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Random set conditioning: conditional interior and conditional closure

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Abstract: In this short note, we present two new concepts. On a complete probability space, we consider two σ -algebras $\mathcal{H} \subseteq \mathcal{F}$ and a \mathcal{F} -graph-measurable random set $\Gamma \subseteq \mathbb{R}^d$. We show the existence of a largest \mathcal{H} -measurable open set contained in X , we call conditional interior and a smallest \mathcal{H} -measurable closed set containing X , we call conditional closure. We then deduce that a conditional essential supremum of real-valued random variables is actually a pointwise supremum over a closed random set.

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1. Introduction

On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let \mathcal{H} be a sub σ -algebra of \mathcal{F} . In this section, we generalise the concept of conditional support of a random variable to random sets, i.e. we construct the conditional closure of a \mathcal{F} -measurable random set Γ as the smallest closed \mathcal{H} -measurable set $\text{cl}(\Gamma|\mathcal{H})$ which contains Γ a.s. When Γ is a singleton $\{X\}$ composed of a \mathcal{F} -measurable random variable X , $\text{cl}(\Gamma|\mathcal{H})$ coincides with the conditional support of X . To do so, we first construct the conditional interior $\mathbf{o}(\Gamma|\mathcal{H})$ of Γ such that $\text{cl}(\Gamma|\mathcal{H})$ is the complement of $\mathbf{o}(\Gamma^c|\mathcal{H})$.

2. Random set conditioning

Definition 2.1. *Let (T, \mathcal{T}) be a topological space. A random set Γ is a set-valued mapping that assigns to each $\omega \in \Omega$ a subset $\Gamma(\omega)$ of T . We say that*

Γ is \mathcal{H} -measurable if

$$\text{graph } \Gamma := \{(\omega, x) : x \in \Gamma(\omega)\} \in \mathcal{H} \otimes \mathcal{T}.$$

When Γ is closed (resp. open) almost surely, we say that Γ is closed (resp. open).

Definition 2.2. Let (T, \mathcal{T}) be a topological space and $\Gamma : \Omega \rightarrow T$, be a random set. We say that the \mathcal{F} -measurable random variable $\xi : \Omega \rightarrow T$ is an \mathcal{F} -measurable selection of Γ if $\xi(\omega) \in \Gamma(\omega)$ for almost all $\omega \in \Omega$. The set of such selections is denoted by $\mathcal{L}(\Gamma, \mathcal{F})$.

We recall that any non empty \mathcal{H} -measurable closed random set Γ admits a Castaing representation $(\gamma_n)_{n \geq 1} \subseteq L^0(\Gamma, \mathcal{H})$ such that $\Gamma(\omega) = \text{cl} \{\gamma_n(\omega) : n \geq 1\}$ a.s., see [1].

In the following, if $x \in \mathbb{R}^d$ and $r \geq 0$, $B(x, r)$ denotes the open ball in \mathbb{R}^d of center x and radius r while $\bar{B}(x, r)$ is its closure. For any $A \subseteq \mathbb{R}^d$ and $\lambda \in \mathbb{R}$, we use the convention that $\lambda \times A = \{\lambda a : a \in A\}$. In particular, $0 \times A = \{0\}$. Recall the following definition, see [1]:

Definition 2.3. The \mathcal{H} -core (also called conditional core), $\mathbf{m}(\Gamma|\mathcal{H})$, of a set-valued mapping Γ is the largest \mathcal{H} -measurable random set Γ' such that $\Gamma' \subseteq \Gamma$ a.s.

