Isokinetic solutions of the Navier-Stokes equation.

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Courbes isocinétiques de l'équation de Navier- Stokes

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Résumé

Une courbe isocinétique est une courbe où la vitesse reste constante. On présente les conditions d'existence des courbes isocinétiques solutions de l'équation de Navier-Stokes lorsque le domaine est borné et que les forces extérieures ne dépendent pas du temps. Le problème se décompose alors en deux équations : Une équation de Poisson pour la pression et une équation différentielle ordinaire pour la vitesse dépendant fortement de la distribution connue à l'instant initial. En général, quand des approximations polynomiales sont justifiées, la solution de l'équation différentielle ordinaire est donnée par des séries entières de fonctions de Weierstrass-Mandelbrot. On peut aisément étendre la méthode à des équations différentielles beaucoup plus générales.

Abstract.

An isokinetic curve is a curve where velocity is constant In this paper, we exhibit the conditions to have **isokinetic curves** solutions of the Navier-Stokes equation, in particular when the domain is bounded and the external forces don't depend on time. Locally, the problem can be decomposed in two equations: A Poisson's equation for the pressure which implies Dirichlet's conditions and an ordinary differential equation on the velocity depending strongly on the known initial distribution at the origin. In general, when polynomial approximations are justified, the solution is given by power series of Weierstrass-Mandelbrot functions. This method gives many information about the behaviour of the potential solution.

Keyword

Navier-Stokes equation, isokinetic curves, ω-time, bounded polynomial iterations, Weierstrass-Mandelbrot functions.

1 - Introduction:
Presentation of the equation of Navier Stokes

Since the important J. Leray's paper on the Navier-Stokes equation (NSE) in 1934, many problems remain about the existence and the construction of a solution, at such a point that this existence constitutes one of the millennium's problems posed by Fefferman. But all the methods used to solve the question are too much dependant of the linearization of the equation. An oversee of the question can be found in the Lemarie-Rieusset's book.

1 –The millennium presentation

Fefferman [6], writer of the problem of the equation of Navier for the Clay's foundation, seems very embarrassed to formulate it: in fact, he proposes four problems A, B, C, D. Carefully, he seems accept periodical solutions to represent the physically reasonable movement. As we will see, we are not sure it is true.
On other hand, he exposes very clearly many ambiguities of the question. But, his paper is singularly imprisoned by many physical considerations, with many parameters and derivatives in relation with the non linearity of the equations.

The Navier's equations concern the unknown velocity vector $\boldsymbol{u} \in R^n$ and the pressure $p \in R^+$ of an incompressible fluid in movement. The variables are the position $\boldsymbol{x}$ and the time *t:* $\boldsymbol{x}=(\boldsymbol{x}, t)$; $\boldsymbol{x} \in R^n$ and $t \in R^+$. At time t, the velocity vector $\boldsymbol{u}$ is function of $(\boldsymbol{x}, t)$: $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_t(\boldsymbol{x})$
We have as many equations as unknown functions (n+1):

$$\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{t}} = \nu\Delta u_i - \sum_{j=1}^{j=n} u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_j} + f_i(\boldsymbol{x}, t),$$
$$i = 1,..n$$

with the null divergence:

$$\text{div}\, \boldsymbol{u} = \sum_{i=1}^{i=n} \frac{\partial u_i}{\partial x_i} = 0$$

and the known function: $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$ at the initial time*:* $t = 0$.

2 - Vector notation

- Let $\partial \boldsymbol{p} = \partial p / \partial \boldsymbol{x}$

For $i = 1, \dots, n$, we note: $N_i(\boldsymbol{x}, \boldsymbol{u}, \partial \boldsymbol{p}) = \nu\Delta u_i - \sum_{j=1}^{j=n} u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_j} + f_i(\boldsymbol{x})$

And the vectors: $\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \partial \boldsymbol{p}) = [N_1, N_2, .. N_n]$

Then: $\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \partial \boldsymbol{p}) = \nu\Delta \boldsymbol{u} - \sum_{j=1}^{j=n} u_j \frac{\partial \boldsymbol{u}}{\partial x_j} - \partial \boldsymbol{p} + \boldsymbol{f}(\boldsymbol{x})$

