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To cite this version:

Meriam El Mansour, Emmanuel Lépinette. Conditional interior and conditional closure of random sets. Journal of Optimization Theory and Applications, 2020, 187 (2), pp.356-369. 10.1007/s10957-020-01768-w . hal-01914299

HAL Id: hal-01914299

https://hal.science/hal-01914299v1

Submitted on 6 Nov 2018

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Random set conditioning: conditional interior and conditional closure

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Abstract: In this short note, we present two new concepts. On a complete probability space, we consider two σ-algebras $\mathcal{H} \subseteq \mathcal{F}$ and a $\mathcal{F}$-graph-measurable random set $\Gamma \subseteq \mathbb{R}^d$. We show the existence of a largest $\mathcal{H}$-measurable open set contained in X, we call conditional interior and a smallest $\mathcal{H}$-measurable closed set containing X, we call conditional closure. We then deduce that a conditional essential supremum of real-valued random variables is actually a pointwise supremum over a closed random set.

Keywords and phrases: Random sets, Conditioning, Conditional support, Essential supremum.
2000 MSC: 60G44, G11-G13.

1. Introduction

On a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$, let $\mathcal{H}$ be a sub σ-algebra of $\mathcal{F}$. In this section, we generalise the concept of conditional support of a random variable to random sets, i.e. we construct the conditional closure of a $\mathcal{F}$-measurable random set Γ as the smallest closed $\mathcal{H}$-measurable set $\mathrm{cl}\,(\Gamma|\mathcal{H})$ which contains Γ a.s. When Γ is a singleton $\{X\}$ composed of a $\mathcal{F}$-measurable random variable X, $\mathrm{cl}\,(\Gamma|\mathcal{H})$ coincides with the conditional support of X. To do so, we first construct the conditional interior $\mathbf{o}(\Gamma|\mathcal{H})$ of Γ such that $\mathrm{cl}\,(\Gamma|\mathcal{H})$ is the complement of $\mathbf{o}(\Gamma^c|\mathcal{H})$.

2. Random set conditionning

Definition 2.1. *Let $(T, \mathcal{T})$ be a topological space. A random set Γ is a set-valued mapping that assigns to each $\omega \in \Omega$ a subset $\Gamma(\omega)$ of T. We say that*

Γ *is* $\mathcal{H}$*-measurable if*

$$\operatorname{graph}\Gamma := \{(\omega, x) : \ x \in \Gamma(\omega)\} \in \mathcal{H} \otimes \mathcal{T}.$$

When Γ *is closed (resp. open) almost surely, we say that* Γ *is closed (resp. open).*

Definition 2.2. *Let* $(T, \mathcal{T})$ *be a topological space and* $\Gamma : \Omega \twoheadrightarrow T$*, be a random set. We say that the* $\mathcal{F}$*-measurable random variable* $\xi \ : \Omega \to T$ *is an* $\mathcal{F}$*-measurable selection of* Γ *if* $\xi(\omega) \in \Gamma(\omega)$ *for almost all* $\omega \in \Omega$*. The set of such selections is denoted by* $\mathcal{L}(\Gamma, \mathcal{F})$*.*

We recall that any non empty $\mathcal{H}$-measurable closed random set Γ admits a Castaing representation $(\gamma_n)_{n\geq 1} \subseteq L^0(\Gamma, \mathcal{H})$ such that $\Gamma(\omega) = \operatorname{cl}\{\gamma_n(\omega) : \ n \geq 1\}$ a.s., see [1].

In the following, if $x \in \mathbb{R}^d$ and $r \geq 0$, $B(x, r)$ denotes the open ball in $\mathbb{R}^d$ of center x and radius r while $\bar{B}(x, r)$ is its closure. For any $A \subseteq \mathbb{R}^d$ and $\lambda \in \mathbb{R}$, we use the convention that $\lambda \times A = \{\lambda a : \ a \in A\}$. In particular, $0 \times A = \{0\}$. Recall the following definition, see [1]:

Definition 2.3. *The* $\mathcal{H}$-core *(also called conditional core),* $\mathbf{m}(\Gamma|\mathcal{H})$*, of a set-valued mapping* Γ *is the largest* $\mathcal{H}$*-measurable random set* Γ' *such that* $\Gamma' \subseteq \Gamma$ *a.s.*

