

Some Sharp Circular and Hyperbolic Bounds of $\exp(x^2)$ with Applications

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Abstract. This article is devoted to obtain some sharp lower and upper bounds for $\exp(x^2)$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. The bounds are of the type $\left[\frac{a+f(x)}{a+1}\right]^\alpha$ where $f(x)$ is cosine or hyperbolic cosine. The results are then used to obtain and refine some known Cusa-Huygens type inequalities. In particular, new simple proof of Cusa-Huygens type inequalities is presented as an application. For other interesting applications of the main results, sharp bounds of the truncated Gaussian sine integral and error function are established. They can be useful in probability theory.

Keywords: circular; hyperbolic; exponential; truncated Gaussian sine integral; Cusa-Huygens inequality; error function.

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1 Introduction

The bounds of exponential function $\exp(x^2)$ can be useful in many areas of mathematics where it appears. Therefore it becomes very natural to find its sharp bounds. Recently Chesneau [1, 2] gave tight lower bounds of $\exp(x^2)$ over the real line. For some other sharp bounds see [14, 15], where the bounds are obtained in $(0, 1)$ using circular and hyperbolic functions. This type of bounds can in fact be obtained naturally in $(0, \pi/2)$ (see [3]). Interested readers are referred to [1, 2, 8, 10, 12] and the references therein. The goal of this paper is to present more tight bounds for $\exp(x^2)$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. For the applications, these bounds are then used to refine some known Cusa-Huygens type inequalities and to exhibit new sharp bounds for Gaussian type integrals, including the so called error function, opening new perspectives in many applied areas, including statistics, probability, physics and engineering.

1.1 First Result

We state the first main result of this paper as follows:

Theorem 1. For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\left(\frac{1 + \cos x}{2}\right)^a \leq \exp(-x^2) \leq \left(\frac{1 + \cos x}{2}\right)^b \quad (1.1)$$

and

$$\left(\frac{2 + \cos x}{3}\right)^c \leq \exp(-x^2) \leq \left(\frac{2 + \cos x}{3}\right)^d \quad (1.2)$$

with the best possible constants $a = 4, b = \frac{-(\pi/2)^2}{\ln(1/2)} \approx 3.559707, c = \frac{-(\pi/2)^2}{\ln(2/3)} \approx 6.08536, d = 6$ and the inequalities hold as equalities at $x = 0$.

Note: The right inequality in (1.2) has been proved in [17, Theorem 2]. In fact it holds for $x \in (0, \infty)$. However, it is not sharp for large values of x . Again our proof will use different method.

Some graphical and numerical illustrations: The inequalities (1.1) are illustrated in Figures 1 and 2. We see the sharpness of the obtained bounds. After a graphical investigation, the inequalities (1.2) seem more sharp; the curves of the functions of the bounds are almost visually confounded, even with a reasonable zoom. To illustrate this point, let us investigate the global L_2 error : $e(h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\exp(-x^2) - h(x))^2 dx$, where $h(x)$ denotes a function in the bounds (1.1) and (1.2). The results are set in Table 1. From this numerical point of view, we then see that the bounds in (1.2) are sharper to those in (1.1).

Table 1: Global L_2 errors $e(h)$ for the functions $h(x)$ in the bounds of (1.1) and (1.2).

	Inequality (1.1)		Inequality (1.2)	
$h(x)$	$\left(\frac{1+\cos x}{2}\right)^a$	$\left(\frac{1+\cos x}{2}\right)^b$	$\left(\frac{2+\cos x}{3}\right)^c$	$\left(\frac{2+\cos x}{3}\right)^d$
$e(h)$	≈ 0.000629229	≈ 0.001120559	$\approx 2.791112 \times 10^{-5}$	$\approx 4.605539 \times 10^{-6}$

