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More Aspects of Arbitrarily Partitionable Graphs[☆]

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Abstract

A graph G of order n is arbitrarily partitionable (AP) if, for every sequence (n_1, \dots, n_p) partitioning n , there is a partition (V_1, \dots, V_p) of $V(G)$ such that $G[V_i]$ is a connected graph of order n_i for $i = 1, \dots, p$. The property of being AP is related to other well-known graph notions, such as perfect matchings and Hamiltonian cycles, with which it shares several properties. This work is dedicated to studying two aspects behind AP graphs.

On the one hand, we consider algorithmic aspects of AP graphs, which received some attention in previous works. We first establish the NP-hardness of the problem of partitioning a graph into connected subgraphs following a given sequence, for various new graph classes of interest. We then prove that the problem of deciding whether a graph is AP is in NP for several classes of graphs, confirming a conjecture of Barth and Fournier for these.

On the other hand, we consider the weakening to APness of sufficient conditions for Hamiltonicity. While previous works have suggested that such conditions can sometimes indeed be weakened, we here point out cases where this is not true. This is done by considering conditions for Hamiltonicity involving squares of graphs, and claw- and net-free graphs.

Keywords: arbitrarily partitionable graphs; partition into connected subgraphs; Hamiltonicity.

1. Introduction

1.1. Partitioning Graphs into Connected Subgraphs

Throughout this work, we call a graph of order G an n -graph. By a *graph partition* of an n -graph G , it is meant a vertex-partition (V_1, \dots, V_p) of $V(G)$ such that each $G[V_i]$ has particular properties. This work is about partitions into connected subgraphs, which we deal with through the following terminology.

Consider a *sequence* $\pi = (n_1, \dots, n_p)$ being a partition of n , *i.e.*, $n_1 + \dots + n_p = n$. We will sometimes call π an n -*sequence* to make clear which integer it is a partition of. When writing $|\pi|$, we mean the *size* of π (*i.e.*, p), while, by writing $\|\pi\|$, we refer to the sum of its elements (*i.e.*, n). The *spectrum* $\text{sp}(\pi)$ of π is the set of values appearing in π .

By a *realization* of π of G , we mean a partition (V_1, \dots, V_p) of $V(G)$ such that $G[V_i]$ has order n_i and is connected for $i = 1, \dots, p$. In other words, π indicates the number of connected subgraphs we want (p), and their respective order (n_1, \dots, n_p) . In case G admits a realization of every n -sequence, and is hence partitionable into arbitrarily many connected subgraphs with arbitrary order, we call G *arbitrarily partitionable* (AP for short). AP graphs can also be found in the literature under different names, such as “arbitrarily

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38 vertex-decomposable graphs” [1, 2, 4] or “fully decomposable graphs” [10] (but these terms
39 might mislead the readers, as the term “decomposition” has multiple other meanings in
40 graph theory).

41 Although the notion of AP graphs is relatively recent (it was introduced in 2002 by
42 Barth, Baudon and Puech, to deal with a practical resource allocation problem [1]), the
43 problem of partitioning graphs into connected subgraphs is much older. Perhaps the most
44 influential result is the one from the 70’s due to Lovász and Gyóri [21, 17], who indepen-
45 dently proved that an n -graph G is k -connected if and only if every n -sequence with size k
46 is realizable in G , even if the k parts are imposed to contain any k vertices chosen before-
47 hand. We refer the interested reader to the literature (such as the Ph.D. thesis of the first
48 author [5]) for many more results of this kind.

49 As a warm up, let us start by raising notable properties of AP graphs. First, it should be
50 noted that, in every AP n -graph, any realization of the n -sequence $(2, \dots, 2)$ (or $(2, \dots, 2, 1)$
51 if n is odd) corresponds to a perfect matching (resp. quasi-perfect matching). Note also
52 that every AP graph being spanned by an AP graph is also AP. Since paths are obviously
53 AP, this implies that every traceable graph (*i.e.*, graph having a Hamiltonian path) is AP.
54 So, in a sense, the class of AP graphs lies in between the class of graphs with a (quasi-
55) perfect matching and the class of traceable graphs (which itself contains the class of
56 Hamiltonian graphs), which are well-studied classes of graphs.

57 Several aspects of AP graphs have been investigated to date. In this work, we focus
58 on two such aspects, namely the *algorithmic aspects* and the *Hamiltonian aspects*, which
59 are to be developed below. Again, we refer the interested reader to e.g. [5] for an in-depth
60 overview of other aspects of interest (such as the structural aspects of AP graphs).

61 1.2. Algorithmic Aspects of AP Graphs

62 From the previous definitions related to graph partitions into connected subgraphs, the
63 following two decision problems naturally arise:

64 **REALIZATION**

65 **Instance:** An n -graph G , and an n -sequence π .

66 **Question:** Is π realizable in G ?

67 **AP**

68 **Instance:** A graph G .

69 **Question:** Is G AP?

70 The NPness of REALIZATION is obvious. Its NP-hardness was investigated by several
71 authors, who established it for numerous restrictions on either G or π . Concerning re-
72 strictions on π , REALIZATION remains NP-hard for instances where $\text{sp}(\pi) = \{k\}$ for every
73 $k \geq 3$, as shown by Dyer and Frieze [12]. Note that finding a realization of any sequence
74 π with $\text{sp}(\pi) \subseteq \{1, 2\}$ is equivalent to finding a sufficiently large matching, which can be
75 done in polynomial time using Edmonds’ Blossom Algorithm [13]; hence, such instances
76 of REALIZATION can be solved in polynomial time. In [6], the first author showed that
77 REALIZATION remains NP-hard for instances where $|\pi| = k$, for any $k \geq 2$.

78 Concerning restrictions of REALIZATION on G , Barth and Fournier proved, in [2], that
79 REALIZATION remains NP-hard for trees with maximum degree 3. In [10], Broersma,
80 Kratsch and Woeginger proved that REALIZATION remains NP-hard for split graphs. In [7],
81 the first author proved that REALIZATION remains NP-hard when restricted to graphs with
82 about a third universal vertices (*i.e.*, vertices neighbouring all other vertices).

83 The status of the AP problem is quite intriguing. A first important point to raise is
 84 that, contrarily to what one could naively think, the NP-hardness of REALIZATION does
 85 not imply that of AP. That is, in all reductions imagined by the authors above, the reduced
 86 n -graph G needs to have a very restricted structure so that a particular n -sequence π is
 87 realizable under particular circumstances only; this very restricted structure makes many
 88 other n -sequences not realizable in G , implying that it is far from being AP.

89 It is actually not even clear whether the AP problem is in NP or co-NP. For AP to be
 90 in NP, one would need to provide a polynomial certificate attesting that all n -sequences are
 91 realizable in G , while the number of such n -sequences is $p(n)$, the partition number of n ,
 92 which is exponential in n . For AP to be in co-NP, one would need to provide a polynomial-
 93 time algorithm for checking that a given sequence is indeed not realizable in G , while the
 94 number of possible partitions of G into connected subgraphs is clearly exponential.

95 On the other hand, though, as pointed out in [2, 6], the AP problem matches the typical
 96 structure of Π_2^p problems (“for every sequence, is there a realization?”), and thus belongs
 97 to Π_2^p (recall that Π_2^p problems are, simply put, those decision problems where we are given
 98 two sets X, Y of elements, each of these elements ranging in some respective sets of values,
 99 and the question is whether, whatever the values of the elements in X are, the elements in
 100 Y can always be assigned values so that a particular property holds). However, it is still
 101 not known whether AP is Π_2^p -hard. Also, we have no evidence that AP is NP-hard.

102 Regarding those questions, the main conjecture is due to Barth and Fournier [2]:

103 **Conjecture 1.1** (Barth, Fournier [2]). *The AP problem is in NP.*

104 Conjecture 1.1 relies on *polynomial kernels of sequences*, which are presumed to exist for
 105 every graph. For an n -graph G , a *kernel* (of n -sequences) is a set \mathcal{K} of n -sequences such
 106 that G is AP if and only if all sequences of \mathcal{K} are realizable in G . That is, a kernel for
 107 G is a (preferably small) set of sequences attesting the APness of G . We say that \mathcal{K} is a
 108 *polynomial kernel* if its size is a polynomial function of n .

