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An APOS perspective of the understanding of the concept of vector subspace

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The content of Linear algebra is often considered to be difficult because of the formal logic required as well as the lack of connections to previous courses such as calculus. The purpose of this study was to explore the conceptions of vector subspace concepts, of 73 in-service mathematics teachers as revealed in their written responses to two tasks. The action, process, object, schema (APOS) theory was used to structure the analysis of the responses. The findings revealed that the teachers struggled with the vector sub-space concepts mainly because of poorly developed conceptions of prerequisite concepts of sets and binary operations.

Keywords: APOS, vector subspace, binary operations, vector space.

Introduction

Linear algebra is considered to be one of the most widely applicable subjects for students in the field of mathematics in that it can be applied to many different content areas, such as engineering and statistics and can be studied for mathematical abstraction. However, we noted that when the students take their first linear algebra course, they seem to encounter cognitive barriers. Dorier, Robert, Robinet and Rogalski (2000) noted that the teaching of vector spaces have completely disappeared in the secondary schools and the teaching has become less formal as there are no studies on algebraic structures. Some criticisms given by students about linear algebra concern the use of formalism and the lack of connections with what they already know since this is not done at secondary level. Dorier et al., (2000) elaborated that the formalism is experienced when students need to learn new definitions, symbols, words and theorems. Stewart and Thomas (2010) noted that many students in the first years cope well with the procedural aspects of solving systems of linear equations but struggle to understand the crucial concepts underpinning the material involving the study of vector space concepts such as subspace, linear independence and spanning. The teachers complain that the students have no skills in elementary cartesian geometry, and display an inconsistent use of the basic tools of logic or set theory (Dorier et al.,2000).

An APOS study set up to explore pre-service teachers' mental constructions of matrix algebra concepts, found that most of the participants were operating at the action and process level, with a few operating at the object level (Ndlovu & Brijlall, 2015). The authors argued that the lack of background knowledge of basic algebra schema hampered the teachers efforts to develop adequate schemas at the object level. Many preservice teachers could not manipulate numbers correctly when multiplying matrices and some of them failed to use notations correctly. They confused the notations A^T and A^{-1} . The goal of mathematics teaching is that students understand mathematical concepts presented to them or information that they discover for themselves. This is also supported by Hiebert and Carpenter (1992) who asserted that one of the most widely accepted ideas in mathematics education is that students should understand mathematics. The research described in this paper is concerned with students' difficulties with conceptual understanding of vector subspace.

Britton and Henderson (2009) in studying student's conceptual understanding of a subspace argued that the abstract "obstacle of formalism" and the theoretical nature of linear algebra are the root cause of the difficulties experienced. They believed that lecturers teach students for procedural rather than conceptual understanding and students have poor backgrounds of the concepts on proofs, logic and set theory. One of the questions asked the students to show that the set of scalar multiples of a vector formed a subspace of \mathbb{R}^3 . Most of the students could show that the set, V , is non empty but failed to prove the aspect on the closure property. Students chose particular vectors instead of arbitrary vectors, and some student worked out the sum of two vectors and assumed that the sum belonged to the set V . Some students had misconceptions about the definition of a subspace, while others mixed up concepts and showed rote learning of the concepts on vector subspace. The researchers also noted that students had problems with logic and set theory, moving from abstract to algebraic mode algebraic mode and failing to write a convincing proof.

Theoretical framework: APOS theory

We use the action–process–object–schema (APOS) theory as a framework to make sense of the data. According to Arnon, Cottrill, Dubinsky, Oktaç, Fuentes, Trigueros and Weller (2014) APOS theory is based on the extension of Piaget's principle that an individual learns mathematics by applying certain mental mechanisms to build specific mental structures. The main mental mechanisms for building the mental structures include interiorisation, coordination and encapsulation. The mental structures refer to the action, process, object and schema. As actions are repeated and reflected on, the student moves from relying on external cues to having internal control over them. This is characterized by an ability to imagine carrying out the steps without necessarily having to perform each one explicitly. Interiorisation is the mechanism that makes this mental shift possible. Encapsulation occurs when an individual becomes aware of a process as a totality upon which transformations can act. At this stage the student can analyse properties of the object and compare objects arising from the same process. (Arnon et al., 2014)