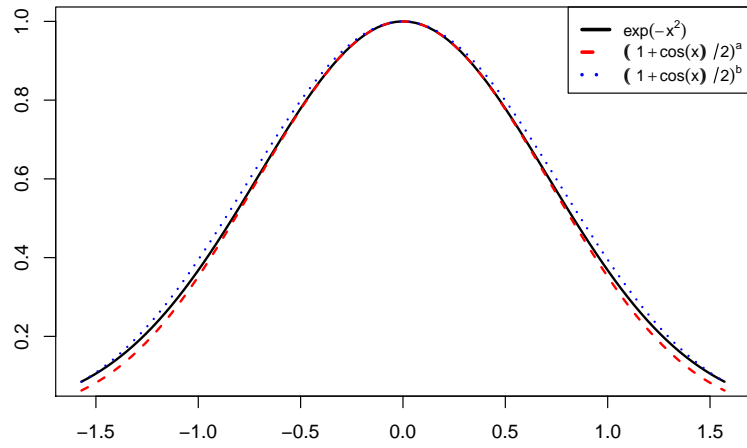


Figure 1: Graphs of the functions of the bounds (1.1) for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

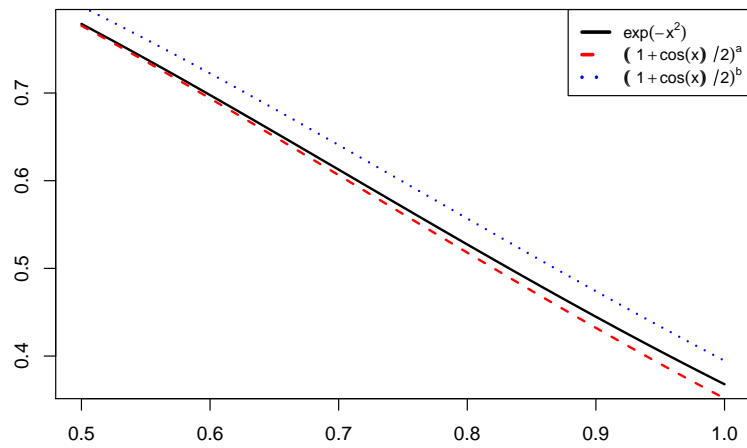


Figure 2: Graphs of the functions of the bounds (1.1) for $x \in (0.5, 1)$.

1.2 Second Result

The hyperbolic variants are given in the following theorem. The bounds of $\exp(x^2)$ given in (1.4) are very sharp. Moreover they are simple and better than the corresponding bounds of $\exp(x^2)$ given in [1, 2] as far as x is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Theorem 2. For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\left(\frac{1 + \cosh x}{2}\right)^\alpha \leq \exp(x^2) \leq \left(\frac{1 + \cosh x}{2}\right)^\beta \quad (1.3)$$

and

$$\left(\frac{2 + \cosh x}{3}\right)^\theta \leq \exp(x^2) \leq \left(\frac{2 + \cosh x}{3}\right)^\gamma \quad (1.4)$$

with the best possible constants $\alpha = 4, \beta = \frac{(\pi/2)^2}{\ln[(1+\cosh(\pi/2))/2]} \approx 4.38856, \theta = 6, \gamma = \frac{(\pi/2)^2}{\ln[(2+\cosh(\pi/2))/3]} \approx 6.054932$ and the inequalities hold as equalities at $x = 0$.

Some graphical and numerical illustrations: The inequalities (1.3) are illustrated in Figures 3 and 4, showing the sharpness of the obtained bounds. After a graphical investigation, the inequalities (1.4) seem more sharp. To illustrate this point, as the previous numerical study, let us consider the global L_2 error : $e_*(h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\exp(x^2) - h(x))^2 dx$, where $h(x)$ denotes a function in the bounds of (1.3) and (1.4). The results are set in Table 2. From this numerical point of view, we then see that the bounds in (1.4) are sharper to those in (1.3).

Table 2: Global L_2 errors $e_*(h)$ for the functions $h(x)$ in the bounds of (1.3) and (1.4).