In the case where Γ is closed, then the conditional core $\mathbf{m}(\Gamma|\mathcal{H})$ exists [1] and $\mathcal{L}(\mathbf{m}(\Gamma|\mathcal{H}), \mathcal{H}) = \mathcal{L}(\Gamma, \mathcal{H})$ in the case where $\mathcal{L}(\Gamma, \mathcal{H}) \neq \emptyset$. If $\gamma \in \mathcal{L}(\mathbb{R}^d, \mathcal{H})$ and $r \in \mathcal{L}(\mathbb{R}_+, \mathcal{F})$, it is easy to see that $\mathbf{m}(\bar{B}(\gamma, r)|\mathcal{H}) = \bar{B}(\gamma, \text{ess inf}_{\mathcal{H}} r)$ where $\text{ess inf}_{\mathcal{H}} r$ designates the conditional essential infimum of r knowing \mathcal{H} . Up to a negligible set, this is the largest \mathcal{H} -measurable random variable which is smaller than r , see [1]. Notice that the conditional core does not depend on the σ -algebra \mathcal{F} so that we may choose \mathcal{F} as the largest one composed of all the subparts of Ω . We introduce an open version of the conditional core:

Definition 2.4. The \mathcal{H} -measurable interior $\mathbf{o}(\Gamma|\mathcal{H})$ of a set-valued mapping Γ is the largest open and \mathcal{H} -measurable random set Γ' such that $\Gamma' \subseteq \Gamma$ a.s. if existence holds.

Recall the following:

Lemma 2.5. Let \mathcal{O} be an open set in a normed space. For every $x \in \mathcal{O}$, $B(x, d(x, \partial\mathcal{O})) \subseteq \mathcal{O}$.

Proof. Let us consider $r^* = \sup R$ where R is the non empty set of all $r > 0$ such that $B(x, r) \subseteq \mathcal{O}$. It is trivial that $B(x, r^*) \subseteq \mathcal{O}$. So, for all $o \in \partial\mathcal{O}$,

$d(x, o) \geq r^*$ hence $r^* \leq d(x, \partial\mathcal{O})$. Moreover, by definition of r^* , for all n there exists $z^n \in B(x, r^* + n^{-1})$ such that $z^n \notin \mathcal{O}$ hence $r^* \leq \|z^n - x\| \leq r^* + n^{-1}$. As the sequence $(z_n)_n$ is bounded, we deduce by a compactness argument that for a subsequence $z_n \rightarrow z$ as $n \rightarrow \infty$. Then, $\|z - x\| = r^*$ hence $z \in \text{cl } \mathcal{O}$. In the case where $z \in \mathcal{O}$, $z^n \in \mathcal{O}$ for n large enough since \mathcal{O} is open, which yields a contradiction. Therefore, $z \in \partial\mathcal{O}$. This implies that $r^* = d(x, z) \geq d(x, \partial\mathcal{O})$ and finally $r^* = d(x, \partial\mathcal{O})$. \square

Theorem 2.6. *Let us consider a \mathcal{F} -measurable open random set \mathcal{O} . Then, the \mathcal{H} -graph-measurable interior of \mathcal{O} exists.*

Proof. By [1, Proposition 2.6], $\text{cl } \mathcal{O}$ and $\partial\mathcal{O} := \text{cl } \mathcal{O} \setminus \mathcal{O}$ are closed \mathcal{F} -measurable random sets. When $\partial\mathcal{O} = \emptyset$, we set $d(x, \partial\mathcal{O}) = \infty$. Otherwise, since $\partial\mathcal{O}$ admits a Castaing representation on $\{\partial\mathcal{O} \neq \emptyset\}$, we deduce that the random continuous mapping $x \mapsto d(x, \partial\mathcal{O})$ is \mathcal{F} -measurable. We deduce that the sets $F^n := \{x : d(x, \partial\mathcal{O}) \geq 1/n\} \cap \text{cl } \mathcal{O}$ are closed \mathcal{F} -measurable random subsets of \mathcal{O} and we have $\mathcal{O} = \cup_n F^n$. Let us define the \mathcal{H} -measurable open random set

$$\mathbf{o}(\mathcal{O}|\mathcal{H}) = \text{int}(\cup_n \mathbf{m}(F^n|\mathcal{H})) \subseteq \mathcal{O},$$

and let us show that this is the largest \mathcal{H} -measurable open set of \mathcal{O} . To do so, let $\mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ be a \mathcal{H} -measurable open subset of \mathcal{O} . As previously, we may write $\mathcal{O}_{\mathcal{H}} = \cup_n H^n$ where $H^n := \{x : d(x, \partial\mathcal{O}_{\mathcal{H}}) \geq 1/n\} \cap \text{cl } \mathcal{O}_{\mathcal{H}}$ are \mathcal{H} -measurable closed subsets of $\mathcal{O}_{\mathcal{H}}$.