109 Note that the existence of a polynomial kernel \mathcal{K} for a given graph class indeed implies
 110 the NPness of the AP problem for that class. All results towards Conjecture 1.1 so far are
 111 based on proving the existence of such kernels for particular graph classes. The first result
 112 of this kind was given by Barth, Baudon and Puech [1], who proved that, for subdivided
 113 claws of order n (*i.e.*, trees where the unique vertex of degree more than 2 has degree 3),
 114 the set of all n -sequences of the form (k, \dots, k, r) or $(k, \dots, k, k + 1, \dots, k + 1, r)$ (where
 115 $r < k$) is a polynomial kernel. Later on [24], Ravoux showed that, still for subdivided claws
 116 of order n , the set of n -sequences π with $|\text{sp}(\pi)| \leq 7$ is an alternative polynomial kernel
 117 (although of bigger size, proving that this second set is indeed a kernel for the considered
 118 graph class is much easier than proving that the first set is). Concerning other polynomial
 119 kernels, Broersma, Kratsch and Woeginger proved in [10] that the set of n -sequences π with
 120 $\text{sp}(\pi) \subseteq \{1, 2, 3\}$ is a polynomial kernel for split n -graphs. The first author also provided
 121 more examples of polynomial kernels in [7]; in particular, for complete multipartite n -
 122 graphs, n -sequences π with $\text{sp}(\pi) \subseteq \{1, 2\}$ form a kernel.

123 An interesting side aspect is that, from the existence of these polynomial kernels, some
 124 of the authors above also derived the polynomiality of the AP problem in certain graph
 125 classes. In particular, Barth and Fournier proved that AP is polynomial-time solvable
 126 when restricted to subdivided stars [2], while Broersma, Kratsch and Woeginger proved it
 127 is polynomial-time solvable when restricted to split graphs [10].

128 1.3. Hamiltonian Aspects of AP Graphs

129 As mentioned in Section 1.1, the AP property can be regarded as a weakening of
 130 traceability/Hamiltonicity. In particular, all sufficient conditions implying traceability also

131 imply APness. An interesting line of research is thus to investigate whether such conditions
 132 can be weakened for the AP property.

133 To the best of our knowledge, only a few results of this sort can be found in the
 134 literature. The first series of such results is related to the following parameter, defined for
 135 any given graph G :

$$136 \quad \sigma_k(G) = \min\{d(v_1) + \dots + d(v_k) : v_1, \dots, v_k \text{ are non-adjacent vertices of } G\}.$$

137 A well-known result of Ore [23] states that every n -graph G ($n \geq 3$) with $\sigma_2(G) \geq n - 1$ is
 138 traceable. In [22], Marczyk proved that every n -graph G ($n \geq 8$) having a (quasi-) perfect
 139 matching and verifying $\sigma_2(G) \geq n - 3$ is AP. Later on, this result was improved by Horňák,
 140 Marczyk, Schiermeyer and Woźniak [18], who proved that every n -graph G ($n \geq 20$) having
 141 a (quasi-) perfect matching and verifying $\sigma_2(G) \geq n - 5$ is AP. A similar result for graphs
 142 G with large $\sigma_3(G)$ was also claimed by Brandt [9].

143 The last weakening we are aware of, deals with the number of edges guaranteeing
 144 APness. From known results, it can be established that connected n -graphs with more
 145 than $\binom{n-2}{2} + 2$ edges are traceable (see [20], Proposition 19). An analogous sufficient
 146 condition for APness was given by Kalinowski, Piłśniak, Schiermeyer and Woźniak [20],
 147 who proved that, a few exceptions apart, all n -graphs ($n \geq 22$) with more than $\binom{n-4}{2} + 12$
 148 edges are AP.

149 1.4. Our Results

150 In this work, we establish more results on AP graphs regarding the two aspects devel-
 151 oped above. More precisely:

- 152 • Regarding the algorithmic aspects, we provide, in Sections 2 and 3, both positive and
 153 negative results. We start, in Section 2, by providing an easy NP-hardness reduction
 154 framework, for showing, through slight modifications, that REALIZATION is NP-hard
 155 when restricted to many graph classes (see, for instance, Theorems 2.3, 2.4, 2.5). In
 156 Section 3, we provide more polynomial kernels for several graph classes excluding
 157 particular patterns as induced subgraphs.
- 158 • In Section 4, we consider the weakening of more Hamiltonian conditions for AP
 159 graphs. Although this line of research seems quite appealing, the results we get
 160 show that the distance between traceable graphs and AP graphs is more tenuous
 161 than one could hope. This is done by considering the notions of *squares* of graphs,
 162 and *claw-free* and *net-free graphs*. More precisely, we show that classical results on
 163 Hamiltonicity and these notions do not weaken to the AP property (in the obvious
 164 way, to the least).

165 We conclude this work with Section 5, in which we raise some open questions.

166 2. An NP-hardness Reduction Framework for REALIZATION

167 In this section we introduce another yet natural reduction for showing the NP-hardness
 168 of REALIZATION. Via several modifications of this reduction, we will, in the next sections,
 169 establish the NP-hardness of REALIZATION for several of the graph classes we consider.
 170 The reduction is from the 3-PARTITION problem, which can be stated as follows (see [16],
 171 and [11] for more properties of 3-PARTITION):

172 3-PARTITION

173 **Instance:** A set $A = \{a_1, \dots, a_{3m}\}$ of $3m$ elements, a bound $B \in \mathbb{N}^*$, and a size $s : A \rightarrow \mathbb{N}^*$
 174 such that:

175 • $\frac{B}{4} < s(a) < \frac{B}{2}$ for every $a \in A$, and

176 • $\sum_{a \in A} s(a) = mB$.

177 **Question:** Can A be partitioned into m parts $A_1 \cup \dots \cup A_m$ such that we have $\sum_{a \in A_i} s(a) =$
178 B for every $i = 1, \dots, m$?

179 In some of our proofs, we will use the fact that 3-PARTITION remains NP-complete in
180 the contexts below, which obviously hold:

181 **Observation 2.1.** Let $\langle A, B, s \rangle$ be an instance of 3-PARTITION where:

182 • $\frac{B}{4} < s(a) < \frac{B}{2}$ for every $a \in A$, and

183 • $\sum_{a \in A} s(a) = mB$.

184 The following instances of 3-PARTITION are equivalent to $\langle A, B, s \rangle$:

185 • $\langle A, B', s' \rangle$, where $s'(a) = s(a) + 1$ for every $a \in A$, and $B' = B + 3$;

186 • $\langle A, B'', s'' \rangle$, where (for any $\alpha \geq 1$) $s''(a) = \alpha \cdot s(a)$ for every $a \in A$, and $B'' = \alpha \cdot B$.

187 Furthermore, we have:

188 • $\frac{B'}{4} < s'(a) < \frac{B'}{2}$ and $\frac{B''}{4} < s''(a) < \frac{B''}{2}$ for every $a \in A$, and

189 • $\sum_{a \in A} s'(a) = mB'$ and $\sum_{a \in A} s''(a) = mB''$.

190 The key argument behind our NP-hardness reduction framework from 3-PARTITION to
191 REALIZATION is the following straightforward equivalence between the two problems:

192 **Theorem 2.2.** REALIZATION is NP-hard when restricted to disconnected graphs.

193 *Proof.* Consider an instance $\langle A, B, s \rangle$ of 3-PARTITION, where $|A| = 3m$ and $A = \{a_1, \dots, a_{3m}\}$.

194 We produce an instance $\langle G, \pi \rangle$ (with G being a disconnected graph) of REALIZATION such
195 that $\langle A, B, s \rangle$ admits a solution if and only if π is realizable in G .

196 Consider, as G , the disjoint union of m complete graphs K_B on B vertices. So, we
197 have $|V(G)| = mB = \sum_{a \in A} s(a)$. As π , consider the $|V(G)|$ -sequence $(s(a_1), \dots, s(a_{3m}))$.
198 The equivalence between the two instances is then easy to visualize. Consider any part V_i
199 with size $s(a_i)$ from a realization of π in G . Then V_i includes vertices from one connected
200 component of G only since otherwise $G[V_i]$ would not be connected. Furthermore, since
201 every connected component is complete, actually V_i can be any subset (with the required
202 size) of its vertices. So, basically, in any realization of π in G , each of the connected
203 components of G is covered by three parts with size $s(a_{i_1})$, $s(a_{i_2})$ and $s(a_{i_3})$, and thus
204 $s(a_{i_1}) + s(a_{i_2}) + s(a_{i_3}) = B$. A solution to $\langle A, B, s \rangle$ can then be directly deduced from a
205 realization of π in G , and conversely by similar arguments. \square

206 Note that, in the reduction given in the proof of Theorem 2.2, we can replace the
207 disjoint union of m complete graphs K_B by the disjoint union of any m AP graphs on B
208 vertices. For instance, one can consider any disjoint union of m traceable graphs of order
209 B .