	Inequality (1.3)		Inequality (1.4)	
$h(x)$	$\left(\frac{1+\cosh x}{2}\right)^\alpha$	$\left(\frac{1+\cosh x}{2}\right)^\beta$	$\left(\frac{2+\cosh x}{3}\right)^\theta$	$\left(\frac{2+\cosh x}{3}\right)^\gamma$
$e_*(h)$	≈ 1.011738	≈ 0.05904132	≈ 0.01013854	≈ 0.001456429

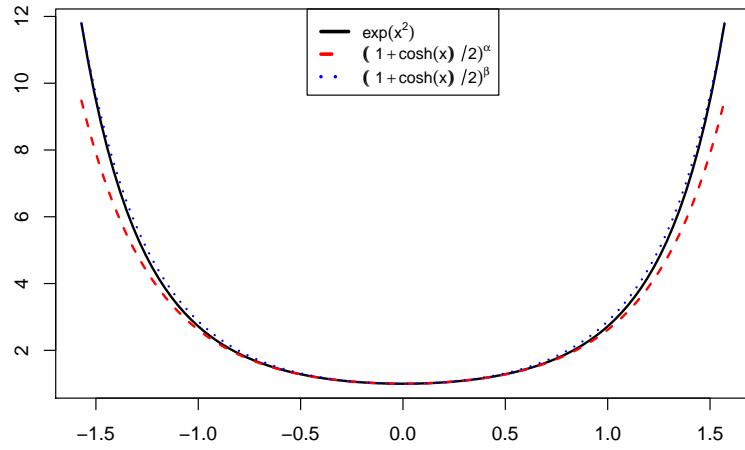


Figure 3: Graphs of the functions of the bounds (1.3) for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

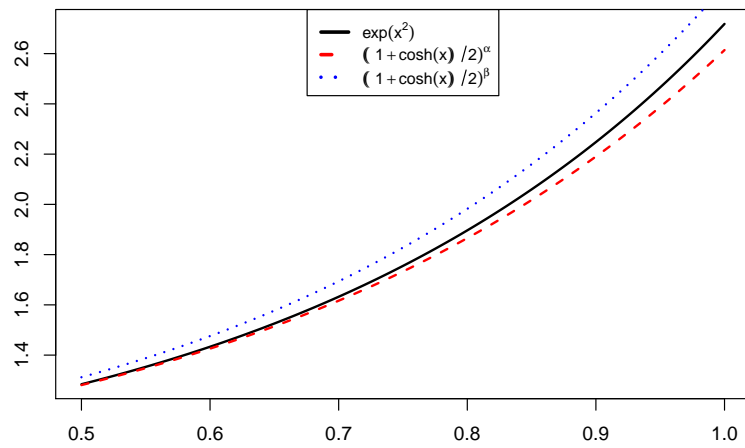


Figure 4: Graphs of the functions of the bounds (1.3) for $x \in (0.5, 1)$.

Note: It follows from Theorem 2 that, for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$\left(\frac{2 + \cosh x}{3}\right)^{-\gamma} \leq \exp(-x^2) \leq \left(\frac{2 + \cosh x}{3}\right)^{-\theta}. \quad (1.5)$$

It is natural to address the following question : what are the best bounds for $\exp(-x^2)$ between those in (1.2) and (1.5) ? An element of answer can be given numerically. By considering again the global L_2 error : $e(h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\exp(-x^2) - h(x))^2 dx$, where $h(x)$ denotes a function in the bounds (1.2) and (1.5). The results are set in Table 4. From this numerical point of view, we then see that the bounds in (1.5) are near twice sharper to those in (1.2).

Table 3: Global L_2 errors $e(h)$ for the functions $h(x)$ in the bounds of (1.2) and (1.5).

	Inequality (1.2)		Inequality (1.5)	
$h(x)$	$\left(\frac{2+\cos x}{3}\right)^c$	$\left(\frac{2+\cos x}{3}\right)^d$	$\left(\frac{2+\cosh x}{3}\right)^{-\gamma}$	$\left(\frac{2+\cosh x}{3}\right)^{-\theta}$
$e(h)$	$\approx 2.791112 \times 10^{-5}$	$\approx 4.605539 \times 10^{-6}$	$\approx 1.068113 \times 10^{-5}$	$\approx 2.338449 \times 10^{-6}$

2 Preliminaries and Lemmas

We now present two lemmas which will be useful for the proofs of our theorems.