So, the **NSE** is: $\partial \boldsymbol{u} / \partial t = \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \partial \boldsymbol{p})$

- Let $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = -(\partial \boldsymbol{u} / \partial \boldsymbol{x})^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}, \partial \boldsymbol{p})$

And: $\partial \boldsymbol{u} \otimes \partial \boldsymbol{u} = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}$

3 - Main hypothesis $\overline{\Omega}$

In order to simplify the problem, we suppose that the fluid is located in a bounded open domain Ω such as its frontier ∂Ω is empty of fluid: so, all *quantities concerning the fluid are null on $\partial\Omega$. This frontier $\partial\Omega$ is smooth and don't move with the time.*

That means : at the time 0 and during all the time t, as Ω is open, all the fluid is far from the frontier $\partial\Omega$: $\{\boldsymbol{u}(\boldsymbol{x},t), \forall \boldsymbol{x} \in \Omega, t\}$ for this finite time t, especially at time $t = 0$; the pressure is null on $\partial\Omega$ and verifies the Dirichlet's conditions.

We can have problems with the non autonomous externally forces $\boldsymbol{f}(\boldsymbol{x},t)$. So, ***we suppose $\boldsymbol{f}(\boldsymbol{x},t)$ free of*** *t:*

$$\boldsymbol{f}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x}).$$

is at least of C^1 class.

If $\boldsymbol{u}(\boldsymbol{x},t)$ reach $\partial\Omega$, the problem change completely.

3 - Isokinetic curve

Let at time 0, the velocity $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x}_0,0)$ is known for the position $\boldsymbol{x}_0$. From this starting point $\boldsymbol{x}_0$ at time 0, we suppose that the NSE gives a velocity $\boldsymbol{u}_t = \boldsymbol{u}(\boldsymbol{x}_0,t)$ at time $t \geq 0$.

Definition

An isokinetic curve is such as:

$$\boldsymbol{u}_t = \boldsymbol{u}_0.$$

We search in this paper all the isokinetic curves except near the frontier $\partial\Omega$. This approach gives some new reflexions about the conditions and the understanding of the existence of the solutions of the Navier-Stokes equation (NSE). Note that the problem of the isokinetic curves is different from the well-known stationary solutions where $\partial\boldsymbol{u}/\partial t = 0$. (see Leray).

To find the isokinetic curves, we decompose this equation in two problems when the external forces don't depend on the time:

- The first problem is the existence of the initial pressure defined by a Poisson's equation when the velocity is known; this question seems well-known under Dirichlet's conditions, especially if the frontier $\partial\Omega$ is smooth and bounded.
- The second is the resolution of an ordinary differential equation defining the position of the velocity in function of the time. This equation is completely defined by initial conditions. The Dirichlet's conditions imply that the domain Ω is bounded and relatively compact. The technics of iteration with the Borh's theorem give conditions to obtain an interesting solution.

2 - Computation of the pressure

First, we recall:

1 - Lemma 1

If $\boldsymbol{u}(\boldsymbol{x})$ is a known function of $L^2(\Omega)$, the condition $div\,\boldsymbol{u} = 0$ implies that the pressure p follows the Poisson's equation:

$$\Delta p = \operatorname{div}\boldsymbol{f}(\boldsymbol{x}) - \partial\boldsymbol{u} \otimes \partial\boldsymbol{u} \quad \forall \boldsymbol{x} \in R^n \;, t_o \in R^+$$

With the Dirichlet's conditions. Then, the pressure is the unique solution in the *space H(Ω) of the infinitely derivable functions with compact support in Ω.*

■ The divergence operator commutes with all derivations. If we apply this operator to the first *n* equations defining $\partial\boldsymbol{u}/\partial t$, we obtain:

$$\text{div}\Delta\boldsymbol{u} = \Delta\text{div}\boldsymbol{u} = 0\ ;\ \text{div}\frac{\partial\boldsymbol{u}}{\partial t} = \frac{\partial}{\partial t}\text{div}\boldsymbol{u} = 0$$