Many actions, objects and processes are interconnected in the individual's mind and these will be organised to form a coherent framework called a schema. An object can be assimilated by an existing schema, thus extending the span of the schema. According to Piaget, schema development also passes through stages of development. The Intra level is the preliminary level and is characterised by analysing particular events or objects in an isolated manner in terms of their properties, where explanations are local and not global and relationships between objects may not be perceived. At the Inter level, comparison and reflection upon properties of objects lead to the establishment of relationships. The individual can coordinate two different interpretations of the concept to mean the same thing. During the Trans stage, the student reflects upon and coordinates the relations and is aware of the complete structure. Using these definitions, we now present a genetic decomposition of the vector space concept.

Genetic decomposition of the Vector Space concept

We draw upon the work of Parraguez and Oktac (2010) and Arnon et al., (2014) to present a summarised description of the genetic decomposition of the vector space concept. The construction of the vector space concept is developed as the coordination of the prerequisite concepts of set and

binary operations. Hence, we refer to the set and binary operations Schema as abstraction layers of the vector space concept.

Set Schema. At an Action level, an individual conceives of a set when given a specific listing if a particular condition of set membership. The Action of gathering and putting objects together in a collection according to some condition is interiorised into a Process. This is encapsulated into an Object when an individual can apply actions or processes to the Process such as compare two sets, consider a set to be an element of another and analyse properties of the set. (Arnon et al., 2014)

Binary operation Schema. A binary operation is a function of two variables defined on a single set or on a Cartesian product of two sets. At an Action level, given a binary operation, an individual can take two specific elements of the sets and apply the formula. The individual interiorises the action into a Process that takes two objects (elements) and acts on these to produce a new Object (element) that is the result of the binary operation. At the Object level, an individual can distinguish between two binary operations, check whether a binary operation satisfies an axiom and compare Objects arising from two different binary operations. (Arnon et al., 2014)

Parraguez and Oktac (2010) describe how these two schema are drawn together to form the concept of vector space:

The Objects that are sets with two kinds of operations (addition and multiplication by a scalar) can be coordinated through the related processes and the vector space axioms that involve both operations, to give rise to a new Object that can be called a vector space. At the Intra level the object of vector space stays isolated from other actions, processes, objects and schemas. For example the student can verify different sets as being vector spaces or not, but does not see the vector space structure inherent in all of them. At the Inter level the object of vector space starts having relationships with other concepts such as subspace, linear transformations, basis, etc. When the student reflects upon these relations, through synthesis they can be recognized as part of a whole structure that makes up a vector space schema. This implies that the Trans level is reached and the student can recognize and work with non-standard examples of vector spaces and can invoke her/his schema when needed. (Parraguez & Oktac, 2010, p. 2116)

This description emphasises the complexity of the construction of the vector space concept which is built upon layers of abstraction. Firstly, the binary operation and set concepts are developed through to higher levels of abstraction via the Action-Process-Object path. The vector space concept is then constructed on these layers, forming an even higher layer of abstraction and as the vector space schema develops, at each stage the previous layer is re-organised as increasing coordination and coherence across the objects and relationships develops. The vector subspace concept is built upon this schema – students will not be able to see the connections between a vector space and vector subspace if they have not developed the vector space schema up to at least an Inter level.