Lemma 1. *The following inequalities hold:*

$$\frac{\sin x}{x} > \frac{1 + 2 \cos x}{2 + \cos x}; \quad x \in (0, \pi) \quad (2.1)$$

and

$$\frac{x}{\sinh x} + \cosh x > 2; \quad x \neq 0. \quad (2.2)$$

Proof: For (2.1), let $f(x) = \sin x (2 + \cos x) - x (1 + 2 \cos x)$. Then simple computation yields that

$$\begin{aligned} f'(x) &= -\sin^2 x + \cos^2 x + 2x \sin x - 1 = 2x \sin x - 2 \sin^2 x \\ &= 2 \sin x (x - \sin x) > 0 \end{aligned}$$

in $(0, \pi)$.

Hence, $f(x)$ is strictly increasing in $(0, \pi)$. Thus $f(x) > f(0)$ for any $x \in (0, \pi)$ implying that

$$\sin x (2 + \cos x) > x (1 + 2 \cos x).$$

For (2.2), by symmetry of the function, we need to consider only positive values of x .

Let, $g(x) = 2 \sinh x - \sinh x \cosh x - x$. Differentiation gives

$$\begin{aligned} g'(x) &= 2 \cosh x - \sinh^2 x - \cosh^2 x - 1 = 2 \cosh x - 2 \cosh^2 x \\ &= 2 \cosh x (1 - \cosh x) < 0. \end{aligned}$$

Therefore $g(x)$ is strictly decreasing in $(0, \infty)$. So, $g(x) < 0$ for every $x \in (0, \infty)$. That means $x + \sinh x \cosh x > 2 \sinh x$. This completes the proof.

□

Note: For hyperbolic version of (2.1) one can see [13, Remark 1].

Lemma 2. (*The L'Hospital's monotonicity rule [9]*) : Let $f, g : [p, q] \rightarrow \mathbb{R}$ be two continuous functions which are derivable on (p, q) and $g'(x) \neq 0$ for any $x \in (p, q)$. If f'/g' is increasing (or decreasing) on (p, q) , then the functions $\frac{f(x)-f(p)}{g(x)-g(p)}$ and $\frac{f(x)-f(q)}{g(x)-g(q)}$ are also increasing (or decreasing) on (p, q) . If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.

3 Proofs of the Theorems

In this section we prove our main results.

Proof of Theorem 1: Clearly for $x = 0$ equalities hold. We need to consider only positive values of x in $(-\pi/2, \pi/2)$ as bounds and $\exp(-x^2)$ are even functions.

$$\begin{aligned} \text{For (1.1) as } a &> \frac{-x^2}{\ln\left(\frac{1+\cos x}{2}\right)} > b, \\ \text{let } f(x) &= \frac{-x^2}{\ln\left(\frac{1+\cos x}{2}\right)} = \frac{f_1(x)}{f_2(x)} \end{aligned}$$

where $f_1(x) = -x^2$ and $f_2(x) = \ln\left(\frac{1+\cos x}{2}\right)$ with $f_1(0) = f_2(0) = 0$. By differentiating

$$\frac{f_1'(x)}{f_2'(x)} = \frac{2x(1+\cos x)}{\sin x} = 2 \frac{x}{\sin x} (1 + \cos x) = 2 F(x)$$

where $F(x) = \frac{x}{\sin x} (1 + \cos x)$. By differentiation

$$\begin{aligned} F'(x) &= -x + \frac{(\sin x - x \cos x)}{\sin^2 x} (1 + \cos x) \\ &= \frac{1}{\sin^2 x} [-x \sin^2 x + \sin x + \sin x \cos x - x \cos x - x \cos^2 x] \\ &= \frac{1}{\sin^2 x} [-x(1 + \cos x) + \sin x(1 + \cos x)] \\ &= \frac{1}{\sin^2 x} [(1 + \cos x)(\sin x - x)] < 0, \end{aligned}$$

since $\sin x - x < 0$ in $(0, \pi/2)$. Therefore $F(x)$ is strictly decreasing in $(0, \pi/2)$ and so is $f(x)$ by Lemma 2. Consequently, $a = f(0+) = 4$ by L'Hospital's rule and $b = f(\pi/2) = \frac{-(\pi/2)^2}{\ln(1/2)} \approx 3.559707$.

