Bolzano, (the appropriate) relevant logic and ground-ing rules for implication

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Abstract

In this paper our goal is double. First of all we aim to show that there is a deep link between Bolzano's notion of exact derivability and Tennant's relevant logic **CR**. Secondly, we aim to argue that Tennant's relevant logic **CR** is an adequate framework for developing interesting grounding rules for the implication connective.

Keyword Bolzano, Tennant, grounding, implication, relevant logic.

1 Introduction

Although neglected for many decades, the great Bohemian thinker Bernard Bolzano is today the center of a renewed interest and enjoys the respect that he deeply deserves: he is indeed considered an outstanding mathematician, an exceptional philosopher and one of the greatest logicians who ever lived (see Morscher (2012)). This renewed enthusiasm towards Bolzano seems to go hand in hand with the contemporary attention to the notion of grounding, or metaphysical explanation (e.g. see Correia and Schnieder (2012); Fine (2010); Poggiolesi (2016b); Schnieder (2011)); Bernard Bolzano can indeed be seen as one of the major figures in the history of the philosophy of this notion. His conception of grounding, together with his attempt at logically characterizing it, are milestones in the analysis of this concept.

As pointed out in several studies, in Bolzano's conception of grounding, the notion of *exact derivability* (*Ableitbarkeit*),[1] that Bolzano himself introduced

[1] In some texts (e.g. Roski (2014)) the term *Ableitbarkeit* is translated with *deducibility*, rather than *derivability*. Since the notion of deducibility involves the requirement of discharge of assumptions, which is absent in the notion of derivability as well as in the original notion of *Ableitbarkeit*, we prefer this second translation.

(*Wissenschaftslehre* II, §155), plays a special role. Roughly speaking the general idea is that the notion of complete grounds[2] of some truth A always presupposes a relation of exact derivability (under certain restrictions) of A from the set of its complete grounds.

In the first part of this paper the main aim is to have a closer look at exact derivability and its relations to contemporary logic. More precisely, we will show that there exists a particular connection between Bolzano's notion of exact derivability and the relevant logic **CR** introduced by Tennant (1984). These reflections will not only clarify the notion of exact derivability *per se*, but also, in virtue of the connection between exact derivability and grounding, shed further light on the general Bolzanian conception of metaphysical grounding.

In the second part of the paper we will focus on the recent studies on the logic of grounding and in particular on the grounding rules for implication, a topic that, as far as we know, has received relatively little treatment. We will try to argue that the logic **CR** can again play a role: it is indeed an useful and interesting framework for formulating the grounding rules for implication.

The paper is organized as follows. In *Section 2* we will recall Bolzano's notions of derivability and exact derivability emphasizing their logical properties. In *Section 3* we will introduce the logic **CR**, while in *Section 4* we will explain why this is the adequate contemporary counterpart of Bolzano's notion of exact derivability. We will use *Section 5* to discuss the issue of the grounding rules for implication. We will show that even in this context the logic **CR** has a role to play.

2 Deducibility and exact deducibility

In the *Wissenschaftslehre* Bolzano famously introduced two concepts that are now part of the history of logic, namely the notion of *derivability* and the notion of *exact derivability*. Many valuable studies (e.g. Morscher (2012); Sebestik (1992)) have been dedicated to the analysis and reconstruction of these concepts: amongst other things, they have highlighted some important properties of derivability and exact derivability.

In this section our aim is double: on the one hand, we will introduce the concepts of derivability and exact derivability, on the other hand we will try to summarize the properties and the logical principles that have been associated with these concepts. In accomplishing these two tasks, we will mainly summarize some recent studies on Bolzano's work (e.g. Berg (1962); Morscher (2012); Rusnock and George (2015)) without however entering into the details of their different interpretations.

Before starting dealing with our double task, let us underline two points. The first point concerns Bolzano's conceptual framework that is behind the notions of derivability and exact derivability, that is thus useful for understanding them but which, nonetheless, for reasons of space, we omit to present here. The reader

[2] For a definition of the notion of complete grounds, as opposed to full grounds, see Poggiolesi (2016a).

who is not acquainted with such a framework is referred to the Introduction of this volume. The second point concerns the choice of not being entirely faithful to Bolzano's original work; in the following aspects we will indeed opt for a more modern and elegant formulation than the one used by Bolzano. (i) We follow the standard conception of propositions and not that of Bolzano according to which all propositions have the form [A has B], see Betti (2012); Casari (1992); (ii) we work with sets of propositions, while Bolzano uses collections of propositions; (iii) we allow empty set of premises, while Bolzano doesn't.