210 In the current paper, most of our proofs for showing that REALIZATION is NP-hard
211 for some graph class rely on implicitly getting the situation described in the proof of
212 Theorem 2.2. Namely, we use the fact that if, for some graph G and some $|V(G)|$ -sequence
213 π , in any realization of π in G some particular parts V_1, \dots, V_k have to contain particular

214 subgraphs in such a way that $G - (V_1 \cup \dots \cup V_k)$ is a disjoint union of m AP graphs with
 215 the same order B , then we essentially get an instance of REALIZATION that is NP-hard.

216 We illustrate this fact with three easy examples. We start off by considering the class
 217 of *subdivided stars* (trees with a unique vertex with degree at least 3). In [2], Barth and
 218 Fournier proved that REALIZATION is NP-hard for trees with maximum degree 3 (but hav-
 219 ing many degree-3 vertices). Using our reduction scheme, we provide an easier proof that
 220 REALIZATION is NP-hard for subdivided stars, hence for trees with unbounded maximum
 221 degree but only one large-degree vertex.

222 In the context of AP graphs, subdivided stars, which played a central role towards
 223 understanding the structure of AP trees, have been also called *multipodes* (see e.g. [1, 2, 4,
 224 19]). In the next proof, when writing $P_k(a_1, \dots, a_k)$, we refer to the subdivided star with
 225 k branches where the i th branch has (not counting the center vertex) order $a_i \geq 1$.

226 **Theorem 2.3.** REALIZATION is NP-hard when restricted to subdivided stars.

227 *Proof.* The reduction is from 3-PARTITION. Given an instance $\langle A, B, s \rangle$ of 3-PARTITION,
 228 we construct a subdivided star G and a $|V(G)|$ -sequence π such that $\langle A, B, s \rangle$ admits a
 229 solution if and only if π is realizable in G .

230 According to Observation 2.1, we may assume that $s(a) > 1$ for every $a \in A$. Let
 231 $s_m = \max\{s(a) : a \in A\}$. Consider, as G , the subdivided star $P_{s_m+m}(1, \dots, 1, B, \dots, B)$
 232 with s_m branches of order 1 and m branches of order B . As π , consider $\pi = (s_m +$
 233 $1, s(a_1), \dots, s(a_{3m}))$.

234 The keystone of the reduction is that, because no element of π is equal to 1, in every
 235 realization of π in G the part containing the center vertex of G necessarily also contains
 236 all vertices from the branches of order 1. Since there are s_m branches of order 1 in G , the
 237 part containing the center vertex must thus have size at least $s_m + 1$. So basically the part
 238 with size $s_m + 1$ of every realization of π in G must include the center vertex of G as well
 239 as all the vertices from its s_m branches of order 1.

240 Once this part is picked, what remains is a forest of m paths P_B (on B vertices) and
 241 the sequence $(s(a_1), \dots, s(a_{3m}))$. Hence, finding a realization of π in G is equivalent to the
 242 problem of finding a realization of $(s(a_1), \dots, s(a_{3m}))$ in a forest of m paths P_B , while this
 243 problem is equivalent to solving $\langle A, B, s \rangle$ according to the arguments given in the proof
 244 of Theorem 2.2. The result then follows. \square

245 In the next result, we consider *series-parallel graphs*, for which several NP-hard problems
 246 are known to be polynomial-time solvable. These graphs, each of which contains two special
 247 vertices (*source* and *sink*), can be defined inductively as follows:

- 248 • K_2 is a series-parallel graph, its two vertices being its source and sink, respectively.
- 249 • Let G and H be two series-parallel graphs with sources s_G and s_H , respectively, and
 250 sinks t_G and t_H , respectively. Then:
 - 251 – the *series-composition* of G and H , obtained by identifying t_G and s_H , is a
 252 series-parallel graph with source s_G and sink t_H ;
 - 253 – the *parallel-composition* of G and H , obtained by identifying s_G and s_H , and
 254 identifying t_G and t_H (with keeping the graph simple, *i.e.*, omitting all multiple
 255 edges, if any is created), is a series-parallel graph with source $s_G = s_H$ and sink
 256 $t_G = t_H$.

257 In the context of AP graphs, a particular class of series-parallel graphs, called *balloons*,
 258 has been investigated towards understanding the structure of 2-connected AP graphs (see
 259 e.g. [4, 3]). The *balloon* (or *k-balloon*, to make the parameter k clear) $B_k(b_1, \dots, b_k)$ is
 260 the series-parallel graph obtained as follows. Start from two vertices r_1 and r_2 . Then, for
 261 every $i = 1, \dots, k$, we join r_1 and r_2 via a *branch* being a new path with b_i internal vertices
 262 having r_1, r_2 as end-vertices. By the *order* of the i th branch, we mean b_i .

263 **Theorem 2.4.** REALIZATION is NP-hard when restricted to series-parallel graphs.

Proof. We use the same reduction scheme as that in the proof of Theorem 2.3. This time,
 consider, as G , the $(2s_m + m)$ -balloon $B_{2s_m+m}(1, \dots, 1, B, \dots, B)$ with $2s_m$ branches of
 order 1, and m branches of order B . As π , consider

$$\pi = (s_m + 1, s_m + 1, s(a_1), \dots, s(a_{3m})).$$

264 Because every vertex from a branch of order 1 of G only neighbours r_1 and r_2 (who
 265 have degree $2s_m + m$), it has to belong, in every realization of π in G , to a same part as
 266 one of r_1 or r_2 . Said differently, the at most two parts covering r_1 and r_2 also have to cover
 267 all of the vertices from the branches of order 1. Because there are $2s_m$ branches of order 1,
 268 these at most two parts must cover at least $2s_m + 2$ vertices. In view of the values in π ,
 269 in every realization of π in G we necessarily have to use the two parts with size $s_m + 1$ to
 270 cover all these vertices. Once these two parts have been picked, what remains is a forest
 271 of m paths P_B of order B and the sequence $(s(a_1), \dots, s(a_{3m}))$. We thus have the desired
 272 equivalence. \square

273 As mentioned in the introductory section, recall that, by the result of Győri and
 274 Lovász [21, 17], all k -connected graphs can always be partitioned into k connected sub-
 275 graphs with arbitrary orders. In the following result, we prove, generalizing the arguments
 276 from the previous proofs, that partitioning a k -connected graph into more than k connected
 277 subgraphs is an NP-hard problem.

278 **Theorem 2.5.** For every $k \geq 1$, REALIZATION is NP-hard when restricted to k -connected
 279 graphs.

Proof. The reduction is similar to that used in the previous two proofs. Let $k \geq 1$ be fixed,
 and construct G as follows. Add k vertices r_1, \dots, r_k to G , as well as ks_m copies of K_1 and
 m copies of K_B . Finally, for every $i = 1, \dots, k$, add an edge between r_i and every vertex
 of $V(G) \setminus \{r_1, \dots, r_k\}$. Note that G is indeed k -connected, and $\{r_1, \dots, r_k\}$ is a k -cutset.
 As π , consider

$$\pi = (s_m + 1, \dots, s_m + 1, s(a_1), \dots, s(a_{3m})),$$

280 where the value $s_m + 1$ appears exactly k times at the beginning of π .