In the case where Γ is closed, then the conditional core $\mathbf{m}(\Gamma|\mathcal{H})$ exists [1] and $\mathcal{L}(\mathbf{m}(\Gamma|\mathcal{H}), \mathcal{H}) = \mathcal{L}(\Gamma, \mathcal{H})$ in the case where $\mathcal{L}(\Gamma, \mathcal{H}) \neq \emptyset$. If $\gamma \in \mathcal{L}(\mathbb{R}^d, \mathcal{H})$ and $r \in \mathcal{L}(\mathbb{R}_+, \mathcal{F})$, it is easy to see that $\mathbf{m}(\bar{B}(\gamma, r)|\mathcal{H}) = \bar{B}(\gamma, \operatorname{ess\,inf}_{\mathcal{H}} r)$ where $\operatorname{ess\,inf}_{\mathcal{H}} r$ designates the conditional essential infinimum of r knowing $\mathcal{H}$. Up to a negligible set, this is the largest $\mathcal{H}$-measurable random variable which is smaller than r, see [1]. Notice that the conditional core does not depend on the σ-algebra $\mathcal{F}$ so that we may choose $\mathcal{F}$ as the largest one composed of all the subparts of Ω. We introduce an open version of the conditional core:

Definition 2.4. *The* $\mathcal{H}$*-measurable interior* $\mathbf{o}(\Gamma|\mathcal{H})$ *of a set-valued mapping* Γ *is the largest open and* $\mathcal{H}$*-measurable random set* Γ' *such that* $\Gamma' \subseteq \Gamma$ *a.s. if existence holds.*

Recall the following:

Lemma 2.5. *Let* $\mathcal{O}$ *be an open set in a normed space. For every* $x \in \mathcal{O}$*,* $B(x, d(x, \partial\mathcal{O})) \subseteq \mathcal{O}$*.*

Proof. Let us consider $r^* = \sup R$ where R is the non empty set of all $r > 0$ such that $B(x, r) \subseteq \mathcal{O}$. It is trivial that $B(x, r^*) \subseteq \mathcal{O}$. So, for all $o \in \partial\mathcal{O}$,

$d(x,o) \geq r^*$ hence $r^* \leq d(x, \partial\mathcal{O})$. Moreover, by definition of r^*, for all n there exists $z^n \in B(x, r^* + n^{-1})$ such that $z^n \notin \mathcal{O}$ hence $r^* \leq \|z^n - x\| \leq r^* + n^{-1}$. As the sequence $(z_n)_n$ is bounded, we deduce by a compactness argument that for a subsequence $z_n \to z$ as $n \to \infty$. Then, $\|z - x\| = r^*$ hence $z \in \operatorname{cl}\mathcal{O}$. In the case where $z \in \mathcal{O}$, $z^n \in \mathcal{O}$ for n large enough since $\mathcal{O}$ is open, which yields a contradiction. Therefore, $z \in \partial\mathcal{O}$. This implies that $r^* = d(x,z) \geq d(x, \partial\mathcal{O})$ and finally $r^* = d(x, \partial\mathcal{O})$. □

Theorem 2.6. *Let us consider a $\mathcal{F}$-measurable open random set $\mathcal{O}$. Then, the $\mathcal{H}$-graph-measurable interior of $\mathcal{O}$ exists.*

Proof. By [1, Proposition 2.6], $\operatorname{cl}\mathcal{O}$ and $\partial\mathcal{O} := \operatorname{cl}\mathcal{O} \setminus \mathcal{O}$ are closed $\mathcal{F}$-measurable random sets. When $\partial\mathcal{O} = \emptyset$, we set $d(x, \partial\mathcal{O}) = \infty$. Otherwise, since $\partial\mathcal{O}$ admits a Castaing representation on $\{\partial\mathcal{O} \neq \emptyset\}$, we deduce that the random continuous mapping $x \mapsto d(x, \partial\mathcal{O})$ is $\mathcal{F}$-measurable. We deduce that the sets $F^n := \{x : \ d(x, \partial\mathcal{O}) \geq 1/n\} \cap \operatorname{cl}\mathcal{O}$ are closed $\mathcal{F}$-measurable random subsets of $\mathcal{O}$ and we have $\mathcal{O} = \cup_n F^n$. Let us define the $\mathcal{H}$-measurable open random set

$$\mathbf{o}(\mathcal{O}|\mathcal{H}) = \operatorname{int}\ (\cup_n \mathbf{m}(F^n|\mathcal{H})) \subseteq \mathcal{O},$$

and let us show that this is the largest $\mathcal{H}$-measurable open set of $\mathcal{O}$. To do so, let $\mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ be a $\mathcal{H}$-measurable open subset of $\mathcal{O}$. As previously, we may write $\mathcal{O}_{\mathcal{H}} = \cup_n H^n$ where $H^n := \{x : \ d(x, \partial\mathcal{O}_{\mathcal{H}}) \geq 1/n\} \cap \operatorname{cl}\mathcal{O}_{\mathcal{H}}$ are $\mathcal{H}$-measurable closed subsets of $\mathcal{O}_{\mathcal{H}}$.

