Didactic transposition in school algebra: The case of writing equations of parallel and perpendicular lines

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Texas A &M University-Corpus Christi, USA; Valentina.Postelnicu@tamucc.edu

A study was conducted with a high school teacher and her 58 Algebra 1 students, with the purpose of gaining insight into students' difficulties with writing equations of parallel and perpendicular lines. Chevallard's theory of didactic transposition was employed in order to account for the relativity of the mathematical knowledge with respect to the institutions where the knowledge was created. During the process of didactic transposition, the mathematical knowledge lost its essential feature, the proof, with dramatic consequences for the school algebra curriculum. What remained to be taught and learned was how to execute tasks. As predicted by mathematicians, this utilitarian view of the curriculum affected the actual process of teaching and learning by focusing on executing basic tasks, and resulted in teachers and students having difficulty executing those tasks.

Keywords: Didactic transposition, school algebra, writing equations of lines.

Introduction

Students' difficulties with aspects of linear functions, like the rate of change, slope, *y*-intercept, or writing equations of parallel lines have been studied in the United States by many researchers, among them the group from Berkeley led by Schoenfeld (Schoenfeld, Smith, & Arcavi, 1993). In spite of a wealth of research, the issue of students' difficulties with linear functions has remained relevant (Postelnicu, 2013). The study reported here has the purpose to account for the students' difficulties with writing equations of parallel and perpendicular lines, and to advance a plausible explanation on the persistence of students' difficulties. We propose another way to look at this issue, by paying attention to the difference between the mathematics as a body of knowledge (scholarly mathematics created by mathematicians) and mathematics as a subject matter to be taught and learned (school mathematics created by textbook authors or taught by teachers) (Bosch & Gascón, 2006).

About the theory of didactic transposition

For this study we employ Chevallard's theory of didactic transposition because it takes into account both the mathematics as a scholarly body of knowledge and as a subject to be taught and learned. By *didactic transposition* of knowledge, we mean "the transition from knowledge regarded as a tool to be put to use to knowledge as something to be taught and learnt" (Chevallard, 1988). As pointed out by Bosch and Gascón (2006), when studying any didactic problem, like the teaching and learning of writing equations of parallel and perpendicular lines, we must account for all the steps of the process of didactic transposition:

i) from scholarly knowledge created by mathematicians (e.g., analytic geometry) to school mathematics written by textbook authors (e.g., high school textbooks); ii) from textbook knowledge (e.g., Algebra I textbook written by Larson et al., 2007) to school mathematics taught by teachers in classrooms (e.g., the mathematics taught by the teacher in the study reported here); and iii) from school mathematics taught by the teacher in this study to mathematics learned by her students.

Each society and institution has a certain way of creating mathematical knowledge and using it, thus bringing to life a *praxeology*, "an organized way of doing and thinking contrived within a given society" (Chevallard, 2006). A praxeology is composed of *praxis* and *logos*, each with two components (Bosch & Gascón, 2014):

i) *praxis* - **tasks**/problems that can be solved employing a certain **technique** ("ways of doing"/executing the tasks/solving the problems), and

ii) *logos*- **technology** (the discourse of the technique, justifies the technique), and **theory** (general discourse or abstract set of constructs and arguments, justifies the technology).

A praxeology can be "*point praxeology*" with only one type of task, "*local praxeology*" with a set of tasks sharing a technological discourse, and "*regional praxelogy*" with all the point and local praxeologies sharing a theory (Bosch & Gascón, 2014). For example, a point praxeology may contain only the task of sketching a "quick" graph of a line with an equation written in slope-intercept form. A local praxelogy may contain all the tasks requiring writing equations of lines given certain conditions, including tasks like Tasks 1 and 2 given to students by the teacher from our study:

Task 1*:* Sketch the graph of the line that is parallel to $y = \frac{1}{2}x - 3$ and goes through $(2,-1)$.

Task 2: Sketch the graph of the line that is perpendicular to $y = \frac{3}{4}x - 2$ and goes through the point $(2,-6)$. What is the equation of the new line you created?