281 Consider any realization of π in G . Because the $s(a_i)$'s are strictly greater than 1, and
 282 the ks_m copies of K_1 are joined to the r_i 's only, every of these K_1 's has to belong to the
 283 same part as one of the r_i 's. Following these arguments, the k parts with size $s_m + 1$ of a
 284 realization of π in G must each include one of the r_i 's and s_m of the K_1 's. What remains
 285 once these k parts have been picked is m vertex-disjoint connected components isomorphic
 286 to K_B , as well as the sequence $(s(a_1), \dots, s(a_{3m}))$. This concludes the proof. \square

287 3. Polynomial Kernels for Graphs Without Forbidden Subgraphs

288 For two graphs G and H , we denote by $G + H$ the *disjoint union* of G and H , which
 289 is the disconnect graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. When

290 writing kG for some $k \geq 1$, we refer to the disjoint union $G + \dots + G$ of k copies of G . If
 291 \mathcal{G} is a family of graphs, we write by $k\mathcal{G}$ the class of graphs that are the disjoint union of k
 292 members of \mathcal{G} . That is, $G \in k\mathcal{G}$ if there exist $G_1, \dots, G_k \in \mathcal{G}$ such that $G = G_1 + \dots + G_k$.

293 We denote by $G \times H$ the *complete join* of G and H , which is the graph with vertex set
 294 $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup V(G) \times V(H)$. For a family (set) of graphs \mathcal{F} ,
 295 we say that a graph G is \mathcal{F} -free if G has no member of \mathcal{F} as an induced subgraph.

296 For $k \geq 1$, we denote by \mathcal{G}_k the class of all connected k -graphs, while we denote by $\mathcal{G}_{\leq k}$
 297 the class of all connected graphs of order at most k .

298 In this section, we exhibit polynomial kernels of sequences for some families of \mathcal{F} -
 299 free graphs. We also discuss some consequences of the existence of these kernels on the
 300 complexity of the REALIZATION and AP problems for the considered classes of graphs.

301 3.1. k -Sequential Graphs

302 Let $k \geq 1$ be fixed, and \mathcal{F} be a class of graphs. An \mathcal{F} -*sequential graph* is a graph that
 303 can be defined inductively as follows:

- 304 • Graphs in \mathcal{F} are \mathcal{F} -sequential graphs;
- 305 • For an \mathcal{F} -sequential graph G and a graph $H \in \mathcal{F}$, the graphs $G + H$ and $G \times H$ are
 306 \mathcal{F} -sequential graphs.

307 The construction of such graphs can equivalently be seen as a sequence of steps, at each
 308 of which a new graph of \mathcal{F} is added, its vertices being possibly joined to all vertices added
 309 during the previous steps. Following that analogy, for any given vertex v of a sequential
 310 graph, we denote by $\text{level}(v)$ the *level* of v , where $\text{level}(v) = i$ if v belongs to the graph that
 311 was added during the i th step. A vertex with level i is called a *join vertex* if a complete
 312 join was performed at the end of the i th step.

313 In what follows, $\mathcal{G}_{\leq k}$ -sequential graphs are also called *k -sequential graphs*. It is worth-
 314 while noting that sequential graphs encapsulate known families of graphs; for instance,
 315 *threshold graphs* ($\{P_4, C_4, 2K_2\}$ -free graphs) are precisely the 1-sequential graphs.

316 We start by proving that, for every k -sequential n -graph, the set

$$317 \quad \mathcal{K}_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \text{sp}(\pi) \subseteq \{1, \dots, 2k + 1\}\}$$

318 is a kernel. When k is fixed, this provides a polynomial kernel of sequences for k -sequential
 319 graphs, which implies that the AP problem is in NP for this class of graphs.

320 **Theorem 3.1.** *For every k, n , the set $\mathcal{K}_{k,n}$ is a kernel for k -sequential n -graphs.*

321 *Proof.* Let G be a k -sequential n -graph. We need to show that G is AP if and only if all
 322 sequences of $\mathcal{K}_{k,n}$ are realizable in G . If we assume that G is AP, then, by definition, all
 323 sequences of $\mathcal{K}_{k,n}$ are realizable in G . We thus have to focus on the converse direction only.

324 Assume that all sequences of $\mathcal{K}_{k,n}$ are realizable in G , and consider any n -sequence
 325 $\pi = (n_1, \dots, n_p) \notin \mathcal{K}_{k,n}$. We have to prove that π admits a realization in G . We build an
 326 n -sequence $\pi' \in \mathcal{K}_{k,n}$ in the following way: we consider every element n_i of π in turn, and:

- 327 • if $n_i \in \{1, \dots, 2k + 1\}$, then we add n_i to π' ;
- 328 • otherwise, we add all elements of any n_i -sequence (m_1, \dots, m_x) where $k + 1 \leq m_i \leq$
 329 $2k + 1$ for every $i = 1, \dots, x$ (such exists since $n_i \geq 2k + 2$).

330 Note that indeed π' is an n -sequence, and $\pi' \in \mathcal{K}_{k,n}$. Let thus R' be a realization of π' in
 331 G , which exists by assumption.

332 We obtain a realization R of π in G in the following way. We consider every $n_i \in \pi$ in
 333 turn. If $n_i \leq 2k + 1$, then there is a corresponding element with value n_i in π' , and thus a
 334 connected subgraph of order n_i in R' , which we add to R . Now, if $n_i \in \pi \geq 2k + 2$, then
 335 there are corresponding elements m_1, \dots, m_x with value in $\{k + 1, \dots, 2k + 1\}$ in π' (such
 336 that $m_1 + \dots + m_x = n_i$), and thus connected subgraphs G_1, \dots, G_x of order m_1, \dots, m_x
 337 in R' . By the definition of a k -sequential graph (in particular, because all graphs added
 338 sequentially to construct G have order at most k), each of the G_i 's has to contain a
 339 join vertex. The join vertex with maximum level implies that G_1, \dots, G_x , in G , form a
 340 connected subgraph of order n_i ; then we add it to R as the part of size n_i .

341 Once every n_i has been considered, R is a realization of π in G . □

342 Regarding Theorem 3.1, it is worth mentioning that, in general, the maximum magni-
 343 tude $2k + 1$ of the elements in $\mathcal{K}_{k,n}$ cannot be lowered. Rephrased differently, there are
 344 cases where n -sequences π with $\text{sp}(\pi) \subseteq \{1, \dots, x\}$ for some $x < 2k + 1$ do not form a
 345 kernel for k -sequential graphs. A straight example is that of $K_2 + K_1$: for this 1-sequential
 346 graph, the sequence (3) has no realization, while all sequences with spectrum from $\{1, 2\}$
 347 are realizable. Hence, for 1-sequential graphs, the APness follows from the realizability of
 348 sequences with elements in $\{1, 2, 3\}$, and this is, in a sense, best possible.

349 There are cases, however, where better kernels can be obtained. For instance, when
 350 $k \geq 2$, the set

$$351 \quad \mathcal{K}'_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \text{sp}(\pi) \subseteq \{1, \dots, \lfloor 3k/2 \rfloor + 1\}\}$$

352 is a better kernel for k -sequential n -graphs.

353 **Theorem 3.2.** *For every $k \geq 2$ and n , the set $\mathcal{K}'_{k,n}$ is a kernel for k -sequential n -graphs.*

354 *Proof.* Let G be a k -sequential n -graph. First assume that G is not connected, *i.e.*, the
 355 last connected component added during the construction of G was not joined to the rest
 356 of the graph. Let H be a connected component of G with smallest order h . By definition,
 357 $k \geq h$. Also, $n \geq 2h$ since G is not connected.

Suppose first that $h \geq 5$. If h is odd, then let

$$A = \left\{ \frac{h+1}{2}, \frac{h+3}{2}, \dots, h-1 \right\}.$$

If h is even, then let

$$A = \left\{ \frac{h}{2} + 1, \frac{h}{2} + 2, \dots, h-1 \right\}.$$

358 Note that every integer at least $2h$ can be expressed as the sum of elements in A . Let thus
 359 $\pi \in \mathcal{K}'_{k,n}$ be an n -sequence taking values from A . It can easily be seen that h cannot be
 360 expressed as the sum of elements in A ; therefore, there is no realization of π in G .

361 Now assume that $h \leq 4$. As above, we consider some set A of elements, depending on
 362 the value of h :

- 363 • if $h = 4$, then consider $A = \{3, 5\}$;
- 364 • if $h = 3$, then consider $A = \{2, 5\}$;
- 365 • if $h = 2$, then consider:

- 366 – $A = \{3, 4\}$ if $n \geq 6$;
- 367 – $A = \{1, 4\}$ if $n = 5$;
- 368 – $A = \{1, 3\}$ if $n = 4$;
- 369 • if $h = 1$, then consider $A = \{2, 3\}$.