We claim that $H^n \subseteq F^n$. Indeed, if $x \in H^n$, it suffices to show that $d(x, \partial\mathcal{O}_{\mathcal{H}}) \leq d(x, \partial\mathcal{O})$ a.s. In the contrary case, on a non null set, there exists $o \in \partial\mathcal{O}$ such that $o \in B(x, d(x, \partial\mathcal{O}_{\mathcal{H}}))$. By Lemma 2.5, this implies that $o \in \mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ hence a contradiction.

As H^n is $\mathcal{H}$-measurable and closed, $H^n \subseteq \mathbf{m}(F^n|\mathcal{H})$ for all n. We deduce that $\mathcal{O}_{\mathcal{H}} \subseteq \cup_n \mathbf{m}(F^n|\mathcal{H})$ hence $\mathcal{O}_{\mathcal{H}} \subseteq \mathbf{o}(\mathcal{O}|\mathcal{H})$. □

Corollary 2.7. *For any $\mathcal{F}$-measurable random set Γ, the $\mathcal{H}$-measurable interior of Γ exists and $\mathbf{o}(\Gamma|\mathcal{H}) = \mathbf{o}(\operatorname{int}\Gamma|\mathcal{H})$.*

Corollary 2.8. *For any $\mathcal{F}$-measurable random set Γ, there exists a smallest $\mathcal{H}$-measurable closed random set containing Γ a.s. denoted by $\operatorname{cl}(\Gamma|\mathcal{H})$ and called the $\mathcal{H}$-measurable conditional closure of Γ. We have*

$$\operatorname{cl}(\Gamma|\mathcal{H}) = \mathbb{R}^d \setminus \mathbf{o}(\mathbb{R}^d \setminus \Gamma|\mathcal{H}).$$

Moreover, for all $\gamma \in \operatorname{cl}(\Gamma|\mathcal{H})$ and $\epsilon \in \mathcal{L}((0,\infty), \mathcal{H})$, for all $H \in \mathcal{H}$ such that $\mathrm{P}(H) > 0$, $\mathrm{P}(\{\Gamma \cap B(\gamma, \epsilon) \neq \emptyset\} \cap H) > 0$.

Proof. The first part is a direct consequence of Corollary 2.7. Suppose that $\mathrm{P}(\{\Gamma \cap B(\gamma,\epsilon) \neq \emptyset\} \cap H) = 0$. Therefore, by definition of the conditional support as a smallest element,

$$\mathrm{cl}\,(\Gamma|\mathcal{H}) = \mathrm{cl}\,(\Gamma|\mathcal{H})1_{\Omega\setminus H} + \mathrm{cl}\,(\Gamma|\mathcal{H}) \cap \left(\mathbb{R}^d \setminus B(\gamma,\epsilon)\right) 1_H.$$

Indeed, the $\mathcal{H}$-measurable set above contains Γ by assumption hence it contains $\mathrm{cl}\,(\Gamma|\mathcal{H})$. We get a contradiction since $\gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}),\mathcal{H})$ a.s. by assumption. □

Corollary 2.9. *Let $X \in \mathcal{L}(\mathbb{R}^d,\mathcal{F})$. There exists a smallest $\mathcal{H}$-measurable closed random set denoted by $\mathrm{supp}_{\mathcal{H}}(X)$ with $\mathrm{P}(X \in \mathrm{supp}_{\mathcal{H}}(X)) = 1$. This is the $\mathcal{H}$-conditional support of X. It is given by $\mathrm{supp}_{\mathcal{H}}(X) = \mathrm{cl}\,(\{X\}|\mathcal{H})$. Moreover, for all $\gamma \in \mathcal{L}(\mathrm{supp}_{\mathcal{H}}(X),\mathcal{H})$ and $\epsilon \in \mathcal{L}((0,\infty),\mathcal{H})$, for all $H \in \mathcal{H}$ such that $\mathrm{P}(H) > 0$, $\mathrm{P}(\{X \in B(\gamma,\epsilon)\} \cap H) > 0$.*

Notice that the conditional closure of a non empty set is necessarily non empty a.s. hence admits a measurable selection. In the following, we adopt the following notation $kA = \{ka : \ a \in A\}$ for any subset $A \subseteq \mathbb{R}^d$ and $k \in A$.

