ${}^p\phi_i^i$ is compatible with the factorization

$${}^p\mathcal{H}^0(Rk_{v!}k_v^*K[i]) \to {}^p\mathcal{H}^0(K[i]) \xrightarrow{{}^p\mathcal{H}^0(\rho^i)} {}^p\mathcal{H}^0(Rk_{v*}k_v^*K[i])$$

Since ${}^p\phi_i^i$ factors through ${}^p\mathcal{H}^i(K)$, its image $k_{v!*}k_v^*\, {}^p\mathcal{H}^i(K)$ is contained in the image of ${}^p\mathcal{H}^i(K)$ (formula 2.1):

$$k_{v!*}k_v^*\, {}^p\mathcal{H}^i(K) = \oplus_{S_l \subset V-v} i_{S_l!*}\mathcal{L}^i_{S_l}[l] \subset \mathrm{Im}\, {}^p\rho^i.$$

ii) To show that ${}^p\rho^i$ factors through $\oplus_{S_l\subset V-v} i_{S_l!*}\mathcal{L}^i_{S_l}[l] \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*K)$, we prove, in view of lemma (2.3, ii), that the induced morphism $\rho_0^i := H^0(i_v^*\, {}^p\rho^i)$ vanish in degree 0.

Lemma 2.4. *The morphism ρ_0^i induced by $i_v^*\, {}^p\rho^i$ on the cohomology in degree zero vanish*

$$\rho_0^i : H^0(i_v^*\, {}^p\mathcal{H}^i(K)) \longrightarrow H^0(i_v^*\, {}^p\mathcal{H}^i(Rk_{v*}k_v^*K))$$

This is a basic argument where Hodge theory is needed in the proof. We assume that f is a fibration by NCD over the strata (section 3), in which case we refer to section 4 for Hodge theory and to section 1.8 below for the semi-purity theorem needed here.

First, we remark the following geometrical interpretation in terms of a small ball $B_v \subset V$ and its inverse $B_{X_v} \subset X$. By lemma (2.3 i), we have

$$H^0(i_v^*\, {}^p\mathcal{H}^0(Rk_{v*}k_v^*\, {}^p\mathcal{H}^i(K))) \simeq H^0(\oplus_{S_l\subset V-v} k_v^* i_{S_l!*}\mathcal{L}^i_l[l]) \xrightarrow{\sim} \mathbb{H}^0(B_v - v,\, {}^p\mathcal{H}^i(K))$$

Second we deduce from the triangle:

$${}^p\tau_{<0}(K[i]) \to {}^p\tau_{\leq 0}(K[i]) \to {}^p\mathcal{H}^i(K) \xrightarrow{[1]}$$

an exact sequence:

$$\mathbb{H}^0(B_v,\, {}^p\tau_{<0}(K[i])) \to \mathbb{H}^0(B_v,\, {}^p\tau_{\leq 0}K[i]) \to \mathbb{H}^0(B_v,\, {}^p\mathcal{H}^i(K)) \to \mathbb{H}^1(B_v,\, {}^p\tau_{<0}K[i])$$

where $\mathbb{H}^r(B_v,\, {}^p\tau_{<0}(K[i])) = 0$ for $r = 0, 1$, hence

$$\mathbb{H}^0(B_v,\, {}^p\tau_{\leq i}(K)) \simeq \mathbb{H}^0(B_v,\, {}^p\mathcal{H}^i(K)) \simeq H^0(i_v^*\, {}^p\mathcal{H}^i(K))$$

which enables the factorization of ρ_0^i as a morphism

$$H^0(i_v^*\, {}^p\tau_{\leq i}K) = \mathbb{H}^0(B_v,\, {}^p\tau_{\leq i}K) \to \mathbb{H}^0(B_v - v,\, {}^p\tau_{\leq i}K) \longrightarrow \mathbb{H}^0(B_v - v,\, {}^p\mathcal{H}^i(K)).$$

In other terms, we have an interpretation of ρ_0^i as a composition morphism via ρ'

$$\mathbb{H}^0(B_v,\, {}^p\tau_{\leq i}K) \xrightarrow{\rho'} {}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v} - X_v, j_{!*}\mathcal{L}) \longrightarrow Gr_i^{{}^p\tau}\mathbb{H}^i(B_{X_v} - X_v, j_{!*}\mathcal{L}).$$

and through $\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$ as follows:

$\mathbb{H}^0(B_v, {}^p\tau_{\leq i}K) \to {}^p\tau_{\leq i}\mathbb{H}^i(X_v, j_{!*}\mathcal{L}) \to {}^p\tau_{\leq i}\mathbb{H}^i(B_{X_v} - X_v, j_{!*}\mathcal{L})$

where the space $\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$ has weight $w \leq a+i$ (corollary 5.9). By assumption the semi purity apply to $Gr_i^{{}^p\tau}\mathbb{H}^i(B_{X_v} - X_v, j_{!*}\mathcal{L})$ of weight $w > a+i$ (proposition 1.9), hence we deduce $\rho_0^i = 0$.

iii) *Proof of* $Im\, {}^p\alpha^i \xrightarrow{\sim} i_{v,*}\mathcal{L}_v^i$. We deduce from lemma (2.3 ii) as above:

$$H^0({}^p\mathcal{H}^{i-1}(Rk_{v*}K_{|V-\{v\}})) \xrightarrow{\sim} R^{i-1}k_{v*}k_v^*({}^p\tau_{\leq i-1}K).$$

We deduce from the formula (2.1) and the computation of ${}^p\rho^{i-1}$ above, a sub-exact sequence of perverse sheaves :

$$0 \to R^{i-1}k_{v*}k_v^*({}^p\tau_{\leq i-1}K) \overset{{}^p\delta^{i-1}}{\to} \mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) \overset{{}^p\alpha^i}{\to} {}^p\mathcal{H}^i(K) \overset{{}^p\rho^i}{\to}$$

where the perverse cohomology of $i_v^!(K)$ coincides with its cohomology ${}^p\mathcal{H}^i(i_v^!K) = H^i_{X_v}(X, j_{!*}\mathcal{L})$ as a complex of vector spaces, hence $\operatorname{Im}{}^p\alpha^i = \operatorname{Coker}{}^p\delta^{i-1}$.

On the other hand, by definition $\mathcal{L}_v^i$ is the image of I^i in the exact sequence

$$H^{i-1}(i_v^*Rk_{v*}K_{|V-\{v\}}) \overset{\delta^{i-1}}{\to} \mathbb{H}^i_v(V, K) \overset{I^i}{\to} H^i(i_v^*K) \overset{\rho^i}{\to} H^i(i_v^*Rk_{v*}K_{|V-\{v\}}). \tag{2.4}$$

hence $\mathcal{L}_v^i := \operatorname{Im} I^i = \operatorname{Coker}\delta^{i-1}$.

We need to prove $\operatorname{Im}\delta^{i-1} = \operatorname{Im}{}^p\delta^{i-1}$, to deduce the result in the form of an exact sequence

$$0 \to i_{v,*}\mathcal{L}_v^i \to {}^p\mathcal{H}^i(K) \to \oplus_{S_l \subset V-v} i_{S_l!*}\mathcal{L}^j_{S_l}[l] \to 0. \tag{2.5}$$

We deduce from (lemma 2.3 ii) and the decomposition on $V - v$:

$R^{i-1}k_{v*}k_v^*({}^p\tau_{\leq i-1}K) \xrightarrow{\sim} H^0({}^p\mathcal{H}^{i-1}(Rk_{v*}K_{|V-\{v\}})) = ({}^p\tau_{\leq i-1}\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}))$

and the following interpretation of the image:

$${}^p\delta^{i-1}(R^{i-1}k_{v*}k_v^*({}^p\tau_{\leq i-1}K)) \xrightarrow{\sim} \delta^{i-1}({}^p\tau_{\leq i-1}\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})).$$

Since $H^{i-1}(i_v^*Rk_{v*}K_{|V-\{v\}}) = \mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}))$, we need to prove

Lemma 2.5. $\delta^{i-1}({}^p\tau_{\leq i-1}\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})) = \delta^{i-1}(\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}))$

The proof is based again on the semi purity theorem. Indeed, the quotient space $\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}))/{}^p\tau_{\leq i-1}$ has weight $w < a+i$ by the semi purity theorem, hence the image of ${}^p\delta^{i-1}$ and δ^{i-1} in $\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L})$ of weight $w \geq a+i$, are the same.

iv) *Proof of the splitting* ${}^p\mathcal{H}^i(K) \xrightarrow{\sim} i_{v*}\mathcal{L}_v^i \oplus (\oplus_{S_l \subset V-v} i_{S_l!*}\mathcal{L}^i_{S_l}[l])$. We consider the morphisms

$${}^p\mathcal{H}^i(Rk_{v!}k_v^*K) \overset{{}^p\beta^i}{\to} {}^p\mathcal{H}^i(K) \overset{{}^p\rho^i}{\to} \operatorname{Im}{}^p\rho^i \subset {}^p\mathcal{H}^i(Rk_{v*}k_v^*K) \tag{2.6}$$

where β^i is dual to ρ^{-i} as $\beta : Rk_{v!}k_v^*K \to K$ is dual to ρ. We deduce from lemma (2.3, iii) a dual argument to the proof in ii) above to assert that β^i factors through $\oplus_{S_l \subset V-v} i_{S_l!*}\mathcal{L}^i_{S_l}[l]$, hence ${}^p\rho^i$ induces an isomorphism $\operatorname{Im}{}^p\beta^i \to \operatorname{Im}{}^p\rho^i \circ {}^p\beta^i := \oplus_{S_l \subset V-v} i_{S_l!*}\mathcal{L}^i_{S_l}[l]$ and defines a splitting.

In the sequence of morphisms: $\mathbb{H}^i_v(V, K) \xrightarrow{{}^p\alpha^i} {}^p\mathcal{H}^i(K) \xrightarrow{{}^p\gamma^i} H^i(i_v^*K)$ we have $\operatorname{Im}{}^p\alpha^i = \operatorname{Ker}{}^p\rho^i$ and $\operatorname{Ker}{}^p\gamma^i = \operatorname{Im}{}^p\beta^i$, hence ${}^p\gamma^i$ induces on $\operatorname{Im}{}^p\alpha^i$ an isomorphism to $\operatorname{Im}{}^p\gamma^i \circ {}^p\alpha^i := i_{v*}\mathcal{L}_v^i$.

Remark 2.6. i) We deduce from the above proof (formula 2.3 and 2.4)
$\mathrm{Im}\ {}^p\delta^{i-1} \simeq \oplus_{S_l\subset V-v, i-j>0} H_v^{i-j}(i_{S_l\,!*}\mathcal{L}^j_{S_l}[l])$, $\mathrm{Im}\ {}^p\delta^{i-1} = \mathrm{Ker}\ {}^p\alpha^i = \mathrm{Ker} I^i = \mathrm{Im}\ \delta^{i-1}$
and the following isomorphism induced by ${}^p\delta^{i-1}$:
$\oplus_{S_l\subset V-v, i-1-j\geq 0} R^{i-1-j} k_{v*}k_v^* i_{S_l\,!*}\mathcal{L}^j[l] \xrightarrow{\sim} \oplus_{S_l\subset V-v, i-j>0} H_v^{i-j}(i_{S_l\,!*}\mathcal{L}^j_{S_l}[l])$

ii) Hodge theory is used in the proof of the three lemmas 2.1, 2.4, 2.5. *We use semi purity at X_v or equivalently local purity at v (section 2.3) to extend the decomposition property across v.*

2.2. **Hard Lefschetz.** We check for all $i \geq 0$, $\eta^i : {}^p\mathcal{H}^{-i}(K) \xrightarrow{\sim} {}^p\mathcal{H}^i(K)$ is an isomorphism. By assumption the restriction of η^i to $V - v$ on $k_v^*\,{}^p\mathcal{H}^{-i}(K)$ is an isomorphism, hence it remains an isomorphism on the intermediate extension

$$(k_v)_{!*}k_v^*\,{}^p\mathcal{H}^{-i}(K) \xrightarrow{\sim} \oplus_{S_l\subset V-v, j} i_{S_l\,!*}\mathcal{L}^j_{S_l}[l] \subset {}^p\mathcal{H}^{-i}(K)$$

across the point v in the strata V_0 of dimension 0. It remains to prove Hard Lefschetz $\mathcal{L}_v^{-i} \xrightarrow{\sim} \mathcal{L}_v^i$ where $\mathcal{L}_v^i := \mathrm{Im}\,(\mathbb{H}^i_{Y_v}(X, j_{!*}\mathcal{L}) \xrightarrow{I^i} \mathbb{H}^i(Y_v, j_{!*}\mathcal{L}))$.

Lemma 2.7. *i) The cup-product with the class of an hyperplane section induces isomorphisms $\eta^i : \mathcal{L}_v^{-i} \to \mathcal{L}_v^i$ for $i > 0$.*
ii) The HS on $\mathcal{L}_v^i$ is Poincaré dual to $\mathcal{L}_v^{-i}$ for $i \geq 0$.

In the case of a big strata S, we have a local system $\mathcal{L}_S^i = R^i f_* j_{!*}\mathcal{L}$ and the lemma follows from Hard Lefschetz for intersection cohomology then the decomposition follows by Deligne's argument in [De 68] applied to the VHS on $\mathcal{L}_S^i = R^i f_* j_{!*}\mathcal{L}$ (subsection 4.6). The general statement of Verdier duality for proper morphisms apply to prove the duality between $\mathcal{L}_S^i$ and $\mathcal{L}_S^{-i}$.

At a point v of a lower dimensional strata, Verdier duality between perverse cohomology sheaves in degree i and $-i$ is compatible with the decomposition established above and apply to $(k_v)_{!*}k_v^*\,{}^p\mathcal{H}^i(K)$ by induction. The duality between $\mathcal{L}_v^i$ and $\mathcal{L}_v^{-i}$ in ii) is deduced from the duality of $Ri^!_{X_v}$ and $Ri^*_{X_v}$ and Verdier duality on X_v in the definition of $\mathcal{L}_v^i$ (formula 1.2).

To prove i) at the point v we consider an hyperplane section H intersecting all NCD in X normally and proceeds by induction. Let i_H denote the closed embedding of H in X; the cup product defines a morphism η equal to the composition of the morphisms $j_{!*}\mathcal{L} \xrightarrow{\rho} i_{H*}i_H^* j_{!*}\mathcal{L} \xrightarrow{\sim} i_{H*}Ri^!_H(j_{!*}\mathcal{L})[2] \xrightarrow{G} j_{!*}\mathcal{L}[2]$. We apply the functors $Ri^!_{X_v}$ and $i^*_{X_v}$ to the above morphisms ρ, G, and η as in the commutative diagram

$$\begin{array}{ccccc}
\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) & \xrightarrow{\rho_i^!} & \mathbb{H}^i_{X_v\cap H}(H, j_{!*}\mathcal{L}) & \xrightarrow{G^!_{i+2}} & \mathbb{H}^{i+2}_{X_v}(X, j_{!*}\mathcal{L}) \\
I^i\downarrow & & I^i_H\downarrow & & I^{i+2}\downarrow \\
\mathbb{H}^i(X_v, j_{!*}\mathcal{L}) & \xrightarrow{\rho_i^*} & \mathbb{H}^i(X_v\cap H, j_{!*}\mathcal{L}) & \xrightarrow{G^*_{i+2}} & \mathbb{H}^{i+2}(X_v, j_{!*}\mathcal{L})
\end{array}$$

where on the first line: $(\eta_v^!)_i = G^!_{i+2}\circ\rho_i^!$ and on the second line: $(\eta_v^*)_i = G^*_{i+2}\circ\rho_i^*$ are functorially induced by ρ and G, while by definition $\mathcal{L}_v^i := \mathrm{Im}\, I^i$, $\mathcal{L}(H)_v^i := \mathrm{Im}\, I^i_H$, $\mathcal{L}_v^{i+2} := \mathrm{Im}\, I^{i+2}$ are the images of the vertical maps induced by I from the top line to the bottom line.

The morphisms ρ_i^* and the dual morphisms G^*_{i+2} induce ρ_i' and G'_{i+2} below

$$\mathcal{L}_v^i \xrightarrow{\rho_i'} \mathcal{L}(H)_v^i \xrightarrow{G'_{i+2}} \mathcal{L}_v^{i+2} \quad \text{where } \mathcal{L}(H)_v^i := \mathrm{Im}\, I^i_H$$

with composition equal to η' induced by η. The proof continue by induction on H. A striking point however, there is no crucial case as in (Weil II [De 80], théorème 4.1.1) unless v is on the generic strata.

Lemma 2.8. *The induced morphism $\rho'_i : \mathcal{L}^i_v \to \mathcal{L}(H)^i_v$ is an isomorphism for $i < 0$ and by duality $G'_{i+2} : \mathcal{L}(H)^i_v \to \mathcal{L}^{i+2}_v$ is an isomorphism for $i + 2 > 0$.*

We prove first the isomorphisms for $i < 0$

$$\rho^!_i : \mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) \to \mathbb{H}^i_{X_v\cap H}(H, j_{!*}\mathcal{L}) \tag{2.7}$$

We apply Artin- Lefschetz theorem ([BBDG 83], corollary 4.1.5) to the affine open subset $X_v - (H \cap X_v)$ with coefficients in a complex of sheaves $\mathcal{K} \in {}^pD^{\leq 0}(X_v)$ to prove the vanishing of $\mathbb{H}^i(X_v - (H \cap X_v), \mathcal{K})$ for $i > 0$ and its dual statement: $\mathbb{H}^i_c(X_v - (H \cap X_v), \mathcal{K}') = 0$ with coefficients in a complex of sheaves $\mathcal{K}' \in {}^pD^{\geq 0}(X_v)$ for $i < 0$.

Since $i^*_{X_v}(j_{!*}\mathcal{L})[-1] \in {}^pD^{\leq 0}(X_v)$ and its dual $i^!_{X_v}(j_{!*}\mathcal{L})[1] \in {}^pD^{\geq 0}(X_v)$ we deduce $\mathbb{H}^i_c(X_v - (H \cap X_v), i^!_{X_v}(j_{!*}\mathcal{L})[1]) = 0$ for $i < 0$, hence we have for $i < 0$

$$\begin{array}{ccc} \mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) & = & \mathbb{H}^{i-1}(X_v, i^!_{X_v} j_{!*}\mathcal{L}[1]) \\ \rho^!_i \downarrow \simeq & & \downarrow \simeq \\ \mathbb{H}^i_{X_v\cap H}(H, j_{!*}\mathcal{L}) & = & \mathbb{H}^{i-1}(X_v \cap H, i^!_{X_v} j_{!*}\mathcal{L}[1]) \end{array}$$

The last equality is the main point and follows from the isomorphisms

$$i^*_{X_v\cap H} R i^!_{X_v}(j_{!*}\mathcal{L}) \xrightarrow{\sim} R i^!_{X_v\cap H} i^*_H(j_{!*}\mathcal{L}), \quad i^*_H(j_{!*}\mathcal{L}[-1]) \xrightarrow{\sim} j_{H!*} i^*_{H\cap(X-Y)}\mathcal{L}[-1])$$

due to the transversality of the intersection of H and X_v.

Proof of the lemma. We extend the diagram above by introducing the kernel of the Intersection morphisms I to get two columns of short exact sequences

$$\begin{array}{ccc} H^{i-1}(i^*_v R(k_v)_* K_{|V-\{v\}}) & \xrightarrow{\rho^*_{i-1}} & H^{i-1}(i^*_v R(k_v)_* K_{H|V-\{v\}}) \\ \delta^{i-1} \downarrow & & \delta^{i-1}_H \downarrow \\ \mathbb{H}^i_v(V, K) \xrightarrow{\sim} \mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) & \xrightarrow{\rho^!_i} & \mathbb{H}^i_{X_v\cap H}(H, j_{!*}\mathcal{L}) \xrightarrow{\sim} \mathbb{H}^i_v(V, K_H) \\ I^i \downarrow & & I^i_H \downarrow \\ \mathcal{L}^i_v = \operatorname{Im} I^i & \xrightarrow{\rho'_i} & \mathcal{L}(H)^i_v = \operatorname{Im} I^i_H \end{array}$$

where $K_H = R(f_{|H})_*(j_{|H})_{!*} j_{!*}\mathcal{L}_{|H}$, $\mathcal{L}^i_v \subset \mathbb{H}^i(i^*_v K) = \mathbb{H}^i(X_v, j_{!*}\mathcal{L})$ and $\mathcal{L}(H)^i_v \subset \mathbb{H}^i(X_v \cap H, j_{!*}\mathcal{L}) = \mathbb{H}^i(i^*_v K_H)$.

On the right column the short exact sequence is defined by the perverse shifted restriction $\mathcal{L}_{|H}[-1]$ of $\mathcal{L}$, for which we suppose Hard Lefschetz by induction.

We prove that for $i < 0$, the morphism $\rho^!_i$, which is an isomorphism by the lemma, induces isomorphisms

$$\rho^!_{i|} : Im\, \delta^{i-1} = \operatorname{Ker} I^i \xrightarrow{\sim} \operatorname{Im} \delta^{i-1}_H = \operatorname{Ker} I^i_H \tag{2.8}$$

hence induces isomorphisms: $\rho'_i : \mathcal{L}^i_v \xrightarrow{\sim} (\mathcal{L}|H)^i_v$ for $i < 0$.

Proof of the isomorphism $\rho^!_{i|} : \operatorname{Ker} I^i \xrightarrow{\sim} \operatorname{Ker} I^i_H$. By the remark (2.6 i)

$$\operatorname{Ker}(I^i) = Im(\delta^{i-1}) = Im(\oplus_{S_l\subset V-v, j<i} H^{i-1-j}(i^*_v R k_{v*} k^*_v i_{S_l!*}\mathcal{L}^j_{S_l}[l]))$$
$$\operatorname{Ker}(I^i_H) = Im(\delta^{i-1}_H) = Im(\oplus_{S_l\subset V-v, j<i} H^{i-1-j}(i^*_v R(k_v)_* k^*_v i_{S_l!*}(\mathcal{L}_{|H})^j_{S_l}[l])).$$

The proof is reduced to the comparison of $\mathcal{L}^j_{S_l}$ and $(\mathcal{L}_{|H})^j_{S_l}$ on each component S_l.

We consider for each S_l, a normal section N_{v_l} to S_l at a general point $v_l \in S_l$, then:

$$(R^{-l+j} f_* i^!_{X_{S_l}} j_{!*}\mathcal{L})_{v_l} \xrightarrow{\sim} \mathbb{H}^{-l+j}_{X_{v_l}}(X_{N_{v_l}}, j_{!*}\mathcal{L}) \xrightarrow{\sim} \mathbb{H}^{j}_{X_{v_l}}(X_{N_{v_l}}, j_{!*}\mathcal{L}[-l])$$

where $j_{!*}\mathcal{L}[-l]$ restricts to a perverse sheaf on $X_{N_{v_l}}$ of dimension $\dim.X - l$. Since $i < 0$ and $j < i$ we have $j < -1$, moreover $X_{v_l} - (H \cap X_{v_l})$ is affine, hence the restriction of $\rho^!_j$ to $X_{N_{v_l}}$ is an isomorphism by the hyperplane section theorem on the fibres of f over S_l: $\rho^!_{j|} : \mathbb{H}^j_{X_{v_l}}(X_{N_{v_l}}, j_{!*}\mathcal{L}[-l]) \to \mathbb{H}^j_{X_{v_l}\cap H}(X_{N_{v_l}} \cap H, j_{!*}\mathcal{L}[-l])$ where the last term is isomorphic to $(R^{-l+j} f_* i^!_{H\cap X_{S_l}} j_{!*}\mathcal{L}_{|H})_{v_l}$.

Corollary 2.9. *The iterated cup-product with the class of a relative hyperplane section induces isomorphisms $\eta^i : {}^p\mathcal{H}^{-i}(K) \to {}^p\mathcal{H}^i(K)$ for $i \geq 0$.*

Corollary 2.10. *For each strata S of V, the local system $\mathcal{L}^i_S$ is a polarized variation of Hodge structure on the smooth variety S.*

We consider here the case of a fibration $f : X \to V$ by NCD over the strata with X smooth, we consider at a point $v \in S$ open neighborhoods B_v of v in V and S_v in S, and a projection $p_v : B_v \to S_v$, inducing the identity on S_v, such that $p_v \circ f : X_{B_v} \to S_v$ is smooth.

The fibers $N_a := p_v^{-1}(a)$ at points $a \in S_v$ form a family of normal sections to S_v such that $X_{N_a} := f^{-1}(N_a)$ is a smooth sub-variety of X_{B_v}. By an argument based on the relative complex $\Omega^*_{X_{B_v}/S_v}\mathcal{L}$ as in section 4.6, the families $R^k f_*(i^!_{X_{S_v}}(j_{!*}\mathcal{L})$ (resp. $R^k f_*(i^*_{X_{S_v}}(j_{!*}\mathcal{L}))$ form variations of MHS on S of weight $\geq a + k + l$ (resp. $\leq a + k + l$) (corollary 5.9).

Then the image of the Intersection morphisms $\mathcal{L}^k_{S_v}$ are pure variation of HS of weight $a + k + l$ on S_v.

2.3. **Proof of the local purity theorem 1.8.** We did assume in the statement of the proposition 2.2 the local purity at v to simplify the exposition. We show now the local purity at v follows in fact from the decomposition on $V - v$.

We assume here that f is a fibration by NCD over the strata in order to apply Hodge theory in sections $4 - 5$. We refer precisely to the subsection (5.2.4) for the construction of the mixed Hodge structure used in this theorem and to the compatibility with the perverse filtration.

The proof given here of the equivalent statement, called the semi-purity at X_v (proposition 1.9), is based on the global result of the polarization of the Intersection cohomology of X. The proof in [DeG 81] is based on the local invariant cycle theorem.

2.3.1. *Inductive hypothesis. Let $j_{!*}\mathcal{L}$ be a shifted polarized VHS of weight a on the smooth and compact variety X, $X_v := f^{-1}(v)$ the fiber at $v \in V$, and $B_{X_v} = f^{-1}(B_v)$ the inverse of a small neighborhood B_v of v. We assume by induction the decomposition theorem on $V - v$, hence we have an isomorphism:*

$$Gr_i^{p_\tau}\mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L}) \xrightarrow{\sim} \mathbb{H}^{r-i}(B_v - \{v\}, {}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L})). \tag{2.9}$$

This isomorphism is used to carry the MHS from the left term (subsection 5.3) to the right.

Under such isomorphism, we prove the following inequalities on the weights w of the MHS

i) $w > a + r$ on ${}^p\tau_{\leq r}\mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L})$, and dually:
i') $w \leq a + r$ on $\mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L})/{}^p\tau_{\leq r}\mathbb{H}^r(B_{X_v} - X_v, j_{!*}\mathcal{L})$.
or equivalently for $j = r - i$:
ii) $w > a + i + j$ on $H^j(B_v - v, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}))$ if $j \geq 0$, and dually:
ii') $w \leq a + i + j$ on $H^j(B_v - v, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}))$ if $j \leq -1$.

Example. The dual of $Gr^W_{a+l}\mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L}))$ for all i and $l \leq i - 1$ in the assertion ii'), is $Gr^W_{a-l}\mathbb{H}^0(B_v - v, {}^p\mathcal{H}^{-i}(Rf_*j_{!*}\mathcal{L}))$ for all i and $a - l \geq a - i + 1$ in the assertion ii); we remark the change of the variable i into the variable $-i$.

2.3.2. *Duality.* Let $K := Rf_*j_{!*}\mathcal{L}$ and $k_v : V - \{v\} \to V$. We have:

$$K(*v) := i_v^* Rk_{v*}k_v^* K \xrightarrow{\sim} R\Gamma(B_v - v, K) \xrightarrow{\sim} R\Gamma(B_{X_v} - X_v, j_{!*}\mathcal{L}).$$

The duality isomorphism $D(K(*v))[1] \xrightarrow{\sim} K(*v)$ where D stands for Verdier dual, is deduced from the duality between Rk_{v*} and $Rk_{v!}$ (resp. i_v^* and $Ri_v^!$) as follows. We apply to k_v^*K two sequences of functors $Rk_{v!} \to Rk_{v*} \to i_{v*}i_v^*Rk_{v*}$ and $Rk_{v!} \to Rk_{v*} \to i_{v*}Ri_v^!Rk_{v!}[1]$, defining dual triangles, from which we deduce

$$D(K(*v))[1] \xrightarrow{\sim} i_{v*}Ri_v^!Rk_{v!}k_v^*K[1] \xrightarrow{\sim} K(*v).$$

which corresponds to the duality of the Intesection cohomology of the "link": $\partial B_{X_v} = f^{-1}(\partial B_v)$ with coefficients in the restriction of $j_{!*}\mathcal{L}$.

Since the perverse filtration is compatible with the MHS over $V - v$, we deduce

$$D(Gr^W_{a-q}H^{-j}(K(*v))) \xrightarrow{\sim} Gr^W_{a+q}H^j(DK(*v)) \xrightarrow{\sim} Gr^W_{a+q}H^{j-1}(K(*v)).$$

and since the duality: $D({}^p\mathcal{H}^{-i}(K)) \xrightarrow{\sim} {}^p\mathcal{H}^i(K)$ follows from the auto-duality of $j_{!*}\mathcal{L}$ by Verdier's direct image theorem for f proper, we deduce

$$D(Gr^W_{a-q}Gr^{{}^p\tau}_{-i}H^{-j}(K(*v))) \xrightarrow{\sim} Gr^W_{a+q}Gr^{{}^p\tau}_{i}H^{j-1}(K(*v)).$$

Hence, the proof is reduced by the above duality, to one of the two cases ii) or ii') in degree $j \geq 0$ or $j \leq -1$.

2.3.3. *Proof by induction on dim.X.* Let H be a general hyperplane section of X transversal to all strata, $i_H : H \to X$ and $j_H : (X - H) \to X$. The restriction to H of the intermediate extension $i_H^*j_{!*}\mathcal{L}$ is equal to $j_{!*}(\mathcal{L}_{|H})$ by transversality.

We assume the local purity theorem for the perverse cohomology sheaves of $Rf_*j_{!*}(\mathcal{L}_{|H}[-1])$, and we use Artin-Lefschetz vanishing theorem to deduce the local purity for the perverse cohomology shaves of K in degree $i \neq 0$ as follows.

Lemma 2.11. *Let $K_c = Rf_*(j_H)_!j_H^*j_{!*}\mathcal{L}$ (resp. $K(*) = Rf_*(j_H)_*j_H^*j_{!*}\mathcal{L}$).*
i) The complex $K_c \in {}^pD_V^{\geq 0}$ (${}^p\mathcal{H}^i(K_c) = 0$ for $i < 0$).
ii) Dually $K(*) \in {}^pD_V^{\leq 0}$ (${}^p\mathcal{H}^i(K(*)) = 0$ for $i > 0$).