370 In every case, it can be noted that every integer at least $2h$ can be expressed as the sum
 371 of elements in A , while h cannot. Thus, as in the previous case, there exists an n -sequence
 372 of $\mathcal{K}'_{k,n}$ that is not realizable in G . So G has to be connected.

373 So now assume that G is connected, and that G is not AP. According to Theorem 3.1,
 374 graph G is AP if and only if all n -sequences of $\mathcal{K}_{k,n}$, *i.e.*, with spectrum from $\{1, \dots, 2k+1\}$,
 375 are realizable in G . To prove the claim, it is sufficient to prove that, under the assumption
 376 that all n -sequences of $\mathcal{K}'_{k,n}$ are realizable in G , all n -sequences of $\mathcal{K}_{k,n}$ also are.

377 Let $\pi \notin \mathcal{K}'_{k,n}$ be an n -sequence of $\mathcal{K}_{k,n}$ not realizable in G . Our goal is to show that
 378 there is another n -sequence $\pi' \in \mathcal{K}'_{k,n}$ that is also not realizable in G . Among all possible
 379 sequences as π , we choose one that minimizes the maximum element value t appearing in
 380 it. Subject to that condition, we also choose such sequence π that minimizes the number
 381 of occurrences of t . So $\frac{3k+3}{2} \leq t \leq 2k+1$, and every n -sequence with maximum element
 382 value at most $t-1$ is realizable in G .

383 First assume that the value t appears at least twice in π . Clearly, $3k+3 \leq 2t \leq 3t-3$.
 384 Let thus π' be an n -sequence obtained from π by removing two elements with value t and
 385 adding three new elements r_1, r_2, r_3 such that $k+1 \leq r_1, r_2, r_3 \leq t-1$ and $r_1+r_2+r_3 = 2t$.
 386 By our choice of π , there is a realization R of π' in G . Since $r_1, r_2, r_3 \geq k+1$ note that
 387 each of the three parts V_1, V_2, V_3 with size r_1, r_2, r_3 of R contains a join vertex. From this,
 388 we can partition $G[V_1 \cup V_2 \cup V_3]$ into two connected subgraphs of order t , which yields a
 389 realization of π in G , a contradiction. More precisely, denote by a, b, c the vertices with
 390 maximum level of V_1, V_2, V_3 . If, say, $\text{level}(a) < \text{level}(b) \leq \text{level}(c)$, then it suffices to move
 391 vertices from V_1 to V_2, V_3 so that two connected parts with size t are obtained. Now assume
 392 $\text{level}(a) = \text{level}(b) = \text{level}(c)$. Note that V_1 has to contain two vertices u', u'' being not
 393 of maximum level. Then we can partition $G[V_1]$ into two convenient subgraphs V'_1, V''_1
 394 having the number of vertices we would like to move to V_2 and V_3 : start from V'_1 and V''_1
 395 containing u' and u'' , respectively, add at least one vertex with level $\text{level}(a)$ to V'_1 and
 396 V''_1 if possible (there might be only one such vertex, but this is not an issue), and add the
 397 remaining vertices of V_1 to V'_1, V''_1 arbitrarily. Note that $G[V'_1]$ and $G[V''_1]$ might be not
 398 connected, but, due to the vertices they and V_2, V_3 include, $G[V_2 \cup V'_1]$ and $G[V_3 \cup V''_1]$ are
 399 connected.

400 So now assume that the value t appears exactly once in π . Here, consider, as π' , the
 401 n -sequence obtained by removing the value t from π , and adding two elements with value 1
 402 and $t-1$, respectively. Again, π' has a realization R in G . Since $t-1$ is the maximum
 403 value appearing in π' , we may assume that the part V_1 of R with size $t-1$ contains a
 404 vertex with maximum level. Indeed, if this is not the case, then we can consider a part V_2
 405 containing such a vertex, and move any $|V_1| - |V_2|$ vertices from V_1 to V_2 in such a way that
 406 the vertices remaining in V_1 still induce a connected subgraph. This is possible because,
 407 by the connectedness of G , the vertices of maximum level are join vertices. We now claim
 408 that a part V_1 with size $t-1$ and a part V_2 with size 1 of R are adjacent, hence yielding a
 409 part with size t , and a realization of π in G , a contradiction. This is because $t-1 > k$ and
 410 V_1 includes a join vertex v with maximum level: either the vertex u of V_2 does not have
 411 maximum level and is thus adjacent to v , or u has maximum level and is thus adjacent of
 412 a vertex in V_1 being not of maximum level. □

413 We note that the requirement $k \geq 2$ in Theorem 3.2 is best possible, as stars with an
414 even number of vertices are 1-sequential graphs, but they cannot be partitioned following
415 $(2, \dots, 2)$. On the other hand, the value $\lfloor 3k/2 \rfloor + 1$ is best possible for some values of k .
416 Namely, $K_2 + K_3$ is a 2-sequential graph such that all 5-sequences with spectrum from
417 $\{1, 2, 3\}$ are realizable, while $(1, 4)$ is not. Also, $K_3 + K_4$ is a 3-sequential graph such that
418 all 7-sequences with spectrum from $\{1, 2, 3, 4\}$ are realizable, while $(2, 5)$ is not. For $k \geq 4$,
419 we do not know whether the value $\lfloor 3k/2 \rfloor + 1$ can be lowered in Theorem 3.2; we believe
420 this would be an interesting aspect to study further on.

421 In general, it is worth mentioning that REALIZATION is NP-hard when restricted to
422 k -sequential n -graphs and sequences of $\mathcal{K}_{k,n}$ and $\mathcal{K}'_{k,n}$. Thus, Theorems 3.1 and 3.2 do not
423 imply the polynomiality of the AP problem when restricted to k -sequential graphs. Note,
424 though, that this does not imply the NP-hardness of AP when restricted to those graphs,
425 as there may exist other polynomial kernels for k -sequential n -graphs whose realizability
426 is easy to check.

427 **Theorem 3.3.** REALIZATION is NP-hard when restricted to k -sequential n -graphs and
428 sequences of $\mathcal{K}_{k,n}$ and $\mathcal{K}'_{k,n}$.

429 *Proof.* The proof is similar to that of Theorem 2.5; we prove the result for $k = B$ (where
430 B is part of the input of the given instance of 3-PARTITION). As G , we consider the
431 k -sequential graph constructed, through $s_m + m + 1$ steps, as follows:

- 432 • at step i with $i \in \{1, \dots, s_m + 1\}$, we add a new isolated vertex u_i to G ;
- 433 • at step i with $i \in \{s_m + 2, \dots, s_m + m\}$, we add a new isolated copy of K_B ;
- 434 • at step $s_m + m + 1$, we add a new vertex v^* joined to all previously-added ones.

435 Note that indeed G is a k -sequential graph for $k = B$. The sequence π we consider is $(s_m +$
436 $1, s(a_1), \dots, s(a_{3m})) \in \mathcal{K}_{k,n}, \mathcal{K}'_{k,n}$. The result follows from the same arguments as earlier:
437 because the $s(a_i)$'s are strictly greater than 1, each u_i has to belong to the same connected
438 part as v^* , which must thus be of size $s_m + 1$. Once this part has been picked, what remains
439 is a disjoint union of m copies of K_B , and the sequence $(s(a_1), \dots, s(a_{3m}))$. \square

440 3.2. $2\mathcal{G}_k$ -Free Graphs

441 Recall that $2\mathcal{G}_k$ -free graphs are $\{\bigcup_{G,H \in \mathcal{G}_k} G + H\}$ -free graphs. So, in every $2\mathcal{G}_k$ -free
442 graph G , for every two disjoint subsets V_1, V_2 of k vertices we take, the graph $G[V_1 \cup V_2]$
443 is connected as soon as $G[V_1]$ and $G[V_2]$ are. Note, in particular, that $2\mathcal{G}_1$ -free graphs are
444 exactly complete graphs (graphs with independence number 1), while $2\mathcal{G}_2$ -free graphs are
445 split graphs ($\{2K_2\}$ -free graphs).

446 In what follows, we prove that the set

$$447 \mathcal{Q}_{k,n} = \{\pi : \pi \text{ is an } n\text{-sequence and } \text{sp}(\pi) \subseteq \{1, \dots, 2k - 1\}\}$$

448 is a kernel for $2\mathcal{G}_k$ -free n -graphs, which proves that the AP problem is in NP for these
449 graphs (for fixed k). The proof is essentially a generalization of the proof that a split
450 graph is AP if and only if all sequences with 1's, 2's and 3's are realizable.