Methodology

This study was conducted with 73 underqualified mathematics teachers who were enrolled in a part-time in-service course at a Zimbabwean university that was designed to upgrade them. The design of the program was such that the teachers would complete the equivalent of an undergraduate three-year degree program. However, the lectures were offered in two intensive block sessions for each

semester and held from 8 Am to 6 Pm every day. The participant teachers have already taken a first course in matrix algebra, a course in mathematical discourse and structures together with two courses in calculus. The second course in linear algebra (during which they participated in the study) includes the concepts of vector spaces, linear independence, linear transformation and diagonalization, eigenvalues and eigenvectors. The research question that underpins this study is: *What does an APOS perspective suggest about the conceptions of vector subspace concepts held by 73 in-service mathematics teachers as revealed in their written responses to two tasks?*

The data was collected from the teachers' written responses to an activity sheet consisting of nine items which were intended to probe their understanding of vector spaces and vector sub-spaces. In this short paper, we focus on two tasks which were set within the vector space of 2×2 matrices. These appear below.

Item	Comments
1. Let \mathbf{V} be the vector space over of all 2×2 matrices over the real field \mathbb{R} . Show that \mathbf{W} is not a subspace of \mathbf{V} , where \mathbf{W} is the set of 2×2 matrices which have a zero determinant.	For this, teachers were expected to find a counter-example to show that the set \mathbf{W} is not closed under vector addition.
2. Show that the set of all $M_{2 \times 2}$ matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a vector space.	For this, teachers could argue that since $M_{2 \times 2}$ is already a vector space, then it was only required to show that the given subset formed a vector subspace of $M_{2 \times 2}$ or, they could show that the eight axioms for a vector space were satisfied.

Table 1: Research Tasks

Results and discussion

The teachers' responses were numbered from one to 73, where the order held no significance, for example, R11 is the response of Teacher number 11 on the list. We found that the teachers displayed different levels of engagement with the vector space concept. On the one hand some teachers held extremely limited conceptions of binary operations and of set that were not even at Action levels, while on the other hand some teachers had developed strong enough conceptions of vector space schema at an Inter level. The details of some of these levels of engagement are described now.

Binary Operation Layer

The teachers' responses to the two tasks, suggested that the teachers were reasoning at various levels about binary operations, ranging from those who did not show evidence of even Action-level conceptions, while some displayed Object-level conceptions. Responses which illustrate different levels of reasoning are discussed below.

Not yet at Action levels

It was clear that some teachers had not developed an Action conception of binary operations, as illustrated by the responses of R13 to Task 1 and that of R1 in Task 2. The response by R13 below shows that the teacher has not been able to carry out the Action of scalar multiplication because he has been sidetracked by thinking about the sign of the scalar.

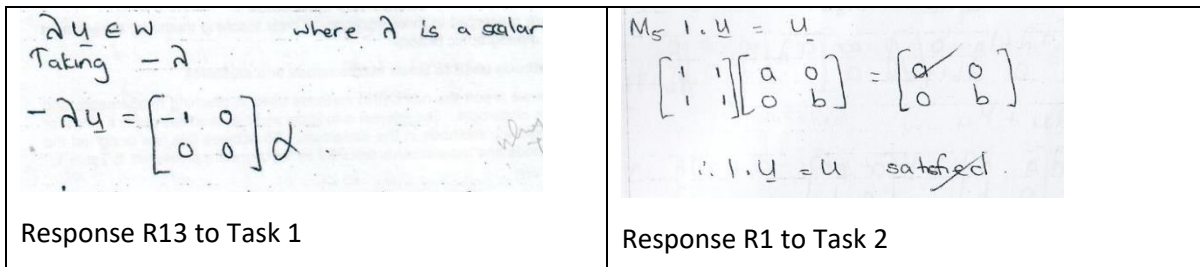


Figure 1: Responses suggesting action level conceptions of the binary operation have not developed

The response R1 showed that the teacher was not clear about what “1” in the axiom referred to in the scalar multiplication $1.u$ and took it as the “identity” matrix, consisting of 1’s in all the entries. The teacher then proceeded to carry out a pairwise multiplication of the corresponding elements in the two matrices. This is shown in the response of R1 in Figure 3.

Process- level engagement with the binary operations

Some teachers’ responses suggested that they were able to engage in Process- level reasoning with the binary operations. However, for Task 2, if they had not developed an Object-level conception they were unable to apply the axioms correctly to the binary operations, as illustrated in Figure 2 below.