In the *Wissenschaftslehre* §147-168 Bolzano introduces his *theory of variation* which is one of its central contributions to the field of logic. The heart of the theory analyses the semantic links that arise amongst propositions as soon as certains of their elements are substituted in a homogeneous way. In this framework Bolzano singles out several relationships, among which the relation of *derivability* stands out. He defines this notion in the following way.

Definition 2.1. Propositions A_1, ..., A_n are derivable from propositions B_1, ..., B_m with respect to the sequence of components $< P_1, ..., P_r >$, which occur in the propositions A_1, ..., A_n as well as B_1, ..., B_m[3], if, and only if:

(1) the premisses B_1, ..., B_m are *compatible* with respect to $< P_1, ..., P_r >$, which is to say that there exists a sequence of components $< Q_1, ..., Q_r >$ that, if substituted with $< P_1, ..., P_r >$, make simultaneously true all the premisses.

(2) each sequence that, substituted with $< P_1, ..., P_r >$, make true all the premisses, make true all the conclusions too.

Let us make a few comments about this definition. First of all, the definition is composed of two items: the first expresses the idea of the premisses being compatible, the second expresses the idea of truth-preservation from premisses to conclusion. Truth-preservation as well as compatibility are relativized to a sequence of components of propositions; it is important to underline what kind of components might be part of that sequence. Translated in contemporary terms variation can concern the sequence of all individual, functional and predicative symbols, all molecular terms as well as all subsentences contained in propositions (in this case we have a notion of *logical derivability*). Variation can also concern a proper subsequence of that sequence (and in that case the descriptive symbols are constants as the logical symbols).[4]

Let us first give an exemple of a set of propositions which are compatible with respect to a sequence of components. We use Bolzano's exemple and we thus consider the propositions "this flower is red", "this flower has a pleasant

[3]As the editors made us notice, although not explicit about this point, it seems that Bolzano would require the sequence $< P_1, ..., P_r >$ to actually occur in the premisses as well as in the conclusion of the derivability relation.

[4]We owe to Paoli (1991) the translation in contemporary terms of the Bolzanian domain of variation. Let us also underline that logical constants do not belong to this domain of variation. The question is however complex since Bolzano could not decide how to draw the distinction between logical and not-logical ideas, see Rusnock and George (2015), II, §148.

fragrance", and "this flower belongs to the twelfth class of Linne's system": these propositions are compatible with respect to the variable idea "this flower", because all three are true if we substitute "this flower" with "this rose".

Let us now move to propositions which stand in a truth-preserving relation from premisses to conclusion with respect to a certain sequence of components. Let us consider the propositions "the triangle ABC is isosceles", "the triangle ABC is equilateral". Each variable idea that, substituted with ABC, makes true "the triangle ABC is equilateral", makes true "the triangle ABC is isosceles" as well. Thus "the triangle ABC is equilateral" is the premiss and "the triangle ABC is isosceles" the conclusion of a relation which is truth-preserving with respect to the variable-idea ABC. (The two propositions nevertheless do not stand in a logical derivability relation.)

As many (e.g. Etchemendy (1990); Sebestik (1999)) have suggested, Bolzano's notion of derivability anticipates in many respects the Tarskian notion of logical consequence: a common idea that underlies both definitions is truth-preservation from premises to a conclusion under certain variations. While from the Tarskian perspective, variation concerns all non-logical elements, from the Bolzanian's perspective variation might vary and non-logical elements may remain constant. In this respect Bolzano's notion is much more general than the Tarski's idea of logical consequence. In another respect, however, the notion is narrower. For contrary to Tarski, Bolzano requires the premises to be compatible with respect to a certain sequence of components.

