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# Random set conditioning: conditional interior and conditional closure

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**Abstract:** In this short note, we present two new concepts. On a complete probability space, we consider two  $\sigma$ -algebras  $\mathcal{H} \subseteq \mathcal{F}$  and a  $\mathcal{F}$ -graph-measurable random set  $\Gamma \subseteq \mathbb{R}^d$ . We show the existence of a largest  $\mathcal{H}$ -measurable open set contained in  $X$ , we call conditional interior and a smallest  $\mathcal{H}$ -measurable closed set containing  $X$ , we call conditional closure. We then deduce that a conditional essential supremum of real-valued random variables is actually a pointwise supremum over a closed random set.

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## 1. Introduction

On a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , let  $\mathcal{H}$  be a sub  $\sigma$ -algebra of  $\mathcal{F}$ . In this section, we generalise the concept of conditional support of a random variable to random sets, i.e. we construct the conditional closure of a  $\mathcal{F}$ -measurable random set  $\Gamma$  as the smallest closed  $\mathcal{H}$ -measurable set  $\text{cl}(\Gamma|\mathcal{H})$  which contains  $\Gamma$  a.s. When  $\Gamma$  is a singleton  $\{X\}$  composed of a  $\mathcal{F}$ -measurable random variable  $X$ ,  $\text{cl}(\Gamma|\mathcal{H})$  coincides with the conditional support of  $X$ . To do so, we first construct the conditional interior  $\mathbf{o}(\Gamma|\mathcal{H})$  of  $\Gamma$  such that  $\text{cl}(\Gamma|\mathcal{H})$  is the complement of  $\mathbf{o}(\Gamma^c|\mathcal{H})$ .

## 2. Random set conditioning

**Definition 2.1.** *Let  $(T, \mathcal{T})$  be a topological space. A random set  $\Gamma$  is a set-valued mapping that assigns to each  $\omega \in \Omega$  a subset  $\Gamma(\omega)$  of  $T$ . We say that*

$\Gamma$  is  $\mathcal{H}$ -measurable if

$$\text{graph } \Gamma := \{(\omega, x) : x \in \Gamma(\omega)\} \in \mathcal{H} \otimes \mathcal{T}.$$

When  $\Gamma$  is closed (resp. open) almost surely, we say that  $\Gamma$  is closed (resp. open).

**Definition 2.2.** Let  $(T, \mathcal{T})$  be a topological space and  $\Gamma : \Omega \rightarrow T$ , be a random set. We say that the  $\mathcal{F}$ -measurable random variable  $\xi : \Omega \rightarrow T$  is an  $\mathcal{F}$ -measurable selection of  $\Gamma$  if  $\xi(\omega) \in \Gamma(\omega)$  for almost all  $\omega \in \Omega$ . The set of such selections is denoted by  $\mathcal{L}(\Gamma, \mathcal{F})$ .

We recall that any non empty  $\mathcal{H}$ -measurable closed random set  $\Gamma$  admits a Castaing representation  $(\gamma_n)_{n \geq 1} \subseteq L^0(\Gamma, \mathcal{H})$  such that  $\Gamma(\omega) = \text{cl} \{\gamma_n(\omega) : n \geq 1\}$  a.s., see [1].

In the following, if  $x \in \mathbb{R}^d$  and  $r \geq 0$ ,  $B(x, r)$  denotes the open ball in  $\mathbb{R}^d$  of center  $x$  and radius  $r$  while  $\bar{B}(x, r)$  is its closure. For any  $A \subseteq \mathbb{R}^d$  and  $\lambda \in \mathbb{R}$ , we use the convention that  $\lambda \times A = \{\lambda a : a \in A\}$ . In particular,  $0 \times A = \{0\}$ . Recall the following definition, see [1]:

**Definition 2.3.** The  $\mathcal{H}$ -core (also called conditional core),  $\mathbf{m}(\Gamma|\mathcal{H})$ , of a set-valued mapping  $\Gamma$  is the largest  $\mathcal{H}$ -measurable random set  $\Gamma'$  such that  $\Gamma' \subseteq \Gamma$  a.s.