Proof. Since the morphism $f \circ j_H : (X - H) \to X$ is affine, the functor $Rf_* \circ j_{H!} = R(f \circ j_H)_!$ transforms ${}^pD^{\geq 0}_{X-H}$ into ${}^pD^{\geq 0}_V$ (left t-exact, [BBDG 83], corollary 4.1.2). When applied to $Rf_*(j_H)_!j_H^*j_{!*}\mathcal{L}$ we deduce that $K_c \in {}^pD_V^{\geq 0}$. The result is a version of Artin-Lefschetz vanishing theorem. The statement ${}^p\mathcal{H}^i(K(*)) = 0$ for $i > 0$ follows by duality. □

Let $K = Rf_*j_{!*}\mathcal{L}$, and $K_H = R(f \circ i_H)_*i_H^*j_{!*}\mathcal{L}$. The restriction morphism $\rho : K \to K_H$, and dually the Gysin morphism: $G : K_H[-2] \to K$, induce morphisms compatible with the ${}^p\tau$ filtration

$$\rho_i : {}^p\mathcal{H}^i(K) \to {}^p\mathcal{H}^i(K_H),\ G_i : {}^p\mathcal{H}^{i-2}(K_H) \to {}^p\mathcal{H}^i(K),\ L_i : {}^p\mathcal{H}^i(K) \longrightarrow {}^p\mathcal{H}^{i+2}(K),$$

where $L_i = G_{i+2} \circ \rho_i$ (resp. $L : K \longrightarrow K[2]$), is defined by the cup product with the Chern class c_1 of a relative ample bundle.

Corollary 2.12. *The restriction* $\rho_i : {}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L}) \to {}^p\mathcal{H}^i(Rf_*(i_H)_* i_H^* j_{!*}\mathcal{L})$ *is an isomorphism for each integer* $i < -1$ *and injective for* $i = -1$.
Dually, the Gysin morphism $G_i : {}^p\mathcal{H}^{i-2}(Rf_*(i_H)_* i_H^* j_{!*}\mathcal{L}) \to {}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L})$ *is an isomorphism for* $i > 1$, *and it is surjective for* $i = 1$.

We deduce from the triangle $(j_H)_! j_H^* j_{!*}\mathcal{L} \to j_{!*}\mathcal{L} \to i_{H*} i_H^* j_{!*}\mathcal{L} \xrightarrow{(1)}$, and its direct image by Rf_*, an exact sequence of perverse cohomology

$$\cdots \to {}^p\mathcal{H}^i(K_c) \xrightarrow{\mathrm{can}_i} {}^p\mathcal{H}^i(K) \xrightarrow{\rho_i} {}^p\mathcal{H}^i(K_H) \xrightarrow{\partial_i} {}^p\mathcal{H}^{i+1}(K_c) \to \cdots \tag{2.10}$$

The corollary follows from lemma 2.11 as the complex $K_c \in {}^pD_{\overline{V}}^{\geq 0}$, or equivalently ${}^p\mathcal{H}^i(K_c) = 0$ for $i < 0$.

2.3.4. *The crucial case for* $i = 0$. We use the inductive assumptions over $V - v$.

Lemma 2.13. *If we suppose the relative Hard Lefschetz theorem for the restriction of the morphisms* f *and* $f \circ i_H$ *to* $V - v$, *then* G_0 *is injective and* ρ_0 *is surjective on* $V - v$. *Moreover, we have a decomposition*

$${}^p\mathcal{H}^0(Rf_* j_{!*}\mathcal{L})_{|V-v} \xrightarrow{\sim} \mathrm{Im}\, G_0 \oplus \ker\, \rho_0. \tag{2.11}$$

Respectively, ρ_{-1} *is injective,* G_1 *is surjective and we have a decomposition*

$${}^p\mathcal{H}^{-1}(Rf_*(i_H)_* i_H^* j_{!*}\mathcal{L})_{|V-v} \xrightarrow{\sim} \mathrm{Im}\, \rho_{-1} \oplus \ker\, G_1 \tag{2.12}$$

Moreover $G_0 : {}^p\mathcal{H}^{-2}(Rf_*(i_H)_* i_H^* j_{!*}\mathcal{L})_{|V-v} \xrightarrow{\sim} \mathrm{Im}\, G_0 \subset {}^p\mathcal{H}^0(Rf_* j_{!*}\mathcal{L})_{|V-v}$ *is an isomorphism onto its image.*

Proof. Recall that $\mathcal{L}$ is defined on $X - Y$. Let $j' : (H - H \cap Y) \to H$. We have by transversality: $i_H^* j_{!*}\mathcal{L} \xrightarrow{\sim} j'_{!*}(i_H^*\mathcal{L})$, hence ${}^p\mathcal{H}^i(K_H) = {}^p\mathcal{H}^i(Rf_* j'_{!*}(i_H^*\mathcal{L}))$. Since by the inductive hypothesis, Hard Lefschetz theorem apply on $V - v$ to $K_H[-1]$, image of the intermediate extension $j'_{!*} i_H^*\mathcal{L}[-1]$ of the shifted restriction of $\mathcal{L}$ on H, we deduce that the composition morphism $\smile c_1 = \rho_0 \circ G_0$ in the diagram is an isomorphism on $V - v$

$${}^p\mathcal{H}^{-2}(K_H)_{|V-v} \xrightarrow{G_0} {}^p\mathcal{H}^0(K)_{|V-v} \xrightarrow{\rho_0} {}^p\mathcal{H}^0(K_H)_{|V-v}$$

where the first term is ${}^p\mathcal{H}^{-1}(Rf_* j'_{!*} i_H^*\mathcal{L}[-1])$ and the last ${}^p\mathcal{H}^1(Rf_* j'_{!*} i_H^*\mathcal{L}[-1])$. Hence ρ_0 is surjevtive, G_0 is injective and the decomposition 2.11 follows.

Respectively, the composition morphism $\smile c_1 = G_1 \circ \rho_{-1}$ in the diagram

$${}^p\mathcal{H}^{-1}(K)_{|V-v} \xrightarrow{\rho_{-1}} {}^p\mathcal{H}^{-1}(K_H)_{|V-v} \xrightarrow{G_1} {}^p\mathcal{H}^1(K)_{|V-v}$$

is an isomorphism and the decomposition 2.12 follows. □

The next result is based on Hodge theory as in (proposition 1.7, definition 5.11 and lemma 5.13)

Corollary 2.14. *If the local purity theorem applies for* ${}^p\mathcal{H}^i(K_H)$, *then it applies for* ${}^p\mathcal{H}^i(K)$, *except eventually the component* $(\mathrm{Ker}\, \rho_0)_{|V-v} \subset {}^p\mathcal{H}^0(Rf_* j_{!*}\mathcal{L})_{|V-v}$.

Proof. The restriction morphisms ρ_i are compatible with MHS for all i and j

$$H^j(B_v - v, {}^p\mathcal{H}^i(K)) \to H^j(B_v - v, {}^p\mathcal{H}^i(K_H))$$

and they are isomorphisms for $i < -1$ by the corollary above. Then, the conditions of local purity on K are satisfied for $i < -1$, since they are satisfied on K_H by

induction. The dual argument for G_i apply, and we are left with the cases $i = 0, -1, 1$.

The case $i = -1$ follows from the decomposition (formula 2.12), as ρ_{-1} is an isomorphism onto a direct summand $\operatorname{Im}\rho_{-1}$ of ${}^p\mathcal{H}^{-1}(Rf_*(i_H)_* i_H^* j_{!*}\mathcal{L})_{|V-v}$.

By Lefschetz isomorphism on $V - v$, we deduce the case $i = 1$ from the case $i = -1$. For $i = 0$, as G_0 is an isomorphism by the lemma, the inequality holds for $\operatorname{Im} G_0$. Only the case of $\operatorname{Ker}\rho_0$ can not be deduced by induction from K_H. □

2.4. **The crucial case.** $w \leq a + j$ on $H^j(B_v - v, {}^p\mathcal{H}^0(Rf_* j_{!*}\mathcal{L}))$ for $j < 0$. The result is local at v, so we can suppose V affine, then choose a projective embedding to allow the use of the polarization of the cohomology.

The proof is subdivided in many steps. We apply the next lemma for ${}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L})$, in the case $i = 0$, but it is equallly proved for all i .

Lemma 2.15. *Let $j_{!*}\mathcal{L}$ be of weight a, then the MHS on*

$$Gr_i^{{}^p\tau} H^{i+j}(B_{X_v} - X_v, j_{!*}\mathcal{L}) \simeq H^j(B_v - v, {}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L}))$$

is of weight $\omega > a + i + j$ for $j > 0$ and dually of weight $\omega \leq a + i + j$ for $j < -1$.

Proof. Let H be a general hyperplane section of V containing v, $H_v = B_v \cap H$ and $K = Rf_* j_{!*}\mathcal{L}$. We suppose H normally embedded outside v so that the perverse truncation commutes with the restriction to $H - v$ up to a shift in degrees; We have a Gysin exact sequence

$$\mathbb{H}^{j-2}(H_v - \{v\}, {}^p\mathcal{H}^i(K))(-1) \xrightarrow{G_j} \mathbb{H}^j(B_v - \{v\}, {}^p\mathcal{H}^i(K)) \to \mathbb{H}^j(B_v - H_v, {}^p\mathcal{H}^i(K))$$

Since ${}^p\mathcal{H}^i(K)$ is in the category ${}^pD_c^{\leq 0}V$, and $B_v - H_v$ is Stein, we apply Artin Lefshetz hyperplane section theorem to show that $\mathbb{H}^j(B_v - H_v, {}^p\mathcal{H}^i(K)) \simeq 0$ for $j > 0$. Then G_j is an isomorphism for $j > 1$ and it is surjective for $j = 1$.

The smooth strict transform H' of H intersects transversally in X the various subspaces Y_l inverse of the strata S_l so that Gysin morphisms are are compatible with the MHS.

Hence, we deduce the statement for $B_v - v$ in the lemma for $j > 0$ from the inductive hypothesis on the local purity of the MHS structure on $H^{j-1}(H_v - v, {}^p\mathcal{H}^i(K)[-1])$ for $j - 1 \geq 0$, since $H^{j-2}(H_v - v, {}^p\mathcal{H}^i(K))(-1) \simeq H^{j-1}(H_v - v, {}^p\mathcal{H}^i(K)[-1])$. □

2.4.1. *The large inequality: $w \leq a$ on $H^{-1}(B_v - v, {}^p\mathcal{H}^0(Rf_* j_{!*}\mathcal{L}))$.* This case also is easily deduced by induction

Lemma 2.16. *i) $\mathbb{H}^0(B_v - \{v\}, {}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L}))$ is of weight $\geq a + i$.*
ii) Dually: $\mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^i(K))$ is of weight $\leq a + i$.

Proof. Let $k_v : (V - v) \to V$ and consider a general hyperplane section H_1 not containing v. We deduce from the triangle of complexes:
$Rk_{v!}k_v^* {}^p\mathcal{H}^i(K) \to Rk_{v*}k_v^* {}^p\mathcal{H}^r(K) \to i_{v*}i_v^* Rk_{v*}k_v^* {}^p\mathcal{H}^i(K)$, the exact sequence

$$\mathbb{H}^0(V - H_1, Rk_{v*}k_v^* {}^p\mathcal{H}^i(K)) \xrightarrow{\gamma_0} H^0(i_v^* Rk_{v*}k_v^* {}^p\mathcal{H}^i(K)) \to \mathbb{H}^1(V - H_1, Rk_{v!}k_v^* {}^p\mathcal{H}^i(K))$$

which shows that γ_0 is surjective, since $\mathbb{H}^1(V - H_1, Rk_{v!}k_v^* {}^p\mathcal{H}^i(K))$ vanishes as $V - H_1$ is affine. We deduce the assertion i) since the weights w of the cohomology $\mathbb{H}^0(V - H_1, Rk_{v*}Rk_v^* {}^p\mathcal{H}^i(K))$ of the open set $V - H_1$ satisfy $w \geq a + i$. □

2.4.2. *The crucial step: $w \neq a$.* It remains to exclude the case $w = a$, which does not follow by induction, that is

$$Gr_a^W Gr_0^{{}^p\tau}\mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}) \xrightarrow{\sim} Gr_a^W \mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^0(K)) \xrightarrow{\sim} 0. \tag{2.13}$$

from which we deduce by duality $Gr_a^W Gr_0^{{}^p\tau}\mathbb{H}^0(B_{X_v} - X_v, j_{!*}\mathcal{L}) \xrightarrow{\sim} 0$.

By corollary 2.14, it remains to consider the case of the sub-perverse sheaf $\mathrm{Ker}\rho_0 \subset {}^p\mathcal{H}^0(K)$ in which case we use the following lifting of cohomology classes

Lemma 2.17. *Let $B_v^* := B_v - v$ and H a general hyperplane section of X. The following map is surjective:*
$W_a\tau_{\leq 0}\mathbb{H}^{-1}(B_v^*, Rf_*j_{H!}j_H^*j_{!*}\mathcal{L}) \to Gr_a^W Gr_0^{{}^p\tau}\mathbb{H}^{-1}(B_v^*, \mathrm{Ker}\,\rho_0)$.

The proof relies on Hodge theory appied to *cohomology with compact support of the fibers of $f_{|X-H}$*as follows.

By transversality of H with all strata of X, the restriction to H of $j_{!*}\mathcal{L}$ is defined in terms of complexes of logarithmic type $\rho : (IC^*\mathcal{L}, F) \to i_{H*}i_H^*(IC^*\mathcal{L}, F)$, where $(IC^*(\mathcal{L}_{|H}[-1]), F)[1] = i_H^*(IC^*\mathcal{L}, F)$ (subsection 4.5.1).
We introduce the mixed cone $(C(\rho)[-1], W, F)$ satisfying

$$(Gr_i^W C(\rho)[-1], F) = Gr_i^W((IC^*\mathcal{L}, F) \oplus Gr_{i+1}^W(i_{H*}i_H^*(IC^*\mathcal{L}, F)[-1])$$

Let $(K_c, F) := Rf_*(C(\rho)[-1], W, F)$. We deduce the exact sequence of perverse cohomology (formula 2.10) from the triangle $K_c \to K \to K_H \xrightarrow{(1)}$.

In the local case, we consider a neighborhood B_v of v and the long exact sequence

$$\to \mathbb{H}^{i-1}(B_v^*, K_H) \to \mathbb{H}^i(B_v^*, K_c) \xrightarrow{\gamma} \mathbb{H}^i(B_v^*, K) \to \mathbb{H}^i(B_v^*, K_H) \to \cdots$$

The terms of this sequence are endowed with MHS (subsection 5.2.4, definition 5.11 and lemma 5.13). Such MHS is compatible with the perverse filtration (proposition 1.7, sections 5.3, 5.3.2), that is the subspaces
$\mathbb{H}^i(B_v^*, {}^p\tau_{\leq}jK)$, $\mathbb{H}^i(B_v^*, {}^p\tau_{\leq}jK_H)$, $\mathbb{H}^i(B_v^*, {}^p\mathcal{H}^j(K))$, and $\mathbb{H}^i(B_v^*, {}^p\mathcal{H}^j(K_H))$ underly corresponding MHS, as well $\mathbb{H}^i(B_v^*, {}^p\tau_{\leq}jK_c)$ and $\mathbb{H}^i(B_v^*, {}^p\mathcal{H}^j(K_c))$.

We apply the previous results to the exact sequence (lemma 2.13)

$$\cdots \to \mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K_c)) \xrightarrow{\gamma} \mathbb{H}^{-1}(B_v^*, \mathrm{Ker}\rho_0) \xrightarrow{\partial} \mathbb{H}^0(B_v^*, \mathrm{Coker}\rho_{-1}) \to$$

Taking Gr_a^W, we have an exact squence

$$Gr_a^W\mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K_c)) \xrightarrow{\gamma_1} Gr_a^W\mathbb{H}^{-1}(B_v^*, \mathrm{Ker}\rho_0) \xrightarrow{\partial} Gr_a^W\mathbb{H}^0(B_v^*, \mathrm{Coker}\rho_{-1}) \to$$

By the inductive hypothesis on H, $Gr_a^W\mathbb{H}^0(B_v^*, \mathrm{Coker}\rho_{-1}) = 0$, hence γ_1 is surjective so that elements of $Gr_a^W\mathbb{H}^{-1}(B_v^*, \mathrm{Ker}\rho_0)$ can be lifted to elements in $W_a\mathbb{H}^{-1}(B_v^*, {}^p\mathcal{H}^0(K_c))$. Moreover, $\tau_{\leq 0}K_c = {}^p\mathcal{H}^0(K_c)$ as $\tau_{<0}K_c = 0$, hence we can lift the elements to $W_a\mathbb{H}^{-1}(B_v^*, \tau_{\leq 0}K_c)$. This ends the proof of lemma 2.17.

2.4.3. *Polarization.* In this step we use the polarization of Intersection cohomology. The following proof is based on the idea that the cohomology of $B_v - v$ fits in two exact sequences issued from the two triangles for $K := Rf_*j_{!*}\mathcal{L}$

$$i_v^*Rk_{v!}k_v^*K \to i_v^*K \to i_v^*Rk_{v*}k_v^*K, \quad Rk_v^!K \to i_v^*K \to i_v^*Rk_{v*}k_v^*K \tag{2.14}$$

from which we deduce the commutative diagram (see also the diagram 5.6)

$$\begin{tikzcd}
\mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}) \arrow[r, "\partial_X"] \arrow[d, "\partial"] \arrow[dr, "\gamma"] & \mathbb{H}^0_c(X - X_v, j_{!*}\mathcal{L}) \arrow[d, "\alpha_X"] \\
\mathbb{H}^0_{X_v}(X, j_{!*}\mathcal{L}) \arrow[r, "A"] & \mathbb{H}^0(X, j_{!*}\mathcal{L})
\end{tikzcd}$$

Let γ denotes the composition morphisms $\gamma = \alpha_X \circ \partial_X = A \circ \partial$.

Lemma 2.18. *Let $u \in Gr_a^W \mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})$ be in the image of the canonical morphism $Gr_a^W \mathbb{H}^{-1}(B_{X_v} - X_v, j_{H!}j_H^* j_{!*}\mathcal{L}) \xrightarrow{\text{can}} Gr_a^W \mathbb{H}^{-1}(B_{X_v} - X_v, j_{!*}\mathcal{L})$, then $\gamma(u) = 0$ in $Gr_a^W \mathbb{H}^0(X, j_{!*}\mathcal{L}) = \mathbb{H}^0(X, j_{!*}\mathcal{L})$.*

Proof. 1) *$\gamma(u)$ is primitive.* Let $B^*_{X_v} := B_{X_v} - X_v$, $b \in Gr_a^W \mathbb{H}^{-1}(B^*_{X_v}, j_{H!}j_H^* j_{!*}\mathcal{L})$ such that $u = \text{can}(b)$, and consider the diagram corresponding to an hyperplane section H

$$Gr_a^W \mathbb{H}^{-1}(B^*_{X_v}, j_{H!}j_H^* j_{!*}\mathcal{L}) \xrightarrow{\partial_!} Gr_a^W \mathbb{H}^0_c(X^*, j_{H!}j_H^* j_{!*}\mathcal{L}) \xrightarrow{\alpha_!} Gr_a^W \mathbb{H}^0(X, j_{H!}j_H^* j_{!*}\mathcal{L})$$

Let $\gamma_! := \alpha_! \circ \partial_!$, then $\gamma(u) = \gamma(\text{can}\, b) = \text{can}\, \gamma_!(b)$, hence the restriction to H: $\rho_H(\gamma(u)) = \rho_H(\text{can}\, \gamma_!(b)) = 0$, and $\gamma(u)$ is a primitive element.

2) $\gamma(u) = 0$. We consider the diagram

$$\begin{tikzcd}
& Gr_a^W \mathbb{H}^0(X, j_{!*}\mathcal{L}) \arrow[dr, "A^*"] & \\
Gr_a^W \mathbb{H}^0_{X_v}(X, j_{!*}\mathcal{L}) \arrow[ur, "A"] \arrow[rr, "I"] & & Gr_a^W \mathbb{H}^0(X_v, j_{!*}\mathcal{L})
\end{tikzcd}$$

where A^* is the dual of A. Let P denotes the scalar product defined by Poincaré duality on $\mathbb{H}^0(X, j_{!*}\mathcal{L})$ and C the Weil operator defined by the HS. The polarization Q on the primitive part of $\mathbb{H}^0(X, j_{!*}\mathcal{L})$ is defined by $Q(a, b) := P(Ca, \overline{b})$.

A non-degenerate pairing P_v

$$P_v : Gr_a^W \mathbb{H}^0_{X_v}(X, j_{!*}\mathcal{L}) \otimes Gr_a^W \mathbb{H}^0(X_v, j_{!*}\mathcal{L}) \to \mathbb{C}.$$

is also defined by duality. The duality between A and A^* is defined for all $b \in Gr_a^W \mathbb{H}^0_{X_v}(X, j_{!*}\mathcal{L})$ and $c \in Gr_a^W \mathbb{H}^0(X, j_{!*}\mathcal{L})$ by the formula :

$$P(Ab, c) = P_v(b, A^* c).$$

Let C be the Weil operator defined by the HS on $Gr^W_{a+i}\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L})$ as well on $Gr^W_{a+i}\mathbb{H}^i(X, j_{!*}\mathcal{L})$, then:

$$P(C.A(\partial u), A(\overline{\partial u})) = P_v(C\partial u, A^* \circ A(\overline{\partial u})) = P_v(C.\partial(u), I(\overline{\partial u}))$$

and since $I(\partial u) = A^* \circ A \circ \text{can}(b) = 0$, we deduce $P(C.\gamma(u), \overline{\gamma(\partial u)})) = P(C.A(\partial u), A(\overline{\partial u})) = 0$, hence $\gamma(u) = 0$ by polarization as $\gamma(u)$ is primitive. □

2.4.4. The last step of the proof of the crucial case is based on two lemma.

Lemma 2.19. *The connecting morphism $\mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}) \xrightarrow{\partial_X} \mathbb{H}^i_c(X - X_v, j_{!*}\mathcal{L})$ induced by the triangles in formula (2.14) is injective on Gr^W_{a+i}*
$Gr^W_{a+i} Gr_i^{p_\tau} \mathbb{H}^{i-1}(B_{X_v} - X_v, j_{!*}\mathcal{L}) \xrightarrow{\partial_X} Gr^W_{a+i} Gr_i^{p_\tau} \mathbb{H}^i_c(X - X_v, j_{!*}\mathcal{L})$.

We prove equivalently $Gr^W_{a+i}\mathbb{H}^{-1}(B_v - v, {}^p\mathcal{H}^i(K)) \xrightarrow{\partial_V} Gr^W_{a+i}\mathbb{H}^0_c(V - \{v\}, {}^p\mathcal{H}^i(K))$ (precisely $Gr^W_{a+i}(\partial_V)$) is injective, by considering the long exact sequence:

$$\mathbb{H}^{-1}(V-\{v\},\, {}^p\mathcal{H}^i(K)) \overset{\partial_V}{\to} H^0_c(V-\{v\},\, {}^p\mathcal{H}^i(K)) \to \mathbb{H}^0(V-v,\, {}^p\mathcal{H}^i(K)).$$

It is enough to prove that $H^{-1}(V-\{v\},\, {}^p\mathcal{H}^i(K))$ is pure of weight $a+i-1$ or by duality $H^1_c(V-\{v\},\, {}^p\mathcal{H}^i(K))$ is pure of weight $a+i+1$. We consider as above the hyperplane section H_1 not containing v and the exact sequence:
$\mathbb{H}^1_{H_1}(V, Rk_{v!}k_v^*\, {}^p\mathcal{H}^i(K)) \overset{\varphi}{\to} \mathbb{H}^1(V, Rjk_{v!}k_v^*\, {}^p\mathcal{H}^i(K)) \to \mathbb{H}^1(V-H_1, Rk_{v!}k_v^*\, {}^p\mathcal{H}^i(K))$
where φ is surjective since the last term vanish; the space
$\mathbb{H}^{-1}(H_1, Rk_{v!}k_v^*\, {}^p\mathcal{H}^i(K))(-1) \xrightarrow{\sim} \mathbb{H}^1_{H_1}(V, Rk_{v!}k_v^*\, {}^p\mathcal{H}^i(K))$
is a pure HS of weight $a+i+1$ (as a sub-quotient of the pure intersection cohomology of the inverse image H_1' smooth in X).

Lemma 2.20. *The following morphism is injective*
$Gr^W_{a+i}\mathbb{H}^i_c(X-X_v, j_{!*}\mathcal{L}) \overset{\alpha_X}{\to} Gr^W_{a+i}\mathbb{H}^i(X, j_{!*}\mathcal{L}) \xrightarrow{\sim} \mathbb{H}^i(X, j_{!*}\mathcal{L})$.

In the long exact sequence $H^{i-1}(X_v, j_{!*}\mathcal{L}) \to \mathbb{H}^i_c(X-X_v, j_{!*}\mathcal{L}) \overset{\alpha_X}{\to} \mathbb{H}^i(X, j_{!*}\mathcal{L})$, the weight w of $H^{i-1}(X_v, j_{!*}\mathcal{L})$ satisfy $w < a+i$ since X_v is closed, then the morphism α_X (precisely $Gr^W_{a+i}(\alpha_X)$) is injective and $H^i(X, j_{!*}\mathcal{L})$ is pure of weight $a+i$.

2.4.5. The morphisms ∂_X and α_X are compatible with ${}^p\tau$ and W. We still denote by ∂_X (resp. α_X) their restriction to ${}^p\tau_r\mathbb{H}^{r-1}(B_{X_v}-X_v, j_{!*}\mathcal{L})$ (resp. ${}^p\tau_r\mathbb{H}^r_c(X-X_v, j_{!*}\mathcal{L})$), then the composed morphism has value in $\mathbb{H}^r(X, j_{!*}\mathcal{L})$ (in fact in the subspace ${}^p\tau_r$ but we avoid the filtration ${}^p\tau$ as we have no information at the point v yet).

Corollary 2.21. $Gr^W_a Gr_0^{{}^p\tau}\mathbb{H}^{-1}(B_{X_v}-X_v, j_{!*}\mathcal{L}) = 0$.

It is enough to apply the lemma 2.17 to $H^{-1}(B_v-v, \operatorname{Ker}\rho_0)$, since
$Gr^W_a Gr_0^{{}^p\tau}\mathbb{H}^{-1}(B_{X_v}-X_v, j_{!*}\mathcal{L}) \simeq Gr^W_a H^{-1}(B_v-v, \operatorname{Ker}\rho_0 \oplus \operatorname{Im} G_0)$
splits (formula 2.11) and $Gr^W_a H^{-1}(B_v-v, \operatorname{Im} G_0) = 0$ by induction.

By lemma 2.17, each element $\overline{u} \in Gr_0^{{}^p\tau}Gr^W_a\mathbb{H}^{-1}(B_v^*, \operatorname{Ker}\rho_0)$ is the class modulo ${}^p\tau_{-1}$ of an element $u \in ({}^p\tau_0 \cap W_a)\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}\mathcal{L})$, where $u = \operatorname{can}(b)$ with $b \in ({}^p\tau_0 \cap W_a)\mathbb{H}^{-1}(B^*_{X_v}, j_{H!}j_H^*j_{!*}\mathcal{L})$.

We use the commutative diagram to lift the elements as needed

$$\begin{array}{ccc}
W_a\mathbb{H}^{-1}(B_v-v, \tau_{\leq 0}Rf_*j_{H!}j_H^*j_{!*}\mathcal{L}) & \xrightarrow{\text{can}} & Gr^W_a H^{-1}(B_v-v, \operatorname{Ker}\rho_0) \\
\downarrow & & \downarrow \\
W_a\tau_{\leq 0}\mathbb{H}^{-1}(B_{X_v}-X_v, j_{H!}j_H^*j_{!*}\mathcal{L}) & \xrightarrow{\text{can}} & Gr^W_a Gr_0^{\tau}\mathbb{H}^0(B_{X_v}-X_v, j_{!*}\mathcal{L})
\end{array}$$

Since $\gamma(u) = \alpha_X(\partial_X u) = 0$, we deduce $\partial_X u = 0$ in $Gr^W_a\mathbb{H}^0_c(B^*_{X_v}, j_{!*}\mathcal{L})$ by lemma (2.20), hence the class $\overline{\partial_X u} = 0$ in $Gr_0^{{}^p\tau}Gr^W_a\mathbb{H}^0_c(B^*_{X_v}, j_{!*}\mathcal{L})$. Finally, since $\partial_X(\overline{u}) = \overline{\partial_X(u)} = 0$, we deduce $\overline{u} = 0$ by lemma (2.19). This ends the proof of the crucial case and the proposition 2.2 for f a fibration by NCD.

Remark 2.22. The following relation follows from the proof of the proposition 2.2

$$\operatorname{Im}\left({}^p\mathcal{H}^i(Rk_{v!}k_v^*K) \to {}^p\mathcal{H}^i(K)\right) = k_{v!*}k_v^*\, {}^p\mathcal{H}^i(K)$$

2.4.6. The inductive step theorem 2 and most of the corollary 1.2 follow for f a fibration by NCD. We complete now the proof of equation 1.7.

Lemma 2.23. *On a projective variety V, we have an orthogonal decomposition of a polarized HS of weight $a+i+j$*

$$Gr_i^{p_\tau}\mathbb{H}^{i+j}(X, j_{!*}\mathcal{L}) \simeq \mathbb{H}^j(V, {}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})) \xrightarrow{\sim} \oplus_{l\leq n}^{S_l\subset V_l^*} \mathbb{H}^j(V, i_{S_l!*}\mathcal{L}^i_{S_l}[l]).$$

The proof is by induction on $dim.$V and for fixed V on $dim.$X. The case of $dim.$V = 1 is clear and may be proved by the technique of the inductive step. We suppose the result true for $dim.\mathrm{V} = n-1$. For $dim.\mathrm{V} = n$, by corollary 2.12, the case of ${}^p\mathcal{H}^i(Rf_*j_{!*}\mathcal{L})$ for $i<0$ follows from the case ${}^p\mathcal{H}^i(Rf_*(i_H)_*i_H^*j_{!*}\mathcal{L})$ by restriction to a general ample hyperplane H, and by duality the case for $i>0$. It remains to prove the crucial case of ${}^p\mathcal{H}^0(Rf_*j_{!*}\mathcal{L})$. First we prove that for each non generic component S_l with $l<n$, the component $\mathbb{H}^j(V, i_{S_l!*}\mathcal{L}^0_{S_l}[l])$ is a sub HS.

We may suppose there exists a projection $\pi : V \to V_l$ to a projective variety V_l inducing a finite morphism on $\overline{S_l}$. By induction, the property apply for the decomposition of $\pi\circ f$ as $dim.V_l < n$, which contains the component $\pi_* i_{S_l!*}\mathcal{L}^0_{S_l}[l]$.

