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## Aspects of proficiency in elementary algebra

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*This paper provides insight in the process of developing a comprehensive and concise summary of the most prominent aspects of proficiency in elementary algebra at the end of secondary grade. It will serve as a theoretical frame of reference for devising or validating instruments for diagnosing the mastery of elementary algebra at the transit from school to university. A first draft had been based on literature, which then was presented to experts for further evaluation. The model now comprises ten aspects of proficiency which are allocated into a table of two dimensions, one referring to elements of algebraic language, the other referring to the cognitive actions performed on them.*

*Keywords: Elementary algebra, basic skills, secondary school mathematics, STEM Education.*

### Introduction

For being successful in subjects from science, technology, engineering or mathematics at high school or university (STEM), a good mastery of formal algebra is indispensable. But what is a good mastery of algebra, and how does it show? To provide a theoretical base for devising an instrument for diagnosis, we have been working on a comprehensive yet concise overview of the important aspects of proficiency in elementary algebra as it has been covered by relevant research. For this, the model is meant to be summative, i.e. it contains aspects of proficiency that are expected by the end of secondary school. And, when referring to elementary algebra we mean symbolic algebra as it appears at the end of secondary school. A first draft of the overview was based on relevant literature from national and international publications, which was then presented to experts for validation. In the following we will first give a short report on the main outcomes of the literature review of the first stage, after that we will present the main outcomes of the expert survey. This paper's main part then contains a detailed description of the present state of our model.

### Model development

#### First draft based on literature

The aim of this first stage of model development was to give a structured compilation of the important aspects of proficiency in elementary formal algebra as it is present in research publications of nearly 40 years, starting with Küchemann's paper on children's understanding of variables (1978). A loose series of unconnected aspects would not be of much help where a concise summary is needed. So we decided to build categories from the various aspects found in literature, based on two a-priori dimensions that served as a first theoretical frame for categorization: It starts with listing all relevant mathematical objects of the domain in question, which here are variables and expressions as elements of the algebraic language. The second dimension focusses on the mental and real activities associated with these objects that all represent some stage of "making

sense of algebra” (Arcavi, 1994). Arcavi’s approach to describing what is meant by understanding algebra seemed a suitable frame, or better: attitude for collecting and arranging research findings from the past years. “Symbol sense” is, in his own words, “a complex and multifaceted 'feel' for symbols” (Arcavi, 1994, p.31) which invites to searching for aspects of quickly grasping a situation where symbolic algebra is involved. Hence the second of the two dimensions of our model comprises a range of various activities that in some way “make sense” of variables and expressions, ranging from stating correct manipulation rules and identifying expression to which these rules can be applied to modeling realistic situations by means of algebra.

making sense of...  
...elements of algebra

	reciting	manipulating	structuring	innermath. interpreting	extramath. interpreting
variable	no meaningful aspects			(4) interpret variables	
parameter				(6) identify parameters (among variables)	
term	(1) formulate rules for manipulating expressions	(2) manipulate expression by applying given rules	(3) identify expression type	(7) formulate equation	
equation			(4) immediately identify equivalence	(8) switch between representations	(9) switch between expression and realistic situation
			(10) interpret equation signs		

**Table 1: Literature based draft of important aspects of proficiency in elementary algebra**

This first draft resulted in formulating ten aspects of proficiency, arranged in a tableau along the two dimensions (table 1) introduced above. Here, these ten aspects were defined as single elementary “blocks” of which more complex activities could be constructed by combination. For example, combining three abilities which we then called “(1) formulate rules for manipulating expressions”, “(2) identify expression type” and “(3) manipulate expression by applying rules” would sum up to what is generally meant by the ability to manipulate algebraic expressions correctly. While this approach seemed sensible for a detailed diagnosis, it proved to be unhandy in discussions even between experts as the following sections shows.