We claim that $H^n \subseteq F^n$. Indeed, if $x \in H^n$, it suffices to show that $d(x, \partial\mathcal{O}_{\mathcal{H}}) \leq d(x, \partial\mathcal{O})$ a.s. In the contrary case, on a non null set, there exists $o \in \partial\mathcal{O}$ such that $o \in B(x, d(x, \partial\mathcal{O}_{\mathcal{H}}))$. By Lemma 2.5, this implies that $o \in \mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ hence a contradiction.

As H^n is \mathcal{H} -measurable and closed, $H^n \subseteq \mathbf{m}(F^n|\mathcal{H})$ for all n . We deduce that $\mathcal{O}_{\mathcal{H}} \subseteq \cup_n \mathbf{m}(F^n|\mathcal{H})$ hence $\mathcal{O}_{\mathcal{H}} \subseteq \mathbf{o}(\mathcal{O}|\mathcal{H})$. \square

Corollary 2.7. *For any \mathcal{F} -measurable random set Γ , the \mathcal{H} -measurable interior of Γ exists and $\mathbf{o}(\Gamma|\mathcal{H}) = \mathbf{o}(\text{int } \Gamma|\mathcal{H})$.*

Corollary 2.8. *For any \mathcal{F} -measurable random set Γ , there exists a smallest \mathcal{H} -measurable closed random set containing Γ a.s. denoted by $\text{cl}(\Gamma|\mathcal{H})$ and called the \mathcal{H} -measurable conditional closure of Γ . We have*

$$\text{cl}(\Gamma|\mathcal{H}) = \mathbb{R}^d \setminus \mathbf{o}(\mathbb{R}^d \setminus \Gamma|\mathcal{H}).$$

Moreover, for all $\gamma \in \text{cl}(\Gamma|\mathcal{H})$ and $\epsilon \in \mathcal{L}((0, \infty), \mathcal{H})$, for all $H \in \mathcal{H}$ such that $\mathbb{P}(H) > 0$, $\mathbb{P}(\{\Gamma \cap B(\gamma, \epsilon) \neq \emptyset\} \cap H) > 0$.

Proof. The first part is a direct consequence of Corollary 2.7. Suppose that $P(\{\Gamma \cap B(\gamma, \epsilon) \neq \emptyset\} \cap H) = 0$. Therefore, by definition of the conditional support as a smallest element,

$$\text{cl}(\Gamma|\mathcal{H}) = \text{cl}(\Gamma|\mathcal{H})1_{\Omega \setminus H} + \text{cl}(\Gamma|\mathcal{H}) \cap (\mathbb{R}^d \setminus B(\gamma, \epsilon))1_H.$$

Indeed, the \mathcal{H} -measurable set above contains Γ by assumption hence it contains $\text{cl}(\Gamma|\mathcal{H})$. We get a contradiction since $\gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$ a.s. by assumption. \square

Corollary 2.9. *Let $X \in \mathcal{L}(\mathbb{R}^d, \mathcal{F})$. There exists a smallest \mathcal{H} -measurable closed random set denoted by $\text{supp}_{\mathcal{H}}(X)$ with $P(X \in \text{supp}_{\mathcal{H}}(X)) = 1$. This is the \mathcal{H} -conditional support of X . It is given by $\text{supp}_{\mathcal{H}}(X) = \text{cl}(\{X\}|\mathcal{H})$. Moreover, for all $\gamma \in \mathcal{L}(\text{supp}_{\mathcal{H}}(X), \mathcal{H})$ and $\epsilon \in \mathcal{L}((0, \infty), \mathcal{H})$, for all $H \in \mathcal{H}$ such that $P(H) > 0$, $P(\{X \in B(\gamma, \epsilon)\} \cap H) > 0$.*

Notice that the conditional closure of a non empty set is necessarily non empty a.s. hence admits a measurable selection. In the following, we adopt the following notation $kA = \{ka : a \in A\}$ for any subset $A \subseteq \mathbb{R}^d$ and $k \in A$.