In the case where  $\Gamma$  is closed, then the conditional core  $\mathbf{m}(\Gamma|\mathcal{H})$  exists [1] and  $\mathcal{L}(\mathbf{m}(\Gamma|\mathcal{H}), \mathcal{H}) = \mathcal{L}(\Gamma, \mathcal{H})$  in the case where  $\mathcal{L}(\Gamma, \mathcal{H}) \neq \emptyset$ . If  $\gamma \in \mathcal{L}(\mathbb{R}^d, \mathcal{H})$  and  $r \in \mathcal{L}(\mathbb{R}_+, \mathcal{F})$ , it is easy to see that  $\mathbf{m}(\bar{B}(\gamma, r)|\mathcal{H}) = \bar{B}(\gamma, \text{ess inf}_{\mathcal{H}} r)$  where  $\text{ess inf}_{\mathcal{H}} r$  designates the conditional essential infimum of  $r$  knowing  $\mathcal{H}$ . Up to a negligible set, this is the largest  $\mathcal{H}$ -measurable random variable which is smaller than  $r$ , see [1]. Notice that the conditional core does not depend on the  $\sigma$ -algebra  $\mathcal{F}$  so that we may choose  $\mathcal{F}$  as the largest one composed of all the subparts of  $\Omega$ . We introduce an open version of the conditional core:

**Definition 2.4.** The  $\mathcal{H}$ -measurable interior  $\mathbf{o}(\Gamma|\mathcal{H})$  of a set-valued mapping  $\Gamma$  is the largest open and  $\mathcal{H}$ -measurable random set  $\Gamma'$  such that  $\Gamma' \subseteq \Gamma$  a.s. if existence holds.

Recall the following:

**Lemma 2.5.** Let  $\mathcal{O}$  be an open set in a normed space. For every  $x \in \mathcal{O}$ ,  $B(x, d(x, \partial\mathcal{O})) \subseteq \mathcal{O}$ .

*Proof.* Let us consider  $r^* = \sup R$  where  $R$  is the non empty set of all  $r > 0$  such that  $B(x, r) \subseteq \mathcal{O}$ . It is trivial that  $B(x, r^*) \subseteq \mathcal{O}$ . So, for all  $o \in \partial\mathcal{O}$ ,

$d(x, o) \geq r^*$  hence  $r^* \leq d(x, \partial\mathcal{O})$ . Moreover, by definition of  $r^*$ , for all  $n$  there exists  $z^n \in B(x, r^* + n^{-1})$  such that  $z^n \notin \mathcal{O}$  hence  $r^* \leq \|z^n - x\| \leq r^* + n^{-1}$ . As the sequence  $(z_n)_n$  is bounded, we deduce by a compactness argument that for a subsequence  $z_n \rightarrow z$  as  $n \rightarrow \infty$ . Then,  $\|z - x\| = r^*$  hence  $z \in \text{cl } \mathcal{O}$ . In the case where  $z \in \mathcal{O}$ ,  $z^n \in \mathcal{O}$  for  $n$  large enough since  $\mathcal{O}$  is open, which yields a contradiction. Therefore,  $z \in \partial\mathcal{O}$ . This implies that  $r^* = d(x, z) \geq d(x, \partial\mathcal{O})$  and finally  $r^* = d(x, \partial\mathcal{O})$ .  $\square$

**Theorem 2.6.** *Let us consider a  $\mathcal{F}$ -measurable open random set  $\mathcal{O}$ . Then, the  $\mathcal{H}$ -graph-measurable interior of  $\mathcal{O}$  exists.*

