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Ibrahim M Elbatal, Farrukh Jamal, Christophe Chesneau, Mohammed Elgarhy, Sharifah Alrajhi

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Ibrahim Elbatal[a], Farrukh Jamal[b], Christophe Chesneau[c], Mohammed Elgarhy[d], Sharifah Alrajhi[e]

[a] *Department of Mathematics and Statistics College of Science, Al Imam Mohammad Ibn Saud Islamic University, Saudi Arabia.*

[b] *Department of Statistics, The Islamia University Bahawalpur, Pakistan.*

[c] *LMNO, University of Caen, France.*

[d] *University of Jeddah, Jeddah, Kingdom of Saudi Arabia.*

[e] *Statistics Department, Faculty of Science, King AbdulAziz University, Jeddah, Kingdom of Saudi Arabia.*

Abstract

In this paper, we introduce a new continuous probability distribution with five parameters called the modified beta Gompertz distribution. It is derived from the modified beta generator proposed by Nadarajah, Teimouri and Shih (2014) and the Gompertz distribution. By investigating its mathematical and practical aspects, we prove that it is quite flexible and can be used effectively in modeling a wide variety of real phenomenon. Among others, we provide useful expansions of crucial functions, quantile function, moments, incomplete moments, moment generating function, entropies and order statistics. We discuss the estimation of the model parameters by the obtained maximum likelihood method. We also present a simulation study in order to test validity of maximum likelihood estimators. Finally, we illustrate the flexibility of the distribution by means of two applications to real data.

Keywords: Modified beta generator, Gompertz distribution, Maximum likelihood estimation.

2000 MSC: 60E05, 62E15, 62F10.

1. Introduction

The Gompertz distribution was initially introduced by Gompertz (1825) to describe human mortality and provide actuarial tables. The literature about the use of the Gompertz distribution in applied areas is enormous. A nice review can be found in

Email address: [a]iielbatal@imamu.edu.sa, [b]farrukhjamalmphil@gmail.com, [c]christophe.chesneau@unicaen.fr, [d]m_elgarhy85@yahoo.com, [e]saalrajhi@kau.edu.sa
(Ibrahim Elbatal[a], Farrukh Jamal[b], Christophe Chesneau[c], Mohammed Elgarhy[d], Sharifah Alrajhi[e])

Tjørve and Tjørve (2017), and the references there in. From a mathematical point of view, the cumulative probability density function (cdf) of the Gompertz distribution with parameters $\lambda > 0$ and $\alpha > 0$ is given by

$$G(x) = 1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}, \quad x > 0.$$

The related probability density function (pdf) is given by

$$g(x) = \lambda e^{\alpha x} e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}, \quad x > 0.$$

It can be viewed as a generalization of the exponential distribution (obtained with $\alpha \to 0$) and thus an alternative to the gamma or Weibull distribution. A feature of the Gompertz distribution is that $g(x)$ is unimodal and has positive skewness whereas the related hazard rate function (hrf) given by $h(x) = g(x)/(1 - G(x))$ is increasing. In order to increase the flexibility of the Gompertz distribution, further extensions have been proposed. A natural one is the generalized Gompertz distribution introduced by El-Gohary *et al.* (2013). By introducing an exponent parameter $a > 0$, the related cdf is given by

$$F(x) = \left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)^a, \quad x > 0.$$

The related applications show that a plays an important role in term of model flexibility. This idea was then extended by Jafari *et al.* (2014) by using the so-called beta generator introduced by Eugene *et al.* (2002). The related cdf is given by

$$\begin{aligned} F(x) &= \frac{1}{B(a,b)} \int_0^{1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}} t^{a-1}(1-t)^{b-1} dt \\ &= I_{1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}(a,b), \quad x > 0. \end{aligned} \tag{1}$$

where $a, b > 0$, $B(a,b)$ denotes the beta function defined by $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ and $I_x(a,b)$ denotes the incomplete beta function ratio defined by $I_x(a,b) = (1/B(a,b)) \int_0^x t^{a-1}(1-t)^{b-1}dt$, $x \in [0,1]$. This distribution has been recently extended by Lazhar (2018) with a five parameters distribution. It is based on the beta generator and the generalized Gompertz distribution.

Motivated by the emergence of complex data from many applied areas, other extended Gompertz distributions have been proposed in the literature. See for instance, El-Damcese *et al.* (2015) who consider the Odd Generalized Exponential generator introduced by Tahir *et al.* (2015), Roozegar *et al.* (2017) who use the the McDonald generator introduced by Alexander *et al.* (2012), Moniem and Seham (2015) and Khan *et al.* (2017) who apply the transmuted generator introduced by Shaw and Buckley (2007), Lima *et al.* (2015) and Chukwu and Ogunde (2016) who use the Kumaraswamy generator, and Lazhar (2017), Yaghoobzadeh (2017) who consider the Marshall-Olkin generator introduced by Marshall and Olkin (1997) and Shadrokh and Yaghoobzadeh (2018) who consider the Beta-G and Geometric generators.

In this paper, we present and study a distribution with five parameters extending the Gompertz distribution. It is based on the modified beta generator developed by Nadarajah *et al.* (2014) (which can also be viewed as a modification of the beta

Marshall-Olkin generator developed by Alizadeh *et al.* (2015)). The advantage of this generator is to nicely combine the advantages of the beta generator of Eugene *et al.* (2002) and the Marshall-Olkin generator of Marshall and Olkin (1997). To the best of our knowledge, its application to the Gompertz distribution has never been considered before. We provide a comprehensive description of its general mathematical properties (expansions of the cdf and pdf, quantile function, various kinds of moments, moment generating function, entropies and order statistics). The estimation of the model parameters by maximum likelihood is then discussed. Finally, we explore applications to real data sets that illustrate the usefulness of the proposed model.

The rest of the paper is organized as follows. Section 2 describes the considered modified beta Gompertz distribution. Some mathematical properties are investigated in Section 3. Section 4 provides the necessary to the estimation of the unknown parameters with the maximum likelihood method. A simulation study is performed in order to test validity of the obtained maximum likelihood estimators. To illustrate the flexibility of the resulting model, applications to two real life data sets are also given.

