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Students' and teachers' mental solving of algebraic equations: From differences to challenges

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This paper presents students' and teachers' strategies for mentally solving algebraic equations. The enactivist notions of problem-posing offer conceptual grounds to engage in analysis of students' and teachers' strategies, and in their comparisons, leading to the exploration of differences in the nature and origin between the solving processes of students and those of teachers. Final remarks reflect on the potential of being sensitized to the nature of these differences in solving processes.

Keywords: Algebraic equations, mental mathematics, problem-solving.

Introduction

This paper is in continuity with the one presented at CERME-8 in WG3 about the mental solving of algebraic equations (Proulx, 2013a). Work has been conducted with secondary-level students and with teachers on mentally solving algebraic equations of the form $Ax+B=C$, $Ax+B=Cx+D$, $Ax/B=C/D$ without paper and pencil or any other material aid. The main research objective is to gain better understandings of the nature of the strategies developed when solving these algebraic equations mentally. From our analysis, students' and teachers' ways of solving highlight significant differences to which it is worth paying attention in terms of their meaning and nature as well as the issues that they raise about the teaching and learning of algebraic equation-solving. This paper reports on analyses of the strategies developed by students and teachers for similar if not identical algebraic equations, and grounds it in a discussion of the nature of solving processes.

Solving processes in mental mathematics: Emergence and problem-posing

As mentioned in Proulx (2013a), recent work in mental mathematics points to a continued need for investigating and conceptualizing how students develop mental mathematics strategies. Researchers have begun criticizing the notion that students “choose” from a toolbox of predetermined strategies to solve problems in mental mathematics. Threlfall (2002), for example, insists rather on the organic emergence and contingency of strategies in relation to the tasks *and* the solver (e.g., what s/he understands, prefers, knows, experienced, is confident with; see Butlen & Peizard, 1992). This view is aligned with Lave's (1988) situated cognition perspective that conceives of mental strategies as flexible emergent responses, adapted and linked to specific contexts and situations.

In mathematics education, the enactivist theory of cognition has been concerned with issues of emergence, adaptation and contingency of learners' mathematical activity (e.g., Maturana & Varela, 1992; Varela, Thompson, & Rosch, 1991). In particular, Varela's (1996) distinction between problem-posing and problem-solving offers insights for conceptualizing the generation of strategies. For Varela, problem-solving implies an understanding that problems are already in the world, lying “out there” waiting to be solved independent of us. He explains, in contrast, that we specify – we pose moment by moment – the problems that we encounter through the meanings we make of the world. We do not “choose” or “take” problems as if they were lying out there objectively and independent of our actions: we bring them forth. In short, for Varela, we *pose* our problems.

This perspective underlines Simmt's (2000) argument that it is not tasks that are given to solvers, but rather prompts that are taken up by solvers, who by posing them in a specific mathematical context create tasks with them. Prompts become tasks when solvers engage with them, when they pose them as tasks. Hence solvers make the prompt a multiplication task, a ratio task, a function task, an algebra task, and so forth, and solve it in relation to this posing. And, this posed task is not static or fixed once and for all, because the posing triggers a solving process that in turn transforms the posed task in an ongoing and dynamic process. It is with/in this process that the task emerges, organically, constantly becoming, being re-solved and re-posed (see Proulx, 2013b). By way of an example of this interaction between the posing and the solving, here is a strategy taken from a study on mental calculations (Proulx et al., 2014). To solve $741-75$, one solver explained:

- (a) $741 - 75$ is like $700 - 75 + 41$.
- (b) $700 - 75$ is like having 7 dollars and subtracting 3 quarters. I am left with \$6.25.
- (c) 6.25 is six-twenty-five, so I add 41 to 625. I do $5+1$ is 6, $4+2$ is 6, and I have 600, so 666.

When $741-75$ was given, the first step was to find a way of solving, of entering, of posing it as a task. This prompt was then posed as a decomposition task, leading the solver to decompose 741 in 700 and 41, in order to subtract 75 from 700. This decomposition produced in return a new prompt for the solver, that is, to solve $700-75$, which was posed as a monetary task (7 dollars minus 75 cents). This other solving step led to another prompt, $625+41$, which the solver posed anew as a decomposition task of each digit in each number in relation to their position (hundreds, tens, units) and its successive addition. Hence each solving step led to the posing of a task to solve, necessitating a way of entering into it, continuing to solve it, etc., producing an entire solution path.