*Proof.* By [1, Proposition 2.6],  $\text{cl } \mathcal{O}$  and  $\partial\mathcal{O} := \text{cl } \mathcal{O} \setminus \mathcal{O}$  are closed  $\mathcal{F}$ -measurable random sets. When  $\partial\mathcal{O} = \emptyset$ , we set  $d(x, \partial\mathcal{O}) = \infty$ . Otherwise, since  $\partial\mathcal{O}$  admits a Castaing representation on  $\{\partial\mathcal{O} \neq \emptyset\}$ , we deduce that the random continuous mapping  $x \mapsto d(x, \partial\mathcal{O})$  is  $\mathcal{F}$ -measurable. We deduce that the sets  $F^n := \{x : d(x, \partial\mathcal{O}) \geq 1/n\} \cap \text{cl } \mathcal{O}$  are closed  $\mathcal{F}$ -measurable random subsets of  $\mathcal{O}$  and we have  $\mathcal{O} = \cup_n F^n$ . Let us define the  $\mathcal{H}$ -measurable open random set

$$\mathbf{o}(\mathcal{O}|\mathcal{H}) = \text{int}(\cup_n \mathbf{m}(F^n|\mathcal{H})) \subseteq \mathcal{O},$$

and let us show that this is the largest  $\mathcal{H}$ -measurable open set of  $\mathcal{O}$ . To do so, let  $\mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$  be a  $\mathcal{H}$ -measurable open subset of  $\mathcal{O}$ . As previously, we may write  $\mathcal{O}_{\mathcal{H}} = \cup_n H^n$  where  $H^n := \{x : d(x, \partial\mathcal{O}_{\mathcal{H}}) \geq 1/n\} \cap \text{cl } \mathcal{O}_{\mathcal{H}}$  are  $\mathcal{H}$ -measurable closed subsets of  $\mathcal{O}_{\mathcal{H}}$ .

We claim that  $H^n \subseteq F^n$ . Indeed, if  $x \in H^n$ , it suffices to show that  $d(x, \partial\mathcal{O}_{\mathcal{H}}) \leq d(x, \partial\mathcal{O})$  a.s. In the contrary case, on a non null set, there exists  $o \in \partial\mathcal{O}$  such that  $o \in B(x, d(x, \partial\mathcal{O}_{\mathcal{H}}))$ . By Lemma 2.5, this implies that  $o \in \mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$  hence a contradiction.

As  $H^n$  is  $\mathcal{H}$ -measurable and closed,  $H^n \subseteq \mathbf{m}(F^n|\mathcal{H})$  for all  $n$ . We deduce that  $\mathcal{O}_{\mathcal{H}} \subseteq \cup_n \mathbf{m}(F^n|\mathcal{H})$  hence  $\mathcal{O}_{\mathcal{H}} \subseteq \mathbf{o}(\mathcal{O}|\mathcal{H})$ .  $\square$

**Corollary 2.7.** *For any  $\mathcal{F}$ -measurable random set  $\Gamma$ , the  $\mathcal{H}$ -measurable interior of  $\Gamma$  exists and  $\mathbf{o}(\Gamma|\mathcal{H}) = \mathbf{o}(\text{int } \Gamma|\mathcal{H})$ .*

**Corollary 2.8.** *For any  $\mathcal{F}$ -measurable random set  $\Gamma$ , there exists a smallest  $\mathcal{H}$ -measurable closed random set containing  $\Gamma$  a.s. denoted by  $\text{cl}(\Gamma|\mathcal{H})$  and called the  $\mathcal{H}$ -measurable conditional closure of  $\Gamma$ . We have*

$$\text{cl}(\Gamma|\mathcal{H}) = \mathbb{R}^d \setminus \mathbf{o}(\mathbb{R}^d \setminus \Gamma|\mathcal{H}).$$

Moreover, for all  $\gamma \in \text{cl}(\Gamma|\mathcal{H})$  and  $\epsilon \in \mathcal{L}((0, \infty), \mathcal{H})$ , for all  $H \in \mathcal{H}$  such that  $\mathbb{P}(H) > 0$ ,  $\mathbb{P}(\{\Gamma \cap B(\gamma, \epsilon) \neq \emptyset\} \cap H) > 0$ .