451 **Theorem 3.4.** For every k, n , the set $\mathcal{Q}_{k,n}$ is a kernel for $2\mathcal{G}_k$ -free n -graphs.

452 *Proof.* The proof goes the same way as that of Theorem 3.1. Consider any n -sequence
453 $\pi = (n_1, \dots, n_p) \notin \mathcal{Q}_{k,n}$. We prove that π is realizable in G . This time, we consider an
454 n -sequence $\pi' \in \mathcal{Q}_{k,n}$ obtained from π as follows. For every element n_i of π :

- 455 • if $n_i \in \{1, \dots, 2k - 1\}$, then we add n_i to π' ;
- 456 • otherwise, we add all elements of any n_i -sequence (m_1, \dots, m_x) where $k \leq m_i \leq 2k - 1$
457 for every $i = 1, \dots, x$ (such exists since $n_i \geq 2k$).

458 We obtain a realization R of π in G in the following way. Let R' be a realization of π'
459 in G . Consider every $n_i \in \pi$. If $n_i \leq 2k - 1$, then we directly get a connected subgraph
460 of order n_i in R' , which we add to R . Otherwise, $n_i \geq 2k$, and there are corresponding
461 elements m_1, \dots, m_x with value in $\{k, \dots, 2k - 1\}$ in π' (that is, $m_1 + \dots + m_x = n_i$), and
462 thus connected subgraphs G_1, \dots, G_x of order m_1, \dots, m_x in R' . Since all G_i 's include a
463 connected subgraph of order k , and G is $2\mathcal{G}_k$ -free, necessarily every set $V(G_i) \cup V(G_j)$ for
464 $i \neq j$ induces a connected subgraph in G . So $V(G_1) \cup \dots \cup V(G_x)$ induces a connected
465 subgraph of order n_i of G , which we add to R as the part of size n_i .

466 Once every n_i has been considered, R is a realization of π in G . □

467 The value $2k - 1$ in the statement of Theorem 3.4 is best possible for $k = 1, 2$ (consider
468 the graph $K_1 + K_2$). However, it might be not optimal for larger values of k . Let us further
469 mention that the NP-hardness of REALIZATION for $2\mathcal{G}_k$ -free n -graphs and sequences of $\mathcal{Q}_{k,n}$
470 might also be established from the reduction in the proof of Theorem 3.3.

471 4. Weakening Hamiltonian Properties

472 In this section, we consider two graph notions behind some known sufficient conditions
473 for Hamiltonicity. Namely, we consider *squares* of graphs and graphs that are *claw-free*
474 and *net-free*. We show that the most obvious way for weakening these sufficient conditions
475 for Hamiltonicity does not yield sufficient conditions for the AP property.

476 4.1. Fleischner's Theorem

477 The *square* G^2 of a given graph G is the graph on vertex set $V(G)$ obtained by adding
478 an edge between every two vertices at distance at most 2 in G . We also say that G^2 was
479 obtained by *squaring* G (*i.e.*, applying the square operation on G).

480 In this section, we consider a well-known result of Fleischner on squares of graphs [15].

481 **Theorem 4.1** (Fleischner's Theorem). *If G is a 2-connected graph, then G^2 is Hamilto-*
482 *nian.*

483 Naturally, Fleischner's Theorem yields that the square of every 2-connected graph is AP.
484 Let us point out, however, that this result cannot be weakened to traceability; namely,
485 one can easily come up with connected graphs G such that G^2 is not traceable. Due to
486 the connection between AP graphs and traceable graphs, one could nevertheless wonder
487 whether such a statement holds for the AP property. We here prove that this is not
488 the case. In particular, we show that REALIZATION remains NP-hard when restricted to
489 squared graphs.

490 To establish that result, we will make use of copies of the gadget H depicted in Figure 1,
491 which will be attached to other graphs in a particular fashion. Namely, let G be a graph
492 with a vertex z . Add a disjoint copy of H to G , and identify z and the (white) vertex u of
493 H . In the resulting graph, we say that there is a *copy of H rooted* in z . Equivalently, we
494 say that the graph was obtained by rooting a copy of H at z .

495 The property of interest of this rooting operation is the following.

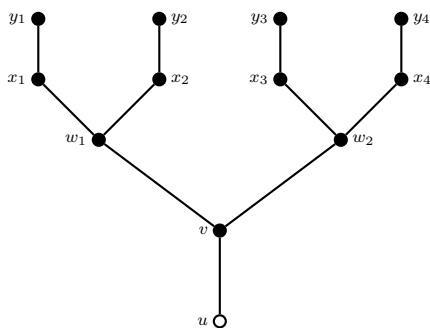


Figure 1: The gadget H needed for the proof of Theorem 4.3.

496 **Lemma 4.2.** *Let G be a graph of order $3n$ having a copy of H rooted at u . Then, in*
 497 *any realization R of $\pi = (3, \dots, 3)$ in G^2 , the 12 vertices of H are covered by exactly four*
 498 *distinct parts.*

Proof. We deal with the vertices of the copy of H following the terminology indicated in Figure 1. First off, let us note that H^2 itself admits a realization of $(3, \dots, 3)$; for instance,

$$(\{y_1, x_1, x_2\}, \{y_2, w_1, v\}, \{y_3, x_3, x_4\}, \{y_4, w_2, u\})$$

499 is one such.

500 Assume now the claim is wrong, and assume there exists a realization R of π in G^2
 501 such that (at least) one of the parts containing u or v contains a vertex of $V(G) \setminus V(H)$
 502 (only these parts can have this property). Note that there cannot be only one such part
 503 as it would otherwise cover only one or two vertices of H , while it has order 12 (hence
 504 the remaining subgraph of H^2 cannot be partitioned into connected subgraphs of order 3).
 505 So there are exactly two parts of R that contain both vertices in $V(H)$ and vertices in
 506 $V(G) \setminus V(H)$. Since H^2 is connected, in G^2 , to the rest of the graph only through u
 507 and v , one of these two parts includes u , w_1 (without loss of generality) and a vertex of
 508 $V(G) \setminus V(H)$, while the second part includes v and two vertices of $V(G) \setminus V(H)$ (as otherwise
 509 the remaining subgraph of H^2 would have order 8 and thus could not be partitioned into
 510 connected subgraphs of order 3). But then we reach a contradiction, as it can easily be
 511 checked that $H^2 - \{u, v, w_1\}$ admits no realization of $(3, 3, 3)$. \square

512 We are now ready to prove the following result, which, in a sense, indicates that the
 513 natural weakening of Fleischner's Theorem to the AP property does not hold in general.

514 **Theorem 4.3.** *REALIZATION is NP-hard when restricted to squared bipartite graphs.*

515 *Proof.* The proof is by reduction from REALIZATION when restricted to instances where
 516 $\pi = (3, \dots, 3)$, which was proved to be NP-hard by Dyer and Frieze [12]. From a given
 517 graph G , we construct, in polynomial time, another graph G' such that π is realizable in G
 518 if and only if $\pi' = (3, \dots, 3)$ is realizable in G'^2 . Furthermore, the graph G' we construct
 519 is bipartite.

520 We start from G' being exactly G . We then consider every edge e of G , and subdivide
 521 it in G' ; we call v_e the resulting vertex in G' . Finally, for every such vertex v_e in G' , we add
 522 a copy of the gadget H from Figure 1, and root it at v_e . Note that G' is indeed bipartite
 523 (due to the subdivision process and because H is a tree), and, because both G and H have
 524 order divisible by 3, so does G' .

