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‘Father Woodland’: A learning environment to facilitate the development of algebraic thinking

Ioannis Papadopoulos¹ and Nafsika Patsiala²

¹Aristotle University of Thessaloniki, Greece; ypapadop@eled.auth.gr

²Aristotle University of Thessaloniki, Greece; nafspats@eled.auth.gr

In this paper, the contribution of the use of the “Father Woodland” learning environment in Grade-3 students’ algebraic thinking is examined. Four types of thinking were identified indicating a progressive movement towards the use of symbolic language that seems to have a rather developmental character. In their solutions the students induced rules for solving equations that will later be introduced formally to them.

Keywords: Early algebraic thinking, puzzle-like environments.

Introduction

Given the central position of algebra in the secondary curriculum which led to a separation between arithmetic and algebra with the first being the main focus of elementary mathematics curriculum, mathematics educators try to cope with the challenge of managing the transition from arithmetic to symbolic algebra. Numerous researchers admit that this separation deprives children of powerful schemes of thinking in the early grades and makes it more difficult to learn algebra in the later years (e.g., Kieran, 2007). One way to address this issue might be to study the impact of certain learning environments in the students’ development of algebraic thinking. Papadopoulos, Kindini and Tsakalaki (2016) working with a mobile puzzle environment found that sixth graders exhibit a progressive movement towards algebraic thinking. In this context, we try to explore the potential contribution of another specific learning environment called ‘Father Woodland’ in young students’ algebraic thinking that would allow us to identify certain steps in this movement from arithmetic to algebra. This is based on two of the algebra goals specified by NCTM (2000) standards, i.e., (i) represent and analyze mathematical situations and structures using algebraic symbols, and (ii) use mathematical models to represent and understand quantitative relationships. Therefore, we try to examine whether this environment facilitates the achievement of these goals through the identification of the types of thinking that the students followed in order to cope with the given tasks.

Early algebraic thinking and ‘Father Woodland’ environment.

Cai and Knuth (2011) do not limit algebraic thinking in earlier grades to simply mastering arithmetic and computational fluency but it goes deeper in identifying the underlying structure of mathematics which includes the development of particular ways of thinking, analysis of relationships between quantities, noticing structure, generalization, problem solving, justifying, proving and predicting. Cai et al. (2005), in a cross-cultural comparative study talk about multiple representations (pictures, diagrams, tables, graphs, and equations) that are used to represent functional relationships between two quantities and more specifically about ‘pictorial equations’ used to represent quantitative relationships providing thus a means for developing students’ algebraic ideas. This raises the necessity to make the distinction between the external and internal representations in the sense of considering at a minimum configurations of symbols or objects external to the individual learner or

problem solver (i.e., concrete materials, pictures/diagrams, spoken words, written symbols) that can be described mathematically and configurations internal to the individual (i.e., mental models and cognitive representations of the mathematical ideas underlying the external representations) respectively (Goldin, 2002). Such internal representations are inferred from the way the students express their aspects of the process of mathematical thinking in their written responses. It seems that certain learning environments can be in favor of introducing young learners to these aspects of algebraic thinking (Papadopoulos et al., 2016). In the current study a specific learning environment has been chosen. It is called ‘Father Woodland’ and is about a Czech fairy-tale figure owning a farm who organizes tug-of-war games among the animals living in the farm (Hejný, Jirotková, & Kratochvílová, 2006). The weakest animal is the mouse. Two mice are as strong as a cat, a cat and a mouse equal a goose and a goose and a mouse equal a dog (Fig. 1). The strength of each animal is represented by a picture and an icon (symbol) and the students are asked mainly to decide between two groups the stronger one or to add some animals to the weaker group in order to create two equivalent groups, or to reveal the identity of hidden animals so as to obtain equity.



Figure 1: Equivalences in the Father Woodland environment

It is a rich environment. Hejny and his colleagues use it in a series of textbooks they produced. The relevant tasks within these textbooks are connected with the development -among others- of an early number sense, pre-concept of divisibility, the lowest common multiple and greatest common divisor as well as the solving of equations. Hejný, et al. (2006) used this environment with Grade 1-3 students focusing on how the environment facilitated the identification and acceptance of the association between animals and quantities. Marchini and Back (2010), used also a modified version of the environment to fit in the Italian schools and worked with Grade-1 students focusing on how the variety of “ways for representing the same mathematical concept together with *treatment* inside a register and *conversion* between registers facilitate pupils’ understanding and the construction of concepts” (p. 55). In this study we focus on the use of this environment as a way to smooth the transition from arithmetic to algebra (in the sense of using pictorial equations as a means or developing algebraic ideas, see Cai et al., 2005) by considering the various types of students’ thinking that would show a progressive movement towards algebraic thinking.

