

# More Aspects of Arbitrarily Partitionable Graphs<sup>☆</sup>

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## Abstract

A graph  $G$  of order  $n$  is arbitrarily partitionable (AP) if, for every sequence  $(n_1, \dots, n_p)$  partitioning  $n$ , there is a partition  $(V_1, \dots, V_p)$  of  $V(G)$  such that  $G[V_i]$  is a connected graph of order  $n_i$  for  $i = 1, \dots, p$ . The property of being AP is related to other well-known graph notions, such as perfect matchings and Hamiltonian cycles, with which it shares several properties. This work is dedicated to studying two aspects behind AP graphs.

On the one hand, we consider algorithmic aspects of AP graphs, which received some attention in previous works. We first establish the NP-hardness of the problem of partitioning a graph into connected subgraphs following a given sequence, for various new graph classes of interest. We then prove that the problem of deciding whether a graph is AP is in NP for several classes of graphs, confirming a conjecture of Barth and Fournier for these.

On the other hand, we consider the weakening to APness of sufficient conditions for Hamiltonicity. While previous works have suggested that such conditions can sometimes indeed be weakened, we here point out cases where this is not true. This is done by considering conditions for Hamiltonicity involving squares of graphs, and claw- and net-free graphs.

*Keywords:* arbitrarily partitionable graphs; partition into connected subgraphs; Hamiltonicity.

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## 1. Introduction

### 1.1. Partitioning Graphs into Connected Subgraphs

Throughout this work, we call a graph of order  $G$  an  $n$ -graph. By a *graph partition* of an  $n$ -graph  $G$ , it is meant a vertex-partition  $(V_1, \dots, V_p)$  of  $V(G)$  such that each  $G[V_i]$  has particular properties. This work is about partitions into connected subgraphs, which we deal with through the following terminology.

Consider a *sequence*  $\pi = (n_1, \dots, n_p)$  being a partition of  $n$ , *i.e.*,  $n_1 + \dots + n_p = n$ . We will sometimes call  $\pi$  an  $n$ -*sequence* to make clear which integer it is a partition of. When writing  $|\pi|$ , we mean the *size* of  $\pi$  (*i.e.*,  $p$ ), while, by writing  $\|\pi\|$ , we refer to the sum of its elements (*i.e.*,  $n$ ). The *spectrum*  $\text{sp}(\pi)$  of  $\pi$  is the set of values appearing in  $\pi$ .

By a *realization* of  $\pi$  of  $G$ , we mean a partition  $(V_1, \dots, V_p)$  of  $V(G)$  such that  $G[V_i]$  has order  $n_i$  and is connected for  $i = 1, \dots, p$ . In other words,  $\pi$  indicates the number of connected subgraphs we want ( $p$ ), and their respective order  $(n_1, \dots, n_p)$ . In case  $G$  admits a realization of every  $n$ -sequence, and is hence partitionable into arbitrarily many connected subgraphs with arbitrary order, we call  $G$  *arbitrarily partitionable* (AP for short). AP graphs can also be found in the literature under different names, such as “arbitrarily

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38 vertex-decomposable graphs” [1, 2, 4] or “fully decomposable graphs” [10] (but these terms  
39 might mislead the readers, as the term “decomposition” has multiple other meanings in  
40 graph theory).

41 Although the notion of AP graphs is relatively recent (it was introduced in 2002 by  
42 Barth, Baudon and Puech, to deal with a practical resource allocation problem [1]), the  
43 problem of partitioning graphs into connected subgraphs is much older. Perhaps the most  
44 influential result is the one from the 70’s due to Lovász and Gyóri [21, 17], who indepen-  
45 dently proved that an  $n$ -graph  $G$  is  $k$ -connected if and only if every  $n$ -sequence with size  $k$   
46 is realizable in  $G$ , even if the  $k$  parts are imposed to contain any  $k$  vertices chosen before-  
47 hand. We refer the interested reader to the literature (such as the Ph.D. thesis of the first  
48 author [5]) for many more results of this kind.

49 As a warm up, let us start by raising notable properties of AP graphs. First, it should be  
50 noted that, in every AP  $n$ -graph, any realization of the  $n$ -sequence  $(2, \dots, 2)$  (or  $(2, \dots, 2, 1)$   
51 if  $n$  is odd) corresponds to a perfect matching (resp. quasi-perfect matching). Note also  
52 that every AP graph being spanned by an AP graph is also AP. Since paths are obviously  
53 AP, this implies that every traceable graph (*i.e.*, graph having a Hamiltonian path) is AP.  
54 So, in a sense, the class of AP graphs lies in between the class of graphs with a (quasi-  
55 ) perfect matching and the class of traceable graphs (which itself contains the class of  
56 Hamiltonian graphs), which are well-studied classes of graphs.

57 Several aspects of AP graphs have been investigated to date. In this work, we focus  
58 on two such aspects, namely the *algorithmic aspects* and the *Hamiltonian aspects*, which  
59 are to be developed below. Again, we refer the interested reader to e.g. [5] for an in-depth  
60 overview of other aspects of interest (such as the structural aspects of AP graphs).

## 61 1.2. Algorithmic Aspects of AP Graphs

62 From the previous definitions related to graph partitions into connected subgraphs, the  
63 following two decision problems naturally arise:

64 REALIZATION

65 **Instance:** An  $n$ -graph  $G$ , and an  $n$ -sequence  $\pi$ .

66 **Question:** Is  $\pi$  realizable in  $G$ ?

67 AP

68 **Instance:** A graph  $G$ .

69 **Question:** Is  $G$  AP?

70 The NPness of REALIZATION is obvious. Its NP-hardness was investigated by several  
71 authors, who established it for numerous restrictions on either  $G$  or  $\pi$ . Concerning res-  
72 trictions on  $\pi$ , REALIZATION remains NP-hard for instances where  $\text{sp}(\pi) = \{k\}$  for every  
73  $k \geq 3$ , as shown by Dyer and Frieze [12]. Note that finding a realization of any sequence  
74  $\pi$  with  $\text{sp}(\pi) \subseteq \{1, 2\}$  is equivalent to finding a sufficiently large matching, which can be  
75 done in polynomial time using Edmonds’ Blossom Algorithm [13]; hence, such instances  
76 of REALIZATION can be solved in polynomial time. In [6], the first author showed that  
77 REALIZATION remains NP-hard for instances where  $|\pi| = k$ , for any  $k \geq 2$ .

78 Concerning restrictions of REALIZATION on  $G$ , Barth and Fournier proved, in [2], that  
79 REALIZATION remains NP-hard for trees with maximum degree 3. In [10], Broersma,  
80 Kratsch and Woeginger proved that REALIZATION remains NP-hard for split graphs. In [7],  
81 the first author proved that REALIZATION remains NP-hard when restricted to graphs with  
82 about a third universal vertices (*i.e.*, vertices neighbouring all other vertices).

83 The status of the AP problem is quite intriguing. A first important point to raise is  
 84 that, contrarily to what one could naively think, the NP-hardness of REALIZATION does  
 85 not imply that of AP. That is, in all reductions imagined by the authors above, the reduced  
 86  $n$ -graph  $G$  needs to have a very restricted structure so that a particular  $n$ -sequence  $\pi$  is  
 87 realizable under particular circumstances only; this very restricted structure makes many  
 88 other  $n$ -sequences not realizable in  $G$ , implying that it is far from being AP.

89 It is actually not even clear whether the AP problem is in NP or co-NP. For AP to be  
 90 in NP, one would need to provide a polynomial certificate attesting that all  $n$ -sequences are  
 91 realizable in  $G$ , while the number of such  $n$ -sequences is  $p(n)$ , the partition number of  $n$ ,  
 92 which is exponential in  $n$ . For AP to be in co-NP, one would need to provide a polynomial-  
 93 time algorithm for checking that a given sequence is indeed not realizable in  $G$ , while the  
 94 number of possible partitions of  $G$  into connected subgraphs is clearly exponential.

95 On the other hand, though, as pointed out in [2, 6], the AP problem matches the typical  
 96 structure of  $\Pi_2^p$  problems (“for every sequence, is there a realization?”), and thus belongs  
 97 to  $\Pi_2^p$  (recall that  $\Pi_2^p$  problems are, simply put, those decision problems where we are given  
 98 two sets  $X, Y$  of elements, each of these elements ranging in some respective sets of values,  
 99 and the question is whether, whatever the values of the elements in  $X$  are, the elements in  
 100  $Y$  can always be assigned values so that a particular property holds). However, it is still  
 101 not known whether AP is  $\Pi_2^p$ -hard. Also, we have no evidence that AP is NP-hard.

102 Regarding those questions, the main conjecture is due to Barth and Fournier [2]:

103 **Conjecture 1.1** (Barth, Fournier [2]). *The AP problem is in NP.*

104 Conjecture 1.1 relies on *polynomial kernels of sequences*, which are presumed to exist for  
 105 every graph. For an  $n$ -graph  $G$ , a *kernel* (of  $n$ -sequences) is a set  $\mathcal{K}$  of  $n$ -sequences such  
 106 that  $G$  is AP if and only if all sequences of  $\mathcal{K}$  are realizable in  $G$ . That is, a kernel for  
 107  $G$  is a (preferably small) set of sequences attesting the APness of  $G$ . We say that  $\mathcal{K}$  is a  
 108 *polynomial kernel* if its size is a polynomial function of  $n$ .

