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# Key ideas as guiding principles to support algebraic thinking in German primary schools

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*German mathematics teaching-units in primary school lack explicit algebra learning environments; however, the topics which are taught address algebraic thinking if seen from a new perspective. Teachers and children are mostly unaware of the algebraic potentials of tasks –especially in the scope of the content area patterns and structures. The project presented here submits a suggestion of algebraic key ideas as guiding principles to rethink ‘arithmetical’ topics and to design learning environments on algebraic thinking. Additionally, effects of implementing and evaluating such tasks are illustrated by one example.*

*Keywords: Algebraic thinking, awareness, key ideas, patterns and structures, properties.*

## Introduction

The approach presented here is worked out for the particular situation in German primary schools and accordingly addresses one of the borderlines in the scope of algebraic thinking pointed out by Hodgen, Oldenburg, Postelnicu, & Strømskag (2015), i.e. “differences between teaching cultures in different countries (and within countries) are enormous and restrict generality of results very much” (p. 386). In German primary school algebraic topics have no tradition and still no explicit place in curricula, textbooks, and teaching-units on the one hand (KMK, 2004). On the other hand, recent research on early algebra or algebraic thinking is emerging (e.g. Akinwunmi, 2012; Gerhard, 2013). Furthermore, the daily classroom interaction and common teachers’ beliefs reveal bright opportunities for early algebra (e.g. Krauthausen & Scherer, 2007). Yet, implementing promising approaches and algebraic tasks in daily school life still is a great issue.

The main aim of the project is making algebraic learning chances possible for children. Opportunities to get to know algebraic ideas and ways of algebraic thinking depend on tasks presented in the classroom. These tasks are offered by teachers. Hence, the focus has to be on tasks and on teachers’ awareness of the potential of these tasks. In so doing early algebra can be supported via a detour that influences classroom interaction and therefore children’s awareness and abilities.

## Theoretical framework

### Algebraic thinking and core areas

Algebraic thinking is assigned to special thinking habits. Current research identifies mainly four algebraic thinking practices, which lay in generalising, representing (incl. symbol use), justifying, and, reasoning with generalisations or relations (e.g. Kaput, 2008; Kieran, Pang, Schifter, & Ng, 2016; Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015). The focus of attention needs to shift from numerical solutions to mathematical structures behind the given patterns or equations. This shift allows seeing the generality as reification (Sfard, 1991) and therefore creating new objects (Mason, 1989). “By attending to relations and fundamental properties of arithmetic operations (what we call relational thinking) rather than focusing exclusively on procedures for calculating answers” (Carpenter, Levi, Franke, & Zeringue, 2005, p. 53) procedural thinking is not erased but expanded.

Algebraic thinking as conceptual (Tall & Gray, 2001), relational or structural thinking can be applied to various topics or as Sfard puts it, “any mathematical activity may be seen as an intricate interplay between the operational and the structural versions of the same mathematical ideas” (Sfard, 1991, p. 27). Although in most of the research studies certain topics are outlined to be particularly relevant for algebraic thinking, different content-orientated registers identifying the strands or core areas of algebra can be found in many of them (Table 1).

		pattern and formulas	patterns	patterns (& structures)
generalised arithmetic	generalised arithmetic	restrictions	arithmetical structures	property structures
	equations			equivalence structures
functional thinking	functional thinking	functions	relations	functional structures
	proportional reasoning			
application of modelling languages	variable	symbolic language		
Kaput 2008	Blanton et al. 2015	Drijvers, Goddijn & Kindt 2011	Kieran et al. 2016	Steinweg 2016

**Table 1: Algebraic strands and core areas**

These lists overlap in various topics as shown in the re-assigned order in Table 1. The core areas therefore are more or less universally acknowledged and differences can be identified in details only. For instance, some authors differentiate between generalised arithmetic and equations whilst others exclude patterns or list symbolic language (variables) within its own section. My suggestion (Table 1 rightmost column) and basis of this paper takes into account the current situation in Germany, because it “is important to indicate that any curriculum has a complex relationship to what actually occurs in classrooms” (Cai et al., 2005, p. 14). The German primary curriculum includes no algebra area, even though a single door for the implementation of algebraic thinking opens up. This possible link is the content area ‘patterns and structures’, which is given in the national standards (KMK, 2004). Moreover, patterns and structure can be regarded as generic field for the different algebraic topic strands (Drijvers et al., 2011).