Lemma 2.10. *Let $H \in \mathcal{H}$. Then, for every $\mathcal{F}$-measurable set Γ, $\mathrm{cl}\,(\Gamma 1_H|\mathcal{H}) = 1_H \mathrm{cl}\,(\Gamma|\mathcal{H})$.*

Proof. First observe that $P(\Gamma 1_H \subseteq 1_H \mathrm{cl}\,(\Gamma|\mathcal{H})) = 1$ as $1_H \mathrm{cl}\,(\Gamma|\mathcal{H}) = \{0\}$ on $\Omega\setminus H$. We deduce that $\mathrm{cl}\,(\Gamma 1_H|\mathcal{H}) \subseteq 1_H \mathrm{cl}\,(\Gamma|\mathcal{H})$ a.s. and $\mathrm{cl}\,(\Gamma 1_H|\mathcal{H}) = \{0\}$ on $\Omega\setminus H$. Let us define $Z = \mathrm{cl}\,(\Gamma 1_H|\mathcal{H})1_H + \mathrm{cl}\,(\Gamma|\mathcal{H})1_{\Omega\setminus H}$. As $\Gamma \subseteq \Gamma 1_H$ on H, we deduce that $\Gamma \subseteq Z$ a.s. hence $\mathrm{cl}\,(\Gamma|\mathcal{H}) \subseteq Z$ a.s. Therefore, $\mathrm{cl}\,(\Gamma|\mathcal{H})1_H \subseteq Z 1_H$ and finally $\mathrm{cl}\,(\Gamma|\mathcal{H})1_H \subseteq \mathrm{cl}\,(\Gamma 1_H|\mathcal{H})$ as the latter set is $\{0\}$ on $\Omega \setminus H$. □

3. Application: Essential supremum as a pointwise supremum

Definition 3.1. *We say that a function f is sequentially lower semi-continuous (s.l.s.c.) on a domain $D \subseteq \mathbb{R}^d$ if for all convergent sequence $(x_n)_{n\geq 1}$ of D, we have $f(\liminf_n x_n) \leq \liminf_n f(x_n)$.*

Definition 3.2. *We say that a function f is sequentially upper semi-continuous (s.u.s.c.) on a domain $D \subseteq \mathbb{R}^d$ if for all convergent sequence $(x_n)_{n\geq 1}$ of D, we have $f(\limsup_n x_n) \geq \limsup_n f(x_n)$.*

Lemma 3.3. *Let f be sequentially lower semi-continuous (s.l.s.c.) on a domain $D \subseteq \mathbb{R}^d$. Let $x_0 \in D$ and $\epsilon > 0$. Then, there exists $r > 0$ such that for all $x \in D$ in the closed ball $B(x_0, r)$, we have $f(x_0) - f(x) \leq \epsilon$.*

Proof. Suppose by contradiction that for all $r > 0$, there exists $x \in B(x_0, r) \cap D$ such that $f(x_0) - f(x) > \epsilon$. Then, for all $n \geq 1$, there exists $x_n \in D$ with $|x_0 - x_n| \leq n^{-1}$ such that $f(x_0) > f(x_n) + \epsilon$. Therefore, $f(x_0) \geq \liminf_n f(x_n) + \epsilon$. As $\lim_n x_n = x_0$, this contradicts the lower semi-continuity of f. □

Corollary 3.4. *A function f is s.l.s.c. on a domain $D \subseteq \mathbb{R}^d$ if and only if it is lower semi continuous, i.e. for every $x \in D$, we have $f(x) = \liminf_{y \to x} f(x)$, where $\liminf_{y \to x} f(x) = \sup_{r>0} \inf_{z \in B(x,r) \cap D} f(z)$.*