*Proof.* The first part is a direct consequence of Corollary 2.7. Suppose that  $P(\{\Gamma \cap B(\gamma, \epsilon) \neq \emptyset\} \cap H) = 0$ . Therefore, by definition of the conditional support as a smallest element,

$$\text{cl}(\Gamma|\mathcal{H}) = \text{cl}(\Gamma|\mathcal{H})1_{\Omega \setminus H} + \text{cl}(\Gamma|\mathcal{H}) \cap (\mathbb{R}^d \setminus B(\gamma, \epsilon))1_H.$$

Indeed, the  $\mathcal{H}$ -measurable set above contains  $\Gamma$  by assumption hence it contains  $\text{cl}(\Gamma|\mathcal{H})$ . We get a contradiction since  $\gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$  a.s. by assumption.  $\square$

**Corollary 2.9.** *Let  $X \in \mathcal{L}(\mathbb{R}^d, \mathcal{F})$ . There exists a smallest  $\mathcal{H}$ -measurable closed random set denoted by  $\text{supp}_{\mathcal{H}}(X)$  with  $P(X \in \text{supp}_{\mathcal{H}}(X)) = 1$ . This is the  $\mathcal{H}$ -conditional support of  $X$ . It is given by  $\text{supp}_{\mathcal{H}}(X) = \text{cl}(\{X\}|\mathcal{H})$ . Moreover, for all  $\gamma \in \mathcal{L}(\text{supp}_{\mathcal{H}}(X), \mathcal{H})$  and  $\epsilon \in \mathcal{L}((0, \infty), \mathcal{H})$ , for all  $H \in \mathcal{H}$  such that  $P(H) > 0$ ,  $P(\{X \in B(\gamma, \epsilon)\} \cap H) > 0$ .*

Notice that the conditional closure of a non empty set is necessarily non empty a.s. hence admits a measurable selection. In the following, we adopt the following notation  $kA = \{ka : a \in A\}$  for any subset  $A \subseteq \mathbb{R}^d$  and  $k \in A$ .

**Lemma 2.10.** *Let  $H \in \mathcal{H}$ . Then, for every  $\mathcal{F}$ -measurable set  $\Gamma$ ,  $\text{cl}(\Gamma 1_H|\mathcal{H}) = 1_H \text{cl}(\Gamma|\mathcal{H})$ .*

*Proof.* First observe that  $P(\Gamma 1_H \subseteq 1_H \text{cl}(\Gamma|\mathcal{H})) = 1$  as  $1_H \text{cl}(\Gamma|\mathcal{H}) = \{0\}$  on  $\Omega \setminus H$ . We deduce that  $\text{cl}(\Gamma 1_H|\mathcal{H}) \subseteq 1_H \text{cl}(\Gamma|\mathcal{H})$  a.s. and  $\text{cl}(\Gamma 1_H|\mathcal{H}) = \{0\}$  on  $\Omega \setminus H$ . Let us define  $Z = \text{cl}(\Gamma 1_H|\mathcal{H})1_H + \text{cl}(\Gamma|\mathcal{H})1_{\Omega \setminus H}$ . As  $\Gamma \subseteq \Gamma 1_H$  on  $H$ , we deduce that  $\Gamma \subseteq Z$  a.s. hence  $\text{cl}(\Gamma|\mathcal{H}) \subseteq Z$  a.s. Therefore,  $\text{cl}(\Gamma|\mathcal{H})1_H \subseteq Z 1_H$  and finally  $\text{cl}(\Gamma|\mathcal{H})1_H \subseteq \text{cl}(\Gamma 1_H|\mathcal{H})$  as the latter set is  $\{0\}$  on  $\Omega \setminus H$ .  $\square$