Design of the study

Seventy 3rd graders (8-9 years old) participated in the study. They were the total population of three classes from two primary schools and they represent a sample of an ordinary Greek primary school. They had no previous experience working with this kind of environments and they had not been taught any of the basic concepts of algebra such as equations or variables. When the students were introduced to the ‘Father Woodland’ environment, each tug-of-war game was presented using both the pictorial and symbolic representations of the animals. The whole study (part of a broader one) lasted five weeks. The students were initially introduced to ‘Father Woodland’ and then on a regular basis they were given tasks to solve individually. The whole project took part in parallel to the normal

teaching and was not integrated in the content of their math lessons. This paper focuses on the first 10 tasks due to the limited number of pages. There are 3 collections of tasks. In the first, there were two groups of animals in each task and the students were asked to add a mouse to the weaker group in order to make both groups equivalent (Fig. 2). This demands comparison and relational thinking connected to the notion of equality as an equivalence. In the second, the tug-of-war game took place but one (or some) of the animals wore a mask. The students were invited to find the animal(s) behind the mask (Fig. 3, left and middle). The aim was to exploit relational thinking in the form of using alternative ways for representing the unknown quantity. Finally, in the third, the students were asked to create two equally strong teams using any combination of the farm animals (Fig. 3, right). The aim was to see whether the students exploit the experience gained before and how intuitive mathematical ideas are embedded in their creations. For each task, the students were asked to explain their answer in a separate textbox. During the study, no feedback was given to the students about their answers. The students' worksheets constituted the data for this study. These data were examined in order to identify evidence of early algebraic thinking and possible formal mathematical concepts, which are informally used in the students' answers. In the context of qualitative content analysis, inductive category development was used to organize the categories.

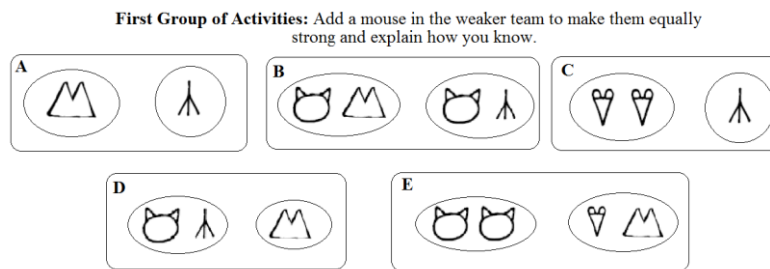


Figure 2: The first group of activities

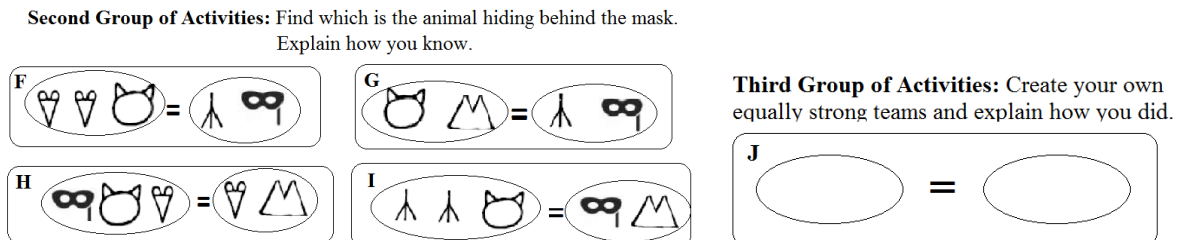


Figure 3: The second (left, middle) and the third (right) group of activities

Results and discussion

After the data examination, the answers were categorized in four types. The criteria for this categorization were the ways students chose to express the underlying structure in each task (i.e., using pictures, words or symbols), the relationships among the given quantities (i.e., using the given or new (invented) relationships), and the mathematical information contained within the pictorial representation (i.e., identifying a basic unit, substituting animals with their equivalents, adding/subtracting the same quantity in both sides, etc.). The four types are: (i) using pictorial language, (ii) using words to express relationships, (iii) combining words and symbolic representations and (iv) using 'symbolic' language to express relationships. Obviously not all the

students applied all the types. This is why it was decided to choose a proper sample of students to show the diversity of the approaches taken.