From now on we adopt the following notations. Let $M, N, ...$ denote sets of propositions. We write:

- $M \models_{<P_1,...,P_r>} N$ to denote that the set N is derivable from the set M with respect to the sequence of components $< P_1, ..., P_r >$.
- $M \not\models_{<P_1,...,P_r>} N$ to denote that the set N is not derivable from the set M with respect to the sequence of components $< P_1, ..., P_r >$.
- $M \not\models N$ to denote that the there exist no sequence of components $< P_1, ..., P_r >$ with respect to which the set M is derivable from the set N.

Let us now move to the main properties that this relation of derivability satisfies. These properties have been emphasized by (Roski, 2014, Ch. 3.3.4), Siebel (2002), Stelzner (2002).

Restricted Monotonicity For all M, M' and N, if $M \models_{<P_1,...,P_r>} N$, $M \subseteq M'$ and the propositions contained in M' are compatible with respect to $< P_1, ..., P_r >$, then $M' \models_{<P_1,...,P_r>} N$.

The relation of derivability is monotonic up to compatibility; indeed, if $M \models_{<P_1,...,P_r>} N$ and the set M is enriched with some propositions A such that A is not compatible with M with respect to $< P_1, ..., P_r >$, then N is not derivable from $M \cup A$ with respect to $< P_1, ..., P_r >$.

Restricted Cut For all M, N, S, T and A, if $M \models_{<P_1,...,P_r>} N, A$ and $A, S \models_{<P_1,...,P_r>}$

T, and M and S are compatible with respect to $< P_1, ..., P_r >$, then $M, S \models_{<P_1,...,P_r>} N, T$.

As it was the case for non-monotonicity, even what is known as cut only holds in a restricted form: one can cut a formula and thus unite two different derivability relations as long as the new set of premisses still satisfies compatibility with respect to a certain sequence of components.

Transitivity For all M, N, S, if $M \models_{<P_1,...,P_r>} N$ and $N \models_{<P_1,...,P_r>} S$, then $M \models_{<P_1,...,P_r>} S$.

Bolzano's derivability relation is transitive without limitation. Let us remind the reader, since it will be useful in what follows, that transitivity is nothing but a special case of cut: it indeed amounts to cut with no added premisses in the middle step.

Let us now move to two logical principles that are associated with the derivability relation. These logical principles have been investigated by Berg (1962); George (1983); Siebel (2003); Stelzner (2002). They will be of crucial importance in the next section.

1. $A \vee B, \neg A \models_{<A,B>} B$
2. $A, \neg A \not\models B$

The first principle is quite well-known and is standardly called *disjunctive syllogism*; it is also a valid principle in classical and intuitionistic logic. It says that if a disjunction is true and one of its disjunct is false, then the other disjunct must be true. It is quite straightforward to see its validity according to Definition 2.1.

Let us analyse the second principle which amounts to the block of famous *Lewis's first paradox* (see Lewis and Langford (1959)); it indeed says that there exists no sequence of components with respect to which B is derivable from propositions A and $\neg A$. This principle has been the object of debate. Indeed, although everybody agrees that Bolzano would have never accepted the passage from A and $\neg A$ to B as valid, there are two different interpretations concerning what Bolzano's reasons might have been. An accurate presentation of these different interpretations would go beyond the scope of the paper and thus we leave it aside; let us however underline that it is a very interesting debate (in particular, see George (1983) and Stelzner (2002) for one interpretation of the Bolzanian block of Lewis's first paradox; and Berg (1962) and Siebel (2003) for the other interpretation).

It is now time to move to the notion of exact derivability. Bolzano introduces two different notions of exact derivability: one in the *Wissenschaftslehre* §155.26, the other in the *Von der mathematischen Lehrart*, §8. According to (Rusnock and George, 2015, Vol. II, p. xxxvi), the latter notion of exact derivability is an improved version of the first: Bolzano spotted a flaw in the definition of exact derivability given in the *Wissenschaftslehre.* We thus embrace the definition of exact derivability given in *Von der mathematischen Lehrart.*

Definition 2.2. Propositions A_1, ..., A_n are exactly derivable from propositions B_1, ..., B_m with respect to the sequence of components $< P_1$, ..., $P_r >$, , which occur in the propositions A_1, ..., A_n as well as B_1, ..., B_m, if, and only if:

- $A_1, ..., A_n \models_{<P_1,...,P_r>} B_1, ..., B_m$
- no A_i, $1 \leqslant i \leqslant n$, may be removed from the set $A_1, ..., A_n$ without the loss of derivability of $B_1, ..., B_m$ from the remainder with respect to $< P_1, ..., P_r >$.