### 3. Application: Essential supremum as a pointwise supremum

**Definition 3.1.** *We say that a function  $f$  is sequentially lower semi-continuous (s.l.s.c.) on a domain  $D \subseteq \mathbb{R}^d$  if for all convergent sequence  $(x_n)_{n \geq 1}$  of  $D$ , we have  $f(\liminf_n x_n) \leq \liminf_n f(x_n)$ .*

**Definition 3.2.** *We say that a function  $f$  is sequentially upper semi-continuous (s.u.s.c.) on a domain  $D \subseteq \mathbb{R}^d$  if for all convergent sequence  $(x_n)_{n \geq 1}$  of  $D$ , we have  $f(\limsup_n x_n) \geq \limsup_n f(x_n)$ .*

**Lemma 3.3.** *Let  $f$  be sequentially lower semi-continuous (s.l.s.c.) on a domain  $D \subseteq \mathbb{R}^d$ . Let  $x_0 \in D$  and  $\epsilon > 0$ . Then, there exists  $r > 0$  such that for all  $x \in D$  in the closed ball  $B(x_0, r)$ , we have  $f(x_0) - f(x) \leq \epsilon$ .*

*Proof.* Suppose by contradiction that for all  $r > 0$ , there exists  $x \in B(x_0, r) \cap D$  such that  $f(x_0) - f(x) > \epsilon$ . Then, for all  $n \geq 1$ , there exists  $x_n \in D$  with  $|x_0 - x_n| \leq n^{-1}$  such that  $f(x_0) > f(x_n) + \epsilon$ . Therefore,  $f(x_0) \geq \liminf_n f(x_n) + \epsilon$ . As  $\lim_n x_n = x_0$ , this contradicts the lower semi-continuity of  $f$ .  $\square$

**Corollary 3.4.** *A function  $f$  is s.l.s.c. on a domain  $D \subseteq \mathbb{R}^d$  if and only if it is lower semi continuous, i.e. for every  $x \in D$ , we have  $f(x) = \liminf_{y \rightarrow x} f(y)$ , where  $\liminf_{y \rightarrow x} f(y) = \sup_{r > 0} \inf_{z \in B(x, r) \cap D} f(z)$ .*

*Proof.* Suppose that  $f$  is s.l.s.c. We have  $\inf_{z \in B(x, r) \cap D} f(z) = \lim_n \downarrow f(z_n)$  where we may assume by a compactness argument that  $z_n \rightarrow \hat{z}_r \in B(x, r)$ . By the assumption,  $f(\hat{z}_r) \leq \liminf_n f(z_n) \leq \inf_{z \in B(x, r) \cap D} f(z)$ . We deduce that  $\inf_{z \in B(x, r) \cap D} f(z) = f(\hat{z}_r)$ . Similarly, with  $r = k^{-1}$  and  $k \in \mathbb{N}$ , we may assume  $\hat{z}_{k^{-1}} \rightarrow \hat{z} = x$  as  $k \rightarrow \infty$ . Therefore,  $\liminf_{y \rightarrow x} f(y) = \lim_{r \rightarrow 0} \uparrow f(\hat{z}_r) \geq f(x)$  by assumption. Moreover, it is clear that  $\inf_{z \in B(x, r) \cap D} f(z) \leq f(x)$  hence  $\liminf_{y \rightarrow x} f(y) \leq f(x)$  so that the equality holds. Reciprocally, suppose that  $f$  is l.s.c. Then, consider a sequence  $x_n \in D$  which converges to  $x \in D$ . By assumption, we know that  $f(x) = \lim_{r \rightarrow 0} \inf_{z \in B(x, r) \cap D} f(z)$  for all  $r > 0$ . Moreover, for each  $r > 0$ , there exists  $n_r$  large enough such that  $x_n \in B(x, r)$  if  $n \geq n_r$ . Therefore,  $\inf_{z \in B(x, r) \cap D} f(z) \leq \inf_{n \geq n_r} f(x_n) \leq \liminf_n f(x_n)$ . We conclude that  $f(x) \leq \liminf_n f(x_n)$ .  $\square$

