

# Harmonic morphisms on $\mathbb{S}^4$

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## Abstract

In this paper we study examples of harmonic morphisms due to Burel from  $(\mathbb{S}^4, g_{k,l})$  into  $\mathbb{S}^2$  where  $(g_{k,l})$  is a family of conformal metrics on  $\mathbb{S}^4$ . To do this construction we define two maps,  $F$  from  $(\mathbb{S}^4, g_{k,l})$  to  $(\mathbb{S}^3, g_{\bar{k},l})$  and  $\phi_{k,l}$  from  $(\mathbb{S}^3, g_{\bar{k},l})$  to  $(\mathbb{S}^2, can)$ ; the two maps are both horizontally conformal and harmonic. Let  $\Phi_{k,l} = \phi_{k,l} \circ F$ . It follows from Baird-Eells that the regular fibres of  $\Phi_{k,l}$  for every  $k, l$  are minimal. If  $|k| = |l| = 1$ , the set of critical points is given by the preimage of the north pole : it consists in two 2-spheres meeting transversally at 2 points. If  $k, l \neq 1$  the set of critical points are the preimages of the north pole (the same two spheres as for  $k = l = 1$  but with multiplicity  $l$ ) together with the preimage of the south pole (a torus with multiplicity  $k$ ).

## 1 Introduction

A harmonic morphism  $F : M \rightarrow N$  between two Riemannian manifolds  $(M, g)$  and  $(N, h)$  is a map which pulls back local harmonic functions on  $N$  to local harmonic functions on  $M$ . Although harmonic morphisms can be traced back to Jacobi, their study in modern times was initiated by Fuglede and Ishihara who characterized them using the notion of horizontal weak conformality, or semiconformality:

**Definition 1.** (see [B-W] p.46)

Let  $F : (M, g) \rightarrow (N, h)$  be a smooth map between Riemannian manifolds and let  $x \in M$ . Then  $F$  is called horizontally weakly conformal at  $x$  if either

1)  $dF_x = 0$

2)  $dF_x$  maps the space  $Ker(dF_x)^\perp$  conformally onto  $T_{F(x)}N$ , i.e. there exists a number  $\lambda(x)$  called the dilation of  $F$  at  $x$  such that

$$\forall X, Y \in Ker(dF_x)^\perp, h(dF_x(X), dF_x(X)) = \lambda^2(x)g(X, Y).$$

The space  $Ker(dF_x)$  (resp.  $Ker(dF_x)^\perp$ ) is called the vertical (resp. horizontal) space at  $x$ .

Fuglede and Ishihara proved independently

**Theorem 1.** ([Fu],[Is])

Let  $F : (M, g) \rightarrow (N, h)$  be a smooth map between Riemannian manifolds. The following two statements are equivalent:

- 1) For every harmonic function  $f : V \rightarrow \mathbb{R}$  defined on an open set  $V$  of  $N$ , the function  $f \circ F$  defined on the open set  $F^{-1}(V)$  of  $M$  is harmonic.
- 2) The map  $F$  is harmonic and horizontally weakly conformal.  
Such a map is called a harmonic morphism

When the target is 2-dimensional, Baird and Eells proved.

**Theorem 2.** ([B-E])

Let  $F : (M^m, g) \rightarrow (N^2, h)$  be a smooth non constant horizontally weakly conformal map between a Riemannian manifold  $(M^m, g)$  and a Riemannian 2-surface  $(N^2, h)$ . Then  $F$  is harmonic (hence a harmonic morphism) if and only if the fibres of  $F$  at regular points are minimal submanifolds of  $M$ .

**Remark 1.** In Makki-Ville ([Ma-Vi]) we extend Th.2 to the singular fibres if  $M$  is compact.

**Remark 2.** There is no non constant harmonic morphisms from  $(\mathbb{S}^4, \text{can})$  to  $\mathbb{S}^2$  ([Wo, Vi]).

So Burel [Bu] endows  $\mathbb{S}^4$  with metrics  $g$  conformal to the canonical metric  $\sigma$ , for which he constructed many harmonic morphisms from  $(\mathbb{S}^4, g)$  to  $\mathbb{S}^2$ .

## 2 Motivation

Let  $C$  be a complex curve in a complex compact manifold  $M$  of complex dimension two. The adjunction formula [G-H] which relates the tangent bundle, normal bundle and homology class of a complex curve in  $\mathbb{C}P^2$  is given by

$$c_1(TC) + c_1(NC) = c_1(T\mathbb{C}P^2)|_C$$

and  $c_1(TC) + c_1(NC)$  depends only on the homology class of  $C$  in  $\mathbb{C}P^2$ .

In particular, let  $(C_n)$  be a family of complex curves in  $\mathbb{C}P^2$  such that, for  $n \neq 0$ ,  $C_n$  is smooth and  $C_n \rightarrow C_0$  and  $C_0$  has one branch point. Then

$$c_1(TC_n) + c_1(NC_n) = c_1(TC_0) + c_1(NC_0) \tag{2.1}$$

**Exemple 1.** Let  $(C_\epsilon)$  given by  $z_1 z_2 = \epsilon z_0^2$  be a family of complex curves in  $\mathbb{C}P^2$  and  $(C_0)$  given by  $z_1 z_2 = 0$  the union of  $S_1 = \{z_1 = 0\}$  and  $S_2 = \{z_2 = 0\}$ . Then we have:

$$c_1(TC_\epsilon) = 2$$

because  $C_\epsilon$  is defined by a polynomial of degree two ( $C_\epsilon$  is a sphere), and

$$c_1(NC_\epsilon) = [C_\epsilon] \cdot [C_\epsilon] = 4.$$

because it is embedded and of degree two.

On the other hand, since  $C_0$  is the union of two spheres,

$$c_1(TC_0) = 2 + 2 = 4,$$

and since  $C_0$  has a positive self-intersection point :

$$c_1(NC_0) = [C_0] \cdot [C_0] - 2 = [C_\epsilon] [C_\epsilon] - 2 = 4 - 2 = 2.$$

So that

$$c_1(TC_\epsilon) + c_1(NC_\epsilon) = 6$$

and

$$c_1(TC_0) + c_1(NC_0) = 6.$$

By contrast, let  $M^4$  be an oriented manifold.

We ask here what happens if  $(\Sigma_n)$  is a sequence of minimal surfaces which degenerates to  $(\Sigma_0)$  with a branch point ? Here  $(\Sigma_n)$  verify ([Vi2]),[C-T])

$$c_1(T\Sigma_n) + c_1(N\Sigma_n) \leq c_1(T\Sigma_0) + c_1(N\Sigma_0) \quad (2.2)$$

**Remark 3.** If we change the orientation on  $M^4$ , but not on the  $\Sigma'_n$ 's,  $c_1(T\Sigma_n)$  is unchanged and  $c_1(N\Sigma_n)$  becomes  $-c_1(N\Sigma_n)$ . Hence (2.2) yields the following

$$c_1(T\Sigma_n) - c_1(N\Sigma_n) \leq c_1(T\Sigma_0) - c_1(N\Sigma_0). \quad (2.3)$$

When a singularity appears, we cannot have equality both in (2.2) and (2.3) because  $c_1(T\Sigma_0) \neq c_1(T\Sigma_n)$ .

In particular the complex curves  $C_n$ 's converging in  $\mathbb{C}P^2$  to  $C_0$  as above satisfy the strict inequality (2.3):

$$c_1(TC_n) - c_1(NC_n) < c_1(TC_0) - c_1(NC_0). \quad (2.4)$$

Now if we change the orientation on  $\mathbb{C}P^2$ , the  $(C_n)$  will still be minimal surfaces in  $\mathbb{C}P^2$  and they will verify for the new orientation

$$c_1(TC_n) + c_1(NC_n) < c_1(TC_0) + c_1(NC_0).$$

So we ask

**Question 1.** When do we have a strict inequality **both** in (2.2) or (2.3) for the same orientation ?

**Exemple 2.** Consider the Burel map  $\Phi_{1,1}$  and let  $(\Sigma_n)$  be a family of regular fibres in  $\mathbb{S}^4$  which converges to the singular fibre  $\Sigma_0$ . We shall see below that the  $\Sigma_n$ 's are embedded tori and that  $\Sigma_0$  is the union of two spheres  $S_1$  and  $S_2$  with two tranverse intersection points of opposite signs. We have

$$c_1(T\Sigma_n) = 0$$

and

$$c_1(N\Sigma_n) = [\Sigma_n] \cdot [\Sigma_n] = 0.$$

On the other hand:

$$c_1(T\Sigma_0) = 4,$$

and

$$c_1(N\Sigma_0) = [\Sigma_n] \cdot [\Sigma_n] - 2(1 - 1) = 0.$$

Thus

$$c_1(T\Sigma_n) \pm c_1(N\Sigma_n) = 0$$

and

$$c_1(T\Sigma_0) \pm c_1(N\Sigma_0) = 4.$$

### 3 Burel's construction

Burel was building upon previous constructions on product of spheres by Baird and Ou ([B-O]). He constructs a horizontally conformal map  $\Phi_{k,l}$  with  $k, l \in \mathbb{N}^*$  from  $\mathbb{S}^4$  into  $\mathbb{S}^2$  by the composition of two horizontally conformal maps  $F$  from  $\mathbb{S}^4$  into  $\mathbb{S}^3 = \mathbb{S}^0 * \mathbb{S}^2$  and  $\phi_{k,l}$  from  $\mathbb{S}^3 = \mathbb{S}^1 * \mathbb{S}^1$  into  $\mathbb{S}^2$ .

The key-point of this construction is the change of variable that allows to identify the joint  $\mathbb{S}^0 * \mathbb{S}^2$  and the joint  $\mathbb{S}^1 * \mathbb{S}^1$ .

First we are going to define the Hopf fibration  $H$  from  $\mathbb{S}^3$  into  $\mathbb{S}^2$  and then use it to define the map  $F$  from  $\mathbb{S}^4$  into  $\mathbb{S}^3$ .