Proof. Suppose that f is s.l.s.c. We have $\inf_{z \in B(x,r) \cap D} f(z) = \lim_n \downarrow f(z_n)$ where we may assume by a compactness argument that $z_n \to \hat{z}_r \in B(x, r)$. By the assumption, $f(\hat{z}_r) \leq \liminf_n f(z_n) \leq \inf_{z \in B(x,r) \cap D} f(z)$. We deduce that $\inf_{z \in B(x,r) \cap D} f(z) = f(\hat{z}_r)$. Similarly, with $r = k^{-1}$ and $k \in \mathbb{N}$, we may assume $\hat{z}_{k^{-1}} \to \hat{z} = x$ as $k \to \infty$. Therefore, $\liminf_{y \to x} f(x) = \lim_{r \to 0} \uparrow f(\hat{z}_r) \geq f(x)$ by assumption. Moreover, it is clear that $\inf_{z \in B(x,r) \cap D} f(z) \leq f(x)$ hence $\liminf_{y \to x} f(x) \leq f(x)$ so that the equality holds. Reciprocally, suppose that f is l.s.c. Then, consider a sequence $x_n \in D$ which converges to $x \in D$. By assumption, we know that $f(x) = \lim_{r \to 0} \inf_{z \in B(x,r) \cap D} f(z)$ for all $r > 0$. Moreover, for each $r > 0$, there exists n_r large enough such that $x_n \in B(x, r)$ if $n \geq n_r$. Therefore, $\inf_{z \in B(x,r) \cap D} f(z) \leq \inf_{n \geq n_r} f(x_n) \leq \liminf_n f(x_n)$. We conclude that $f(x) \leq \liminf_n f(x_n)$. □

Remark 3.5. *Consider the function $f = 1_{(1,\infty)}$. This function is continuous at any point $x \neq 1$. Moreover, if $x_n \to 1$, then $\liminf_n f(x_n) \geq f(1) = 0$. Therefore, f is l.s.c. Nevertheless, $x_n = 1 + n^{-1} \to 1$ but $f(1) < \liminf_n f(x_n) = 1$, i.e. the equality $f(\liminf_n f(x_n)) = \liminf_n f(x_n)$ does not hold in general for a countable sequence $(x_n)_{n \in \mathbb{N}}$.*

Remark 3.6. *Recall that the l.s.c. convex envelope $\overline{\text{co}}(f)$ of a function f dominating an affine function is defined as the largest function smaller than f which is both convex and l.s.c. It is well known that $\overline{\text{co}}(f)$ coincides with the supremum of the affine functions smaller than f, see [].*

Theorem 3.7. *Let $h(\omega, x)$, $x \in \mathbb{R}^d$, be a real-valued mapping such that h is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable and is l.s.c in x and let Γ be a closed $\mathcal{F}$-measurable set of $\mathbb{R}^d$. Then,*

$$\begin{aligned} \operatorname{ess\,sup}_{\mathcal{H}} \{h(\gamma) : \ \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} &= \sup_{x \in \text{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n), \text{ a.s.,} \\ &= \operatorname{ess\,sup}_{\mathcal{H}} \{h(\gamma), \ \gamma \in \mathcal{L}(\text{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})\}, \text{ a.s.,} \end{aligned}$$

where $(\gamma_n)_{n\in\mathbb{N}}$ *is a Castaing representation of* $\mathrm{cl}\,(\Gamma|\mathcal{H})$.

Proof. As $\mathrm{cl}\,(\Gamma|\mathcal{H})$ is $\mathcal{H}$-measurable and closed, it admits a Castaing representation $\mathrm{cl}\,(\Gamma|\mathcal{H}) = \mathrm{cl}\{\gamma_n : n \in \mathbf{N}\}$ where $\gamma_n \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})$ for all n. We first prove that

$$\sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n).$$

Let $(\gamma_n)_n \subset \mathrm{cl}\{\gamma_n : n \in \mathbf{N}\} = \mathrm{cl}\,(\Gamma|\mathcal{H})$, then $h(\gamma_n) \leq \sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$ and thus $\sup_n h(\gamma_n) \leq \sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$. Let $x \in \mathrm{cl}\,(\Gamma|\mathcal{H})$. Then by lower semi-continuity, $h(x) \leq \liminf_k h(\gamma_{n_k})$ for some random subsequence (γ_{n_k}) such that $\gamma_{n_k} \to x$. Thus $h(x) \leq \sup_n h(\gamma_n)$ and $\sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) \leq \sup_n h(\gamma_n)$ hence the equality holds.