Analogously to our notation for derivability, we write:

- $M \Vdash_{<P_1,...,P_r>} N$ to denote that the set N is exactly derivable from the set M with respect to the sequence of components $< P_1, ..., P_r >$;
- $M \nVdash_{<P_1,...,P_r>} N$ to denote that the set N is not exactly derivable from the set M with respect to the sequence of components $< P_1, ..., P_r >$;
- $M \nVdash N$ to denote that there is no sequence of components with respect to which the set N is exactly derivable from the set M.

The idea behind the notion of exact derivability is rather simple: exact derivability is derivability plus the condition that no premiss can be omitted without losing derivability itself. On the one hand, exact derivability keeps the most salient ingredients of derivability, namely compatibility of premisses and preservation of truth both relative to a sequence of components. On the other hand, all the properties of derivability discussed above are lost. As is easily seen, exact derivability does not enjoy monotonicity nor restricted monotonicity; it does not enjoy cut[5] and thus, since transitivity is a special case of cut, as we have said above, it is not even a transitive relation.

As for the logical principles associated with exact derivability, they include the following three:

1. $A \vee B$, $\neg A \Vdash_{<A,B>} B$
2. A, $\neg A \nVdash B$
3. B $\nVdash A \vee \neg A$

The first two principles are those that were associated with derivability. Principle 1, namely disjunctive syllogism, still holds: both $\neg A$ and $A \vee B$ are indispensable premisses to derive B. As for principle 2, namely the block of Lewis's first paradox, since exact derivability is a special case of derivability, it obviously still holds. Principle 3 is new and amounts to the block of *Lewis's second paradox* (see Lewis and Langford (1959)); it can be thought of as the direct consequence of the second item of Definition 2.2.

[5] Let us underline that Bolzano himself realized that cut is not valid for exact derivability, see (Rusnock and George, 2015, II, §155.32).

We have thus finished our summary of the two Bolzanian notions of derivability and exact derivability, that are milestones in the history of logic. In what follows we will only focus on the notion of exact derivability. In particular we will try to answer the following question: which modern notion of derivability best approximates Bolzano's notion of exact derivability?

Of course we are not the first to be intrigued by this issue, e.g. see George (1983); Siebel (2003); Stelzner (2002); several parallels have been drawn between the notion of exact derivability and the relavant logic **R** introduced by Anderson and Belnap (1975). In the following section we will show that, although the parallel between relevant logic and Bolzano is appropriate, there is another relavant logic that matches the properties of the notion of exact derivability far better than Anderson and Belnap's: it is the logic **CR** introduced by Tennant (1984). As we will show, the resemblance between Tennant's approach and Bolzano's approach is so striking that it is actually astonishing that nobody has noticed it before.

3 Tennant's relevant logic

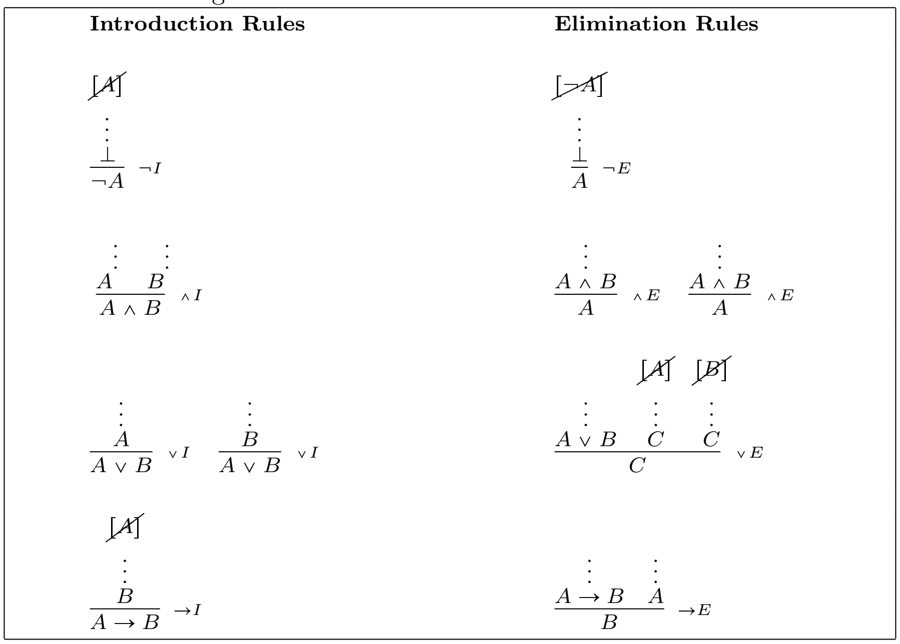
Let us start with classical logic, probably the best known of all logics, characterized by a classical logical consequence relation and the corresponding material implication connective. Classical logic has been the object of several criticisms and revisions occasioned by the so-called *fallacies of relevance*. These fallacies can be expressed either in an implicational form or in a deductive form. Lewis's first paradox can thus either be expressed as the theorem $(A \wedge \neg A) \rightarrow B$, or as the fact that B is deducible from the premisses A and $\neg A$. Analogously, Lewis's second paradox can either be expressed as the theorem $B \rightarrow (A \vee \neg A)$ or as the fact that $(A \vee \neg A)$ is deducible from the premise B.