**Definition 2.** *The Hopf fibration  $H : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$  of the 3-sphere over the 2-sphere is defined by*

$$H(z_0, z_1) = (|z_0|^2 - |z_1|^2, 2z_0z_1). \quad (3.1)$$

Let  $x = (\cos t e^{ia}, \sin t e^{ib})$  a point in  $\mathbb{S}^3$  where  $t \in [0, \pi/2]$  and  $a, b \in [0, 2\pi]$  then

$$\begin{aligned} H(x) &= (\cos^2 t - \sin^2 t, 2 \cos t \sin t e^{i(a+b)}) \\ &= (\cos 2t, \sin 2t e^{i(a+b)}) \end{aligned} \quad (3.2)$$

We define the map  $F : \mathbb{S}^4 \rightarrow \mathbb{S}^3$  for  $s \in [0, \pi]$ ,  $t \in [0, \pi/2]$  and  $a, b \in [0, 2\pi]$  by

$$\begin{aligned} F \left( \cos s, \sin s \left( \cos t e^{ia}, \sin t e^{ib} \right) \right) &= (\cos \alpha(s), \sin \alpha(s) H(x)) \\ &= \left( \cos \alpha(s), \sin \alpha(s) \cos 2t, \sin \alpha(s) \sin 2t e^{i(a+b)} \right). \end{aligned} \quad (3.3)$$

where  $\alpha$  is a increasing regular function such that  $\alpha(0) = 0$  and  $\alpha(\pi) = \pi$ , with  $\alpha(s)$  chosen so that  $F$  is semi-conformal *i.e.*

$$\alpha(s) = 2 \arctan \left( \tan^2 \left( \frac{s}{2} \right) \right).$$

Now for  $s$  fixed we have a geodesic sphere centred at the north pole of  $\mathbb{S}^4$  of radius  $\sin s$ . The map  $F$  sends it to a geodesic sphere centered at the north pole of  $\mathbb{S}^3$  of radius  $\sin \alpha(s)$ . Between the 3-sphere and the 2-sphere the map  $F$  is the *Hopf* map.

We now define the map  $\varphi_{k,l}$  from  $\mathbb{S}^3$  to  $\mathbb{S}^2$ . We need to define a new coordinate system on an open dense subset of  $\mathbb{S}^3$  which allows us to go from  $\mathbb{S}^3 = \mathbb{S}^0 * \mathbb{S}^2$  into  $\mathbb{S}^3 = \mathbb{S}^1 * \mathbb{S}^1$ , and this by supposing :

$$\cos \alpha(s) + i \sin \alpha(s) \cos 2t = \cos u(s, t) e^{i\psi(s, t)} \quad (3.4)$$

$$\sin \alpha(s) \sin 2t e^{i(a+b)} = \sin u(s, t) e^{i(a+b)}. \quad (3.5)$$

By changing the variable the point now is of the form

$$\left( \cos u(s, t) e^{i\psi(s, t)}, \sin u(s, t) e^{i(a+b)} \right) \text{ in } \mathbb{S}^3. \quad (3.6)$$

For simplification we write  $u, \psi, \alpha$  instead of  $u(s, t), \psi(s, t), \alpha(s, t)$ .

Now let  $\beta : [0, \frac{\pi}{2}] \longrightarrow [0, \pi]$  be a regular function of  $u$  such that

$$\beta(0) = 0 \text{ and } \beta \left( \frac{\pi}{2} \right) = \pi.$$

Note that the domain of  $\beta$  is  $[0, \frac{\pi}{2}]$  and not  $[0, \pi]$  as stated in [Bu]. In the new coordinate system, we define the application  $\phi_{k,l} : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  by:

$$\phi_{k,l} \left( \cos u e^{i\psi}, \sin u e^{i(a+b)} \right) = \left( \cos \beta(u), \sin \beta(u) e^{i(k\psi+l(a+b))} \right) \quad (3.7)$$

where  $\beta(u)$  is chosen so that  $\phi_{k,l}$  is horizontally conformal. For this,  $\beta$  must satisfy the following equation:

$$\frac{\beta'_u}{\sin \beta} = \sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}}.$$

This equation has an explicit solution given by ([B-O]) see next section

$$\beta(u) = 2 \arctan \left\{ \left| \frac{l - p(u)}{l + p(u)} \right|^{\frac{l}{2}} \left| \frac{k + p(u)}{k - p(u)} \right|^{\frac{k}{2}} \right\} \quad (3.8)$$

with  $p(u) = \sqrt{k^2 \sin^2 u + l^2 \cos^2 u}$ .

Notice that the absolute value in the equation is missing in [Bu].

## 4 Computation of $\beta$

We now compute the function  $\beta$  and prove (3.8) following the hints of [B-O]. We begin by quoting a result of [B-O].

**Lemma 1.** *Let  $F : (r_1 S^1) \times \dots \times (r_p S^p) \rightarrow a S^1$  be the map from the product of  $p$  circles of radius  $r_1, \dots, r_p$ , respectively, into a circle of radius  $a$ , given by*

$$F(r_1 e^{i\theta_1}, \dots, r_p e^{i\theta_p}) = a e^{i(k_1 \theta_1 + \dots + k_p \theta_p)}, \text{ for integers } k_1, \dots, k_p. \quad (4.1)$$

Then  $F$  is a harmonic morphism with dilation  $\lambda$  given by

$$\lambda^2 = a^2 \left( \frac{k_1^2}{r_1^2} + \dots + \frac{k_p^2}{r_p^2} \right) \quad (4.2)$$

We define a map  $\phi : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  as follows:

$$\mathbb{S}^3 \ni \left( \cos u e^{i\psi}, \sin u e^{iA} \right) \rightarrow \left( \cos \beta(u), \sin \beta(u) e^{i(k\psi+lA)} \right)$$

where  $\psi, A \in [0, 2\pi]$ ,  $k, l$  are non-zero integers and  $u \in [0, \pi/2]$ .

We begin by solving the horizontal conformality condition for  $\phi$ .

For fixed  $u_0 \in (0, \pi/2)$ , by lemma 1, the restriction of  $\phi$  to the product of circles:

$$\cos u S^1 \times \sin u S^1 \rightarrow \sin \beta S^1$$

$$\left( \cos u e^{i\psi}, \sin u e^{iA} \right) \rightarrow \sin \beta e^{i(k\psi+lA)}$$

is a harmonic morphism with dilation given by

$$\lambda^2 = \sin^2 \beta \left( \frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u} \right). \quad (4.3)$$

The metric on  $\mathbb{S}^3$  is induced by the metric on  $\mathbb{R}^4$ . By taking derivatives along  $u, \psi$  and  $A$ , we get the following orthonormal basis of tangent vectors to  $\mathbb{S}^3$ :

$$\begin{aligned}\epsilon_1 &= (-\sin u \cos \psi, -\sin u \sin \psi, \cos u \cos A, \cos u \sin A) \\ \epsilon_2 &= (-\sin \psi, \cos \psi, 0, 0) \\ \epsilon_3 &= (0, 0, -\sin A, \cos A)\end{aligned}$$

Note that  $\frac{\partial}{\partial u} = \epsilon_1$ ,  $\frac{\partial}{\partial \psi} = \cos u \epsilon_2$ ,  $\frac{\partial}{\partial A} = \sin u \epsilon_3$ .

We compute  $\frac{\partial \phi}{\partial \psi}$  and  $\frac{\partial \phi}{\partial A}$  and we derive that

$$d\phi(l \cos u \epsilon_2 - k \sin u \epsilon_3) = 0$$

hence the horizontal space  $H$  in  $\mathbb{S}^3$  w.r.t.  $\phi$  consists in the vectors tangent to  $\mathbb{S}^3$  and orthogonal to  $V = l \cos u \epsilon_2 - k \sin u \epsilon_3$ . It is generated by

$$H_1 = \frac{\partial}{\partial u} = \epsilon_1, \quad H_2 = k \sin u \epsilon_2 + l \cos u \epsilon_3$$

We compute in  $\mathbb{R}^3$  that  $\langle d\phi(H_1), d\phi(H_2) \rangle = 0$ . So the horizontal conformality of  $\phi$  reduces to requiring that

$$\|d\phi(H_1)\|^2 = \left\| \frac{\partial \phi}{\partial u} \right\|^2 = \frac{\|d\phi(H_2)\|^2}{\|H_2\|^2} = \lambda^2$$

where  $\lambda$  is given by (4.3).

$$\frac{\partial \phi}{\partial u} = \left( \frac{\partial \beta}{\partial u} \right) (-\sin \beta, \cos \beta e^{i(k\psi + lA)}). \quad (4.4)$$

Then the condition for  $\phi$  to be horizontally conformal is

$$\left( \frac{\partial \beta}{\partial u} \right)^2 = \sin^2 \beta \left( \frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u} \right) \quad (4.5)$$

Case 1:  $|k| = |l|$ . Then Eq. (4.5) takes the form

$$\frac{1}{\sin^2 \beta} \left[ \left( \frac{\partial \beta}{\partial u} \right)^2 \right] = \frac{4k^2}{\sin^2 2u}, \quad (4.6)$$

which can be solved explicitly as follows. Set

$$v = \frac{\partial}{\partial u}.$$

We have that

$$\begin{aligned}v \left( \log \tan \frac{\beta}{2} \right) &= \frac{\partial}{\partial \beta} \left( \log \tan \frac{\beta}{2} \right) v(\beta) \\ &= \frac{1}{\tan \frac{\beta}{2}} \frac{1}{2 \cos^2 \frac{\beta}{2}} v(\beta) \\ &= \frac{1}{2 \sin \frac{\beta}{2} \cos \frac{\beta}{2}} v(\beta) \\ &= \frac{1}{\sin \beta} v(\beta) \\ &= \frac{1}{\sin \beta} \frac{\partial \beta}{\partial u}.\end{aligned}$$