2. The Modified Beta Gompertz Distribution

Let $c > 0$, $G(x)$ be a cdf and $g(x)$ be a related pdf. The modified beta generator introduced by Nadarajah *et al.* (2014) is characterized by the cdf given by

$$F(x) = I_{\frac{cG(x)}{1-(1-c)G(x)}}(a, b), \tag{2}$$

By differentiation of $F(x)$, a pdf is given by

$$f(x) = \frac{c^a g(x) \left[G(x)\right]^{a-1} \left[1 - G(x)\right]^{b-1}}{B(a, b) \left[1 - (1-c)G(x)\right]^{a+b}}, \qquad x \in \mathbb{R}. \tag{3}$$

The hrf is given by

$$h(x) = \frac{c^a g(x) \left[G(x)\right]^{a-1} \left[1 - G(x)\right]^{b-1}}{B(a, b) \left[1 - (1-c)G(x)\right]^{a+b} \left(1 - I_{\frac{cG(x)}{1-(1-c)G(x)}}(a, b)\right)}, \qquad x \in \mathbb{R}.$$

Let us now present our main distribution of interest. Using the cdf $G(x)$ of the Gompertz distribution with parameters $\lambda > 0$ and $\alpha > 0$ as baseline, the cdf given by (2) becomes

$$F(x) = I_{\frac{c\left(1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)}{1-(1-c)\left(1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)}}(a, b), \quad x > 0. \tag{4}$$

The related distribution will be call the modified beta Gompertz distribution (MBGz distribution for short), also denoted by MBGz(λ, α, a, b, c). The related pdf (3) is given by

$$f(x) = \frac{c^a \lambda e^{\alpha x} e^{-\frac{\lambda b}{\alpha}(e^{\alpha x}-1)} \left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)^{a-1}}{B(a, b) \left[1 - (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)\right]^{a+b}}, \qquad x > 0. \tag{5}$$

The hrf is given by

$$h(x) = \frac{c^a \lambda e^{\alpha x} e^{-\frac{\lambda b}{\alpha}(e^{\alpha x}-1)} \left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)^{a-1}}{B(a,b)\left[1-(1-c)\left(1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)\right]^{a+b}} \left[1 - I_{\frac{c\left(1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)}{1-(1-c)\left(1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)}}(a,b)\right],$$

$x > 0.$ (6)

Figure 1 shows the plots for $f(x)$ and $h(x)$ for selected parameter values λ, α, a, b, c. We observe that these functions can take various curvature forms depending on the parameter values, showing the increasing of the flexibility of the former Gompertz distribution.

(a) (b)

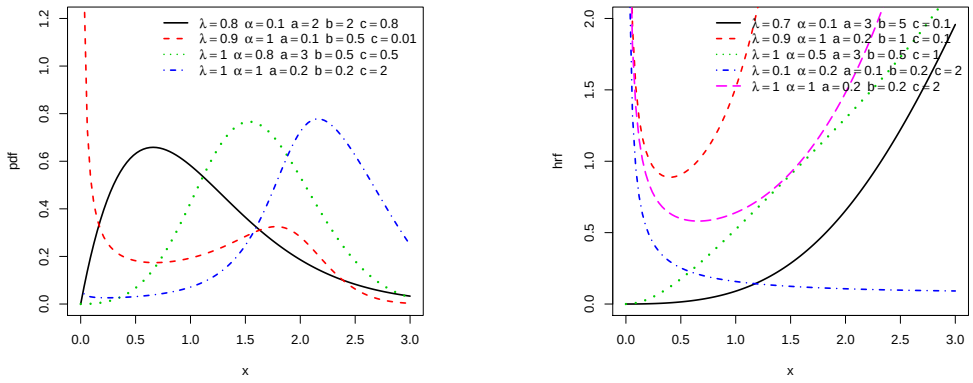

Figure 1: Some plots of the pdf $f(x)$ (a) and some plots for the hrf $h(x)$ (b).

A strong point of the MBGz distribution is to contain different useful distributions in the literature. The most popular of them are listed below.

- When $c = 1/(1-\theta)$ with $\theta \in (0,1)$ (θ is a proportion parameter), we obtain the beta Gompertz geometric distribution introduced by Shadrokh and Yaghoobzadeh (2018), i.e. with cdf

$$F(x) = I_{\frac{1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}{1-\theta e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}}(a,b), \quad x > 0.$$

  However, this distribution excludes the case $c \in (0,1)$ by construction. The importance of small values for c can also be determinant in the applications (see Section 4).

- When $c = 1$, we get the beta Gompertz distribution with four parameters introduced by Jafari *et al.* (2014), i.e. with cdf

$$F(x) = I_{1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}(a, b), \quad x > 0.$$

- When $c = b = 1$, we get the generalized Gompertz distribution studied by El-Gohary *et al.* (2013), i.e. with cdf

$$F(x) = \left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)^a, \quad x > 0.$$

- When $a = b = 1$ and $c = \frac{1}{\theta}$ with $\theta > 1$ we get the a particular case of the Marshall-Olkin extended generalized Gompertz distribution introduced by Lazhar (2017), i.e. with cdf

$$F(x) = \frac{1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}{\theta + (1-\theta)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)}, \quad x > 0.$$

- When $a = b = c = 1$, we get the Gompertz distribution introduced by Gompertz (1825), i.e. with cdf

$$F(x) = 1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}, \quad x > 0.$$

- When $c = 1$ and $\alpha \to 0$, we get beta Exponential distribution studied by Nadarajah and Kotz (2006), i.e. with cdf

$$F(x) = I_{1-e^{-\lambda x}}(a, b), \quad x > 0.$$

- When $b = c = 1$ and $\alpha \to 0$, we get the generalized exponential distribution studied by Gupta and Kundu (1999), i.e. with cdf

$$F(x) = \left(1 - e^{-\lambda x}\right)^a, \quad x > 0.$$

- When $a = b = c = 1$ and $\alpha \to 0$ we get the exponential distribution, i.e. with cdf

$$F(x) = 1 - e^{-\lambda x}, \quad x > 0.$$

3. Some mathematical properties

3.1. On the shapes of the pdf

The shapes of $f(x)$ given by (5) can be described analytically. As usual, the critical points x_* of the pdf $f(x)$ satisfies $\frac{\partial}{\partial x}\ln(f(x_*)) = 0$, with

$$\frac{\partial}{\partial x}\ln(f(x)) = \alpha - b\lambda e^{\alpha x} + (a-1)\frac{\lambda e^{\alpha x} e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}{1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}} + (a+b)(1-c)\frac{\lambda e^{\alpha x} e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}{c + (1-c)e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}.$$

A point x_* corresponds to a local maximum if $\frac{\partial^2}{\partial x^2}\ln(f(x_*)) < 0$, a local minimum if $\frac{\partial^2}{\partial x^2}\ln(f(x_*)) > 0$ and a point of inflection if $\frac{\partial^2}{\partial x^2}\ln(f(x_*)) = 0$.