By the term *relevant logics* one generally indicates those substructural logics that attempt to block the fallacies of relevance. This has been done in two different ways. Relevant logic in the tradition of Anderson and Belnap attacks the problem of relevance from an implication point of view: their main concern is to describe a relevant conditional. Relevant logic in the tradition of Tennant attacks the problem of relevance from a deductive point of view: his main concern is indeed to describe a relevant deduction. Let us note that:

> Solving the problem of relevance in the deductive form is arguably a precondition for solving it for the conditional, if we suppose (as is reasonable) that the relevant conditional is to be governed by anything like the rule of conditional proof. To assert the conditional $A \rightarrow B$, one will have to be able *relevantly to prove* B from A; and characterizing the notion of relevant deduction appealed to here is no more than what I have called the problem of relevance in its deductive form. (Tennant, 1987, p. 665)

As emphasized above, a parallel has already been drawn between Bolzano's notion of exact derivability and relevant logic; in particular, it has been said

Figure 1: Classical Natural Deduction Calculus



that Bolzano's notion of exact derivability captures an idea remarkably similar to the one that motivates the relevant logic **R** of Anderson and Belnap. But in view of what has been said in the paragraph above, we can already see that this is quite imprecise. Exact derivability is indeed a relation, Anderson and Belnap are mainly concerned with conditionals and it is Tennant who focuses on relevant derivation. Thus, if an analogy is to be drawn, it is far more appropriate to compare Tennant and Bolzano. As far as we know, this has never been done, but seems to be the correct direction to look at. Let us continue to explore this parallel and discover whether it is really appropriate.

In order to accomplish our task, let us consider the proof of Lewis's first paradox in its deductive form. We follow Tennant in first presenting it in an informal way and then turn to its natural deduction tree presentation. We want to show that from A and $\neg A$, B can be deduced. Thus we accomplish the following four steps:

1. Assume A
2. By introduction of disjunction, obtain $A \vee B$
3. Assume $\neg A$
4. By disjunctive syllogism, obtain B

Let us now present the inference from A and $\neg A$ to B by using the rules of the classical natural deduction calculus as well as the properties of the classical derivability relation. For doing this, let us first of all introduce the language of classical logic and then the classical calculus of natural deduction which is formulated in that language.

Definition 3.1. The classical language $\mathcal{L}^c$ is composed of a denumerable stock of propositional atoms (p, q, r, ...), the logical operators $\neg$, $\wedge$, $\vee$ and $\rightarrow$, the parentheses (,) and the square brackets [,]. The connective $\leftrightarrow$ is defined as usual; the symbol $\perp$ stands for the absurd and is defined as $A \wedge \neg A$. Propositional formulas are standardly constructed.

Definition 3.2. The classical calculus of natural deduction **C**, formulated in the language $\mathcal{L}^c$, is composed of the rules of Figure 1. We write $M \vdash_C A$ to denote that the formula A is derivable in the classical calculus of natural deduction **C** from the set of premisses M.