Tasks 1 and 2 have the same target knowledge, an algorithm for solving the class of problems that require the writing of the equation of a line passing through a given point and parallel or perpendicular to a given line. Such tasks are similar to those proposed in the textbook used by the participants in this study. The technique used to execute Task 1 may employ algorithms like the one used by the teacher in this study:

1. Draw a Cartesian systems of coordinates and plot the given point $(2,-1)$.
2. Identify the slope of the line parallel with the given line, $m = \frac{1}{2}$.
3. Use a "quick graph" to obtain a second point on the line (from the point $(2,-1)$ move 1 unit up and 2 units to the right, and obtain and plot the point $(4,0)$).
4. Draw the line passing through the given point $(2,-1)$ and the newly found point $(4,0)$.

Part of the technological discourse for this local praxeology may include the justification for the fact that the second point $(4,0)$ obtained in the way described above is indeed situated on the line with the slope $m = \frac{1}{2}$ and passing through $(2,-1)$, or the justification that the line parallel to the line $y = \frac{1}{2}x - 3$ has the same slope, $m = \frac{1}{2}$. A theoretical justification of the fact that parallel lines have the same slope and, conversely (i.e., a theorem and its proof in analytic geometry, based on similarity of triangles) may belong to a regional praxeology.

Method

Participants in this study were an Algebra 1 teacher from a public high school in the United States and her 58 students who agreed to participate in this study. The teacher had a Bachelor degree in Mathematics and five years of teaching experience. Classroom observations (Erickson, 1985) were conducted by the researcher/author of this paper for all six Algebra 1 classes of 50 minutes each, taught by the participating teacher, during the same day of school. The researcher took notes referring to the way the teacher and her students interacted with the mathematical content of the tasks. The teacher taught the same lesson, about writing equations of parallel and perpendicular lines, to each of her classes. Prior to the day of observation, the teacher introduced to her students the equation of a line in slope-intercept form and point-slope form, and the notions of parallel and perpendicular lines. The teacher started each lesson with Task 1 (described above), discussed it with her students, and then administered Task 2 (described above). Two raters scored students' answers to Task 2 ("1" for correct answer, and "0" for incorrect or incomplete answer). Using techniques from grounded theory (Strauss & Corbin, 1998), students' algorithms were split into two categories, algorithms with a graphical approach (teacher's approach, described in the section referring to the theory of didactic transposition, and in the section referring to teacher knowledge), or algorithms with an algebraic approach (textbook approach, described in the section referring to textbook knowledge). The inter-rater agreement (Cohen, 1960) was very high, k = .92 (p < .001), 95 % CI [.83, .98].

Analysis and results

Figure 1, below, is adapted from Bosch & Gascón (2006) and illustrates the steps of the process of didactic transposition specific to the study presented in this paper.

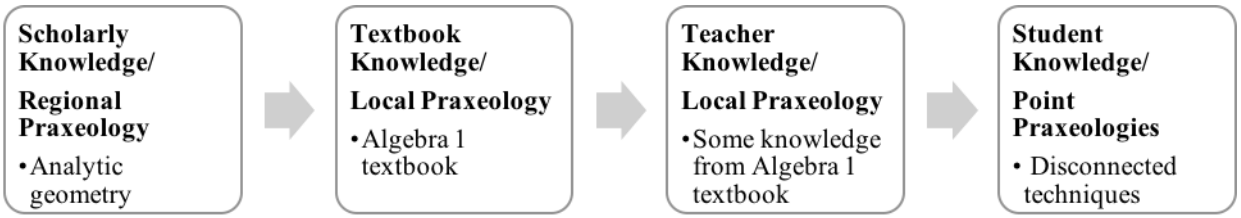

Figure 1. The process of didactic transposition

We will describe each of the types of knowledge/praxeologies in Figure 1, starting from the scholarly knowledge, in the direction of the arrows. Worth mentioning, in determining the regional, local or point praxeologies, one starts from the task and what the task entails, in our case writing linear equations of parallel or perpendicular lines to a given line, and passing through a given point.