Then the left-hand side of Eq. (4.6) is equal to

$$v\left(\log \tan \frac{\beta}{2}\right) v\left(\log \tan \frac{\beta}{2}\right). \quad (4.7)$$

On the other hand we have that

$$\begin{aligned} v\left(\log \tan^k u\right) &= v(k \log \tan u) \\ &= \frac{k}{\cos^2 u \tan u} \\ &= \frac{k}{\cos u \sin u} \\ &= \frac{2k}{\sin 2u}. \end{aligned}$$

Then the right-hand side of Eq. (4.6) is equal to

$$v\left(\log \tan^k u\right) v\left(\log \tan^k u\right).$$

by the substitution in Eq. (4.6) we obtain

$$v\left(\log \tan \frac{\beta}{2}\right) = v\left(\log \tan^k u\right)$$

yielding the solution

$$\beta(u) = 2 \arctan\left(\tan^k u\right) \quad (4.8)$$

Case 2:  $|k| \neq |l|$ . Now the reduction equation for horizontal conformality becomes

$$\frac{1}{\sin^2 \beta} \left[ \left( \frac{\partial \beta}{\partial u} \right)^2 \right] = \frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}. \quad (4.9)$$

In order to proceed as before, we must write  $\sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}}$  as a derivative. We pose

$$\sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}} = \frac{\partial I}{\partial u}, \quad (4.10)$$

then we find an explicit formula for  $I$ . First we must evaluate the integral

$$I = \int \sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}} du.$$

There are two cases:

(a)  $l^2 > k^2$ .

We have

$$\sqrt{\frac{k^2}{\cos^2 u} + \frac{l^2}{\sin^2 u}} = \frac{l}{\cos u \sin u} \sqrt{1 - \frac{l^2 - k^2}{l^2} \sin^2 u}$$

First make the substitution:

$$\sin \theta = \frac{\sqrt{l^2 - k^2}}{l} \sin u. \quad (4.11)$$

For the derivative we obtain:

$$\cos \theta d\theta = \frac{\sqrt{l^2 - k^2}}{l} \cos u du. \quad (4.12)$$

Then

$$\begin{aligned}
\sin^2 \theta &= \frac{l^2 - k^2}{l^2} \sin^2 u \\
\cos^2 \theta &= 1 - \frac{l^2 - k^2}{l^2} \sin^2 u \\
\cos \theta &= \sqrt{1 - \frac{l^2 - k^2}{l^2} \sin^2 u} \\
\cos^2 u &= 1 - \frac{l^2}{l^2 - k^2} \sin^2 \theta = \frac{l^2 - k^2 - l^2 \sin^2 \theta}{l^2 - k^2}
\end{aligned} \tag{4.13}$$

Then by using (4.11),(4.12) and (4.13) we obtain the integral

$$\begin{aligned}
I &= \int \frac{l^2 \cos^2 \theta}{\cos^2 u \sin u \sqrt{l^2 - k^2}} d\theta \\
&= \int \frac{l \cos^2 \theta}{\cos^2 u \sin \theta} d\theta \\
&= \int \frac{l \cos^2 \theta (l^2 - k^2)}{\sin \theta (l^2 - k^2 - l^2 \sin^2 \theta)} d\theta.
\end{aligned}$$

On the other hand we have the following equality

$$\frac{l}{\sin \theta} + \frac{lk^2 \sin \theta}{l^2 - k^2 - l^2 \sin^2 \theta} = \frac{l \cos^2 \theta (l^2 - k^2)}{\sin \theta (l^2 - k^2 - l^2 \sin^2 \theta)}$$

then we obtain

$$I = l \int \frac{1}{\sin \theta} d\theta + lk^2 \int \frac{\sin \theta}{l^2 - k^2 - l^2 \sin^2 \theta} d\theta.$$

The second of these integrals is easily evaluated after substituting  $\phi = \cos \theta$  and we obtain

$$I = l \int \frac{-d\phi}{1 - \phi^2} + lk^2 \int \frac{-d\phi}{l^2 \left( \phi^2 - \left( \frac{k}{l} \right)^2 \right)} = \frac{1}{2} l \log \left| \frac{1 - \phi}{1 + \phi} \right| + \frac{1}{2} k \log \left| \frac{k + l\phi}{k - l\phi} \right|.$$

Let

$$p(u) = \sqrt{l^2 \cos^2 u + k^2 \sin^2 u},$$

then

$$\begin{aligned}
p(u)^2 &= l^2 \cos^2 u + k^2 \sin^2 u \\
&= (k^2 - l^2) \sin^2 u + l^2 \\
&= -l^2 \sin^2 \theta + l^2 \\
&= l^2 \cos^2 \theta
\end{aligned}$$

We thus obtain :

$$\left| \frac{k + p}{k - p} \right| = \left| \frac{k + l \cos \theta}{k - l \cos \theta} \right| = \left| \frac{k + l\phi}{k - l\phi} \right|$$

and

$$\left| \frac{l + p}{l - p} \right| = \left| \frac{1 + \phi}{1 - \phi} \right|.$$



Hence

$$\begin{aligned} I &= \frac{1}{2}l \log \left| \frac{l-p}{l+p} \right| + \frac{1}{2}k \log \left| \frac{k+p}{k-p} \right| \\ &= \log \left| \frac{l-p}{l+p} \right|^{\frac{l}{2}} + \log \left| \frac{k+p}{k-p} \right|^{\frac{k}{2}} \\ &= \log \left\{ \left| \frac{l-p}{l+p} \right|^{\frac{l}{2}} \left| \frac{k+p}{k-p} \right|^{\frac{k}{2}} \right\}. \end{aligned}$$

By the substitution of the two side of the Eq. (4.9) and from (4.10) we obtain

$$v \left( \log \tan \frac{\beta}{2} \right) = \frac{\partial I}{\partial u} = v \left( \log \left\{ \left| \frac{l-p}{l+p} \right|^{\frac{l}{2}} \left| \frac{k+p}{k-p} \right|^{\frac{k}{2}} \right\} \right)$$

yielding the solution

$$\beta(u) = 2 \arctan \left\{ \left| \frac{l-p}{l+p} \right|^{\frac{l}{2}} \left| \frac{k+p}{k-p} \right|^{\frac{k}{2}} \right\} \quad (4.14)$$

(b)  $l^2 < k^2$ .

Similarly, we suppose

$$\sinh \theta = \frac{\sqrt{k^2 - l^2}}{l} \sin u$$

now involving hyperbolic functions, that gives us

$$\begin{aligned} I &= l \int \frac{1}{\sinh \theta} d\theta + lk^2 \int \frac{\sinh \theta}{l^2 - k^2 - l^2 \sinh^2 \theta} d\theta \\ &= lI_1 + lk^2 I_2. \end{aligned}$$

It is easily evaluated after substituting  $\phi = \cosh \theta$  and we obtain

$$\begin{aligned} I_1 &= \int \frac{d\phi}{\phi^2 - 1} \\ &= \int \frac{d\phi}{2(\phi - 1)} - \int \frac{d\phi}{2(\phi + 1)} \\ &= \frac{1}{2} \log |\phi - 1| - \frac{1}{2} \log |\phi + 1| \\ &= \frac{1}{2} \log \left| \frac{\phi - 1}{\phi + 1} \right| \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int \frac{d\phi}{l^2 \left( \left( \frac{k}{l} \right)^2 - \phi^2 \right)} \\ &= \int \frac{1}{2lk} \log \left| \frac{k + l\phi}{k - l\phi} \right| \end{aligned}$$

then

$$I = \frac{1}{2}l \log \left| \frac{\phi - 1}{\phi + 1} \right| + \frac{1}{2}k \log \left| \frac{k + l\phi}{l\phi - k} \right|$$

Using the following two equalities

$$\frac{\phi - 1}{\phi + 1} = \frac{p - l}{p + l}$$

and

$$\frac{p + k}{p - k} = \frac{l\phi + k}{l\phi - k} \text{ with } p = l\phi,$$

we obtain

$$I = \log \left| \frac{p - l}{p + l} \right|^{\frac{l}{2}} \left| \frac{k + p}{p - k} \right|^{\frac{k}{2}}$$

By Eq. (4.9) we obtain

$$v \left( \log \tan \frac{\beta}{2} \right) = v \left( \log \left| \frac{p - l}{p + l} \right|^{\frac{l}{2}} \left| \frac{k + p}{p - k} \right|^{\frac{k}{2}} \right)$$

yielding the solution

$$\beta(u) = 2 \arctan \left\{ \left| \frac{p - l}{p + l} \right|^{\frac{l}{2}} \left| \frac{k + p}{p - k} \right|^{\frac{k}{2}} \right\}. \quad (4.15)$$

## 5 The Preimages of $\Phi_{k,l}$

In this section, we take a point  $P$  in  $\mathbb{S}^2$  and we look for its preimage in  $\mathbb{S}^4$  by  $\Phi_{k,l}$ . First, we look for the preimage of this point in  $\mathbb{S}^3$  by the map  $\phi_{k,l}$  and then we fix a point on this preimage and look for its preimage in  $\mathbb{S}^4$  by the map  $F$ .