109 Note that the existence of a polynomial kernel  $\mathcal{K}$  for a given graph class indeed implies  
 110 the NPness of the AP problem for that class. All results towards Conjecture 1.1 so far are  
 111 based on proving the existence of such kernels for particular graph classes. The first result  
 112 of this kind was given by Barth, Baudon and Puech [1], who proved that, for subdivided  
 113 claws of order  $n$  (*i.e.*, trees where the unique vertex of degree more than 2 has degree 3),  
 114 the set of all  $n$ -sequences of the form  $(k, \dots, k, r)$  or  $(k, \dots, k, k + 1, \dots, k + 1, r)$  (where  
 115  $r < k$ ) is a polynomial kernel. Later on [24], Ravoux showed that, still for subdivided claws  
 116 of order  $n$ , the set of  $n$ -sequences  $\pi$  with  $|\text{sp}(\pi)| \leq 7$  is an alternative polynomial kernel  
 117 (although of bigger size, proving that this second set is indeed a kernel for the considered  
 118 graph class is much easier than proving that the first set is). Concerning other polynomial  
 119 kernels, Broersma, Kratsch and Woeginger proved in [10] that the set of  $n$ -sequences  $\pi$  with  
 120  $\text{sp}(\pi) \subseteq \{1, 2, 3\}$  is a polynomial kernel for split  $n$ -graphs. The first author also provided  
 121 more examples of polynomial kernels in [7]; in particular, for complete multipartite  $n$ -  
 122 graphs,  $n$ -sequences  $\pi$  with  $\text{sp}(\pi) \subseteq \{1, 2\}$  form a kernel.

123 An interesting side aspect is that, from the existence of these polynomial kernels, some  
 124 of the authors above also derived the polynomiality of the AP problem in certain graph  
 125 classes. In particular, Barth and Fournier proved that AP is polynomial-time solvable  
 126 when restricted to subdivided stars [2], while Broersma, Kratsch and Woeginger proved it  
 127 is polynomial-time solvable when restricted to split graphs [10].

### 128 1.3. Hamiltonian Aspects of AP Graphs

129 As mentioned in Section 1.1, the AP property can be regarded as a weakening of  
 130 traceability/Hamiltonicity. In particular, all sufficient conditions implying traceability also

131 imply APness. An interesting line of research is thus to investigate whether such conditions  
 132 can be weakened for the AP property.

133 To the best of our knowledge, only a few results of this sort can be found in the  
 134 literature. The first series of such results is related to the following parameter, defined for  
 135 any given graph  $G$ :

$$136 \quad \sigma_k(G) = \min\{d(v_1) + \dots + d(v_k) : v_1, \dots, v_k \text{ are non-adjacent vertices of } G\}.$$

137 A well-known result of Ore [23] states that every  $n$ -graph  $G$  ( $n \geq 3$ ) with  $\sigma_2(G) \geq n - 1$  is  
 138 traceable. In [22], Marczyk proved that every  $n$ -graph  $G$  ( $n \geq 8$ ) having a (quasi-) perfect  
 139 matching and verifying  $\sigma_2(G) \geq n - 3$  is AP. Later on, this result was improved by Horňák,  
 140 Marczyk, Schiermeyer and Woźniak [18], who proved that every  $n$ -graph  $G$  ( $n \geq 20$ ) having  
 141 a (quasi-) perfect matching and verifying  $\sigma_2(G) \geq n - 5$  is AP. A similar result for graphs  
 142  $G$  with large  $\sigma_3(G)$  was also claimed by Brandt [9].

143 The last weakening we are aware of, deals with the number of edges guaranteeing  
 144 APness. From known results, it can be established that connected  $n$ -graphs with more  
 145 than  $\binom{n-2}{2} + 2$  edges are traceable (see [20], Proposition 19). An analogous sufficient  
 146 condition for APness was given by Kalinowski, Piłśniak, Schiermeyer and Woźniak [20],  
 147 who proved that, a few exceptions apart, all  $n$ -graphs ( $n \geq 22$ ) with more than  $\binom{n-4}{2} + 12$   
 148 edges are AP.

#### 149 1.4. Our Results

150 In this work, we establish more results on AP graphs regarding the two aspects devel-  
 151 oped above. More precisely:

- 152 • Regarding the algorithmic aspects, we provide, in Sections 2 and 3, both positive and  
 153 negative results. We start, in Section 2, by providing an easy NP-hardness reduction  
 154 framework, for showing, through slight modifications, that REALIZATION is NP-hard  
 155 when restricted to many graph classes (see, for instance, Theorems 2.3, 2.4, 2.5). In  
 156 Section 3, we provide more polynomial kernels for several graph classes excluding  
 157 particular patterns as induced subgraphs.
- 158 • In Section 4, we consider the weakening of more Hamiltonian conditions for AP  
 159 graphs. Although this line of research seems quite appealing, the results we get  
 160 show that the distance between traceable graphs and AP graphs is more tenuous  
 161 than one could hope. This is done by considering the notions of *squares* of graphs,  
 162 and *claw-free* and *net-free graphs*. More precisely, we show that classical results on  
 163 Hamiltonicity and these notions do not weaken to the AP property (in the obvious  
 164 way, to the least).

165 We conclude this work with Section 5, in which we raise some open questions.

## 166 2. An NP-hardness Reduction Framework for REALIZATION

167 In this section we introduce another yet natural reduction for showing the NP-hardness  
 168 of REALIZATION. Via several modifications of this reduction, we will, in the next sections,  
 169 establish the NP-hardness of REALIZATION for several of the graph classes we consider.  
 170 The reduction is from the 3-PARTITION problem, which can be stated as follows (see [16],  
 171 and [11] for more properties of 3-PARTITION):

172 3-PARTITION

173 **Instance:** A set  $A = \{a_1, \dots, a_{3m}\}$  of  $3m$  elements, a bound  $B \in \mathbb{N}^*$ , and a size  $s : A \rightarrow \mathbb{N}^*$   
 174 such that:

175 •  $\frac{B}{4} < s(a) < \frac{B}{2}$  for every  $a \in A$ , and

176 •  $\sum_{a \in A} s(a) = mB$ .

177 **Question:** Can  $A$  be partitioned into  $m$  parts  $A_1 \cup \dots \cup A_m$  such that we have  $\sum_{a \in A_i} s(a) =$   
178  $B$  for every  $i = 1, \dots, m$ ?

179 In some of our proofs, we will use the fact that 3-PARTITION remains NP-complete in  
180 the contexts below, which obviously hold:

181 **Observation 2.1.** Let  $\langle A, B, s \rangle$  be an instance of 3-PARTITION where:

182 •  $\frac{B}{4} < s(a) < \frac{B}{2}$  for every  $a \in A$ , and

183 •  $\sum_{a \in A} s(a) = mB$ .

184 The following instances of 3-PARTITION are equivalent to  $\langle A, B, s \rangle$ :

185 •  $\langle A, B', s' \rangle$ , where  $s'(a) = s(a) + 1$  for every  $a \in A$ , and  $B' = B + 3$ ;

186 •  $\langle A, B'', s'' \rangle$ , where (for any  $\alpha \geq 1$ )  $s''(a) = \alpha \cdot s(a)$  for every  $a \in A$ , and  $B'' = \alpha \cdot B$ .

187 Furthermore, we have:

188 •  $\frac{B'}{4} < s'(a) < \frac{B'}{2}$  and  $\frac{B''}{4} < s''(a) < \frac{B''}{2}$  for every  $a \in A$ , and

189 •  $\sum_{a \in A} s'(a) = mB'$  and  $\sum_{a \in A} s''(a) = mB''$ .

190 The key argument behind our NP-hardness reduction framework from 3-PARTITION to  
191 REALIZATION is the following straightforward equivalence between the two problems:

192 **Theorem 2.2.** REALIZATION is NP-hard when restricted to disconnected graphs.

193 *Proof.* Consider an instance  $\langle A, B, s \rangle$  of 3-PARTITION, where  $|A| = 3m$  and  $A = \{a_1, \dots, a_{3m}\}$ .

194 We produce an instance  $\langle G, \pi \rangle$  (with  $G$  being a disconnected graph) of REALIZATION such  
195 that  $\langle A, B, s \rangle$  admits a solution if and only if  $\pi$  is realizable in  $G$ .

196 Consider, as  $G$ , the disjoint union of  $m$  complete graphs  $K_B$  on  $B$  vertices. So, we  
197 have  $|V(G)| = mB = \sum_{a \in A} s(a)$ . As  $\pi$ , consider the  $|V(G)|$ -sequence  $(s(a_1), \dots, s(a_{3m}))$ .  
198 The equivalence between the two instances is then easy to visualize. Consider any part  $V_i$   
199 with size  $s(a_i)$  from a realization of  $\pi$  in  $G$ . Then  $V_i$  includes vertices from one connected  
200 component of  $G$  only since otherwise  $G[V_i]$  would not be connected. Furthermore, since  
201 every connected component is complete, actually  $V_i$  can be any subset (with the required  
202 size) of its vertices. So, basically, in any realization of  $\pi$  in  $G$ , each of the connected  
203 components of  $G$  is covered by three parts with size  $s(a_{i_1})$ ,  $s(a_{i_2})$  and  $s(a_{i_3})$ , and thus  
204  $s(a_{i_1}) + s(a_{i_2}) + s(a_{i_3}) = B$ . A solution to  $\langle A, B, s \rangle$  can then be directly deduced from a  
205 realization of  $\pi$  in  $G$ , and conversely by similar arguments.  $\square$

206 Note that, in the reduction given in the proof of Theorem 2.2, we can replace the  
207 disjoint union of  $m$  complete graphs  $K_B$  by the disjoint union of any  $m$  AP graphs on  $B$   
208 vertices. For instance, one can consider any disjoint union of  $m$  traceable graphs of order  
209  $B$ .