Scholarly knowledge/Regional praxeology

As can be seen in Figure 1, we chose analytic geometry as our regional praxeology. Given the space constraints, we refer here only to perpendicular lines. When writing equations of perpendicular lines, we use the following **theorem**:

> Two nonvertical, nonhorizontal lines l_1, l_2 with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$ (Kay, 2001, p. 303).

Worth mentioning, the proof chosen for the above theorem is specific to the participants' institutionalized knowledge. Our regional praxeology/theory contains all the axioms, definitions, theorems and their proofs needed to prove the above theorem. An example of a path through the theory is given by Kay (2001) in his textbook, College Geometry, where he starts constructing the geometry with the foundations of absolute geometry (points, lines, segments, angles, triangles, quadrialterals, circles), and continues with the Euclidean geometry (trigonometry, coordinates, vectors). This path of knowledge includes the definition of a right angle, the definition of perpendicular lines, the Pythagorean theorem and its converse together with its proof based on similarity of triangles, and the distance formula between two points, given their coordinates. Within this regional praxeology, "a right angle is any angle having measure 90" and "two (distinct) lines l_1, l_2 are perpendicular if and only if l_1, l_2 contain the sides of a right angle" (Kay, 2001, p. 97). A proof of the theorem stated above is simple (see Figure 2).

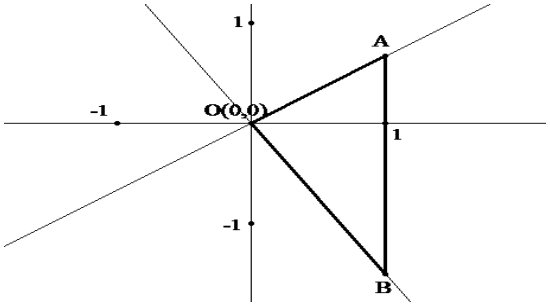

Figure 2. Perpendicular lines and their slopes

As can be seen in Figure 2, the system of coordinates has been specially chosen, without loss of generality, so that its origin, *O*, coincides with the point of intersection of the two lines, l_1 (with slope m_1, containing the segment *OA*, with *A* chosen such that its *x*-coordinate, $x_A = 1$) and l_2 (with slope m_2, containing the segment *OB* with *B* chosen such that its *x*-coordinate, $x_B = 1$). The line l_1 passes through $O(0,0)$ and has the slope m_1, hence all its points (x, y) satisfy the equation $y = m_1 x$. Similarly, all the points (x, y) situated on l_2 satisfy the equation $y = m_2 x$. As such, the points *A* and *B* have the coordinates: $A(1, m_1)$ and $B(1, m_2)$, respectively. To prove the direct implication of the theorem, we assume that the lines are perpendicular therefore they contain the sides of a right angle, hence the triangle AOB is right, and according to the Pythagorean theorem we have $OA^2 + OB^2 = AB^2$. Using the distance formula to calculate *OA*, *OB*, and *AB* function of their coordinates, we obtain $m_1^2 + 1 + m_2^2 + 1 = (m_2 - m_1)^2$ and after we simplify, we have $m_1 m_2 = -1$. Conversely, if $m_1 m_2 = -1$, then $m_1^2 + 1 + m_2^2 + 1 = (m_2 - m_1)^2$, therefore $OA^2 + OB^2 = AB^2$. Using the converse of the Pythagorean theorem, it follows that the triangle AOB is right, hence the lines l_1 and l_2 containing its legs, *OA* and *OB*, respectively, are perpendicular.

The observation that any vertical line with the equation $x = a$ is perpendicular to any horizontal line with the equation $y = b$ takes care of the exception stated in the theorem ("nonvertical, nonhorizontal lines").