**Remark 3.5.** *Consider the function  $f = 1_{(1, \infty)}$ . This function is continuous at any point  $x \neq 1$ . Moreover, if  $x_n \rightarrow 1$ , then  $\liminf_n f(x_n) \geq f(1) = 0$ . Therefore,  $f$  is l.s.c. Nevertheless,  $x_n = 1 + n^{-1} \rightarrow 1$  but  $f(1) < \liminf_n f(x_n) = 1$ , i.e. the equality  $f(\liminf_n f(x_n)) = \liminf_n f(x_n)$  does not hold in general for a countable sequence  $(x_n)_{n \in \mathbb{N}}$ .*

**Remark 3.6.** *Recall that the l.s.c. convex envelope  $\overline{\text{co}}(f)$  of a function  $f$  dominating an affine function is defined as the largest function smaller than  $f$  which is both convex and l.s.c. It is well known that  $\overline{\text{co}}(f)$  coincides with the supremum of the affine functions smaller than  $f$ , see [].*

**Theorem 3.7.** *Let  $h(\omega, x)$ ,  $x \in \mathbb{R}^d$ , be a real-valued mapping such that  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable and is l.s.c in  $x$  and let  $\Gamma$  be a closed  $\mathcal{F}$ -measurable set of  $\mathbb{R}^d$ . Then,*

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} &= \sup_{x \in \text{cl}(\Gamma | \mathcal{H})} h(x) = \sup_n h(\gamma_n), \text{ a.s.}, \\ &= \text{ess sup}_{\mathcal{H}} \{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma | \mathcal{H}), \mathcal{H})\}, \text{ a.s.}, \end{aligned}$$

where  $(\gamma_n)_{n \in \mathbf{N}}$  is a Castaing representation of  $\text{cl}(\Gamma|\mathcal{H})$ .

*Proof.* As  $\text{cl}(\Gamma|\mathcal{H})$  is  $\mathcal{H}$ -measurable and closed, it admits a Castaing representation  $\text{cl}(\Gamma|\mathcal{H}) = \text{cl}\{\gamma_n : n \in \mathbf{N}\}$  where  $\gamma_n \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$  for all  $n$ . We first prove that

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n).$$

Let  $(\gamma_n)_n \subset \text{cl}\{\gamma_n : n \in \mathbf{N}\} = \text{cl}(\Gamma|\mathcal{H})$ , then  $h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$  and thus  $\sup_n h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ . Let  $x \in \text{cl}(\Gamma|\mathcal{H})$ . Then by lower semi-continuity,  $h(x) \leq \liminf_k h(\gamma_{n_k})$  for some random subsequence  $(\gamma_{n_k})$  such that  $\gamma_{n_k} \rightarrow x$ . Thus  $h(x) \leq \sup_n h(\gamma_n)$  and  $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) \leq \sup_n h(\gamma_n)$  hence the equality holds.

Therefore, since  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable,  $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n)$  is  $\mathcal{H}$ -measurable. Moreover, as  $\Gamma \subseteq \text{cl}(\Gamma|\mathcal{H})$  a.s., for every  $\gamma \in \mathcal{L}(\Gamma, \mathcal{F})$ ,  $h(\gamma) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$  hence  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$  a.s. To show the reverse inequality, we first show the equality

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

To see it, notice that

$$\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n) \leq \text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

Moreover, it is trivial that  $h(\gamma) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$  for all  $\gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$ . Since  $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$  is  $\mathcal{H}$ -measurable, we deduce that  $\text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\} \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)$ . Since the family  $\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}$  is directed upward, we also deduce that  $\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \lim_n \uparrow h(\gamma^n)$  where  $\gamma^n \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})$ .