It follows that the direct sum $\oplus_{l<n}^{S_l\subset V_l^*} \mathbb{H}^j(V, i_{S_l!*}\mathcal{L}^i_{S_l}[l]) \subset Gr_0^{p_\tau}\mathbb{H}^j(X, j_{!*}\mathcal{L})$ is a sub-HS. Then, it is a standard argument to show that its orthogonal subspace, which coincides with $\mathbb{H}^j(V, i_{S_n!*}\mathcal{L}^i_{S_n}[l])$ (the generic component) is a sub-HS. This is a clarification of the proof for constant coefficients in [CaMi 5] and a generalization to coefficients.

Remark 2.24. For a quasi-projective variety, we must consider MHS. This apply in general for coefficients in a mixed Hodge complex generalizing the notion of variation of MHS. The pure case in this paper is the building bloc.

2.4.7. *Proof for any projective morphism.* The above results establish the case of fibrations by NCD. Given $j_{!*}\mathcal{L}$ on X and a diagram $X' \xrightarrow{\pi} X \xrightarrow{f} V$ such that the desingularization π and $f' := f\circ\pi$ are fibrations by NCD over the strata. We apply the above result to π and the extension $j'_{!*}\mathcal{L}$ on X' to deduce the decomposition of $K := R\pi_* j'_{!*}\mathcal{L}$ on X into a direct sum of intermediate extensions.

We indicate here, how we can extend Hodge theory and deduce the structure of polarized VHS on $\mathcal{L}^i_S$ in all cases without reference to the the special fibration case.

Let $\mathcal{L}$ be defined on a smooth open set Ω of X, $j : \Omega \to X$ and $j' : \Omega \to X'$ an open embedding into a desingularization of X.

Lemma 2.25. *Let $\pi : X' \to X$ be a desingularization defined by a fibration by NCD over the strata, $j'_{!*}\mathcal{L}$ the intermediate extension of $\mathcal{L}$ on X'. If X is projective, $\mathbb{H}^i(X, j_{!*}\mathcal{L})$ is a sub-Hodge structure of $Gr_0^{p_\tau}\mathbb{H}^i(X', j'_{!*}\mathcal{L})$. The induced Hodge structure on $\mathbb{H}^i(X, j_{!*}\mathcal{L})$ is independent of the choice of X'.*

Proof. We apply the above result to $\pi : X' \to X$ to deduce the decomposition of ${}^p\mathcal{H}^0(R\pi_* j'_{!*}\mathcal{L}) \simeq \oplus_{S\in\mathcal{S}}\mathcal{L}^0_S$ on X into a direct sum consisting of intermediate extensions of polarized VHS.
It follows from lemma 2.23, that the decomposition

$$Gr_0^{p_\tau}\mathbb{H}^i(X', j'_{!*}\mathcal{L}) \simeq \mathbb{H}^i(X, {}^p\mathcal{H}^0(R\pi_* j'_{!*}\mathcal{L})) \simeq \oplus_{S\in\mathcal{S}}\mathbb{H}^i(X, i_{S!*}\mathcal{L}^0_S)$$

is compatible with HS. Moreover, on the big strata U (as we can suppose V irreducible), we have $j_{!*}\mathcal{L} = i_{U!*}\mathcal{L}^0_U$; from which we deduce a HS on $\mathbb{H}^i(X, j_{!*}\mathcal{L})$ as a a sub-quotient HS of $\mathbb{H}^i(X', j'_{!*}\mathcal{L})$.

The uniqueness is deduced by the method of comparison of the two desingularizations X'_1 and X'_2 with a common desingularization X' of the fiber product $X'_1 \times_X X'_2$. □

2.4.8. *Proof of the corollary 2.10 and lemma 2.23 for any projective morphism.*

We describe the underlying structure of polarized VHS on $\mathcal{L}^i_v$. Let $f : X \to V$, $\pi : X' \to X$ such that π and $f \circ \pi$ are fibrations, $v \in S$ a general point of a strata S on V, and N_v a general normal section to S at v such that the inverse image X_{N_v} is normally embedded in X (resp. X'_{N_v} in X'). By definition the fiber $\mathcal{L}^i_{S,v}$ is the image of the Intersection morphism: $\mathbb{H}^i_{X_v}(X_{N_v}, j_{!*}\mathcal{L}) \xrightarrow{I} \mathbb{H}^i(X_v, j_{!*}\mathcal{L})$.

By transversality, the perverse truncation ${}^p\tau_X$ on $K := R\pi_* j'_{!*}\mathcal{L}$ on X, is compatible with the restriction of K to X_{N_v}. Hence, we can suppose v in the zero dimensional strata. As the decomposition is established for π and $f \circ \pi$, it follows for f, and $\mathbb{H}^i_{X_v}(X, {}^p\mathcal{H}^0(K)) = Gr_0^{{}^p\tau_X}\mathbb{H}^i_{X'_v}(X', j'_{!*}\mathcal{L})$ carry an induced MHS (see subsection 5.3).

The natural decomposition on ${}^p\mathcal{H}^0(K)$ induce a decomposition of $\mathbb{H}^i_{X_v}(X, {}^p\mathcal{H}^0(K))$ into direct sum of MHS where $\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L})$ is a summand, on which the MHS is realized as sub-quotient of the MHS on $\mathbb{H}^i_{X'_v}(X', j'_{!*}\mathcal{L})$.

A similar argument puts on $\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$ a MHS as a sub-quotient of the MHS on $Gr_0^{{}^p\tau_X}\mathbb{H}^i(X'_v, j'_{!*}\mathcal{L})$ such that the image MHS of the intersection morphism I is pure on $\mathcal{L}^i_v$.

Verdier duality between $\mathcal{L}^{-k}_S$ and $\mathcal{L}^k_S$ follows from the auto-duality of $j_{!*}\mathcal{L}$ by Verdier formula for the proper morphism f. Since Hard Lefschetz isomorphisms between $\mathcal{L}^{-k}_S$ and $\mathcal{L}^k_S$ are also satisfied, we deduce that the VHS $\mathcal{L}^k_S$ are polarized.

Similarly, the decomposition for ${}^p\mathcal{H}^i(Rf_*\,{}^p\mathcal{H}^0(R\pi_* j'_{!*}\mathcal{L}))$ is part of the decomposition for ${}^p\mathcal{H}^i(R(f \circ \pi)_* j'_{!*}\mathcal{L})$ and contains the decomposition of ${}^p\mathcal{H}^i(Rf_* j_{!*}\mathcal{L})$. *This ends the proof of lemma 2.23 and altogether the proof of corollary 2.10.*

2.5. **Local purity and Grauert generalized Intersection isomorphism.** Let v be a point in V and consider the morphisms $j_{X_v} : (X - X_v) \to X$, $i_{X_v} : X_v \to X$, $i_v : v \to V$ and $k_v : (V - v) \to V$. The perverse filtration ${}^p\tau$ on $K := Rf_* j_{!*}\mathcal{L}$ on V induces a filtration ${}^p\tau$ on $Ri^!_v K$ defined functorially by $Ri^!_v\,{}^p\tau K$ and computed by $i^!_v$-acyclic filtered representative of $(K, {}^p\tau)$. Similarly, we define ${}^p\tau$ on $i^*_v K$ and $i^*_v Rk_{v*}k^*_v K$, from which we deduce increasing filtrations ${}^p\tau$ on $\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L})$, $\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$ and $\mathbb{H}^i(B^*_{X_v}, j_{!*}\mathcal{L})$.

Lemma 2.26. *The induced perverse filtration ${}^p\tau$ satisfy the properties:*

$$ {}^p\tau_i\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) = \mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}),\ {}^p\tau_{i-1}\mathbb{H}^i(X_v, j_{!*}\mathcal{L}) = 0. $$

We have converging spectral sequences with respect to the induced filtration ${}^p\tau$

$$ {}_{p\tau}E_1^{pq} := \mathbb{H}^{2p+q}_v(V, {}^p\mathcal{H}^{-p}(K)) \Rightarrow Gr_{-p}^{{}^p\tau}\mathbb{H}^{p+q}_{X_v}(X, j_{!*}\mathcal{L}) $$
$$ {}_{p\tau}E_1^{pq} := \mathbb{H}^{2p+q}(v, {}^p\mathcal{H}^{-p}(K)) \Rightarrow Gr_{-p}^{{}^p\tau}\mathbb{H}^{p+q}(X_v, j_{!*}\mathcal{L}) $$
$$ {}_{p\tau}E_1^{pq} := \mathbb{H}^{2p+q}(B^*_v, {}^p\mathcal{H}^{-p}(K)) \Rightarrow Gr_{-p}^{{}^p\tau}\mathbb{H}^{p+q}(B^*_{X_v}, j_{!*}\mathcal{L}). $$

Since $\mathbb{H}^i_v(V, {}^p\mathcal{H}^{-p}(K)) = 0$ for $i < 0$ as ${}^p\mathcal{H}^{-p}(K)$ is perverse, we have

$$ {}_{p\tau}E_1^{pq} := \mathbb{H}^{2p+q}_v(V, {}^p\mathcal{H}^{-p}(K)) \text{ for } 2p + q < 0 $$

hence $Gr_j^{{}^p\tau}\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) = 0$ for $i < j$ or ${}^p\tau_i\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) = \mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L})$.

Since $\mathbb{H}^i(v, {}^p\mathcal{H}^{-p}(K)) = 0$ for $i > 0$, we deduce similarly ${}^p\tau_{i-1}\mathbb{H}^i(X_v, j_{!*}\mathcal{L}) = 0$.

Proposition 2.27 (Generalized Grauert Intersection isomorphism)**.**
The following isomorphism induced by the Intersection morphism I for each r, follows from the local purity theorem (subsection 2.3.1):

$$Gr_r^{{}^p\tau}\mathbb{H}^r_{X_v}(X, j_{!*}\mathcal{L}) \xrightarrow{I\simeq} Gr_r^{{}^p\tau}\mathbb{H}^r(X_v, j_{!*}\mathcal{L}) \tag{2.15}$$

and the induced morphism by I vanish on $Gr_i^{{}^p\tau}\mathbb{H}^r_{X_v}(X, j_{!*}\mathcal{L})$ for all indices $i \neq r$.

Reciprocally, if the spectral sequence with respect to the perverse filtration on K degenerates at rank 1, we can deduce the local purity theorem from Grauert isomorphism I.

Lemma 2.28. *i) The morphisms of the exact sequence defined by the Intersection morphism I*

$$\mathbb{H}^{i-1}(B^*_{X_v}, j_{!*}\mathcal{L}) \xrightarrow{\partial_i} \mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L}) \xrightarrow{I} \mathbb{H}^i(X_v, j_{!*}\mathcal{L}) \xrightarrow{\gamma_i} \mathbb{H}^i(B^*_{X_v}, j_{!*}\mathcal{L}) \xrightarrow{[1]} \cdots$$

are strictly compatible with the induced ${}^p\tau$ filtrations and we have

$H^j(i_v^! \, {}^p\mathcal{H}^i(K)) \simeq Gr_i^{{}^p\tau}\mathbb{H}^{i+j}_{X_v}(X, j_{!*}\mathcal{L})$, $H^j(i_v^* \, {}^p\mathcal{H}^i(K)) \simeq Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(X_v, j_{!*}\mathcal{L})$,

$H^j(B_v^*, {}^p\mathcal{H}^i(K)) \simeq Gr_i^{{}^p\tau}\mathbb{H}^{i+j}(B^*_{X_v}, j_{!*}\mathcal{L})$.

ii) In particular, by lemma 2.28, the following induced morphisms ∂_i^p and γ_i^p on $Gr_*^{{}^p\tau}$ satisfy the relations :

$\partial_i^p : Gr_p^{{}^p\tau}\mathbb{H}^{i-1}(B^*_{X_v}, j_{!*}\mathcal{L}) \to Gr_p^{{}^p\tau}\mathbb{H}^i_{X_v}(X, j_{!*}\mathcal{L})$ is an isomorphism for $p < i-1$ and it is surjective for $p = i-1$;

$\gamma_i^p : Gr_p^{{}^p\tau}\mathbb{H}^i(X_v, j_{!*}\mathcal{L}) \simeq Gr_p^{{}^p\tau}\mathbb{H}^i(B^*_{X_v}, j_{!*}\mathcal{L})$ is an isomorphism for $p > i+1$ and it is injective for $p = i+1$.

The exact sequence is deduced from the following triangle

$$Ri^!_{X_v}j_{!*}\mathcal{L} \xrightarrow{I} i^*_{X_v}j_{!*}\mathcal{L} \longrightarrow i^*_{X_v}Rj_{X_v*}j^*_{X_v}j_{!*}\mathcal{L}$$

The strict compatibility follows from the decomposition theorem.

Proof of the proposition. We deduce first Grauert isomorphism (formula 2.15). The decomposition theorem follows from local purity. We have:

$Gr_r^{{}^p\tau}\mathbb{H}^r_{X_v}(X, j_{!*}\mathcal{L}) \simeq H^0(i_v^! \, {}^p\mathcal{H}^r(K)) \simeq \mathcal{L}_v^r$ and $Gr_r^{{}^p\tau}\mathbb{H}^r(X_v, j_{!*}\mathcal{L}) \simeq H^0(i_v^* \, {}^p\mathcal{H}^r(K)) \simeq \mathcal{L}_v^r$.

Reciprocally, in the exact sequence

$$Gr_0^{{}^p\tau}\mathbb{H}^{-1}_{X_v}(X, j_{!*}\mathcal{L}) \to Gr_0^{{}^p\tau}\mathbb{H}^{-1}(X_v, j_{!*}\mathcal{L}) \to Gr_0^{{}^p\tau}\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}\mathcal{L}) \to$$
$$Gr_0^{{}^p\tau}\mathbb{H}^0_{X_v}(X, j_{!*}\mathcal{L}) \xrightarrow{I} Gr_0^{{}^p\tau}\mathbb{H}^0(X_v, j_{!*}\mathcal{L})$$

where $Gr_0^{{}^p\tau}\mathbb{H}^{-1}_{X_v}(X, j_{!*}\mathcal{L}) = 0$, if $I : Gr_0^{{}^p\tau}\mathbb{H}^{-1}(X_v, j_{!*}\mathcal{L}) \simeq Gr_0^{{}^p\tau}\mathbb{H}^{-1}(B^*_{X_v}, j_{!*}\mathcal{L})$ is an isomorphism, then by compatibility with MHS,

$$0 = Gr_a^W Gr_0^{{}^p\tau}\mathbb{H}^{-1}(X_v, j_{!*}\mathcal{L}) \simeq Gr_a^W Gr_0^{{}^p\tau}\mathbb{H}^{-1}((B^*_{X_v}, j_{!*}\mathcal{L})$$

which proves the crucial case of the local purity theorem.

Remark 2.29. i) From the duality $D : Gr_{-r}^{{}^p\tau}\mathbb{H}^{-r}_{X_v}(X, j_{!*}\mathcal{L}) \xrightarrow{\sim} Gr_r^{{}^p\tau}\mathbb{H}^r(X_v, j_{!*}\mathcal{L})^*$ and Grauert Intersection isomorphism above, we deduce a non degenerate Intersection form

$$Gr_r^{{}^p\tau}\mathbb{H}^r_{X_v}(X, j_{!*}\mathcal{L}) \otimes Gr_{-r}^{{}^p\tau}\mathbb{H}^{-r}_{X_v}(X, j_{!*}\mathcal{L}) \xrightarrow{IP} \mathbb{Q},\ (a, b) \mapsto D(b)(I(a))$$

When X_v is a NCD on a smooth surface and in the case of Intersection cohomology with rational coefficients viewed as homology, Grauert contractibility theorem states that the Intersection form defined on the direct sum of the components of X_v is non degenerate if and only if X_v contracts to a point v on a normal surface ([BPV 84] Theorem 2.1). We refer to ([CaMi 5] section 6.3) for more details in the case of constant coefficients $\mathbb{Q}$ and to ([Na 85] proposition 5.1) for the isolated singularity case.

ii) We remark that the decomposition theorem may be proved directly as in [BBDG 83] without the assumption of local purity. This argument is sketched in ([CaMi 5] lemma 5.1.1), however it is important to state the semi-simplicity of the perverse cohomology there, in the category of pure perverse sheaves and not all perverse sheaves.

3. Fibration by normal crossing divisors

The proof of the decomposition is reduced to the case of a fibration by NCD (definition 1.4) over the strata, in which case the proof along a strata is reduced to the zero dimensional strata by intersecting with a normal section, such that we can rely on logarithmic complexes in all arguments based on Hodge theory.

The proof of the proposition 1.5 is divided into many steps, *first we transform the morphism* f and then, simultaneously, a desingularization $\pi : X' \to X$.

3.0.1. *Thom-Whitney stratification* [GMacP 88]. Let $f : X \to V$ be an algebraic map, and $\mathcal{S} = (S_\alpha)$ a Whitney stratification by a family of strata S_α of V. The subspaces $V_l = \cup_{\dim S_\alpha \leq l} S_\alpha$ form an increasing family of closed algebraic sub-sets of V of dimension $\leq l$, with index $l \leq n$, where n is the dimension of V,

$$V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

The inverse of a sub-space $Z \subset V$, is denoted $X_Z := f^{-1}(Z)$. We are interested in the inverse $X_S = f^{-1}(S)$ of strata S of the stratification $\mathcal{S}$.

Let us recall what is a Thom-Whitney stratification:

Definition 3.1. A Thom-Whitney stratification of f consists of two Whitney stratifications one for V and one for X such that the inverse of each stratum S of V is a union of connected strata of X and the restriction of f to X_S is a topological fibration.

Using Thom-Mather first isotopy theorem ([Mat 12]) and the properties of Whitney stratifications ([GMacP 88]) one can prove the following properties:

- (T) Over a stratum S of V the morphism $f_| : X_S \to S$ induced by f, is a locally trivial topological fibration.
- (W) The link at any point of a stratum is a locally constant topological invariant of the stratum [Mat 12], [LeT 83].

We can suppose that the various strata S of V are algebraic and the restriction of f to a stratum of X_S smooth.

Lemma 3.2. *Let $f : X \to V$ be a projective morphism and Y a closed algebraic strict subset of X containing the singularities of X. There exists a commutative*

diagram:

$$\begin{array}{ccccccccccc} X & \xleftarrow{\pi_1} & X_1 & \cdots & \xleftarrow{\pi_i} & X_i & \cdots & \xleftarrow{\pi_k} & X_k & \xleftarrow{\pi_{k+1}} & X_{k+1} \\ f\downarrow & & f_1\downarrow & & & f_i\downarrow & & & f_k\downarrow & & f_{k+1}\downarrow \\ V & \xleftarrow{id} & V & \cdots & \xleftarrow{id} & V & \cdots & \xleftarrow{id} & V & \xleftarrow{id} & V \end{array}$$

where X_1 is a desingularization of X by modifications over Y such that $Y_1 := \pi_1^{-1}(Y)$ is a NCD, such that there exists $V^1 \subset V$ and inductively a decreasing sequence V^i for $0 < i \leq k$ of closed algebraic subspaces of V of dimension $d_i > 0$ such that for $j \leq i$, the inverse image $f_{i+1}^{-1}(V^j)$ are NCD in the non-singular variety X_{i+1}, and $f_{i+1}^{-1}(V^j) - f_{i+1}^{-1}(V^{j+1})$ is a relative NCD over $V^j - V^{j+1}$.

The morphism π_{i+1} is a modification over $f_i^{-1}(V^i)$ for all $i \geq 1$, obtained by blowing-ups over $f_i^{-1}(V^i)$ in X_i and transforms $f_i^{-1}(V^i)$ into a NCD such that π_{i+1} induces an isomorphism:

$$(\pi_{i+1})_| : (X_{i+1} - f_{i+1}^{-1}(V^i)) \xrightarrow{\sim} X_i - f_i^{-1}(V^i).$$

The X_i are smooth for $i \geq 1$ and $f_{i+1} := f_i \circ \pi_{i+1}$. Moreover, there exists a Whitney stratification $\mathcal{S}^i$ of V adapted to V^j for $j \leq i$, satisfying the following property: the strata of $\mathcal{S}^i$ and of $\mathcal{S}^{i-1}$ coincide outside V^i.
The open subset $\Omega := f^{-1}(V - V^1) \subset X$ is dense in X.
Let $\rho_i := \pi_1 \circ \cdots \circ \pi_i$ for all $i \geq 1$. We can suppose that ρ_i induces an isomorphism on $\rho_i^{-1}(\Omega - Y) \subset X_i$, and that the morphism f_i induces on $\rho_i^{-1}(Y \cap \Omega)$ a fibration by relative NCD over the strata of $V - V^1$ for $i \geq 1$.

Remark 3.3. Let $d_i := \dim V^i$, the union V_{d_i} of strata of dimension $\leq i$ is equal to V^i for $0 < i \leq k$ and $f_{k+1}^{-1}(V_{d_i})$ is a NCD in X_{k+1} fibred by NCD over $V^i - V^{i+1}$.

Moreover, we can suppose that the family of subspaces V^i is maximal in the following sense: the dimension of V^i is $n - i$ for $0 < i \leq k = n$.

We refer to the morphism $f_{k+1} : X_{k+1} \to V$ as an admissible fibration with respect to the family Vi.

Proof of Lemma 3.2. Since f is projective, we can always suppose $V = f(X)$. Let $\pi_1 : X_1 \to X$ be a desingularisation morphism of X with NCD as exceptional divisor such that $Y_1 := \pi_1^{-1}(Y)$, as well as the inverse image of the irreductible components of Y, are sub-NCD in X_1.

Let $f_1 := f \circ \pi_1$. Since X_1 is smooth, there exists an open subset $U \subset V$ such that the restriction of f_1 over U is smooth.

If the dimension of $f(Y)$ is strictly smaller than $n := \dim V$, let $U \subset V - f(Y)$ over which the restriction of f_1 is smooth and let $V^1 := V - U$.

In the case $f(Y) = V$, there exists U such that the restriction of f_1 to $Y_1 \cap f_1^{-1}(U)$ is a fibration by relative NCD over U. Let $V^1 := V - U$, then $d_1 := \dim V^1$ is strictly smaller than $\dim V = n$. We can always choose U, hence V^1, such that $d_1 = n - 1$.

We consider a Thom-Whitney stratification of the morphism $f_1 : X_1 \to V$; in particular the image by f_1 of a stratum of X_1 is a stratum of V. We suppose also that the stratification of X_1 is compatible with the divisor Y_1. Let $\mathcal{S}^0$ denotes the other stratification of V.

We construct the algebraic sub-spaces V^i of V, the morphisms $f_i = X_i \to V$, and $\pi_i : X_i \to X_{i-1}$ in the lemma by descending induction on $\dim V^i$. Let $\mathcal{S}^1$ be a Whitney stratification compatible with V^1 which is a refinement of $\mathcal{S}^0$. Let

$D^1 := f_1^{-1}(V^1)$; we introduce the horizontal divisor

$$Y_h^1 := \overline{Y_1 \cap f_1^{-1}(V - V^1)}.$$

We construct $\pi_2 : X_2 \to X_1$ by blowings-up over D^1 such that the inverse image $D^2 := \pi_2^{-1}(D^1)$ of D^1 and the union $D^2 \cup Y_h^2$ with $Y_h^2 := \pi_2^{-1}(Y_h^1)$ are NCD in the smooth variety X_2. We can do this construction without modification of $X_1 - D^1$ because $Y_h^1 - D^1$ is already a NCD.

Let $f_2 := f_1 \circ \pi_2$, hence $D^2 := f_2^{-1}(V^1)$, the next argument allows us to construct a subspace V^2 such that $D^2 \cap f_2^{-1}(V^1 - V^2)$ is a relative NCD over $V^1 - V^2$.

Lemma 3.4. *Let $f : X \to Z$ be a projective morphism on a non-singular space X, T an algebraic subspace of Z such that $D := f^{-1}(T)$ is a NCD in X as well the inverse image of each irreductible component of T.*

Then, there exists a non singular algebraic subset S_0 in T, such that the dimension of $T - S_0$ is strictly smaller than $\dim T$ *and f is a relative NCD over $T - S_0$.*

Proof. i) Let D_{β_i} denote the components of the divisor D and $D_{\beta_1,\dots,\beta_j}$ the intersection of $D_{\beta_1}, \dots, D_{\beta_j}$. Let $d := \dim T$ be the dimension of T; there exists an open subset $S_{\beta_1,\dots,\beta_j}$ complementary of an algebraic sub-space of T of dimension strictly smaller than d, such that $f^{-1}(S_{\beta_1,\dots,\beta_j}) \cap D_{\beta_1,\dots,\beta_j}$ is either empty, or a topological fibration over $S_{\beta_1,\dots,\beta_j}$. Over the open subset $S_0' := \cap_{\beta_1,\dots,\beta_j} S_{\beta_1,\dots,\beta_j}$, the divisor $D \cap f^{-1}(S_0')$ is a topological fibration. This property is required to obtain the big stratum of a Whitney stratification over which we have a fibration by NCD and which satisfies the assertion (2) of the definition 1.4 above.

ii) Still, we need to check the assertion (3) of the definition 1.4 on a dense open subset of S_0'. Let T_1 be an irreductible component of T of dimension d. Since the inverse image of T_1 is also a NCD contained in D by the hypothesis of the Lemma, there exists a dense open subset S_1' in T_1 with inverse image a sub-NCD of D, by the above hypothesis. As the argument is local, we consider an open affine subset $\mathcal{U}$ of Z with non empty intersection $S_1 := T_1 \cap \mathcal{U} \subset S_1'$ and a projection $q : \mathcal{U} \to \mathbb{C}^d$ whose restriction to S_1 is a finite projection. There exists an open affine dense subset U_1 of $\mathbb{C}^d$ such that $q \circ f$ induces a smooth morphism over U_1.

Considering S_0' defined in i) above, we notice that f induces over $q^{-1}(U_1) \cap S_0'$ a fibration by NCD: indeed, let x be a point in $q^{-1}(U_1) \cap S_0' \cap S_1$. There exists an open neighborhood $\mathcal{U}_x$ of x in $(q^{-1}(U_1) \cap S_0') \subset \mathcal{U}$, small enough such that the restriction of q to $\mathcal{U}_x \cap S_1$ is not ramified on its image.

On the other hand, for all $y \in \mathcal{U}_x$, we have: $f^{-1}(q^{-1}(q(y)) \cap \mathcal{U}_x)$ is smooth, since $q(y) \in U_1$. The dimension of $f^{-1}(q^{-1}(q(y) \cap \mathcal{U}_x)$ is $\dim X - d$. As x is in S_0' the dimension of $f^{-1}(y) \cap D$ is $\dim X - 1 - d$ and it is a divisor with normal crossings in $f^{-1}(q^{-1}(q(y) \cap \mathcal{U}_x)$ which has dimension: $\dim X - d$.

We remark that $q^{-1}(q(x)) \cap \mathcal{U}_x$ is a normal section of $q^{-1}(U_1) \cap S_0'$ at x in Z. Hence f induces on $q^{-1}(U_1) \cap S_0'$ a fibration by NCD lying in the normal sections.

The open subset S_0 is the union of open subsets $q^{-1}(U_1) \cap S_0'$ for the various irreductible components T_1 of T of maximal dimension d. The strata which are the connected component of S_0 satisfy the property (3) of the definition 1.4 and this ends the proof of the lemma 3.4.

End of the proof of Lemma 3.2. Let d_1 be the dimension of V^1. According to lemma 3.4 there exists an open algebraic subset S_0 of V^1, over which the restriction

of f_2 induces a relative NCD over S_0, moreover, the dimension d_2 of the algebraic set $V^2 := V^1 - S_0$ is $d_2 < d_1$ strictly less than d_1. If $d_1 = n-1$, we can always choose S_0 such that $d_2 = n-2$.

We define a new Whitney stratification $\mathcal{S}^2$ of V, compatible with V^2 which coincides with $\mathcal{S}^1$ outside V^1, by keeping the same strata outside V^1 and adding Thom-Whitney strata including the connected components of S_0 just defined and strata in the complement of V^2 in V^1.

We complete the proof by repeating this argument for the closed algebraic subspace V^2 of V^1. This process is the beginning of an inductive argument as follows.

Induction hypothesis: Given projective morphisms $f_j : X_j \to V$ for $1 \le j \le i$ as in the diagram above, and $\pi_j : X_j \to X_{j-1}$, $f_j := f_{j-1} \circ \pi_j$ where X_j is non-singular, a Whitney stratification $\mathcal{S}^i$ of V compatible with a family of algebraic sub-spaces $V^j \subset \cdots \subset V^1 \subset V$ for $j \le i$ such that there exists a Whitney stratification on X_i stratifying f_i, and such that $f_{i-1}^{-1}(V^j)$ are NCD in X_i, as well the inverse image of each irreductible component of V^j for $1 \le j < i$, and moreover f_i induces an admissible morphism over $V - V^i$. Also, let Y_h^i be a divisor of X_i such that $f_i^{-1}(V^1) \cup Y_h^i$ is a NCD and $Y_h^i \cap (X_i - f_i^{-1}(V^1))$ is a relative NCD.

Inductive step: A sequence of blowings-up centered over $f_i^{-1}(V^i)$ leads to the construction of $\pi_{i+1} : X_{i+1} \to X_i$ such that X_{i+1} is non-singular and the inverse images of V^j by $f_i \circ \pi_{i+1}$, as well as its irreductible components for $1 \le j \le i$, and their union with $Y_h^{i+1} := \pi_{i+1}^{-1}(Y_h^i)$ are NCD in X_{i+1}.

Let $f_{i+1} := f_i \circ \pi_{i+1}$. The stratification $\mathcal{S}^i$ of V underlies a Thom-Whitney stratification of f_{i+1}. It follows from lemma 3.4 that in each maximal strata S of V^i in $\mathcal{S}^i$, there exists an open dense subset $S_0^i(S)$ over which π_{i+1} is a relative NCD. Let V_0^i be the union of all S_0^i. The complement $V^i - V_0^i$ of V_0^i is a closed algebraic strict sub-space V^{i+1} of V^i, and f_{i+1} is admissible over $V - V^{i+1}$. We construct a refinement of the stratification $\mathcal{S}^i$ and then a Thom-Whitney stratification $\mathcal{S}^{i+1}$ compatible with V^{i+1}, keeping the same strata outside V^i and introducing as new strata open subsets of the connected components of the open subset V_0^i of V^i, then completeing by a Thom-Whitney stratification of the complement in V^i.

The inductive argument ends when $V^{k+1} = \emptyset$, which occurs after a finite number of steps since the family V^i is decreasing.

Corollary 3.5. *Let $f : X \to V$ be a projective morphism, there exists a desingularization $X \xleftarrow{\pi} X'$ of X such that either π or $\pi \circ f$ is a fibration by NCD over the strata for an adequate Thom-Whitney stratification.*

Moreover, for any algebraic subspace Y containing the singularities of X, we can choose π such that $Y' := \pi^{-1}(Y)$ is a NCD in X'.