A posteriori, in light of this entire solution path, one can assert the presence of a strategy, but this is an assertion after the fact because all this unfolds one step at a time when advancing in the solving of the posed task. Interacting with the prompt, engaging in the task, is to take a step and another, and these steps emerge in the solution path. All this happens in continuity, step by step, with each step leading to another posing of the task, to another solving process contingent on and emerging from previous steps, leading in return to another posed task, leading to another step, etc.

[E]ach solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied to particular cases. [...] The 'strategy' (in the holistic sense of the entire solution path) is not decided, it emerges. (Threlfall, 2002, p. 42)

The entire solution path, or strategy in Threlfall's sense¹, is not predetermined, but generated for solving, emerging from the interaction with the prompt. Thus the solver *transforms* the prompt as a mathematical task generating a strategy for the posed task for solving it. It is this dynamic entry on strategies, on solution paths, that characterizes the conception that grounds the analysis of solving processes in this study, that is, of students' and teachers' strategies for solving algebraic equations.

¹ It is in Threlfall's sense that the expression *strategy* is used in this paper, that is, not as a fixed and reified entity, but as the *entire solution path*, in its totality and dynamic nature as it unfolds through the diverse *solving steps*.

Solving algebraic equations without paper and pencil

The context in which the mental mathematics sessions are conducted is simple. A group of solvers, students or teachers, sit at their table without any paper or pencil and attempt to solve the prompts given. The organization takes the following structure: (1) an equation is offered in writing on the board; (2) solvers have 15-20 seconds to solve the equation mentally (without paper-and-pencil or material aid); (3) at the signal, strategies are shared orally and explained to the group.

The data collected come from the strategies explained orally by solvers, taken in note form by at least two research assistants (RAs). These notes are refined with on-the-spot discussions between the research team members (PI and RAs) following data-collection sessions, to produce a report on the various solution paths developed by participants. In this report, each strategy is given a name that describes it for matters of classification (descriptive level of analysis). With these descriptions, various analyses are conducted, depending on the purpose aimed for (meaning making engaged with, nature of strategies produced, difference with paper and pencil, etc.). In this case, as explained in the introduction, the analysis of strategies is conducted with the precise intention of establishing comparisons between students' and teachers' solving processes. Thus the analysis in this paper is not conducted for/on the strategies themselves, but mostly to establish this comparison ground between the solution paths being laid down by students and by teachers.

Students' posed tasks and solution paths

Work was conducted in two Grade-8 classrooms with about 30 students for 75-minute periods. In each classroom, the same five equations were given. The two used in the analysis are "Solve for x the equation $2x+3=5$ " and "Solve for x the equation $\frac{2}{5}x = \frac{1}{2}$ "; the solution paths emerging for these are illustrative of the solving processes developed by students. As such, the analysis is not focused on the occurrences of strategies, but on the nature of the solving processes and the functionality and meaning of the strategies developed. For this, strategies in both classrooms are grouped to offer and set up comparative grounds with teachers' solving processes.

For "Solve for x the equation $2x+3=5$ ", students produced the following:

Inversing operations. One student explained that he did "minus 3" on the right side of the equality to obtain $2x=2$, which directly gave $x=1$ (without needing to divide by 2).

Balancing. One student explained having done the same actions on both sides of the equality, thus subtracting 3 on both sides to obtain $2x=2$ and then dividing by 2 on both sides to get $x=1$.

Direct reading. Some students explained that with $2x+3=5$ he knew right away that x must equal 1, because $2+3=5$. Another explained having first taken the x out of the equation, leaving $2+3=5$. So, when 2 was multiplied by x , it has to remain a value of 2 to fit in the equation, so x needed to be 1. Another student explained that this meant that $x=0$, because in $2+3=5$ the x is unnecessary.