At last, consider any  $\mathcal{H}$ -measurable selection  $\gamma$  of  $\text{cl}(\Gamma|\mathcal{H})$  and a deterministic sequence  $\epsilon_n > 0$  with  $\lim_n \epsilon_n = 0$ . By Corollary 2.8,  $\Lambda_n = \{\Gamma \cap B(\gamma, \epsilon_n) \neq \emptyset\} \in \mathcal{F}$  satisfies  $P(\Lambda_n|\mathcal{H}) > 0$  a.s. Indeed, otherwise, on a non null  $\mathcal{H}$ -measurable set  $\tilde{\Lambda}_n \subseteq \Lambda_n$ , we have  $P(\tilde{\Lambda}_n \cap \Lambda_n|\mathcal{H}) = 0$  hence  $P((\tilde{\Lambda}_n \cap \Lambda_n) = 0)$ , i.e. a contradiction with Corollary 2.8. By a measurable selection argument, it is possible to construct  $\hat{\gamma}_n \in L^0(\Gamma, \mathcal{F})$  such that  $\hat{\gamma}_n \in B(\gamma, \epsilon_n)$  on  $\Lambda_n$ . We define  $\hat{\Lambda}_n = \{\hat{\gamma}_n \in B(\gamma, \epsilon_n)\}$ . Since  $\Lambda_n \subseteq \hat{\Lambda}_n$ , we have  $P(\hat{\Lambda}_n|\mathcal{H}) > 0$  a.s.

Therefore,

$$\begin{aligned} \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\}1_{\hat{\Lambda}_n} &\geq h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}, \\ \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\}P(\hat{\Lambda}_n|\mathcal{H}) &\geq \mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}) \end{aligned} \quad (3.1)$$

The last inequality holds on a set  $A^n \in \mathcal{H}$  such that  $P(A^n) = 1$ . Moreover, there exists  $B^n \in \mathcal{H}$  such that  $P(B^n) = 1$  and

$$\mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}) = \int_{\mathbb{R}^d} h(x)1_{x \in B(\gamma, \epsilon_n)}P(\hat{\gamma}_n \in dx|\mathcal{H}).$$

To obtain this equality, it suffices to use a regular version of the conditional law of  $\hat{\gamma}_n$ . Moreover, we obtain the following pointwise inequality satisfied for each  $\omega \in C_n = A_n \cap B_n$ :

$$\begin{aligned} \int_{\mathbb{R}^d} h(x)1_{x \in B(\gamma, \epsilon_n)}P(\hat{\gamma}_n \in dx|\mathcal{H}) &\geq \inf_{z \in B(\gamma, \epsilon_n)} h(z) \int_{\mathbb{R}^d} 1_{x \in B(\gamma, \epsilon_n)}P(\hat{\gamma}_n \in dx|\mathcal{H}) \\ &\geq \inf_{z \in B(\gamma, \epsilon_n)} h(z)P(\hat{\Lambda}_n|\mathcal{H}). \end{aligned}$$

Note that we do not need for the infimum  $\inf_{z \in B(\gamma, \epsilon_n)} h(z)$  to be  $\mathcal{H}$ -measurable as the integral above is considered for each fixed  $\omega$  with respect to  $x$  and, precisely,  $\inf_{z \in B(\gamma, \epsilon_n)} h(z)$  does not depend on  $x$ . Since  $P(\hat{\Lambda}_n|\mathcal{H}) > 0$ , we then deduce by (3.1) that  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \inf_{z \in B(\gamma, \epsilon_n)} h(z)$  for every  $n \geq 1$  a.s. Therefore,  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \lim_n \inf_{z \in B(\gamma, \epsilon_n)} h(z)$  a.s. Since, by compactness and lower semi-continuity,  $\inf_{z \in B(\gamma, \epsilon_n)} h(z) = h(z_n)$  where  $z_n \in B(\gamma, \epsilon_n)$  converges pointwise to  $\gamma$  as  $n \rightarrow \infty$ , we finally deduce that  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \liminf_n h(z_n) \geq h(\gamma)$  by lower semi-continuity. This inequality holds for any  $\mathcal{H}$ -measurable selection of  $\text{cl}(\Gamma|\mathcal{H})$ . Therefore,  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \geq \text{ess sup}_{\mathcal{H}}\{h(\gamma), \gamma \in \mathcal{L}(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}$ . The conclusion of the lemma follows.  $\square$