Owing to these two reasons my own suggestion stresses the terms patterns and structures. In the following paragraph these terms are theoretically analysed in more depth.

### **Patterns and structures**

Often mathematics itself is described as the science of patterns (Devlin, 1997). In this view, all mathematical theories arise from patterns spotted. Even axioms characterise patterns to build on. Not surprisingly, teaching and learning about patterns and structures is not a special topic but is fundamental for all mathematics lessons:

Pattern is less a topic of mathematics than a defining quality of mathematics itself. Mathematics ‘makes sense’ because its patterns allow us to generalize our understanding from one situation to another. Children who expect mathematics to ‘makes sense’ *look* for patterns. (Brownell, Chen, & Ginet, 2014, p. 84)

Becoming aware of patterns allows us to see sense in mathematics and to appreciate its beauty. This awareness is at least twofold. On the one hand, seeking patterns can be classified as meta-cognitive, on the other hand, there is a cognitive component of awareness which is characterised by “knowledge of structure” (Mulligan & Mitchelmore, 2009, p. 38).

*Patterns* can be described as “any predictable regularity, usually involving numerical, spatial or logical relationships” (Mulligan & Mitchelmore, 2009, p. 34). Constructing a pattern of numbers or shapes by making up a rule or a certain operative variation (Wittmann, 1985) of a given number or task is an individual creative process. If, for instance, the pattern of a number sequences is creatively made up, the regularity then is fixed and can be used, continued, and described (Steinweg, 2001).

In the approach presented here *structure* is understood as mathematical structure and not as a category system to describe the individual pattern awareness of children (Rivera, 2013). Mason, Stephens & Watson (2009) recommend “to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced” (p. 10). They point out the difference between the spotting of (singular) relations and the use of the given example as paradigmatic for certain properties of a general structure (Mason et al., 2009). Thus, detecting structures, in contrast to patterns, requires mathematical knowledge about objects and operations. The relation between mathematical objects is essentially determined by mathematical structures (Wittmann & Müller, 2007). Awareness of structures often suffers from the fact that structures are mentioned only briefly and only formulated in rules, like  $a+b=b+a$ , in mathematics lessons. Unfortunately, these condensed statements are not an appropriate tool to become aware of the logical structures and properties of mathematical objects and relations which are fundamental for mathematics. In summary structure is the crucial term in the twosome patterns and structures. The approach described in this paper therefore puts emphasise on structure, and mentions structure purposefully in each key idea in order to call attention to it.

### **Algebraic key ideas – a suggestion**

The key ideas outlined in the approach presented here focus on (1) patterns (& structures), (2) property structures, (3) equivalence structures, and (4) functional structures (Steinweg, 2016). The first idea differentiates between patterns and structures. Patterns are not a priori structures but may eventually generate products following mathematical properties and relations. Hence, the expression ‘structures’ is given in brackets to indicate this substantial difference. The second key idea lies in the properties of numbers and operations: Numbers can be divided into odd and even, divisibility can be explored, etc. Daily used –supposedly arithmetical– operations follow structures because of their properties (commutativity, associativity, distributivity). One example of this key idea is presented below. The third key idea holds learning opportunities in evaluating, preserving or construing equivalence in given correct or incorrect equations by sorting terms, etc. The main issue here is to overcome the urge to calculate the given terms and to solely compare the results but to focus on the relation of given numbers, sums, differences, products, or quotients (Kieran, 1981; Steinweg, 2006). This key idea goes hand in hand with the currently commonly used and fostered individual strategies in arithmetic, which can be found in Germany (also cf. Mason et al., 2009). The last key idea sums up learning environments on functional structures, (i.e. mainly proportional), relations, and co-variation aspects. One example is a task called ‘number & partner number’

(Akinwunmi, 2012). The structural relationship can be described by a rule (functional term) which assigns a partner number to each given number.

As mentioned above, the key ideas presented here are ordered by mathematical core areas and put emphasis on structures as one of many feasible approaches. Sufficient knowledge of mathematical structures is crucial for both teachers and children. Only well trained teachers are able to understand the mathematical structures and to make them accessible for children. One possible strategy to get access to mathematical structures lies in implementing especially designed tasks which enable children to explore, use, describe, and even prove mathematical structures (Steinweg, 2001).