Let us now study the asymptotic properties of $f(x)$. We have

$$f(x) \sim \frac{c^a}{B(a,b)}\lambda^a x^{a-1}, \quad x \to 0.$$

So, for $a \in (0,1)$, we have $\lim_{x\to 0} f(x) = +\infty$, for $a = 1$, we have $\lim_{x\to 0} f(x) = bc\lambda$ and for $a > 1$, we have $\lim_{x\to 0} f(x) = 0$. We have

$$f(x) \sim \frac{1}{c^b B(a,b)}\lambda e^{\alpha x} e^{\frac{\lambda b}{\alpha}} e^{-\frac{\lambda b}{\alpha}e^{\alpha x}}, \quad x \to +\infty.$$

Thus $\lim_{x\to+\infty} f(x) = 0$ in all cases. Figure 1 (a) illustrates these points for selected parameters.

3.2. On the shapes of the hrf

Similarly to the pdf, the critical points x_* of the hrf $h(x)$ given by (6) satisfies $\frac{\partial}{\partial x}\ln(h(x_*)) = 0$, with

$$\frac{\partial}{\partial x}\ln(h(x)) = \alpha - b\lambda e^{\alpha x} + (a-1)\frac{\lambda e^{\alpha x} e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}{1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}} + (a+b)(1-c)\frac{\lambda e^{\alpha x} e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}{c + (1-c)e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}}$$
$$+ \frac{c^a \lambda e^{\alpha x} e^{-\frac{\lambda b}{\alpha}(e^{\alpha x}-1)} \left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)^{a-1}}{B(a,b)\left[1 - (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)\right]^{a+b}} \left[1 - I_{\frac{c\left(1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)}{1-(1-c)\left(1-e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right)}}(a,b)\right].$$

A point x_* corresponds to a local maximum if $\frac{\partial^2}{\partial x^2}\ln(h(x_*)) < 0$, a local minimum if $\frac{\partial^2}{\partial x^2}\ln(h(x_*)) > 0$ and a point of inflection if $\frac{\partial^2}{\partial x^2}\ln(h(x_*)) = 0$.

We also have

$$h(x) \sim \frac{c^a}{B(a,b)}\lambda^a x^{a-1}, \quad x \to 0.$$

So, for $a \in (0,1)$, we have $\lim_{x\to 0} h(x) = +\infty$, for $a = 1$, we have $\lim_{x\to 0} h(x) = bc\lambda$ and for $a > 1$, we have $\lim_{x\to 0} h(x) = 0$. We have

$$h(x) \sim b\lambda e^{\alpha x}, \quad x \to +\infty.$$

Thus $\lim_{x\to+\infty} h(x) = +\infty$ in all cases. Figure 1 (b) illustrates these points for selected parameters.

3.3. Linear representation

Let us determine useful linear representations for $F(x)$ given by (4) and $f(x)$ given by (5). First of all, let us suppose that $c \in (0,1)$. It follows from the generalized binomial formula, i.e. $(1+z)^\gamma = \sum_{k=0}^{+\infty} \binom{\gamma}{k} z^k$ for $|z| < 1$ and $\gamma \in \mathbb{R}$, with $\binom{\gamma}{k} = \frac{\gamma(\gamma-1)\dots(\gamma-k+1)}{k!}$, that

$$\begin{aligned} F(x) &= \frac{1}{B(a,b)} \int_0^{\frac{cG(x)}{1-(1-c)G(x)}} t^{a-1}(1-t)^{b-1} dt \\ &= \frac{1}{B(a,b)} \sum_{k=0}^{+\infty} \binom{b-1}{k} (-1)^k \int_0^{\frac{cG(x)}{1-(1-c)G(x)}} t^{a+k-1} dt \\ &= \frac{1}{B(a,b)} \sum_{k=0}^{+\infty} \binom{b-1}{k} \frac{(-1)^k}{a+k} \left[\frac{cG(x)}{1-(1-c)G(x)}\right]^{a+k}. \end{aligned}$$

On the other hand, using again the generalized binomial formula, we obtain

$$\left[\frac{cG(x)}{1-(1-c)G(x)}\right]^{a+k} = c^{a+k} \sum_{\ell=0}^{+\infty} \binom{-(a+k)}{\ell} (-1)^\ell (1-c)^\ell [G(x)]^{\ell+a+k}. \tag{7}$$

In a similar manner, we have

$$[G(x)]^{\ell+a+k} = \left[1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}\right]^{\ell+a+k} = \sum_{m=0}^{+\infty} \binom{\ell+a+k}{m} (-1)^m (1 - H_m(x)),$$

where $H_m(x) = 1 - e^{-\frac{m\lambda}{\alpha}(e^{\alpha x}-1)}$ is the cdf of a Gompertz distribution with parameters $m\lambda$ and α. Combining these equalities, we obtain the following series expansion:

$$F(x) = \sum_{m=0}^{+\infty} v_m (1 - H_m(x)), \tag{8}$$

where

$$v_m = \frac{(-1)^m}{B(a,b)} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \binom{\ell+a+k}{m} \binom{-(a+k)}{\ell} \binom{b-1}{k} c^{a+k} (-1)^{k+\ell} (1-c)^\ell \frac{1}{a+k}.$$

By derivation of $F(x)$, $f(x)$ can be expressed as

$$f(x) = \sum_{m=0}^{+\infty} w_m h_m(x), \tag{9}$$

where $w_m = -v_m$ and $h_m(x)$ is the pdf of a Gompertz distribution with parameters $m\lambda$ and α.