525 The equivalence between partitioning G and G'^2 (following $(3, \dots, 3)$) follows from the
 526 fact that, according to Lemma 4.2, in every realization R' of $(3, \dots, 3)$ in G'^2 , the 12 vertices

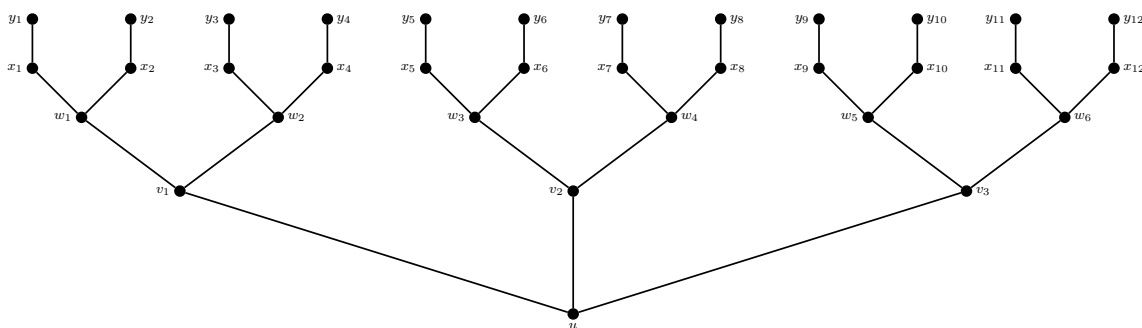


Figure 2: The tree T described in the proof of Theorem 4.5.

527 from any copy of H are included in exactly four parts. By construction, when removing
 528 the copies of H from G'^2 , the graph we obtain is exactly G . Hence, when removing from
 529 R' the parts covering the copies of H , what remains are parts covering the vertices of G
 530 only, and inducing connected subgraphs. These parts thus form a realization of π in G .
 531 Hence, a realization of π in G exists if and only if G'^2 admits one of $(3, \dots, 3)$. \square

532 We proved Theorem 4.3 for squared bipartite graphs, but we do think it would be
 533 interesting knowing whether REALIZATION remains NP-hard when restricted to squared
 534 trees. We leave this question open for now.

535 **Question 4.4.** *Is REALIZATION NP-hard when restricted to squared trees?*

536 It is worthwhile pointing out that squared trees without the AP property do exist,
 537 which makes Question 4.4 legitimate.

538 **Theorem 4.5.** *There exist trees T with $\Delta(T) = 3$ such that T^2 is not AP.*

539 *Proof.* We give a single example illustrating the claim, but it naturally generalizes to an
 540 infinite family of such trees. Also, considering trees with larger maximum degree might
 541 simplify the proof a lot, but we think having the result for subcubic trees is more significant.

542 Consider, as T , the following tree (see Figure 2):

- 543 • T has a degree-3 vertex u with neighbours v_1, v_2, v_3 ;
- 544 • each of v_1, v_2, v_3 has two other degree-3 neighbours, w_1, \dots, w_6 ;
- 545 • each of w_1, \dots, w_6 has two other degree-2 neighbours, call these x_1, \dots, x_{12} ;
- 546 • each of x_1, \dots, x_{12} has another degree-1 neighbour, call these y_1, \dots, y_{12} .

547 In what follows, we deal with the vertices of T labelled as depicted in Figure 2. Note
 548 that $n = |V(T)| = 34$. To prove the claim, we show that T^2 has not realization of the
 549 n -sequence $\pi = (1, 3, \dots, 3)$. Towards a contradiction, assume this is not true, and consider
 550 a realization $R = (V_1, \dots, V_p)$ of π in T^2 , where V_1 is the unique part with size 1.

551 First, we note that it is not possible that $V_1 = \{u\}$. Indeed, in that case, because $T - \{u\}$
 552 has three connected components of order 11, necessarily, in R , one of the parts with size 3,
 553 say V_2 , has to contain at least two of v_1, v_2, v_3 . No matter which three vertices are contained
 554 in V_2 , we note that, in all cases, the graph $T^2 - V_1 - V_2$ has at least one connected component
 555 of order 10, which thus cannot be partitioned into connected subgraphs of order 3.

556 So we may assume that u belongs to a part of R with size 3. Then $V_1 \neq \{u\}$ includes a
 557 vertex from one of the three connected components of $G - \{u\}$. The other two connected

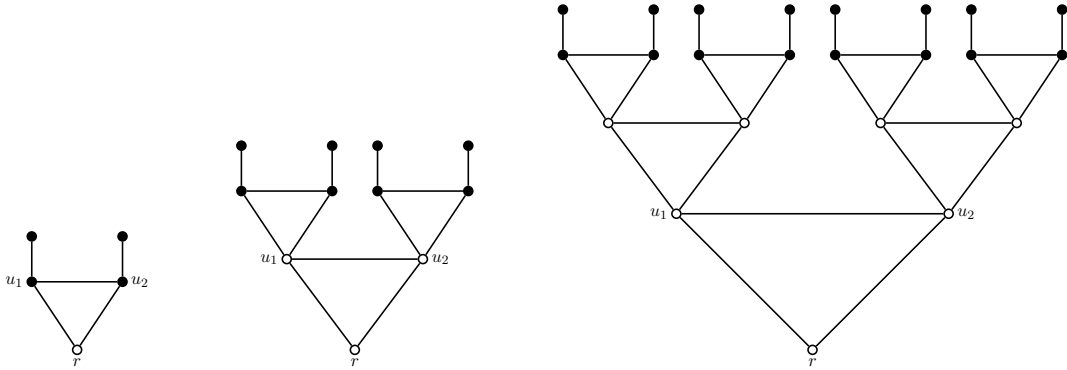


Figure 3: The graphs S_1 (left), S_2 (middle) and S_3 (right).

558 components, together with u , induce two copies, both rooted at u , of the gadget depicted in
 559 Figure 1. According to Lemma 4.2, each of these two gadgets must entirely be covered by
 560 parts of size 3 of R . This is not possible, since they share the same root u ; a contradiction.
 561 □

562 4.2. Forbidding Claws and Nets

563 Another condition guaranteeing Hamiltonicity of graphs is the absence of two induced
 564 subgraphs, the *claw* and the *net*. The claw is the complete bipartite graph $K_{1,3}$, while the
 565 net Z_1 is the graph obtained by attaching a pendant vertex to every vertex of a triangle.

566 **Theorem 4.6** (e.g. [14]). *Every 2-connected (resp. connected) $\{K_{1,3}, Z_1\}$ -free graph is*
 567 *Hamiltonian (resp. traceable).*

568 One could again wonder how Theorem 4.6 could be weakened to the AP property. In this
 569 section, we point out that such a sufficient condition for APness cannot be obtained by
 570 just dropping any of $K_{1,3}$ or Z_1 from the equation.

571 Let us first point out that the reduction in the proof of Theorem 2.2 yields disconnected
 572 graphs that are $\{K_{1,3}, Z_1\}$ -free. From this, we directly get that REALIZATION is NP-hard
 573 for such disconnected graphs. This is not satisfactory, however, as, in the context of AP
 574 graphs, it makes more sense considering connected graphs.

575 The counterpart of that result for connected net-free graphs, though, follows directly
 576 from the proof of Theorem 2.3, as subdivided stars are clearly net-free graphs.

577 **Theorem 4.7.** *REALIZATION is NP-hard when restricted to connected net-free graphs.*

578 Unfortunately, the similar result for claw-free graphs does not follow immediately from
 579 another of the reductions we have introduced in the previous sections. Below, we thus pro-
 580 vide another reduction for establishing such a claim (upcoming Theorem 4.9). We actually
 581 even establish the NPness of REALIZATION for *line graphs* (graphs of edge adjacencies), a
 582 well-known subclass of claw-free graphs.

583 The proof is another implementation of the reduction framework introduced in Sec-
 584 tion 2, which relies on the use of the following infinite family \mathcal{S} of claw-free gadgets. \mathcal{S}
 585 contains graphs S_1, S_2, \dots defined inductively as follows (see Figure 3 for an illustration).
 586 Each S_i contains a unique degree-2 vertex which we call the *root* of S_i . S_1 is the graph
 587 obtained by considering a triangle ru_1u_2r , then joining u_1 to a pendant vertex, and then
 588 joining u_2 to a pendant vertex. The root of S_1 is r . Now consider any $i \geq 2$ such that S_{i-1}
 589 can be constructed. Then S_i is obtained from a triangle ru_1u_2r by adding two disjoint

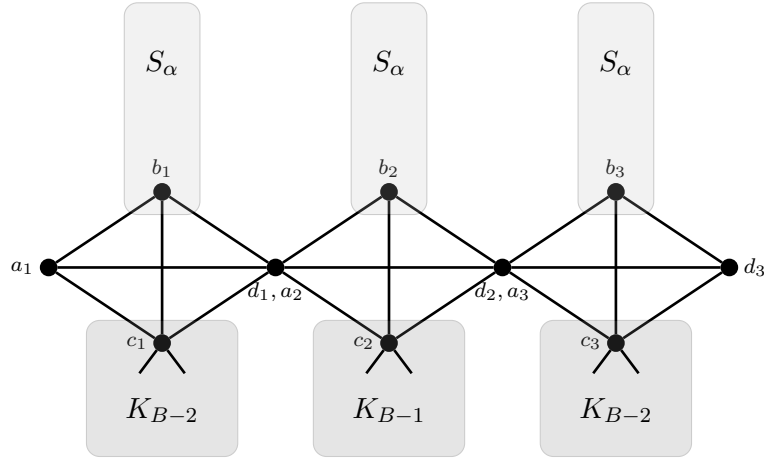


Figure 4: Illustration of the reduced graph constructed in the proof of Theorem 4.9, for $m = 3$.