Indeed in the lemma 3.2, the morphism f_{k+1} is a fibration by NCD over the strata, moreover we can suppose $k = n$. To construct π as a fibration by NCD, we apply the lemma to the case $f = Id : X \to X$.

We will need the following improved version of lemma 3.2

Lemma 3.6 (Fibrations relative to a subspace)**.** *With the notations of lemma 3.2, let Z be a strict algebraic subspace of V. There exists a diagram*

$$X \xleftarrow{\pi'} X' \xrightarrow{f'} V, \quad f' := f \circ \pi'$$

and a Thom-Whitney stratification of f' compatible with Z such that $f'^{-1}(Z) = \pi'^{-1}(f^{-1}(Z))$ is a relative NCD over the strata in Z.

with an index i such that $Z \subset V^i$, $\dim V^i = \ell$ and $Z' := f_{i+1}^{-1}(Z)$ is a NCD in X_{i+1} relative over $V^i - V^{i+1}$.

proof Let ℓ be the dimension of Z. In the proof of lemma 3.2, we check that we can add a subspace X^i with index i fired by NCD over an open Zariski subset of a subspace $V^i \supset Z$ of dimension ℓ. We can suppose that all sub-spaces V^r of dimension $d_r \geq \ell$ contain Z.

We are interested in V^k and V^{k+1} in the diagram, of dimension d_k and d_{k+1} respectively, such that $d_k \geq \ell > d_{k+1}$. If $d_k = \ell$ we set $i = k+1$. Otherwise, we add the space X_i obtained by blowing up the subspace $f_k^{-1}(V^i := Z \cup V^{k+1}) \subset X_k$. Hence, we add a sub-diagram $X_k \xleftarrow{\pi_i} X_i \xleftarrow{\pi_{k+1}} X_{k+1}$ where X_{k+1} is the blowing up of $f_i^{-1}(V^{k+1}) \subset X_i$ and $f_i := f_k \circ \pi_i$.

Proof of the proposition 1.5 We need to construct a diagram

$$X' \xrightarrow{\pi} X \xrightarrow{f} V$$

such that π and $f \circ \pi$ are simultaneously fibrations by NCD over the strata.

Going back to the inductive argument for f in the lemma 3.2, we apply at each step of the induction the corollary 3.5 to construct π. In particular we start with the desingularization by an admissible modification.

Induction hypothesis. We suppose there exists:

1) A diagram $\mathcal{D}_i$ of morphisms where π'_i is admissible:

$$X \xleftarrow{\pi'_i} X_i \xrightarrow{f_i} V, \quad f_i = f \circ \pi'_i$$

2) A decreasing family of algebraic subspaces $V^i \subset V^j$ for $j < i$ with inverse image $f_i^{-1}(V^j)$ consisting of NCD in X_i for $j < i$, and a stratification $\mathcal{S}^i$ of V compatible with the family V^j, such that the restriction of f_i to $X_i - f_i^{-1}(V^i)$ over $V - V^i$ is a fibration by relative NCD over the strata.

Inductive step. Let d_i (resp. $n-i$) be the dimension of V^i. We want to define a sub-space $V^{i+1} \subset V^i$ of dimension strictly smaller $d_{i+1} < d_i$ (resp. $n-i-1$) and to extend the diagram over the open subset $V^i - V^{i+1}$, that is, to construct a diagram $\mathcal{D}_{i+1}$ of morphisms:

$$X \xleftarrow{\pi'_{i+1}} X_{i+1} \xrightarrow{f_{i+1}} V, \quad f_{i+1} = f \circ \pi'_{i+1}$$

such that:

1) π'_{i+1} is admissible and defined as a composition map $\pi'_{i+1} : X_{i+1} \xrightarrow{\pi_{i+1}} X_i \xrightarrow{\pi'_i} X$ of $\pi'_i : X_i \to X$ with a modification π_{i+1} inducing an isomorphism:

$$X_{i+1} - f_{i+1}^{-1}(V^i) \xrightarrow{\sim} X_i - f_i^{-1}(V^i).$$

2) $f_{i+1}^{-1}(V^i)$ is a relative NCD over the open subset $V^i - V^{i+1}$.

To achieve this step, we remark that the variety $f_i^{-1}(V^i)$ in X_i is over $f^{-1}(V^i)$ in X. Then, we apply the lemma (3.6) to $\pi'_i : X_i \to X$ and $f^{-1}(V^i) \subset X$ to construct the admissible morphism π'_{i+1}

$$f^{-1}(V^i) \subset X \xleftarrow{\pi'_i} X_i \xleftarrow{\pi_{i+1}} X_{i+1}, \quad \pi'_{i+1} := \pi'_i \circ \pi_{i+1} : X_{i+1} \to X$$

That is we develop the constructions of the lemma 3.2 to construct π'_{i+1} over X *by modification only of sub-spaces over* $f_i^{-1}(V^i)$ to transform $f_i^{-1}(V^i)$ into a NCD, hence X_{i+1} in the diagram $\mathcal{D}_{i+1}$ differs from X_i in the diagram $\mathcal{D}_i$ only over

$f_i^{-1}(V^i)$. The stratification of X is modified inside the subset $f^{-1}(V^i)$, hence we modify adequately the stratification of X inside the subset V^i.

There exists an open algebraic subset $U \subset V_i$ in the big strata of V_i over which f_{i+1} is a fibration by NCD (lemma 3.4). Let $V^{i+1} := V_i - U$ of dimension strictly smaller than d_i. Then, the Thom-Whitney stratifications of f and π'_i outside V_i extend over U into Thom-whitney stratifications of f and π'_{i+1} . Hence, we deduce a diagram $\mathcal{D}_{i+1}$ which extend the diagram $\mathcal{D}_i$ such that f_{i+1} (resp. π'_{i+1}) is a fibration by NCD over the strata of $f_{i+1}^{-1}(V^i - V^{i+1})$ (resp. $f^{-1}(V^i - V^{i+1})$.

Thus we complete the construction of the diagram $\mathcal{D}_{i+1}$ and the inductive step. At the end we define π'_m and f_m both admissible for some index m.

Corollary 3.7. *The decomposition theorem for f can be deduced from both cases π' and f' in the proposition 1.5.*

Let $\mathcal{L}$ be a local system on $X - Y$, there exists an open algebraic set $\Omega \subset X - Y$ dense in X such that $\Omega' := \pi'^{-1}(\Omega) \subset X'$ is isomorphic to Ω, which carry the local system $\mathcal{L}$. Let $j' : \Omega' \to X'$, then the decomposition theorem for $j_{!*}\mathcal{L}$ with respect to the orginal proper algebraic morphism f follows from both cases of f' and π' ([De 68] proposition 2.16).

4. Logarithmic complexes

We develop the construction of Hodge theory by logarithmic complexes with coefficients in an admissible graded polarized variation of mixed Hodge structure (VMHS): $(\mathcal{L}, W, F)$ with singularities along a normal crossing divisor Y. We refer to [Ka 86] and [EY 14] for basic computations and to [Ka 86, ICTP 14, StZ 85] for admissibility (definition 4.2 below).

In this section we construct the logarithmic complex $\Omega^*\mathcal{L}$ in the subsection 4.1 and the sub-complex $IC^*\mathcal{L}$ in the subsection 4.3. The theorem 4.3 states the existence of MHS on $\mathbb{H}^*(X - Y, \mathcal{L}, W, F)$. The definition of the weight filtration in section 4.4 and theorem 4.31 is motivated by a long explanation in the local case.

The admissibility on $X^* := X - Y$ refers to asymptotic properties of $(\mathcal{L}, W, F)$ along the NCD. Such asymptotic properties are expressed on Deligne's extension $\mathcal{L}_X$ of $\mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_{X^*}$, defined in terms of the "multivalued" horizontal sections of ∇ on X^*, on which the connection ∇ extends with logarithmic singularities along Y.

The extension $\mathcal{L}_X$ is a locally free analytic sheaf of modules, hence algebraic if X is projective. It is uniquely characterized by the residues of the logarithmic singularities of the connexion ∇, and defined in terms of a choice of the logarithm of the eigenvalues of the monodromy ([De 70] Théorme d'existence Proposition 5.2).

The fibre of the vector bundle $\mathcal{L}(x) := \mathcal{L}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ is viewed as the space of the " multivalued" horizontal sections of $\mathcal{L}$ at x (sections of a universal covering of the complementary of Y in a ball B_x at x).

The extension over Y of the Hodge filtration of a polarizable variation of HS constructed by Schmid [Sc 73, GrSc 73, CaKSc 86] is a fundamental asymptotic property, that is required by assumption in the case of a graded polarizable variation of MHS, as a condition to admissibility.

The local monodromy T_i around a component Y_i of Y, defines a nilpotent endomorphism $N_i := Log\, T_i^u$ logarithm of the unipotent part of T preserving the extension of the filtration W of $\mathcal{L}$ by sub-bundles $\mathcal{W}_X \subset \mathcal{L}_X$.

Deligne pointed out the problem of the existence of the relative monodromy filtration $M(\sum_i N_i, W))$ in ([De 80], I.8.15). The required properties are proved in the case of geometric variation of MHS over a punctured disc in [E 83] and studied axiomatically as conditions of admissibility in [StZ 85]. The definition of admissibility in [Ka 86] along a NCD is by reduction to the case of a punctured disc.

Definition 4.1 (Preadmissible VMHS)**.** A variation of mixed Hodge structure $(\mathcal{L}, W, \mathcal{F})$ graded polarizable over the punctured unit disc D^* with local monodromy T, is called *preadmissible* if the following conditions are satisfied:

i) The Hodge filtration $\mathcal{F} \subset \mathcal{L}_{\mathcal{O}_{D^*}}$ extends to a filtration $\mathcal{F}$ on Deligne's extension $\mathcal{L}_{\mathcal{O}_D}$ by subbundles inducing for each k on $Gr_k^{\mathcal{W}}\mathcal{L}_{\mathcal{O}_D}$, Schmid's extension of the Hodge filtration.

ii) Let $W^0 := \mathcal{W}(0)$, $F_0 := \mathcal{F}(0)$ denote the filtrations on the fiber $L_0 := \mathcal{L}_{\mathcal{O}_D}(0)$ at $0 \in D$ and T the local monodromy at 0, $N = \log T$; then the following two conditions must be satisfied:

1) $NF_0^p \subset F_0^{p-1}$ for all $p \in \mathbb{Z}$.

2) The weight filtration $M(N, W^0)$ relative to W^0 exists.

Notice that the extension of the filtration $\mathcal{F}$ to $\mathcal{L}_{\mathcal{O}_D}$ cannot be deduced in general from the various Schmid extensions to $Gr_k^{\mathcal{W}}\mathcal{L}_{\mathcal{O}_D}$.

We remark that the filtrations $M := M(N, W^0)$ and F_0 at the origin define an MHS.

The admissibility property in the next definition by Kashiwara coincides over D^* with the above definition in the unipotent case (but not the quasi-unipotent case) as proved in [Ka 86].

Definition 4.2 (Admissible VMHS)**.** Let X be a complex analytic space and $U \subset X$ a nonsingular open subset, complement of a closed analytic subset. A graded polarizable variation of mixed Hodge structure $(\mathcal{L}, W, \mathcal{F})$ on U is called *admissible* if for every analytic morphism $f : D \to X$ on a unit disc which maps D^* to U, the inverse $(f_{|D^*})^*(\mathcal{L}, W, \mathcal{F})$ is a preadmissible variation on D^*.

In the case of locally unipotent admissible VMHS, Kashiwara noticed that preadmissible VMHS in the unit disc are necessarily admissible.

We construct below the weight filtration directly on the logarithmic complex (4.1), generalizing the case of constant coefficients in [De 72].

We remark that the construction of the intermediate extension (or the Intersection complex) of a local system on $X - Y$ (4.3.1), as well the development of mixed Hodge theory, involve the behaviour at "infinity", along the NCD Y.

4.0.1. *Notations.* Let $Y := \cup_{i\in I} Y_i$ be a NCD, union of smooth irreducible components with index I, and for $J \subset I$, set $Y_J := \cap_{i\in J} Y_i$, $Y_J^* := Y_J - \cup_{i\notin J}(Y_i \cap Y_J)$ ($Y_\emptyset^* := X^* = X - Y$). We denote uniformly the various embeddings by $j : Y_J^* \to X$.

The local system $\mathcal{L}$ on X^* is defined by a connection ∇ on the fibre bundle $\mathcal{L}_{X^*} = \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_{X^*}$ with horizontal sections given by a vector space L with monodromy action T_j for each $j \in I$. The extension of $\mathcal{L}_{X^*}$ with a regular singular connection is a couple consisting of a fibre bundle $\mathcal{L}_X$ and a connection $\nabla : \mathcal{L}_X \to \Omega_X^1(LogY) \otimes \mathcal{L}_X$ ([De 70], [Ma 87] definition 3.1). The residue of ∇ is defined along a component Y_i of the NCD Y as an endomorphism of the restriction $\mathrm{Res}_{Y_i}\nabla : \mathcal{L}_{Y_i} \to \mathcal{L}_{Y_i}$, of $\mathcal{L}_X$ to Y_i.

The eigenvalues of the residue are constant along a connected component of Y_j and related to the local monodromy T_j of $\mathcal{L}$ at a general point of Y_j by the formula: $Log\, T_j = -2i\pi \mathrm{Res}_{Y_j}\nabla$ ([De 70], theorem 1.17, proposition 3.11).

The construction of $\mathcal{L}_X$ is local. Deligne's idea is to fix the choice of the residues of the connection by the condition that the eigenvalues of the residue belong to the image of a section of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$, determined by fixing the real part of $z : n \leq \mathcal{R}(z) < n+1$, hence fixing the determination of the logarithm $Log : \mathbb{C}^* \to \mathbb{C}$, and forcing the uniqueness of the construction. Hence the local constructions glue into a global bundle. In Deligne's extension case $n = 0$.

Let $X(x) \xrightarrow{\sim} D^{n+l}$ be a neighborhood of a point $x \in Y$ isomorphic to a product of complex discs, such that $X(x)^* = X(x) \cap (X - Y) \xrightarrow{\sim} (D^*)^n \times D^l$ where D^* is the disc with the origin deleted. For simplification, *we often set* $l = 0$ and we suppose $X := D^n$ is a ball in $\mathbb{C}^n$ with Y defined by $y_1 \cdots y_n = 0$. The fundamental group $\pi_1(X(x)^*)$ is a free abelian group generated by n elements representing classes of closed paths around the hypersurface Y_i, defined locally by the equation $y_i = 0$, at a general point of Y_i, one for each index i.

The restriction of the local system $\mathcal{L}$ to $X(x)^*$ corresponds to a representation of $\pi_1(X(x)^*)$ in a vector space L, hence to the action of commuting automorphisms T_i of L for $i \in [1, n]$ indexed by the local components Y_i of Y and called local monodromy action around Y_i.

Classically L is viewed as the fibre of $\mathcal{L}$ at the base point of the fundamental group $\pi_1(X^*)$, however to represent the fibre of Deligne's extended bundle at x, we view L *as the vector space of multivalued horizontal sections of* $\mathcal{L}$, that is the sections of the inverse of $\mathcal{L}$ on a universal cover $\pi : \widetilde{D}^n \to D^n := X(x)^*$ defined for $z = (z_1, \cdots, z_n) \in \widetilde{D}^n \subset \mathbb{C}^n$ by $y = \pi(z)$ with components $y_j := e^{2i\pi z_j}$, then $L := H^0(\widetilde{D}^n, \pi^{-1}\mathcal{L})$.

In the case of a VMHS, we suppose L defined over $\mathbb{Q}$. The automorphisms T_i defined over $\mathbb{Q}$, decompose as a product of semi-simple T_i^s and unipotent T_i^u commuting automorphisms $T_i = T_i^s T_i^u$. On the complex vector space L, T_i^s is diagonalizable and represented over $\mathbb{C}$ by the diagonal matrix of its eigenvalues. The logarithm of T_i is defined as the sum $Log\, T_i = Log\, T_i^s + Log\, T_i^u = D_i + N_i$, where $D_i = Log\, T_i^s$ is diagonalizable over $\mathbb{C}$ with entries $Log\, a_i$ on the diagonal for all eigenvalues a_i of T_i^s and for a fixed determination of Log on $\mathbb{C}^*$, while $N_i := Log\, T_i^u$ is a nilpotent endomorphism, defined by $N_i = -\Sigma_{k\geq 1}(1/k)(I - T_i^u)^k$ as a polynomial function of the nilpotent morphisms $(I - T_i^u)$, where the sum is finite. At a point x on $\cup_{i \in J \subset I} Y_i$ a product of closed paths corresponds to a sum of various N_i.

Locally, at a point $x \in Y$ on the intersection of n-components $Y_i, i \in [1, n]$, a spectral decomposition into a finite direct sum is defined on $L := \mathcal{L}(x)$

$$L = \oplus_{(a.)} L^{a.}, \quad L^{a.} = \cap_{i \in [1,n]} (\cup_{j>0} \ker\, (T_i - a_i I)^j)$$

with sequences of eigenvalues $(a.)$ as indices, one component a_i for each T_i.

4.0.2. For a detailed description of $\mathcal{L}_X$ at $x \in Y$, let $\alpha_j \in [0, 1[$ for $j \in [1, n]$ such that $e^{-2i\pi\alpha_j} = a_j$ is an eigenvalue for T_j, then the fiber $\mathcal{L}_{X,x}$ is generated as an $\mathcal{O}_{X,x}$-submodule of $(j_*\mathcal{L}_{X^*})_x$ by the image of the embedding of the space L of multivalued horizontal sections into $\mathcal{L}_{X,x}$ by the correspondence $v \to \widetilde{v}$, defined for

y near x by

$$\tilde{v}(y) = (exp(\Sigma_{j\in J}(logy_j)(\alpha_j - \frac{1}{2i\pi}N_j))).v = \Pi_{j\in J}{y_j}^{\alpha_j}\exp(\Sigma_{j\in J} - \frac{1}{2i\pi}(logy_j)N_j).v,$$

in ([De 70], 5.2.1- 5.2.3), then $\tilde{v}(y)$ is a uniform analytic section on $X(x)^*$. It is important to stress that a basis v_a of L is sent onto a basis $\tilde{v}_a$ of $\mathcal{L}_{X,x}$, and if X is projective $\mathcal{L}_X$ is an algebraic bundle by Serre's general correspondence. In the text, we omit in general the analytic notations X^{an} as we consider projective varieties. The action of N_i on L determines the connection ([De 70], theorem 1.17, proposition 3.11) as

$$\nabla\tilde{v} = \Sigma_{j\in J}[\widetilde{(\alpha_j v)} - \frac{1}{2i\pi}\widetilde{(N_j v)}] \otimes \frac{dy_j}{y_j}.$$

4.1. **The logarithmic complex $\Omega^*\mathcal{L} := \Omega^*_X(LogY)\otimes\mathcal{L}_X$.**

Let $\tilde{\mathcal{L}}$ be an admissible polarized variation of Hodge structures (VHS) on a smooth open subset Ω of a smooth complex algebraic variety X of dimension m, complement of a NCD Y in X and $j : \Omega \to X$ the embedding. We denote by $\mathcal{L} := \tilde{\mathcal{L}}[m]$ the complex of sheaves reduced to $\tilde{\mathcal{L}}$ in degree $-m$ (*referred to as a shifted VHS*). Let $(\mathcal{L}_X, \nabla)$ be Deligne's (shifted) extension of $\mathcal{L}\otimes\mathcal{O}_\Omega$ with logarithmic singularities. The connection $\nabla : \mathcal{L}_X \to \Omega^1_X(LogY)\otimes\mathcal{L}_X$ extends naturally into a complex, called the (shifted by m) logarithmic complex $\Omega^*_X(LogY)\otimes\mathcal{L}_X$.

When $\mathcal{L}$ underlies a variation of MHS $(\mathcal{L}, W, F)$, the filtration by sub-local systems W of $\mathcal{L}$ extends as a filtration by canonical sub-analytic bundles $\mathcal{W}_X \subset \mathcal{L}_X$. By the condition of admissibility the filtration $\mathcal{F}_U$ extends by sub-bundles $\mathcal{F}_X \subset \mathcal{L}_X$. Both $\mathcal{W}_X$ and $\mathcal{F}_X$ are combined here to define the structure of mixed Hodge complex.

Theorem 4.3. *Let $\mathcal{L}$ be a shifted admissible graded polarized variation of MHS on $X - Y$. There exists a weight filtration W on the logarithmic complex with coefficients $\mathcal{L}_X$ by perverse sheaves, and a Hodge filtration F by complexes of analytic sub-sheaves such that the bi-filtered complex*

$$\Omega^*\mathcal{L} := (\Omega^*_X(LogY)\otimes\mathcal{L}_X, W, F) \tag{4.1}$$

underly a structure of mixed Hodge complex and induces a canonical MHS on the cohomology groups $\mathbb{H}^i(X - Y, \mathcal{L})$.

The weight is defined by constructible sub-complexes, although it consists in each degree, of analytic sub-sheaves of $\Omega^i_X(LogY)\otimes\mathcal{L}_X$.

The filtration F is classically deduced on the logarithmic complex from the sub-bundles $\mathcal{F}^p_X$ in $\mathcal{L}_X$ satisfying Griffith's transversality:

$$F^p = 0 \to \mathcal{F}^p\mathcal{L}_X \cdots \to \Omega^i_X(LogY)\otimes\mathcal{F}^{p-i}\mathcal{L}_X \to \cdots$$

In the rest of this section the direct definition of the weight filtration W as well its properties in the case of a NCD is based on the local study in [Ka 86] *and* [EY 14].

4.2. **The direct image $Rj_*\mathcal{L} \simeq \Omega^*\mathcal{L}$.** To represent the complex $Rj_*\mathcal{L}\otimes\mathbb{C}$ in the derived category, we use its de Rham realization $\Omega^*\mathcal{L} := \Omega^*_X(LogY)\otimes\mathcal{L}_X$. Indeed, the quasi-isomorphism $Rj_*\mathcal{L} \xrightarrow{\sim} \Omega^*\mathcal{L}$ follows from Grothendieck's algebraic de Rham cohomology [Gro 66] and its generalization to local systems by Deligne ([De 70], definition 3.1).

We also describe a sub-complex $IC^*\mathcal{L}$ representing the intermediate extension [GMacP 83, Br 82, BBDG 83]. Various related definitions given here in terms of local coordinates are independent of the choice of coordinates. This approach is fit for calculus.

4.2.1. *The (higher) direct image $Rj_*\mathcal{L}$.* The residue of the connection ∇ on the analytic restriction $\mathcal{L}_{Y_i}$ of $\mathcal{L}_X$ decomposes into Jordan sum $D_i - \frac{1}{2i\pi}N_i$ where N_i is nilpotent and D_i diagonal with eigenvalues $\alpha_i \in [0,1[$ such that the eigenvalues of the monodromy T_j are $a_j = \mathbf{e}(\alpha_j) := e^{-2i\pi\alpha_j}$. The de Rham complex with coefficients $\mathcal{L}_X$, is quasi-isomorphic to $Rj_*\mathcal{L}$ ([De 70], section II.3):

$$Rj_*\mathcal{L} \xrightarrow{\sim} \Omega^*\mathcal{L} := \Omega^*_X(LogY) \otimes \mathcal{L}_X \tag{4.2}$$

Let $m = n + l$, in the local situation (4.0.1) with Y defined near a point x at the origin in D^m by $y_1 \cdots y_n = 0$, the fiber of the complex $Rj_*\mathcal{L}$ is quasi-isomorphic to a Koszul complex. We associate to the component of the spectral decomposition, a strict simplicial vector space $(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i), i \in [1,n])$ such that for all sequences $(i.) = (i_1 < \cdots < i_p)$: $L(\alpha., i.) = L^{\mathbf{e}(\alpha.)}$, $\alpha_{i_j} Id - \frac{1}{2i\pi}N_{i_j} : L(\alpha., i. - i_j) \to L(\alpha., i.)$.

The Koszul complex is the sum of this simplicial vector space; it is denoted by $s(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i)_{i\in[1,n]}$.

Definition 4.4. The direct sum of the complexes $s(L^{\mathbf{e}(\alpha.)}, \alpha_j Id - \frac{1}{2i\pi}N_j)_{j\subset[1,n]}$ over all sequences $(\alpha.)$ is denoted by $s(L, \alpha.Id - \frac{1}{2i\pi}N.)$.
It is also denoted as an exterior algebra

$$\Omega^*L := \Omega(L, \alpha.Id - \frac{1}{2i\pi}N.) = \oplus_{\alpha.}\Omega(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i), i \in [1,n]). \tag{4.3}$$

where $\mathbf{e}(\alpha_j) = e^{-2i\pi\alpha_j} = a_j$ is an eigenvalue for T_j.

4.2.2. *The tilda embedding.* For $M \subset I$ of length $n := |M|$ and $x \in Y^*_M$, *the above correspondence $v \mapsto \tilde{v}$, from L to $\mathcal{L}_{X,x}$, extends to an embedding*:

$$L(i_1, \ldots, i_j) \to (\Omega^*_X(LogY) \otimes \mathcal{L}_X)_x \text{ by } v \mapsto \tilde{v}\frac{dy_{i_1}}{y_{i_1}} \wedge \ldots \wedge \frac{dy_{i_j}}{y_{i_j}}.$$

It induces quasi-isomorphisms

$$(Rj_*\mathcal{L})_x \cong (\Omega^*_X(LogY) \otimes \mathcal{L}_X)_x \cong \Omega^*L \cong s(L, \alpha_. - \frac{1}{2i\pi}N.). \tag{4.4}$$

The endomorphisms $\alpha_j Id$ and N_j correspond to endomorphisms denoted by the symbols $\alpha_j Id$ and $\mathcal{N}_j$ on the image sections $\widetilde{v}$ in $\mathcal{L}_X)_x$. We recall below a proof of the following result: the subcomplex $\Omega(L^{\mathbf{e}(\alpha.)}, \alpha_i Id - \frac{1}{2i\pi}N_i), i \in [1,n])$ is acyclic if there exists an index i such that $\alpha_i \neq 0$.

Proposition 4.5. *Let $M \subset I$, $x \in Y^*_M$ be a general smooth point of $\cap_{i\in M}Y_i$, $L := \mathcal{L}_X(x)$, and $L^u \subset L$ the subspace on which the action of each monodromy T_i is unipotent. The complex of $\mathcal{O}_{X,x}$-modules $(\Omega^*\mathcal{L})_x$ is quasi-isomorphic to the complex of exterior algebra Ω^*L^u defined by (L^u, T_i):*

$$\Omega^*L^u := \left(0 \to L^u \to \cdots \to \oplus_{\{i_1<\ldots<i_{k-1}\}}L^u \to \oplus_{\{i_1<\ldots<i_k\}}L^u \cdots \to 0\right) \simeq (\Omega^*\mathcal{L})_x$$

with differential in degree $k-1$: $(d_{k-1}v_\bullet)_{\{i_1<\ldots<i_k\}}) = \sum_j (-1)^j \frac{N_{i_j}}{2i\pi} v_{\{i_1<\ldots\hat{i}_j\ldots<i_k\}}$.

Proof. We show that for each power of the maximal ideal $\mathfrak{m}^r$ at x with $r > 0$, the subcomplex $\mathfrak{m}^r(\Omega^*\mathcal{L})_x$ is acyclic, and for $r = 0$ only the component $\Omega^* L^u$ is not acyclic. As x is at the intersection of n components of Y, we associate to each monomial $y^{m\cdot} := y^{m_1} \cdots y^{m_n}$ of degree r a complex vector subspace $sL \subset \mathcal{L}_x$ by the correspondence $v \to y^{m\cdot}\tilde{v}$ (4.2.2) to get a complex of vector subspaces $\Omega^* L y^{m\cdot}$ of $\mathfrak{m}^r(\Omega^*\mathcal{L})_x$

$$\Omega^* L y^{m\cdot} := \left(0 \to sL \to \cdots \to \oplus_{\{i_1<\ldots<i_k\}} y^{m\cdot} L \frac{dy_{i_1}}{y_{i_1}} \wedge \cdots \wedge \frac{dy_{i_k}}{y_{i_k}} \cdots \to 0\right) \tag{4.5}$$

with differential defined by: $y^{m\cdot}\widetilde{v} \to y^{m\cdot}([(m_i+\alpha_i)Id - \frac{1}{2i\pi}N_i].\widetilde{v}) \otimes \frac{dy_i}{y_i}$. The complex $\mathfrak{m}^r(\Omega^*\mathcal{L})_x$ is an inductive limit of the sub-complexes $\Omega^* L y^{m\cdot}$ for various sections $y^{m\cdot}$ such that $\sum_i m_i \leq r$. We show in particular that each complex $\Omega^* L y^{m\cdot}$ is acyclic for $r > 0$.

Lemma 4.6. *For each monomial $y^{m\cdot} := y^{m_1} \cdots y^{m_n}$, the sub-complex $\Omega^* L y^{m\cdot}$ of $(\Omega^*\mathcal{L})_x$ (4.5) is acyclic if the degree $r := \sum_{i\leq n} m_i$ of $y^{m\cdot}$ is strictly positive, hence it is quasi-isomorphic to the complex*

$$\Omega^* L := \left(0 \to L \to \oplus_{i\in[1,n]} L \to \cdots \to \oplus_{\{i_1<\ldots<i_k\}} L \cdots \to 0\right)$$

with differential in degree $k-1$:

$$\forall v_\bullet,\ d_{k-1}(v_\bullet)_{\{i_1<\ldots<i_k\}} = \sum_j (-1)^j ((m_{i_j} + \alpha_{i_j})Id - \tfrac{1}{2i\pi}N_{i_j}) v_{\{i_1<\ldots\hat{i}_j\ldots<i_k\}}.$$

Since $\nabla y^{m\cdot}\tilde{v} = \sum_i m_i y^{m\cdot}\tilde{v} \otimes \frac{dy_i}{y_i} + y^{m\cdot}\Sigma_i[\widetilde{\alpha_i v} - \frac{1}{2i\pi}\widetilde{N_i v}] \otimes \frac{dy_i}{y_i} = y^{m\cdot}[(m_i + \alpha_i)Id - \frac{1}{2i\pi}N_i]\widetilde{v} \otimes \frac{dy_i}{y_i}$

The complex $\Omega^* L y^{m\cdot}$ is isomorphic to the complex $\Omega^* L$ in the lemma where the differentials appear as given by morphisms $(m_i+\alpha_i)Id - \frac{1}{2i\pi}N_i : L \to L$, hence it is acyclic if one of such morphisms is an isomorphism, that is at least one $m_j + \alpha_j \neq 0$ in which case $(m_j + \alpha_j)Id - \frac{1}{2i\pi}N_j$ is an isomorphism of L as N_j is nilpotent; indeed, the complex may be written as a cone over such morphism. We deduce the proposition as $\alpha_j \notin \mathbb{Z}$ unless $\alpha_j = 0$, then $m_j + \alpha_j = 0$ iff $m_j = 0$ and $\alpha_j = 0$. □

Remark 4.7 (Reduction to the locally unipotent case)**.** It follows from the proposition that the cohomology of the restriction to Y is determined locally by the unipotent subspace under the monodromy actions. This is a good reason to reduce the study of the weight filtration to local systems with locally unipotent monodromy.