With: $\text{div}\sum_{j=1}^{j=n} u_j\frac{\partial}{\partial x_j}\boldsymbol{u} = \partial\boldsymbol{u}\otimes\partial\boldsymbol{u} + \sum_{j=1}^{j=n} u_j\frac{\partial}{\partial x_j}\text{div}\boldsymbol{u}$

$$\text{div}\sum_{j=1}^{j=n} u_j\frac{\partial}{\partial x_j}\boldsymbol{u} = \partial\boldsymbol{u}\otimes\partial\boldsymbol{u}$$

And: $\text{div}\,\partial\boldsymbol{p} = \text{div}\nabla p = \Delta p$

So: $0 = \partial\boldsymbol{u}\otimes\partial\boldsymbol{u} + \Delta p - \text{div}\boldsymbol{f}(\boldsymbol{x})$

The pressure is only function of $\boldsymbol{x}$ and of $\boldsymbol{u}(\boldsymbol{x})$. If $\boldsymbol{u}(\boldsymbol{x})$ is known, the unicity of p depends on the unicity of the solution of the Poisson's equation:

$$\Delta p = \text{div}\boldsymbol{f}(\boldsymbol{x}) - \partial\boldsymbol{u}\otimes\partial\boldsymbol{u}$$

But, the solution of the Poisson's equation is given modulo a function satisfying the Laplace's equation. Limit conditions such as the Dirichlet's conditions are necessary to prove the existence and the uniqueness of this solution. This condition doesn't appear clearly in the text of Fefferman, but many authors, as Tao, know it. ■

The Dirichlet's condition implies that *Ω* is bounded.

2 - Lemma 2

If, at any arbitrary fixed time t ≧0, the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ is a $L^2(\Omega)$ known function of the position $\boldsymbol{x}$: $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_t(\boldsymbol{x})$ and if frontier ∂Ω is smooth and don't move with the time, then, the pressure is the unique solution in the space H(Ω), which is the adherence of the infinitely derivable functions with compact support in Ω .

If the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ is a smooth known $C^\infty(\Omega)$ function of the position $\boldsymbol{x}$ and smooth on ∂Ω, then, the pressure is $C^\infty(\Omega)$.

■ We obtain the same Poisson's equation with the same limit conditions:

$$\Delta p = \text{div}\boldsymbol{f}(\boldsymbol{x}) - \partial\boldsymbol{u}\otimes\partial\boldsymbol{u}$$

So, at time t ≧0, the existence and the uniqueness of the solution is valid. ■

3 – Isokinetic curves in a small neighbourhood of 0

First, we present some properties of the isokinetic curves if they exist.

Lemma 3

A differentiable isokinetic curve can exist only under the condition:

$$|\partial\boldsymbol{u}_t/\partial\boldsymbol{x}| \neq 0$$

Then, we have: $d\boldsymbol{x}/dt = -(\partial\boldsymbol{u}(\boldsymbol{x})/\partial\boldsymbol{x})^{-1}\ \partial\boldsymbol{u}(\boldsymbol{x})/\partial t$

■ If $\boldsymbol{u}_t$ is a differentiable isokinetic curve, we have $\boldsymbol{u}(\boldsymbol{x}+d\boldsymbol{x}, t+dt) = \boldsymbol{u}(\boldsymbol{x},t)$.

That means: $\frac{\partial \boldsymbol{u}(x)}{\partial t} dt + \frac{\partial \boldsymbol{u}(x)}{\partial x} d\boldsymbol{x} = 0$. Then an unique isokinetic curve can exist only if $\partial \boldsymbol{u}_t(\boldsymbol{x}) / \partial \boldsymbol{x}$ is invertible. So, we must have $|\partial \boldsymbol{u}_t / \partial \boldsymbol{x}| \neq 0$. ■

1 - Hypothesis H0 at a fixed time $t=0$:

- $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$ *is a known smooth function of* $\boldsymbol{x} \in R^n$ *, at the fixed time* 0.
- *The jacobian matrix* $\partial \boldsymbol{u}_{to} / \partial \boldsymbol{x}$ *is invertible. If* $\partial \Omega_0 = \{\boldsymbol{x} | \; |\partial \boldsymbol{u}_0 / \partial \boldsymbol{x}| = 0\}$ *, that means: if* $\boldsymbol{x} \notin \partial \Omega_0$*, the domain* $\Omega \cap \overleftarrow{\Omega}_0$ *of* $\boldsymbol{x}$ *is (Dirichlet) bounded under the previous condition for* 0.