Let us underline a couple of things relative to the calculus **C** and the corresponding derivability relation. First of all, since we have assumed the symbol $\perp$ to be defined as a conjunction of two contradictory formulas, in order to introduce it, we will use the rule $\wedge I$, e.g.

$$\frac{A \quad \neg A}{\perp} \wedge I$$

Secondly, in the rules $\neg I$, $\vee E$ and $\rightarrow I$ we have used the notation $[\not{A}]$. This notation means that *all* (for this stand the square brackets) occurrences of the formula A are discharged (for this stands the bar), once the rule is used. If there is no occurrence of the formula A to discharge, each of the rules above can be used anyway: it is an application of the rule *with vacuous discharge of assumptions*. For the sake of clarity, let us make some examples of application of the rules $\neg I$, $\vee E$ and $\rightarrow I$ with vacuous discharges.

Exemple 1

$$\frac{B}{A \rightarrow B} \rightarrow I$$

In this application of the rule $\rightarrow I$ there is no discharge of assumptions: in the implication that occurs as conclusion of the rule the antecedent A is not previously used as a premise.

Exemple 2

$$\cfrac{\cfrac{A, \neg A}{\perp} \wedge I}{B} \neg E$$

In this derivation, let us call it d, first we introduce the absurd by means of the rule $\wedge I$ and afterwards we introduce the formula B by means of the rule $\neg E$: in this application of the rule $\neg E$ no assumption is discharged since the formula $\neg B$ does not occur previously in the derivation.

We close this paragraph by making a third and final remark about the derivability relation of the classical calculus: this relation is notoriously transitive and enjoys cut.

We now have all the elements to clearly present the inference from A and $\neg A$ to B by using the rules of the classical natural deduction calculus as well as one property of the classical derivability relation. First of all, we consider an application of the rule introducing disjunction, namely the rule $\lor I$:

$$\frac{A}{A \lor B}\ {\scriptstyle \lor I}$$

Secondly, using the rules $\land I$, $\neg E$ (with vacuous discharge, as illustrated in the previous page) and $\lor E$, from $A \lor B$ and $\neg A$, we infer B by disjunctive syllogism:

$$\cfrac{A \lor B \qquad \cfrac{\cfrac{\not{A}, \neg A}{\bot}\ {\scriptstyle \land E}}{B}\ {\scriptstyle \neg E} \qquad \not{B}}{B}\ {\scriptstyle \lor E}$$

Finally, by cut applied on the previous two derivations, we obtain the desired result. We call this derivation d':

$$\cfrac{\cfrac{A}{A \lor B}\ {\scriptstyle \lor I} \qquad \cfrac{\cfrac{\not{A}, \neg A}{\bot}\ {\scriptstyle \land I}}{B}\ {\scriptstyle \neg E} \qquad \not{B}}{B}\ {\scriptstyle \lor E}$$

Let us assume that we accept the rule that introduces disjunction since it is clearly not problematic. Then, in order to block the paradox, only two strategies are available: either disjunctive syllogism is rejected or cut is rejected. While Anderson and Belnap's relevant logic is based on the rejection of disjunctive syllogism, Tennant's logic leads to the rejection of cut. Since Tennant's rejection of cut is elegant and subtle, let us better explain it.

In proof theory particular attention is and has always been devoted to redundancies in derivations. A formula occurring in a derivation (in natural deduction calculi) is said to be redundant, or more technically *maximal*, when it is both a conclusion of an introduction rule and a major premise of the corresponding elimination rule (e.g. see Prawitz (1965)). So for example in the following derivation d

$$\cfrac{\cfrac{A, B}{A \land B}\ {\scriptstyle \land I}}{A}\ {\scriptstyle \land E}$$

the formula $A \land B$ is redundant because it is at the same time the conclusion of the rule $\land I$ and the major premise of the rule $\land E$. A derivation containing at least one maximal formula is said to be *not-normal*; a derivation containing no maximal formula is said to be *normal*. Normal derivations are particularly attractive because they are derivations where each formula is relevant to derive the conclusion (e.g. see Tennant (1984)). In the example d above, the formula $A \wedge B$ is clearly not relevant to derive the conclusion A (from the assumptions A and B). In order to conclude A, we could have just stopped to the assumptions A and B. Thus d is not a normal derivation.