## Methodology

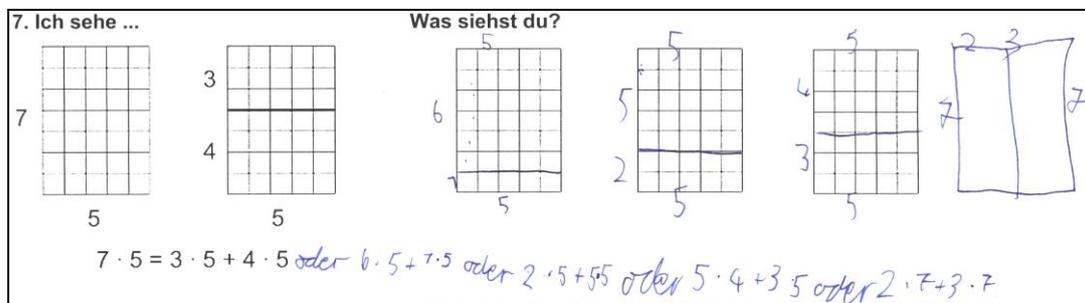
In the research project learning environments suitable for the four key ideas outlined above are designed (Wittmann, 1995) and evaluated in order to uncover the algebraic potential of common tasks and to give tangible examples in the algebraic core areas within the field patterns and structures. Each learning environment includes various tasks in a booklet to be handed out to the children and information for teachers in a teacher's guide (Steinweg, 2013). The teachers participated in an introductory meeting in which the tasks and possible teaching arrangements – given in the guidelines– were discussed. They committed themselves to implement all of the tasks in daily classroom work with the intensity and depth of the use of the learning environments being in their hands. This means that there was no specific focus on the child-teacher-interaction while working on the tasks –with the exception of some mathematics lessons randomly visited by the author. The research therefore focusses on the question: *Does the implementation of the designed tasks show any effects on children's algebraic competencies?* Six German primary school classes with 144 children from 2<sup>nd</sup> to 4<sup>th</sup> grade (on average 7- to 9-year-olds) participated in the project.

## Research results on the example of distributivity

This paper exemplarily illustrates the research idea on distributivity as one element of the key idea 'property structures'. The main challenge is to see the structure of equations and terms in a meta-perspective way. For instance in the term  $2 \cdot 8 + 5 \cdot 8$  children have to spot the specific 'internal semantic' (Kieran, 2006, p. 32). Only if the equal factor is identified as an important component in the products can the 'variable' factors be summed up. The two products have to be identified as objects in a sum and then the two different factors can be added to create a new product ( $7 \cdot 8$ ). The additive combination of products and the decomposing of products into a sum of two products with one equal factor in each case seems a tough challenge for the children. The shift of attention to elements of the equation as objects and to identify the mathematical structure is essential for algebraic thinking. Most likely, the children participating in the project had already experienced derive-and-combine-strategies solving multiplication tasks in class. The actual approach to the multiplication tables in German mathematics in primary school is peculiar. There is no longer 'doing tables' but working on core tasks (e.g. doubles, times 5, times 10) and derive-and-combine-strategies to solve other multiplications. Only core tasks should be known by heart as facts (sometimes known as 'helping facts' in Anglo-Saxon literature). Unfortunately, an arithmetical perspective –calculate terms to determine the specific result– is normally supported by teachers in primary mathematics. The out of the common change of perception of the structure of equations is

therefore challenging. Only by a shift of attention can the structure of the maths behind the equation, i.e. distributivity, be recognised.

The tasks implemented in the project try to support the identification of structure. For instance columns of variations of one equation are given to allow focussing on both the constant and the changing elements, like  $3 \cdot 2 + 6 \cdot 2 = \dots \cdot 2$ ,  $3 \cdot 3 + 6 \cdot 3 = \dots \cdot 3$ ,  $3 \cdot 4 + 6 \cdot 4 = \dots \cdot 4$ , etc. Alongside tasks in symbolic representations, rectangle areas as representation for multiplications (length by width) can be used as well. If rectangles are accepted as multiplication representation, manipulating these rectangles by cutting and re-interpreting the two part-rectangles as multiplications can be the next step to explore and understand distributivity. The children were given one example and then asked to find three more possibilities to decompose the product  $7 \cdot 5$  (Figure 1). Such rectangles can be provided by the teachers as representations on worksheets or ‘actively’ made up by the children by cutting out sections of grid paper. As an instance of possible developments in algebraic thinking by simply working on the tasks Philipp’s solution of one exemplary task of one worksheet given in the booklet is interpreted in Figure 1.