We note **H**t^* if **H0** is true at each time between 0 and t.

2 – Proposition 1

If the isokinetic solution $\boldsymbol{u}_t$ *of the NSE for* $\forall t \geq 0$ *exists under* **H0**, *for all* $\boldsymbol{x}_t \in \Omega \cap \Omega_0$*, , and this solution is:*

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_0(\boldsymbol{x}(t)) = \boldsymbol{u}_0(\boldsymbol{x}_t)$$

where $\boldsymbol{u}_0$ *is the known initial velocity, then* $\boldsymbol{x}_t$ *must be the solution of the ordinary differential equation with the known function* $\boldsymbol{F}_0$ *of* $\boldsymbol{x}_t$*:*

$$\mathrm{d}\boldsymbol{x}_t / \mathrm{d}t = \boldsymbol{F}_0(\boldsymbol{x}_t) = -(\partial \boldsymbol{u}_0(\boldsymbol{x}_t) / \partial \boldsymbol{x}_t)^{-1} \boldsymbol{N}_0(\boldsymbol{x}_t)$$

for each initial condition: $\boldsymbol{x}_0 = \boldsymbol{x}(\boldsymbol{x}_0, 0) \in \Omega \cap \overleftarrow{\Omega}_0$

The pressure verifies the Poisson equation at $\boldsymbol{x}_t$*:*

$$\Delta p_t = \mathrm{div} f(\boldsymbol{x}_t) - \partial \boldsymbol{u}_t \otimes \partial \boldsymbol{u}_t$$

If $\boldsymbol{f}$ *does not depend of* t*, the isokinetic curves are isobaric curves.*

■ At time 0 , the pressure p_0 must satisfy the Poisson's equation for $\boldsymbol{f}(\boldsymbol{x})$ fixed and the known function $\boldsymbol{u}_0(\boldsymbol{x})$ at $\boldsymbol{x}$:

$$\Delta p_0 = \mathrm{div} \boldsymbol{f}(\boldsymbol{x}) - \partial \boldsymbol{u}_0 \otimes \partial \boldsymbol{u}_0 = h_0(\boldsymbol{x})$$

If $\boldsymbol{x}$ has a good smooth frontier (as under Dirichlet's conditions), the solution is unique. And $\boldsymbol{N}_0(\boldsymbol{x})$ is a known function of $\boldsymbol{x}$ **.**

Now, we study $\boldsymbol{x}_t = \boldsymbol{x}_t(\boldsymbol{x}_0, t)$ solution of the NSE for $\forall t \geq 0$ and a initial position $\boldsymbol{x}_0$. As $\boldsymbol{u}_0(\boldsymbol{x}_t)$ is known but $\boldsymbol{x}_t = \boldsymbol{x}(\boldsymbol{x}_0, t)$ is unknown, the NSE becomes in a neighbourhood $\mathcal{V}^+(0)$:

$$\partial \boldsymbol{u}_0(\boldsymbol{x}_t) / \partial t = \boldsymbol{N}_0(\boldsymbol{x}_t)$$

But

$$\partial \boldsymbol{u}_0(\boldsymbol{x}_t) / \partial t = -\partial \boldsymbol{u}_0(\boldsymbol{x}_t) / \partial \boldsymbol{x}_t . \mathrm{d}\boldsymbol{x}_t / \mathrm{d}t$$

Then:

$$\mathrm{d}\boldsymbol{x}_t / \mathrm{d}t = -(\partial \boldsymbol{u}_0 / \partial \boldsymbol{x}_t)^{-1} \boldsymbol{N}_0(\boldsymbol{x}_t).$$

$$\mathrm{d}\boldsymbol{x}_t / \mathrm{d}t = \boldsymbol{F}_0(\boldsymbol{x}_t)$$

Where $\boldsymbol{F}_0(\boldsymbol{x}_t)$ is a known function of only $\boldsymbol{x}_t$ if $\partial \boldsymbol{u}_0(\boldsymbol{x}_t) / \partial \boldsymbol{x}_t$ is invertible. The NSE is reduced to an ordinary differential equation (ONSE) near 0 with the initial position $\boldsymbol{x}_0 = \boldsymbol{x}(\boldsymbol{0})$.