Another excellent example of a not-normal derivation is represented by the derivation d' from A and $\neg A$ to B that we have seen above. Let us consider this derivation again:

$$\dfrac{\dfrac{A}{A \vee B}\,\vee I \qquad \dfrac{\dfrac{\not{A}, \neg A}{\bot}\,\wedge I}{B}\,\neg E \qquad \not{B}}{B}\,\vee E$$

The derivation contains a redundancy, namely the (maximal) formula $A \vee B$ which is at the same time the conclusion of the rule $\vee I$ and the major premise of the rule $\vee E$. The formula $A \vee B$ has no relevant role to play in deriving the conclusion B. B could have been derived from the premisses A and $\neg A$ by means of the derivation d, that we have seen before, where there is no occurrence of the formula $A \vee B$.[6]

Now that we have clarified what normal derivations are, we can explain Tennant's strategy for eliminating cut. Tennant's idea is indeed very simple: in his logic **CR** only derivations that are in normal form, i.e. that do not contain redundancies (or equivalently maximal formulas), are allowed. As a direct consequence, only a cut amongst derivations that do not create non-normal derivations is allowed. To see this, consider again the derivation from A and $\neg A$ to B that contains the maximal formula $A \vee B$. Such a maximal formula has been created by applying cut to the derivation from A to $A \vee B$ and the derivation from $A \vee B, \neg A$ to B. This application of cut is not permitted in **CR** since it creates a maximal formula, i.e. a non-normal derivation, and, as already explained, only normal derivations are allowed.

The insistence on normality of proofs together with the consequent failure of unrestricted cut is thus the first characteristic of Tennant's logic **CR**; there are two more properties that need to be mentioned in order to fully understand this logic. First of all, in the logic **CR**, in all applications of rules in which discharge is indicated, discharge is *obligatory*. In other terms, no vacuous discharge is allowed.

> [...] there must be an undischarged occurrence of the assumption of the indicated form on which the subordinate conclusion depends. Upon application of the rule all such occurrences must be discharged. (Tennant, 1987, p. 672)

For illustrating the point, let us consider again the *Examples 1* and *2* above where applications of rules with vacuous discharge occurred. Let us start with *Example 2*. In this case the inference from A and $\neg A$ to B was obtained by

[6]Thus, d and d' are two different derivations from the formulas A and $\neg A$ to the formula B; d is normal, while d' is not-normal.

applying first the rule $\wedge I$ and secondly the rule $\neg E$. In Tennant's logic this passage is no longer valid because it involves an application of the rule $\neg E$ with vacuous discharge.[7] Let us now move to *Exemple 1* where we examined the inference from B to $A \to B$. Once again in Tennant's logic this inference is no longer valid because it involves an application of the rule $\to I$ with vacuous discharge of assumptions.

Note that by preventing the possibility of vacuous discharge in the rule that introduces the implication connective, i.e. rule $\to I$, Tennant's logic straightforwardly contains a relevant implication rather than the material one of classical logic. Indeed as Dunn and Restall observe:

> In the rule for introducing implication, a proviso has been attached which has the effect of requiring that the hypothesis A was actually used in obtaining B. *This is precisely what makes the implication relevant*. Dunn and Restall (1996)

Let us finally move to the third characteristic of Tennant's logic; the rule that eliminates disjunction has the following form:

$$\dfrac{\begin{matrix} \\ \vdots \\ A \vee B\end{matrix} \quad \begin{matrix}[\not A]\\ \vdots \\ \bot/C\end{matrix} \quad \begin{matrix}[\not B]\\ \vdots \\ \bot/C\end{matrix}}{\bot/C}\ {\scriptstyle \vee E'}$$

where the slash notation $\bot/C$ is to be understood as follows: we allow a subordinate conclusion of either one of the cases to be brought down as the main conclusion, if the other subordinate conclusion is $\bot$. If both subordinate conclusions are of the same form, the main conclusion has the same form. Also, in line with what has been said before, no vacuous discharge is allowed.