Therefore, since h is $\mathcal{H}\otimes\mathcal{B}(\mathbb{R})$-measurable, $\sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n)$ is $\mathcal{H}$-measurable. Moreover, as $\Gamma \subseteq \mathrm{cl}\,(\Gamma|\mathcal{H})$ a.s., for every $\gamma \in \mathcal{L}(\Gamma, \mathcal{F})$, $h(\gamma) \leq \sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$ hence $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \leq \sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$ a.s. To show the reverse inequality, we first show the equality

$$\sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma),\ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

To see it, notice that

$$\sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n) \leq \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma),\ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

Moreover, it is trivial that $h(\gamma) \leq \sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$ for all $\gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})$. Since $\sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$ is $\mathcal{H}$-measurable, we deduce that $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma),\ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})\} \leq \sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$. Since the family $\{h(\gamma),\ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H})\}$ is directed upward, we also deduce that $\sup_{x\in\mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \lim_n \uparrow h(\gamma^n)$ where $\gamma^n \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})$.

At last, consider any $\mathcal{H}$-measurable selection γ of $\mathrm{cl}\,(\Gamma|\mathcal{H})$ and a deterministic sequence $\epsilon_n > 0$ with $\lim_n \epsilon_n = 0$. By Corollary 2.8, $\Lambda_n = \{\Gamma \cap B(\gamma, \epsilon_n) \neq \emptyset\} \in \mathcal{F}$ satisfies $P(\Lambda_n|\mathcal{H}) > 0$ a.s. Indeed, otherwise, on a non null $\mathcal{H}$-measurable set $\tilde{\Lambda}_n \subseteq \tilde{\Lambda}_n$, we have $P(\tilde{\Lambda}_n \cap \Lambda_n|\mathcal{H}) = 0$ hence $P((\tilde{\Lambda}_n \cap \Lambda_n) = 0$, i.e. a contradiction with Corollary 2.8. By a measurable selection argument, it is possible to construct $\hat{\gamma}_n \in L^0(\Gamma, \mathcal{F})$ such that $\hat{\gamma}_n \in B(\gamma, \epsilon_n)$ on Λ_n. We define $\hat{\Lambda}_n = \{\hat{\gamma}_n \in B(\gamma, \epsilon_n)\}$. Since $\Lambda_n \subseteq \hat{\Lambda}_n$, we have $P(\hat{\Lambda}_n|\mathcal{H}) > 0$ a.s.

Therefore,

$$\begin{aligned} \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} 1_{\hat{\Lambda}_n} &\geq h(\hat{\gamma}_n) 1_{\hat{\Lambda}_n}, \\ \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \mathrm{P}(\hat{\Lambda}_n|\mathcal{H}) &\geq \mathbb{E}\left(h(\hat{\gamma}_n) 1_{\hat{\Lambda}_n}|\mathcal{H}\right) \end{aligned} \tag{3.1}$$

The last inequality holds on a set $A^n \in \mathcal{H}$ such that $\mathrm{P}(A^n) = 1$. Moreover, there exists $B^n \in \mathcal{H}$ such that $P(B_n) = 1$ and

$$\mathbb{E}\left(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}\right) = \int_{\mathbb{R}^d} h(x)1_{x\in B(\gamma,\epsilon_n)}\mathrm{P}(\hat{\gamma}_n \in dx|\mathcal{H}).$$

To obtain this equality, it suffices to use a regular version of the conditional law of $\hat{\gamma}_n$. Moreover, we obtain the following pointwise inequality satisfied for each $\omega \in C_n = A_n \cap B_n$:

$$\begin{aligned}\int_{\mathbb{R}^d} h(x)1_{x\in B(\gamma,\epsilon_n)}\mathrm{P}(\hat{\gamma}_n \in dx|\mathcal{H}) &\geq \inf_{z\in B(\gamma,\epsilon_n)} h(z)\int_{\mathbb{R}^d} 1_{x\in B(\gamma,\epsilon_n)}\mathrm{P}(\hat{\gamma}_n \in dx|\mathcal{H}) \\ &\geq \inf_{z\in B(\gamma,\epsilon_n)} h(z)\mathrm{P}(\hat{\Lambda}_n|\mathcal{H}).\end{aligned}$$