Suppose that $\boldsymbol{x}_t = \boldsymbol{x}_t(\boldsymbol{x}_0, t)$ be the solution of ONSE for the initial position $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ at time t. The velocity at time $t \geqq 0$ for the known initial position $\boldsymbol{x}_0$ is:

$$\boldsymbol{u}(\boldsymbol{x}_0, t) = \boldsymbol{u}_0\big(\boldsymbol{x}(\boldsymbol{x}_o, t)\big)$$

and the NSE at time t remains yet valid for the new velocity $\boldsymbol{u}_0(\boldsymbol{x}_t)$ in a neighbourhood $\mathcal{V}^+(0)$ of 0 , if this solution exists.
As the velocity is constant along an isokinetic curve: $\boldsymbol{u}_t = \boldsymbol{u}_0$; then the pressure obeys the same Poisson's equation at time t and at time 0:

$$\Delta p_t = \mathrm{div}\boldsymbol{f}(\boldsymbol{x}) - \partial\boldsymbol{u}_t \otimes \partial\boldsymbol{u}_t$$

$$\Delta p_t = \mathrm{div}\boldsymbol{f}(\boldsymbol{x}) - \partial\boldsymbol{u}_0 \otimes \partial\boldsymbol{u}_0 = \Delta p_O$$

The unicity of the solution implies that the pressure remains constant. ■

4 – Existence and unicity of $\boldsymbol{u}(\boldsymbol{x}, t)$ far from the turbulences and the frontiers

Now, let $\boldsymbol{g}(\boldsymbol{x}, t)$ be a continuous function defined on $O\times I$, where O is an open set of R^n and I an open set of R. $(\boldsymbol{x}_0, t_0) \in O\times I$. **We recall** well-known Cauchy-Lipchitz results for the ODE:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{g}(\boldsymbol{x}, t)$$

1 - Proposition 2

Let $\boldsymbol{g}(\boldsymbol{x}, t)$ a continuous function defined on $O\times I$, where O is an open set of R^n and I an open set of R. $(\boldsymbol{x}_0, t_0) \in O\times I$. We suppose that $\boldsymbol{g}$ is locally lipchitzian with respect to $\boldsymbol{x}$. Then, the Cauchy's problem:
$\frac{dx}{dt} = \boldsymbol{g}(\boldsymbol{x}, t)$ with the initial condition $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$,
has unique solution in a neighbourhood of t_0.
If $\boldsymbol{g}$ is of class C^1, the unique solution $\boldsymbol{x}$ is C^1 and maximal.

2 - Remark

If $|\partial\boldsymbol{u}_o/\partial\boldsymbol{x}| = 0$, the corresponding surface is a singularity for the NSE and we have risk of turbulences all along this surface. So, before to do anything, we have to note the points such as $\Omega_o = \{x|\ |\partial\boldsymbol{u}_o/\partial\boldsymbol{x}| = 0\}$, then, we study the behaviour of the process far from these points as possible.
So, we define the time:

$$\omega = t_{\Omega 0} = \frac{\inf t}{\boldsymbol{x}_0 \in \Omega}(t|\ \boldsymbol{x}_t(\boldsymbol{x}_0, t) \in \Omega_0).$$

But, we note that $\boldsymbol{x}_t$ remains bounded.
We will clarify this condition in the next proposition.

3 – Proposition 3

Let $\boldsymbol{u}_o(\boldsymbol{x})$ be the known smooth function of the position $\boldsymbol{x}$ at time $t = 0$.
*In a neighbourhood of 0 : $0 \leqq t < \omega$, under **H**0, the unique maximal solution of the NSE, is for each initial condition $\boldsymbol{x}_0 = \boldsymbol{x}(0)$:*

$$\boldsymbol{u}(\boldsymbol{x}_t, t) = \boldsymbol{u}_o(\boldsymbol{x}(t))$$

$\boldsymbol{x}_t = \boldsymbol{x}(\boldsymbol{x}_o, t)$ *is solution of the ONSE with* $\boldsymbol{F}_o$ *known for an initial position* $\boldsymbol{x}_o$*:*

$$\mathrm{d}\boldsymbol{x}_t/\mathrm{d}t = \boldsymbol{F}_o(\boldsymbol{x}_t);$$

- Then: $\Delta p_t = \operatorname{div} f(\boldsymbol{x}) - \partial \boldsymbol{u}_t \otimes \partial \boldsymbol{u}_t = h_t(\boldsymbol{x}_t)$