Similarly for (1.2) as $c > \frac{-x^2}{\ln(\frac{2+\cos x}{3})} > d$,

$$\text{let } g(x) = \frac{-x^2}{\ln(\frac{2+\cos x}{3})} = \frac{g_1(x)}{g_2(x)}$$

where $g_1(x) = -x^2$ and $g_2(x) = \ln(\frac{2+\cos x}{3})$ with $g_1(0) = g_2(0) = 0$.

$$\text{Then } \frac{g_1'(x)}{g_2'(x)} = \frac{2x(2+\cos x)}{\sin x} = 2G(x)$$

where $G(x) = \frac{x}{\sin x}(2 + \cos x)$. Differentiation gives

$$\begin{aligned} G'(x) &= -x + \frac{(\sin x - x \cos x)(2 + \cos x)}{\sin^2 x} \\ &= \frac{1}{\sin^2 x} [\sin x(2 + \cos x) - x(1 + 2 \cos x)] > 0, \end{aligned}$$

by virtue of Lemma 1 i.e. (2.1). Now $g(x)$ is strictly increasing in $(0, \pi/2)$ by Lemma 2. Thus, $c = g(\pi/2) = \frac{-(\pi/2)^2}{\ln(2/3)} \approx 6.08536$ and $d = g(0+) = 6$. This completes the proof. \square

Remark 1. For $x \in (-\epsilon, \epsilon]$ where $\epsilon \in (0, \pi)$, we can actually see that, the inequalities in Theorem 1 hold with the best possible constants $a = 4, b = \frac{-\epsilon^2}{\ln(\frac{1+\cos \epsilon}{2})}, c = \frac{-\epsilon^2}{\ln(\frac{2+\cos \epsilon}{3})}$ and $d = 6$.

Proof of Theorem 2: Equalities hold for $x = 0$. As in the proof of Theorem 1, we need to consider only positive values of x in $(-\pi/2, \pi/2)$.

$$\text{For (1.3), let } f(x) = \frac{x^2}{\ln(\frac{1+\cosh x}{2})} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = x^2$ and $f_2(x) = \ln\left(\frac{1+\cosh x}{2}\right)$ with $f_1(0) = 0 = f_2(0)$. Differentiating

$$\frac{f_1'(x)}{f_2'(x)} = \frac{2x(1+\cosh x)}{\sinh x} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = 2x(1 + \cosh x)$ and $f_4(x) = \sinh x$ with $f_3(0) = f_4(0) = 0$. Differentiation yields

$$\begin{aligned} \frac{f_3'(x)}{f_4'(x)} &= 2 \left[\frac{x \sinh x + 1 + \cosh x}{\cosh x} \right] \\ &= 2 [x \tanh x + \operatorname{sech} x + 1] \\ &= 2 F(x) \end{aligned}$$

where $F(x) = x \tanh x + \operatorname{sech} x + 1$. By differentiating

$$\begin{aligned} F'(x) &= x \operatorname{sech}^2 x + \tanh x - \operatorname{sech} x \tanh x \\ &= \tanh x \operatorname{sech} x \left[\frac{x}{\sinh x} + \cosh x - 1 \right] > 0, \end{aligned}$$

since $\frac{x}{\sinh x} + \cosh x > 2$ by (2.2) of Lemma 1. Therefore $F(x)$ is strictly increasing, which implies that $f(x)$ is also strictly increasing by Lemma 2. Thus, $\alpha = f(0+) = 4$ and $\beta = f(\pi/2) = \frac{(\pi/2)^2}{\ln\left[\frac{1+\cosh(\pi/2)}{2}\right]} \approx 4.38856$.