### Expert survey

For validating the results of the literature review an expert survey was conducted. By choosing the experts we focused on professors and seminar teachers of mathematics education in German speaking areas who published primarily about teaching and learning elementary algebra within recent years. Twenty-four colleagues were contacted via e-mail, one of them declined but recommended a colleague as a substitute who, after checking our requirements, was contacted too. All experts were asked to complete an online-questionnaire which was implemented in the free online-tool SoSci Survey ([www.soscisurvey.de](http://www.soscisurvey.de)). The experts were expected to comment on two fields: firstly, on the contents and the structure of the model, while each of the ten aspects was illustrated by one exemplary task, and, secondly, on a battery of additional 48 items that was devised to operationalize the ten aspects. Because giving feedback on all 48 items would have taken too much time we created 6 different versions of the online-questionnaire and reduced the extent of each version by an overlapping multi-matrix-design (cf. Zandler, Vogel & Spannagel, 2013). All in

all, an expert had to view 22 or 23 tasks. Since the response rate was 50%, each task was processed and judged by an average of about 6 randomly assigned experts.

In the first section of the survey the experts were presented an interactive version of Table 1 where explanations of the entries were shown together with exemplary tasks in a pop-up window. Here, they were asked to familiarize themselves with the table entries and explanations while making notes on which aspects they thought needed to be reworked or were unessential for basic mastery of algebra, or which aspects were missing in their view. There also was place for free comments on the model as a whole. In the second section of the survey the experts were asked to assign each of the 22 or 23 task assigned to them to three of the ten aspects of the model and comment on their decision. Here, a smaller version of the interactive table was present with each assignment question. By combining assignment questions and open question types the question format can be considered as being half-open. This type of questions is suitable for experts who are considered to have a differentiated self-awareness, expressiveness, motivation and fidelity (cf. Gerl & Pehl, 1983). The open question format is to be seen as the qualitative part of the online-questionnaire which complies with a written survey in sense of an expert interview (Bogner, Littig, & Menz, 2002). Qualitative methods are especially suitable for purposes of theory-based exploration (cf. Bortz & Döring, 2006). In the following we focus on the results of this qualitative part of the survey.

### **Data analysis and results of the expert survey**

The aim of the expert survey was to validate the first, literature based draft of the model. For data-reduction by clustering analogous comments we followed an open-ended approach to qualitative evaluation as provided by Grounded Theory (Corbin & Strauss, 1990) along these four steps: (1st) All comments from the open question parts of the survey were rephrased by the authors into single conclusive statements about the model or a task. (2nd) All these statements were pasted into the free digital mindmap programm xmind ([www.xmind.com](http://www.xmind.com)) where, for each expert, they were arranged in the order as they appeared in the survey. (3rd) Categorisation took place first with the aim of clustering comments of analogous content – now renamed as “contributions to model revision” and then, (4th), with the aim of categorizing these contributions according to how each of them would contribute to modifying the model or the test battery.

Since many of the statements – though sometimes commenting on single items of the battery – were of a general kind, the process of categorizing eventually detached from the original structure of the questionnaire, so that the following four categories emerged: (a) contributions to clarifying the theoretical frame of the model, (b) contributions to restructuring the model, (c) contributions to reformulating definitions or exemplary tasks of single aspects of the model, (d) contributions to reformulating or deleting a task from the test battery. Among the contributions assigned to category (a), some experts asked how the various aspects of “making sense of algebra” relate to existing models of mathematical understanding. Other experts expressed their uncertainty of how our model relates functions. And other experts were missing activities of preformal algebra. Among the contributions assigned to category (b), one expert pointed out that the activities of transforming or interpreting algebraic expression would imply the activity to identify the structure of an expression so that “structuring” needs to be given a more prominent role in our model’s layout. Some experts mentioned “substituting” as one of the central activities in doing algebra. Additionally, from their task assignments it became apparent, that many experts were misinterpreting the elementary activity

“to transform with given rules” from our first draft as meaning “being able to manipulate expressions correctly” regardless whether a rule is given or not. This contribution also was assigned to category (c), as it would not only lead to introducing a new aspect but to reformulating existing elementary activities too. There were also further contributions to reformulating definitions or exemplary tasks, and to reformulating or deleting tasks from the test battery.