For the case $c > 1$, we must do some transformation for the equation (7) in order to apply the generalized binomial formula. We can write

$$\begin{aligned}\left[\frac{cG(x)}{1-(1-c)G(x)}\right]^{a+k} &= \left[\frac{G(x)}{1-(1-\frac{1}{c})(1-G(x))}\right]^{a+k} \\ &= [G(x)]^{a+k}\sum_{\ell=0}^{+\infty}\binom{-(a+k)}{\ell}(-1)^{\ell}(c-1)^{\ell}c^{-\ell}[1-G(x)]^{\ell} \\ &= \sum_{\ell=0}^{+\infty}\sum_{q=0}^{\ell}\binom{-(a+k)}{\ell}\binom{\ell}{q}(-1)^{\ell+q}(c-1)^{\ell}c^{-\ell}[G(x)]^{q+a+k}.\end{aligned}$$

On the other hand, we have

$$[G(x)]^{q+a+k} = \sum_{m=0}^{+\infty}\binom{q+a+k}{m}(-1)^{m}(1-H_m(x)).$$

Therefore, we can write $F(x)$ as (8) with

$$v_m^* = \frac{(-1)^m}{B(a,b)}\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{q=0}^{\ell}\binom{-(a+k)}{\ell}\binom{\ell}{q}\binom{q+a+k}{m}\binom{b-1}{k}(-1)^{\ell+q+k}(c-1)^{\ell}c^{-\ell}\frac{1}{a+k},$$

and $f(x)$ as (9) with $w_m = -v_m^*$ (and still $h_m(x)$ is the pdf of a Gompertz distribution with parameters $m\lambda$ and α). For the sake of simplicity, we shall refer to the form (9) far all series representation of $f(x)$, whatever $c \in (0,1)$ or $c > 1$.

Hereafter, we denote by X a random variable having the cdf $F(x)$ given by (4) (and the pdf $f(x)$ given by (5)) and by Y_m a random variable following the Gompertz distribution with parameters $m\lambda$ and α, i.e. having the cdf $H_m(x)$ (and the pdf $h_m(x)$).

3.4. Quantile function

The quantile function of X is given by

$$Q(u) = \frac{1}{\alpha}\ln\left(1-\frac{\alpha}{\lambda}\ln\left(1-\frac{I_u^{-1}(a,b)}{c+(1-c)I_u^{-1}(a,b)}\right)\right), \quad u \in (0,1),$$

where $I_u^{-1}(a,b)$ denotes the inverse of $I_u(a,b)$. It satisfies $F(Q(u)) = Q(F(u)) = u$. Using Nadarajah *et al.* (2014), one can show that

$$Q(u) \sim \frac{1}{\lambda c}a^{\frac{1}{a}}B(a,b)^{\frac{1}{a}}u^{\frac{1}{a}}, \quad u \to 0.$$

From $Q(u)$, we can simulate the MBGz distribution. Indeed, let U be a random variable following the uniform distribution over $(0,1)$. Then the random variable $X = Q(U)$ follows the MBGz distribution.

The median of X is given by $M = Q(1/2)$. We can also use $Q(u)$ to define skewness measures. Let us just introduce the Bowley skewness based on quartiles and the Moors kurtosis respectively defined by

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}, \qquad M_o = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

Contrary to γ_1 and γ_2, these quantities have the advantage to be always defined. We refer to Kenney and Keeping (1962) and Moors (1988).

3.5. Moments

Let r be a positive integer. The r-th ordinary moment of X is defined by $\mu'_r = \mathbb{E}(X^r) = \int_{-\infty}^{+\infty} x^r f(x)dx$. Using the linear representation given by (9), we can express μ'_r as

$$\mu'_r = \sum_{m=0}^{+\infty} w_m \int_{-\infty}^{+\infty} x^r h_m(x)dx = \sum_{m=0}^{+\infty} w_m \mathbb{E}(Y_m^r).$$

By doing the change of variables $u = e^{\alpha x}$, we obtain

$$\mathbb{E}(Y_m^r) = \frac{m\lambda}{\alpha^{r+1}} e^{\frac{m\lambda}{\alpha}} \int_1^{+\infty} (\ln u)^r e^{-\frac{m\lambda}{\alpha}u} du.$$

This integral has connections with the so-called generalized integro-exponential function. Further developments can be found in Milgram (1985) and Lenart (2014). Therefore we have

$$\mu'_r = \sum_{m=0}^{+\infty} w_m \frac{m\lambda}{\alpha^{r+1}} e^{\frac{m\lambda}{\alpha}} \int_1^{+\infty} (\ln u)^r e^{-\frac{m\lambda}{\alpha}u} du.$$

Obviously, the mean of X is given by $\mathbb{E}(X) = \mu'_1$ and the variance of X is given by $\mathbb{V}(X) = \mu'_2 - (\mu'_1)^2$.

3.6. Skewness

The r-th central moment of X is given by $\mu_r = \mathbb{E}[(X - \mu'_1)^r]$. It follows from the binomial formula that

$$\mu_r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k (\mu'_1)^k \mu'_{r-k}.$$

On the other side, the r-th cumulants of X can be obtained via the equation:

$$\kappa_r = \mu'_r - \sum_{k=1}^{r-1} \binom{r-1}{k-1} \kappa_k \mu'_{r-k},$$

with $\kappa_1 = \mu'_1$. The skewness of X is given by $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and the kurtosis of X is given by $\gamma_2 = \kappa_4/\kappa_2^2$. One can also introduce the MacGillivray skewness given by

$$\rho(u) = \frac{Q(1-u) + Q(u) - 2Q(1/2)}{Q(1-u) - Q(u)}, \quad u \in (0,1).$$

It illustrates the effects of the parameters a, b, α and λ on the skewness. Further details can be found in MacGillivray (1986).