Recall that a set  $\Lambda$  of measurable random variables is said  $\mathcal{F}$ -decomposable if for any finite partition  $(F_i)_{i=1, \dots, n} \subseteq \mathcal{F}$  of  $\Omega$ , and for every family  $(\gamma_i)_{i=1, \dots, n}$  of  $\Lambda$ , we have  $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$ . In the following, we denote by  $\Sigma(\Lambda)$  the  $\mathcal{F}$ -decomposable envelop of  $\Lambda$ , i.e. the smallest  $\mathcal{F}$ -decomposable family containing  $\Lambda$ . Notice that

$$\Sigma(\Lambda) = \left\{ \sum_{i=1}^n \gamma_i 1_{F_i} : n \geq 1, (\gamma_i)_{i=1, \dots, n} \subseteq \Lambda, (F_i)_{i=1, \dots, n} \subseteq \mathcal{F} \text{ s.t. } \sum_{i=1}^n F_i = \Omega \right\}. \quad (3.2)$$

The closure  $\bar{\Sigma}(\Lambda)$  in probability of  $\Sigma(\Lambda)$  is decomposable even if  $\Lambda$  is not decomposable. By [1, ?], there exists a  $\mathcal{F}$ -measurable closed random set  $\sigma(\Lambda)$  such that  $\bar{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$  is the set of all measurable selectors of  $\sigma(\Lambda)$ . If  $\Gamma$  is a  $\mathcal{F}$ -measurable random set, then  $\Lambda = L^0(\Gamma, \mathcal{F})$  is  $\mathcal{F}$ -measurable. Therefore,  $\bar{\Sigma}(\Lambda) = L^0(\bar{\Gamma}, \mathcal{F})$  by [1, ?].



**Corollary 3.8.** *Let  $h(\omega, x)$ ,  $x \in \mathbb{R}^d$ , be a real-valued mapping such that  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable and is l.s.c in  $x$ . Let us consider a family  $\Lambda$  of measurable random variables so that  $\bar{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$  is the set of all measurable selectors of some  $\mathcal{F}$ -measurable closed random set  $\sigma(\Lambda)$ . Then,*

$$\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} = \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \bar{\Sigma}(\Lambda)\} = \sup_{x \in \text{cl}(\sigma(\Lambda)|\mathcal{H})} h(x).$$

*Proof.* Notice that for any finite partition  $(F_i)_{i=1, \dots, n} \subseteq \mathcal{F}$  of  $\Omega$ ,  $n \geq 1$ , and for every family  $(\gamma_i)_{i=1, \dots, n}$  of  $\Lambda$ , we have

$$h\left(\sum_{i=1}^n \gamma_i 1_{F_i}\right) = \sum_{i=1}^n h(\gamma_i) 1_{F_i}.$$

Therefore, as  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$  a.s. for any  $\gamma \in \Lambda$ , we deduce that  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$  a.s. for any  $\gamma \in \Sigma(\Lambda)$ , and finally for all  $\gamma \in \bar{\Sigma}(\Lambda)$  since  $h$  is l.s.c. We deduce that  $\text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \Lambda\} \geq \text{ess sup}_{\mathcal{H}}\{h(\gamma) : \gamma \in \bar{\Sigma}(\Lambda)\}$  and the equality holds since  $\Lambda \subseteq \bar{\Sigma}(\Lambda)$ . The last statement is deduced from Theorem 3.7.  $\square$

## References

- [1] Lépinette, E. and Molchanov I. Conditional cores and conditional convex hulls of random sets. <https://arxiv.org/abs/1711.10303>