In particular, if $Z \subset Y$ is a component with non locally unipotent monodromy (the monodromy T_Z around Z has no eigenvalue equal to 1), then $Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z}) = j_{Z!}(j_{!*}\mathcal{L}_{|X-Z}) = j_{!*}\mathcal{L}$, where $j_Z : X - Z \to X$ denotes the open embedding. In general, only the locally unipotent summand of $\mathcal{L}$ is interesting.

The above description of $(Rj_*\mathcal{L})_x$ is the model for the next description of various perverse sheaves in the rest of the section.

4.3. **The intermediate extension $j_{!*}\mathcal{L} \simeq IC^*\mathcal{L}$.** We describe the intermediate extension $j_{!*}\mathcal{L}$ at a point at "infinity" along the NCD Y, by a sub-complex $IC^*\mathcal{L} \subset \Omega^*\mathcal{L}$ containing the submodule $\Omega^*_X \otimes \mathcal{L}_X$.

The NCD Y is defined by the ideal $\mathcal{I}_Y$, product of the ideals of the components Y_i. The complex of $IC^*\mathcal{L}$ contains the product $\mathcal{I}_Y\Omega^*\mathcal{L}$ as an acyclic sub-complex by lemma 4.6. The quotient complex $\Omega^*\mathcal{L}/IC^*\mathcal{L}$ is supported by Y. We state the results for locally unipotent local system to simplify the notations (remark 4.7).

4.3.1. *Definition of $IC^*\mathcal{L}$ for a locally unipotent local system.* In the local situation (4.0.1), at a general point $x \in Y_M^*$ of the intersection of Y_i for $i \in M$, we set for all $K = \{i_1, \dots, i_k\} \subset M = [1, n]$, $A := \mathcal{O}_x \xrightarrow{\sim} \mathbb{C}\{\{x\}\}$ denote the analytic local ring, $y_K := y_{i_1} \cdots y_{i_k}$ and $dy_K := dy_{i_1} \wedge \cdots \wedge dy_{i_k}$. Let $\widetilde{L}$ denote the vector space defined by sections $\widetilde{v}$ for all flat vectors $v \in L$, generating the fiber $\mathcal{L}_{X,x} := A\widetilde{L}$ of the sheaf $\mathcal{L}_X$ at x. The fiber $(\Omega^*\mathcal{L})_x$ of the sheaf $\Omega^*\mathcal{L}$ at x, is generated as an $(\Omega_X^*)_x$-module by $\sum_{K\subset M} A\widetilde{L}\frac{dy_K}{y_K}$.

Let $N_J = \Pi_{j\in J} N_j$ denotes a composition of endomorphisms of L, and consider the strict simplicial sub-complex of the de Rham logarithmic complex (4.1) defined by $ImN_J := N_J L$ in $L(J) = L$.

More generally, each subspace $V \subset L$, defines a subspace $\widetilde{V} \subset \widetilde{L}$ generating an A-submodule of the fiber at x; for $K, B \subset M$ we are interested in the generating subspaces $V = N_K L$ image of N_K and $V = \operatorname{Ker} N_B$ the kernel N_B.

Definition 4.8 (Stalk of the Intersection complex $(IC^*\mathcal{L})_x$)**.**

Let $x \in Y_M^*$ be a general point in Y_M and $A := \mathcal{O}_x$. The sub-complex $(IC^*\mathcal{L})_x \subset (\Omega^*\mathcal{L})_x$ is generated as an $\Omega^*_{X,x}$-algebra by the $\mathcal{O}_x$-sub-modules:

$$\sum_{B\subset K\subset M} y_B A(\widetilde{N_{K-B}L})\frac{dy_K}{y_K}$$

sum for all $K \subset M$ and for each K for all $B \subset K$.

As an $\mathcal{O}_{X,x}$-module, $(IC^*\mathcal{L})_x$ is generated by $\sum_{B\subset K} y_B A(\widetilde{N_{K-B}L})\frac{dy_K}{y_K} \wedge \Omega^*_{X,x}$. By reduction to a normal section to Y_M at x in the proof, we skip $\Omega^*_{X,x}$.

Lemma 4.9 (Independence of the coordinates)**.** *The definition of $(IC^*\mathcal{L})_x$ is independent of the choice of the coordinates defining Y.*

Proof. We check the independence, after restriction to a normal section, for a one variable change at a time: $z_j = fy_j$ for $j \in K$ with f invertible at x. We can suppose $j = 1$, and we write $IC^*L(z_1)$ (resp. $IC^*L(y_1)$) for the stalk at x of the complex when defined with the coordinate z_1 and y_j for $j > 1$ (resp. with the coordinate y_1 instead of z_1). We prove $IC^*L(z_1) \subset IC^*L(y_1)$.

For $K \subset M = [1, n]$ of length $|K| = k$, and $K_i \subset K$ of length $|K_i| = k - i$, since $\frac{dz_1}{z_1} = \frac{dy_1}{y_1} + \frac{df}{f}$, where $\frac{df}{f}$ is regular at x, a section $s \in z_{K_i} A(\widetilde{N_{K-K_i}L})\frac{dz_K}{z_K} \subset IC^*L(z_1)$ is transformed into a sum

$$s = w_1 + w_2, \quad w_1 \in IC^*L(y_1), \quad w_2 \in y_{K_i} A(\widetilde{N_{K-K_i}L})\ \frac{df}{f} \wedge_{j\in K-\{1\}} \frac{dy_j}{y_j}$$

to show that w_2 is also in $IC^*L(y_1)$, it is enough to check that $N_{K-K_i}L \subset N_{K-1-K_i}L$ if $1 \notin K_i$, as $\frac{df}{f}$ is regular. The proof of $IC^*L(y_1) \subset IC^*L(z_1)$ is similar. □

The following proposition is admitted and well known.

Proposition 4.10 (IC^*L)**.** *The intersection complex $(IC^*\mathcal{L})_x$ at a point $x \in Y_M^*$, for a locally unipotent local system $\mathcal{L}$, is quasi-isomorphic to the complex:*

(4.6)

$$IC^*L := \left(0 \to L \to \oplus_{i\in[1,n]} N_i L \cdots \to \oplus_{\{i_1<\ldots<i_k\subset[1,n]\}} N_{\{i_1<\ldots<i_k\}} L \cdots \to 0\right)$$

with differentials induced by the embedding into Ω^*L.

4.3.2. *Local definition of* $(IC^*\mathcal{L})_x$ *for a general local system.* We introduce in terms of the spectral decomposition of L, for each set $\alpha.$, the composition of endomorphisms of $L^{\mathbf{e}(\alpha.)}$: $(\alpha.Id - \frac{1}{2i\pi}N.)_J = \Pi_{j\in J}(\alpha_j Id - \frac{1}{2i\pi}N_j)$.

The strict simplicial sub-complex of the de Rham logarithmic complex (4.1) is defined by $Im(\alpha.Id - \frac{1}{2i\pi}N.)_J$ in $L^{\mathbf{e}(\alpha.)}(J) = L^{\mathbf{e}(\alpha.)}$.

Definition 4.11 (IC^*L)**.** The simple complex defined by the above simplicial sub-vector spaces is the intersection complex $IC^*L := \oplus_{\alpha.} IC^*L^{\mathbf{e}(\alpha.)}$ where

$$IC^*L^{\mathbf{e}(\alpha.)}: = s((\alpha.Id - \frac{1}{2i\pi}N.)_J L^{\mathbf{e}(\alpha.)}, \alpha.Id - \frac{1}{2i\pi}N.)_{J\subset[1,n]}. \tag{4.7}$$

We introduce for each set $\alpha.$ the subset $M(\alpha.) \subset [1,n]$ such that $j \in M(\alpha.)$ if and only if $\alpha_j = 0$. Let $N_{J\cap M(\alpha.)} = \Pi_{j\in J\cap M(\alpha.)} N_j$ (it is the identity if $J \cap M(\alpha.) = \emptyset$). For each $J \subset [1,n]$, we have the equality of the image subspaces: $(\alpha.Id - \frac{1}{2i\pi}N.)_J L^{\mathbf{e}(\alpha.)} = N_{J\cap M(\alpha.)} L^{\mathbf{e}(\alpha.)}$ since the endomorphism $(\alpha_j Id - \frac{1}{2i\pi}N_j)$ is an isomorphism on $L^{\mathbf{e}(\alpha.)}$ whenever $\alpha_j \neq 0$, hence

$$IC^*L \simeq \oplus_{\alpha.} s(N_{J\cap M(\alpha.)} L^{\mathbf{e}(\alpha.)})_{J\subset M} \tag{4.8}$$

Lemma 4.12. *Let L^u denote the subspace of L such that all local monodromies are unipotent, then we have quasi-isomorphisms $IC^*L^u \simeq IC^*L \simeq (IC^*\mathcal{L})_x$.*

Indeed, if there exists an index k such that $\alpha_k \neq 0$, then $IC^*L^{\mathbf{e}(\alpha.)}$ may be written as a cone over the quasi-isomorphism
$\alpha_k Id - \frac{1}{2i\pi}N_k : s(N_{J\cap M(\alpha.)} L^{\mathbf{e}(\alpha.)})_{J\subset M-k} \to s(N_{J\cap M(\alpha.)} L^{\mathbf{e}(\alpha.)})_{J\subset M-k}$.

4.3.3. *Global definition of the Intersection complex.* The local definition of IC^*L is well adapted to computations. As it is independent of the local coordinates defining Y (4.9), we deduce a global defintion of the Intersection complex as follows.

A filtration W of $\mathcal{L}$ by sub-local systems, extends to a filtration of $\mathcal{L}_X$ by sub-bundles $\mathcal{W}_X$. For all subsets M of I, the decomposition of the restriction to Y_M is global: $\mathcal{L}_{Y_M} = \oplus_{(\alpha.)} \mathcal{L}_{Y_M}^{\mathbf{e}(\alpha.)}$. The endomorphisms $\mathcal{N}_i$ for i in M are defined on $\mathcal{L}_{Y_M}$, and $\mathcal{N}_M = \Pi_{i\in M}\mathcal{N}_i$ is compatible with the decomposition. The image $\mathcal{N}_M\mathcal{L}_{Y_M}$ is an analytic sub-bundle of $\mathcal{L}_{Y_M}$. The residue of the connection ∇ along each Y_j defines an endomorphism $\alpha_j Id - \frac{1}{2i\pi}\mathcal{N}_j$ on the component $\mathcal{L}_{Y_j}^{\mathbf{e}(\alpha.)}$ of $\mathcal{L}_{Y_j}$ compatible with the filtration by sub-analytic bundles $\mathcal{W}_{Y_j}$. For subsets $J \subset I$ of the set I of indices of the components of Y, let:

$$(\alpha.Id - \frac{1}{2i\pi}\mathcal{N}.)_J := (\alpha_{i_1}Id - \frac{1}{2i\pi}\mathcal{N}_{i_1})\cdots(\alpha_{i_j}Id - \frac{1}{2i\pi}\mathcal{N}_{i_j}) : \mathcal{L}_{Y_J}^{\mathbf{e}(\alpha.)} \to \mathcal{L}_{Y_J}^{\mathbf{e}(\alpha.)}$$

Definition 4.13. The Intersection complex $IC^*\mathcal{L}$ is the sub-analytic complex of $\Omega_X^*(LogY)\otimes\mathcal{L}_X \supset \Omega_X^*\otimes\mathcal{L}_X$ whose fibre at a point $x \in Y_M^*$ is defined, in terms of a set of coordinates y_i defining equations of Y_M for $i \in M$, as an $\Omega_{X,x}^*$ sub-module generated by the sections $\widetilde{v} \wedge_{j\in J} \frac{dy_j}{y_j}$ for all $v \in$ Im $(\alpha.Id - \frac{1}{2i\pi}N.)_J \subset \oplus_{\alpha.} L^{\mathbf{e}(\alpha.)}$ and $J \subset M$ where $\widetilde{v} \in \mathcal{L}_{X,x}$ and $L = \mathcal{L}_X(x)$.

This definition is independent of the choice of coordinates (lemma 4.9) and a section of $IC^*\mathcal{L}$ restricts to a section in $IC^*\mathcal{L}$ near x, since $(\alpha.Id - \frac{1}{2i\pi}N.)_J L \subset (\alpha.Id - \frac{1}{2i\pi}N.)_{J-i} L$ for all $i \in J$.

For example in the unipotent case, for $M = \{1,2\}$ and $x \in Y_M^*$, the sections in $(IC^*\mathcal{L})_x \subset (\Omega_X^2(LogY)\otimes\mathcal{L}_X)_x$ are generated by $\tilde{v}\frac{dy_1}{y_1}\wedge\frac{dy_2}{y_2}$ for $v \in N_M L$, $\tilde{v}\frac{dy_1}{y_1}\wedge dy_2$ for $v \in N_1 L$, $\tilde{v}dy_1 \wedge \frac{dy_2}{y_2}$ for $v \in N_2 L$ and $\tilde{v}dy_1 \wedge dy_2$ for $v \in L$.

We remark that the notation $IC(X, \mathcal{L})$ instead of $IC^*\mathcal{L}$ is used in ([ICTP 14], chapter 8). The intermediate extension $j_{!*}\mathcal{L}$ of $\mathcal{L}$ is topologically defined by an explicit formula in terms of the stratification ([Br 82], §3). Locally its fiber $(j_{!*}\mathcal{L})_x$ at a point $x \in Y_M^*$ is quasi-isomorphic to the above complex IC^*L.

4.4. **Weight filtration.** The description of the weight filtration on the logarithmic complex is based on the work of Kashiwara [Ka 86] and [EY 14]. Although the declared purpose of the paper [Ka 86] is a criteria of admissibility in codimension one, various local statements and proofs in [Ka 86] are useful to define the weight filtration ([ICTP 14], ch.8).

We start with a graded polarized variation of MHS on the underlying shifted local system $\mathcal{L}$ on the complement of a NCD Y, satisfying certain asymptotic properties at points of Y, summarized under the condition of the admissibility property assuming as axioms the properties of the asymptotic behavior of variation of MHS of geometric origin. In particular we ask for the filtration F on $\mathcal{L}_{X^*}$ to extend into a filtration of $\mathcal{L}_X$ by sub-bundles F, and the existence of the monodromy filtration relative to the natural extension of the filtration W along Y.

Let $x \in Y$ at the intersection of n locally irreducible components with local monodromy action T_i, and let $L := \mathcal{L}_X(x) = (\mathcal{L}_X)_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ denote the space of "multiform" sections of $\mathcal{L}$ (to emphasize this point the notation $L := \psi_{x_1} \cdots \psi_{x_n}\mathcal{L}$ is also used). The conditions of admissibility, stated in terms of the extension of F and the nilpotent endomorphisms $N_i := LogT_i^u$ logarithm of the unipotent monodromy T_i^u, are defined by the structure of Infinitesimal mixed Hodge structure (IMHS) on L, recalled below.

4.4.1. *Mixed nilpotent orbit and Infinitesimal mixed Hodge structure (IMHS).* We deduce from a variation of MHS on the complement of a NCD Y, locally at a point $x \in Y$, the data: $(L, W, F, N_1, \ldots, N_n)$ with an increasing filtration W s.t. $N_jW_k \subset W_k$, a Hodge filtration F and its conjugate $\overline{F}$ with respect to the rational structure inherited from $\mathcal{L}$. It is called here mixed nilpotent orbit (pre-infinitesimal mixed Hodge module in ([Ka 86], 4.2).

The filtration W is an extension of the weight W on $\mathcal{L}$. We may call it the finite filtration in opposition to the relative monodromy filtration to be introduced later called the limit filtration.

Nilpotent orbit. We consider an integer w and a data $(L, F, S; N_1, \ldots, N_n)$ where L is a finite dimensional complex vector space with a $\mathbb{Q}$-structure, F is a decreasing $\mathbb{C}$-filtration, such that: $N_jF^p \subset F^{p-1}$, S is a non-degenerate rational bilinear form satisfying

$$S(x, y) = (-1)^w S(y, x) \quad \text{for}\, x, y \in L \quad \text{and} \quad S(F^p, F^q)) = 0 \text{ for } p + q > w.$$

and N_i are $\mathbb{Q}$-nilpotent endomorphisms for all i, mutually commuting, such that $S(N_ix, y) = S(x, N_iy)$ and $N_iF^p \subset F^{p-1}$.

Definition 4.14 (Nilpotent orbit)**.** ([Ka 86], 4.1) The above data is called a (polarized) nilpotent orbit of weight w if :

i) The monodromy filtration M of the nilpotent endomorphism $N = \sum_j t_jN_j$ does not depend on the various t_j for $t_j > 0$ and all j.

ii) The data (L, M, F) is a MHS on L of weight w (a MHS (L, M, F) is of weight w, or shifted by w, if (Gr_k^M, F) is a HS of weight $w + k$). The bilinear form S_k

such that $S_k(x,y) = S(x, N^k y)$ polarizes the primitive subspace $P_k = \mathrm{Ker}(N^{k+1} : Gr_k^M \to Gr_{-k-2}^M)$ with its induced HS of weight $w+k$.

Henceforth, all nilpotent orbits are polarized.

Remark 4.15 (Nilpotent orbit)**.** i) As N_i acts on F, the exponential morphism $e^{i\sum t_j N_j}$ acts on F for $t_i \in \mathbb{R}$. The nilpotent orbit theorem proved by Schmid states that the above definition is equivalent to the statement:

There exists $c > 0$ such that $(L, e^{i\sum t_j N_j} F, e^{-i\sum t_j N_j}\overline{F})$ is a Hodge structure of weight w polarized by S for $t > c$.

The HS is viewed as variation of HS for variable t_i and M is called the limit MHS.

ii) On this formula, it is clear that for each subset of indices $K \subset [1,n]$, the filtrations $F_K := e^{i\sum_{j\in K} t_j N_j} F, \overline{F}_K := e^{-i\sum_{j\in K} t_j N_j}\overline{F}$ define a variation of HS on L such that $N_i F_K^p \subset F_K^{p-1}$ with $i \notin K$. Then $(L, W(\sum_{i\notin K} N_i), F_K)$ is a MHS for fixed $t_j, j \in K$.

Example. In the case of a polarized variation of HS of pure weight w on the product n-times of a small punctured disc D^*, the limit Hodge filtration exists on the finite dimensional vector space $L := \psi_{x_1}\cdots\psi_{x_n}\mathcal{L}$. It defines with the monodromy filtration M a nilpotent orbit approximating the variation of HS ([CaK 82], [CaKSc 86]).

Mixed nilpotent orbit. We consider a data $(L, W, F, N_1, \dots, N_n)$, where L is a finite dimensional complex vector space with a $\mathbb{Q}$-structure, N_i are $\mathbb{Q}$-nilpotent endomorphisms for all i, W (resp. F) is an increasing $\mathbb{Q}$-filtration (resp. decreasing $\mathbb{C}$-filtration), such that: $N_j F^p \subset F^{p-1}$ and $N_j W_k \subset W_k$.

Definition 4.16 (Mixed nilpotent orbit)**.** The above data is called a mixed nilpotent orbit (graded polarized) if the data with restricted structures

$$(Gr_i^W L, F_|, (N_1)_|, \dots, (N_n)_|)$$

is a nilpotent orbit for each integer i, of weight i with some polarization S_i (it is called pre-infinitesimal mixed Hodge module in ([Ka 86], 4.2).

Definition 4.17 (IMHS)**.** ([Ka 86], 4.3)

A mixed nilpotent orbit $(L, W, F, N_1, \dots, N_n)$ is called an infinitesimal mixed Hodge structure (IMHS) if the following conditions are satisfied:
i) For each $J \subset I = \{1, \dots, n\}$, the monodromy filtration $M(J)$ of $\sum_{j\in J} N_j$ relative to W exists and satisfy $N_j M_i(J) \subset M_{i-2}(J)$ for all $j \in J$ and $i \in \mathbb{Z}$.
ii) The filtrations $M(I), F$ define a graded polarized MHS. The filtrations W and $M(J)$ are compatible with the MHS and the morphisms N_i are of type $(-1,-1)$.

IMHS are called IMHM in [Ka 86]; Deligne remarked, the fact that if the relative monodromy filtration $M(\sum_{i\in I} N_i, W)$ exists in the case of a mixed nilpotent orbit, then it is necessarily the weight filtration of a MHS.

Kashiwara proves in ([Ka 86], theorem 4.4.1) the main theorem: it is enough to check the existence of $W(N_i)$ (in cxdimension 1). Moreover, IMHS form an abelian category for which the filtrations W, F, and M are strict ([Ka 86], prop. 5.2.6).

The morphism N_i is not an auto-morphism of IMHS as it shifts F but not W. However, when combined with the descent lemma (lemma 4.25) it leads to an important application below (subsection 4.4.5).

4.4.2. **Definition of the filtration** $N * W$. ([Ka 86], 3.4)

Let (L, W, N) denote an increasing filtration W on a vector space L with a nilpotent endomorphism N compatible with W s.t. the relative monodromy filtration $M(N, W)$ exists. Then a new filtration $N * W$ of L is defined by the formula ([Ka 86], 3.4)

$$(N * W)_k := NW_{k+1} + M_k(N, W) \cap W_k = NW_{k+1} + M_k(N, W) \cap W_{k+1} \tag{4.9}$$

where the last equality follows from ([Ka 86], Prop 3.4.1).
The endomorphism $N : L \to L$ and the identity I on L induce morphisms

$$N : W_k \to (N * W)_{k-1}, \quad I : (N * W)_{k-1} \to W_k.$$

We mention two important properties of $N * W$ ([Ka 86], lemma 3.4.2):
The monodromy filtration relative to $N * W$ exists and $M(N, N * W) = M(N, W)$.
Moreover, the following decomposition property is satisfied

$$\begin{aligned} Gr_k^{N*W} L &\simeq \mathrm{Im}\,(N : Gr_{k+1}^W L \to Gr_k^{N*W} L) \oplus \ker\,(I : Gr_k^{N*W} L \to Gr_{k+1}^W L) \\ & Im\,(N : Gr_{k+1}^W L \to Gr_k^{N*W} L) \xrightarrow{\sim} Im\,(N : Gr_{k+1}^W L \to Gr_{k+1}^W L). \end{aligned} \tag{4.10}$$

To refer to this decomposition, the couple $(L, N * W)$ is said to form a (graded) distinguished pair.

4.4.3. **The filtration W^J associated to an IMHS**. Let $(L, W, F, N_i, i \in M)$ denote the IMHS at $x \in Y_M^*$ and $J \subset M$. The relative weight filtration

$$M(J, W) := M(N, W) \text{ for } N \in C(J) := \{\Sigma_{j \in J} t_j N_j, t_j > 0\}, \tag{4.11}$$

is well defined by assumption. A basic lemma ([Ka 86], cor. 5.5.4) asserts that:

Lemma 4.18. *$(L, N_1 * W, F, N_i, i \in M)$ and $(L, M(N_1, W), F, N_i, i \neq 1)$ are IMHS.*

Remark 4.19. Geometrically, $N_1 * W$ and $M(N_1, W)$ may be viewed on $\mathcal{L}_{Y_1}$ near x. In term of the equation z_1 of Y_1, the filtration $N_1 * W$ viewed on $\psi_{z_1}\mathcal{L}$ is not a variation of MHS.
In ([Ka 86], lemma 3.4.2 ii and proposition 5.2.5) the following equality is proved:

$$M(\textstyle\sum_{i \in M} N_i, N_1 * W) = M(\sum_{i \in M - \{1\}} N_i, M(N_1, W)) = M(\sum_{i \in M} N_i, W).$$

We deduce a recursive definition of an increasing filtration W^J of L by the star operation

$$W^J := N_{i_1} * (\ldots (N_{i_j} * W) \ldots) \text{ for } J = \{i_1, \ldots, i_j\} \tag{4.12}$$

It is denoted by $\Psi_J * W$ in ([Ka 86], 5.8.2) as it is interpreted as defined repeatedly on Ψ_{x_i} for $x_i \in J$ in the theory of nearby cycles.

It followws by induction that $(L, W^J, F, N_i, i \in M)$ is an IMHS. The filtration W^J does not depend on the order of composition of the respective transformations $N_{i_k} *$ since $N_{i_p} * (N_{i_q} * W) = N_{i_q} * (N_{i_p} * W)$ for all $i_p, i_q \in J$ according to a basic result in ([Ka 86], proposition 5.5.5). The star operation has the following properties ([Ka 86], formula 5.8.5 and 5.8.6):
$\forall J_1, J_2 \subset M : M(J_1, W^{J_2}) = M(J_1, W)^{J_2}$, $\forall J \subset K \subset M : M(K, W^J) = M(K, W)$.
The filtrations W^J fit together to define the weight filtration on $\Omega^* L$:

Definition 4.20. The filtrations W and F on the de Rham complex associated to an IMHS $(L, W, F, N_i, i \in M)$ are defined as simplicial complexes

$$W_k \Omega^* L := s(W_{k-|J|}^J, N.)_{J \subset M},\ F^p \Omega^* L := s(F^{k-|J|}, N.)_{J \subset M}$$

Remark 4.21. The induced morphism $N_i : W^J_k \to (N_i * W^J)_{k-1}$ drops the degree of the filtration W^J. For further use, it is important to add to the above data, the canonical inclusion $I : (N_i * W^J)_{k-1} \to W^J_k$.

Beware that the weight filtrations W on $\mathcal{L}$ underlying the variation of MHS, extends as a constant filtration W' on $\Omega^* L$. The weight W on $\Omega^* L$ is the local definition of a global weight on the de Rham complex. The relation between W and W' on the terms of $\Omega^* L$ is described by ([Ka 86] lemma 3.4.2 iii) and corollary 3.4.3).

4.4.4. *Local decomposition of* $(\Omega^* L, W)$. The various decompositions in each degree, as in the formula 4.10, result into the decomposition of the whole complex $Gr^W_*(\Omega^* L)$. The proof involves the whole de Rham data $DR(L)$ attached to an IMHS ([Ka 86], section 5.6):

$$DR(L) := \{L_J, W^J, F^J, I_{J,K} : L_J \to L_K, N_{J-K} : L_K \to L_J\}_{K \subset J \subset M}$$

where for all $J \subset M$, $L_J := L$, $F_J := F$, W^J is the filtration defined above (formula 4.12), $N_{J-K} := \prod_{i \in J-K} N_i$ and $I_{J,K} := Id : L \to L$.

A set of properties of this data stated by Kashiwara in seven formula (5.6.1)-(5.6.7) follow from results proved in various sections of ([Ka 86]). In particular, for each $J \subset M$, the data: $(L_J, W^J, F_J, \{N_j, j \in M\})$ is an IMHS ([Ka 86], formula 5.6.6), which follows from ([Ka 86], corollary 5.5.4), while formula (5.6.7) with adapted notations states that for each $K \subset J \subset M$,

$$GrN_{J-K} : Gr^{W^K}_a L_K \to Gr^{W^J}_{a-|J-K|} L_J,\ GrI_{J,K} : Gr^{W^J}_{a-|J-K|} L_J \to Gr^{W^K}_a L_K$$

form a graded distinguished pair:

$$Gr^{W^J}_{a-|J-K|} L_J \simeq \operatorname{Im} GrN_{J-K} \oplus \operatorname{Ker} GrI_{J,K}.$$

Now, we are in position to define the ingredients of the local decomposition of $\Omega^* L$

Lemma 4.22 ([Ka 86], proposition 2.3.1 and lemma 5.6.2)**.** *i) Set for each $J \subset M$*

$$P^J_k(L) := \cap_{K \subset J, K \neq J} Ker(GrI_{J,K} : Gr^{W^J}_k L \to Gr^{W^K}_{k+|J-K|} L) \subset Gr^{W^J}_k L$$

then $P^J_k(L)$ has pure weight k with respect to the weight filtration $M(\sum_{j \in J} N_j, L)$. ii) We have: $Gr^{W^J}_k L \xrightarrow{\sim} \oplus_{K \subset J} N_{J-K} P^K_{k+|J-K|}(L)$.

In the definition of $P^J_k(L)$, we can suppose $|K| = |J| - 1$ in i).

For $i \in J$, let $K = J - i$, $N_i : Gr^{W^K}_{k+1} L \to Gr^{W^J}_k L$ and $I_i : Gr^{W^J}_k L \to Gr^{W^K}_{k+1} L$, then N_i on $Gr^{W^J}_k L$ is equal to $N_i \circ I_i$, hence N_i vanish on $P^J_k \subset \operatorname{Ker} N_i$ for all $i \in J$, and $N := \sum_{i \in J} N_i$ vanish on P^J_k. Then the weight filtration $W(N)$ on the component $P^J_K(L)$ of $Gr^{W^J}_k L$ coincides with the weight filtration of the morphism 0, hence $P^J_K(L) \subset Gr^{W(N)}_k \operatorname{Ker} N \subset Gr^{W(N)}_k Gr^{W^J}_k L$ is a polarized pure HS.
Remark that $W(N)_{k-1} \operatorname{Ker} N$ is a subset of the component $\operatorname{Im} N_i \subset Gr^{W^J}_k L$.

The statement (ii) is proved in ([Ka 86], proposition 2.3.1) with a general terminology. We illustrate here the proof in the case of two nilpotent endomorphisms.