Lemma 2.10. *Let $H \in \mathcal{H}$. Then, for every \mathcal{F} -measurable set Γ , $\text{cl}(\Gamma 1_H|\mathcal{H}) = 1_H \text{cl}(\Gamma|\mathcal{H})$.*

Proof. First observe that $P(\Gamma 1_H \subseteq 1_H \text{cl}(\Gamma|\mathcal{H})) = 1$ as $1_H \text{cl}(\Gamma|\mathcal{H}) = \{0\}$ on $\Omega \setminus H$. We deduce that $\text{cl}(\Gamma 1_H|\mathcal{H}) \subseteq 1_H \text{cl}(\Gamma|\mathcal{H})$ a.s. and $\text{cl}(\Gamma 1_H|\mathcal{H}) = \{0\}$ on $\Omega \setminus H$. Let us define $Z = \text{cl}(\Gamma 1_H|\mathcal{H})1_H + \text{cl}(\Gamma|\mathcal{H})1_{\Omega \setminus H}$. As $\Gamma \subseteq \Gamma 1_H$ on H , we deduce that $\Gamma \subseteq Z$ a.s. hence $\text{cl}(\Gamma|\mathcal{H}) \subseteq Z$ a.s. Therefore, $\text{cl}(\Gamma|\mathcal{H})1_H \subseteq Z 1_H$ and finally $\text{cl}(\Gamma|\mathcal{H})1_H \subseteq \text{cl}(\Gamma 1_H|\mathcal{H})$ as the latter set is $\{0\}$ on $\Omega \setminus H$. \square

3. Application: Essential supremum as a pointwise supremum

Definition 3.1. *We say that a function f is sequentially lower semi-continuous (s.l.s.c.) on a domain $D \subseteq \mathbb{R}^d$ if for all convergent sequence $(x_n)_{n \geq 1}$ of D , we have $f(\liminf_n x_n) \leq \liminf_n f(x_n)$.*

Definition 3.2. *We say that a function f is sequentially upper semi-continuous (s.u.s.c.) on a domain $D \subseteq \mathbb{R}^d$ if for all convergent sequence $(x_n)_{n \geq 1}$ of D , we have $f(\limsup_n x_n) \geq \limsup_n f(x_n)$.*

Lemma 3.3. *Let f be sequentially lower semi-continuous (s.l.s.c.) on a domain $D \subseteq \mathbb{R}^d$. Let $x_0 \in D$ and $\epsilon > 0$. Then, there exists $r > 0$ such that for all $x \in D$ in the closed ball $B(x_0, r)$, we have $f(x_0) - f(x) \leq \epsilon$.*

Proof. Suppose by contradiction that for all $r > 0$, there exists $x \in B(x_0, r) \cap D$ such that $f(x_0) - f(x) > \epsilon$. Then, for all $n \geq 1$, there exists $x_n \in D$ with $|x_0 - x_n| \leq n^{-1}$ such that $f(x_0) > f(x_n) + \epsilon$. Therefore, $f(x_0) \geq \liminf_n f(x_n) + \epsilon$. As $\lim_n x_n = x_0$, this contradicts the lower semi-continuity of f . \square

Corollary 3.4. *A function f is s.l.s.c. on a domain $D \subseteq \mathbb{R}^d$ if and only if it is lower semi continuous, i.e. for every $x \in D$, we have $f(x) = \liminf_{y \rightarrow x} f(y)$, where $\liminf_{y \rightarrow x} f(y) = \sup_{r > 0} \inf_{z \in B(x, r) \cap D} f(z)$.*