For "Solve for x the equation $\frac{2}{5}x = \frac{1}{2}$ ", students produced the following:

Transforming in equivalent fractions and decimals. One student explained having transformed $\frac{1}{2}$ in $\frac{10}{20}$ to make the $\frac{1}{2}$ divided by $\frac{2}{5}$ simpler, then repeating the same thing for $\frac{2}{5}$ to obtain $\frac{20}{50}x = \frac{10}{20}$. He explained that this is equivalent to $0.4x=0.5$ and thus the response is $x=0.5/0.4$.

Inversing and transforming in decimals. One student explained inversing the equation to $\frac{5}{2}x = 2$.

He then transformed in decimals to obtain $2.5x=2$, and dividing by 2 got $1.25x=1$ so that $x=1.25$.

Cross multiplying. After making the equation $\frac{2x}{5} = \frac{1}{2}$, the student explained having cross multiplied, where 5 times $\frac{1}{2}$ gave $2x= 2.5$ and thus $x=1.25$.

Halving. One student explained that half of 5 is 2.5, and because one looks for $\frac{1}{2}$, so x is 1.25.

Finding a scalar. One student explained looking for the value of x that made $2/5$ equal $\frac{1}{2}$. Placing fractions over 10, he explained that x is 1.25 because 4 times 1.25 is 5 and $5/10=\frac{1}{2}$.

Finding common denominator and adding. The student explained having placed fractions over 10, obtaining $\frac{4}{10}x = \frac{5}{10}$. Subtracting $4/10$ and $5/10$ gave $-1/10$, so then x is worth $1/10$.

All these solution paths, and their underlying solving steps, are not necessarily “adequate” or “standard”, but illustrate an emergent solving process geared toward finding a value for x that satisfies the equality. Through these examples of solution paths emerges a diversity of solving steps, of entries for solving. As explained in Proulx (2013c), the “mental” dimension provokes the search for an entry point, a way of posing the prompt, of getting in, of making a step. The solving context is thus created by the solver, producing an adapted way of entering into the problem. This diversity, which is translated in a variety of solution paths for solving the “same” equation, transforms that “same” equation, which is differently contextualized or posed differently from one solver to the next as each develops his/her own ways of posing and solving. The diversity of solution paths illustrates well how the various ways of posing the task led to varied ways of solving by solvers, hence diverse strategies or solution paths. In other words, the “same” equation gives rise to the emergence of a variety of posing, which leads to the development of a variety of strategies.

Teachers’ posed tasks and solution paths

The work with the group of 20 secondary-level teachers was conducted during a day-long session, where the solving of algebraic equations was carried out in the first half of the day. Similar equations given to students, even identical ones, were given to teachers to solve (about 10).

In general, most if not all teachers’ strategies can be described as efficient and errorless for solving the algebraic equations, e.g., through balancing strategies and isolating x . However, at specific moments during the sessions, some strategies of a more arithmetical nature were proposed. For example, some teachers explained having done what they described as “recovering”, where, e.g., for “Solve for x the equation $x^2 - 4 = -3$ ”, one teacher explained:

[Hiding x^2 with his hand] I look for the number which, when subtracted 4, gives -3 . I know it is 1. So then, what number squared gives 1? It is $+/-1$.

This solution path is similar to “inversing” methods discussed in Filloy and Rojano (1989), or Nathan and Koedinger’s (2000) unwinding, where operations are undone to obtain a value for x . As Filloy and Rojano explain, in order to solve an algebraic equation this way “[i]t is not necessary to operate on or with the unknown” (p. 20), because it comes back to an arithmetical context of operating on numbers. However, these arithmetic strategies were occasional for teachers. For example, for “Solve for x the equation $\frac{2}{5}x = \frac{1}{2}$ ”, teachers produced the following:

Equating middle and extreme products. One teacher explained having acted like with ratios, multiplying middle and extreme terms together, obtaining $4x=5$, hence $x=5/4$.

Multiplying by the inverse. One teacher explained having divided by $2/5$ on each side of the equality, leading to multiply by $5/2$ to get the same answer, giving $x=5/4$.

Isolating x in two steps. One teacher explained having multiplied by 5 on each side of the equality, obtaining $2x=5/2$, and then dividing all by 2 to obtain $x=5/4$.

Following these solution paths, one teacher offered another entry:

Simplifying the equation. The teacher explained having aimed to get rid of $1/2$ by multiplying the entire equation by 2, giving “4 fifths of x equals 1”. He then multiplied by $5/4$ to find x .