3.7. Moment generating function

The moment generating function of X is given by $M_X(t) = \mathbb{E}\left(e^{tX}\right) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx$. Using (9), we have

$$M_X(t) = \sum_{m=0}^{+\infty} w_m \int_{-\infty}^{+\infty} e^{tx} h_m(x)dx = \sum_{m=0}^{+\infty} w_m M_{Y_m}(t),$$

where $M_{Y_m}(t) = \mathbb{E}(e^{tY_m})$, the moment generating function of Y_m. Doing successively the change of variables $u = e^{\alpha x}$ and the change of variable $v = \frac{m\lambda}{\alpha}u$, we obtain

$$\begin{aligned} M_{Y_m}(t) &= \frac{m\lambda}{\alpha} e^{\frac{m\lambda}{\alpha}} \int_1^{+\infty} u^{\frac{t}{\alpha}} e^{-\frac{m\lambda}{\alpha}u} du = e^{\frac{m\lambda}{\alpha}} \left(\frac{\alpha}{m\lambda}\right)^{\frac{t}{\alpha}} \int_{\frac{m\lambda}{\alpha}}^{+\infty} v^{\frac{t}{\alpha}} e^{-v} dv \\ &= e^{\frac{m\lambda}{\alpha}} \left(\frac{\alpha}{m\lambda}\right)^{\frac{t}{\alpha}} \Gamma\left(\frac{t}{\alpha} + 1, \frac{m\lambda}{\alpha}\right), \end{aligned}$$

where $\Gamma(d, x)$ denotes the complementary incomplete gamma function defined by $\Gamma(d, x) = \int_x^{+\infty} t^{d-1} e^{-t} dt$. Therefore we can write

$$M_X(t) = \sum_{m=0}^{+\infty} w_m e^{\frac{m\lambda}{\alpha}} \left(\frac{\alpha}{m\lambda}\right)^{\frac{t}{\alpha}} \Gamma\left(\frac{t}{\alpha} + 1, \frac{m\lambda}{\alpha}\right).$$

Alternatively, using the moments of X, one can write

$$M_X(t) = \sum_{r=0}^{+\infty} \frac{t^r}{r!} \mu'_r = \sum_{r=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{t^r}{r!} w_m \frac{m\lambda}{\alpha^{r+1}} e^{\frac{m\lambda}{\alpha}} \int_1^{+\infty} (\ln u)^r e^{-\frac{m\lambda}{\alpha}u} du.$$

3.8. Incomplete moments and mean deviations

The r-th incomplete moment of X is defined by $m_r(t) = \mathbb{E}\left(X^r 1_{\{X \leq t\}}\right) = \int_{-\infty}^{t} x^r f(x)dx$. Using (9), we can express $m_r(t)$ as

$$m_r(t) = \sum_{m=0}^{+\infty} w_m \int_{-\infty}^{t} x^r h_m(x)dx.$$

Doing successively the change of variables $u = e^{\alpha x}$, we obtain

$$\int_{-\infty}^{t} x^r h_m(x)dx = \frac{m\lambda}{\alpha^{r+1}} e^{\frac{m\lambda}{\alpha}} \int_1^{e^{\alpha t}} (\ln u)^r e^{-\frac{m\lambda}{\alpha}u} du.$$

The mean deviation of X about the mean is given by

$$\delta_1 = \mathbb{E}(|X - \mu'_1|) = 2\mu'_1 F(\mu'_1) - 2m_1(\mu'_1),$$

where $m_1(t)$ denote the first incomplete moment. The mean deviation of X about the median $M = Q(1/2)$ is given by

$$\delta_2 = \mathbb{E}(|X - M|) = \mu'_1 - 2m_1(M).$$

3.9. Entropies

Let us now investigate different kinds of entropy measures. The Rényi entropy of X is defined by

$$\mathcal{I}_\gamma(X) = \frac{1}{1-\gamma} \ln \left[\int_{-\infty}^{+\infty} [f(x)]^\gamma \, dx \right],$$

with $\gamma > 0$ and $\gamma \neq 1$. It follows from (3) that

$$[f(x)]^\gamma = \frac{c^{a\gamma} [g(x)]^\gamma \, [G(x)]^{\gamma(a-1)} \, [1-G(x)]^{\gamma(b-1)}}{B(a,b)^\gamma \, [1-(1-c)G(x)]^{\gamma(a+b)}}.$$

The generalized binomial formula implies that

$$\frac{[G(x)]^{\gamma(a-1)}}{[1-(1-c)G(x)]^{\gamma(a+b)}} = \sum_{k=0}^{+\infty} \binom{-\gamma(a+b)}{k} (-1)^k (1-c)^k [G(x)]^{k+\gamma(a-1)}.$$

Similarly, we have

$$[G(x)]^{k+\gamma(a-1)} = \sum_{\ell=0}^{+\infty} \binom{k+\gamma(a-1)}{\ell} (-1)^\ell [1-G(x)]^\ell.$$

Therefore

$$[f(x)]^\gamma = \frac{c^{a\gamma}}{B(a,b)^\gamma} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \binom{-\gamma(a+b)}{k} \binom{k+\gamma(a-1)}{\ell} (-1)^{k+\ell} (1-c)^k [1-G(x)]^{\ell+\gamma(b-1)} [g(x)]^\gamma.$$

By doing the change of variable $u = e^{\alpha x}$ and the change of variable $v = (\ell+\gamma b)\frac{\lambda}{\alpha} u$, we get

$$\begin{aligned}
\int_{-\infty}^{+\infty} [1-G(x)]^{\ell+\gamma(b-1)} [g(x)]^\gamma dx &= \int_0^{+\infty} e^{-(\ell+\gamma b)\frac{\lambda}{\alpha}(e^{\alpha x}-1)} \lambda^\gamma e^{\alpha\gamma x} dx \\
&= \lambda^\gamma \frac{1}{\alpha} e^{(\ell+\gamma b)\frac{\lambda}{\alpha}} \int_1^{+\infty} u^{\gamma-1} e^{-(\ell+\gamma b)\frac{\lambda}{\alpha} u} du \\
&= \frac{\alpha^{\gamma-1}}{(\ell+\gamma b)^\gamma} e^{(\ell+\gamma b)\frac{\lambda}{\alpha}} \int_{(\ell+\gamma b)\frac{\lambda}{\alpha}}^{+\infty} v^{\gamma-1} e^{-v} dv \\
&= \frac{\alpha^{\gamma-1}}{(\ell+\gamma b)^\gamma} e^{(\ell+\gamma b)\frac{\lambda}{\alpha}} \Gamma\left(\gamma, (\ell+\gamma b)\frac{\lambda}{\alpha}\right).
\end{aligned}$$

By putting the above equalities together, we have

$$\mathcal{I}_\gamma(X) = \frac{1}{1-\gamma} \left[\alpha\gamma \ln(c) - \gamma \ln(B(a,b)) + (\gamma-1)\ln(\alpha) + \frac{\gamma b \lambda}{\alpha} + \ln\left[\sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \binom{-\gamma(a+b)}{k} \binom{k+\gamma(a-1)}{\ell} (-1)^{k+\ell} (1-c)^k \frac{e^{\ell\frac{\lambda}{\alpha}}}{(\ell+\gamma b)^\gamma} \Gamma\left(\gamma, (\ell+\gamma b)\frac{\lambda}{\alpha}\right) \right] \right].$$