### 5.1 The preimage of $F$

We recall definition of the map  $F$  in (3.3)

$F : \mathbb{S}^4 \rightarrow \mathbb{S}^3$  for  $s \in [0, \pi]$ ,  $t \in [0, \pi/2]$  and  $a, b \in [0, 2\pi]$  and

$$F \left( \cos s, \sin s \left( \cos t e^{ia}, \sin t e^{ib} \right) \right) = \left( \cos \alpha(s), \sin \alpha(s) \cos 2t, \sin \alpha(s) \sin 2t e^{i(a+b)} \right).$$

where  $\alpha$  is an increasing regular function such that  $\alpha(0) = 0$  and  $\alpha(\pi) = \pi$

**Proposition 1.** *Let  $P \in \mathbb{S}^3$ ,*

- 1) *If  $P \neq (1, 0, 0, 0)$ , then  $F^{-1}(P)$  is a closed loop.*
- 2)  $F^{-1}(\pm 1, 0, 0, 0) = \{(\pm 1, 0, 0, 0)\}$

*Proof.* We fix  $Z \in \mathbb{S}^2$  and let  $P = (\cos \alpha_0, \sin \alpha_0 Z)$  with  $Z \in \mathbb{S}^2$  and  $\alpha_0 \in [0, \pi]$ . Now we look for its preimage in  $\mathbb{S}^4$ . There exists a unique  $s_0$  such that  $\alpha_0 = \alpha(s_0)$ .

- 1) If  $\sin \alpha_0 \neq 0$ ,

$$F^{-1}(P) = \{(\cos s_0, \sin s_0 x) : H(x) = Z\} \quad (5.1)$$

- 2) If  $\sin \alpha_0 = 0$ , then  $P = (\pm 1, 0, 0, 0)$ . Moreover if  $\alpha_0 = 0$  (resp.  $\alpha_0 = \pi$ ) then  $s_0 = 0$  (resp.  $s_0 = \pi$ ) and 2) follows.

□

## 5.2 The preimage of $\phi_{k,l}$

We denote by  $N_{\mathbb{S}^2}$  (resp  $S_{\mathbb{S}^2}$ ) the north pole  $(1, 0, 0)$  (resp. south pole  $(-1, 0, 0)$ ).

We also recall definition of  $\phi_{k,l} : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  given in (3.7) :

$$\phi_{k,l} \left( \cos u e^{i\psi}, \sin u e^{i(a+b)} \right) = \left( \cos \beta(u), \sin \beta(u) e^{i(k\psi+l(a+b))} \right) \quad (5.2)$$

**Proposition 2.** *The map  $\phi_{1,1}$  is the Hopf map so  $\phi_{1,1}^{-1}(Q)$  is a great circle in  $\mathbb{S}^3$ . More generally,  $\phi_{k,l}^{-1}(\{Q\})$  is a  $(k,l)$  torus-knot if  $Q \neq N_{\mathbb{S}^2}, S_{\mathbb{S}^2}$  and  $\phi_{k,l}^{-1}(\{N_{\mathbb{S}^2}\})$  and  $\phi_{k,l}^{-1}(\{S_{\mathbb{S}^2}\})$  are great circles in  $\mathbb{S}^3$ .*

*Proof.* Let  $Q = (\cos v_0, \sin v_0 e^{i\mu_0})$  with  $v_0 \in [0, \pi]$  and  $\mu_0 \in [0, 2\pi]$ .

There exists a unique  $u_0 \in [0, \frac{\pi}{2}]$  such that  $v_0 = \beta(u_0)$ .

If  $v_0 = 0$  (resp.  $v_0 = \pi$ ) i.e.  $Q = N_{\mathbb{S}^2}$  (resp.  $Q = S_{\mathbb{S}^2}$ ), then  $u_0 = 0$  (resp.  $u_0 = \frac{\pi}{2}$ ) and

$$\phi_{k,l}^{-1}(\{N_{\mathbb{S}^2}\}) = \{(e^{i\Psi}, 0) : \Psi \in [0, 2\pi]\},$$

$$\text{resp. } \phi_{k,l}^{-1}(\{S_{\mathbb{S}^2}\}) = \{(0, e^{iA}) : A \in [0, 2\pi]\}.$$

Now assume  $Q = (\cos v_0, \sin v_0 e^{i\mu_0})$  with  $v_0 \in ]0, \pi[$  and  $\mu_0 \in [0, 2\pi]$ .

The preimage of  $Q$  is

$$\phi_{k,l}^{-1}(Q) = \{(\cos u_0 e^{i\psi}, \sin u_0 e^{iA}) : \mu_0 = k\psi + lA, \psi, A \in [0, 2\pi]\}.$$

We obtain a torus knot of type  $(k, l)$ ; it is included in the torus on  $\mathbb{S}^3$  given by

$$T_{u_0} = \{(\cos(u_0) e^{i\psi}, \sin(u_0) e^{iA}) : \psi \in [0, 2\pi] \text{ and } A \in [0, 2\pi]\}.$$

□

## 5.3 The preimage of the North pole $N_{\mathbb{S}^2} = (1, 0, 0)$ of $\mathbb{S}^2$ by $\Phi_{k,l}$

In this section, we find the preimage of the North pole  $N_{\mathbb{S}^2} = (1, 0, 0)$  by the map  $\Phi_{k,l}$ . We also recall the definition of  $\Phi_{k,l} = \phi_{k,l} \circ F$  : where

$$\phi_{k,l} \left( \cos u e^{i\psi}, \sin u e^{i(a+b)} \right) = \left( \cos \beta(u), \sin \beta(u) e^{i(k\psi+l(a+b))} \right) \quad (5.3)$$

and

$$F \left( \cos s, \sin s \left( \cos t e^{ia}, \sin t e^{ib} \right) \right) = \left( \cos \alpha(s), \sin \alpha(s) \cos 2t \sin \alpha(s) \sin 2t e^{i(a+b)} \right).$$

**Proposition 3.** *The preimage of the north pole  $N_{\mathbb{S}^2} = (1, 0, 0)$  in  $\mathbb{S}^2$  by the map  $\Phi_{k,l}$  is the union of the two totally geodesic 2-spheres*

$$S_1 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_4 = x_5 = 0\}$$

and

$$S_2 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = 0\},$$

with  $S_1, S_2 \subset \mathbb{S}^4 \subset \mathbb{R}^5$ .

The spheres  $S_1$  and  $S_2$  intersect at each pole  $N_{\mathbb{S}^4} = (1, 0, 0, 0, 0)$  and  $S_{\mathbb{S}^4} = (-1, 0, 0, 0, 0)$  with opposite signs of intersection.

*Proof.* We look for a point of the form  $(\cos s, \sin s \cos t e^{ia}, \sin s \cos t e^{ib}) \in \mathbb{S}^4$ .

Let  $Q = (\cos u e^{i\psi}, \sin u e^{iA}) \in \mathbb{S}^3$  with  $\phi_{k,t}(Q) = N_{\mathbb{S}^2}$ . Then  $\beta(u) = 0$  hence  $u = 0$ . The preimage of  $N_{\mathbb{S}^2}$  in  $\mathbb{S}^3$  is given for  $u = 0$  by  $\{(e^{i\psi}, 0)\} \in \mathbb{S}^3 \subset \mathbb{C}^2$ .

We fix  $\psi$  and we look for the preimage of  $(e^{i\psi}, 0)$  in  $\mathbb{S}^4$ .

Looking at the two equations (3.4) and (3.5), we obtain by a small calculation the following

$$\sin \alpha(s) \sin 2t = 0. \quad (5.4)$$

$$\cos \alpha(s) + i \sin \alpha(s) \cos 2t = e^{i\psi(s,t)} \quad (5.5)$$

Then,  $\sin \alpha(s) = 0$  or  $\sin 2t = 0$ .

- 1) If  $\sin \alpha(s) = 0$  then  $\sin s = 0$  therefore  $s = 0$  or  $s = \pi$ . Using (5.5), we have  $e^{i\psi(s,t)} = \pm 1$  then,  $\psi = 0$  or  $\psi = \pi$ . Here, we obtain the two poles  $N_{\mathbb{S}^4} = (1, 0, 0, 0, 0)$  and  $S_{\mathbb{S}^4} = (-1, 0, 0, 0, 0)$ .
- 2) On the other hand, if  $\sin 2t = 0$  then  $t = 0$  or  $t = \frac{\pi}{2}$ . Using (5.5), we obtain for  $\psi \neq 0$  and  $\psi \neq \pi$  two cases :
  - a) If  $0 < \psi < \pi$ , then  $\alpha = \psi$  and  $t = 0$ , then, we obtain in  $\mathbb{S}^4$ ,  $(\cos s, \sin s(e^{ia}, 0))$  where  $a \in [0, 2\pi]$  and  $s \in ]0, \pi[$ .  
Here we have the sphere  $S_1$  punctured at the two poles.
  - b) If  $\pi < \psi < 2\pi$ , then  $\alpha = 2\pi - \psi$  and  $t = \frac{\pi}{2}$  therefore we obtain in  $\mathbb{S}^4$ ,  $(\cos s, \sin s(0, e^{ib}))$  where  $b \in [0, 2\pi]$  and  $s \in ]0, \pi[$ .  
Here we have the sphere  $S_2$  punctured at the two poles.

In case one we obtain the two poles  $N_{\mathbb{S}^4}$  and  $S_{\mathbb{S}^4}$ .

In case two we obtain the two great spheres  $S_1$  and  $S_2$  minus the poles  $N_{\mathbb{S}^4}$  and  $S_{\mathbb{S}^4}$ .

Putting cases 1. and 2. together shows that the preimage of  $N_{\mathbb{S}^2}$  consists of two 2-spheres  $S_1$  and  $S_2$  intersecting transversally at the poles  $N_{\mathbb{S}^4}$  and  $S_{\mathbb{S}^4}$ .