$$\text{To prove (1.4), let } g(x) = \frac{x^2}{\ln\left(\frac{2+\cosh x}{3}\right)} = \frac{g_1(x)}{g_2(x)}$$

where $g_1(x) = x^2$ and $g_2(x) = \ln\left(\frac{2+\cosh x}{3}\right)$ with $g_1(0) = g_2(0) = 0$. Differentiation gives

$$\frac{g_1'(x)}{g_2'(x)} = \frac{2x(2+\cosh x)}{\sinh x} = \frac{g_3(x)}{g_4(x)}$$

where $g_3(x) = 2x(2 + \cosh x)$ and $g_4(x) = \sinh x$ with $g_3(0) = g_4(0) = 0$. Therefore

$$\begin{aligned} \frac{g_3'(x)}{g_4'(x)} &= 2 \left[\frac{x \sinh x + 2 + \cosh x}{\cosh x} \right] \\ &= 2 G(x) \end{aligned}$$

where $G(x) = x \tanh x + 2 \operatorname{sech} x + 1$. By differentiation, we get

$$\begin{aligned} G'(x) &= x \operatorname{sech}^2 x + \tanh x - 2 \operatorname{sech} x \tanh x \\ &= \tanh x \operatorname{sech} x \left[\frac{x}{\sinh x} + \cosh x - 2 \right] > 0, \end{aligned}$$

due to second inequality (2.2) of Lemma 1. So $G(x)$ is strictly increasing and hence $g(x)$ in $(0, \pi/2)$ by Lemma 2. Therefore, $\theta = g(0+) = 6$ and $\gamma = g(\pi/2) = \frac{(\pi/2)^2}{\ln\left[\frac{2+\cosh(\pi/2)}{3}\right]} \approx 6.054932$. This proves Theorem 2. \square

Remark 2. For $x \in (-\epsilon, \epsilon)$ where $\epsilon > 0$, it's easy to see that, the inequalities in Theorem 2 hold with the best possible constants $\alpha = 4, \beta = \frac{\epsilon^2}{\ln(\frac{1+\cosh\epsilon}{2})}, \gamma = \frac{\epsilon^2}{\ln(\frac{2+\cosh\epsilon}{3})}, \theta = 6$.

4 Some Applications

Three applications of Theorems 1 and 2 are presented below.

4.1 Application 1: On Cusa-Huygens Type Inequalities

The famous Cusa-Huygen's inequality [4, 5, 6, 7, 11] is known as

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}; \quad 0 < x < \frac{\pi}{2} \quad (4.1)$$

and its hyperbolic version, sometimes called hyperbolic Cusa-Huygen's inequality [7] is stated as follows:

$$\frac{\sinh x}{x} < \frac{2 + \cosh x}{3}; \quad x \neq 0. \quad (4.2)$$

Some researchers have tried to obtain extended sharp versions of the inequalities (4.1) and (4.2) in recent years. In [6, 11] the following inequalities have been established:

$$\left(\frac{2 + \cos x}{3}\right)^\lambda < \frac{\sin x}{x} < \frac{2 + \cos x}{3}; \quad x \in \left(0, \frac{\pi}{2}\right) \quad (4.3)$$

with the best possible constants $\lambda \approx 1.11374$ and 1.

The authors of [6, 11] proved double inequality (4.3) in a complex way. In 2013, a simple proof of it was claimed by Sun and Zhu [18]; but later it was found that the proof was logically incorrect [16]. We present here very simple and lucid proof of (4.3).