Textbook knowledge/ Local praxeology

In the United States, the Algebra 1 course, usually taught in the first year of high school, contains topics like linear equations with one and two variables, linear functions, linear inequalities, systems of linear equations, quadratic equations and functions, and introductions to polynomials, exponential equations and functions. For historical reasons (Kilpatrick & Izsák, 2008), the American students take the Algebra 1 course before Geometry, thus they learn about the slope of a line, or slopes of parallel and perpendicular lines before learning about similarity or how to prove the Pythagorean theorem. As such, there is no expectation for justifications or proofs for the "key facts" stated in Algebra 1 textbooks. The textbook knowledge for this study comes from the Algebra 1 textbook used by the participants' school district (Larson et al., 2007). The lesson referring to the writing of parallel and perpendicular lines offers some techniques and the technological discourse to justify the techniques (e.g., definition of perpendicular lines and a "key concept" without proof - the theorem referring to the slopes of perpendicular lines):

> PERPENDICULAR LINES. Two lines in the same plane are perpendicular if they intersect to form a right angle. Horizontal and vertical lines are perpendicular to each other.
>
> KEY CONCEPT. If two nonvertical lines in the same plane have slopes that are negative reciprocals, then the lines are perpendicular. If two nonvertical lines in the same plane are perpendicular, then the slopes are negative reciprocals (Larson et al., 2007, p. 320).

A similar definition and a similar theorem about parallel lines precede the definition and theorem about perpendicular lines. The technique/algorithm described in the textbook (Larson et al., 2007, pp. 319-321) for executing tasks similar to Task 1 and 2 has an algebraic approach (Knuth, 2000):

1. Identify the slope *m* of the new line based on the "key concepts" referring to the slopes of parallel or perpendicular lines (parallel lines have the same slope, perpendicular lines have slopes negative reciprocals).
2. Use the slope-intercept form of an equation $y = mx + b$, the newly found slope *m*, and the given point (x_1, y_1) to find *b* (the *y*-intercept).
3. Write the equation of the newly found line $y = mx + b$.

Teacher knowledge/Local praxeology

We continue describing Figure 1 with teacher knowledge, as observed. After discussions with her students, the teacher executed Task 1 using the algorithm with the graphical approach described above, in the section about the theory of didactic transposition. She drew a system of coordinates, plotted the point $(2,-1)$, identified the slope of the parallel line $m = \frac{1}{2}$ and used it to obtain the second point $(4,0)$ (from $(2,-1)$ moved 1 unit up and 2 units to the right), and drew the line connecting the points $(2,-1)$ and $(4,0)$. To check the execution of Task 1, the teacher proposed to her students to graph the first line $y = \frac{1}{2}x - 3$ as well, on the same system of coordinates like the newly obtained line, and make a judgment regarding their parallelism, based on visual inspection. When one of the students proposed a technique with an algebraic approach for Task 1, the teacher allowed the student to carry it out, but then she asked the class for another way to execute Task 1, and led the students to use her graphical approach technique. Part of the observed technological

discourse for the teacher's local praxeology also included the definitions of parallel lines ("lines that do not meet") and perpendicular lines ("two lines in a plane that intersect at a 90° angle"), and the "key concepts" that parallel lines have the same slope and perpendicular lines have the slopes opposite reciprocals. In short, the teacher's local praxeology, as observed, consisted of some basic tasks, a technique with a graphical approach, and some of the technological discourse necessary to justify the technique. In the teacher's view, as expressed during the class discussions, Tasks 1 and 2 were part of the same class of problems, requiring the same technique, and Task 2 could be approached in the same way as she approached Task 1, i.e., graphically.

Student knowledge/Point praxeologies

Executing Task 2 with the algorithm from the textbook implies: i) identify the slope of the perpendicular line, $m = -\frac{4}{3}$; ii) substitute $(2,-6)$ and $m = -\frac{4}{3}$ in $y = mx + b$, and determine b from $-6 = -\frac{4}{3}(2) + b$; iii) write the equation of the newly found line $y = -\frac{4}{3}x - \frac{10}{3}$. Only two students (3.4 %) of those 58 participating in the study executed Task 2 successfully, and both employed the technique with an algebraic approach described in their textbook. About one out of five students (20.6 %) tried to solve the problem employing the technique with the graphical approach used by the teacher – they determined the slope of the perpendicular line, $m = -\frac{4}{3}$, started from the given point $(2,-6)$ and used the "quick graph" technique to obtain the second point on the line, then drew the line connecting those two points and determined incorrectly, by visual inspection, the value of the *y*-intercept of the newly created line, and finally wrote the equation of the line (see Figure 3).