Type 1 – Using pictorial language

This type refers to the students who preferred drawing pictures in detail rather than using a symbolic representation (Task-E, Fig. 4). The answer is correct. The missing mouse must join the group on the left to get two equal teams but the reasoning is weak since it is limited to merely transfer the ‘abstract’ information into a more ‘realistic’ version and it lacks an explicit explanation of the ‘underlying’ thought. It seems that the student fails to shift the focus to the existing relationships between the values of the participating animals.



Figure 4: Use of drawings

Type 2 – Using words to express relationships

This type was used by the majority of the students and it proved more convenient for most of them to express their solution of the problem. Actually, in this type, the students made a step forward by trying to use words for expressing relationships among the quantities as it can be seen in the next two examples. This choice in many of the cases was described by the students in detail revealing thus their line of thought. In Task-J, one student created two equivalent groups by placing 5 dogs on the left and 20 mice on the right (Fig. 5). His explanation was: “*I thought that 5 dogs are as strong as 5 geese and 5 mice. But, 5 geese = 5 mice and 5 cats and 5 cats are as strong as 10 mice. So 20 mice*”.

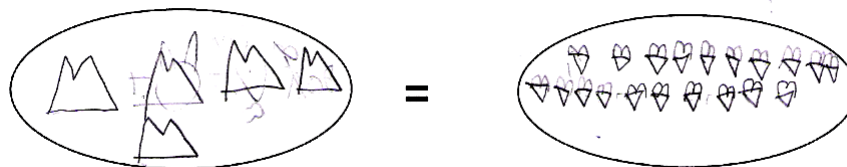


Figure 5: Use of mouse as the basic unit

The student exploited all the default information given by the pictures in Figure 1, e.g., $1d(og)=1g(oose)+1m(ouse)$, $1g(oose)=1c(at)+1m(ouse)$, and $1c(at)=2m(ice)$. Then, the whole process can be presented on a more formal way as $1d=1g+1m \Rightarrow 5d=5(g+m)$ [*multiply both parts by the same number*] $\Rightarrow 5d=5g+5m$ [*distributive property*] $\Rightarrow 5d=5(c+m)+5m$ [*substitute with equivalent*] $\Rightarrow 5d=5c+5m+5m \Rightarrow 5d=5 \times 2m$ [*substitute with equivalent*] $+10m \Rightarrow 5d=10m+10m \Rightarrow 5d=20m$. It must be said that this is not explicitly outlined by the child. But this enables us to identify in the student’s explanation the seeds of the mathematical reasoning described above. In the same way we will try to see the possible formal way of expressing the students’ answers in the remaining part of the paper. Actually, we make inferences about students’ internal representations on the basis of their production of external representations (Goldin & Shteingold, 2001).

The second example concerns Task-H which asked the students to identify the animal hidden behind the mask. The student's answer was (Fig. 6, left): "*There is a cat hiding behind the mask. Because, a mouse and a dog are as strong as a cat and two mice. I figured it out because a mouse and a dog are as strong as 5 mice so behind the mask is a cat since 2 cats and 1 mouse are as strong as 5 mice too*".

The student's starting point was the right part of the equation and she chose the mouse as the building block to replace all the involved animals. It is necessary to mention here that the students did not restrict themselves on the given relationships that were given during the first session (Fig. 1). They were able, during the next sessions, to identify new relationships based on the given ones. This student made use of one of these invented relationships by claiming implicitly that a dog has the same strength with 4 mice. This claim results to the total amount of 5 mice in the right part. Given that a cat has the same strength with 2 mice, it means that there are 3 mice in the left part of the equation plus the hidden animal. Two mice are needed to obtain equality; therefore, a cat must be placed behind the mask.

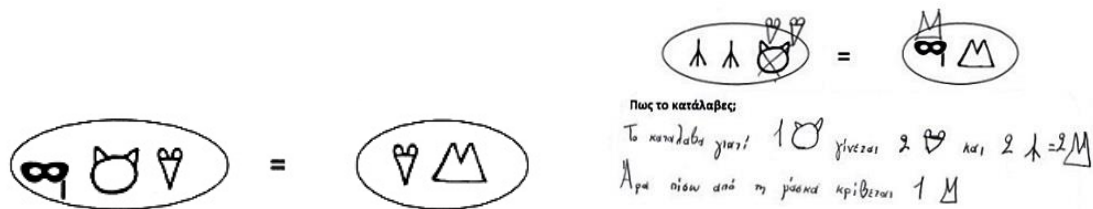


Figure 6: Use of words (left) and combination of symbols and words (right) to explain relationships