Since,  $H_2(\mathbb{S}^4, \mathbb{Z}) = 0$ ,  $S_1$  and  $S_2$  have a zero total number of intersection points (counted with sign). Hence,  $N_{\mathbb{S}^4}$  and  $S_{\mathbb{S}^4}$  are intersection points of opposite signs. □

**Remark 4.** *In fact we can check by hand that the two intersection points have different signs. For that we choose a positive orthonormal basis  $\{e_1, e_2, e_3, e_4, e_5\}$  of  $\mathbb{R}^5$  where  $e_1 = N_{\mathbb{S}^4}$ . We can see clearly that  $S_1$  and  $S_2$  are the intersection of  $\mathbb{S}^4$  with the two subspaces of  $\mathbb{R}^5$  generated by  $\{e_1, e_2, e_3\}$  (resp.  $\{e_1, e_4, e_5\}$ ).*

*Let  $N_{\mathbb{S}^4} = (1, 0, 0, 0, 0)$  and  $S_{\mathbb{S}^4} = (-1, 0, 0, 0, 0)$  be the two intersection points of  $S_1$  and  $S_2$ . First, for  $N_{\mathbb{S}^4} \in S_1 \cap S_2$  we have :*

- $T_N S_1$  and  $T_N S_2$  are generated by the two positive bases  $\{e_2, e_3\}$  resp.  $\{e_4, e_5\}$  and  $T_N \mathbb{S}^4$  is generated by the positive basis  $\{e_2, e_3, e_4, e_5\}$ . So the orientation at  $N$  is positive.

*Now, we take  $S_{\mathbb{S}^4} \in S_1 \cap S_2$  we have :*

- $T_P S_1$  and  $T_P S_2$  are generated by the two positive bases  $\{-e_2, e_3\}$  resp.  $\{-e_4, e_5\}$  and  $T_P \mathbb{S}^4$  is generated by the positive basis  $\{-e_2, e_3, e_4, e_5\}$ . So the orientation at this point  $P$  is negative.

*Therefore, the two intersection points have opposite signs.*

## 5.4 The preimage of the South pole $S_{\mathbb{S}^2} = (-1, 0, 0)$ in $\mathbb{S}^2$ by $\Phi_{k,l}$

In this section we look for the preimage of the second pole  $S_{\mathbb{S}^2} = (-1, 0, 0)$  by the map  $\Phi_{k,l}$ .

**Proposition 4.** *The preimage of the pole south  $S_{\mathbb{S}^2} = (-1, 0, 0)$  in  $\mathbb{S}^2$  is a Clifford torus in the equator of  $\mathbb{S}^4$ .*

*Proof.* If  $\beta(u) = \pi$ , we have

$$\cos \beta(u) = -1 \text{ and } u = \pi/2.$$

The preimage of this pole in  $\mathbb{S}^3$ , is given for  $u = \frac{\pi}{2}$  by

$$\{(0, e^{iA})\} \in \mathbb{S}^3 \subset \mathbb{C}^2.$$

We fix  $A$  and we look for the preimage of  $(0, e^{iA})$  in  $\mathbb{S}^4$ . Looking at (3.4) and (3.5), as above we get the two equations:

$$\cos \alpha(s) + i \sin \alpha(s) \cos 2t = 0 \tag{5.6}$$

$$\sin \alpha(s) \sin 2t = 1 \tag{5.7}$$

Therefore,

$$\alpha(s) = \pi/2 \text{ and } 2t = \pi/4.$$

By a small computation  $\alpha\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$ ; since  $\alpha$  is strictly increasing we conclude that  $s = \frac{\pi}{4}$ .

We conclude that the preimage in  $\mathbb{S}^4$  of the south pole  $(-1, 0, 0)$  is a Clifford torus  $T$  in the equator  $\mathbb{S}^3$  of  $\mathbb{S}^4$  :

$$T := \left\{ \left( 0, \frac{\sqrt{2}}{2} e^{ia}, \frac{\sqrt{2}}{2} e^{ib} \right) : (a, b) \in [0, 2\pi] \times [0, 2\pi] \right\}$$

□

## 6 Critical points of $\Phi_{k,l}$

In this section we are going to find the critical points of the map  $\Phi_{k,l}$ . To do this, we need to prove the following theorem:

**Theorem 3.** *The set of critical points of  $\Phi_{k,l}$  for  $k = l = 1$  is given by two 2-spheres having the two poles as intersection points. Otherwise, if  $k, l \neq 1$  the set of critical points are the preimages of the north pole (the same two spheres as for  $k = l = 1$ ) together with the preimage of the south pole (a torus).*

### 6.1 Critical points of $F$

We investigate the map  $F$  from  $\mathbb{S}^4$  into  $\mathbb{S}^3$  given by (3.3). For  $0 < s < \pi$ ,  $\alpha'(s) \neq 0$ . It follows that all points of  $\mathbb{S}^4$  are regular for  $F$ , outside of the poles. We now investigate what happens at the North and South poles.

We look at a neighbourhood of the pole  $N_{\mathbb{S}^4} = (1, 0, 0, 0, 0)$ . Near the pole  $N_{\mathbb{S}^4} = (1, 0, 0, 0, 0)$ , the parameter  $s$  is close to 0 so we identify a neighborhood of  $N_{\mathbb{S}^4}$  with a 4-ball centred at  $N_{\mathbb{S}^4}$ .

$$\mathfrak{B}_\epsilon = \left\{ sx : (s, x) \in [0, \epsilon] \times \mathbb{S}^3 \right\},$$

By projection on the last two coordinates we identify a neighborhood of the north pole  $N_{\mathbb{S}^2}$  in  $\mathbb{S}^2$  to a disc  $D$  of  $\mathbb{R}^2$ .

Now consider the regular function

$$\alpha(s) = 2 \arctan \left( \tan^2 \left( \frac{s}{2} \right) \right).$$

For  $s \sim 0$ , we have

$$\alpha(s) \sim 2 \arctan \frac{s^2}{4}.$$

Consequently,

$$\alpha(s) \sim \frac{s^2}{2}.$$

Hence

$$(\cos \alpha(s), \sin \alpha(s)H(x)) \sim \left( 1 - \frac{s^4}{4}, \frac{s^2}{2}H(x) \right). \quad (6.1)$$

Under the above identifications we write  $F$  as

$$sx \longrightarrow \frac{s^2}{2}H(x).$$

It follows that the North pole  $N_{\mathbb{S}^4}$  is a critical point for  $F$ .

In the second step we look at a neighbourhood of the pole  $S_{\mathbb{S}^4} = (-1, 0, 0, 0, 0)$ , here we are going to use the same procedure that we use for the other pole.

So we identify a neighborhood of  $S$  with a 4-ball centred at  $S_{\mathbb{S}^4}$ .

Near the pole  $S_{\mathbb{S}^4} = (-1, 0, 0, 0, 0)$ , the parameter  $s$  is close to  $\pi$ , for  $s \sim \pi$ , we put  $s' = \pi - s$ , now  $s' \sim 0$ , then

$$\sin s = \sin(\pi - s') = \sin s' \sim s' = \pi - s.$$

For a small  $\epsilon > 0$ , the set  $\{(\pi - s)x : (\pi - s, x) \in [0, \epsilon] \times \mathbb{S}^3\}$  parametrizes a neighborhood of the south pole  $S_{\mathbb{S}^4}$ .

The function

$$\alpha(s) = \alpha(\pi - s') \sim 2 \arctan \frac{4}{s'^2}(1 + o(s')) \sim \frac{4}{s'^2}(1 + o(s')) \sim \frac{4}{(\pi - s)^2}(1 + o(\pi - s))$$

or

$$\alpha(s) \sim \frac{4}{(\pi - s)^2}(1 + o(\pi - s)).$$

We can write  $F$  in this neighborhood as

$$(\cos \alpha(s), \sin \alpha(s)H(x)) \sim \left( (1 + o(\pi - s)), \frac{(s - \pi)^2}{2}H(x) \right). \quad (6.2)$$

Under the above identification we write  $F$  as

$$s'x \longrightarrow \frac{s'^2}{2}H(x).$$

It's clear that the south pole is a critical point of the map  $F$ .

## 6.2 Estimate of $\beta$ near the endpoints of $\left[0, \frac{\pi}{2}\right]$

We recall that  $\beta : \left[0, \frac{\pi}{2}\right] \rightarrow [0, \pi]$  is a regular function of  $u$  such that  $\beta(0) = 0$  and  $\beta\left(\frac{\pi}{2}\right) = \pi$ . Given by the formula for  $k \neq l$

$$\beta(u) = 2 \arctan \left\{ \left| \frac{l-p(u)}{l+p(u)} \right|^{\frac{1}{2}} \left| \frac{k+p(u)}{k-p(u)} \right|^{\frac{k}{2}} \right\}, \quad (6.3)$$

with

$$p(u) = \sqrt{l^2 \cos^2 u + k^2 \sin^2 u}. \quad (6.4)$$

For  $k = l$  the formula is

$$\beta(u) = 2 \arctan \left( \tan^k u \right). \quad (6.5)$$

**Lemma 2.** *Let  $\beta : \left[0, \frac{\pi}{2}\right] \rightarrow [0, \pi]$  as above*

1) *For  $u \sim 0$  we have*

$$\beta(u) = Cu^l + o(u^l) \text{ with } C \in \mathbb{R}^+. \quad (6.6)$$

2) *For  $u \sim \frac{\pi}{2}$ , let  $v = u - \frac{\pi}{2}$  we have*

$$\beta(u) = \pi - C \left( u - \frac{\pi}{2} \right)^k + o(v^k) \text{ with } C \in \mathbb{R}^+. \quad (6.7)$$

*Proof.*

1) We shall examine its behavior near a critical point, for this we use Taylor's Formula. We have

$$p^2 = l^2 \cos^2 u + k^2 \sin^2 u.$$

Then, in a neighborhood of 0 we have :

$$\begin{aligned} p^2 &= l^2 \left( 1 - \frac{u^2}{2} \right)^2 + k^2 u^2 + o(u^2) \\ &= l^2 + (k^2 - l^2)u^2 + o(u^2) \\ &= l^2 \left( 1 + \frac{k^2 - l^2}{l^2} u^2 \right) + o(u^2) \end{aligned}$$

Then,

$$\begin{aligned} p &= l \sqrt{1 + \frac{k^2 - l^2}{l^2} u^2 + o(u^2)} \\ &= l \left( 1 + \frac{k^2 - l^2}{2l^2} u^2 \right) + o(u^2) \end{aligned}$$