210 In the current paper, most of our proofs for showing that REALIZATION is NP-hard  
211 for some graph class rely on implicitly getting the situation described in the proof of  
212 Theorem 2.2. Namely, we use the fact that if, for some graph  $G$  and some  $|V(G)|$ -sequence  
213  $\pi$ , in any realization of  $\pi$  in  $G$  some particular parts  $V_1, \dots, V_k$  have to contain particular

214 subgraphs in such a way that  $G - (V_1 \cup \dots \cup V_k)$  is a disjoint union of  $m$  AP graphs with  
 215 the same order  $B$ , then we essentially get an instance of REALIZATION that is NP-hard.

216 We illustrate this fact with three easy examples. We start off by considering the class  
 217 of *subdivided stars* (trees with a unique vertex with degree at least 3). In [2], Barth and  
 218 Fournier proved that REALIZATION is NP-hard for trees with maximum degree 3 (but hav-  
 219 ing many degree-3 vertices). Using our reduction scheme, we provide an easier proof that  
 220 REALIZATION is NP-hard for subdivided stars, hence for trees with unbounded maximum  
 221 degree but only one large-degree vertex.

222 In the context of AP graphs, subdivided stars, which played a central role towards  
 223 understanding the structure of AP trees, have been also called *multipodes* (see e.g. [1, 2, 4,  
 224 19]). In the next proof, when writing  $P_k(a_1, \dots, a_k)$ , we refer to the subdivided star with  
 225  $k$  branches where the  $i$ th branch has (not counting the center vertex) order  $a_i \geq 1$ .

226 **Theorem 2.3.** REALIZATION is NP-hard when restricted to subdivided stars.

227 *Proof.* The reduction is from 3-PARTITION. Given an instance  $\langle A, B, s \rangle$  of 3-PARTITION,  
 228 we construct a subdivided star  $G$  and a  $|V(G)|$ -sequence  $\pi$  such that  $\langle A, B, s \rangle$  admits a  
 229 solution if and only if  $\pi$  is realizable in  $G$ .

230 According to Observation 2.1, we may assume that  $s(a) > 1$  for every  $a \in A$ . Let  
 231  $s_m = \max\{s(a) : a \in A\}$ . Consider, as  $G$ , the subdivided star  $P_{s_m+m}(1, \dots, 1, B, \dots, B)$   
 232 with  $s_m$  branches of order 1 and  $m$  branches of order  $B$ . As  $\pi$ , consider  $\pi = (s_m +$   
 233  $1, s(a_1), \dots, s(a_{3m}))$ .

234 The keystone of the reduction is that, because no element of  $\pi$  is equal to 1, in every  
 235 realization of  $\pi$  in  $G$  the part containing the center vertex of  $G$  necessarily also contains  
 236 all vertices from the branches of order 1. Since there are  $s_m$  branches of order 1 in  $G$ , the  
 237 part containing the center vertex must thus have size at least  $s_m + 1$ . So basically the part  
 238 with size  $s_m + 1$  of every realization of  $\pi$  in  $G$  must include the center vertex of  $G$  as well  
 239 as all the vertices from its  $s_m$  branches of order 1.

240 Once this part is picked, what remains is a forest of  $m$  paths  $P_B$  (on  $B$  vertices) and  
 241 the sequence  $(s(a_1), \dots, s(a_{3m}))$ . Hence, finding a realization of  $\pi$  in  $G$  is equivalent to the  
 242 problem of finding a realization of  $(s(a_1), \dots, s(a_{3m}))$  in a forest of  $m$  paths  $P_B$ , while this  
 243 problem is equivalent to solving  $\langle A, B, s \rangle$  according to the arguments given in the proof  
 244 of Theorem 2.2. The result then follows.  $\square$

245 In the next result, we consider *series-parallel graphs*, for which several NP-hard problems  
 246 are known to be polynomial-time solvable. These graphs, each of which contains two special  
 247 vertices (*source* and *sink*), can be defined inductively as follows:

- 248 •  $K_2$  is a series-parallel graph, its two vertices being its source and sink, respectively.
- 249 • Let  $G$  and  $H$  be two series-parallel graphs with sources  $s_G$  and  $s_H$ , respectively, and  
 250 sinks  $t_G$  and  $t_H$ , respectively. Then:
  - 251 – the *series-composition* of  $G$  and  $H$ , obtained by identifying  $t_G$  and  $s_H$ , is a  
 252 series-parallel graph with source  $s_G$  and sink  $t_H$ ;
  - 253 – the *parallel-composition* of  $G$  and  $H$ , obtained by identifying  $s_G$  and  $s_H$ , and  
 254 identifying  $t_G$  and  $t_H$  (with keeping the graph simple, *i.e.*, omitting all multiple  
 255 edges, if any is created), is a series-parallel graph with source  $s_G = s_H$  and sink  
 256  $t_G = t_H$ .

257 In the context of AP graphs, a particular class of series-parallel graphs, called *balloons*,  
 258 has been investigated towards understanding the structure of 2-connected AP graphs (see  
 259 e.g. [4, 3]). The *balloon* (or *k-balloon*, to make the parameter  $k$  clear)  $B_k(b_1, \dots, b_k)$  is  
 260 the series-parallel graph obtained as follows. Start from two vertices  $r_1$  and  $r_2$ . Then, for  
 261 every  $i = 1, \dots, k$ , we join  $r_1$  and  $r_2$  via a *branch* being a new path with  $b_i$  internal vertices  
 262 having  $r_1, r_2$  as end-vertices. By the *order* of the  $i$ th branch, we mean  $b_i$ .

263 **Theorem 2.4.** REALIZATION is NP-hard when restricted to series-parallel graphs.

*Proof.* We use the same reduction scheme as that in the proof of Theorem 2.3. This time,  
 consider, as  $G$ , the  $(2s_m + m)$ -balloon  $B_{2s_m+m}(1, \dots, 1, B, \dots, B)$  with  $2s_m$  branches of  
 order 1, and  $m$  branches of order  $B$ . As  $\pi$ , consider

$$\pi = (s_m + 1, s_m + 1, s(a_1), \dots, s(a_{3m})).$$

264 Because every vertex from a branch of order 1 of  $G$  only neighbours  $r_1$  and  $r_2$  (who  
 265 have degree  $2s_m + m$ ), it has to belong, in every realization of  $\pi$  in  $G$ , to a same part as  
 266 one of  $r_1$  or  $r_2$ . Said differently, the at most two parts covering  $r_1$  and  $r_2$  also have to cover  
 267 all of the vertices from the branches of order 1. Because there are  $2s_m$  branches of order 1,  
 268 these at most two parts must cover at least  $2s_m + 2$  vertices. In view of the values in  $\pi$ ,  
 269 in every realization of  $\pi$  in  $G$  we necessarily have to use the two parts with size  $s_m + 1$  to  
 270 cover all these vertices. Once these two parts have been picked, what remains is a forest  
 271 of  $m$  paths  $P_B$  of order  $B$  and the sequence  $(s(a_1), \dots, s(a_{3m}))$ . We thus have the desired  
 272 equivalence.  $\square$

273 As mentioned in the introductory section, recall that, by the result of Győri and  
 274 Lovász [21, 17], all  $k$ -connected graphs can always be partitioned into  $k$  connected sub-  
 275 graphs with arbitrary orders. In the following result, we prove, generalizing the arguments  
 276 from the previous proofs, that partitioning a  $k$ -connected graph into more than  $k$  connected  
 277 subgraphs is an NP-hard problem.

278 **Theorem 2.5.** For every  $k \geq 1$ , REALIZATION is NP-hard when restricted to  $k$ -connected  
 279 graphs.

*Proof.* The reduction is similar to that used in the previous two proofs. Let  $k \geq 1$  be fixed,  
 and construct  $G$  as follows. Add  $k$  vertices  $r_1, \dots, r_k$  to  $G$ , as well as  $ks_m$  copies of  $K_1$  and  
 $m$  copies of  $K_B$ . Finally, for every  $i = 1, \dots, k$ , add an edge between  $r_i$  and every vertex  
 of  $V(G) \setminus \{r_1, \dots, r_k\}$ . Note that  $G$  is indeed  $k$ -connected, and  $\{r_1, \dots, r_k\}$  is a  $k$ -cutset.  
 As  $\pi$ , consider

$$\pi = (s_m + 1, \dots, s_m + 1, s(a_1), \dots, s(a_{3m})),$$

280 where the value  $s_m + 1$  appears exactly  $k$  times at the beginning of  $\pi$ .

281 Consider any realization of  $\pi$  in  $G$ . Because the  $s(a_i)$ 's are strictly greater than 1, and  
 282 the  $ks_m$  copies of  $K_1$  are joined to the  $r_i$ 's only, every of these  $K_1$ 's has to belong to the  
 283 same part as one of the  $r_i$ 's. Following these arguments, the  $k$  parts with size  $s_m + 1$  of a  
 284 realization of  $\pi$  in  $G$  must each include one of the  $r_i$ 's and  $s_m$  of the  $K_1$ 's. What remains  
 285 once these  $k$  parts have been picked is  $m$  vertex-disjoint connected components isomorphic  
 286 to  $K_B$ , as well as the sequence  $(s(a_1), \dots, s(a_{3m}))$ . This concludes the proof.  $\square$

### 287 3. Polynomial Kernels for Graphs Without Forbidden Subgraphs

288 For two graphs  $G$  and  $H$ , we denote by  $G + H$  the *disjoint union* of  $G$  and  $H$ , which  
 289 is the disconnect graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . When

290 writing  $kG$  for some  $k \geq 1$ , we refer to the disjoint union  $G + \dots + G$  of  $k$  copies of  $G$ . If  
 291  $\mathcal{G}$  is a family of graphs, we write by  $k\mathcal{G}$  the class of graphs that are the disjoint union of  $k$   
 292 members of  $\mathcal{G}$ . That is,  $G \in k\mathcal{G}$  if there exist  $G_1, \dots, G_k \in \mathcal{G}$  such that  $G = G_1 + \dots + G_k$ .