The γ-entropy is defined by

$$H_\gamma(X) = \frac{1}{\gamma - 1} \ln \left[1 - \int_{-\infty}^{+\infty} [f(x)]^\gamma \, dx \right].$$

Using the expansion above, we obtain

$$H_\gamma(X) = \frac{1}{\gamma - 1} \ln \left[1 - \frac{c^{a\gamma} \alpha^{\gamma-1} e^{\gamma b \frac{\lambda}{\alpha}}}{B(a,b)^\gamma} \times \right.$$
$$\left. \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \binom{-\gamma(a+b)}{k} \binom{k+\gamma(a-1)}{\ell} (-1)^{k+\ell} (1-c)^k \frac{e^{\ell \frac{\lambda}{\alpha}}}{(\ell + \gamma b)^\gamma} \Gamma\left(\gamma, (\ell + \gamma b)\frac{\lambda}{\alpha}\right) \right].$$

The Shannon entropy of X is defined by $S(X) = \mathbb{E}(-\ln[f(X)])$ is a particular case of the Rényi entropy when γ tends to 1^+.

3.10. Order statistics

Let $X_1, \ldots, X_n$ be the random sample from X and $X_{i:n}$ be the i-th order statistic. Then the pdf of $X_{i:n}$ is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} \, [1 - F(x)]^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x) [F(x)]^{j+i-1}. \end{aligned}$$

It follows from (8) and (9) that

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{m=0}^{+\infty} w_m h_m(x) \left[\sum_{k=0}^{+\infty} v_k (1 - H_k(x)) \right]^{j+i-1}.$$

Using a result from Gradshteyn and Ryzhik (2000), power series raised to a positive power as follows

$$\left(\sum_{k=0}^{+\infty} a_k x^k \right)^n = \sum_{k=0}^{+\infty} d_{n,k} x^k,$$

where the coefficients $(d_{n,k})_{k \in \mathbb{N}}$ are determined from the recurrence equation: $d_{n,0} = a_0^n$ and, for any $m \geq 1$, $d_{n,m} = (1/(m a_0)) \sum_{k=1}^{m} (k(n+1) - m) a_k d_{n,m-k}$. Therefore, noticing that $1 - H_k(x) = \left(e^{-\frac{\lambda}{\alpha}(e^{\alpha x} - 1)} \right)^k$, we have

$$\left[\sum_{k=0}^{+\infty} v_k (1 - H_k(x)) \right]^{j+i-1} = \sum_{k=0}^{+\infty} d_{j+i-1,k} (1 - H_k(x)),$$

where $d_{j+i-1,0} = v_0^{j+i-1}$ and, for any $m \geq 1$, $d_{j+i-1,m} = \frac{1}{mv_0} \sum_{k=1}^{m} (k(j+i)-m) v_k d_{j+i-1,m-k}$. By combining the equalities above, we obtain

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} w_m d_{j+i-1,k} h_m(x)(1 - H_k(x)).$$

Finally, one can observe that $h_m(x)(1-H_k(x)) = m\lambda e^{\alpha x} e^{-\frac{(m+k)\lambda}{\alpha}(e^{\alpha x}-1)} = \frac{m}{m+k} u_{m+k}(x)$, where $u_{m+k}(x)$ denotes the pdf of the Gompertz distribution with parameters $(m+k)\lambda$ and α. So the pdf of i-th order statistic of the MBGz distribution can be expressed as a linear combination of Gompertz pdfs, i.e.

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} w_m d_{j+i-1,k} \frac{m}{m+k} u_{m+k}(x).$$

Let r be a positive integer. Then the r-th ordinary moment of $X_{i:n}$ can be expressed as

$$\begin{aligned}
\mathbb{E}(X_{i:n}^r) &= \int_{-\infty}^{+\infty} x^r f_{i:n}(x) dx \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} w_m d_{j+i-1,k} \frac{m\lambda}{\alpha^{r+1}} e^{\frac{(m+k)\lambda}{\alpha}} \int_1^{+\infty} (\ln u)^r e^{-\frac{(m+k)\lambda}{\alpha} u} du.
\end{aligned}$$

4. Statistical inference

4.1. Maximum likelihood estimation

We now investigate the estimation of the parameters of the MBGz distribution. Let $x_1, \ldots, x_n$ be n observed values from the MBGz distribution and $\xi = (\lambda, \alpha, a, b, c)$ be the vector of unknown parameters. The log likelihood function is given by

$$\begin{aligned}
\ell(\xi) = an \ln(c) + n \ln(\lambda) + \alpha \sum_{i=1}^{n} x_i - \frac{\lambda b}{\alpha} \sum_{i=1}^{n} (e^{\alpha x_i} - 1) + (a-1) \sum_{i=1}^{n} \ln\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right) \\
- n \ln(B(a,b)) - (a+b) \sum_{i=1}^{n} \ln\left[1 - (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right)\right].
\end{aligned}$$

The maximum likelihood estimators of the parameters are obtained by maximizing the log likelihood function. They can be obtained by solving the non-linear equations: $\frac{\partial}{\partial\lambda}\ell(\xi) = 0$, $\frac{\partial}{\partial\alpha}\ell(\xi) = 0$, $\frac{\partial}{\partial a}\ell(\xi) = 0$, $\frac{\partial}{\partial b}\ell(\xi) = 0$, $\frac{\partial}{\partial c}\ell(\xi) = 0$ with

$$\begin{aligned}
\frac{\partial \ell(\xi)}{\partial \lambda} = \frac{n}{\lambda} - \frac{b}{\alpha} \sum_{i=1}^{n} (e^{\alpha x_i} - 1) + (a-1) \sum_{i=1}^{n} \frac{\frac{1}{\alpha}(e^{\alpha x_i} - 1) e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}}{1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}} \\
+ (a+b) \sum_{i=1}^{n} \frac{(1-c)\frac{1}{\alpha}(e^{\alpha x_i} - 1) e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}}{1 - (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right)},
\end{aligned}$$