Set $W^i := N_i * W$ for $i = 1, 2$, $W^{12} = N_1 * N_2 * W$ in the following diagram

$$\begin{tikzcd}
 & Gr_i^{W^1} L \arrow[dr, "N_2'"] & & & Gr_i^{W^1} L \arrow[dl, "I_1"'] & \\
Gr_{i+1}^{W} L \arrow[ur, "N_1"] \arrow[dr, "N_2"'] & & Gr_{i-1}^{W^{12}} L & Gr_{i+1}^{W} & & Gr_{i-1}^{W^{12}} L \arrow[ul, "I_2'"'] \arrow[dl, "I_1'"] \\
 & Gr_i^{W^2} L \arrow[ur, "N_1'"'] & & & Gr_i^{W^2} L \arrow[ul, "I_2"] &
\end{tikzcd}$$

where I_i' and N_i' are induced by I_i and N_i. The morphisms N_i and I_i (resp. N_i' and I_i') form distinguished pairs. Let $P_{i-1}^{12,1} L := \operatorname{Ker} I_2' : Gr_{i-1}^{W^{12}} L \to Gr_i^{W^1} L$ (resp. $P_{i-1}^{12,2} L := \operatorname{Ker} I_1'$, $P_i^1 L := \operatorname{Ker} I_1$, $P_i^2 L := \operatorname{Ker} I_2$), then

$Gr_i^{W^1} L \simeq N_1 Gr_{i+1}^{W} L \oplus P_i^1 L$, and $Gr_i^{W^2} L \simeq N_2 Gr_{i+1}^{W} L \oplus P_i^2 L$
$Gr_{i-1}^{W^{12}} L \simeq N_2' Gr_i^{W^1} L \oplus P_{i-1}^{12,1} L \simeq N_2' N_1 Gr_{i+1}^{W} L \oplus N_2' P_i^1 L \oplus P_{i-1}^{12,1} L$, and
$Gr_{i-1}^{W^{12}} L \simeq N_1' N_2 Gr_{i+1}^{W} L \oplus N_1' P_i^2 L \oplus P_{i-1}^{12,2} L$,

Let $P_{i-1}^{12} L := P_{i-1}^{12,2} L \cap P_{i-1}^{12,1} L$. It remains to prove

$$P_{i-1}^{12,1} L = N_2' P_i^1 L \oplus P_{i-1}^{12} L \text{ and } P_{i-1}^{12,2} L = N_1' P_i^2 L \oplus P_{i-1}^{12} L.$$

which follows from the linear algebra relations:

$$I_2' \circ N_1' = N_1 \circ I_2, \text{ and } I_2 \circ I_1' = I_1 \circ I_2'.$$

Remark 4.23. In the global decomposition of the logarithmic complex, the HS $P_k^J(L)$ is viewed as a fibre of a VHS on Y_J (corollary 4.24 below, and theorem 4.31).

The following corollary is a basic step to check the structure of MHC on the logarithmic de Rham complex

Corollary 4.24 (local decomposition)**.** *The graded vector space of the filtration W on $\Omega^* L$ (definition 4.20) satisfy the decomposition property into a direct sum of Intersection complexes*

$$Gr_k^W(\Omega^* L) \xrightarrow{\sim} \oplus IC^*(P_{k-|K|}^K(L)[-|K|))_{K \subset M} \tag{4.13}$$

The corollary is the local statement of the structure of MHC on the logarithmic de Rham complex.

It follows from the lemma by definition of IC^* (proposition 4.10) by a general description of perverse sheaves in the normally crossing divisor case ([Ka 86], section 2). The description is in terms of a family of vector spaces and a diagram of morphisms. The minimal extensions are characterized by relations of distinguished pairs as in the formula 4.10.

The family of vector spaces is deduced from a perverse sheaf by repeated application of the functors nearby and vanishing cycles with respect to the coordinates.

4.4.5. *Decomposition of the Intersection complex $IC^* L$.* We need to prove a similar decomposition of the Intersection complex $IC^* L$ associated to $(L, W, F, N_i, i \in M)$ an IMHS. For each $j \in M$, we consider the data $(N_j L, N_j * W, F, N_i, i \in M)$ with induced filtrations from the IMHS $(L, N_i * W, F, N_i, i \in M)$ (we should write

$(N_j * W)_{|N_jL}, F_{|N_jL}, (N_i)_{|N_jL}$ but the symbol of restriction is erased). We have a graded exact sequence defined by the induced filtrations on N_jL:

$$0 \longrightarrow Gr_k^{W^j} N_jL \longrightarrow Gr_k^{W^j} L \longrightarrow Gr_k^{W^j} L/N_jL \longrightarrow 0$$

Lemma 4.25 (descent lemma for IMHS)**.** *i) Let $(L, W, F, N_i, i \in M)$ be an IMHS, then $(N_jL, W^j := ((N_j * W)_{|N_jL}, F, N_i, i \in M)$ is an IMHS for all $j \in M$.*

ii) The couple $(L, W, F, N_i) \xrightarrow{N_j} (N_jL, W^j, F, N_i)$ form a graded distinguished pair: $Gr_k^{W^j} N_jL$ splits into a direct sum

$$Im\,(N_j : Gr_{k+1}^W L \to Gr_k^{W^j} NjL) \oplus \mathrm{Ker}\,(I_j : Gr_k^{W^j} N_jL \to Gr_{k+1}^W L)$$

Proof. The filtration W^j is first constructed on L, then induced on N_jL. The proof is simultaneous for i) and ii). We consider the diagram

$$\begin{array}{ccccc} 0 & \to & Gr_{k+1}^W L & = & Gr_{k+1}^W L \\ & & \downarrow N_j' \uparrow I_j' & & \downarrow N_j \uparrow I_j \\ 0 & \to & Gr_k^{W^j} N_jL & \xrightarrow{s} & Gr_k^{W^j} L \end{array}$$

where s is injective, N_j' and I_j' are other symbols for the restriction of N_j and I_j. Set $P_k := \ker I_j$, $\mathrm{Im}\, N_j$ for the image of N_j, $\mathrm{Im}\, N_j'$ for the image of N_j' and $\widetilde{\mathrm{Im}}\, N_j := \mathrm{Im}\,(N_j : Gr_{k+1}^W L \to Gr_{k+1}^W L)$.

The terms on the right column form a distinguished pair, that is we have a splitting $Gr_k^{W^j} L \simeq \mathrm{Im}\, N_j \oplus \ker I_j$ and an isomorphism $\mathrm{Im}\, N_j \simeq \widetilde{\mathrm{Im}}\, N_j$.

Set $P_k' := (Gr_k^{W^j} N_jL) \cap P_k$. As $\mathrm{Im}\, N_j' \simeq s(\mathrm{Im}\, N_j') = \mathrm{Im}\, N_j$, we deduce the splitting

$$Gr_k^{W^j} N_jL = \mathrm{Im}\, N_j' \oplus P_k'$$

where $P_k' = \ker\,(I_j' : Gr_k^{W^j} N_JL \to Gr_{k+1}^{\tilde{W}} N_KL)$. We prove that both components of this splitting are nilpotent orbits of weight k.

1) We apply the descent lemma ([CaKSc 87] descent lemma 1.16) to $\mathrm{Im}\, N_j'$ since it is isomorphic to the image $\widetilde{\mathrm{Im}}\, N_j$ (the image of N_j acting on the nilpotent orbit $Gr_{k+1}^W L$). We deduce that the data $(\mathrm{Im}\, N_j', F, N_i, i \in M)$ is a nilpotent orbit of weight k.

2) The kernel $P_k = \mathrm{Ker}\, I_j \subset Gr_{k+1}^W N_jL \subset Gr_{k+1}^W L$ is compatible with the action of N_i for all $i \in M$, and since $N_j = N_j \circ I_j$ vanish on P_k we deduce that $P_k \simeq Gr_k^{W(N_j)}(\mathrm{Ker}\, N_j)$ has a pure polarized HS induced by the MHS of $W(N_j) \subset Gr_{k+1}^{W^j} L$. In particular $W(N_j)_{k-1}\mathrm{Ker}\, N_j$ is in $\mathrm{Im}\, N_j \simeq \widetilde{\mathrm{Im}}\, N_j$ which is consistent with the descent lemma.

Moreover the orthogonal subspace to $P_k' = \mathrm{Ker}\, I_j' \subset P_k$ in the primitive part $Gr_k^{W(N_j)}\mathrm{Ker}\, N_j$ (resp. in $Gr_k^{W(N_j)}\mathrm{Ker}\, N_j$) defines a splitting compatible with the action of N_i for all $i \in M$ as the N_i are infinitesimal isometry of the bilinear product $S : S(N_ia, b) + S(a, N_ib) = 0$, hence P_k' is a sub-nilpotent orbit of the nilpotent orbit structure on $Gr_k^{W(N_j)} Gr_{k+1}^{W^j} L$. □

Corollary 4.26. *i) Let $(L, W, F, N_i, i \in M)$ be an IMHS, then $(N_KL, W^K, F, N_i, i \in M)$ is an IMHS for all $K \subset M$, and the embedding $(N_KL, W^K, F) \subset (L, W^K, F)$ in the category of IMHS is strict for the filtrations.*

ii) The couple $(N_K L, W^K, F, N_i) \xrightarrow{N_j} (N_J L, W^j, F, N_i)$ *form a graded distinguished pair:* $Gr_k^{W^j} N_J L$ *splits into a direct sum*

$$\mathrm{Im}\,(N_j : Gr_{k+1}^{W^K} N_K L \to Gr_k^{W^J} N_J L) \oplus \mathrm{Ker}\,(I_j : Gr_k^{W^J} N_J L \to Gr_{k+1}^{W^K} N_K L)$$

and the image $\mathrm{Im}\, N_j$ *is isomorphic to* $\mathrm{Im}\,(N_j : Gr_{k+1}^{W^K} N_K L \to Gr_{k+1}^{W^K} N_K L)$.

We prove the corollary by induction on the length of K. By the lemma, the embedding $(N_j L, W^j, F, N_i) \subset (L, W^j, F, N_i)$ is an injective morphism in the abelian category of IMHS.

We assume by induction that $(N_K L, W^K, F, N_i, i \in M)$ is an IMHS, and let $J := K \cup i$ for $i \notin K$. First we consider the induced morphism $N_j : W^K \to W^K$, the filtration $N_j * W^K_{|N_K L}$ defined by the star operation on $N_K L$ and its induced filtration on $N_J L \subset N_K L$. There is another filtration obtained from W^K on L: $N_j * W^K$ on L and its induced filtration on $N_K L$ such that we have a diagram

$$(N_j * W^K)_{|N_J L} \to N_j * W^K_{|N_K L} \to W^J := N_j * W^K$$

By the lemma the first morphism is in the category of IMHS and by the inductive hypothesis the second morphism is compatible with the IMHS, hence the morphisms are strict and the filtration W^J in L induces $N_j * W^K_{|N_J L}$ on $N_J L$. Hence $(N_J L, W^J, F, N_i, i \in M)$ is an IMHS, and the embedding into $(L, W^J, F, N_i, i \in M)$ is compatible with the IMHS.

4.4.6. **Decomposition of** $IC^* L$. As in the case of $\Omega^* L$ (subsection 4.4.4) we introduce the data $DR(IC^*(L) :=$

$$\{N_J L, W^J, F^J, I_{J,K} : N_J L \to N_K L, N_{J-K} : N_K L \to N_J L\}_{K \subset J \subset M}$$

Set for each $J \subset M$

$$P_k^J(N_J L) := \cap_{K \subset J, K \neq J} \mathrm{Ker}\,(Gr I_{J,K} : Gr_k^{W^J} N_J L \to Gr_{k+|J-K|}^{W^K} N_K L) \subset Gr_k^{W^J} L$$

then $P_k^J(N_J L) = P_k^J(L) \cap Gr_k^{W^J} N_J L$ has pure weight k with respect to the weight filtration $M(\sum_{j \in J} N_j, L)$ and it is polarized.

Corollary 4.27. *i)* $Gr_k^{W^J} N_J L \xrightarrow{\sim} \oplus_{K \subset J} N_{J-K} P^K_{k+|J-K|}(N_K L)$.

ii) $$Gr_k^W(IC^* L) \xrightarrow{\sim} \oplus_{K \subset M} IC(P^K_{k-|K|}(N_K L)[-|K|].$$

4.5. **Global definition and properties of the weight** W**.** The local study ended with the local decomposition into Intersection complexes. We develop now the corresponding global result. The local cohomology of $\Omega^* \mathcal{L}$ is reduced to $\mathcal{L}$ on $X - Y$ and to the various analytic restrictions $\mathcal{L}_{Y_J}$ on Y (proposition 4.5).

In the case of a pure VHS, the weight starts with the subcomplex $IC^* \mathcal{L}$.

Notations in the global case. We suppose $\mathcal{L}$ locally unipotent by the remark 4.7 as other components of the spectral decomposition have acyclic de Rham complexes. We recall that $\mathcal{L} := \widetilde{\mathcal{L}}[n]$ with $n := dim.X$ is a shifted variation of MHS, hence all complexes and bundles below are already shifted by n to the left. For all $J \subset I$, let $\mathcal{L}_{Y_J}$ denote the restriction of the analytic module $\mathcal{L}_X$ to Y_J. The weight filtration W on $\mathcal{L}$ defines a filtration by sub-bundles $\mathcal{W}_X$ of $\mathcal{L}_X$ with restriction $\mathcal{W}_{Y_J}$ to Y_J, such that the relative monodromy filtrations of the endomorphism $\sum_{i \in J} \mathcal{N}_i$ of $(\mathcal{L}_{Y_J}, \mathcal{W}_{Y_J})$ exist for all $J \subset I$. We write $\mathcal{M}(J, \mathcal{W}_{Y_J}) := \mathcal{M}(\sum_{i \in J} \mathcal{N}_i, \mathcal{W}_{Y_J})$, it is a filtration by analytic sub-bundles, so that we can define

$$(\mathcal{N}_i * \mathcal{W}_{Y_i})_k := \mathcal{N}_i \mathcal{W}_{Y_i, k+1} + \mathcal{M}_k(\mathcal{N}_i, \mathcal{W}_{Y_i}) \cap \mathcal{W}_k = \mathcal{N}_i \mathcal{W}_{Y_i, k+1} + \mathcal{M}_k(\mathcal{N}_i, \mathcal{W}_{Y_i}) \cap \mathcal{W}_{k+1}$$

and an increasing filtration $\mathcal{W}^J$ of $\mathcal{L}_{Y_J}$ is recursively defined by the star operation

$$\mathcal{W}^J := \mathcal{N}_{i_1} * (\dots(\mathcal{N}_{i_j} * \mathcal{W})\dots) \text{ for } J = \{i_1, \dots, i_j\}.$$

Definition 4.28 (weight W on $\Omega^*\mathcal{L}$)**.** Let $(\mathcal{L}, W, F)$ be an admissible variation of MHS on $U := X - Y$. The weight filtration by sub-analytic complexes, denoted also by W on $\Omega^*\mathcal{L}$, is defined locally in terms of the various restrictions $\mathcal{W}_{Y_J}$ to the strata Y_J^*.

At a point $y \in Y_M^*$ for $M \subset I$, $W_r(\Omega_X^*(LogY) \otimes \mathcal{L}_X)_y$ is defined locally in a neighborhood of $y \in Y_M^*$, in terms of the IMHS $(L, W, F, N.)$ at y and a set of coordinates y_i for $i \in M$ (including local equations of Y at y), as follows:

The term W_r of the filtration is generated as an $\Omega_{X,y}^*-$ sub-module by the germs of the sections $\widetilde{v} \otimes \wedge_{j\in J} \frac{dy_j}{y_j}$ for $J \subset M$ and $v \in W^J_{r-|J|}L$.

In particular, for $M = \emptyset$, at a point $y \in U$, W_r is generated as an $\Omega_{X,y}^*-$ sub-module by the germs of the sections $v \in W_r\mathcal{L}_y$.

The definition of W above is independent of the choice of coordinates on a neighborhood $U(y)$, since if we change the coordinate into $y_i' = fy_i$ with f invertible holomorphic at y, we check first that the submodule $W^J_{r-|J|}(\mathcal{L}_X)_y$ of $\mathcal{L}_{X,y}$ defined by the image of $W^J_{r-|J|}L$ is independent of the coordinates as in the construction of the canonical extension (see also lemma 4.9). For a fixed $\alpha \in W^J_{r-|J|}\mathcal{L}_{X,y}$, as the difference $\frac{dy_i'}{y_i'} - \frac{dy_i}{y_i} = \frac{df}{f}$ is holomorphic at y, the difference of the sections $\alpha \otimes \wedge_{j\in J}\frac{dy_j'}{y_j'} - \alpha \otimes \wedge_{j\in J}\frac{dy_j}{y_j}$ is still a section of the $\Omega_{X,y}^*-$sub-module generated by the germs of the sections $W^J_{r-|J|}(\mathcal{L}_X)_y \otimes \wedge_{j\in(J-i)}\frac{dy_j}{y_j}$.
Finally, we remark that the sections defined at y restrict to sections defined on $(U(y) - Y_M^* \cap U(y))$.

4.5.1. *Global statements.* The Hodge filtration on the Intersection complex defined algebraically on $IC^*\mathcal{L}$ induces a HS on the Intersection cohomology with coefficients in a polarized VHS.

Theorem 4.29. *Let $(\mathcal{L}, F)$ be a shifted polarized VHS of weight a, then the sub-complex $(IC^*\mathcal{L}, F)$ of the logarithmic complex with induced filtration F is a Hodge complex which defines a pure HS of weight $a+i$ on its hypercohomology $\mathbb{H}(X, IC^*\mathcal{L})$, equal to the Intersection cohomology $IH^i(X, \mathcal{L})$.*

We rely on the proof by Kashiwara ([KaK 86], Theorem 1 and Proposition 3).

Remark 4.30. i) The proof of the local purity theorem here relies on the polarization of the intersection cohomology in the case of the complement of a NCD in X.

The original proof of the purity theorem is local at v [DeG 81]. As noted by the referee of the notes, the MHS on the boundary of the exceptional divisor at a point $v \in V$ (definition 5.11, lemma 5.13) is independent of the global variety X but depends only on the components of the NCD X_v and their embedding in X.

=G-N A. V. Navarro, F. Guillén, Sur le théorème local des cycles invariants. = It is interesting and possible to adapt such local proof, then the algebraic-geometric constructions of W and F leads to an alternative construction of Hodge theory as follows.

We start with the case of curves studied in [Zu 79], then proceed by induction on dim.X. We assume the purity of $IC^*\mathcal{L}$ in dimension $n-1$ in the inductive step. Given a smooth algebraic variety X of dimension n and a morphism $f : X \to V$ to a smooth curve V, we deduce the decomposition theorem with just a local proof of purity, from which we deduce that the intersection cohomology $\mathbb{H}^i(X, \mathcal{L})$ carries a pure HS induced by the Hodge filtration.

ii) The subtle proof of the comparison in [KaK 86] is based on the auto-duality of the Intersection cohomology.

We are not aware of a direct comparison of the filtration F on $IC^*\mathcal{L}$ defined here in an algebro-geometric way, with the filtration F in terms of L^2-cohomology in [KaK 87] and [CaKSc 86].

4.5.2. *The bundles $\mathcal{P}^J_k(\mathcal{L}_{Y_J})$.* Given a subset $J \subset I$, the filtration $\mathcal{W}^J$ induces a filtration by sub-analytic bundles of $\mathcal{L}_{Y_J}$. We introduce the following bundles

$$\mathcal{P}^J_k(\mathcal{L}_{Y_J}) := \cap_{K\subset J, K\neq J}\mathrm{Ker}\,(GrI_{J,K} : Gr^{\mathcal{W}^J}_k \mathcal{L}_{Y_J} \to Gr^{\mathcal{W}^K}_{k+|J-K|}\mathcal{L}_{Y_J}) \subset Gr^{\mathcal{W}^J}_k \mathcal{L}_{Y_J}$$

where $I_{J,K}$ is defined as in the local case.

In particular $Y_\emptyset := X$, $\mathcal{P}^\emptyset_k(\mathcal{L}_X) = Gr^{\mathcal{W}}_k \mathcal{L}_X$ and $\mathcal{P}^J_k(\mathcal{L}_{Y_J}) = 0$ if $Y^*_J = \emptyset$.

Theorem 4.31. *i) The weight W is a filtration by sub-complexes of $Rj_*\mathcal{L}$ consisting of perverse sheaves defined over $\mathbb{Q}$.*

ii) The bundles $\mathcal{P}^J_k(\mathcal{L}_{Y_J})$ are Deligne's extensions of polarized VHS $\mathcal{P}^J_k(\mathcal{L})$ on Y^*_J of weight k with respect to the weight induced by $\mathcal{M}(\sum_{j\in J}\mathcal{N}_j, \mathcal{L}_{Y^*_J})$.

iii) The graded perverse sheaves for the weight filtration, satisfy the decomposition property into intermediate extensions for all k

$$Gr^W_k(\Omega^*\mathcal{L}) := Gr^W_k(\Omega^*_X(LogY) \otimes \mathcal{L}_X) \xrightarrow{\sim} \oplus_{J\subset I}(i_{Y_J})_* j_{!*}\mathcal{P}^J_{k-|J|}(\mathcal{L})[-|J|].$$

where j denotes uniformly the inclusion of Y^*_J into Y_J for each $J \subset I$.

We remark that $\mathcal{P}^J_{k-|J|}(\mathcal{L})[-|J|]$ are local systems on Y^*_J shifted by $[m - |J|]$ the dimension of Y^*_J. The proof is essentially based on the local study above. The various graded complexes $Gr^W_k(\Omega^*_X(LogY) \otimes \mathcal{L}_X)$ are direct sum of Intersections complexes defined over $\mathbb{Q}$. In particular, we deduce that the whole filtration W_k on the de Rham complex is defined over $\mathbb{Q}$, and W_k is perverse with respect to the stratification defined by Y.

The local rational structure of the complexes W_k glue into a global rational structure, since perverse sheaves may be glued as the usual sheaves, although they are not concentrated in a unique degree ([BBDG 83] corollary 2.1.23 and 2.2.19).

Another proof of the existence of the rational structure is based on Verdier's specialization [ELM 10].

The next result on weights satisfy a property cited in ([De 80], cor 3.3.5).

Corollary 4.32. *The de Rham logarithmic mixed Hodge complex $(\Omega^*\mathcal{L}, W, F)$ of an admissible variation of MHS of weight $\omega \geq a$ induces on the hypercohomolgy $\mathbb{H}^i(X - Y, \mathcal{L})$ a MHS of weight $\omega \geq a + i$.*

Indeed, $W_k = 0$ on the logarithmic complex for $k \leq a$.

Proposition 4.33. *The Intersection complex $(IC(X, \mathcal{L}), W, F)$ of an admissible variation of MHS, as an embedded sub-complex of $(\Omega^*\mathcal{L}, W, F)$ with induced filtrations, is a mixed Hodge complex satisfying for all k:*

$$(IC^*(Gr^W_k\mathcal{L}), F) \simeq (Gr^W_k IC^*\mathcal{L}, F)$$

The proposition follows from corollary 5.2. In general the intersection complex of an extension of two local systems, is not the extension of their intersection complex.

4.6. **The relative logarithmic complex $\Omega^*_f\mathcal{L} := \Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X$.**

Let $f : X \to V$ be a smooth proper morphism of smooth complex algebraic varieties, and let Y be an "horizontal" NCD in X, that is a relative NCD with smooth components over V. For each point $v \in V$, the fiber Y_v is a NCD in the smooth fiber X_v. In this case, the various intersections $Y_{i_1,\cdots,i_p}$ of p- components are smooth over V and $Y \to V$ is a topological fiber bundle.

Let $U := X - Y$, $j : U \to X$. The sheaf of modules $i^*_{X_v}\mathcal{L}_X$ induced on each fiber X_v, by the canonical extension $\mathcal{L}_X$, is isomorphic to the canonical extension $(i^*_{U_v}\mathcal{L})_{X_v}$ of the induced local system $i^*_{U_v}\mathcal{L}$. The family of cohomology spaces $H^i(U_v,\mathcal{L})$ (resp.$\mathbb{H}^i(X_v, j_{!*}\mathcal{L})$) for $v \in V$, form a variation of MHS. The logarithmic complex $\Omega^*_f\mathcal{L} := \Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X$ ([De 70], section 2.22) satisfy, in the case of an horizontal NCD: $i^*_{X_v}\Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X \simeq \Omega^*_{X_v}(LogY_v)\otimes(i^*_{U_v}\mathcal{L})_{X_v}$.

When $\mathcal{L}$ underlies an admissible graded polarized variation of MHS $(\mathcal{L}, W, F)$, its restriction to the open subset U_v in X_v is also admissible.

4.6.1. *The relative logarithmic complex with coefficients $\Omega^*_f\mathcal{L}$ for a smooth f.* The image of the filtrations W and F, by the map $\Omega^*\mathcal{L} \to \Omega^*_f\mathcal{L} := \Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X$

$$F := \mathrm{Im}(R^if_*F_X \to R^if_*(\Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X)),$$
$$W := \mathrm{Im}(R^if_*W_X \to R^if_*(\Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X))$$

define a variation of MHS on $R^i(f\circ j)_*\mathcal{L}$ inducing at each point $v \in V$ the corresponding weight W and Hodge F filtrations of the MHS on $(\mathbb{H}^i(U_v,\mathcal{L}), W, F)$.

Proposition 4.34. *i) The direct image $R^i(f\circ j)_*\mathcal{L}$ is a local system on V and*

$$R^i(f\circ j)_*\mathcal{L}\otimes\mathcal{O}_V \simeq R^if_*(\Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X)$$

ii) Moreover, the connecting morphism in Katz-Oda's construction [KaOd 68] *coincides with the connection on V defined by the local system $R^i(f\circ j)_*\mathcal{L}$*

$$R^if_*(\Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X) \xrightarrow{\nabla_V} \Omega^1_V\otimes R^if_*(\Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X)$$

iii) The filtration F, is horizontal with respect to ∇_V, while W is locally constant, and $(R^i(f\circ j)_*\mathcal{L}, W, F)$ is a graded polarized variation of MHS on V.

Deligne's proof of ([De 70], proposition 2.28) extends in (i), as well the connecting morphism [KaOd 68] in (ii).

Remark 4.35. We refer to the remark 4.30 for the case of a "vertical" NCD, that is when f is smooth over the complement of a NCD $W \subset V$ and $Y := f^{-1}W$.

4.6.2. *The relative Intersection complex $IC^*_f\mathcal{L}$.*

We deduce from the bi-filtered complex $(IC^*\mathcal{L}, W, F)$ the i-th direct image

$$(\mathcal{L}^i, F, W) := (R^if_*j_{!*}\mathcal{L},\ Im\,(R^if_*F \to R^if_*IC^*\mathcal{L}), Im\,(R^if_*W \to R^if_*IC^*\mathcal{L}))$$

To prove that *the filtrations W and F defined on $\mathcal{L}^i := R^if_*j_{!*}\mathcal{L}$, form a structure of a graded polarized variation of MHS on V,* we consider the image complex of $IC^*\mathcal{L}$,

$IC^*_f\mathcal{L} := \mathrm{Im}\Big(IC^*\mathcal{L} \to \Omega^*_{X/V}(LogY)\otimes\mathcal{L}_X\Big)$ with image filtrations W and F, then:

$W_V := Im(R^if_*W \to R^if_*(IC^*_f\mathcal{L}))$ is locally constant on V, and

$F_V := Im(R^if_*F \to R^if_*(IC^*_f\mathcal{L}))$ on $R^if_*j_{!*}\mathcal{L}\otimes\mathcal{O}_V \simeq R^if_*(IC^*_f\mathcal{L})$ is horizontal with respect to ∇_V.

Proposition 4.36. *i) The direct image $R^if_*j_{!*}\mathcal{L}$ of the intersection complex is a local system on V and*

$$R^if_*j_{!*}\mathcal{L}\otimes\mathcal{O}_V \simeq R^if_*(IC^*_f\mathcal{L})$$

ii) The connecting morphism in Katz-Oda's construction coincides with the connection on V defined by the local system $R^if_*j_{!*}\mathcal{L}$

$$R^if_*(IC^*_fj_{!*}\mathcal{L}) \xrightarrow{\nabla_V} \Omega^1_V\otimes R^if_*(IC^*_f\mathcal{L})$$

ii) The de Rham complex defined by the connection on $(R^if_*IC^*_f\mathcal{L}, W_V, F_V)$ underlies a structure of mixed Hodge complex on V defined by the variation of MHS induced on $R^if_*j_{!*}\mathcal{L}$.

iii) The truncation filtration τ induces a filtration on $\mathbb{H}^{i+j}(X, j_{!*}\mathcal{L})$ compatible with the MHS and $Gr^\tau_i\mathbb{H}^{i+j}(X, j_{!*}\mathcal{L}) \simeq \mathbb{H}^j(V, R^if_*IC^*_f\mathcal{L})$.

5. Logarithmic Intersection complex for an open algebraic variety

Let $Z = \cup_{i\in I_Z\subset I}Y_i \subset Y$ be a sub-divisor of Y, union of components of Y with index in a subset I_Z of I, $j_Z := (X-Z) \to X$, and $i_Z : Z \to X$.

We construct a logarithmic sub-complex $IC^*\mathcal{L}(LogZ) \subset \Omega^*\mathcal{L}$ which is a realization of the direct image $Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})$ (denoted $\Omega^*(\mathcal{L}, Z)$ in [ICTP 14] definition 8.3.31); from which we deduce by duality various logarithmic complexes with weight and Hodge filtrations, realizing cohomological constructions associated to Z in Y such as the functors:

$j_{Z!}((j_{!*}\mathcal{L})_{|X-Z}), Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z}), i^*_ZRj_{Z*}((j_{!*}\mathcal{L})_{|X-Z}), Ri^!_Zj_{!*}\mathcal{L}$ and $i^*_Zj_{!*}\mathcal{L}$.

We describe a structure of mixed Hodge complexes (MHC) on $IC^*\mathcal{L}(LogZ)$ that is $Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})$, and $i_{Z*}i^*_ZRj_{Z*}((j_{!*}\mathcal{L})_{|X-Z}) := i_{Z*}i^*_ZIC^*\mathcal{L}(LogZ)$ (def . 5.11 and lemma 5.13 on cohomology of the boundary of a tubular neighborhood of Z).

5.1. $IC^*\mathcal{L}(LogZ) \simeq Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})$.

We suppose $\mathcal{L}$ locally unipotent as we refer to the remark 4.7 in general. The subcomplex of analytic sheaves $IC^*\mathcal{L}(LogZ)$ of the logarithmic de Rham complex $\Omega^*_X(LogY)\otimes\mathcal{L}_X$ is defined below locally at a point $x \in Y^*_M$ in terms of a set of coordinates y_i for $i \in M$ defining a set of equations of Y_M.

5.1.1. *The Logarithmic Intersection complex: $IC^*L(LogZ)$.*

Given an IMHS $(L, W, F, N_i, i \in M)$ and a subset $M_Z \subset M$, we consider for each $J \subset M$ the subsets $J_Z := J \cap M_Z$ and $J'_Z = J - J_Z$ such that $J = J_Z \cup J'_Z$, in particular $M'_Z := M - M_Z$.

The correspondence which attach to each index $J \subset M$ the subset $N_{J'_Z}L$ of L, define a sub-complex $IC^*L(LogZ)$ of Ω^*L as a sum over $J \subset M$

$$IC^*L(LogZ) := s(N_{J'_Z}L)_{J\subset M} \subset \Omega^*L \tag{5.1}$$

Example. On the 3-dimensional disc $D^3 \subset \mathbb{C}^3$, let $Y = Y_1 \cup Y_2 \cup Y_3$ be the NCD defined by the coordinates $y_1y_2y_3 = 0$. A unipotent local system $\mathcal{L}$ is defined by a vector space L with the action of 3 nilpotent endomorphisms $(L, N_i, i \in [1,3])$.