293 We denote by  $G \times H$  the *complete join* of  $G$  and  $H$ , which is the graph with vertex set  
 294  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H) \cup V(G) \times V(H)$ . For a family (set) of graphs  $\mathcal{F}$ ,  
 295 we say that a graph  $G$  is  $\mathcal{F}$ -free if  $G$  has no member of  $\mathcal{F}$  as an induced subgraph.

296 For  $k \geq 1$ , we denote by  $\mathcal{G}_k$  the class of all connected  $k$ -graphs, while we denote by  $\mathcal{G}_{\leq k}$   
 297 the class of all connected graphs of order at most  $k$ .

298 In this section, we exhibit polynomial kernels of sequences for some families of  $\mathcal{F}$ -  
 299 free graphs. We also discuss some consequences of the existence of these kernels on the  
 300 complexity of the REALIZATION and AP problems for the considered classes of graphs.

### 301 3.1. $k$ -Sequential Graphs

302 Let  $k \geq 1$  be fixed, and  $\mathcal{F}$  be a class of graphs. An  $\mathcal{F}$ -sequential graph is a graph that  
 303 can be defined inductively as follows:

- 304 • Graphs in  $\mathcal{F}$  are  $\mathcal{F}$ -sequential graphs;
- 305 • For an  $\mathcal{F}$ -sequential graph  $G$  and a graph  $H \in \mathcal{F}$ , the graphs  $G + H$  and  $G \times H$  are  
 306  $\mathcal{F}$ -sequential graphs.

307 The construction of such graphs can equivalently be seen as a sequence of steps, at each  
 308 of which a new graph of  $\mathcal{F}$  is added, its vertices being possibly joined to all vertices added  
 309 during the previous steps. Following that analogy, for any given vertex  $v$  of a sequential  
 310 graph, we denote by  $\text{level}(v)$  the *level* of  $v$ , where  $\text{level}(v) = i$  if  $v$  belongs to the graph that  
 311 was added during the  $i$ th step. A vertex with level  $i$  is called a *join vertex* if a complete  
 312 join was performed at the end of the  $i$ th step.

313 In what follows,  $\mathcal{G}_{\leq k}$ -sequential graphs are also called  *$k$ -sequential graphs*. It is worth-  
 314 while noting that sequential graphs encapsulate known families of graphs; for instance,  
 315 *threshold graphs* ( $\{P_4, C_4, 2K_2\}$ -free graphs) are precisely the 1-sequential graphs.

316 We start by proving that, for every  $k$ -sequential  $n$ -graph, the set

$$317 \quad \mathcal{K}_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \text{sp}(\pi) \subseteq \{1, \dots, 2k + 1\}\}$$

318 is a kernel. When  $k$  is fixed, this provides a polynomial kernel of sequences for  $k$ -sequential  
 319 graphs, which implies that the AP problem is in NP for this class of graphs.

320 **Theorem 3.1.** *For every  $k, n$ , the set  $\mathcal{K}_{k,n}$  is a kernel for  $k$ -sequential  $n$ -graphs.*

321 *Proof.* Let  $G$  be a  $k$ -sequential  $n$ -graph. We need to show that  $G$  is AP if and only if all  
 322 sequences of  $\mathcal{K}_{k,n}$  are realizable in  $G$ . If we assume that  $G$  is AP, then, by definition, all  
 323 sequences of  $\mathcal{K}_{k,n}$  are realizable in  $G$ . We thus have to focus on the converse direction only.

324 Assume that all sequences of  $\mathcal{K}_{k,n}$  are realizable in  $G$ , and consider any  $n$ -sequence  
 325  $\pi = (n_1, \dots, n_p) \notin \mathcal{K}_{k,n}$ . We have to prove that  $\pi$  admits a realization in  $G$ . We build an  
 326  $n$ -sequence  $\pi' \in \mathcal{K}_{k,n}$  in the following way: we consider every element  $n_i$  of  $\pi$  in turn, and:

- 327 • if  $n_i \in \{1, \dots, 2k + 1\}$ , then we add  $n_i$  to  $\pi'$ ;
- 328 • otherwise, we add all elements of any  $n_i$ -sequence  $(m_1, \dots, m_x)$  where  $k + 1 \leq m_i \leq$   
 329  $2k + 1$  for every  $i = 1, \dots, x$  (such exists since  $n_i \geq 2k + 2$ ).



330 Note that indeed  $\pi'$  is an  $n$ -sequence, and  $\pi' \in \mathcal{K}_{k,n}$ . Let thus  $R'$  be a realization of  $\pi'$  in  
 331  $G$ , which exists by assumption.

332 We obtain a realization  $R$  of  $\pi$  in  $G$  in the following way. We consider every  $n_i \in \pi$  in  
 333 turn. If  $n_i \leq 2k + 1$ , then there is a corresponding element with value  $n_i$  in  $\pi'$ , and thus a  
 334 connected subgraph of order  $n_i$  in  $R'$ , which we add to  $R$ . Now, if  $n_i \in \pi \geq 2k + 2$ , then  
 335 there are corresponding elements  $m_1, \dots, m_x$  with value in  $\{k + 1, \dots, 2k + 1\}$  in  $\pi'$  (such  
 336 that  $m_1 + \dots + m_x = n_i$ ), and thus connected subgraphs  $G_1, \dots, G_x$  of order  $m_1, \dots, m_x$   
 337 in  $R'$ . By the definition of a  $k$ -sequential graph (in particular, because all graphs added  
 338 sequentially to construct  $G$  have order at most  $k$ ), each of the  $G_i$ 's has to contain a  
 339 join vertex. The join vertex with maximum level implies that  $G_1, \dots, G_x$ , in  $G$ , form a  
 340 connected subgraph of order  $n_i$ ; then we add it to  $R$  as the part of size  $n_i$ .

341 Once every  $n_i$  has been considered,  $R$  is a realization of  $\pi$  in  $G$ . □

342 Regarding Theorem 3.1, it is worth mentioning that, in general, the maximum magni-  
 343 tude  $2k + 1$  of the elements in  $\mathcal{K}_{k,n}$  cannot be lowered. Rephrased differently, there are  
 344 cases where  $n$ -sequences  $\pi$  with  $\text{sp}(\pi) \subseteq \{1, \dots, x\}$  for some  $x < 2k + 1$  do not form a  
 345 kernel for  $k$ -sequential graphs. A straight example is that of  $K_2 + K_1$ : for this 1-sequential  
 346 graph, the sequence (3) has no realization, while all sequences with spectrum from  $\{1, 2\}$   
 347 are realizable. Hence, for 1-sequential graphs, the APness follows from the realizability of  
 348 sequences with elements in  $\{1, 2, 3\}$ , and this is, in a sense, best possible.

349 There are cases, however, where better kernels can be obtained. For instance, when  
 350  $k \geq 2$ , the set

$$351 \quad \mathcal{K}'_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \text{sp}(\pi) \subseteq \{1, \dots, \lfloor 3k/2 \rfloor + 1\}\}$$

352 is a better kernel for  $k$ -sequential  $n$ -graphs.

353 **Theorem 3.2.** *For every  $k \geq 2$  and  $n$ , the set  $\mathcal{K}'_{k,n}$  is a kernel for  $k$ -sequential  $n$ -graphs.*

354 *Proof.* Let  $G$  be a  $k$ -sequential  $n$ -graph. First assume that  $G$  is not connected, *i.e.*, the  
 355 last connected component added during the construction of  $G$  was not joined to the rest  
 356 of the graph. Let  $H$  be a connected component of  $G$  with smallest order  $h$ . By definition,  
 357  $k \geq h$ . Also,  $n \geq 2h$  since  $G$  is not connected.

Suppose first that  $h \geq 5$ . If  $h$  is odd, then let

$$A = \left\{ \frac{h+1}{2}, \frac{h+3}{2}, \dots, h-1 \right\}.$$

If  $h$  is even, then let

$$A = \left\{ \frac{h}{2} + 1, \frac{h}{2} + 2, \dots, h-1 \right\}.$$

358 Note that every integer at least  $2h$  can be expressed as the sum of elements in  $A$ . Let thus  
 359  $\pi \in \mathcal{K}'_{k,n}$  be an  $n$ -sequence taking values from  $A$ . It can easily be seen that  $h$  cannot be  
 360 expressed as the sum of elements in  $A$ ; therefore, there is no realization of  $\pi$  in  $G$ .

361 Now assume that  $h \leq 4$ . As above, we consider some set  $A$  of elements, depending on  
 362 the value of  $h$ :

- 363 • if  $h = 4$ , then consider  $A = \{3, 5\}$ ;
- 364 • if  $h = 3$ , then consider  $A = \{2, 5\}$ ;
- 365 • if  $h = 2$ , then consider:

- 366           –  $A = \{3, 4\}$  if  $n \geq 6$ ;
- 367           –  $A = \{1, 4\}$  if  $n = 5$ ;
- 368           –  $A = \{1, 3\}$  if  $n = 4$ ;
- 369           • if  $h = 1$ , then consider  $A = \{2, 3\}$ .