If we translate the reasoning of the student into its formal version, considering x the unknown, we obtain the following series of equations. $x + c + m = m + d \xrightarrow{d=4m} x + c + m = 5m$ [substitute with equivalent] $\xrightarrow{c=2m} x + 3m = 5m$ [substitute with equivalent] $\Rightarrow x = 2m$ [subtract the same amount from both sides] $\xrightarrow{c=2m} x = c$ [substitute with equivalent]

The strategy that led to successful solution here was: (i) choose the basic unit (e.g., the mouse), (ii) translate the picture to equation, (iii) substitute the dog by its equivalent number of mice and execute the operation, (iv) substitute the cat by its equivalent number of mice and execute the operation, (iv) find the unknown (x).

Type 3 – Combining words and symbolic language

This type starts -as a first step towards symbolic language- to combine words and symbols to show relationships between the participating animals. This is one answer for Task-I: "*I figured it out because 1 cat becomes (equals) 2 mice plus 2 dogs = 4 mice. So there is 1 mouse hiding behind the mask*".

The student in the first half of her answer used the word 'becomes' to denote the equality between cat and mice (Fig. 6, right). But, in the second half she used the sign of '=' to denote again the relationship between mice and geese (left part) and dogs (right part). Firstly, she substituted the cat (c) with 2 mice (m). Now, the left part consists of two identical sub-groups (a goose and a mouse per subgroup) which if substituted by their equivalence in terms of dogs reveal the identity of the unknown. So, starting from the left part: $2g + c = x + d \xrightarrow{c=2m} 2g + 2m = x + d$ [substitute with equivalent] $\Rightarrow 2(g + m) = x + d$ [distributive property] $\xrightarrow{g+m=d} 2d =$

$x + d[\text{substitute with equivalent}] \Rightarrow x = d [\text{subtract the same quantity}]$. The knowledge that 2 mice plus 2 geese is the same as 2 dogs -which is based on the given relationship that a mouse plus a goose is the same as a dog- reveals an implicit understanding of the above mentioned distributive property.

Type 4 – Using symbolic language to express relationships

The last type used by the students abandons the use of words and the reasoning is mainly symbolic. The first example is an answer from Task-F dealing with the animal behind the mask. The student started with the left group, using a symbolic expression to show its substitution by a dog (

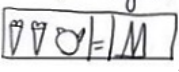
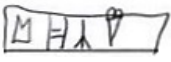
). Then, based on this expression she wrote another one to show the solution ( , see Fig.7 left). The first claim of the student seems arbitrary but if seen carefully it makes use of known relationships in order to obtain new ones. The left group represents the sentence $m+m+c$. Given that $m+c=g$ (oose) the sentence becomes $m+g$ which, according to the given relationships, equals with a dog. Then, it is obvious that what is needed in the right part of the equation is a mouse.



Figure 7: Usage of symbolic representations (left and right)

The second example shows a solution for Task-C. This solution is considered more elaborated since the student made use of the sign of the required operation to show the equivalence. This is indicative of understanding both the operation that takes place and the correct use of the sign of this operation. The missing animal is the mouse that must join the left group to get equality (Fig. 7, right). The sum of the strength of the three mice equals the strength of the goose. Again, this is a relation different than the given ones and it is important that so young students exhibit this ability, to use and combine given situations in order to get new ones. So, it is interesting to follow the thought of the student. Two mice equal with a cat. Then, a cat and a mouse have the same strength with a goose. Consequently (transitivity) three mice are equivalent with a goose. Expressing relationships using this symbolic language to solve a problem constitutes an important step towards the development of algebraic thinking. Besides, all the answers show an explicit focus of the students to the underlying structure of each equivalence in order to reach a solution.

It is interesting to examine now the findings of this study in the light of a previous one. Papadopoulos et al. (2016) in their study based on the use of mobile puzzles with 6th graders, distinguished mainly four types of students' thinking (translating the picture to equality expressions, using words to show the relationship, using symbolic language to show the relationship, and combination of more than one of the previous types). This means that there is a match between the types of thinking in these two studies and this strengthens the possible positive contribution of puzzle-like learning

environments to the development of young students' algebraic thinking. The feeling from the first study (no numerical data available) was that the order of these types is rather developmental in the sense that types 3 and 4 are more advanced (and thus less frequent), and are met at the end of the project (an indication that they are connected with the accumulated experience). One could attribute this finding to the teachers' appreciation of the symbolic answers instead of textual ones. However, this is not the case, since the teachers were not involved in the project and in the meanwhile the students did not receive any feedback about their answers.