**Figure 1: Philipp explores distributive structures by interpreting rectangles as multiplications**

Philipp’s solution is stunning in some ways. The task asks him to find three further decompositions of the given product. He marks his ideas in the given three rectangle areas and writes down matching symbolic representations of the product-sums. The little dots in the first rectangle grid point at the fact that Philipp might have counted the number of squares. The other solutions do not show these presumed counting dots. Philipp apparently becomes aware of the structure and the main idea. The relation between the terms is understood individually. Philipp connects the different solution by using the verbally form of ‘oder’, which means ‘or’. This habit indicates that he is aware of the possibility of different compositions of the product but not yet sure about the equality-relation between these terms. The worksheet invites the children to find three products by giving three blank rectangles. Philipp extended the task spontaneously by drawing a fourth rectangle. This add-on again is remarkable. Philipp sketches a rectangle without drawing the grid. The countable squares obviously are no longer necessary for him. The fourth solution is the only one which decomposes the factor 5 instead of the factor 7. Philipp applies the main structure of splitting up the factor flexible for either factor now. The example of Philipp shows the multi-faceted possibilities to gain access to mathematical structures by working on challenging tasks.

As the main research question aims at evaluating effects of the implementation of the learning environments, results of a pre- and post-test are of interest. In this paper the results of the test item  $10 \cdot 5 - 4 \cdot 5 = \_\_\_ \cdot \_\_\_$  (corresponding to distributivity) are documented exemplarily (Table 2).

Category	pre-test (n = 135)	post-test (n = 133)
Algebraic	1.5 %	32 %
Procedural	32.5 %	35 %
no answer given	66 %	33 %

**Table 2: Results solving  $10 \cdot 5 - 4 \cdot 5 = \underline{\quad} \cdot \underline{\quad}$**

The task is quite hard to handle for the participating children in the pre-test even so the curriculum expects teachers to work on derive-and-combine-strategies in multiplication. Two thirds have no idea what to fill in the blanks. Only in very few cases are children able to combine the two multiplications into  $6 \times 5$  and thereby make use of the structure (algebraic perspective). After participating in the project one third is now able to give this answer. Another third places a multiplication like  $3 \times 10$ , which is fitting because of an equivalent result (procedural, arithmetical perspective). The figures suggest a developmental step of the procedural thinkers to the algebraic thinkers apparently. This assumption actually cannot be confirmed by the data. The developments are very much individual. For instance, some children who gave no answer prior to the project are now able to see the structural relation or calculate to find matching terms and others still have no answer at all. Despite the fact that these results are still far from being satisfactory, the increase in numbers of children using an algebraic perspective is considerable.

## Discussion

The project gives an initial indication that it is possible to foster algebraic thinking by providing sound learning environments without explicit variable use in the scope of the content field of patterns and structures. The challenges offered support effects on understanding and on performance in algebraic tasks. Yet, the impact of learning environments alone is not enough to support all children. As mentioned above, the project provides no binding specifications to teachers of how, for example, to focus on distributivity, but offers different opportunities to explore this mathematical structure via the designed tasks. As a “good balance between skill and insight, between acting and thinking, is [...] crucial” (Drijvers et al., 2011, p. 22), further effort should focus on exploring the differences between procedural and structural work on tasks. Teachers’ instructions and interaction in classroom discussions as well as the specific role of representations have to be focused on in further studies.

The hope is that the developed key ideas function as bridges between arithmetical topics and algebraic ones and also as guiding principles for classroom interaction. If common arithmetical strategies –like derive-and-combine– are seen from a different angle, they actually are algebraic ones. From a meta-perspective view the procedures performed are determined by mathematical structure and the properties of operations. The shift of attention towards structures has to be made explicit to both teachers and children. Only if teachers appreciate algebraic structures can they offer effective support and take up children’s algebraic ideas. In the particular situation in Germany awareness of the multi-faceted potential of the usually underestimated core area patterns and structures is crucial. Last but not least, the sensibility implies a win-win-situation: “Awareness of structure of expressions helps students understand these better, thus leading to a better understanding of rules and procedures” (Banerjee & Subramaniam, 2012, p. 364).

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