370 In every case, it can be noted that every integer at least  $2h$  can be expressed as the sum  
 371 of elements in  $A$ , while  $h$  cannot. Thus, as in the previous case, there exists an  $n$ -sequence  
 372 of  $\mathcal{K}'_{k,n}$  that is not realizable in  $G$ . So  $G$  has to be connected.

373       So now assume that  $G$  is connected, and that  $G$  is not AP. According to Theorem 3.1,  
 374 graph  $G$  is AP if and only if all  $n$ -sequences of  $\mathcal{K}_{k,n}$ , *i.e.*, with spectrum from  $\{1, \dots, 2k+1\}$ ,  
 375 are realizable in  $G$ . To prove the claim, it is sufficient to prove that, under the assumption  
 376 that all  $n$ -sequences of  $\mathcal{K}'_{k,n}$  are realizable in  $G$ , all  $n$ -sequences of  $\mathcal{K}_{k,n}$  also are.

377       Let  $\pi \notin \mathcal{K}'_{k,n}$  be an  $n$ -sequence of  $\mathcal{K}_{k,n}$  not realizable in  $G$ . Our goal is to show that  
 378 there is another  $n$ -sequence  $\pi' \in \mathcal{K}'_{k,n}$  that is also not realizable in  $G$ . Among all possible  
 379 sequences as  $\pi$ , we choose one that minimizes the maximum element value  $t$  appearing in  
 380 it. Subject to that condition, we also choose such sequence  $\pi$  that minimizes the number  
 381 of occurrences of  $t$ . So  $\frac{3k+3}{2} \leq t \leq 2k+1$ , and every  $n$ -sequence with maximum element  
 382 value at most  $t-1$  is realizable in  $G$ .

383       First assume that the value  $t$  appears at least twice in  $\pi$ . Clearly,  $3k+3 \leq 2t \leq 3t-3$ .  
 384 Let thus  $\pi'$  be an  $n$ -sequence obtained from  $\pi$  by removing two elements with value  $t$  and  
 385 adding three new elements  $r_1, r_2, r_3$  such that  $k+1 \leq r_1, r_2, r_3 \leq t-1$  and  $r_1+r_2+r_3 = 2t$ .  
 386 By our choice of  $\pi$ , there is a realization  $R$  of  $\pi'$  in  $G$ . Since  $r_1, r_2, r_3 \geq k+1$  note that  
 387 each of the three parts  $V_1, V_2, V_3$  with size  $r_1, r_2, r_3$  of  $R$  contains a join vertex. From this,  
 388 we can partition  $G[V_1 \cup V_2 \cup V_3]$  into two connected subgraphs of order  $t$ , which yields a  
 389 realization of  $\pi$  in  $G$ , a contradiction. More precisely, denote by  $a, b, c$  the vertices with  
 390 maximum level of  $V_1, V_2, V_3$ . If, say,  $\text{level}(a) < \text{level}(b) \leq \text{level}(c)$ , then it suffices to move  
 391 vertices from  $V_1$  to  $V_2, V_3$  so that two connected parts with size  $t$  are obtained. Now assume  
 392  $\text{level}(a) = \text{level}(b) = \text{level}(c)$ . Note that  $V_1$  has to contain two vertices  $u', u''$  being not  
 393 of maximum level. Then we can partition  $G[V_1]$  into two convenient subgraphs  $V'_1, V''_1$   
 394 having the number of vertices we would like to move to  $V_2$  and  $V_3$ : start from  $V'_1$  and  $V''_1$   
 395 containing  $u'$  and  $u''$ , respectively, add at least one vertex with level  $\text{level}(a)$  to  $V'_1$  and  
 396  $V''_1$  if possible (there might be only one such vertex, but this is not an issue), and add the  
 397 remaining vertices of  $V_1$  to  $V'_1, V''_1$  arbitrarily. Note that  $G[V'_1]$  and  $G[V''_1]$  might be not  
 398 connected, but, due to the vertices they and  $V_2, V_3$  include,  $G[V_2 \cup V'_1]$  and  $G[V_3 \cup V''_1]$  are  
 399 connected.

400       So now assume that the value  $t$  appears exactly once in  $\pi$ . Here, consider, as  $\pi'$ , the  
 401  $n$ -sequence obtained by removing the value  $t$  from  $\pi$ , and adding two elements with value 1  
 402 and  $t-1$ , respectively. Again,  $\pi'$  has a realization  $R$  in  $G$ . Since  $t-1$  is the maximum  
 403 value appearing in  $\pi'$ , we may assume that the part  $V_1$  of  $R$  with size  $t-1$  contains a  
 404 vertex with maximum level. Indeed, if this is not the case, then we can consider a part  $V_2$   
 405 containing such a vertex, and move any  $|V_1| - |V_2|$  vertices from  $V_1$  to  $V_2$  in such a way that  
 406 the vertices remaining in  $V_1$  still induce a connected subgraph. This is possible because,  
 407 by the connectedness of  $G$ , the vertices of maximum level are join vertices. We now claim  
 408 that a part  $V_1$  with size  $t-1$  and a part  $V_2$  with size 1 of  $R$  are adjacent, hence yielding a  
 409 part with size  $t$ , and a realization of  $\pi$  in  $G$ , a contradiction. This is because  $t-1 > k$  and  
 410  $V_1$  includes a join vertex  $v$  with maximum level: either the vertex  $u$  of  $V_2$  does not have  
 411 maximum level and is thus adjacent to  $v$ , or  $u$  has maximum level and is thus adjacent of  
 412 a vertex in  $V_1$  being not of maximum level. □

413 We note that the requirement  $k \geq 2$  in Theorem 3.2 is best possible, as stars with an  
414 even number of vertices are 1-sequential graphs, but they cannot be partitioned following  
415  $(2, \dots, 2)$ . On the other hand, the value  $\lfloor 3k/2 \rfloor + 1$  is best possible for some values of  $k$ .  
416 Namely,  $K_2 + K_3$  is a 2-sequential graph such that all 5-sequences with spectrum from  
417  $\{1, 2, 3\}$  are realizable, while  $(1, 4)$  is not. Also,  $K_3 + K_4$  is a 3-sequential graph such that  
418 all 7-sequences with spectrum from  $\{1, 2, 3, 4\}$  are realizable, while  $(2, 5)$  is not. For  $k \geq 4$ ,  
419 we do not know whether the value  $\lfloor 3k/2 \rfloor + 1$  can be lowered in Theorem 3.2; we believe  
420 this would be an interesting aspect to study further on.

421 In general, it is worth mentioning that REALIZATION is NP-hard when restricted to  
422  $k$ -sequential  $n$ -graphs and sequences of  $\mathcal{K}_{k,n}$  and  $\mathcal{K}'_{k,n}$ . Thus, Theorems 3.1 and 3.2 do not  
423 imply the polynomiality of the AP problem when restricted to  $k$ -sequential graphs. Note,  
424 though, that this does not imply the NP-hardness of AP when restricted to those graphs,  
425 as there may exist other polynomial kernels for  $k$ -sequential  $n$ -graphs whose realizability  
426 is easy to check.

427 **Theorem 3.3.** REALIZATION is NP-hard when restricted to  $k$ -sequential  $n$ -graphs and  
428 sequences of  $\mathcal{K}_{k,n}$  and  $\mathcal{K}'_{k,n}$ .

429 *Proof.* The proof is similar to that of Theorem 2.5; we prove the result for  $k = B$  (where  
430  $B$  is part of the input of the given instance of 3-PARTITION). As  $G$ , we consider the  
431  $k$ -sequential graph constructed, through  $s_m + m + 1$  steps, as follows:

- 432 • at step  $i$  with  $i \in \{1, \dots, s_m + 1\}$ , we add a new isolated vertex  $u_i$  to  $G$ ;
- 433 • at step  $i$  with  $i \in \{s_m + 2, \dots, s_m + m\}$ , we add a new isolated copy of  $K_B$ ;
- 434 • at step  $s_m + m + 1$ , we add a new vertex  $v^*$  joined to all previously-added ones.

435 Note that indeed  $G$  is a  $k$ -sequential graph for  $k = B$ . The sequence  $\pi$  we consider is  $(s_m +$   
436  $1, s(a_1), \dots, s(a_{3m})) \in \mathcal{K}_{k,n}, \mathcal{K}'_{k,n}$ . The result follows from the same arguments as earlier:  
437 because the  $s(a_i)$ 's are strictly greater than 1, each  $u_i$  has to belong to the same connected  
438 part as  $v^*$ , which must thus be of size  $s_m + 1$ . Once this part has been picked, what remains  
439 is a disjoint union of  $m$  copies of  $K_B$ , and the sequence  $(s(a_1), \dots, s(a_{3m}))$ .  $\square$

### 440 3.2. $2\mathcal{G}_k$ -Free Graphs

441 Recall that  $2\mathcal{G}_k$ -free graphs are  $\{\bigcup_{G,H \in \mathcal{G}_k} G + H\}$ -free graphs. So, in every  $2\mathcal{G}_k$ -free  
442 graph  $G$ , for every two disjoint subsets  $V_1, V_2$  of  $k$  vertices we take, the graph  $G[V_1 \cup V_2]$   
443 is connected as soon as  $G[V_1]$  and  $G[V_2]$  are. Note, in particular, that  $2\mathcal{G}_1$ -free graphs are  
444 exactly complete graphs (graphs with independence number 1), while  $2\mathcal{G}_2$ -free graphs are  
445 split graphs ( $\{2K_2\}$ -free graphs).