Let $Z = Y_1 \cup Y_2$ be defined by $y_1y_2 = 0$ and $M_Z = \{1,2\}$, then $IC^*(LogZ)$ is defined by the diagram:

$$\begin{array}{ccccc} L & \xrightarrow{N_1,N_2} & L \oplus L & \xrightarrow{N_1,N_2} & L \\ \downarrow N_3 & & \downarrow N_3 & & \downarrow N_3 \\ N_3L & \xrightarrow{N_1,N_2} & N_3L \oplus N_3L & \xrightarrow{N_1,N_2} & N_3L \end{array}$$

with differentials defined by N_i with + or - sign. Then, we have a quasi-isomorphism $\mathbb{R}\Gamma(D^3 - (D^3 \cap Z), j_{!*}\mathcal{L}) \simeq IC^*L(LogZ)$.

5.1.2. *Weight filtration.*

Definition 5.1. The filtration W^J on the term $N_{J'_Z}L$ of index J in $IC^*L(LogZ) \subset \Omega^*L$ (formula 5.1) is induced by the embedding $(N_{J'_Z}L, W^J) \subset (L, W^J)$

$$IC^*L(LogZ) := s(N_{J'_Z}L)_{J\subset M},\ W_kIC^*L(LogZ) := s(W^J_{k-|J|}N_{J'_Z}L)_{J\subset M}.$$

As in the case of Ω^*L and IC^*L, to compute $Gr^W_kIC^*L(LogZ)$ we introduce the data $DR(IC^*L(LogZ) :=$

$$\{N_{J'_Z}L, W^J, F^J, I_{J,K} : N_{J'_Z}L \to N_{K'_Z}L, N_{J-K} : N_{K'_Z}L \to N_{J'_Z}L\}_{K\subset J\subset M}$$

satisfying various properties as in the subsection 4.4.4, including the properties of distinguished pairs for consecutive terms $N_i : N_{K'_Z}L \to N_{J'_Z}L$ for $J = K \cup i$ (corollary 4.26). Set for each $J \subset M$

$$P^J_k(N_{J'_Z}L) := \cap_{K\subset J, K\neq J}\mathrm{Ker}\,(GrI_{J,K} : Gr^{W^J}_kN_{J'_Z}L \to Gr^{W^K}_{k+|J-K|}N_{K'_Z}L)$$

then $P^J_k(N_{J'_Z}L) \subset Gr^{W^J}_kN_{J'_Z}L$ has pure weight k with respect to the weight filtration $M(\sum_{j\in J}N_j, L)$ and it is polarized.

Corollary 5.2. *We have:* $Gr^{W^J}_kN_{J'_Z}L \xrightarrow{\sim} \oplus_{K\subset J}N_{J-K}P^K_{k+|J-K|}(N_{K'_Z}L)$,
and $Gr^W_k(IC^*L(LogZ)) \xrightarrow{\sim} \oplus_{K\subset M}IC(P^K_{k-|K|}(N_{K'_Z}L)[-|K|]$.

5.1.3. *Global statements.* In terms of the diagram $(X - Y) \xrightarrow{j'} (X - Z) \xrightarrow{j_Z} X$, $j = j_Z \circ j'$ such that $Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z}) \simeq Rj_{Z*}j'_{!*}\mathcal{L}$.

Definition 5.3 ($IC^*\mathcal{L}(LogZ)$)**.** For $J \subset M$, let $J_Z = J \cap I_Z, J'_Z = J - J_Z$. The fiber of $IC^*\mathcal{L}(LogZ)$ is generated at $x \subset Y^*_M$ as an $\Omega^*_{X,x}$-submodule, by the sections $\tilde{v} \wedge_{j\in J}\frac{dy_j}{y_j}$ for $J \subset M$ for all $v \in N_{J'_Z}L$.

Here $N_\emptyset = Id$. In other terms, in degree k, for each subset $K \subset M$ of length $|K| = k$, let $K' = (I - I_Z) \cap K$, and $K_i \subset K'$ denote a subset of length $|K_i| = k - i$ and $A := \mathcal{O}_{X,x}$, the subcomplex $IC^*\mathcal{L}(LogZ)_x \subset \Omega^*\mathcal{L}_x$ is defined with the notations of (4.3.1), as the $\Omega^*_{X,x}$-submodule:

$$\oplus_{|K|=k}\left(y_{K'}A\widetilde{L}\frac{dy_K}{y_K} + \cdots + \sum_{K_i\subset K'} y_{K_i}A(\widetilde{N_{K'-K_i}L})\frac{dy_K}{y_K} + \cdots\right)$$

Next we check the isomorphism $(Rj_{Z*}j'_{!*}\mathcal{L})_x \simeq IC^*L(LogZ)$ of the fiber at a point $x \subset Y^*_M$.

Lemma 5.4. *We have:* $(Rj_{Z*}(j_{!*}\mathcal{L})_{|X-Z})_x \xrightarrow{\sim} IC^*\mathcal{L}(LogZ)_x$

We check the proof on the terms of the spectral sequence with respect to the truncation filtration on $j'_{!*}\mathcal{L}$, that is $E_2^{p,q} = R^p j_{Z*} H^q((j'_{!*}\mathcal{L})_x)$ (in classical notations, after décalage in Deligne's notations) converging to $H^{p+q}((Rj_{Z*}j'_{!*}\mathcal{L})_x)$.

Indeed, the complex $IC^*L(LogZ)$ may be described as the exterior algebra defined by the action of $N_i, i \in M_Z$ on the Koszul complex $s(N_{J'_Z}L)_{J'_Z \subset M'_Z}$. That is the simple complex associated to the double complex $s(s(N_{J'_Z}L)_{J'_Z \subset M'_Z}, N_i)_{i \in M_Z}$

$$IC^*L(LogZ) := s(\Omega(N_{J'_Z}L, N_i, i \in M_Z)_{J'_Z \subset M'_Z} = \Omega(IC^*(L, N_i, i \in M'_Z), N_i, i \in M_Z)$$

then the spectral sequence is associated to the double complex converging to the cohomology of the associated simple complex $H^{p+q}(IC^*L(LogZ))$.

We define as well the modules $P_k^J(N_{J'_Z}L)$ and the corresponding global sheaves $\mathcal{N}_{J'_Z}\mathcal{L}$ with support on $Y^*_{J'_Z}$ as well $\mathcal{P}_k^J(\mathcal{N}_{J'_Z}\mathcal{L})$ with support on $Y_{J'_Z} \cap Y_{J_Z} \cap Y^*_M$. We deduce from the above corollary 5.2

Proposition 5.5. *The induced filtrations W and F on $IC^*\mathcal{L}(LogZ)$ form a structure of MHC such that the graded perverse sheaves for the weight filtration, satisfy the decomposition property into intermediate extensions for all k (corollary 5.2)*

$$Gr_k^W IC^*\mathcal{L}(LogZ) \xrightarrow{\sim} \oplus_{J \subset I}(i_{Y_J})_* j_{!*}\mathcal{P}^J_{k-|J|}(\mathcal{N}_{J'_Z}\mathcal{L})[-|J|].$$

where $Z := \cup_{i \in I_Z} Y_i$, $J'_Z := J - (J \cap I_Z)$, $Y_{J'_Z} := \cap_{i \in J'_Z} Y_i$, $\mathcal{N}_{J'_Z}\mathcal{L}$ on $Y^*_{J'_Z}$, $\mathcal{N}_{J'_Z}$ the composition of $\mathcal{N}_i, i \in J'_Z$ and j denotes uniformly the inclusion of Y^*_J into Y_J for each $J \subset I$. In particular for $J = \emptyset$, $Y_\emptyset = X$ and we have $j_{!*}Gr_k^W\mathcal{L}$; otherwise the summands are supported by Y_J.

The MHS on $\mathbb{H}^*(X - Z, j_{!*}\mathcal{L})$ is defined by $(IC^*\mathcal{L}(LogZ), W, F)$.

We remark that $\mathcal{P}^J_{k-|J|}(\mathcal{N}_{J'_Z}\mathcal{L})[-m]$ are local systems in degree 0 on Y^*_J shifted in the formula by $[m - |J|]$ the dimension of Y^*_J and associated with the image of $\mathcal{L}$ by the endomorphism $\mathcal{N}_{J'_Z}$ on $Y^*_{J'_Z}$.

Remark 5.6. For any NCD Z such that $Z \cup Y$ is still a NCD, we may always suppose that $\mathcal{L}$ is a variation of MHS on $X - (Y \cup Z)$ (by enlarging Y) and consider Z equal to a union of components of $Y \cup Z$.

5.1.4. *Thom-Gysin isomorphism.* Let H be a smooth hypersurface intersecting transversally Y such that $H \cup Y$ is a NCD, $i_H : H \to X$, $j_H : H - (Y \cap H) \to H$, then $i_H^* j_{!*}\mathcal{L}$ is isomorphic to the (shifted) intermediate extension $(j_H)_{!*}(i_H^*\mathcal{L})$ of the restriction of $\mathcal{L}$ to H, with abuse of notations as we should write $((j_H)_{!*}((i_H^*\mathcal{L})[-1]))[1]$. The residue with respect to H, $R_H : i_H^*(IC^*\mathcal{L}(LogH)/IC^*\mathcal{L}) \xrightarrow{\sim} i_H^* IC^*\mathcal{L}[-1]$ induce an isomorphism, inverse to Thom-Gysin isomorphism

$$i_H^* j_{!*}\mathcal{L}[-1] \xrightarrow{\sim} i_H^! j_{!*}\mathcal{L}[1] \xrightarrow{\sim} i_H^*(IC^*\mathcal{L}(LogH)/IC^*\mathcal{L}).$$

Moreover, if H intersects transversally $Y \cup Z$ such that $H \cup Y \cup Z$ is a NCD, we have a triangle

$$(i_H)_* Ri_H^! IC^*\mathcal{L}(LogZ) \to IC^*\mathcal{L}(LogZ) \to IC^*\mathcal{L}(LogZ \cup H) \overset{[1]}{\to}$$

then we deduce an isomorphism of the quotient complex with the cohomology with support: $(IC^*\mathcal{L}(LogZ \cup H)/IC^*\mathcal{L}(LogZ)) \xrightarrow{\sim} Ri_H^! IC^*\mathcal{L}(LogZ)[1]$, induced by the connecting isomorphism.

We have also an isomorphism of the restriction $i_H^* IC^*\mathcal{L}(LogZ)$ with the complex $IC^*(i_H^*\mathcal{L})(LogZ \cap H)$ constructed directly on H.

The residue with respect to H: $IC^*\mathcal{L}(LogZ \cup H)[1] \to i_{H*}IC^* i_H^*\mathcal{L}(LogZ \cap H)$ vanish on $IC^*\mathcal{L}(LogZ)$ and induces an inverse to the Thom-Gysin isomorphism

$i_H^* IC^*\mathcal{L}(LogZ) \xrightarrow{\sim} Ri_H^! IC^*\mathcal{L}(LogZ)[2]$. The above constructions are compatible with the filtrations up to a shift in degrees.

5.2. **Duality and hypercohomlogy of the link.** The link at a point $v \in V$ refers to the boundary of a ball with center v intersecting transversally the strata in a small neighborhood of v. It is a topological invariant, hence its hypercohomology is well defined. In the case of a proper morphism $f : X \to V$, we refer to the hypercohomology of the boundary of a tubular neighboorhood of $Z := f^{-1}(v)$, that is the inverse image of the link at v.

The duality functor D in the derived category of sheaves of vector spaces over $\mathbb{Q}$ (resp. $\mathbb{C}$) is used here (see [BBDG 83] and the references there), as well the cone construction, to deduce various logarithmic complexes from the structure of mixed Hodge complex on $IC^*\mathcal{L}(LogZ)$.

5.2.1. *Duality.* To develop a comprehensive theory of weights, we recall some conventions. The fields $\mathbb{Q}$ and $\mathbb{C}$ form a HS of weight 0. If H is a HS of weight a, we write $H(r)$ for the HS of weight $a - 2r$.

i) A Hodge complex K of $\mathbb{Q}-$vector spaces is of weight a if its cohomology $H^i(K)$ is a HS of weight $a + i$. We set $\mathbb{Q}[r]$ for the complex with $\mathbb{Q}$ in degree $-r$ and zero otherwise, then $H^{-r}(\mathbb{Q}[r]) = \mathbb{Q}$ is a HS of weight 0, hence $\mathbb{Q}[r]$ must be a HC of weight r.

On a smooth proper variety X, the dual of $H^i(X, \mathbb{Q})$ has weight $-i$, while $H^{2n-i}(X, \mathbb{Q}) = \mathbb{H}^{-i}(X, \mathbb{Q}[2n])$ has weight $2n - i$. We write Poincaré duality with value in $\mathbb{Q}[2n](n)$ of weight 0 , such that $H^{2n-i}(X, \mathbb{Q}(n)) = \mathbb{H}^{-i}(X, \mathbb{Q}[2n](n))$ with weight $-i$ corresponds to the dual HS on $H^i(X, \mathbb{Q})$.

Hence, it is convenient to set $\omega_X := \mathbb{Q}_X[2n](n)$ for the dualizing complex in the category of sheaves of $\mathbb{Q}-$HS.

ii) A *pure variation* of HS $\widetilde{\mathcal{L}}$ on a Zariski open subset $U \subset X$ of weight $w(\widetilde{\mathcal{L}}) = b$, has a dual variation $\widetilde{\mathcal{L}}^*$ of HS of weight $w(\widetilde{\mathcal{L}}^*) = -b$. A polarization on $\widetilde{\mathcal{L}}$ induces an isomorphism defined by the underlying non degenerate bilinear product $S : \widetilde{\mathcal{L}} \xrightarrow{\sim} \widetilde{\mathcal{L}}^*(-b)$.

As a complex of sheaves, $\mathcal{L} := \widetilde{\mathcal{L}}[n]$ has weight $w = a = b + n$. Its dual in the derived category $D\mathcal{L} := \mathcal{H}om(\mathcal{L}, \mathbb{Q}_X[2n](n)) \xrightarrow{\sim} \widetilde{\mathcal{L}}^*(n)[n]$, has weight $w = -b + n - 2n = -a$. Then, the polarization induces an isomorphism $S : \mathcal{L} \xrightarrow{\sim} D\mathcal{L}(-a)$. In the category of topological constructible sheaves, this isomorphism extends into:

$j_{!*}\mathcal{L} \simeq j_{!*}(D\mathcal{L}(-a)) \xrightarrow{\sim} D(j_{!*}\mathcal{L})(-a) := R\mathcal{H}om(j_{!*}\mathcal{L}, \mathbb{Q}_X[2n])(-a)$.

Lemma 5.7. *For X compact and a shifted polarized variation of HS $\mathcal{L}$ of weight a, we have the auto-duality isomorphism with equal weights*

$$\mathbb{H}^i(X, j_{!*}\mathcal{L})^* \xrightarrow{\sim} \mathbb{H}^{-i}(X, j_{!*}\mathcal{L})(a) \tag{5.2}$$

where $w(\mathbb{H}^i(X, j_{!*}\mathcal{L})^*) = -(a + i)$ and $w(H^{-i}(X, j_{!*}\mathcal{L})(a)) = a - i - 2a$.

The lemma follows from the polarization $S : j_{!*}\mathcal{L} \xrightarrow{\sim} D(j_{!*}\mathcal{L})(-a)$ and the duality

$\mathbb{H}^i(X, j_{!*}\mathcal{L})^* = H^{-i}(Hom(\mathbb{R}\Gamma(X, j_{!*}\mathcal{L}), \mathbb{Q}) \xrightarrow{\sim} H^{-i}(X, D(j_{!*}\mathcal{L})) \xrightarrow{\sim} H^{-i}(X, j_{!*}\mathcal{L})(a)$

iii) *Dual filtrations.* The dual of a variation of MHS $(\widetilde{\mathcal{L}}, W, F)$ on a smooth Zariski open subset U, is a variation of MHS $(\widetilde{\mathcal{L}}^*, W, F)$, with filtrations dual to W and F on $\mathcal{L}$, such that $Gr^W_{-i}(\widetilde{\mathcal{L}}^*) \simeq (Gr^W_i \widetilde{\mathcal{L}})^*$.

In the bifiltered derived category of an abelian category with a dualizing functor D, the dual of a bifiltered complex (K, W, F), is denoted by (DK, W, F) with dual filtrations defined by:

$$W_{-i}DK := D(K/W_{i-1}), F^{-i}DK := D(K/F^{i+1})$$

such that: $DGr_i^W K \xrightarrow{\sim} Gr_{-i}^W DK$ and $DGr_F^i K \xrightarrow{\sim} Gr_F^{-i} DK$.

As a complex of sheaves, $(\mathcal{L} := \widetilde{\mathcal{L}}[n], W, F)$ has its weight increased by n, such that $W^i(\mathcal{L}) := W^{i-n}(\widetilde{\mathcal{L}})[n]$, and $F_i(\mathcal{L}) := F_i(\widetilde{\mathcal{L}})[n]$. Thus the filtration on its cohomology $H^{-n}(\mathcal{L})$, decreased by n, satisfy $(H^{-n}(\mathcal{L}), W) = (\widetilde{\mathcal{L}}, W)$.

The dual in the bifiltered derived category

$$D(\mathcal{L}, W, F) := \mathbb{R}Hom((\mathcal{L}, W, F), \mathbb{Q}_U[2n](n)),$$

is a shifted variation of MHS $(D\mathcal{L}, W, F)$ such that: $Gr_{-i}^W(D\mathcal{L}, F) \simeq D(Gr_i^W \mathcal{L}, F)$.

Thus, the dual of a mixed Hodge complex on a variety X is a MHC.

A graded polarization on a variation of MHS is defined by a family of polarizations S_i on $(Gr_i^W \mathcal{L}, F)$. It induces isomorphisms:

$$(Gr_i^W \mathcal{L}, F) \xrightarrow{\sim} D(Gr_i^W \mathcal{L}, F)(-i) \xrightarrow{\sim} (Gr_{-i}^W D\mathcal{L}, F)(-i)$$

5.2.2. *Structure of mixed Hodge complex on $Ri_Z^! j_{!*}\mathcal{L}$, $i_Z^* j_{!*}\mathcal{L}$ and $j_{Z!}(j_{!*}\mathcal{L}_{|X-Z})$.* We use the duality $DRi_Z^! j_{!*}\mathcal{L} \simeq i_Z^* Dj_{!*}\mathcal{L} \simeq i_Z^* j_{!*}\mathcal{L}(a)$ to deduce the weight filtrations on $i_Z^* j_{!*}\mathcal{L}$.

Definition 5.8 (Dual filtrations)**.** Let $\mathcal{L}$ be a shifted polarized VHS of weight a on a Zariski open subset $U := X - Y$ where Y is a NCD in a smooth projective variety X, and Z a closed sub-NCD of Y.

i) The filtrations on $Ri_Z^! j_{!*}\mathcal{L}$ are defined by the isomorphism

$$(Ri_Z^! j_{!*}\mathcal{L}, W, F) \simeq i_Z^*(IC^*\mathcal{L}(LogZ)/IC^*\mathcal{L}, W, F)[-1].$$

such that $W_i(i_{Z*}Ri_Z^! j_{!*}\mathcal{L}) = W_{i+1}(IC^*\mathcal{L}(LogZ)/IC^*\mathcal{L})[-1]$.

ii) The filtrations on the complex $i_Z^* j_{!*}\mathcal{L}$ are deduced by duality

$$(i_Z^* j_{!*}\mathcal{L}, W, F) \simeq D(i_Z^! j_{!*}\mathcal{L}(a), W, F)$$

iii) The filtrations on $j_{Z!}(j_{!*}\mathcal{L}_{|X-Z})$ are deduced by duality:

$$(j_{Z!}((j_{!*}\mathcal{L})_{|X-Z}), W, F) \simeq D(IC^*\mathcal{L}(a)(LogZ), W, F)$$

Corollary 5.9. *The weights w satisfy the following inequalities:*

$$w\mathbb{H}^i(X - Z, j_{!*}\mathcal{L}) \geq a + i, w\mathbb{H}_Z^i(X, j_{!*}\mathcal{L}) \geq a + i$$
$$w\mathbb{H}^i(Z, j_{!*}\mathcal{L}) \leq a + i, w\mathbb{H}_c^i(X - Z, j_{!*}\mathcal{L}) \leq a + i$$

The weights of $IC^*\mathcal{L}(LogZ)$ as a MHC, are $\geq a$ by construction. The weights of $Ri_Z^!\mathcal{L}$ as a MHC, are $\geq a$ since the weights of $IC^*\mathcal{L}(LogZ)/IC^*\mathcal{L}$ are $\geq a + 1$. We use duality computations for the weights of $\mathbb{H}^i(Z, j_{!*}\mathcal{L})$:
$\mathbb{H}^i(Z, Gr_j^W i_Z^* j_{!*}\mathcal{L}) \simeq \mathbb{H}^i(Z, Gr_j^W Di_Z^! j_{!*}\mathcal{L}(a)) \simeq \mathbb{H}^i(Z, D(Gr_{-j}^W i_Z^! j_{!*}\mathcal{L}(a))(-j)) \simeq \mathbb{H}^{-i}(Z, (Gr_{-j}^W i_Z^! j_{!*}\mathcal{L}(a))(-j))^*$
where $w(Gr_{-j}^W i_Z^! j_{!*}\mathcal{L}(a))(-j)) = -a-j+2j$, hence $w\mathbb{H}^{-i}(Z, (Gr_{-j}^W i_Z^! j_{!*}\mathcal{L}(a))(-j))^* = a + i - j \leq a + i$ since it vanishes for $j < 0$.

Lemma 5.10. *Let $\mathcal{L}$ be a polarized VHS of weight a on a Zariski open subset $X-Y$ and Z a closed subvariety of X projective.*

i) There exists a long exact sequence of MHS

$$(5.3)\quad \cdots \to \mathbb{H}^i_Z(X, j_{!*}\mathcal{L}) \to \mathbb{H}^i(X, j_{!*}\mathcal{L}) \to \mathbb{H}^i(X-Z, j_{!*}\mathcal{L}) \to \mathbb{H}^{i+1}_Z(X, j_{!*}\mathcal{L}) \to \cdots$$

with weights
$w(\mathbb{H}^i_Z(X, j_{!*}\mathcal{L})) \geq a+i, w(\mathbb{H}^i(X, j_{!*}\mathcal{L})) = a+i, w(\mathbb{H}^i(X-Z, j_{!*}\mathcal{L})) \geq a+i.$

ii) We have a dual exact sequence of MHS

$$(5.4)\quad \to \mathbb{H}^i_c(X-Z, j_{!*}\mathcal{L}) \to \mathbb{H}^i(X, j_{!*}\mathcal{L}) \to \mathbb{H}^i(Z, j_{!*}\mathcal{L}) \to \mathbb{H}^{i+1}_c(X-Z, j_{!*}\mathcal{L}) \to \cdots$$

with weights
$w(\mathbb{H}^i_c(X-Z, j_{!*}\mathcal{L})) \leq a+i, w(\mathbb{H}^i(X, j_{!*}\mathcal{L})) = a+i, w(\mathbb{H}^i(Z, j_{!*}\mathcal{L})) \leq a+i.$

The lemma is proved first for X smooth projective, Y a NCD and $Z \subset Y$ a sub-NCD. The exact sequence is associated to the triangle $Ri^!_Z j_{!*}\mathcal{L} \to j_{!*}\mathcal{L} \to Rj_{Z*}(\mathcal{L}_{|X-Z})$ and the inequalities on the weights follow from the corollary above.

In the general case, the lemma is deduced from the case of NCD by application of the decomposition theorem as in the subsection 5.17.

Example. i) For a pure $\mathcal{L}$ of weight a, $i_{Z*}Ri^!_Z j_{!*}\mathcal{L}$, is supported by Z, of weights $w_i \geq a$, and $i_{Z*}i^*_Z j_{!*}\mathcal{L}$ has weights $w_i \leq a$. The Intersection morphism I induces: $Gr^W_i i_{Z*}Ri^!_Z j_{!*}\mathcal{L} \to Gr^W_i j_{!*}\mathcal{L} \to Gr^W_i i_{Z*}i^*_Z j_{!*}\mathcal{L}$ for $i = a$ and 0 for $i \neq a$.

ii) A polarized VHS on $\mathbb{C}^*$ corresponds to the action of a nilpotent endomorphism N on L. The duality in the assertion ii) at 0 between $Gr^W_{-r+1}i^*_Z(j_{!*}\mathcal{L}))[-1]$ and $Gr^W_r i^*_Z(IC^*\mathcal{L}(LogZ)/IC^*\mathcal{L})$ corresponds to the duality between $Gr^W_{-r+1}\mathrm{Ker}N$ and $Gr^W_{r-1}L/NL$. For higher dimension, one may introduce strict simplicial coverings of a NCD to have such interpretation.

5.2.3. *Mixed cone.* We set $W_{j+1}(K[1]) := (W_jK)[1]$. The mixed cone over a morphism $f : (K, W, F) \to (K', W', F')$ is the cone $(C(f), W, F)$ with the corresponding filtrations on $K[1] \oplus K'$.

5.2.4. *Weight filtration on $i_{Z*}i^*_Z Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})$ and on the cohomology of the boundary of a tubular neighborhood of Z.*

Let $\mathcal{K} := (j_{!*}\mathcal{L})_{|X-Z}$, we have the following triangles

$$(5.5)\qquad j_{Z!}\mathcal{K} \xrightarrow{can} Rj_{Z*}\mathcal{K} \longrightarrow i_{Z*}i^*_Z Rj_{Z*}\mathcal{K}, \quad Ri^!_Z j_{!*}\mathcal{L} \xrightarrow{I} i^*_Z j_{!*}\mathcal{L} \longrightarrow i^*_Z Rj_{Z*}\mathcal{K}$$

where *can* is the composition of $j_{Z!}\mathcal{K} \xrightarrow{p} j_{!*}\mathcal{L} \xrightarrow{i} Rj_{Z*}\mathcal{K}$ and I is the restriction to Z of the composition of $i_{Z*}Ri^!_Z j_{!*}\mathcal{L} \xrightarrow{q} j_{!*}\mathcal{L} \xrightarrow{r} i_{Z*}i^*_Z Rj_{Z*}\mathcal{K}$. From which we deduce two descriptions of the complex $i^*_Z Rj_{Z*}\mathcal{K}$, as the restriction of the cone complex over the morphism *can* or as the cone over I.

We remark that the morphism p is a morphism of perverse sheaves. We have a triangle

$$i^*_Z Rj_{Z*}\mathcal{K}[-1] \to j_{Z!}\mathcal{K} \xrightarrow{p} j_{!*}\mathcal{L} \xrightarrow{[1]}$$

where $\mathrm{Ker}\, p \simeq i_{Z*}i^*_Z Rj_{Z*}\mathcal{K}[-1]$ is dual to $IC^*\mathcal{L}(LogZ)/IC^*\mathcal{L}$. In particular we put on $\mathrm{Ker}\, p$ the dual filtrations W and F.

Definition 5.11 ($i_{Z*}i^*_Z Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})$)**.** Let *can* be defined as a bifiltered morphism of complexes with their filtrations W and F

$$can : j_{Z!}(j_{!*}\mathcal{L}_{|X-Z}) \to Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})$$

The MHS on $\mathbb{R}\Gamma(Z, i^*_Z Rj_{Z*}(j_{!*}\mathcal{L})_{|X-Z}$ is defined by the mixed cone $C(can)$.

Remark 5.12. Let B_Z be a neighborhood of Z, small enough to be a deformation retract of Z. The mixed cone $C(I)$ over the Intersection morphism $I : i_{Z*}Ri_Z^!j_{!*}\mathcal{L} \xrightarrow{I} i_{Z*}i_Z^*j_{!*}\mathcal{L}$ is quasi-isomorphic to $i_{Z*}i_Z^*Rj_{Z*}(j_{!*}\mathcal{L})_{|X-Z}$ and define a MHS on the hypercohomology of $B_Z^* := B_Z - Z$ with $Z \in B_Z$ deleted.

$$\mathbb{H}^i(Z, i_{Z*}i_Z^*Rj_{Z*}((j_{!*}\mathcal{L})_{|X-Z})) \simeq \mathbb{H}^i(B_Z^*, j_{!*}\mathcal{L})$$

We have an exact sequence of MHS

$$\cdots \to \mathbb{H}^i(X - Z, j_{!*}\mathcal{L}) \to \mathbb{H}^i(B_Z^*, j_{!*}\mathcal{L}) \to \mathbb{H}^{i+1}(X, j_!(j_{!*}\mathcal{L}_{|X-Z})) \to \cdots$$

The cone depends only on the neighborhood of Z and not on X, and we have a duality isomorphism

$$i_Z^*Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})[-1] \xrightarrow{\sim} D(i_Z^*Rj_{Z*}(j_{!*}\mathcal{L}_{|X-Z})).$$

To realize the morphism I, we introduce the shifted cone $C(i)[-1]$ over the embedding $i : IC^*\mathcal{L} \to IC^*\mathcal{L}(LogZ)$. We have an isomorphism $C(i)[-1] \simeq i_{X_v*}Ri_Z^!j_{!*}\mathcal{L}$ such that the projection $C(i)[-1] \to IC^*\mathcal{L}$ is well defined and compatible with the filtrations; then I is the composition with the restriction map $IC^*\mathcal{L} \to i_{Z*}i_Z^*IC^*\mathcal{L}$.

The MHS on the hypercohomology of B_Z^* is defined equivalently by the mixed cone over the morphism *can* or the Intersection morphism I

Lemma 5.13. *There exist an isomorphism of mixed cones $C(I) \xrightarrow{\sim} C(can)$ compatible with both filtrations W and F.*

The isomorphism is constructed by comparison with the cone over the morphism $j_{Z!}(j_{!*}\mathcal{L}_{|X-Z}) \oplus i_{Z*}Ri_Z^!j_{!*}\mathcal{L} \to j_{!*}\mathcal{L}$, such that we have a diagram with coefficients in $j_{!*}\mathcal{L}$ where $X^* = X - Z$, relating two exact sequences denoted by (1) and (2)

$$\begin{array}{ccccccccc}
\mathbb{H}_Z^r(X) & & \xrightarrow{1} & & \mathbb{H}^r(Z) & & & & \mathbb{H}_Z^{r+1}(X) \\
 & \searrow & & \nearrow & & \searrow 1 & & \nearrow 1 & \\
 & & \mathbb{H}^r(X) & & & & \mathbb{H}^r(B_Z^*) & & \\
 & \nearrow & & \searrow & & \nearrow 2 & & \searrow 2 & \\
\mathbb{H}_c^r(X^*) & & \xrightarrow{2} & & \mathbb{H}^r(X^*) & & & & \mathbb{H}_c^{r+1}(X^*)
\end{array} \tag{5.6}$$

5.3. **Compatibility of the perverse filtration with MHS.** Let V be a quasi-projective variety and $K \in D_c^b(V, \mathbb{Q})$ a complex with constructible cohomology sheaves. The perverse filtration ${}^p\tau$ on K induces a perverse filtration ${}^p\tau$ on the hypercohomology groups $\mathbb{H}^k(V, K)$. This topological filtration is described by algebraic-geometric techniques in [CaMi 10] as follows.