### The revised model

For model revision, the authors discussed each contribution as to whether to incorporate it and how. Among contributions that were accepted was that the activity of structuring needed a more prominent position within the model. Here we followed recent findings of Ruede (2015) as to which structuring can be understood as an activity of making sense of an expression by identifying relations between parts. Relations are identified by substitution, i.e. parts of an expression are seen as entities that can be related to each other. Thus the activity of “substituting” was accepted as part of the central activity of structuring. Together with Musgrave's et al. (2015) definition of substitution and Kieran’s (1989) distinction between systemic and surface structure it helps to refine activities of recognising the applicability of transformation rules or the operational ordering of an expression. Now, these two activities cover the activity of substituting which, from a cognitive perspective, means to construe parts of an expression as meaningful entities, esp. to replace variables and terms by other variables or terms in writing or in thought.

	making sense of... ...elements of algebra	knowing	acting structuring		
			transforming		interpreting
variables incl. parameters		no meaningful aspects		(3) to recognize the applicability of transformation rules	(4) to recognize the operational ordering
expressions and equations		(1) to specify transformation rules or terminology	(2) to transform following given rules	(5) to compute or to compare	
		(6) to transform (efficiently)			(7) to interpret variables and parameters
				(8) to switch between expressions and inner-mathematical situations	(9) to switch between expressions and tables or graphs
					(10) to switch between expressions and real situations

**Table 2: A concise summary of the important aspects of proficiency in elementary algebra**

Further amendments were applied to the model so that Table 2 now shows its present state. In the following each table entry is explained in detail.

#### Elements of algebra

- Variables including parameters: Variables are signs that represent numbers or quantities. Parameters are variables that vary over sets of values of other variables (Veränderliche vs. Einzelzahl: Malle, 1993, variable vs. metavariable: Drijvers, 2001, values taken by a variable: Bardini et al., 2005). This discrimination arises from the context of the task.

- Expressions and equations: Algebraic expressions are compositions of variables and arithmetic operation signs. When a variable is viewed as representing a range of number values or quantities (variable object: Schoenfeld & Arcavi, 1988; Bereichsaspekt: Malle, 1993) the value of the expression is interpreted as a function of this variable (Malle, 1993; Heid, 1996). Equations are expressions where two terms are compared with regard to their values, symbolized by an equation sign. An equation differs from a computation or transformation of a term in that it is used in a relational sense (notion of equivalence: Kieran, 1981; operational vs. relational view: Baroody & Ginsburg, 1983; Zuweisungs- vs. Vergleichszeichen: Malle, 1993).

### **Making sense of elements of algebra**

- Knowing and acting: a first level of differentiation that differs between declarative knowledge about rules and various forms of “making sense” of algebraic objects. The latter is further differentiated into transformational and generational types of activities following Kieran (2004). Kieran’s third class of “meta-level” type of activities has been omitted as it describes a higher level of mastery that is not considered being a part of basic mastery.
- It seemed appropriate to formulate within the range of Kieran’s (2004) transformational and generational types of activities three central activities: Transforming (to transform an algebraic expression into an equivalent expression of different structure (transformational equivalence: Musgrave et al., 2015; treatment: Duval, 2006)), structuring (to transform or interpret an algebraic expression while maintaining its structure (substitutional equivalence: Musgrave et al., 2015, Rüede 2015)) and interpreting (to describe a non-algebraic situation by formal algebra and vice versa (conversion: Duval, 2006)). Among these, structuring takes on a fundamental role. It describes an activity of recognizing the structure of a present expression or formulating an expression that is structurally equivalent to relations between quantities in a given situation.