446 In what follows, we prove that the set

$$447 \mathcal{Q}_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \text{sp}(\pi) \subseteq \{1, \dots, 2k - 1\}\}$$

448 is a kernel for  $2\mathcal{G}_k$ -free  $n$ -graphs, which proves that the AP problem is in NP for these  
449 graphs (for fixed  $k$ ). The proof is essentially a generalization of the proof that a split  
450 graph is AP if and only if all sequences with 1's, 2's and 3's are realizable.

451 **Theorem 3.4.** For every  $k, n$ , the set  $\mathcal{Q}_{k,n}$  is a kernel for  $2\mathcal{G}_k$ -free  $n$ -graphs.

452 *Proof.* The proof goes the same way as that of Theorem 3.1. Consider any  $n$ -sequence  
453  $\pi = (n_1, \dots, n_p) \notin \mathcal{Q}_{k,n}$ . We prove that  $\pi$  is realizable in  $G$ . This time, we consider an  
454  $n$ -sequence  $\pi' \in \mathcal{Q}_{k,n}$  obtained from  $\pi$  as follows. For every element  $n_i$  of  $\pi$ :

- 455 • if  $n_i \in \{1, \dots, 2k - 1\}$ , then we add  $n_i$  to  $\pi'$ ;
- 456 • otherwise, we add all elements of any  $n_i$ -sequence  $(m_1, \dots, m_x)$  where  $k \leq m_i \leq 2k - 1$
- 457 for every  $i = 1, \dots, x$  (such exists since  $n_i \geq 2k$ ).

458 We obtain a realization  $R$  of  $\pi$  in  $G$  in the following way. Let  $R'$  be a realization of  $\pi'$   
 459 in  $G$ . Consider every  $n_i \in \pi$ . If  $n_i \leq 2k - 1$ , then we directly get a connected subgraph  
 460 of order  $n_i$  in  $R'$ , which we add to  $R$ . Otherwise,  $n_i \geq 2k$ , and there are corresponding  
 461 elements  $m_1, \dots, m_x$  with value in  $\{k, \dots, 2k - 1\}$  in  $\pi'$  (that is,  $m_1 + \dots + m_x = n_i$ ), and  
 462 thus connected subgraphs  $G_1, \dots, G_x$  of order  $m_1, \dots, m_x$  in  $R'$ . Since all  $G_i$ 's include a  
 463 connected subgraph of order  $k$ , and  $G$  is  $2\mathcal{G}_k$ -free, necessarily every set  $V(G_i) \cup V(G_j)$  for  
 464  $i \neq j$  induces a connected subgraph in  $G$ . So  $V(G_1) \cup \dots \cup V(G_x)$  induces a connected  
 465 subgraph of order  $n_i$  of  $G$ , which we add to  $R$  as the part of size  $n_i$ .

466 Once every  $n_i$  has been considered,  $R$  is a realization of  $\pi$  in  $G$ . □

467 The value  $2k - 1$  in the statement of Theorem 3.4 is best possible for  $k = 1, 2$  (consider  
 468 the graph  $K_1 + K_2$ ). However, it might be not optimal for larger values of  $k$ . Let us further  
 469 mention that the NP-hardness of REALIZATION for  $2\mathcal{G}_k$ -free  $n$ -graphs and sequences of  $\mathcal{Q}_{k,n}$   
 470 might also be established from the reduction in the proof of Theorem 3.3.

#### 471 4. Weakening Hamiltonian Properties

472 In this section, we consider two graph notions behind some known sufficient conditions  
 473 for Hamiltonicity. Namely, we consider *squares* of graphs and graphs that are *claw-free*  
 474 and *net-free*. We show that the most obvious way for weakening these sufficient conditions  
 475 for Hamiltonicity does not yield sufficient conditions for the AP property.

##### 476 4.1. Fleischner's Theorem

477 The *square*  $G^2$  of a given graph  $G$  is the graph on vertex set  $V(G)$  obtained by adding  
 478 an edge between every two vertices at distance at most 2 in  $G$ . We also say that  $G^2$  was  
 479 obtained by *squaring*  $G$  (*i.e.*, applying the square operation on  $G$ ).

480 In this section, we consider a well-known result of Fleischner on squares of graphs [15].

481 **Theorem 4.1** (Fleischner's Theorem). *If  $G$  is a 2-connected graph, then  $G^2$  is Hamilto-*  
 482 *nian.*

483 Naturally, Fleischner's Theorem yields that the square of every 2-connected graph is AP.  
 484 Let us point out, however, that this result cannot be weakened to traceability; namely,  
 485 one can easily come up with connected graphs  $G$  such that  $G^2$  is not traceable. Due to  
 486 the connection between AP graphs and traceable graphs, one could nevertheless wonder  
 487 whether such a statement holds for the AP property. We here prove that this is not  
 488 the case. In particular, we show that REALIZATION remains NP-hard when restricted to  
 489 squared graphs.

490 To establish that result, we will make use of copies of the gadget  $H$  depicted in Figure 1,  
 491 which will be attached to other graphs in a particular fashion. Namely, let  $G$  be a graph  
 492 with a vertex  $z$ . Add a disjoint copy of  $H$  to  $G$ , and identify  $z$  and the (white) vertex  $u$  of  
 493  $H$ . In the resulting graph, we say that there is a *copy* of  $H$  *rooted* in  $z$ . Equivalently, we  
 494 say that the graph was obtained by rooting a copy of  $H$  at  $z$ .

495 The property of interest of this rooting operation is the following.

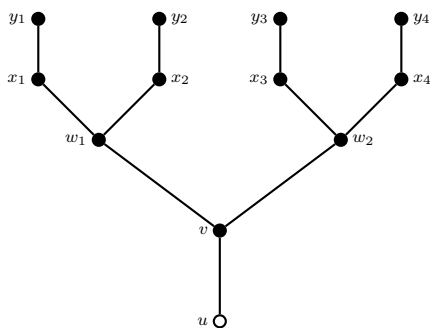


Figure 1: The gadget  $H$  needed for the proof of Theorem 4.3.

496 **Lemma 4.2.** *Let  $G$  be a graph of order  $3n$  having a copy of  $H$  rooted at  $u$ . Then, in*  
 497 *any realization  $R$  of  $\pi = (3, \dots, 3)$  in  $G^2$ , the 12 vertices of  $H$  are covered by exactly four*  
 498 *distinct parts.*

*Proof.* We deal with the vertices of the copy of  $H$  following the terminology indicated in Figure 1. First off, let us note that  $H^2$  itself admits a realization of  $(3, \dots, 3)$ ; for instance,

$$(\{y_1, x_1, x_2\}, \{y_2, w_1, v\}, \{y_3, x_3, x_4\}, \{y_4, w_2, u\})$$

499 is one such.

500 Assume now the claim is wrong, and assume there exists a realization  $R$  of  $\pi$  in  $G^2$   
 501 such that (at least) one of the parts containing  $u$  or  $v$  contains a vertex of  $V(G) \setminus V(H)$   
 502 (only these parts can have this property). Note that there cannot be only one such part  
 503 as it would otherwise cover only one or two vertices of  $H$ , while it has order 12 (hence  
 504 the remaining subgraph of  $H^2$  cannot be partitioned into connected subgraphs of order 3).  
 505 So there are exactly two parts of  $R$  that contain both vertices in  $V(H)$  and vertices in  
 506  $V(G) \setminus V(H)$ . Since  $H^2$  is connected, in  $G^2$ , to the rest of the graph only through  $u$   
 507 and  $v$ , one of these two parts includes  $u$ ,  $w_1$  (without loss of generality) and a vertex of  
 508  $V(G) \setminus V(H)$ , while the second part includes  $v$  and two vertices of  $V(G) \setminus V(H)$  (as otherwise  
 509 the remaining subgraph of  $H^2$  would have order 8 and thus could not be partitioned into  
 510 connected subgraphs of order 3). But then we reach a contradiction, as it can easily be  
 511 checked that  $H^2 - \{u, v, w_1\}$  admits no realization of  $(3, 3, 3)$ .  $\square$

512 We are now ready to prove the following result, which, in a sense, indicates that the  
 513 natural weakening of Fleischner's Theorem to the AP property does not hold in general.

514 **Theorem 4.3.** *REALIZATION is NP-hard when restricted to squared bipartite graphs.*

515 *Proof.* The proof is by reduction from REALIZATION when restricted to instances where  
 516  $\pi = (3, \dots, 3)$ , which was proved to be NP-hard by Dyer and Frieze [12]. From a given  
 517 graph  $G$ , we construct, in polynomial time, another graph  $G'$  such that  $\pi$  is realizable in  $G$   
 518 if and only if  $\pi' = (3, \dots, 3)$  is realizable in  $G'^2$ . Furthermore, the graph  $G'$  we construct  
 519 is bipartite.

520 We start from  $G'$  being exactly  $G$ . We then consider every edge  $e$  of  $G$ , and subdivide  
 521 it in  $G'$ ; we call  $v_e$  the resulting vertex in  $G'$ . Finally, for every such vertex  $v_e$  in  $G'$ , we add  
 522 a copy of the gadget  $H$  from Figure 1, and root it at  $v_e$ . Note that  $G'$  is indeed bipartite  
 523 (due to the subdivision process and because  $H$  is a tree), and, because both  $G$  and  $H$  have  
 524 order divisible by 3, so does  $G'$ .