Let $V \subset \mathbb{P}^N$ be an affine embedding and $n := \dim V$. We consider two families of hyperplanes $\Lambda_* := \Lambda_i$ and $\Lambda'_* := \Lambda'_i$ for $1 \le i \le n$ in $\mathbb{P}^N$, intersecting V in two families H_* and W_* of increasing closed sub-varieties of V, where $H_{-r} := \cap_{1\le i\le r}\Lambda_i \cap V$ and $W_{-r} := \cap_{1\le i\le r}\Lambda'_i \cap V$:

$$H_* : V = H_0 \supset H_{-1} \supset \ldots \supset H_{-n}, \quad W_* : V = W_0 \supset W_{-1} \supset \ldots \supset W_{-n} \tag{5.7}$$

and $H_{-n-1} = \emptyset = W_{-n-1}$. Let $h_i : (V - H_{i-1}) \to V$ denote the open embeddings with indices in $[-n, 0]$. The following decreasing filtration δ

$$\delta^p\mathbb{H}^*(V, K) := \mathrm{Im}\Big(\oplus_{i+j=p}\mathbb{H}^*_{W_{-j}}(V, (h_i)_!h_i^*K) \to \mathbb{H}^*(V, K)\Big) \tag{5.8}$$

defined in ([CaMi 10], remark 3.6.6) is well adapted to computation in Hodge theory. We have ([CaMi 10], Theorem 4.2.1)

Proposition 5.14. *For an affine embedding of V into a projective space and for a general choice of both families H_* and W_* depending on K and the embedding of V, the filtration δ is equal to the perverse filtration ${}^p\tau$ up to a change in indices. Precisely*

$$\delta^i\mathbb{H}^*(V,K) = {}^p\tau_{-i+l}\mathbb{H}^l(V,K)$$

5.3.1. *Proof of the proposition 1.6.* In the case of a projective morphism $f : X \to V$ of complex varieties where V is projective, let $Y \subset X$ be a closed subvariety, $\mathcal{L}$ a local system on $X - Y$, $j : (X - Y) \to X$, W a closed subvariety of V, $Z := f^{-1}W$ the inverse image of W and $j_Z : (X - Z) \to X$.

We apply the above result to $K := Rf_*R(j_Z)_*(j_Z) * j_{!*}\mathcal{L}$ (including $Z = Y$ or $Z = \emptyset$) (the case $Rf_*(j_Z)_!(j_Z) * j_{!*}\mathcal{L}$ is dual). The perverse filtration ${}^p\tau$ on $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L})$ is defined via the isomorphism $\mathbb{H}^k(X - Z, j_{!*}\mathcal{L}) \simeq \mathbb{H}^k(V - W, K)$.

When Z is a NCD in X, we check that the induced filtration ${}^p\tau$ on $\mathbb{H}^k(X-Z, j_{!*}\mathcal{L})$ is compatible with the MHS (that is a filtration by sub-MHS) as stated in the proposition 1.6.

We suppose f a fibration by NCD over the strata, Y a NCD in X and $\mathcal{L}$ an admissible variation of MHS on $X - Y$, W union of strata such that $Z := f^{-1}W$ and $Z \cup Y$ are NCD.

Starting with the complex $(IC^*\mathcal{L}(LogZ), W, F)$ (4.3.1), the system of truncations maps $\cdots \to {}^p\tau_{\leq -i}Rf_*IC^*\mathcal{L}(LogZ) \to {}^p\tau_{\leq -i+1}Rf_*IC^*\mathcal{L}(LogZ)\cdots$ is defined over $\mathbb{Q}$ and it is isomorphic in $D^b_c(V,\mathbb{C})$ to a system of inclusions maps $\cdots \to P^iK \to P^{i-1}K\cdots$ where $P^iK \simeq {}^p\tau_{\leq -i}Rf_*IC^*\mathcal{L}(LogZ)$ is decreasing and where we suppose K, all P^iK and Gr^i_PK injective complexes ([BBDG 83], 3.1.2.7) defined over $\mathbb{Q}$.

The filtration P on K is defined up to unique isomorphism in the category of filtered complexes. Moreover, we represent the filtrations W and F by acyclic resolutions for the global sections functor $\Gamma(V,*)$ on the same complex (K, P, W, F) representative of $(Rf_*IC^*\mathcal{L}(LogZ), P, W, F)$.

To prove that $\Gamma(V, Gr^P_iK, W, F)$ is a MHC and $\mathbb{H}^r(V, Gr^P_iK, W, F)$ has an induced MHS, it is enough to check the following lemma

Lemma 5.15. *Let $f : X \to V$ be a fibration by NCD over the strata, and W a closed subvariety of V such that $Z := f^{-1}(W)$ is a NCD in X. The induced filtration δ (5.8) on $\mathbb{H}^j(X - Z, j_{!*}\mathcal{L})$ corresponds to a filtration δ' by sub-MHS on $\mathbb{H}^j(X, IC^*\mathcal{L}(LogZ))$. Consequently the perverse filtration ${}^p\tau$ on $\mathbb{H}^j(X - Z), j_{!*}\mathcal{L})$ is compatible with the MHS.*

Proof. We reduce the proof to the case of an affine embedding of $U := V - W$. Let $\pi : \widetilde{V} \to V$ be a blowing up of W such that the embedding of $U \hookrightarrow \widetilde{V}$ is affine. Since $f^{-1}(W)$ is a NCD in X, the morphism $f : X \to V$ factors as $\pi \circ g$ with $g : X \to \widetilde{V}$.

We apply the proposition 5.14 to the embedding of U in $\widetilde{V}$, to construct the families H_* and W_* in $\widetilde{V}$ as intersection with two general families of hyperplanes Λ_* and Λ'_* in the ambient projective space with inverse image $H'_* := g^{-1}H_*$ and $W'_* := g^{-1}W_*$ in X, such that the various intersections $H'_i \cap W'_j$ are *transversal and intersect transversally* the various strata in X. Let $h'_i : (X - H'_i) \to X$,

$i_{H'_i} : H'_i \to X$ and $K' := IC^*\mathcal{L}(LogZ)$. The filtration in the formula 5.8 is written on X as

$$\delta^p\mathbb{H}^*(X,K') := \mathrm{Im}(\oplus_{i+j=p}\mathbb{H}^*_{W'_{-j}}(X,(h'_i)_!h_i'^*K') \to \mathbb{H}^*(X,K')) \tag{5.9}$$

Since the inverse image H'_* and W'_* are transversal in X and not critical for $\mathcal{L}$, the construction of the intermediate extension $j_{!*}\mathcal{L}$ of a shifted VHS on $X-Y$, commute with the restriction to each H'_i for various indices i. In particular, $(h'_i)_!h_i'^*K'$ is the mixed cone $C(\rho')$ over the canonical morphism $\rho' : K' \to i_{H'_i} * i^*_{H'_i}K'$.

Since W'_{-j} is not an hypersurface, we introduce the blowing up $\pi_{-j} : \widetilde{X} \to X$ of W'_{-j} in X such that $\widetilde{W'_{-j}} := \pi_{-j}^{-1}(W'_{-j})$ is a smooth hypersurface. Then, we construct the bifiltered logarithmic complex $K'' := IC^*\mathcal{L}(Log(\pi_j^{-1}Z) \cup \widetilde{W'_{-j}})$ on $\widetilde{X}$ and the cone $C(\rho'')$ over the morphism $\rho'' : R\pi_{-j*}K'' \to i_{W'_{-j}*}i^*_{W'_{-j}}K'$, such that $C(\rho'') \simeq i^!_{W'_{-j}}K'$. This construction apply similarly to $i_{H'_i} * i^*_{H'_i}K'$, hence we define $\rho'' : C(\rho') \to C(\rho')$ on the cone $C(\rho')$ and form the mixed cone $C(\rho'')$ to put the structure of bifiltered logarithmic complex on $Ri^!_{W'_{-j}}(h'_i)_!h_i'^*K'$. □

5.3.2. *The proof of the proposition 1.7 is similar but does not follow from* [CaMi 10]. We resume the notations of the proposition 1.7 and consider a local embedding of V at a point v in a ball $B \in \mathbb{C}^N$, inducing a Stein neighborhood $B_v := B \cap V$ of v. We check that the proof of the proposition 5.14 in the case of an algebraic open subset, can be adapted to the case of the punctured ball $B^*_v := B_v - \{v\}$.

We consider two families of hyperplanes Λ_* and Λ'_* containing v in general position in $\mathbb{C}^N$, and define by intersecting with B_v two families of sections H_* and W_* on B_v where $H_{-r} := \cap_{1\le i\le r}\Lambda_i \cap B_v$ and $W_{-r} := \cap_{1\le i\le r}\Lambda'_i \cap B_v$:

$$H_* : V = H_0 \supset H_{-1} \supset \ldots \supset H_{-n}, \quad W_* : V = W_0 \supset W_{-1} \supset \ldots \supset W_{-n} \tag{5.10}$$

with negative indices $-i$ for $i \in [0,n]$ and let $H_{-n-1} = \emptyset = W_{-n-1}$.

To get an affine embedding, we consider the blow up $\pi : \overline{B}_v \to B_v$ of $v \in B_v$. The two families of sections lift *strictly* into two families $\overline{H}_*$ and $\overline{W}_*$ in $\overline{B}_v$.

Let $\widetilde{B}^*_v := \overline{B}_v - \pi^{-1}(v)$, $\widetilde{H}_i := \overline{H}_i \cap \widetilde{B}^*_v$ and $\widetilde{W}_i := \overline{W}_i \cap \widetilde{B}^*_v$ such that $\widetilde{B}^*_v - \widetilde{H}_i \simeq B_v - H_i$ (resp. $\widetilde{B}^*_v - \widetilde{W}_i \simeq B_v - W_i$) is Stein for each index i. This Stein condition with the general position arguments are sufficient for the proof.

5.3.3. Set $h'_j : (\widetilde{B}^*_v - \widetilde{H}_j) \to \widetilde{B}^*_v$ and $i_{\widetilde{W}_j} : \widetilde{W}_j \to \widetilde{B}^*_v$. Let $K \in D^b_c(B^*_v, \mathbb{Q})$ be a complex on B^*_v or equivalently on the isomorphic space $\widetilde{B}^*_v$ and define after H_* and W_* two decreasing filtrations F and G on K by

$$F^pK := (h'_{p-1})_!h'_{p-1}{}^*K \text{ for } p \in [-n,0] \text{ where } F^{-n}K = K.$$

$$G^pK := (i^!_{\widetilde{W}_{-p}})K \text{ for } p \in [0,n] \text{ where } G^0K = K.$$

The filtration $\delta := \delta(F,G)$ is the diagonal filtration

$$\delta^p := \textstyle\sum_{i+j=p} F^iK \cap G^jK \text{ for } [-n,n].$$

Let $Dec(\delta)$ denote the filtration décalé of δ [De 72]. By construction the spectral sequences of the induced filtrations δ and $Dec(\delta)$ on $R\Gamma(\widetilde{B}^*_v, K)$ are related as follows:

$$E^{pq}_r(R\Gamma(\widetilde{B}^*_v, K), Dec(\delta)) \simeq E^{2p+q,-p}_{r+1}(R\Gamma(\widetilde{B}^*_v, K), \delta)$$

Proposition 5.16. *There exist an isomorphism of filtered complexes*

$$(R\Gamma(\widetilde{B}_v^*, K), P) \simeq (R\Gamma(\widetilde{B}_v^*, K), Dec(\delta))$$

Equivalently, there exists a convergent spectral sequence

$$E_1^{2p+q,-p}(R\Gamma(\widetilde{B}_v^*, K), \delta)) \Longrightarrow Gr_{-p}^{{}^p\tau}\mathbb{H}^{p+q}(\widetilde{B}_v^*, K).$$

$$Dec(\delta)^p\mathbb{H}^l(\widetilde{B}_v^*, K) = \delta^{p+l}\mathbb{H}^l(\widetilde{B}_v^*, K) = {}^p\tau_{-p}\mathbb{H}^l(\widetilde{B}_v^*, K)$$

5.3.4. *Proof of the proposition.* We introduce the morphisms $h_i'' : (\widetilde{H}_i - \widetilde{H}_{i-1}) \to \widetilde{B}_v^*$ and $i_j'' : (\widetilde{W}_{-j} - \widetilde{W}_{-j-1}) \to \widetilde{B}_v^*$ to construct the spectral sequences attached to the filtrations. The main argument in the proof is to find conditions on the families $\overline{H}_*$ and $\overline{W}_*$ in $\overline{B}_v$ to have the following equality

$$h_{i!}''\, h_{i-1}''^{*}\, i_j''^{!} K = i_j''^{!}\, h_{i!}''\, h_{i-1}''^{*} K \tag{5.11}$$

To this purpose, we state in terms of Stein open sub-manifolds, a result stated in terms of affine open sub-varieties [CaMi 10] theorem 5.1.2) which seems to go back to an unpublished letter of Deligne [Ar 5].

Lemma 5.17. *Let $j : (\widetilde{B}_v^* - \widetilde{H}) \to \widetilde{B}_v^*$, and $j' : (\widetilde{B}_v^* - \widetilde{H}') \to \widetilde{B}_v^*$. For any two hypersurface sections $\overline{H}$ and $\overline{H}'$ in general position depending on the Stein embedding $\widetilde{B}_v^* \to \overline{B}_v$ and the perverse sheaf K on $\widetilde{B}_v^*$, we have:*

$$\mathbb{H}^r(\widetilde{B}_v^*, j_! j^* j_*' (j')^! K = 0, \textit{ for } r \neq 0.$$

We consider an adequate stratification of $\overline{B_v}$ adapted to K on $\widetilde{B}_v^*$ (the restriction of the cohomology sheaves of K are locally constant on the strata) such that $\pi^{-1}(v)$ is a union of strata.

The condition on the families to be in general position is defined by the conditions on the families $\overline{H}_*, \overline{W}_*$ to intersect transversally and to intersect all strata transversally, then, by ([CaMi 10] lemma 6.1.5)

$$\mathbb{H}^r(B_v^*, Gr_\delta^a Gr_P^b R\Gamma(B_v^*, K) = 0, \forall r \neq a - b.$$

Indeed, this group is equal to $\mathbb{H}^r(B_v^*, \oplus_{i+j=a} h_{i!}''\, h_{i-1}''^{*}(i_j''^{!}\, {}^p\mathcal{H}^{-p}(K)[p]))$ by ([CaMi 10] subsection 3.4, assertion 9) and vanish for $r \neq a-b$ by ([CaMi 10] lemma 6.1.1 *iii'*). Finally, we apply the following criteria ([CaMi 10] prop 5.6.1) to deduce the proposition:

Let A be an abelian category with enough injectives and (L, P, F) a bifiltered complex bounded below such that

$$H^r(Gr_F^a Gr_P^b L) = 0, \forall r \neq a - b. \tag{5.12}$$

Then, there is a natural isomorphism in the filtered derived category $(L, P) \simeq (L, DecF)$ that induces the identity on L. Hence, there is a natural isomorphism of spectral sequences associated with (L, P) and $(L, DecF)$ inducing the identity on the abutments $(H^*(L), P) \simeq (H^*(L), DecF)$.

5.3.5. *End of the proof of proposition 1.7.* The inverse image $f^{-1}(v) := X_v$ is a NCD in $B_{X_v} := f^{-1}(B_v)$. Let $B_{X_v}^* : B_{X_v} - X_v$, $j_{X_v} : B_{X_v}^* \to X$ and $K' := j_{!*}\mathcal{L}_{|B_{X_v}^*}$. The restriction $f' : B_{X_v}^* \to B_v^*$ of the morphism f factors through $g' : B_{X_v}^* \to \widetilde{B}_v^*$. We apply the above results to $K := Rg_*' K' \simeq Rf_*' K'$ on $\widetilde{B}_v^* \simeq B_v^*$.

The inverse image of $\widetilde{H}_*$ and $\widetilde{W}_*$ are two families H'_* and W'_* in $B^*_{X_v}$. Set $h'_j : (B^*_{X_v} - H'_j) \to B^*_{X_v}$ and $i_{W'_j} : W'_j \to B^*_{X_v}$. We define after H'_* and W'_* two decreasing filtrations F and G on K' by

$$F^pK' := (h'_{p-1})_! {h'_{p-1}}^* K' \text{ for } p \in [-n, 0] \text{ where } F^{-n}K' = K'.$$

$$G^pK' := (i^!_{W'_{-p}})K' \text{ for } p \in [0, n] \text{ where } G^0K' = K'.$$

The filtration $\delta := \delta(F, G)$ is the diagonal filtration

$$\delta^p := \textstyle\sum_{i+j=p} F^iK' \cap G^jK' \text{ for } [-n, n].$$

Then the induced filtration δ on $\mathbb{H}^*(B^*_{X_v}, j_{!*}\mathcal{L})$, defined by

$$\delta^p\mathbb{H}^*(B^*_{X_v}, K') := \mathrm{Im}\Big(\oplus_{i+j=p}\mathbb{H}^*_{W'_{-j}}(B^*_{X_v}, (h'_i)_! {h'_i}^* K') \to \mathbb{H}^*(B^*_{X_v}, K')\Big) \tag{5.13}$$

is equal to the perverse filtration up to indices.

5.3.6. *Filtration by sub-MHS.* We prove that δ is a filtration by sub-MHS.
The restriction of the morphism f to B_{X_v}, factors through $g : B_{X_v} \to \overline{B}_v$. The inverse image $\overline{H}'_* := g^{-1}\overline{H}_*$ and $\overline{W}'_* := g^{-1}\overline{W}_*$ form two families in B_{X_v} whose mutual intersections are transversal to X_v and the strata of the stratification induced on B_{X_v}.
In particular, $\overline{W}'_j, \overline{H}'_i$ and $\overline{W}'_j \cap \overline{H}'_i$ intersect X_v transversally in B_{X_v}.

The hypercohomology of $B^*_{X_v}$ with coefficients in $j_{!*}\mathcal{L}$ is defined by the mixed cone $C(I)$ over the morphism $I : i^!_{X_v} IC^*\mathcal{L}(LogX_v) \to i^*_{X_v} IC^*\mathcal{L}(LogX_v)$, as $C(I)$ is quasi-isomorphic to $i_{X_v*}i^*_{X_v}Rj_{X_v*}j_{!*}\mathcal{L}_{|X-X_v}$ (definition 5.11, remark 5.12, and lemma 5.13).

From now on the proof is similar to the proof of lemma 5.15. Let $h_i : (B_{X_v} - H_j) \to B_{X_v}$, $i_{H_i} : H_i \to B_{X_v}$ and $i_{W_{-j}} : W_{-j} \to B_{X_v}$.

We construct a bifiltered logarithmic cone $C(\rho') \simeq (h_j)_! h^*_j(IC^*\mathcal{L}(LogX_v)_{|B_{X_v}})$, then a bifiltered logarithmic cone $C(\rho'') \simeq Ri^!_{W_{-j}}(h_j)_! h^*_j(IC^*\mathcal{L}(LogX_v)_{|B_{X_v}}$ by introducing the blowing up of W_{-j} in B_{X_v} and finally the bifiltered logarithmic cone $C(I)$ over an intersection morphism $I : i^!_{X_v}C(\rho'') \to i^*_{X_v}C(\rho'')$ such that $\mathbb{H}^*(X_v, C(I)) \simeq \mathbb{H}^*_{W'_{-j}}(B^*_{X_v}, (h'_i)_! {h'_i}^* IC^*\mathcal{L})$ has a MHS.

References

[Ar 5] E. Arapoura, The Leray spectral sequence is motivic, *Invent. Math.* 160:567–589, 2005.

[BPV 84] W. Barth, C. Peters, A. Van de Ven, Compact complex surfaces. *Ergebnisse der Mathematik, Springer, Berlin, Heidelberg*, 3.folge. Band 4, 1984.

[BBDG 83] A.A. Beilinson, J. Bernstein, P. Deligne, O. Gabber, Analyse et Topologie sur les espaces singuliers Vol.I, *Astérisque, France* 100, 1983.

[Br 82] J.L. Brylinski, (Co)homologie d'intersection et faisceaux pervers. *Séminaire Bourbaki, Astérisque, France* 585, 92-93: 129–157,1982.

[ICTP 14] E. Cattani, F. El Zein, P. A. Griffiths, D. T. Lê: Hodge theory, *Princeton University Press*, Princeton, NJ, USA, 2014.

[CaK 82] E. Cattani, A. Kaplan, Polarized mixed Hodge structures.*Invent. Math.* 67: 107–115, 1982.

[CaKSc 86] E. Cattani, A. Kaplan, W. Schmid, Degeneration of Hodge structures. *Ann. of Math.* 123: 457–535,1986.

[CaKSc 87] E. Cattani, A. Kaplan, W. Schmid, L^2 and intersection cohomologies for a polarizable variation of Hodge structure. *Invent. Math.* 87(2): 217–252, 1987.

[CaDeK 95] E. Cattani, P. Deligne, A. Kaplan, On the locus of Hodge classes. *J. Amer. Math. Soc.* 8(2): 483–506, 1995.

[CaMi 5] M.A. de Cataldo, L. Migliorini: The Hodge theory of algebraic maps, *Ann. Sci. cole Norm. Sup.*, 38: 693-750, 2005.

[CaMi 10] M.A. de Cataldo, L. Migliorini: The perverse filtration and the Lefschetz hyperplane theorem, arXiv:0805.4634, *Annals of Math.*, 171, No. 3: 2089-2113, 2010.

[DiGr 61] J. Dieudonné, A. Grothendieck, Eléments de Géométrie algébrique III. *Publ. Math. IHES, Paris* 11, 1961.

[De 68] P. Deligne, Théorèmes de Lefschetz et critères de dégénérescence de suites spectrales, *Publ. Math. IHES*, 35: 107–126, 1968.

[De 70] P. Deligne, Equations différentielles à points singuliers réguliers. *Lecture Notes, Springer, Berlin* 163, 1970.

[De 71] P. Deligne, Théorie de Hodge I, *Actes, Congrès intern. math.*, 1: 425–430, 1971.

[De 72] P. Deligne, Théorie de Hodge II. *Publ. Math. IHES, Paris* 40: 5–57, 1972.

[De 75] P. Deligne, Théorie de Hodge III. *Publ. Math. IHES, Paris* 44: 6–77, 1975.

[De 80] P. Deligne, Conjecture de Weil II. *Publ. Math. IHES, Paris* 52, 1980.

[DeK 73] P. Deligne, N. Katz, Groupes de monodromie en géométrie algébrique. *Lecture Notes in Math. 340, Springer, Berlin*, 1973.

[DeG 81] P. Deligne, O. Gabber, *Théorème de pureté d'après Gabber*. Note written by Deligne and distributed at IHES, Paris, 1981.

[Dol 71] P. Dolbeault, Valeurs principales et résidus sur les espaces analytiques. *Séminaire Pierre Lelong, Paris*,1971.

[E 74] F. El Zein, Comparaison des résidus de Grothendieck et de Herrera, *C.R.Acad.Sci. Paris, Ser. A*, 278, 863–866, 1974.

[E 78] F. El Zein, Complexe dualisant, *Bull. Amer. Math. Soc.*, .

[E 83] F. El Zein, Théorie de Hodge des cycles évanescents. *Ann. Scient. Ec. Norm. Sup.* 19: 107–184, 1986. *Notes C.R. Acad. Sc., Paris, Série I* 295: 669-672, 1982; 292:51–54, 1983; 296: 199–202, 1983.

[E 88] F. El Zein, Théorie de Hodge à coefficients: étude locale. *CR Ac. Sc.* t 307 Série I: 593–598, 1988.

[E 2] F. El Zein, A. Nemethi, On the weight filtration of the homology of algebraic varieties: the generalised Leray cycles. *Ann. Scuola Norm.Sup. Pisa* Cl. Sci. (5) Vol. I: 869 -903, 2002.

[ELM 10] F. El Zein, Le D. T., L. Migliorini, A topological construction of the weight filtration. *Manuscripta Mathematica* 133(1–2): 173–182, 2010.

[EY 14] F. El Zein, X. Ye, Filtration perverse et théorie de Hodge, *C.R.Acad.Sci. Paris, Ser. I*, 352: 1045–1049, 2014.

[EL 14] F. El Zein, D.T. Lê, Du théorème de décomposition à la pureté locale, *C.R.Acad.Sci. Paris, Ser. I*, 352: 1051–1055, 2014.

[EL 15] F. El Zein, D.T. Lê, De la pureté locale à la décomposition, *C.R.Acad.Sci. Paris, Ser. I*, 353: 75–80, 2015.

[GMacP 83] M. Goresky, R. MacPherson, Intersection homology theory. *Topology* 19: 135–162, 1980. Intersection homology II. *Inv.Math.* 72: 77–129, 1983.

[GMacP 88] M. Goresky, R. MacPherson, Stratified Morse theory. *Ergebnisse der Mathematik, Springer, Berlin, Heidelberg*, 3.folge. Band 2, 1988.

[Gr] P. Griffiths, Topics in Transcendental Algebraic Geometry. *Ann. of Math. studies*, Princeton Univ. Press.106.

[GrSc 73] P. Griffiths, W. Schmid, Recent developments in Hodge theory. Discrete subgroups of Lie Groups.*Bombay Colloquium, Oxford University Press* 1973.

[Gro 58] A. Grothendieck, The cohomology theory of abstract algebraic varieties. *Proc. Int. Cong. Math., Edimburg* 103–118, 1958.

[Gro 66] A. Grothendieck, On the de Rham cohomology of Algebraic Varieties.*Publ. Math. IHES* 29, 1966.

[Gro 72] A. Grothendieck, Groupes de Monodromie en Géométrie Algébrique. *Springer Lecture Notes in Math.* 288, 1972.

[Ha 66] R. Hartshorne, Residues and Duality. *Lecture Notes in Math. , Springer, Berlin, Heidelberg* 20, 1966.

[Ha 72] R. Hartshorne, Algebraic de Rham cohomology *Manuscripta Math., Berlin* 7: 125–140, 1972.

[Ha 75] R. Hartshorne, On the de Rham cohomology of Algebraic Varieties.*Publ. Math. IHES* 45, 6–98,1975.

[HeL 71] R. Herrera, D. Lieberman, Residues and Principal values on complex spaces. *Mathematisch Annalen.* 194: 259–294, 1971.

[HeLi 71] R. Herrera, D. Lieberman, Duality and the de Rham cohomology of infinitesimal neighborhoods. *Invent. Math., Berlin* 13: 97–124, 1971.

[Ka 86] M. Kashiwara, A study of variation of mixed Hodge structure. *Publ. RIMS, Kyoto University* 22: 991–1024, 1986.

[KaK 86] M. Kashiwara, T. Kawai, Hodge structure and holonomic systems. *Proc. Japan Acad. Ser. A* 62: 1–4, 1986.

[KaK 87] M. Kashiwara, T. Kawai, Poincaré lemma for a variation of Hodge structure. *Publ. RIMS, Kyoto University* 23: 345–407, 1987.

[KaOd 68] N. M. Katz, T. Oda, On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto University* 8: 199–213, 1968.

[La 73] A. Landman, On the Picard Lefschetz transformation acquiring general singularities. *Trans. Amer. Math. Soc.* 181: 89–126, 1973.

[LeT 83] D.T. Lê, B. Teissier, Cycles évanescents et conditions de Whitney II. *Proc. Symp. Pure Math. , Amer. Math. Soc.* 40, part 2: 65–103, 1983.

[Ma 83] B. Malgrange, *Polynôme de Bernstein-Sato et cohomologie évanescente.* Astérisque, 101-102: 243–267, 1983.

[Ma 87] B. Malgrange, Regular connections, after Deligne, in algebraic D-modules, by Borel A. *Perspective in Math. 2, Academic Press, Boston*, 151–173, 1987.

[Mat 12] J. Mather, Notes on topological stability. *Bull. Amer. Math. Soc.* 49(4): 475–506, 2012.

[Na 85] A. V. Navarro, Sur la théorie de Hodge des variétés algébriques à singularités isolées. *Astérisque* 130: 272–307, 1985.

[NaG 90] A. V. Navarro, F. Guillén, Sur le théorème local des cycles invariants. *Duke Math. J.* 61: 133–155, 1990.

[RaR 70] J. Ramis, G. Ruget, Complexe dualisant et théorèmes de dualité en géométrie analytique complexe. *Publ. Math. IHES, Paris* 38: 77–91, 1970.

[RaR 74] J. Ramis, G. Ruget, Résidus et dualité.*Inv.Math.* 26: 89–131, 1974.

[Sa 88] M. Saito, (1) Modules de Hodge polarisables. *Publ. RIMS, Kyoto University* 24: 849–995, 1988. (2) Mixed Hodge modules. *Publ. RIMS, Kyoto University* 26: 221–333, 1990.

[Sc 73] W. Schmid, Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.* 22: 211–319, 1973.

[Se 55] J.P. Serre, Un théorème de dualité.*Com.Math.Helvetici* 29: 9–26, 1955.

[St 76] J. H. M. Steenbrink, Limits of Hodge structures. *Invent. Math.* 31: 229–257, 1976.

[StZ 85] J. H. M. Steenbrink, S. Zucker, Variation of mixed Hodge structures I. *Invent. Math.* 80: 489–542, 1985.

[Ve 65] J. L. Verdier, Dualité dans la cohomologie des espaces localement compacts. *Séminaire Bourbaki, Paris* 300, 1965.

[Ve 76] J. L. Verdier, Stratifications de Whitney et théorème de Bertini–Sard. *Invent. Math.* 36: 295–312, 1976.

[Ve 77] J. L. Verdier, Catégories dérivées *Lecture Notes, Springer, Berlin* 569: 262–312, 1977.

[Ve2 85] J. L. Verdier, Extension of a perverse sheaf over a closed subspace. *Astérisque, Paris*, 130: 210–217, 1985. Prolongement des faisceaux pervers monodromiques. *Astérisque, Paris*, 130: 218–236,1985.

[Vo 7] C. Voisin, Hodge theory and complex algebraic geometry I, II. *Cambridge University Press*, 76, 77, 2007.

[Zu 79] S. Zucker, Hodge theory with degenerating coefficients: L_2-cohomology in the Poincaré-metric, *Ann. Math.*, 109: 415–476, 1979.

Institut de Mathématiques de Jussieu., Paris, France
E-mail address: `fouad.elzein@imj-prg.fr, elzein@free.fr`

Université d'Aix-Marseille LATP, UMR-CNRS 7353 39, rue Joliot-Curie F-13453 Marseille Cedex 13, France
E-mail address: `ledt@ictp.it`

School of Mathematics and Information Science in Guangzhou University, Guangzhou, P.R.China
E-mail address: `yexm3@gzhu.edu.cn`