### **Ten aspects of proficiency in elementary algebra**

- (1) “To specify transformation rules or terminology” – Important technical terms for expressions and rules for manipulating expressions or equations are identified or specified, e.g. names for classes of terms or equations, or rules for simplifying expressions, binomial rules, rules for solving quadratic equations, etc.
- (2) “To transform by following given rules” – Expressions and equations are transformed into equivalent expressions or equations by applying given rules (manipulation skills: Hoch & Dreyfus, 2006).
- (3) “To recognize applicability of transformation rules” – An expression is identified as a representative of a class of structurally equivalent expressions and transformation rules that are associated with this class. This is done by, mentally or explicitly, substituting variables or terms by terms or variables (systemic structure: Kieran, 1989; structure sense: Hoch & Dreyfus, 2006)
- (4) “To recognize the operational ordering” – The logical ordering of the operations within an expression is recognized. This is done by, mentally or explicitly, substituting terms by variables (surface structure, Kieran, 1989; Rechenschema: Vollrath & Weigand, 2006; Rechenhandlung: Malle, 1993)



- (5) “To compute or to compare” – An expression with an equation sign is interpreted in an operational or a relational sense, as it is appropriate in the context (Malle, 1993; operational vs. relational view: Baroody & Ginsburg, 1983; Knuth et al., 2006).
- (6) “To transform (efficiently)” – Expressions and equations are being transformed into equivalent expressions or equations (2,4), by activating existing knowledge about transformation rules (1) which are identified as applicable to the present problem (3). Also, two expressions or equations are identified as being equivalent „on the spot“ without applying rules explicitly (algebraic expectation, Pierce & Stacey, 2001). A transformation is „efficient“ if, among various rules of transformation that are applicable, one is chosen that allows relatively few steps and few computations (strategic flexibility: Rittle-Johnson & Star, 2009; structural relations of second order: Ruede 2015, cp. Malle 1993).
- (7) “To interpret variables and parameters” – Variable signs are interpreted or used as representations of numbers (Einsetzungs-, Gegenstandsaspekt: Malle, 1993; Küchemann, 1978). Within given contexts, appropriate variables are identified or used as parameters.
- (8) “To switch between expressions and innermathematical situations” – A non-algebraic but innermathematical situation (e.g. dot patterns or geometric configurations) is described by a term or an equation, and vice versa (Bauplan: Vollrath & Weigand, 2006).
- (9) “To switch between expressions and tables or graphs” – An expression or equation is translated to a value table or a graph, and vice versa (McGregor & Stacey, 1995), e.g. when viewed as a function (Duval, 2006; Nitsch, 2015), or for solving an equation (Arcavi, 1994).
- (10) “To switch between expressions and real situations” – An expression or equation is translated to a realistic situation, and vice versa (McGregor & Stacey, 1995; Heid, 1996), e.g. when viewed as a function (Nitsch, 2015). This activity involves a higher gradient of abstraction than activities (7,8,9) that results from the need to replace the concrete mental model of the given real situation by an abstract mental model before formulating an expression (Malle, 1993).

## Summary and outlook

In its present state, the model intends to be a concise summary of aspects of proficiency in elementary algebra, based on relevant literature and a survey of maths educators from the German speaking community. It represents a normative view on what ideal schooling can provide at the end of secondary grade, thus serving as a theoretical base for devising instruments for a summative and differentiated diagnosis of proficiency in elementary algebra at the transition from school to university. The model does not cover all aspects of school needs to consider, but it is restricted to

- symbolic algebra, not generic: at the end of secondary school maths, an individual's proficiency in algebra must have reached a stage of being competent with symbolic representations of indeterminate number values and quantities and relations between them,
- a summative view, not formative: the model is meant to comprise all aspects of proficiency at the end of secondary school maths, not while they are being taught,

- algebra, but not functions: a model about algebraic proficiency cannot cover all aspects of the concept of function, but does cover some which are only present in the form of a functional interpretation of an algebraic expression.

Additionally, based on this frame of reference, a test battery is presently being prepared for large scale application. While the model helps to devise tasks that cover most important aspects of proficiency in elementary algebra, the data raised from applying the battery will be used to generate an empirical cognitive model that adds to the theoretical normative view of the present one.

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