Proof. Suppose that f is s.l.s.c. We have $\inf_{z \in B(x, r) \cap D} f(z) = \lim_n \downarrow f(z_n)$ where we may assume by a compactness argument that $z_n \rightarrow \hat{z}_r \in B(x, r)$. By the assumption, $f(\hat{z}_r) \leq \liminf_n f(z_n) \leq \inf_{z \in B(x, r) \cap D} f(z)$. We deduce that $\inf_{z \in B(x, r) \cap D} f(z) = f(\hat{z}_r)$. Similarly, with $r = k^{-1}$ and $k \in \mathbb{N}$, we may assume $\hat{z}_{k^{-1}} \rightarrow \hat{z} = x$ as $k \rightarrow \infty$. Therefore, $\liminf_{y \rightarrow x} f(y) = \lim_{r \rightarrow 0} \uparrow f(\hat{z}_r) \geq f(x)$ by assumption. Moreover, it is clear that $\inf_{z \in B(x, r) \cap D} f(z) \leq f(x)$ hence $\liminf_{y \rightarrow x} f(y) \leq f(x)$ so that the equality holds. Reciprocally, suppose that f is l.s.c. Then, consider a sequence $x_n \in D$ which converges to $x \in D$. By assumption, we know that $f(x) = \lim_{r \rightarrow 0} \inf_{z \in B(x, r) \cap D} f(z)$ for all $r > 0$. Moreover, for each $r > 0$, there exists n_r large enough such that $x_n \in B(x, r)$ if $n \geq n_r$. Therefore, $\inf_{z \in B(x, r) \cap D} f(z) \leq \inf_{n \geq n_r} f(x_n) \leq \liminf_n f(x_n)$. We conclude that $f(x) \leq \liminf_n f(x_n)$. \square

Remark 3.5. *Consider the function $f = 1_{(1, \infty)}$. This function is continuous at any point $x \neq 1$. Moreover, if $x_n \rightarrow 1$, then $\liminf_n f(x_n) \geq f(1) = 0$. Therefore, f is l.s.c. Nevertheless, $x_n = 1 + n^{-1} \rightarrow 1$ but $f(1) < \liminf_n f(x_n) = 1$, i.e. the equality $f(\liminf_n f(x_n)) = \liminf_n f(x_n)$ does not hold in general for a countable sequence $(x_n)_{n \in \mathbb{N}}$.*

Remark 3.6. *Recall that the l.s.c. convex envelope $\overline{\text{co}}(f)$ of a function f dominating an affine function is defined as the largest function smaller than f which is both convex and l.s.c. It is well known that $\overline{\text{co}}(f)$ coincides with the supremum of the affine functions smaller than f , see [].*

Theorem 3.7. *Let $h(\omega, x)$, $x \in \mathbb{R}^d$, be a real-valued mapping such that h is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable and is l.s.c in x and let Γ be a closed \mathcal{F} -measurable set of \mathbb{R}^d . Then,*

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} &= \sup_{x \in \text{cl}(\Gamma | \mathcal{H})} h(x) = \sup_n h(\gamma_n), \text{ a.s.}, \\ &= \text{ess sup}_{\mathcal{H}} \{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma | \mathcal{H}), \mathcal{H})\}, \text{ a.s.}, \end{aligned}$$

where $(\gamma_n)_{n \in \mathbf{N}}$ is a Castaing representation of $\text{cl}(\Gamma|\mathcal{H})$.

Proof. As $\text{cl}(\Gamma|\mathcal{H})$ is \mathcal{H} -measurable and closed, it admits a Castaing representation $\text{cl}(\Gamma|\mathcal{H}) = \text{cl}\{\gamma_n : n \in \mathbf{N}\}$ where $\gamma_n \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$ for all n . We first prove that

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n).$$

Let $(\gamma_n)_n \subset \text{cl}\{\gamma_n : n \in \mathbf{N}\} = \text{cl}(\Gamma|\mathcal{H})$, then $h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ and thus $\sup_n h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$. Let $x \in \text{cl}(\Gamma|\mathcal{H})$. Then by lower semi-continuity, $h(x) \leq \liminf_k h(\gamma_{n_k})$ for some random subsequence (γ_{n_k}) such that $\gamma_{n_k} \rightarrow x$. Thus $h(x) \leq \sup_n h(\gamma_n)$ and $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) \leq \sup_n h(\gamma_n)$ hence the equality holds.