Note that we do not need for the infinimum $\inf_{z\in B(\gamma,\epsilon_n)} h(z)$ to be $\mathcal{H}$-measurable as the integral above is considered for each fixed ω with respect to x and, precisely, $\inf_{z\in B(\gamma,\epsilon_n)} h(z)$ does not depend on x. Since $\mathrm{P}(\hat{\Lambda}_n|\mathcal{H}) > 0$, we then deduce by (3.1) that $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma,\mathcal{F})\} \geq \inf_{z\in B(\gamma,\epsilon_n)} h(z)$ for every $n \geq 1$ a.s. Therefore, $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma,\mathcal{F})\} \geq \lim_n \inf_{z\in B(\gamma,\epsilon_n)} h(z)$ a.s. Since, by compactness and lower semi-continuity, $\inf_{z\in B(\gamma,\epsilon_n)} h(z) = h(z_n)$ where $z_n \in B(\gamma,\epsilon_n)$ converges pointwise to γ as $n \to \infty$, we finally deduce that $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma,\mathcal{F})\} \geq \liminf_n h(z_n) \geq h(\gamma)$ by lower semi-continuity. This inequality holds for any $\mathcal{H}$-measurable selection of $\mathrm{cl}\,(\Gamma|\mathcal{H})$. Therefore, $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma,\mathcal{F})\} \geq \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma),\ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}),\mathcal{H})\}$. The conclusion of the lemma follows. □

Recall that a set Λ of measurable random variables is said $\mathcal{F}$-decomposable if for any finite partition $(F_i)_{i=1,\cdots,n} \subseteq \mathcal{F}$ of Ω, and for every family $(\gamma_i)_{i=1,\cdots,n}$ of Λ, we have $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$. In the following, we denote by $\Sigma(\Lambda)$ the $\mathcal{F}$-decomposable envelop of Λ, i.e. the smallest $\mathcal{F}$-decomposable family containing Λ. Notice that

$$\Sigma(\Lambda) = \left\{\sum_{i=1}^n \gamma_i 1_{F_i} :\ n \geq 1,\ (\gamma_i)_{i=1,\cdots,n} \subseteq \Lambda,\ (F_i)_{i=1,\cdots,n} \subseteq \mathcal{F}\,\text{s.t.}\,\sum_{i=1}^n F_i = \Omega\right\}. \quad (3.2)$$

The closure $\overline{\Sigma}(\Lambda)$ in probability of $\Sigma(\Lambda)$ is decomposable even if Λ is not decomposable. By [1, ?], there exists a $\mathcal{F}$-measurable closed random set $\sigma(\Lambda)$ such that $\overline{\Sigma}(\Lambda) = L^0(\sigma(\Lambda),\mathcal{F})$ is the set of all measurable selectors of $\sigma(\Lambda)$. If Γ is a $\mathcal{F}$-measurable random set, then $\Lambda = L^0(\Gamma,\mathcal{F})$ is $\mathcal{F}$-measurable. Therefore, $\overline{\Sigma}(\Lambda) = L^0(\overline{\Gamma},\mathcal{F})$ by [1, ?].

Corollary 3.8. *Let $h(\omega, x)$, $x \in \mathbb{R}^d$, be a real-valued mapping such that h is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable and is l.s.c in x. Let us consider a family Λ of measurable random variables so that $\overline{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$ is the set of all measurable selectors of some $\mathcal{F}$-measurable closed random set $\sigma(\Lambda)$. Then,*

$$\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \Lambda\} = \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \overline{\Sigma}(\Lambda)\} = \sup_{x \in \mathrm{cl}\,(\sigma(\Lambda)|\mathcal{H})} h(x).$$

Proof. Notice that for any finite partition $(F_i)_{i=1,\cdots,n} \subseteq \mathcal{F}$ of Ω, $n \geq 1$, and for every family $(\gamma_i)_{i=1,\cdots,n}$ of Λ, we have

$$h(\sum_{i=1}^{n} \gamma_i 1_{F_i}) = \sum_{i=1}^{n} h(\gamma_i) 1_{F_i}.$$

Therefore, as $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Lambda$, we deduce that $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Sigma(\Lambda)$, and finally for all $\gamma \in \overline{\Sigma}(\Lambda)$ since h is l.s.c. We deduce that $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \Lambda\} \geq \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \overline{\Sigma}(\Lambda)\}$ and the equality holds since $\Lambda \subseteq \overline{\Sigma}(\Lambda)$. The last statement is deduced from Theorem 3.7. □

References

[1] Lépinette, E. and Molchanov I. Conditional cores and conditional convex hulls of random sets. https://arxiv.org/abs/1711.10303