Where: $\boldsymbol{F}_o(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_o, \partial \boldsymbol{p}_o) = (\partial \boldsymbol{u}_o / \partial \boldsymbol{x})^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}_o, \boldsymbol{p}_o)$

With p_o*:* $\Delta p_o = \operatorname{div} f(\boldsymbol{x}) - \partial \boldsymbol{u}_o \otimes \partial \boldsymbol{u}_o = h_o(\boldsymbol{x})$

More, it exists a local flow in a neighbourhood of $\boldsymbol{u}_o$*.*

■ - We take $t_o = 0$ and write under **H0** for all $0 \leqq t < \omega$,

If $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_o(\boldsymbol{x}(t))$

Is solution of the NSE, $\boldsymbol{x}(t)$ must be solution of:

$$\mathrm{d}\boldsymbol{x}_t/\mathrm{d}t = \boldsymbol{F}_o(\boldsymbol{x}_t)$$

Where: $\boldsymbol{F}_o(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_o, \partial \boldsymbol{p}_o) = (\partial \boldsymbol{u}_o / \partial \boldsymbol{x})^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}_o, \partial \boldsymbol{p}_o)$

And p_o is the unique solution

$$\Delta p_o = \operatorname{div} f(\boldsymbol{x}) - \partial \boldsymbol{u}_o \otimes \partial \boldsymbol{u}_o = h_o(\boldsymbol{x})$$

If we note $\boldsymbol{x}_o = \boldsymbol{x}(0)$ the initial position at time $t_o = 0$, the position $\boldsymbol{x}(t)$ at time t will be $\boldsymbol{x}(t) = \boldsymbol{x}(\boldsymbol{x}_o, t)$. ■

4 - General solution far from the turbulences and the frontiers

As the solution of the ONSE defining $\boldsymbol{x}(t)$ is bounded, we have proved in our book the following result:

- The solution of the ONSE is reduced to the study of the zero of $\boldsymbol{F}_0(\boldsymbol{x})$ *(*$\boldsymbol{F}_o(\boldsymbol{\alpha}) = \boldsymbol{0}$*) and the eigen values at these points. More, if we suppose that* $\boldsymbol{F}_0(\boldsymbol{x})$ *can be uniformly approximated by polynomials, at least locally near fixed points (*$\boldsymbol{F}_o(\boldsymbol{\alpha}) = \boldsymbol{0}$*) , especially if they are hyperbolic, the solution is a power series of Weirestrass - Mandelbrot functions. The equality of Perceval is satisfied.*

■ Hint: If the space Ω of the ONSE is bounded, the image is relatively compact and we can use the Bohr – Cauchy's representation with almost periodic functions. (see my little book). Then, under the hypothesis of the hyperbolicity of the fixed points, the solution is developed in power series of Weierstrass – Mandelbrot's functions. But, here the fixed points and the eigen values at these fixed points are determined by the function $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_o, \boldsymbol{p}_o) = -(\partial \boldsymbol{u}_o / \partial \boldsymbol{x})^{-1} \boldsymbol{N}(\boldsymbol{x}, \boldsymbol{u}_o, \partial \boldsymbol{p}_o)$Anyway, we have seen that the solutions are local near each fixed points and we can meet many other perturbations when the process cross from a basin of a fixed point to an another as in the Lorenz iteration. ■

But, we can have many other turbulences when $|\partial \boldsymbol{u}_t / \partial \boldsymbol{x}| = \boldsymbol{0}$.

5 - Construction of $\boldsymbol{u}(\boldsymbol{x}, t)$ near by the singularities: the turbulences

We stay far from the frontiers

Exercise 1

In R^2, if $\partial \boldsymbol{u}_o / \partial \boldsymbol{x}$ is not invertible, we have $|\partial \boldsymbol{u}_o / \partial \boldsymbol{x}| = 0$ with $\operatorname{div} \boldsymbol{u}_o = 0$.