Here the teacher simplifies the equation, eliminating the $1/2$, in order to find the value of x through multiplying by the inverse coefficient. Numerous teachers were intrigued by this solution path and questioned the teacher about it. He explained that his intention was to get rid of $1/2$ to obtain “1 on one side” of the equation and because “multiplying by 2 is easy here”. Asked about the numbers present in the equation, he also explained that it was not clear for him if other numbers like $3/2$ or $1/6$ would have led him toward similar solving steps and that it was the presence of $1/2$ that triggered his activity. This is thus an example of a local strategy, affected by the concrete “data” in the equation: the solving steps are produced on the spot for this equation and not as a general strategy applicable to all cases (as well as not being a strategy for isolating x , but about simplifying $1/2$). The entry in the solving is done locally with the $1/2$, the task is posed as one implying a $1/2$, and not by the equation taken in its totality independent of its concrete values as could be the case in a cross-multiplying product. Although local, this strategy underlines an entry directly grounded in the data of the equation. The teacher simplifies this equation by doubling $1/2$, because it was “simple” to do so, and then solves it. However, this kind of solution path diverges from most of the strategies that teachers have produced.

Whereas in students one sees more local solving steps of this sort, directly sensitive to the data in the equation to solve, the strategies developed by teachers appear more decontextualized and general, less centered on the direct data of the equation. In short, faced with the same prompts, teachers posed problems different from those posed by students. This difference between teachers and students is well reflected in a comment made, after sharing the “doubling the $1/2$ ” strategy, by one of the teachers about what he perceived as the optimal strategy to solve this equation:

In Grade-8 the winning strategy is really the ratio one [multiplying extreme and middle values]. We work at it so much with them and I encourage them to use it in front of these sorts of equations. [...] I am not against the other strategies, but with my students [waiving his hand in discouragement], I am not sure that it would come out much, especially if we ask them to solve without paper calculations. In mental mathematics it is not obvious, whereas with ratios I think that 2 times $2x$ gives $4x$, and 1 times 5 gives 5, and $4x$ over 5 they know afterwards that they have to divide by 4, these are rules of transformation of the equations.

This comment on the winning strategy and students’ (un)ease, supported by the other teachers, contrasts with the students’ solution paths displayed above. Shortly afterwards, this prompt was given to teachers: “Find the value of $2t$ in $3(2t)+6=18$ ”, for which they produced three strategies:

Balancing. One teacher subtracted 6 on each side of the equation, obtaining $3(2t)=12$, and then dividing by 3 on each side to obtain $2t=4$.

Undoing of operations. One teacher explained having done the opposite operation, by subtracting 6 from 18 and then dividing it by 3.

Recovering. One teacher explained hiding $2t$ to find the number which when added 6 is worth 18. This number is 12. Then he asks which number multiplied by 3 gives 12. This number is 4.

Here again, this recovering strategy provoked questions from teachers, not stemming from misunderstanding but mostly from curiosity about having used this kind of strategy to solve the equation. One teacher explained that even if he himself solves algebraic equations in a variety of ways, he does not teach this variety to his students because he considers it important to proceed step by step in a structured and linear fashion for each side of the equality: something with which other teachers strongly agreed. However, after this comment, another teacher raised the following:

I have a question. I would show it like that to my students [step by step, operating on each side of the equation in the same manner]. However, as a secondary student I was never shown this “balancing” way. It is one of my colleagues who told me “Listen, I teach it like that”. Then, when I taught Grade-7, I started doing this “you do the inverse operation, bing, bang”. And I wonder if it has not become an automatism. Is it because we always do it like that, that students themselves begin doing it also rapidly? Is it OK if they do so, or do they need to continue with their personal strategies? [...] Should we encourage varieties of strategies in students?

This comment is reminiscent of Freudenthal’s (1983) one about automatism in teaching:

I have observed, not only with other people but also with myself [...] that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why [students don’t understand them] is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question. (p. 469)

What Freudenthal underlines as much as the teacher is not about misunderstandings of non-usual solution paths, but about well-ingrained habits of solving that (1) prevent one from stepping outside them, and (2) question the relevance of alternate solution paths for solving equations. The teacher’s question is about this, that is, the relevance or legitimacy of alternate solution paths: Should they be taught? Should they be accepted? These questions sensitize one to the variations in ways of solving, but also to the challenges that this raises for teaching-learning situations.