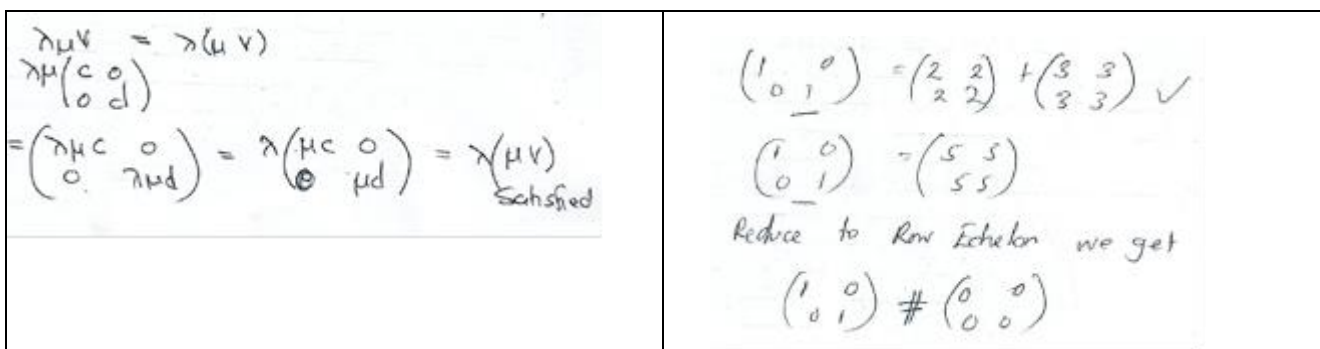


Figure 2: Response R11 working with scalar multiplication and R12 showing confusion about closure

In Figure 2, R11 is trying to show the associativity property of the operation of scalar multiplication $(\lambda\mu)v = \lambda(\mu v)$. The teacher is able to multiply the scalar into the vector (2×2 matrix) without any problems, suggesting that she has developed a Process conception of the binary operation of scalar multiplication. However, her expression on the left hand side of the first line does not have any brackets. This indicates that the teacher has problems with distinguishing between Objects arising from the different binary operations $(\lambda\mu)v$ as opposed to $\lambda(\mu v)$. This suggests that the teacher has

not yet developed the necessary Object-level conception of the binary operation of scalar multiplication.

Similarly some teachers showed evidence of Process-level reasoning about the binary operation of addition but this was not sufficient to enable them to respond correctly to Task 1, which required them to show that the closure condition was not satisfied as in the case of R12 which also appears in Figure 2.

The response R12, shows firstly that the teacher has considered a particular type of 2×2 matrices, that has identical entries. These matrices belong to the given set, because their determinants are zero. The teacher has done the vector addition of the two matrix elements, but was unable to show that the set was not closed under the binary operation. The sum of the matrices does satisfy the condition of having a zero determinant. Although he is able to reason at a Process-level about the binary operation of vector addition of 2×2 matrices, the teacher was confused about what he needed to do, with the result of the addition (sum). He seemed to be trying to show that the sum should be equal to the identity matrix. This shows that he is unable to work with the sum of the vectors as an Object.

Set Layer

Similar to the binary operations layer, teachers' responses to the two tasks, showed reasoning at various levels in the Set layer. Examples of responses which illustrate this are discussed below.

Not yet at Action levels

Many teachers were not clear about what the elements of the subset of Task 1 was. One teacher (R17) used a general 2×2 matrix, U , with variable entries and the zero matrix and tried to show that the closure condition was not satisfied as shown in Figure 3. The teacher has assumed that the matrix U belonged to the set W . This suggests that the teacher has not developed an Action conception of the subset W , of all 2×2 matrices which have a zero determinant.