	Type-1	Type-2	Type-3	Type-4
Tasks A-E	5 (1.6%)	253 (83%)	14 (4.6%)	33 (10.8%)
Tasks F-I	8 (3.74%)	157 (73.36%)	24 (11.22%)	25 (11.68%)
Task J	2 (2.94%)	56 (82.35%)	8 (11.77%)	2 (2.94%)

Table 1: Frequency of Types 1-4

In this study an effort was made to get arithmetical evidence that would shed light on this issue. Table-1 confirms that Types 3 and 4 are indeed the less frequent ones. The low frequency of Type-1 was expected since the pictorial language was already included in the statement of the tasks.

Table-2 presents the distribution of the last two types across the range of the tasks. As it can be seen to a great extent the number of instance for each Type is increased as we move towards the last tasks indicating that these types are connected with the accumulated experience.

	Task-A	Task-B	Task-C	Task-D	Task-E	Task-F	Task-G	Task-H
Type-3	3	2	4	3	3	7	7	7
Type-4	5	6	3	8	10	8	8	8

Table 2: Distribution of Types 3-4 across tasks

Conclusions

The findings of our research indicate that the 'Father Woodland' environment might contribute to the development of students' algebraic thinking. The four types of thinking mirror the rules induced by the students in order to solve the posed problems. Starting from certain external representations of equality sentences the students made an attempt to express their internal representations through the shift from pictorial to symbolic language. Obviously, this is not all that matters with the development of algebraic thinking with young children. However, it cannot be considered trivial. The students had to add or remove the same animal (quantity) from both sides, to substitute certain animals with their equivalence, isolate the unknown animal (variable) trying to maintain the same strength between both groups of animals applying at the same time the distributive law or transitivity. Despite that lack of explicit knowledge about operations and relations hinders a good approach to algebra (Gerhard, 2013), it seemed that there were instances of an implicit knowledge of certain rules for solving equations which will be later introduced formally to the students. However, it still remains to answer questions like: In what way the transition from the animal symbols back to the arithmetical or algebraic equations will be possible? Additionally, the findings support the developmental character of these types of thinking when the students use puzzle-like learning environments aiming to support

algebraic thinking. However, this does not mean that some students did not occasionally move backwards to previous types of thinking. This is in itself a significant finding we aim to explore further since the relatively small number of participants does not allow to generalize our findings.

References

- Cai, J. & Knuth, E. (2011). A global dialogue about early algebraization from multiple perspectives In J. Cai y E. Knuth (Eds.) *Early Algebraization* (pp. vii–x). New York: Springer
- Cai, J., Lew, H. C., Morris, A., Moyer, J. C., Ng, S. F., & Schmittau, J. (2005). The development of students' algebraic thinking in earlier grades. *ZDM*, 37(1), 5–15
- Gerhard, S. (2013). How arithmetic education influences the learning of symbolic algebra. In B. Ubuz, C. Haser & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 430–439). Manavgat - Side, Turkey: Middle East Technical University.
- Goldin, G. (2002). Representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of international research in mathematics education*, (pp. 197–218). London, UK: Lawrence Erlbaum.
- Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A.A. Cuoco and F.R. Curcio (Eds.), *The roles of representation in school mathematics*, (pp. 1–23). Reston, VA: NCTM,
- Hejný, M., Jirotková, D., & Kratochvílová, J. (2006). Early conceptual thinking. In J. Novotná, et al (Eds.), *Proceedings 30th Conferences of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 289–296). Prague: PME
- Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F.K. Lester Jr. (Ed.), *Handbook of research in mathematics education* (pp. 707–762). Greenwich: Information Age Publishing
- Marchini, C. & Back, K. (2010). Teachers' best practices using differentiation. In B. Maj et al (Eds.), *Motivation via natural differentiation in mathematics*, (pp. 47–56). Rzeszow, Poland: Wydawnictwo Uniwersytetu Rzeszowskiego.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author
- Papadopoulos, I., Kindini, T., & Tsakalaki, X. (2016). Using mobile puzzles to develop algebraic thinking. In Csikos, C., et al. (eds.). *Proceedings 40th Conference of the International Group for the Psychology of Mathematics Education*, (Vol.4, pp. 43–50), Szeged, Hungary: PME