We derive

$$\begin{aligned} l - p &= \frac{l^2 - k^2}{2l} u^2 + o(u^2) \\ l + p &= 2l + \frac{k^2 - l^2}{2l} u^2 + o(u^2) \\ \frac{l - p}{l + p} &= \frac{l^2 - k^2}{4l^2} u^2 + o(u^2) \end{aligned}$$

and

$$\begin{aligned} k - p &= k - l - \frac{k^2 - l^2}{2l}u^2 + o(u^2) \\ k + p &= l + k + \frac{k^2 - l^2}{2l}u^2 + o(u^2) \\ \frac{k + p}{k - p} &= \frac{k + l}{k - l} + o(u). \end{aligned}$$

So, we get

$$\left| \frac{l - p}{l + p} \right|^{\frac{l}{2}} = \left| \frac{l^2 - k^2}{4l^2} \right|^{\frac{l}{2}} u^l = C_1 u^l + o(u^l) \quad (6.8)$$

and

$$\left| \frac{k + p}{k - p} \right|^{\frac{k}{2}} = \left| \frac{k + l}{k - l} \right|^{\frac{k}{2}} = C_2 + o(u). \quad (6.9)$$

We put  $C_3 = C_1 C_2$ , then the product of the two estimates above (6.8) and (6.9) gives us the following

$$\left| \frac{l - p}{l + p} \right|^{\frac{l}{2}} \left| \frac{k + p}{k - p} \right|^{\frac{k}{2}} = C_3 u^l + o(u^l). \quad (6.10)$$

Finally, using (6.10) we obtain for  $\beta(u)$  the following

$$\begin{aligned} \beta(u) &= 2 \arctan(C_3 u^l) + o(u^l) \\ &= 2C_3 u^l + o(u^l) \end{aligned}$$

Consequently,

$$\beta(u) = C u^l + o(u^l) \text{ with } C = 2C_3. \quad (6.11)$$

2) Now we are going to use the same procedure as in the proof of 1) but this time in a neighborhood of  $\frac{\pi}{2}$ . Let  $v = \frac{\pi}{2} - u \geq 0$ . Then, using the trigonometric formulas, we obtain the following

$$\begin{aligned} p^2 &= l^2 \cos^2 u + k^2 \sin^2 u \\ &= l^2 \cos^2\left(\frac{\pi}{2} - v\right) + k^2 \sin^2\left(\frac{\pi}{2} - v\right) \\ &= l^2 \sin^2 v + k^2 \cos^2 v. \end{aligned} \quad (6.12)$$

Then, for  $v$  in a neighborhood of 0, we have

$$\begin{aligned} p^2 &= l^2 v^2 + k^2 \left(1 - \frac{v^2}{2}\right)^2 + o(v^2) \\ &= k^2 + (l^2 - k^2)v^2 + o(v^2) \\ &= k^2 \left(1 + \frac{l^2 - k^2}{k^2} v^2\right) + o(v^2). \end{aligned}$$

Then,

$$\begin{aligned} p &= k \sqrt{1 + \frac{l^2 - k^2}{k^2} v^2} + o(v^2) \\ &= k \left(1 + \frac{l^2 - k^2}{2k^2} v^2\right) + o(v^2) \end{aligned}$$



We derive,

$$\begin{aligned} k - p &= \frac{k^2 - l^2}{2k} v^2 + o(v^2) \\ k + p &= 2k + \frac{l^2 - k^2}{2k} v^2 + o(v^2) \\ \frac{k + p}{k - p} &= \frac{4k^2}{k^2 - l^2} \frac{1}{v^2} (1 + o(v)) \end{aligned}$$

and

$$\begin{aligned} l - p &= l - k + \frac{k^2 - l^2}{2k} v^2 + o(v^2) \\ l + p &= l + k + \frac{l^2 - k^2}{2k} v^2 + o(v^2) \\ \frac{l - p}{l + p} &= \frac{l - k}{l + k} + o(v^2). \end{aligned}$$

So, we get

$$\left| \frac{l - p}{l + p} \right|^{\frac{l}{2}} = \left| \frac{l - k}{l + k} \right|^{\frac{l}{2}} = C_1 + o(v^2) \quad (6.13)$$

and

$$\left| \frac{k + p}{k - p} \right|^{\frac{k}{2}} = \left| \frac{4k^2}{l^2 - k^2} \right|^{\frac{k}{2}} \frac{1}{v^{2k}} (1 + o(v)) = \frac{C_2(1 + o(v))}{v^{2k}}. \quad (6.14)$$

We put  $C_3 = C_1 C_2$ , then the product of the two estimates above (6.13) and (6.14) gives us the following

$$\left| \frac{l - p}{l + p} \right|^{\frac{l}{2}} \left| \frac{k + p}{k - p} \right|^{\frac{k}{2}} = \frac{C_3 (1 + o(v))}{v^k}. \quad (6.15)$$

We let  $y = \frac{\beta(u)}{2}$  be such that

$$LHS(6.15) = \tan y.$$

We put  $z = y - \frac{\pi}{2}$ , then

$$\frac{\cos y}{\sin y} \sim -z \frac{1}{1 - \frac{z^2}{2}} \sim -z + o(z^2).$$

Therefore,

$$\beta(v) \sim \pi - 2C_3 v^k. \quad (6.16)$$

Finally, using (6.16) we obtain for  $\beta(u)$  the following

$$\beta(u) = \pi - C \left( \frac{\pi}{2} - u \right)^k \quad \text{with } C = 2C_3. \quad (6.17)$$

□

### 6.3 Critical points of $\phi_{k,l}$

We go back to the map from  $\mathbb{S}^3$  into  $\mathbb{S}^2$  given by

$$\phi_{k,l}(\cos u e^{i\psi}, \sin u e^{iA}) = (\cos \beta(u), \sin \beta(u) e^{i(k\psi + lA)}),$$

where  $u \in [0, \frac{\pi}{2}]$  and  $\psi, A \in [0, 2\pi]$ .

We know that the only critical values of  $\phi_{k,l}$  are the south and north poles  $S_{\mathbb{S}^2}, N_{\mathbb{S}^2}$ .

- 1) The North pole  $N_{\mathbb{S}^2} = (1, 0, 0)$ : for  $u = 0$ , the corresponding points of  $\mathbb{S}^3$  are of the form  $P = (e^{i\psi_0}, 0, 0)$ , with  $\phi_{k,l}(P) = (1, 0, 0)$ . We now investigate the behaviour of  $\phi_{k,l}$  in a neighborhood of such a  $P$ .

We take  $\eta > 0$  small. We identify a neighborhood of the point  $P \in \mathbb{S}^3$  with  $[0, \eta] \times [0, 2\pi] \times [0, 2\pi]$  by setting

$$(u, e^{i\psi}, e^{iA}) \mapsto (\cos ue^{i\psi}, \sin ue^{iA}). \quad (6.18)$$

Note that

$$(u, A) \mapsto z = \sin ue^{iA}, \quad (6.19)$$

parametrizes a disk in polar coordinates for  $0 \leq \sin u \leq \eta$  and  $0 \leq A \leq 2\pi$ .

We write the point of  $\mathbb{S}^2$  as  $(\cos v, \sin ve^{i\mu})$  and we identify a neighborhood of  $N_{\mathbb{S}^2} = (1, 0, 0)$  with the disk  $\{\sin ve^{i\mu} : v \in [0, \eta[ \text{ and } e^{i\mu} \in \mathbb{S}^1\}$ .

**Lemma 3.** *In these two coordinates systems  $\phi_{k,l}$  can be written in a neighborhood of the North pole  $N_{\mathbb{S}^2}$  as*

$$(e^{i\psi}, z) \mapsto Cz^l e^{ik\psi} \quad (6.20)$$

*Proof.* For  $u \sim 0$ , the function

$$\beta(u) \sim Cu^l.$$

Then,

$$\sin \beta(u) \sim Cu^l.$$

Now  $|z| = \sin u$ , then

$$\sin \beta(u) \sim C|z|^l.$$

Let  $e^{i\psi} \in \mathbb{S}^1$ , then we have

$$\sin \beta(u) e^{i(k\psi + lA)} \sim C|z|^l e^{ilA} e^{ik\psi}.$$

Therefore,  $\phi_{k,l}$  can be written as

$$\phi_{k,l} : (e^{i\psi}, z) \mapsto Cz^l e^{ik\psi}.$$

□

- 2) The South pole  $S_{\mathbb{S}^2} = (-1, 0, 0)$ : now for  $u = \frac{\pi}{2}$  we get points  $Q \in \mathbb{S}^3$  of the form  $Q = (0, 0, e^{iA})$ , with  $\phi_{k,l}(Q) = (-1, 0, 0)$ . We now investigate the behavior of  $\phi_{k,l}$  in a neighborhood of such a  $Q$ . We proceed as above.

We identify a neighborhood of  $Q$  with  $[0, \eta] \times \mathbb{S}^1 \times [0, 2\pi]$  by setting

$$(u, e^{iA}, e^{i\psi}) \mapsto (\cos ue^{i\psi}, \sin ue^{iA}). \quad (6.21)$$

Note that

$$(u, \psi) \mapsto z = \cos ue^{i\psi}, \quad (6.22)$$

parametrizes a disk in polar coordinates for  $0 \leq \cos u \leq \eta$  and  $0 \leq \psi \leq 2\pi$ .

We write the point of  $\mathbb{S}^2$  as  $(\cos v, \sin ve^{i\mu})$  and we identify a neighborhood of  $S = (-1, 0, 0)$  with the disk  $\{\sin ve^{i\mu} \text{ with } v \in [0, \eta[ \text{ and } e^{i\mu} \in \mathbb{S}^1\}$ .