(i) $0 \in V$ Axiom 1 holds
 (ii) let $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2×2 matrix and $w = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ mat. w 0 determinant
 $\therefore u+w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Axiom 2 holds on addition closure

Figure 3: Response R17, unable to identify non-zero elements of the given Set

Process-level reasoning

Some teachers showed that they were able to recognize whether elements belonged to the set of 2×2 matrices with zero determinants by considering the condition which characterized the set. One teacher (R6) considered a matrix with elements 1, 2, 3 and 4 and showed that the determinant of the matrix is not equal to zero as shown in Figure 4. That is, he identified a 2×2 matrix which did not belong to the given set. This shows that he can work out that the element does not belong to W .

suggesting that he has developed a Process conception of the Set of all 2×2 matrices which have a zero determinant, however he could not prove that the set \mathbf{W} was not a subspace.

Handwritten work showing a counterexample for a set \mathbf{W} of 2×2 matrices with zero determinant. The student defines $u = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \notin \mathbf{W}$ and calculates its determinant as $\det u = 4 - 6 = -2 \neq 0$. An arrow points from the calculation to the conclusion that u is not in \mathbf{W} .

Figure 4: Response R6 considering an example of a 2×2 matrix which did not belong to the given set.

Indications of Inter level conceptions of vector space schema

There were six teachers whose responses suggest that they have may developed an Inter level conception of the vector space schema. R3, for example, considered two 2×2 matrices \mathbf{x} , and \mathbf{y} where \mathbf{x} had 0's in the first column and 1 in the first row of the second column, while \mathbf{y} had had 1's in the first column and 0's in the second column. The sum $(\mathbf{x} + \mathbf{y})$ was therefore a 2×2 matrix with three entries being 1. So the determinant was 1 and hence did not satisfy the zero- determinant condition. The proof involved generating a counter-example of two elements and then showing that the sum did not belong to \mathbf{W} , hence implying that \mathbf{W} was not closed under the binary operation of vector addition. This argument suggests that the student has developed Object-level conception of both binary operation as well as Object-level conception of the set of 2×2 matrices with zero determinants set and the relations between these are perceived. Being able to show that the set \mathbf{W} did not satisfy the conditions for being a subspace of the vector space \mathbf{V} , suggests that these teachers have developed Inter level conceptions of the vector space concepts. These teachers also presented appropriate responses to Task 2.

Conclusion

The use of APOS theory suggests that most of the teachers have not developed the necessary mental constructions that would enable them to reason about the properties of vector subspaces, which requires an Inter level conception of vector space schema. The study showed that for many teachers, their conceptions of the prerequisite schemas of set and binary operations hampered them from developing the vector space schema at an Inter level. Most teachers showed that they were able to generate the vector addition of two matrices, in their responses to Task 1. This suggests that they had developed a Process conception of the binary operation of vector addition using 2×2 matrices. However, to show that the set was not closed under the binary operation, the teachers needed to have encapsulated the process of vector addition into an Object

whose properties they could analyse further. Some teachers had not developed even Action level conceptions of the binary operations. Many teachers struggled with the basics such as trying to identify the elements of the subset and were unable to reason about the kind of elements that produced a zero determinant. These teachers had basic problems with working with the set itself of matrices. These struggles indicate that their conceptions of the set of 2×2 matrices with zero determinants had not progressed past an Action level so they did not have access to the sophisticated Object-level reasoning about properties of 2×2 matrix with zero determinants. It is however important to note that the classification is based on written responses and if interviews

were used, the inferences about their levels of reasoning may differ somewhat depending on how the teachers' responded to questions about their reasoning.

Unlike simpler concepts which require coordination between few Objects and Schema, developing a robust understanding of vector subspace is dependent on a sufficiently strong conception of the various layers underlying the concept. Each of these layers is built upon previous ones and becomes increasingly abstract, requiring the coordination and connection between the various objects and relations. Hence very few teachers were able to cope with both tasks. For the teachers to develop the insight that was necessary, they needed to have access to Object-level conceptions of the set (of 2×2 matrices with zero determinants as well as Object-level conceptions of both binary operations. It is therefore no surprise that only six teachers seemed to be able to cope with the abstraction required to present proofs about why a subset did not form a vector subspace of the vector space.

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