590 copies S' and S'' of S_{i-1} , identifying the root of S' and u_1 , and similarly identifying the
 591 root of S'' and u_2 . The root of S_i is r .

592 For every $i \geq 1$, let n_i denote the number of vertices of S_i . So $n_1 = 5$, and, for every
 593 $i \geq 2$, we have $n_i = 2n_{i-1} + 1$. More precisely, we have $n_i = 5 \times 2^{i-1} + 2^{i-1} - 1$. To every
 594 member $S_i \in \mathcal{S}$, we associate a set I_i of integers defined as follows:

- 595 • $I_1 = \{3\}$, and
- 596 • $I_i = \{n_{i-1} + 1, \dots, 2n_{i-1} - 1\}$ of integers, for every $i \geq 2$.

597 Note that $|I_1| < |I_2| < \dots$. Furthermore, every S_i has the following property regarding I_i :

598 **Observation 4.8.** *Let $i \geq 1$ be fixed. For every $\alpha \in I_i$, the graph S_i has no subset*
 599 *$V_\alpha \subset V(S_i)$ such that $S_i[V_\alpha]$ is a connected α -graph and $S_i - V_\alpha$ is a connected graph*
 600 *containing the root of S_i .*

601 *Proof.* Assume such a part V_α exists. Let r denote the root of S_i . Note that every non-leaf
 602 vertex of S_i different from r is a cut-vertex. Under all assumptions, this yields that, by
 603 the value of α , necessarily the two neighbours r' and r'' of r belong to V_α . Since $r'r''$ is a
 604 cut-edge of S_i , this means that V_α has to cover all vertices different from r ; but this is not
 605 possible due to the value of α . This is a contradiction. \square

606 We are now ready to prove the NP-hardness of REALIZATION for claw-free graphs.

607 **Theorem 4.9.** *REALIZATION is NP-hard when restricted to connected claw-free graphs.*

608 *Proof.* We follow the lines of the proofs of Theorems 2.3, 2.4 and 2.5. Let $\langle A, B, s \rangle$ be an
 609 instance of 3-PARTITION, where we use the same terminology as in these proofs. We may
 610 assume that $s(a_1) \leq \dots \leq s(a_{3m})$. Free to modify this instance following Observation 2.1,
 611 we can assume that there is an α such that $s(a_1), \dots, s(a_{3m}) \in I_\alpha$.

612 We construct G as follows (see Figure 4 for an illustration). We add m disjoint copies
 613 of K_4 to the graph, where the vertices of the i th copy are denoted by a_i, b_i, c_i, d_i . For every
 614 $i = 1, \dots, m - 1$, we then identify the vertices d_i and a_{i+1} , so that the K_4 's form a kind of
 615 path connected via cut-vertices. For every $i = 1, \dots, m - 1$, we then add a copy of S_α to
 616 the graph, and we identify its root with b_i . Finally, we consider every $i = 1, \dots, m$, and:

- 617 • for $i = 1$ or $i = m$, we add a complete graph K_{B-2} to the graph, and we identify one
618 of its vertices and c_i ;
- 619 • for $i \in \{2, \dots, m-1\}$, we add a complete graph K_{B-1} to the graph, and we identify
620 one of its vertices and c_i .

621 Note that G is claw-free (it is actually a line graph). The $|V(G)$ -sequence π we consider
622 for the reduction is $\pi = (mn_\alpha + m - 1, s(a_1), \dots, s(a_{3m}))$.

623 Recall that $n_\alpha > s(a_{3m})$, and that α was chosen so that $s(a_1), \dots, s(a_{3m}) \in I_\alpha$. For this
624 reason, by Observation 4.8, in any realization of π in G , the part V_1 with size $mn_\alpha + m - 1$
625 has to contain the vertices of all S_α 's added to G , and, because $G[V_1]$ must be connected,
626 also all vertices d_1, \dots, d_{m-1} . Then $G - V_1$ is a disjoint union of traceable B -graphs, and
627 we have to find a realization of $(s(a_1), \dots, s(a_{3m}))$ in it. This is equivalent to finding a
628 solution to $\langle A, B, s \rangle$. \square

629 5. Conclusion

630 In this work, we have first considered the algorithmic complexity of the REALIZATION
631 and AP problems. On the one hand, we have mainly established, along all sections, the
632 NP-hardness of REALIZATION for more classes of graphs with various structure. On the
633 other hand, we have provided, in Section 3, new kernels of sequences showing that the AP
634 problem is in NP for a few more classes of graphs. However, we are still far from a proof
635 that 1) every graph has a polynomial kernel of sequences (which would establish the full
636 NPness of AP), and that 2) the AP problem is complete for some complexity class (NP or
637 Π_2^P being candidate classes). More efforts should thus be dedicated to these points.

638 One particular appealing case is the one of *cographs* ($\{P_4\}$ -free graphs), which was
639 mentioned in [10] by Broersma, Kratsch and Woeginger. It can easily be noted that the
640 reduction in our proof of Theorem 3.3 yields cographs, so REALIZATION is NP-hard for
641 these graphs. It is still open, though, whether there is a polynomial kernel of sequences for
642 cographs. Note that Theorem 3.1 makes a step in that direction, as 1-sequential graphs
643 (threshold graphs) form a subclass of cographs.

644 The second line of research we have considered in this work is the weakening, to AP-
645 ness, of well-known sufficient conditions for Hamiltonicity (or traceability). It would be
646 interesting if there were such a weakening for every condition for Hamiltonicity, as it would
647 emphasize the relationship between Hamiltonicity and APness. However, previous investi-
648 gations and some of our results seem to indicate that this connection is not as tight as one
649 could expect.

650 We believe, however, that it would be nice dedicating more attention to this direction;
651 let us thus raise an open question which might be interesting. As mentioned in the intro-
652 ductory section, Ore's well-known condition for Hamiltonicity can be weakened to APness.
653 In particular, all n -graphs G with $\sigma_2(G) \geq n - 2$ having a (quasi-) perfect matching are
654 AP. This result implies one direction of upcoming Question 5.1, which, if true, would stand
655 as a result *à la* Bondy-Chvátal.

656 Namely, for a graph G , the k -closure of G is the (unique) graph obtained by repeatedly
657 adding an edge between two non-adjacent vertices with degree sum at least k . A celebrated
658 result of Bondy and Chvátal states that an n -graph is Hamiltonian if and only if its n -closure
659 is Hamiltonian [8]. Analogously, an n -graph is traceable if and only if its $(n - 1)$ -closure is
660 traceable. However, it is not true that every n -graph is AP if and only if its $(n - 2)$ -closure
661 is AP: In the complete bipartite graph $K_{n/2-1, n/2+1}$, every two non-adjacent vertices have

662 degree sum at least $n - 2$, so its $(n - 2)$ -closure is complete and thus AP; however, note that
 663 $K_{n/2-1, n/2+1}$ has no perfect matching (realization of $(2, \dots, 2)$) and is thus no AP. This is
 664 actually not the only exception. Indeed, let G be a graph of order $n = 4k + 2$ consisting
 665 of two complete graphs $K_{n/2}$ with a common vertex and a pendant edge attached to this
 666 vertex. Then the $(n - 2)$ -closure of this graph is complete, but G has not realization of
 667 $(n/2, n/2)$. We wonder whether there are many such exceptions, in the following sense:

668 **Question 5.1.** *Is there an “easy” class of graphs \mathcal{G} , such that if $G \notin \mathcal{G}$ is an n -graph, then*
 669 *G is AP if and only if the $(n - 2)$ -closure of G is AP?*

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