**Lemma 4.** *In these two coordinates system  $\phi_{k,l}$  can be written in a neighborhood of the South pole  $S_{\mathbb{S}^2}$  as*

$$(e^{iA}, z) \mapsto Cz^k e^{ilA} \quad (6.23)$$

*Proof.* For  $u \sim \frac{\pi}{2}$ , we have

$$\beta(u) \sim \pi - C \left( \frac{\pi}{2} - u \right)^k.$$

Then,

$$\sin \beta(u) \sim C \left( \frac{\pi}{2} - u \right)^k.$$

Now  $|z| = \cos u \sim \frac{\pi}{2} - u$ , then

$$\sin \beta(u) \sim C |z|^k.$$

Let  $e^{iA} \in \mathbb{S}^1$ , then we have

$$\sin \beta(u) e^{i(k\psi + lA)} \sim C |z|^k e^{ik\psi} e^{ilA}.$$

Therefore,  $\phi_{k,l}$  can be written as

$$\phi_{k,l} : (e^{iA}, z) \mapsto C z^k e^{ilA}.$$

□

## 7 Multiple fibres

### 7.1 Smooth multiple fibres

We begin by defining a notion of multiple fibres for harmonic morphisms.

**Definition 3.** Let  $\phi : M^m \rightarrow N^n$  be a harmonic morphism and let  $p_0$  be a critical value of  $\phi$  in  $N^n$  such that  $\Sigma = \phi^{-1}(p_0)$  is smooth, connected and closed. The fibre  $\Sigma = \phi^{-1}(p_0)$  is a multiple fibre of multiplicity  $\mu$  if there exists

- 1) a neighbourhood  $U$  of  $p_0$  in  $N^n$
- 2) a tubular neighbourhood  $\mathcal{T}$  of  $\Sigma$  and a projection  $\pi : \mathcal{T} \rightarrow \Sigma$  such that
  - i)  $\phi^{-1}(U) \subset \mathcal{T}$
  - ii) for every  $p \in U$ ,  $\phi^{-1}(p)$  is connected and compact
  - iii) for every  $X \in \Sigma$  and every  $p \in U$ ,  $\pi^{-1}(X)$  and  $\phi^{-1}(p)$  meet at  $\mu$  points and these intersection points have the same sign.

We let  $[\Sigma]$  be the homology class of  $\Sigma$  in  $H_{m-n}(\mathcal{T}, \mathbb{Z})$ . Then for  $p$  close enough to  $p_0$ , and by iii), the homology class of  $[\phi^{-1}(p)]$  verifies

$$[\phi^{-1}(p)] = \pm \mu [\Sigma] \tag{7.1}$$

## 8 Multiple fibres of $\phi_{kl}$ from $\mathbb{S}^3$ to $\mathbb{S}^2$

**Proposition 5.** *We consider the harmonic morphism  $\phi_{kl} : (\mathbb{S}^3, g_{kl}) \longrightarrow \mathbb{S}^2$ .*

1) *The preimage of  $N_{\mathbb{S}^2} = (1, 0, 0)$  is a multiple fibre of multiplicity  $l$ .*

2) *The preimage of  $S_{\mathbb{S}^2} = (-1, 0, 0)$  is a multiple fibre of multiplicity  $k$ .*

*Proof.* We write the proof for  $N_{\mathbb{S}^2}$  and the proof for  $S_{\mathbb{S}^2}$  is identical. We let  $\Sigma = \phi_{kl}^{-1}(N_{\mathbb{S}^2}) = \{(e^{i\psi}, 0) \in \mathbb{S}^3\}$ . We define a tubular neighbourhood  $\mathcal{T}$  by

$$\mathcal{T} = \{(\cos ue^{i\psi}, \sin ue^{iA}) : \Psi, A \in [0, 2\pi], 0 \leq \sin u < \eta\} \quad (8.1)$$

We identify

$$\mathcal{T} \simeq \Sigma \times \mathbb{D}_\eta = \{(e^{i\psi}, z)\} \quad (8.2)$$

where

$$z = \sin ue^{iA} \in \mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\} \quad (8.3)$$

and the projection  $\pi$  becomes

$$(e^{i\psi}, z) \mapsto z \quad (8.4)$$

We identify a neighbourhood of  $N_{\mathbb{S}^2}$  in  $\mathbb{S}^2$  with a disk  $D$  in  $\mathbb{C}$ ; under this identification,  $N_{\mathbb{S}^2}$  is identified to 0. In the above identification of  $\mathcal{T}$ , we write

$$\phi_{kl} : \mathcal{T} \longrightarrow D \quad (8.5)$$

$$\phi_{kl} : (e^{i\psi}, z) \mapsto C(z)z^l e^{ik\psi} \quad (8.6)$$

where  $C(z)$  is of the form  $C(z) = C + o_1(|z|)$ ,  $C$  being a non-zero complex number. Now let  $w \in D$ . Using (8.6), we can write

$$\phi_{kl}^{-1}(w) = \{(e^{i\psi}, z) : C(z)z^l e^{ik\psi} = w\}$$

If  $(e^{i\psi_0}, 0) \in \Sigma$ , a point  $(e^{i\psi_0}, z)$  belongs to  $\pi^{-1}(e^{i\psi_0}, 0) \cap \phi_{kl}^{-1}(w)$  if

$$C(z)z^l e^{ik\psi_0} = w \quad (8.7)$$

We derive

**Lemma 5.** *The equation (8.7) has  $l$  preimages.*

We now prove

**Lemma 6.** *The intersections of  $\phi_{kl}^{-1}(w)$  with the disk  $\pi^{-1}(e^{i\psi_0}, 0)$  all have the same signs.*

*Proof.* We write the coordinate  $z$  in  $D$  as  $z = x + iy$  and we compute the derivative  $d\phi_{kl}$  on  $D$ ; it verifies

$$\frac{\partial \phi_{kl}}{\partial x} = Clz^{l-1} e^{ik\psi_0} + o(|z|^l) \quad (8.8)$$

$$\frac{\partial \phi_{kl}}{\partial y} = i(Clz^{l-1} e^{ik\psi_0} + o(|z|^l)) \quad (8.9)$$

It follows from (8.8) and (8.9) that  $\text{Ker}(d\phi_{kl})$  does not contain vectors tangent to a fibre of the tubular neighbourhood  $\mathcal{T}$ . Hence  $\phi_{kl}^{-1}(w)$  is always transverse to the fibres of the tubular neighbourhood  $\mathcal{T}$ : since  $\mathcal{T} \setminus \phi_{kl}^{-1}(0)$  is connected, the sign of the intersections of  $\phi_{kl}^{-1}(w)$  with one of the fibres of  $\pi$  will be of the same sign.  $\square$

□

## 9 Singular multiple fibres

We need to adapt Def.3 to fit the case of a singular multiple fibre. First, we replace the tubular neighbourhood by the following object:

**Definition 4.** Let  $\phi : M^m \rightarrow N^n$  be a harmonic morphism and let  $p_0$  be a critical value of  $\phi$  in  $N^n$ ; suppose that  $\Sigma = \phi^{-1}(p_0)$  is smooth except at a singular set  $\mathcal{S}$  of codimension at least 2. An open set  $\mathcal{T}$  together with a surjective map

$$\pi : \mathcal{T} \rightarrow \Sigma \quad (9.1)$$

is called a singular tubular neighbourhood of  $\Sigma$  if there exists a sequence of open neighbourhoods  $U_n$  of  $\mathcal{S}$  such that the following is true:

i)  $\mathcal{S} = \bigcap U_n$

ii) for every  $n$ , the restriction of  $\pi$  to  $\pi^{-1}(\Sigma \setminus (\Sigma \cap U_n))$  is a tubular neighbourhood of  $\Sigma \setminus (\Sigma \cap U_n)$ .

We now give a modified version of Def.3.

**Definition 5.** Let  $\phi : M^m \rightarrow N^n$  be a harmonic morphism and let  $p_0$  be a critical value of  $\phi$  in  $N^n$  such that  $\Sigma = \phi^{-1}(p_0)$  is compact and smooth outside of a subset  $\mathcal{S}$  of codimension at least 2.

The fibre  $\Sigma = \phi^{-1}(p_0)$  is a multiple fibre of multiplicity  $\mu$  if there exists a singular tubular neighbourhood  $\pi : \mathcal{T} \rightarrow \Sigma$  of  $\Sigma$  such that for every  $X \in \Sigma \setminus \mathcal{S}$ , there exists a neighbourhood  $V_X$  of  $p_0$  such that for every  $p \in V_X$ ,  $\pi^{-1}(X)$  and  $\phi^{-1}(p)$  meet at  $\mu$  points and these intersection points all have the same sign.

## 10 Multiple fibres of $\Phi_{kl}$ from $\mathbb{S}^4$ to $\mathbb{S}^2$

**Proposition 6.** We consider the harmonic morphism  $\Phi_{kl} : (\mathbb{S}^4, g_{kl}) \rightarrow \mathbb{S}^2$ .

1) The preimage of  $S_{\mathbb{S}^2} = (-1, 0, 0)$  is a multiple fibre of multiplicity  $l$

2) The preimage of  $N_{\mathbb{S}^2} = (1, 0, 0)$  is a multiple fibre of multiplicity  $k$ .

*Proof.* Since the preimage of  $S_{\mathbb{S}^2}$  is smooth and the preimage of  $N_{\mathbb{S}^2}$  is not, we treat both cases separately.