Discussion and reflections on algebraic equation solving processes

In what ways do these differences in solution paths raise challenges? The discontinuity, the distances between the various strategies developed by students and teachers are without any doubt important sources of challenges for the teaching-learning of algebraic equation solving. One way of addressing these questions is to probe the solving processes as much in students as in teachers. How can this variety be explained, as well as the differences in solution paths of teachers and students?

If we take into account Threlfall’s (2002) views, we can consider that the nature of what emerges for the student is quite different from what emerges for a teacher. A student’s experience is quite different from that of a teacher. For the teacher, one has the impression that some earlier

experiences intensely orient future ways of solving. The teacher's comments above, as well as Freudenthal's, insist on the challenge of stepping out of the frame pretraced by earlier repetitive experiences of teaching the solving of algebraic equations along specific solution paths. Faced with having to teach students and make them learn, teachers make choices that in turn orient the nature of their own mathematical experiences of this specific mathematics thematic. By insisting on some solution paths seen as fruitful for students to solve equations, these solution paths in turn orient teachers' solving processes; a phenomenon that the above expression *automatism* describes well.

It is this experience that plays a major role for teachers. When facing the "same" equations, teachers and students pose quite different tasks. Although all solving steps leading to what are seen as different strategies arise from the posing of different problems by different persons when interacting with the prompt, the solving steps do not share the same origins. Teachers are expert solvers in the sense of what could be called an "overspecialization". They can perceive these equations through the same algebraic lens, steering almost all equations to the same kind of task, posing them as the same tasks. Students are not non-experts, but have, however, not yet lived the same repetitive experiences that cause this overspecialisation. Think of the riverbed metaphor: teachers' riverbed is well dug, quite specialized, and the river runs through it comfortably. That of students looks much more like a stream that deviates at the least change in scenery, but this does not prevent it from unfolding (sometimes in the same place, sometimes elsewhere, unpredicted but adequate, or not).

Final remarks: On differences and challenges

Despite its efficiency in solving, teachers' overspecialisation limits the variety of problems that they pose, being less sensitive to variations (in numbers, unknowns, operations, etc.). This makes it difficult for them to act differently, as one of the teacher expressed, but also to appreciate the variety of students' solution paths; a variety that teachers themselves no longer experience much in their day-to-day mathematics teaching (a situation that also leads to question the validity of these solution paths). These different poses provoke a distance between solution paths in students and in teachers. The challenge for the teaching and learning of algebraic equation-solving can then be seen in teachers' overspecialization – which implicitly imposes a particular pose of the task and thus the strategy – which brings about a distance with students' solution paths. Also, this overspecialisation seems to be generated by the belief that a specifically guided and structured experience of solving will be helpful to students: teachers believe, as expressed above, that students need these structured and specific experiences. This belief may be an important source of reflection, because the challenge of this overspecialisation points to the necessity of not discarding the less common or local solution paths, but of developing a sensitivity toward them and toward the role played by this overspecialisation or these automatisms.

This resonates with Anghileri's (2001) literature review on mental calculations, pointing to students locally tailored strategies, linked to their understanding of problems. However, she explains that these strategies do not last because they are often substituted by standard algorithms taught in classrooms, where students attempt to conform to what happens in everyday mathematics lessons. Anghileri adds that this situation is complex, because without negating the power of standard algorithms for solving, they conflict, provoke a "distance", in the development of students' aptitudes in problem-solving. For her, this amounts to a matter of intentions:

The emphasis in teaching arithmetic has changed from preparation of *disciplined human calculators* to developing children's abilities as *flexible problem solvers*. This change in emphasis requires new approaches in teaching that will develop children confidence in their own methods rather than replicating taught procedures, and that will enable them to understand the methods used by others (Anghileri, 2001, p. 79).

The transition toward flexible problem-solvers represents an invitation to think about algebraic teaching and learning, an invitation directly aligned with the above argument on the importance of developing sensitivities about distances between teachers and students' algebraic solving processes.

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