Therefore, since h is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable, $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n)$ is \mathcal{H} -measurable. Moreover, as $\Gamma \subseteq \text{cl}(\Gamma|\mathcal{H})$ a.s., for every $\gamma \in \mathcal{L}(\Gamma, \mathcal{F})$, $h(\gamma) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ hence $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ a.s. To show the reverse inequality, we first show the equality

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

To see it, notice that

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n) \leq \text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

Moreover, it is trivial that $h(\gamma) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ for all $\gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$. Since $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ is \mathcal{H} -measurable, we deduce that $\text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\} \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$. Since the family $\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}$ is directed upward, we also deduce that $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \lim_n \uparrow h(\gamma^n)$ where $\gamma^n \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$.

At last, consider any \mathcal{H} -measurable selection γ of $\text{cl}(\Gamma|\mathcal{H})$ and a deterministic sequence $\epsilon_n > 0$ with $\lim_n \epsilon_n = 0$. By Corollary 2.8, $\Lambda_n = \{\Gamma \cap B(\gamma, \epsilon_n) \neq \emptyset\} \in \mathcal{F}$ satisfies $P(\Lambda_n|\mathcal{H}) > 0$ a.s. Indeed, otherwise, on a non null \mathcal{H} -measurable set $\tilde{\Lambda}_n \subseteq \Lambda_n$, we have $P(\tilde{\Lambda}_n \cap \Lambda_n|\mathcal{H}) = 0$ hence $P((\tilde{\Lambda}_n \cap \Lambda_n) = 0)$, i.e. a contradiction with Corollary 2.8. By a measurable selection argument, it is possible to construct $\hat{\gamma}_n \in L^0(\Gamma, \mathcal{F})$ such that $\hat{\gamma}_n \in B(\gamma, \epsilon_n)$ on Λ_n . We define $\hat{\Lambda}_n = \{\hat{\gamma}_n \in B(\gamma, \epsilon_n)\}$. Since $\Lambda_n \subseteq \hat{\Lambda}_n$, we have $P(\hat{\Lambda}_n|\mathcal{H}) > 0$ a.s.

Therefore,

$$\begin{aligned} \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\}1_{\hat{\Lambda}_n} &\geq h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}, \\ \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\}P(\hat{\Lambda}_n|\mathcal{H}) &\geq \mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}) \end{aligned} \quad (3.1)$$

The last inequality holds on a set $A^n \in \mathcal{H}$ such that $P(A^n) = 1$. Moreover, there exists $B^n \in \mathcal{H}$ such that $P(B^n) = 1$ and

$$\mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}) = \int_{\mathbb{R}^d} h(x)1_{x \in B(\gamma, \epsilon_n)}P(\hat{\gamma}_n \in dx|\mathcal{H}).$$

To obtain this equality, it suffices to use a regular version of the conditional law of $\hat{\gamma}_n$. Moreover, we obtain the following pointwise inequality satisfied for each $\omega \in C_n = A_n \cap B_n$:

$$\begin{aligned} \int_{\mathbb{R}^d} h(x)1_{x \in B(\gamma, \epsilon_n)}P(\hat{\gamma}_n \in dx|\mathcal{H}) &\geq \inf_{z \in B(\gamma, \epsilon_n)} h(z) \int_{\mathbb{R}^d} 1_{x \in B(\gamma, \epsilon_n)}P(\hat{\gamma}_n \in dx|\mathcal{H}) \\ &\geq \inf_{z \in B(\gamma, \epsilon_n)} h(z)P(\hat{\Lambda}_n|\mathcal{H}). \end{aligned}$$