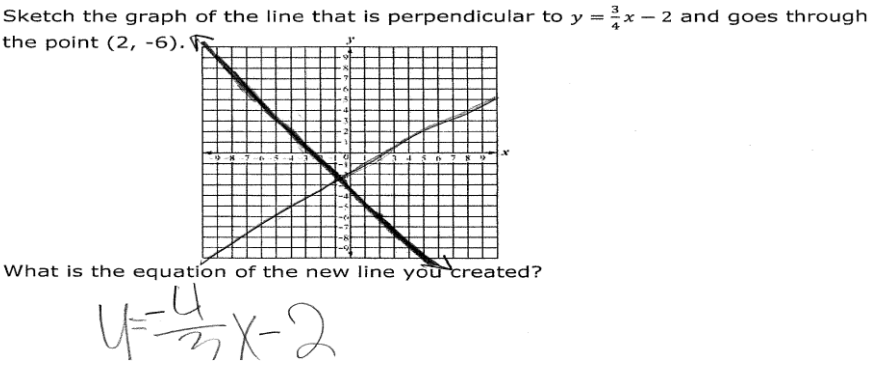

Figure 3. Example of student work – Task 2

It can be seen in Figure 3 that the student incorrectly determined the *y*-intercept of the newly created line as -2, while the correct value is $-\frac{10}{3}$, a value hard to determine with the graphical approach. Moreover, the student tried to check/evaluate her work, and graphed the original line together with the newly created line and it seems that she was satisfied with her visual inspection of the perpendicularity of those two lines she graphed. The rest of the students (76 %) tried to use a graphical approach (strongly suggested by the teacher's approach and the presence of the grid), but could not identify the slope of the perpendicular line, or determine the *y*-intercept. When writing the equation of the perpendicular line, students tried to substitute the point $(2,-6)$ or its coordinates in

the slope-intercept form equation of a line $y = mx + b$, irrespective of their graphical representations or the meanings of the coordinates of the point, slope of the line, and *y*-intercept of the line. They obtained equations, like "$y = -\frac{4}{3}x + (2,-6)$", "$y = -\frac{6}{2}x - 2,$" or "$y = 2x - 6,$" showing a great disconnect with the technique required to execute Task 2. As observed from their written assignments, the students' praxeologies were point praxeologies with only one task and technique (successful or not), without any justification of the technique.

Discussion

The scholarly knowledge constructed from axioms, definitions, and rigorously proven theorems has been replaced by other definitions and axiom-like "key concepts" considered true without proof. In our case, the theorem regarding the slopes of perpendicular lines and its proof has been replaced by a "key concept" without proof, and several "solved examples" presenting the technique for executing basic tasks. What remained to be taught and learned was how to execute basic tasks. The change was dramatic from scholarly mathematics to textbook knowledge, since mathematics was stripped of its essential feature – the proof. The mathematicians warned that this dramatic change in our curriculum would lead to generations of teachers and students with increasing difficulty executing tasks requiring more than one step (Wu, 1997). The textbook knowledge shrank to teacher knowledge, but remained the same in nature, i.e., a set of tasks, techniques, and justifications for techniques. As observed, the teacher from this study used only basic tasks that were similar with the examples solved in the textbook, employed only the graphical approach, and supported her technique with some justifications. The teacher may have favored those tasks and techniques for various reasons like, the end-of-course exams contained mainly basic tasks for which the techniques were appropriate, her students' weak competency with symbolic manipulations, and time constraints. The technique with a graphical approach imposed by the teacher in this study was inadequate for Task 2. Almost all the students (96.6 %) relied on the teacher's knowledge – a subset of the textbook knowledge. The students failed to connect the graphical and symbolic representations of points and lines, and the techniques necessary to carry out Task 2. As predicted by mathematicians (Wu, 1997), the students had difficulty executing tasks with more than one step.

The observed *logos* from the local praxeologies (textbook and teacher knowledge) contained only the technological discourse. There was no observed *logos* in the case of point praxeologies (student knowledge). Without theory based on proof, the knowledge advancement can only be obtained by learning to execute new tasks with new techniques that are not necessarily connected to old ones. As seen in our study, this type of knowledge advancement did not lead to teacher's or students' success.

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