That means $\boldsymbol{u}_o$ belongs in R^1. Then, in R^2, the solution exists.
So we work in R^n with $n \geq 3$.

1 – Computation when $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = 0$

If $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = 0$, the corresponding surface is a singularity for the NSE and we have turbulences all along this surface. So, before to do anything, we have to note the points such as: $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = 0$.
In this case, for all $\boldsymbol{x}$: $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = 0$,we have at least a vector $\alpha(\boldsymbol{x})$ such as $\alpha(\boldsymbol{x})\ \partial \boldsymbol{u}_o/\partial \boldsymbol{x} = 0$ and $\alpha(\boldsymbol{x})\ \boldsymbol{N}_0(\boldsymbol{x}) = 0$. (We don't examine here the other situations). So, $\alpha(\boldsymbol{x})\ \partial \boldsymbol{u}(\boldsymbol{x},t)/\partial t = 0$, and then, in a neighbourhood of *0*:

$$\alpha(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x},t) = \alpha(\boldsymbol{x})\boldsymbol{u}_o(\boldsymbol{x})$$

Then, setting $\boldsymbol{u}(\boldsymbol{x},t) = (u_1(\boldsymbol{x},t), \boldsymbol{v}(\boldsymbol{x},t))$ with a coordinate $u_1(\boldsymbol{x},t) \in R$, $\boldsymbol{v}(\boldsymbol{x},t) \in R^{n-1}$, we can write $u_1(\boldsymbol{x},t)$ in function of the others under the condition $\alpha_1(\boldsymbol{x}) \neq 0$: $u_1(\boldsymbol{x},t)$=$g(\ \boldsymbol{v}(\boldsymbol{x},t))$**.**
So, we obtain a new equation:

$$\partial \boldsymbol{v}/\ \partial t = \ \boldsymbol{N}^{n-1}(\boldsymbol{x}, u_1, \boldsymbol{v}, \partial \boldsymbol{p})$$

where $\boldsymbol{N}^{n-1}$ is the restriction of $\boldsymbol{N}$ to R^{n-1}.

$$\partial \boldsymbol{v}/\ \partial t = \ \boldsymbol{N}^{n-1}(\boldsymbol{x}, g(\boldsymbol{v}), \boldsymbol{v}, \partial \boldsymbol{p})$$

We have to solve this new ODE in R^{n-1}.
But, the ONSE is now completely modified.

1 – Construction in a neighbourhood of $|\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = 0$

Now, study the behaviour of the process near these points. We need this result:

Exercise 2

Let $w \in R$ and g a k -lipchitzian function of w, with k independent of t, then, for all t_o > ω, the Cauchy,s problem of
$dw/dt = h(w,t)/w$ with $w(t_o)$>0has a unique solution.
■ With the change of variable $z = w^2$, the proof is achieved. We note that $w = 0$ is a point of singularity. More, at this point $w = 0$, $g(\sqrt{z}, t)$ is not lipchitzian. ■

Notation

. We denote $w = |\partial \boldsymbol{u}_o/\partial \boldsymbol{x}| = W(x_1, \boldsymbol{y})$, and we can write $\boldsymbol{x} = (x_1, \boldsymbol{y})$ with $x_1 \in R$ a coordinate of $\boldsymbol{x}$ and $\boldsymbol{y} \in R^{n-1}$ all the other coordinates of $\boldsymbol{x}$. W is smooth and we apply the implicit theorem $x_1 = W^{-1}(w, \boldsymbol{y})$**.** To simplify the problem, we choose x_1 such as W will be invertible in a very large domain.

We write the ONSE with the comatrix $Com(W)$of W:

$$dx_1/\mathrm{d}t = -Com(W)_1 F_{o1}(\boldsymbol{x})/w = \partial W^{-1}/\partial w . dw/\mathrm{d}t + \partial W^{-1}/d\boldsymbol{y} . d\boldsymbol{y}/\mathrm{d}t$$

That means:

$$dw/\mathrm{d}t = -h(w, \boldsymbol{y})/w$$

$$d\boldsymbol{y}/\mathrm{d}t = -Com(W)\boldsymbol{F}_o(\boldsymbol{x})/w$$

So, except at time ω where $w = 0$, the ONSE is completely modified.