Note that we do not need for the infimum $\inf_{z \in B(\gamma, \epsilon_n)} h(z)$ to be \mathcal{H} -measurable as the integral above is considered for each fixed ω with respect to x and, precisely, $\inf_{z \in B(\gamma, \epsilon_n)} h(z)$ does not depend on x . Since $P(\hat{\Lambda}_n|\mathcal{H}) > 0$, we then deduce by (3.1) that $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \inf_{z \in B(\gamma, \epsilon_n)} h(z)$ for every $n \geq 1$ a.s. Therefore, $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \lim_n \inf_{z \in B(\gamma, \epsilon_n)} h(z)$ a.s. Since, by compactness and lower semi-continuity, $\inf_{z \in B(\gamma, \epsilon_n)} h(z) = h(z_n)$ where $z_n \in B(\gamma, \epsilon_n)$ converges pointwise to γ as $n \rightarrow \infty$, we finally deduce that $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \liminf_n h(z_n) \geq h(\gamma)$ by lower semi-continuity. This inequality holds for any \mathcal{H} -measurable selection of $\text{cl}(\Gamma|\mathcal{H})$. Therefore, $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}$. The conclusion of the lemma follows. \square

Recall that a set Λ of measurable random variables is said \mathcal{F} -decomposable if for any finite partition $(F_i)_{i=1, \dots, n} \subseteq \mathcal{F}$ of Ω , and for every family $(\gamma_i)_{i=1, \dots, n}$ of Λ , we have $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$. In the following, we denote by $\Sigma(\Lambda)$ the \mathcal{F} -decomposable envelop of Λ , i.e. the smallest \mathcal{F} -decomposable family containing Λ . Notice that

$$\Sigma(\Lambda) = \left\{ \sum_{i=1}^n \gamma_i 1_{F_i} : n \geq 1, (\gamma_i)_{i=1, \dots, n} \subseteq \Lambda, (F_i)_{i=1, \dots, n} \subseteq \mathcal{F} \text{ s.t. } \sum_{i=1}^n F_i = \Omega \right\}. \quad (3.2)$$

The closure $\bar{\Sigma}(\Lambda)$ in probability of $\Sigma(\Lambda)$ is decomposable even if Λ is not decomposable. By [1, ?], there exists a \mathcal{F} -measurable closed random set $\sigma(\Lambda)$ such that $\bar{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$ is the set of all measurable selectors of $\sigma(\Lambda)$. If Γ is a \mathcal{F} -measurable random set, then $\Lambda = L^0(\Gamma, \mathcal{F})$ is \mathcal{F} -measurable. Therefore, $\bar{\Sigma}(\Lambda) = L^0(\bar{\Gamma}, \mathcal{F})$ by [1, ?].

Corollary 3.8. *Let $h(\omega, x)$, $x \in \mathbb{R}^d$, be a real-valued mapping such that h is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable and is l.s.c in x . Let us consider a family Λ of measurable random variables so that $\bar{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$ is the set of all measurable selectors of some \mathcal{F} -measurable closed random set $\sigma(\Lambda)$. Then,*

$$\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} = \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \bar{\Sigma}(\Lambda)\} = \sup_{x \in \text{cl}(\sigma(\Lambda)|\mathcal{H})} h(x).$$

Proof. Notice that for any finite partition $(F_i)_{i=1, \dots, n} \subseteq \mathcal{F}$ of Ω , $n \geq 1$, and for every family $(\gamma_i)_{i=1, \dots, n}$ of Λ , we have

$$h\left(\sum_{i=1}^n \gamma_i 1_{F_i}\right) = \sum_{i=1}^n h(\gamma_i) 1_{F_i}.$$

Therefore, as $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Lambda$, we deduce that $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Sigma(\Lambda)$, and finally for all $\gamma \in \bar{\Sigma}(\Lambda)$ since h is l.s.c. We deduce that $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \bar{\Sigma}(\Lambda)\}$ and the equality holds since $\Lambda \subseteq \bar{\Sigma}(\Lambda)$. The last statement is deduced from Theorem 3.7. \square

References

- [1] Lépinette, E. and Molchanov I. Conditional cores and conditional convex hulls of random sets. <https://arxiv.org/abs/1711.10303>