525 The equivalence between partitioning  $G$  and  $G'^2$  (following  $(3, \dots, 3)$ ) follows from the  
 526 fact that, according to Lemma 4.2, in every realization  $R'$  of  $(3, \dots, 3)$  in  $G'^2$ , the 12 vertices

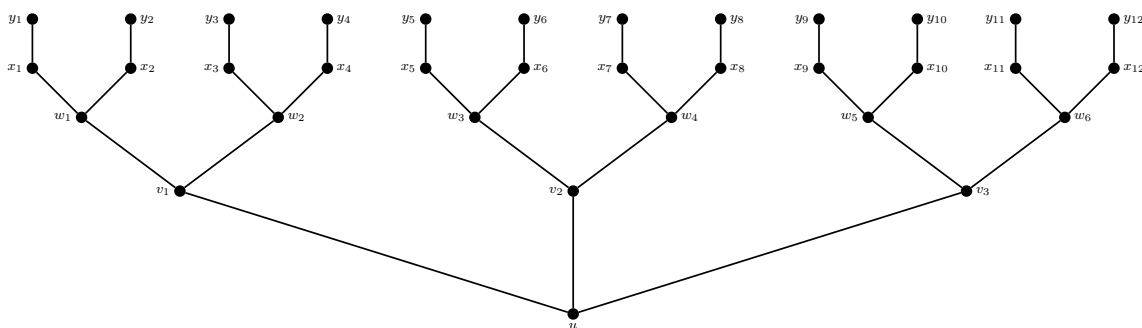


Figure 2: The tree  $T$  described in the proof of Theorem 4.5.

527 from any copy of  $H$  are included in exactly four parts. By construction, when removing  
 528 the copies of  $H$  from  $G'^2$ , the graph we obtain is exactly  $G$ . Hence, when removing from  
 529  $R'$  the parts covering the copies of  $H$ , what remains are parts covering the vertices of  $G$   
 530 only, and inducing connected subgraphs. These parts thus form a realization of  $\pi$  in  $G$ .  
 531 Hence, a realization of  $\pi$  in  $G$  exists if and only if  $G'^2$  admits one of  $(3, \dots, 3)$ .  $\square$

532 We proved Theorem 4.3 for squared bipartite graphs, but we do think it would be  
 533 interesting knowing whether REALIZATION remains NP-hard when restricted to squared  
 534 trees. We leave this question open for now.

535 **Question 4.4.** *Is REALIZATION NP-hard when restricted to squared trees?*

536 It is worthwhile pointing out that squared trees without the AP property do exist,  
 537 which makes Question 4.4 legitimate.

538 **Theorem 4.5.** *There exist trees  $T$  with  $\Delta(T) = 3$  such that  $T^2$  is not AP.*

539 *Proof.* We give a single example illustrating the claim, but it naturally generalizes to an  
 540 infinite family of such trees. Also, considering trees with larger maximum degree might  
 541 simplify the proof a lot, but we think having the result for subcubic trees is more significant.

542 Consider, as  $T$ , the following tree (see Figure 2):

- 543 •  $T$  has a degree-3 vertex  $u$  with neighbours  $v_1, v_2, v_3$ ;
- 544 • each of  $v_1, v_2, v_3$  has two other degree-3 neighbours,  $w_1, \dots, w_6$ ;
- 545 • each of  $w_1, \dots, w_6$  has two other degree-2 neighbours, call these  $x_1, \dots, x_{12}$ ;
- 546 • each of  $x_1, \dots, x_{12}$  has another degree-1 neighbour, call these  $y_1, \dots, y_{12}$ .

547 In what follows, we deal with the vertices of  $T$  labelled as depicted in Figure 2. Note  
 548 that  $n = |V(T)| = 34$ . To prove the claim, we show that  $T^2$  has not realization of the  
 549  $n$ -sequence  $\pi = (1, 3, \dots, 3)$ . Towards a contradiction, assume this is not true, and consider  
 550 a realization  $R = (V_1, \dots, V_p)$  of  $\pi$  in  $T^2$ , where  $V_1$  is the unique part with size 1.

551 First, we note that it is not possible that  $V_1 = \{u\}$ . Indeed, in that case, because  $T - \{u\}$   
 552 has three connected components of order 11, necessarily, in  $R$ , one of the parts with size 3,  
 553 say  $V_2$ , has to contain at least two of  $v_1, v_2, v_3$ . No matter which three vertices are contained  
 554 in  $V_2$ , we note that, in all cases, the graph  $T^2 - V_1 - V_2$  has at least one connected component  
 555 of order 10, which thus cannot be partitioned into connected subgraphs of order 3.

556 So we may assume that  $u$  belongs to a part of  $R$  with size 3. Then  $V_1 \neq \{u\}$  includes a  
 557 vertex from one of the three connected components of  $G - \{u\}$ . The other two connected

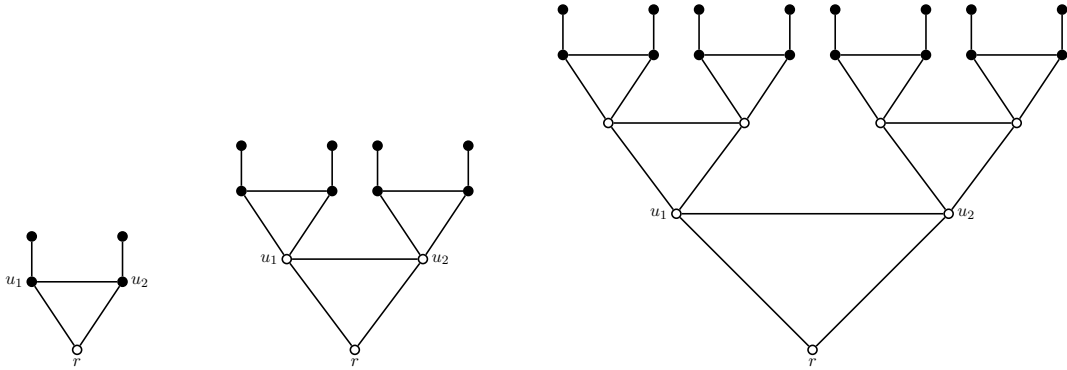


Figure 3: The graphs  $S_1$  (left),  $S_2$  (middle) and  $S_3$  (right).

558 components, together with  $u$ , induce two copies, both rooted at  $u$ , of the gadget depicted in  
 559 Figure 1. According to Lemma 4.2, each of these two gadgets must entirely be covered by  
 560 parts of size 3 of  $R$ . This is not possible, since they share the same root  $u$ ; a contradiction.  
 561 □

#### 562 4.2. Forbidding Claws and Nets

563 Another condition guaranteeing Hamiltonicity of graphs is the absence of two induced  
 564 subgraphs, the *claw* and the *net*. The claw is the complete bipartite graph  $K_{1,3}$ , while the  
 565 net  $Z_1$  is the graph obtained by attaching a pendant vertex to every vertex of a triangle.

566 **Theorem 4.6** (e.g. [14]). *Every 2-connected (resp. connected)  $\{K_{1,3}, Z_1\}$ -free graph is*  
 567 *Hamiltonian (resp. traceable).*

568 One could again wonder how Theorem 4.6 could be weakened to the AP property. In this  
 569 section, we point out that such a sufficient condition for APness cannot be obtained by  
 570 just dropping any of  $K_{1,3}$  or  $Z_1$  from the equation.

571 Let us first point out that the reduction in the proof of Theorem 2.2 yields disconnected  
 572 graphs that are  $\{K_{1,3}, Z_1\}$ -free. From this, we directly get that REALIZATION is NP-hard  
 573 for such disconnected graphs. This is not satisfactory, however, as, in the context of AP  
 574 graphs, it makes more sense considering connected graphs.

575 The counterpart of that result for connected net-free graphs, though, follows directly  
 576 from the proof of Theorem 2.3, as subdivided stars are clearly net-free graphs.

577 **Theorem 4.7.** *REALIZATION is NP-hard when restricted to connected net-free graphs.*

578 Unfortunately, the similar result for claw-free graphs does not follow immediately from  
 579 another of the reductions we have introduced in the previous sections. Below, we thus pro-  
 580 vide another reduction for establishing such a claim (upcoming Theorem 4.9). We actually  
 581 even establish the NPness of REALIZATION for *line graphs* (graphs of edge adjacencies), a  
 582 well-known subclass of claw-free graphs.

583 The proof is another implementation of the reduction framework introduced in Sec-  
 584 tion 2, which relies on the use of the following infinite family  $\mathcal{S}$  of claw-free gadgets.  $\mathcal{S}$   
 585 contains graphs  $S_1, S_2, \dots$  defined inductively as follows (see Figure 3 for an illustration).  
 586 Each  $S_i$  contains a unique degree-2 vertex which we call the *root* of  $S_i$ .  $S_1$  is the graph  
 587 obtained by considering a triangle  $ru_1u_2r$ , then joining  $u_1$  to a pendant vertex, and then  
 588 joining  $u_2$  to a pendant vertex. The root of  $S_1$  is  $r$ . Now consider any  $i \geq 2$  such that  $S_{i-1}$   
 589 can be constructed. Then  $S_i$  is obtained from a triangle  $ru_1u_2r$  by adding two disjoint

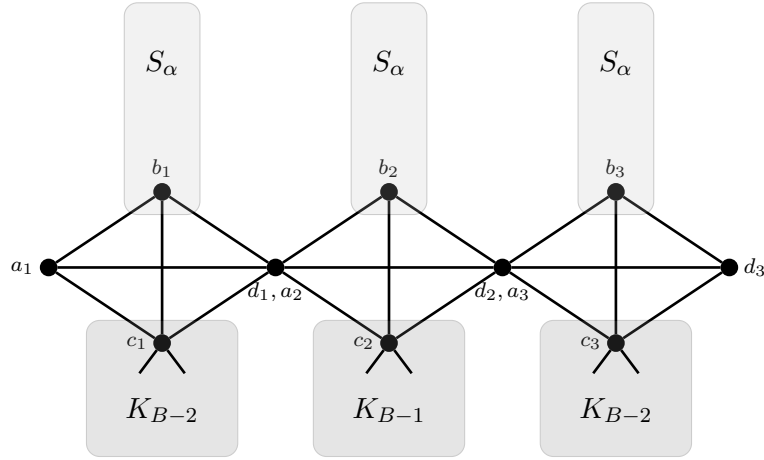


Figure 4: Illustration of the reduced graph constructed in the proof of Theorem 4.9, for  $m = 3$ .