$$\frac{\partial \ell(\xi)}{\partial \alpha} = \sum_{i=1}^{n} x_i - \frac{\lambda b}{\alpha} \sum_{i=1}^{n} \left[x_i e^{\alpha x_i} - \frac{1}{\alpha}(e^{\alpha x_i} - 1) \right]$$
$$+ (a-1) \sum_{i=1}^{n} \frac{\frac{\lambda}{\alpha} e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)} \left[x_i e^{\alpha x_i} - \frac{1}{\alpha}(e^{\alpha x_i} - 1) \right]}{1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}}$$
$$+ (a+b) \sum_{i=1}^{n} \frac{(1-c) e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)} \left[x_i e^{\alpha x_i} - \frac{1}{\alpha}(e^{\alpha x_i} - 1) \right]}{1 - (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right)},$$

by setting $B^{(1,0)}(a,b) = \frac{\partial}{\partial a} B(a,b)$ and $B^{(0,1)}(a,b) = \frac{\partial}{\partial b} B(a,b)$ (one can remark that $B^{(1,0)}(a,b) = \psi(a) - \psi(a+b)$ and $B^{(0,1)}(a,b) = \psi(b) - \psi(a+b)$, where $\psi(x)$ denotes the so called digamma function),

$$\frac{\partial \ell(\xi)}{\partial a} = n \ln c + \sum_{i=1}^{n} \ln \left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right) - n \frac{B^{(1,0)}(a,b)}{B(a,b)}$$
$$- \sum_{i=1}^{n} \ln \left[1 - (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right)\right],$$

$$\frac{\partial \ell(\xi)}{\partial b} = -\frac{\lambda}{\alpha} \sum_{i=1}^{n} (e^{\alpha x_i} - 1) - n \frac{B^{(0,1)}(a,b)}{B(a,b)} - \sum_{i=1}^{n} \ln \left[1 - (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right)\right]$$

and

$$\frac{\partial \ell(\xi)}{\partial c} = \frac{an}{c} - (a+b) \sum_{i=1}^{n} \frac{1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}}{1 + (1-c)\left(1 - e^{-\frac{\lambda}{\alpha}(e^{\alpha x_i}-1)}\right)}.$$

We can solve the above non-linear equations simultaneously. Mathematical package can be used to get the maximum likelihood estimators of the unknown parameters. Also, all the second order derivatives exist. As usual, the normal approximation for the maximum likelihood estimators can be used for constructing approximate confidence intervals, confidence regions and testing hypotheses of λ, α, a, b, c.

4.2. Simulation

It is very difficult to compare the theoretical performances of the different maximum likelihood estimates (MLEs) for the MBGz distribution. Therefore, simulation is needed to compare the performances of the MLE mainly with respect to their mean square errors for different sample sizes. A numerical study is performed using Mathematica 9 software. Different sample sizes are considered through the experiments at size $n = 50$, 100 and 150. The experiment will be repeated 3000 times. In each experiment, the estimates of the parameters will be obtained by maximum likelihood methods of estimation. The means and MSEs for the different estimators will be reported from these experiments. We can see from Table 1 when n is increase MSE is decrease.

Table 1: The MLEs and MSEs of MBGz distribution.

n	Parameters	Initial	MLE	MSE	Initial	MLE	MSE
50	a	3.0	3.0024	0.5057	2.5	2.6424	0.1737
	b	1.5	1.6409	0.1499	1.5	1.5219	0.0400
	c	0.5	0.4941	0.0008	0.5	0.5050	0.0004
	α	0.5	0.5422	0.0198	0.5	0.5291	0.0116
	λ	0.5	0.5241	0.0387	0.5	0.5235	0.0122
100	a	3.0	3.0778	0.2458	2.5	2.5060	0.0754
	b	1.5	1.6083	0.0779	1.5	1.5373	0.0291
	c	0.5	0.4986	0.0003	0.5	0.4991	0.0003
	α	0.5	0.5572	0.0147	0.5	0.5123	0.0029
	λ	0.5	0.4926	0.0126	0.5	0.5035	0.0070
150	a	3.0	2.9041	0.1015	2.5	2.5125	0.0284
	b	1.5	1.6159	0.0485	1.5	1.5232	0.0088
	c	0.5	0.4940	0.0002	0.5	0.5002	0.0001
	α	0.5	0.5477	0.0094	0.5	0.5137	0.0015
	λ	0.5	0.4694	0.0072	0.5	0.4968	0.0015
50	a	1.5	1.4706	0.0325	1.5	1.5435	0.0641
	b	1.8	1.7764	0.0639	1.8	1.7838	0.0955
	c	0.5	0.5054	0.0013	1.5	1.5285	0.0203
	α	0.5	0.4833	0.0029	0.5	0.4895	0.0008
	λ	0.5	0.5488	0.0160	0.5	0.5364	0.0118
100	a	1.5	1.5138	0.0201	1.5	1.5194	0.0224
	b	1.8	1.8177	0.0380	1.8	1.8309	0.0451
	c	0.5	0.5004	0.0007	1.5	1.5010	0.0047
	α	0.5	0.5007	0.0023	0.5	0.5011	0.0005
	λ	0.5	0.5106	0.0059	0.5	0.5036	0.0028
150	a	1.5	1.5313	0.0102	1.5	1.4690	0.0094
	b	1.8	1.8152	0.0194	1.8	1.8396	0.0258
	c	0.5	0.5055	0.0003	1.5	1.4864	0.0017
	α	0.5	0.5173	0.0022	0.5	0.5007	0.0004
	λ	0.5	0.5044	0.0034	0.5	0.4943	0.0009

4.3. Applications

This section provides an application to show how the MBGz distribution can be applied in practice. We compare MBGz to Exponentaited Generalized Weibull-Gompertz distribution (EGWGz) by El-Bassiouny *et al.* (2017) and other well known distributions in literature, Kumaraswamy-Gompertz (Kw-Gz), beta Gompertz(BGz) and Gompertz (Gz) models. The MLEs are computed using Quasi-Newton Code for Bound Constrained Optimization and the log-likelihood function evaluated. The goodness-of-fit measures, Anderson-Darling (A*), Cramer-von Mises (W*), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and log-likelihood ($\hat{\ell}$) values are computed. The lower the values of these criteria, the better the fit. The value for the Kolmogorov Smirnov (KS) statistic and its p-value are also provided. The required computations are carried out in the R software.