2 – Computation in a neighbourhood of $|\partial \boldsymbol{u}_t / \partial \boldsymbol{x}| = \mathbf{0}$
It is easy to check the coherence of the solution $\boldsymbol{u}_t(\boldsymbol{x}) = \boldsymbol{u}_o(\boldsymbol{x}(\boldsymbol{x}_o, t))$, at every time as we have done for all $0 \leqq t < \omega$. But we have to consider all the turbulences where $|\boldsymbol{\partial u_t / \partial x}| = \mathbf{0}$ at each time t.

6 - Conclusion

The paradigm of the problem has completely moved: the equations of the movement depend entirely on the given initial situation $\boldsymbol{u}_o(\boldsymbol{x})$. But the good solution is local near by the initial $\boldsymbol{u_o}(\boldsymbol{x})$ when there is no turbulence. Even in this case, the situation is complicate because the solution is developed in power series of Weierstrass – Mandelbrot's functions and not with periodic functions. The mapping of the space Ω with isokinetic curves seem useful to study the NSE, but we have to study an infinity of equations with possible turbulences instead of the NSE. We note the importance of the isobaric curves, well-known in meteorology, more easy to operate than the velocity because it is a one dimensional parameter. The analysis of the isokinetic curves are more complicated when the problem is not autonomous: that is if $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$. But, the method can be used to integrate in the equation more parameters like the temperature…

New generalised equation $\boldsymbol{N}(\boldsymbol{u}, \boldsymbol{x})$
We see easily that we can take more complicated derivable function $\boldsymbol{N}(\boldsymbol{u}, \boldsymbol{x})$. The method will be the same. In $\boldsymbol{N}(\boldsymbol{u}, \boldsymbol{x})$, we can put any functions (smooth functions, derivatives, even unique solution of an equation as the pressure p) of $\boldsymbol{u}(\boldsymbol{x})$.

Definition
*Let $\boldsymbol{N}(\boldsymbol{u}, \boldsymbol{x})$ be a smooth function of $\boldsymbol{u}(\boldsymbol{x})$, and of (any smooth functions, derivatives, even unique solution of a functional or integro-differential equation of) **u**, where $\boldsymbol{u} \in R^n$ and $\boldsymbol{x} \in R^n$.*
Let: $$\partial \boldsymbol{u}(x, t) / \partial t = \boldsymbol{N}(\boldsymbol{u}, \boldsymbol{x})$$

With the initial condition at time $t = 0$, $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_o$ is a smooth known function.
We define the isokinetic curves as curves such as: $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}_o$.

Theorem
Then, the isokinetic curves solution of:
$$\partial \boldsymbol{u}(x) / \partial t = \boldsymbol{N}(\boldsymbol{u}, \boldsymbol{x})$$
Is, for all $0 \leqq t < \omega$: $\boldsymbol{u}_t(\boldsymbol{x}) = \boldsymbol{u}_o(\boldsymbol{x}(\boldsymbol{t}))$,
where $\boldsymbol{x}(\boldsymbol{t})$ is the solution of the ordinary differential equation with

$$F_o(x) = F(x, u_o) = -\left(\frac{\partial u_o}{\partial x}\right)^{-1} N(u_o, x)$$

function of x:

$$\mathrm{d}x/\mathrm{d}t = F_o(x)\,.$$

■ Hints: As $u_o(x)$ is function of x, every smooth functions, derivatives and unique solutions of a functional or integro-differential equation of u can be explained as functions of x and we can write $N(u_o, x) = N_o(x)$.
The only important things to check are:
- the jacobian $|\partial u_o/\partial x| \neq 0$,
- the unicity of the solution of the functionals in $N(x)$,
- the veracity of $\mathbf{H}t^*$ at each time *t*, and especially the *ω* point,
- the border of the domain of x.
And we have to solve the ordinary differential equation: $\mathrm{d}x/\mathrm{d}t = F(u_o(x))$ ■

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