1) We recall the map  $F$  from the preimage of  $S_{\mathbb{S}^2}$  in  $\mathbb{S}^4$  (which we denote  $\Sigma_S$ ) and  $\mathbb{S}^3$

$$F \left( 0, \frac{\sqrt{2}}{2} \cos a, \frac{\sqrt{2}}{2} \sin a, \frac{\sqrt{2}}{2} \cos b, \frac{\sqrt{2}}{2} \sin b \right) = \left( 0, e^{i(a+b)} \right) \quad (10.1)$$

We introduce the tubular neighbourhood  $\mathcal{T}$  of  $\Sigma_S$  as  $] - \epsilon, \epsilon[ \times ] - \epsilon, \epsilon[ \times \Sigma$ ; we parametrize it as  $(\cos s, y) \times \left( 0, \frac{\sqrt{2}}{2} e^{ia}, \frac{\sqrt{2}}{2} e^{ib} \right) \mapsto T_S \left( \cos s, y, \frac{\sqrt{2}}{2} e^{ia}, \frac{\sqrt{2}}{2} e^{ib} \right)$  with

$$T_S \left( \cos s, y, \frac{\sqrt{2}}{2} e^{ia}, \frac{\sqrt{2}}{2} e^{ib} \right) = \left( \cos s, \sin s \sqrt{1-y} \frac{\sqrt{2}}{2} e^{ia}, \sin s \sqrt{1+y} \frac{\sqrt{2}}{2} e^{ib} \right) \quad (10.2)$$

Thus the fibre of an element of  $\Sigma$  in the tubular neighbourhood  $\mathcal{T}$  is parametrized by

$$(\cos s, y) \in ] - \epsilon, +\epsilon[ \times ] - \epsilon, +\epsilon[.$$

We fix  $w$  close to  $S_{\mathbb{S}^2}$  and we fix  $e^{ia}, e^{ib}$ : we look for  $\cos s, y$  close to 0 such that

$$\Phi_{kl} \left( T_S \left( \cos s, y, \frac{\sqrt{2}}{2} e^{ia}, \frac{\sqrt{2}}{2} e^{ib} \right) \right) = w \quad (10.3)$$

We compute  $F \left( T_S \left( \cos s, y, \frac{\sqrt{2}}{2} e^{ia}, \frac{\sqrt{2}}{2} e^{ib} \right) \right)$

$$= \left( \cos \alpha(s), \sin \alpha(s) H \left( \sqrt{1-y} \frac{\sqrt{2}}{2} e^{ia}, \sin s \sqrt{1+y} \frac{\sqrt{2}}{2} e^{ib} \right) \right)$$

where  $H : \mathbb{S}^3 \rightarrow \mathbb{S}^2$  is the Hopf fibration; thus we translate (10.3) into

$$\phi_{kl} \left( \cos \alpha(s), -y \sin \alpha(s), \sin \alpha(s) \sqrt{1-y^2} e^{i(a+b)} \right) = w \quad (10.4)$$

Similarly to (8.7) or rather its equivalent for  $S_{\mathbb{S}^2}$ , we rewrite (10.4) as

$$C(\cos \alpha(s) - iy \sin \alpha(s))^k (1 + o(\|(\cos s, y)\|)) e^{il(a+b)} = w \quad (10.5)$$

This gives us  $k$  values for the couple  $(\cos s, y)$ .

To see that they are all of the same sign, we proceed as in Lemma 6 and we take the partial derivatives of (10.5) w.r.t.  $s$  and  $y$ . Using the fact that  $\sin \alpha(s)$  is close to 1, we see that these partial derivatives are linearly independent. It follows that the fibres of the tubular neighbourhood  $\mathcal{T}_S$  are always transverse to the preimages of points  $w$  close to  $S$ .

This concludes the proof of Lemma 6 1).

We now prove Lemma 6 2) using the definition above of singular multiple fibres. The preimage of  $N_{\mathbb{S}^2}$  is

$$\Phi_{kl}^{-1}(N) = \{(\cos s, \sin s(\cos t e^{ia}, \sin t e^{ib})) \text{ with } \cos t \sin t = 0\} \quad (10.6)$$

A singular tubular neighbourhood will be given by

$$\mathcal{T}_N = \{(\cos s, \sin s(\cos t e^{ia}, \sin t e^{ib})) \text{ with } |\cos t \sin t| < \eta\} \quad (10.7)$$

for  $\eta$  small enough. Since  $\cos^2 t + \sin^2 t = 1$ ,  $\mathcal{T}_N$  will split into the union of  $\mathcal{T}_1$  and  $\mathcal{T}_2$  where

$$\mathcal{T}_1 \text{ ( resp. } \mathcal{T}_2) = \{(\cos s, \sin s(\cos t e^{ia}, \sin t e^{ib})) \text{ with } |\cos t| < \eta \text{ ( resp. } |\sin t| < \eta)\}. \quad (10.8)$$

Note that  $\mathcal{T}_1$  and  $\mathcal{T}_2$  intersect only at the two poles of  $\mathbb{S}^4$ .

Now fix  $p \in \Phi_{kl}^{-1}(N_{\mathbb{S}^2})$ .

If  $p = (\cos s, \sin s e^{ia}, 0)$  ( resp.  $p = (\cos s, 0, \sin s e^{ib})$ ), then

$$\pi^{-1}(p) = \{(\cos s, \sin s \cos t e^{ia}, \sin s \sin t e^{i\theta}) / |\sin t| < \eta\} \quad (10.9)$$

$$\text{( resp. } \pi^{-1}(p) = \{(\cos s, \sin s \cos t e^{i\theta}, \sin s \sin t e^{ib}) / |\cos t| < \eta\}). \quad (10.10)$$

The fibre  $\pi^{-1}(p)$  is parametrized by

$$z = \sin t e^{i\theta} \text{ ( resp. } z = \cos t e^{i\theta}). \quad (10.11)$$

We now show that if  $w$  is close to  $N_{\mathbb{S}^2}$  and  $p$  is of the form  $p = (\cos s, \sin s e^{ia}, 0)$ , with  $\cos s \neq \pm 1$ , then  $\pi^{-1}(p) \cap \Phi_{kl}^{-1}(w)$  contains  $k$  points and that these intersection points have the same sign. The

proof of the same fact for  $p$  of the form  $p = (\cos s, 0, \sin s e^{ib})$  is identical.

We let  $q \in \pi^{-1}(p)$  be a point of the type (10.9); we have

$$F(q) = (\cos \alpha(s), \sin \alpha(s) \cos 2t, \sin \alpha(s) \sin 2te^{i(\theta+a)}). \quad (10.12)$$

Changing variables in  $\mathbb{S}^3$ , we have

$$\begin{aligned} \sin u e^{i(a+\theta)} &= \sin \alpha(s_0) \sin 2te^{i(\theta+a)} = 2 \sin \alpha(s_0) \cos t e^{ia} \sin t e^{i\theta} \\ &= 2 \sin \alpha(s_0) \cos t e^{ia} z \end{aligned} \quad (10.13)$$

where  $z$  is given by (10.11). It follows that  $|\sin u| < 2\eta$ . We also derive

$$\beta(u) = C(1 + o(t))[2 \sin \alpha(s_0) \cos t]^l \sin^l t. \quad (10.14)$$

On the other hand,

$$\cos \alpha(s) + i \sin \alpha(s) \cos 2t = \cos u e^{i\psi}. \quad (10.15)$$

Since  $t$  is very small, we derive

$$\psi = \alpha(s_0) + o(t). \quad (10.16)$$

We can now write the restriction of  $\Phi_{kl} = \varphi_{kl} \circ F(q)$  to  $\pi^{-1}(p)$ . To do this, we continue using the parameter  $z$  on  $\pi^{-1}(p)$  (cf. (10.11)) and we identify a neighbourhood of  $N_{\mathbb{S}^2}$  with a small disk  $D$  in  $\mathbb{C}$ . Using (10.14) we get

$$z = \sin t e^{i\theta} \mapsto C[2 \sin \alpha(s) \cos t]^l \sin^l t (1 + o(|z|)) e^{i[k\psi + l(a+\theta)]}. \quad (10.17)$$

In other words, there exists a complex number  $Z_0$  (independent of  $s_0$ ) such that we can rewrite (10.17) as

$$z \mapsto Z_0 \sin^l \alpha(s_0) (1 + o(|z|)) z^l. \quad (10.18)$$

Hence, if  $w$  is a small enough non-zero complex number, more precisely, if

$$0 < |w| < \frac{1}{2} \eta^l |Z_0| \sin^l \alpha(s_0),$$

it has  $l$  preimages in  $\pi^{-1}(p)$ .

REMARK. We point out the contrast with the smooth multiple fibre case: the neighbourhood of  $N_{\mathbb{S}^2}$  where we look for points with  $l$  preimages in  $\pi^{-1}(X)$  depends on  $X$  and get smaller and smaller as  $X$  approaches the singularities of the singular fibre.

This being said, we proceed as in the smooth case to show that the  $l$  preimages have the same sign. The map given by (10.18) is a submersion and  $\mathcal{T}_1 \setminus S_1$  and  $\mathcal{T}_2 \setminus S_2$  are connected. Thus all the preimages in  $\mathcal{T}_1$  (resp.  $\mathcal{T}_2$ ) have the same sign. Possibly after changing the orientation on one of the 2-spheres  $S_1$  and  $S_2$ , we can ensure that these signs are all the same.  $\square$

## 11 Appendix

To make this self contained we reproduce the computation of ([Bu]).

The metric  $(g_{k,l})$  is expressed explicitly in terms of  $s, t, a, b$  by

$$g_{k,l} = \frac{2 [ds^2 + \sin s^2 (dt^2 + \cos^2 t da^2 + \sin^2 t db^2)]}{\sqrt{(k^2 \sin^2 2t + l^2 \cos^2 2t) (\sin^4 s) / 4 + l^2 \cos^2 s}}$$

For the map  $F$  to be horizontally conformal of dilation  $\lambda$ , the function  $\alpha$  must satisfy the following equation

$$\alpha'(s)^2 = \frac{4 \sin^2 \alpha(s)}{\sin^2 s} \quad (11.1)$$

the equation (11.1) has an explicit solution, given by

$$\alpha(s) = 2 \arctan \left( \tan^2 \left( \frac{s}{2} \right) \right).$$

The associated metric take the form :

$$\bar{g} = \frac{\sqrt{2}}{(k^2 \sin^2 s + l^2 \cos^2 s)^{1/4}} g_{\mathbb{S}^3}.$$

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