Simple Proof of Inequality (4.3): Using [14, Theorem 2] and [3, Proposition 3], we have

$$\exp(-kx^2) < \frac{\sin x}{x} < \exp(-x^2/6); \quad x \in (0, \pi/2)$$

where $k = \frac{-\ln(2/\pi)}{(\pi/2)^2}$. From this we can write

$$\left(\frac{\sin x}{x}\right)^6 < \exp(-x^2) < \left(\frac{\sin x}{x}\right)^{1/k}; \quad x \in (0, \pi/2) \quad (4.4)$$

where $k = \frac{-4\ln(2/\pi)}{\pi^2}$. From (1.2) and (4.4) it is clear that

$$\left(\frac{2+\cos x}{3}\right)^\lambda < \frac{\sin x}{x} < \frac{2+\cos x}{3}$$

where $\lambda = kc = \frac{-4\ln(2/\pi)}{\pi^2} \cdot \frac{-\pi^2}{4\ln(2/3)} = \frac{\ln(2/\pi)}{\ln(2/3)} \approx 1.11374$. Moreover, λ and 1 are the best possible constants, because k and c are. The proof of (4.3) is complete. \square

Sándor [11] proved that the best positive constants m and n such that

$$\left(\frac{1+\cosh x}{2}\right)^m < \frac{\sinh x}{x} < \left(\frac{1+\cosh x}{2}\right)^n; \quad x > 0 \quad (4.5)$$

are $2/3$ and 1 respectively.

In the following corollary, we refine right inequality of (4.5) in the interval $(0, \pi/2)$.

Corollary 1. *For $x \in (0, \pi/2)$ one has*

$$\frac{\sinh x}{x} < \left(\frac{1+\cosh x}{2}\right)^\mu \quad (4.6)$$

where $\mu = \frac{\pi^2}{24\ln\left[\frac{1+\cosh(\pi/2)}{2}\right]} \approx 0.731427$ is the best possible constant.

Proof: Using [14, Theorem 3], [3] we can actually see that

$$e^{-x^2/6} < \frac{x}{\sinh x} \text{ for } x \in (0, \frac{\pi}{2})$$

which implies

$$\left(\frac{\sinh x}{x}\right)^6 < \exp(x^2); \quad x \in \left(0, \frac{\pi}{2}\right) \quad (4.7)$$

Now (1.3) and (4.7) give us

$$\frac{\sinh x}{x} < \left(\frac{1+\cosh x}{2}\right)^\mu$$

where $\mu = \frac{\beta}{6} = \frac{\pi^2}{24\ln\left[\frac{1+\cosh(\pi/2)}{2}\right]} \approx 0.731427$ is the best possible constant. \square

Other useful applications of (1.1) and (1.2) include the sharp bounds of Gaussian type integrals, with simple analytical expressions. Both of them are described below.

4.2 Application 2: Simple Bounds for a Truncated Sine Gaussian Integral

In Corollary 2, we determine simple bounds for the truncated Gaussian sine integral defined by $\int_0^y \sin x \exp(-x^2) dx$. This function has some connection with the Dawson type integrals.

Corollary 2. *For $y \in (0, \pi/2)$, it is true that*

$$\begin{aligned} \frac{3}{c+1} \left[1 - \left(\frac{2 + \cos y}{3} \right)^{c+1} \right] &\leq \int_0^y \sin x \exp(-x^2) dx \\ &\leq \frac{3}{d+1} \left[1 - \left(\frac{2 + \cos y}{3} \right)^{d+1} \right]. \end{aligned}$$

with the best possible constants $c \approx 6.08536$ and $d = 6$.

Proof: By utilizing (1.2), we can write

$$\int_0^y \sin x \left(\frac{2 + \cos x}{3} \right)^c dx \leq \int_0^y \sin x \exp(-x^2) dx \leq \int_0^y \sin x \left(\frac{2 + \cos x}{3} \right)^d dx.$$

By remarking that $\int_0^y \sin x \left(\frac{2 + \cos x}{3} \right)^c dx = \frac{3}{c+1} \left[1 - \left(\frac{2 + \cos y}{3} \right)^{c+1} \right]$, with the same for d in place of c , we end the assertion. \square

4.3 Application 3: Simple Bounds for the Error Function erf

For the last application, we consider the well known error function defined by

$$erf(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-x^2) dx.$$

For this function also we give sharp explicit bounds in Corollary 3.