590 copies  $S'$  and  $S''$  of  $S_{i-1}$ , identifying the root of  $S'$  and  $u_1$ , and similarly identifying the  
 591 root of  $S''$  and  $u_2$ . The root of  $S_i$  is  $r$ .

592 For every  $i \geq 1$ , let  $n_i$  denote the number of vertices of  $S_i$ . So  $n_1 = 5$ , and, for every  
 593  $i \geq 2$ , we have  $n_i = 2n_{i-1} + 1$ . More precisely, we have  $n_i = 5 \times 2^{i-1} + 2^{i-1} - 1$ . To every  
 594 member  $S_i \in \mathcal{S}$ , we associate a set  $I_i$  of integers defined as follows:

- 595 •  $I_1 = \{3\}$ , and
- 596 •  $I_i = \{n_{i-1} + 1, \dots, 2n_{i-1} - 1\}$  of integers, for every  $i \geq 2$ .

597 Note that  $|I_1| < |I_2| < \dots$ . Furthermore, every  $S_i$  has the following property regarding  $I_i$ :

598 **Observation 4.8.** *Let  $i \geq 1$  be fixed. For every  $\alpha \in I_i$ , the graph  $S_i$  has no subset*  
 599  *$V_\alpha \subset V(S_i)$  such that  $S_i[V_\alpha]$  is a connected  $\alpha$ -graph and  $S_i - V_\alpha$  is a connected graph*  
 600 *containing the root of  $S_i$ .*

601 *Proof.* Assume such a part  $V_\alpha$  exists. Let  $r$  denote the root of  $S_i$ . Note that every non-leaf  
 602 vertex of  $S_i$  different from  $r$  is a cut-vertex. Under all assumptions, this yields that, by  
 603 the value of  $\alpha$ , necessarily the two neighbours  $r'$  and  $r''$  of  $r$  belong to  $V_\alpha$ . Since  $r'r''$  is a  
 604 cut-edge of  $S_i$ , this means that  $V_\alpha$  has to cover all vertices different from  $r$ ; but this is not  
 605 possible due to the value of  $\alpha$ . This is a contradiction.  $\square$

606 We are now ready to prove the NP-hardness of REALIZATION for claw-free graphs.

607 **Theorem 4.9.** *REALIZATION is NP-hard when restricted to connected claw-free graphs.*

608 *Proof.* We follow the lines of the proofs of Theorems 2.3, 2.4 and 2.5. Let  $\langle A, B, s \rangle$  be an  
 609 instance of 3-PARTITION, where we use the same terminology as in these proofs. We may  
 610 assume that  $s(a_1) \leq \dots \leq s(a_{3m})$ . Free to modify this instance following Observation 2.1,  
 611 we can assume that there is an  $\alpha$  such that  $s(a_1), \dots, s(a_{3m}) \in I_\alpha$ .

612 We construct  $G$  as follows (see Figure 4 for an illustration). We add  $m$  disjoint copies  
 613 of  $K_4$  to the graph, where the vertices of the  $i$ th copy are denoted by  $a_i, b_i, c_i, d_i$ . For every  
 614  $i = 1, \dots, m - 1$ , we then identify the vertices  $d_i$  and  $a_{i+1}$ , so that the  $K_4$ 's form a kind of  
 615 path connected via cut-vertices. For every  $i = 1, \dots, m - 1$ , we then add a copy of  $S_\alpha$  to  
 616 the graph, and we identify its root with  $b_i$ . Finally, we consider every  $i = 1, \dots, m$ , and:



- 617 • for  $i = 1$  or  $i = m$ , we add a complete graph  $K_{B-2}$  to the graph, and we identify one  
618 of its vertices and  $c_i$ ;
- 619 • for  $i \in \{2, \dots, m-1\}$ , we add a complete graph  $K_{B-1}$  to the graph, and we identify  
620 one of its vertices and  $c_i$ .

621 Note that  $G$  is claw-free (it is actually a line graph). The  $|V(G)$ -sequence  $\pi$  we consider  
622 for the reduction is  $\pi = (mn_\alpha + m - 1, s(a_1), \dots, s(a_{3m}))$ .

623 Recall that  $n_\alpha > s(a_{3m})$ , and that  $\alpha$  was chosen so that  $s(a_1), \dots, s(a_{3m}) \in I_\alpha$ . For this  
624 reason, by Observation 4.8, in any realization of  $\pi$  in  $G$ , the part  $V_1$  with size  $mn_\alpha + m - 1$   
625 has to contain the vertices of all  $S_\alpha$ 's added to  $G$ , and, because  $G[V_1]$  must be connected,  
626 also all vertices  $d_1, \dots, d_{m-1}$ . Then  $G - V_1$  is a disjoint union of traceable  $B$ -graphs, and  
627 we have to find a realization of  $(s(a_1), \dots, s(a_{3m}))$  in it. This is equivalent to finding a  
628 solution to  $\langle A, B, s \rangle$ .  $\square$

## 629 5. Conclusion

630 In this work, we have first considered the algorithmic complexity of the REALIZATION  
631 and AP problems. On the one hand, we have mainly established, along all sections, the  
632 NP-hardness of REALIZATION for more classes of graphs with various structure. On the  
633 other hand, we have provided, in Section 3, new kernels of sequences showing that the AP  
634 problem is in NP for a few more classes of graphs. However, we are still far from a proof  
635 that 1) every graph has a polynomial kernel of sequences (which would establish the full  
636 NPness of AP), and that 2) the AP problem is complete for some complexity class (NP or  
637  $\Pi_2^P$  being candidate classes). More efforts should thus be dedicated to these points.

638 One particular appealing case is the one of *cographs* ( $\{P_4\}$ -free graphs), which was  
639 mentioned in [10] by Broersma, Kratsch and Woeginger. It can easily be noted that the  
640 reduction in our proof of Theorem 3.3 yields cographs, so REALIZATION is NP-hard for  
641 these graphs. It is still open, though, whether there is a polynomial kernel of sequences for  
642 cographs. Note that Theorem 3.1 makes a step in that direction, as 1-sequential graphs  
643 (threshold graphs) form a subclass of cographs.

644 The second line of research we have considered in this work is the weakening, to AP-  
645 ness, of well-known sufficient conditions for Hamiltonicity (or traceability). It would be  
646 interesting if there were such a weakening for every condition for Hamiltonicity, as it would  
647 emphasize the relationship between Hamiltonicity and APness. However, previous investi-  
648 gations and some of our results seem to indicate that this connection is not as tight as one  
649 could expect.

650 We believe, however, that it would be nice dedicating more attention to this direction;  
651 let us thus raise an open question which might be interesting. As mentioned in the intro-  
652 ductory section, Ore's well-known condition for Hamiltonicity can be weakened to APness.  
653 In particular, all  $n$ -graphs  $G$  with  $\sigma_2(G) \geq n - 2$  having a (quasi-) perfect matching are  
654 AP. This result implies one direction of upcoming Question 5.1, which, if true, would stand  
655 as a result *à la* Bondy-Chvátal.

656 Namely, for a graph  $G$ , the  $k$ -closure of  $G$  is the (unique) graph obtained by repeatedly  
657 adding an edge between two non-adjacent vertices with degree sum at least  $k$ . A celebrated  
658 result of Bondy and Chvátal states that an  $n$ -graph is Hamiltonian if and only if its  $n$ -closure  
659 is Hamiltonian [8]. Analogously, an  $n$ -graph is traceable if and only if its  $(n - 1)$ -closure is  
660 traceable. However, it is not true that every  $n$ -graph is AP if and only if its  $(n - 2)$ -closure  
661 is AP: In the complete bipartite graph  $K_{n/2-1, n/2+1}$ , every two non-adjacent vertices have

662 degree sum at least  $n - 2$ , so its  $(n - 2)$ -closure is complete and thus AP; however, note that  
 663  $K_{n/2-1, n/2+1}$  has no perfect matching (realization of  $(2, \dots, 2)$ ) and is thus no AP. This is  
 664 actually not the only exception. Indeed, let  $G$  be a graph of order  $n = 4k + 2$  consisting  
 665 of two complete graphs  $K_{n/2}$  with a common vertex and a pendant edge attached to this  
 666 vertex. Then the  $(n - 2)$ -closure of this graph is complete, but  $G$  has not realization of  
 667  $(n/2, n/2)$ . We wonder whether there are many such exceptions, in the following sense:

668 **Question 5.1.** *Is there an “easy” class of graphs  $\mathcal{G}$ , such that if  $G \notin \mathcal{G}$  is an  $n$ -graph, then*  
 669  *$G$  is AP if and only if the  $(n - 2)$ -closure of  $G$  is AP?*

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 673 the graph construction that is detailed right before Question 5.1.

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