Data set 1

The second data set represents the time to failure(103 h) of turbocharger of one type of engine given in Xu *et al.* (2003). The data is as follows: 0.0312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.977, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585

Data set 2

The data Set (of size 69) by Badar and Priest (1982) correspond to a single fiber with 20 and 101 mm of gauge length, respectively. The data is as follows: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0
Tables 2 and 4 list the maximum likelihood estimates (and the corresponding standard errors in parentheses) of the unknown parameters of the MBGz distribution for Data Set 1 and Data Set 2 respectively. Tables 3 and 5 show the statistics AIC, BIC, *W**, *A**, *KS*, P-Value values for all the considered models. We then see that the proposed MBGz model fits these data better than the other models. The MBGz model may be an interesting alternative to other models available in the literature for modeling positive real data. To complete this fact, PP, QQ, epdf and ecdf plots of the MBGz distribution given in Figures 2 and 3 for Data Set 1 and Data Set 2 respectively.

Table 2: MLEs (standard errors in parentheses) for Data Set 1.

Distribution	Estimates				
MBGz(λ, α, a, b, c)	0.0085	2.5537	1.0737	1.3153	5.0687
	(0.0067)	(0.5727)	(0.3197)	(0.8933)	(3.3003)
EGWGz(λ, a, b, c, β)	3.2078	2.4598	0.0203	1.8974	0.5460
	(1.2099)	(0.6498)	(0.0531)	(1.8193)	(0.2430)
KwGz(a, b, c, d, θ)	0.1861	1.4948	1.4909	0.9811	
	(0.3130)	(0.5076)	(0.4735)	(2.4368)	
BGz(a, b, θ, v)	0.3144	1.5591	1.4798	0.4966	
	(0.4283)	(0.3658)	(0.4543)	(0.8692)	
Gz(λ, α)	0.0841	1.8811			
	(0.0268)	(0.2043)			

Table 3: The AIC, BIC, W^*, A^*, KS, P-Value values for Data Set 1.

Dist	$\hat{\ell}$	AIC	BIC	W^*	A^*	KS	P-Value
MBGz	50.0387	110.0776	118.2481	0.0328	0.2745	0.0539	0.9889
EGWGz	52.6888	115.3776	126.5482	0.0706	0.5341	0.0785	0.7885
KwGz	51.2042	110.4084	119.3448	0.0529	0.4125	0.0640	0.9396
BGz	51.1518	110.3026	119.2399	0.0518	0.4057	0.0627	0.9484
Gz	53.9686	111.9374	122.4056	0.0819	0.5921	0.0810	0.7547

Table 4: MLEs (standard errors in parentheses) for Data Set 2.

Distribution	Estimates				
MBGz(λ, α, a, b, c)	0.0098	0.5270	0.8768	4.5635	0.1561
	(0.0116)	(0.1599)	(0.3893)	(0.8862)	(0.2442)
EGWGz(λ, a, b, c, β)	0.0101	0.6077	0.1078	1.6929	0.6613
	(0.0141)	(0.1506)	(0.3427)	(1.2539)	(0.3379)
KwGz(a, b, θ, v)	0.0133	0.2923	2.0164	13.7085	
	(0.0120)	(0.1641)	(0.7880)	(7.0208)	
BGz(a, b, c, d, θ)	0.0125	0.1856	3.7622	2.0116	
	(0.0100)	(0.1601)	(2.5635)	(3.3802)	
Gz(λ, α)	0.0074)	(0.6243)			
	(0.0035)	(0.0748)			

Table 5: The AIC, BIC, W^*, A^*, KS, P-Value values for Data Set 2.

Dist	$\hat{\ell}$	AIC	BIC	W^*	A^*	KS	P-Value
MBGz	78.2184	168.1770	176.0214	0.0222	0.1840	0.0707	0.9888
EGWGz	79.3744	168.5489	178.5933	0.0479	0.2922	0.0821	0.9623
KwGz	80.7197	169.4395	176.1950	0.0430	0.3326	0.0966	0.8489
BGz	82.9924	173.9849	180.7404	0.0922	0.6736	0.1080	0.7389
Gz	80.9566	168.9234	177.2911	0.0359	0.2335	0.0903	0.8299

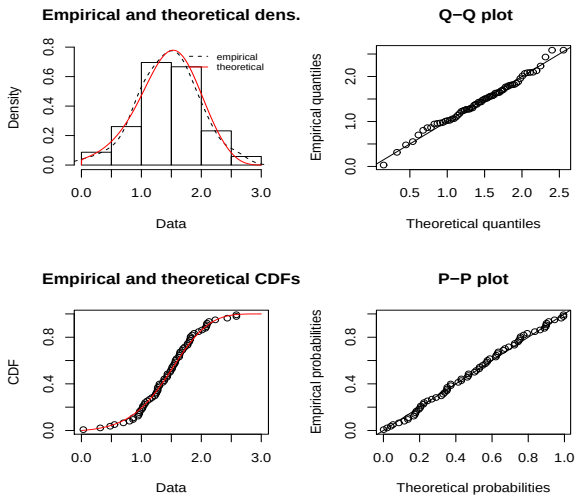

Figure 2: PP, QQ, epdf and ecdf plots of the MBGz distribution for Data Set 1.

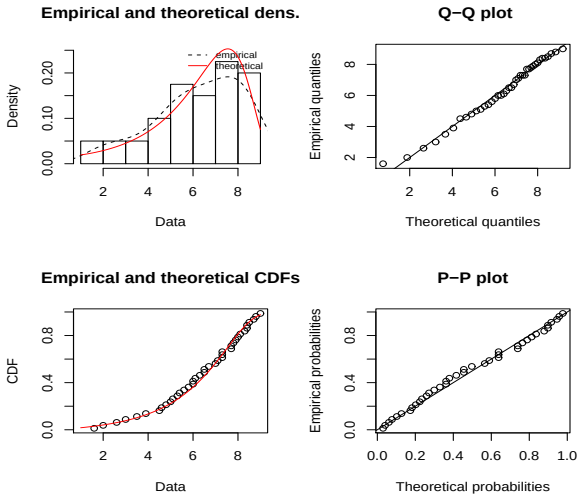

Figure 3: PP, QQ, epdf and ecdf plots of the MBGz distribution for Data Set 2.

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