Corollary 3. *For $y \in (0, \pi/2)$, it holds that*

$$\begin{aligned} \frac{(6\cos^3 y + 32\cos^2 y + 81\cos y + 160) \sin y + 105y}{384} &\leq \frac{\sqrt{\pi} erf(y)}{2} \leq \\ \frac{(40\cos y + 576) \sin^5 y - (3730\cos y + 14720) \sin^3 y + (37965\cos y + 87360) \sin y + 49635y}{174960} & \end{aligned} \tag{4.8}$$

Proof: Using (1.1) and (1.2), we have

$$\left(\frac{1 + \cos x}{2}\right)^4 \leq \exp(-x^2) \leq \left(\frac{2 + \cos x}{3}\right)^6.$$

Therefore

$$\int_0^y \left(\frac{1 + \cos x}{2}\right)^4 dx \leq \frac{\sqrt{\pi} \operatorname{erf}(y)}{2} \leq \int_0^y \left(\frac{2 + \cos x}{3}\right)^6 dx.$$

Using the expansions : $(1 + \cos x)^4 = \frac{1}{8}(56\cos x + 28\cos(2x) + 8\cos(3x) + \cos(4x) + 35)$ and $(2 + \cos x)^6 = \frac{1}{32}(10224\cos x + 4815\cos(2x) + 1400\cos(3x) + 246\cos(4x) + 24\cos(5x) + \cos(6x) + 6618)$ and by integration, we obtain required result. \square

Some graphical and numerical illustrations: The sharpness of the bounds in (4.8) are illustrated in Figures 5 and 6. Let us now investigate the global L_2 error : $e_o(h) = \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\pi} \operatorname{erf}(x)}{2} - h(x)\right)^2 dx$, where $h(x)$ denotes a function in the bounds of (4.8). The results are set in Table 4. We see that the error is negligible, attesting the interest of our findings.

Table 4: Global L_2 errors $e_o(h)$ for the functions $h(x)$ in the bounds of (4.8) ; Boundinf is for the lower bound and Boundsup is for the upper bound.

$h(x)$	Boundinf	Boundsup
$e_o(h)$	$\approx 6.930623 \times 10^{-5}$	$\approx 2.314179 \times 10^{-7}$

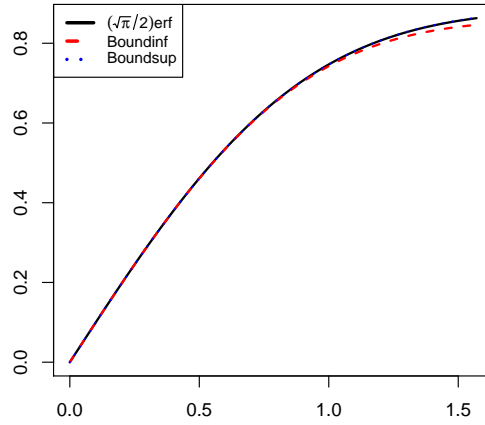


Figure 5: Graphs of the functions of the bounds (4.8) for $x \in (0, \frac{\pi}{2})$.

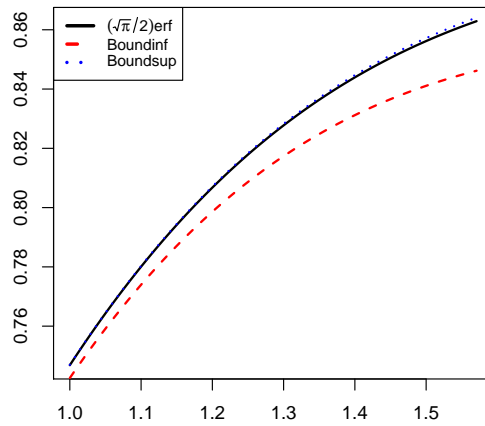


Figure 6: Graphs of the functions of the bounds (4.8) for $x \in (1, \frac{\pi}{2})$.

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