It is interesting to see why Tennant replaces the old rule $\vee E$ of elimination of disjunction with the new rule $\vee E'$. The rule $\vee E$ in a calculus where no vacuous discharge is allowed does not permit derivation of disjunctive syllogism. Indeed, as we have seen above, disjunctive syllogism can be proved to be derivable in classical logic by means of the following derivation:

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\not A, \neg A}{\bot}\ {\scriptstyle \wedge I}}{B}\ {\scriptstyle \neg E} \qquad \not B}{B}\ {\scriptstyle \vee E}$$

But because of the conclusion B in the application of the rule $\neg E$, this derivation is no longer valid in the logic **CR**: vacuous discharge is indeed no longer allowed and B is a vacuous discharge. The new rule $\vee E'$ easily solves this problem in the following way:

[7] Once more the derivation from A and $\neg A$ to B is blocked in the logic **CR**.

Figure 2: **CR** Natural Deduction Calculus

Introduction Rules

$$\dfrac{\begin{matrix}[\cancel{A}]\\ \vdots \\ \bot\end{matrix}}{\neg A}\ {\scriptstyle \neg I}$$

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}\quad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}\ {\scriptstyle \wedge I}$$

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B}\ {\scriptstyle \vee I} \qquad \dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B}\ {\scriptstyle \vee I}$$

$$\dfrac{\begin{matrix}[\cancel{A}]\\ \vdots \\ B\end{matrix}}{A \to B}\ {\scriptstyle \to I}$$

Elimination Rules

$$\dfrac{\begin{matrix}[\cancel{\neg A}]\\ \vdots \\ \bot\end{matrix}}{A}\ {\scriptstyle \neg E}$$

$$\dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\ {\scriptstyle \wedge E} \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\ {\scriptstyle \wedge E}$$

$$\dfrac{\begin{matrix}\vdots \\ A \vee B\end{matrix}\quad \begin{matrix}[\cancel{A}]\\ \vdots \\ \bot/C\end{matrix}\quad \begin{matrix}[\cancel{B}]\\ \vdots \\ \bot/C\end{matrix}}{\bot/C}\ {\scriptstyle \vee E'}$$

$$\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix}\quad \begin{matrix}\vdots \\ A\end{matrix}}{B}\ {\scriptstyle \to E}$$

The overall restrictions are two: (i) every natural deduction is in normal form, (ii) no deduction contains vacuous discharge.

$$\dfrac{A \vee B \qquad \dfrac{A, \neg A}{\bot}\ {\scriptstyle \wedge I} \qquad \cancel{B}}{B}\ {\scriptstyle \vee E'}$$

Let us sum up what we have said up to now in the following definition.

Definition 3.3. The calculus of natural deduction **CR**, formulated in the language $\mathcal{L}^c$, is composed of the rules of Figure 2 (the notation used for discharged assumptions is to be read in the same way as in the classical calculus, except that, as remarked in point (ii), no vacuous discharge is allowed). We write $M \vdash_{CR} A$ to denote that the formula A is derivable in the calculus **CR** from the set of premisses M.

As it has been proved by Tennant (1984) and as it has been (at least partially) explained in this section, in **CR** the following logical principles hold:

1. $A \vee B, \neg A \vdash_{CR} B$
2. $A, \neg A \nvdash_{CR} B$

3. B $\nvdash_{CR} A \vee \neg A$

There also exists a sequent calculus for the logic **CR**, that we do not introduce for a question of space. Tennant has proved the equivalence between the sequent calculus and the natural deduction calculus for the logic **CR**.

4 R, CR and exact derivability

We will use this section to list all the common points between the logic **CR** introduced by Tennant and Bolzano's notion of exact derivability. Moreover we will show the differences with respect to Anderson and Belnap's relevant logic.

As already emphasized at the beginning of the previous section, the first analogy between Bolzano's and Tennant's results consists in the fact that both philosophers focus their attention on the relevance of a derivability relation.[8] Thus, while Anderson and Belnap are mainly concerned in identifying a relevant conditional, Tennant, just like Bolzano, wants to characterize a particular type of derivability relation. This difference is actually quite significant and, as far as we know, has never been highlighted before.

Secondly, Bolzano's notion of exact derivability is non-monotonic (and thus is associated with principle 3. above), avoids Lewis's first paradox (see principle 2.) and does not enjoy cut (nor, as a special case, transitivity). Both Tennant's logic **CR** and Anderson and Belnap's logic **R** are non-monotonic and avoid Lewis's first paradox; but while the latter allows unrestricted cut, the former characterizes itself precisely for the lack of a relation enjoying a full cut. Given that derivability relations which do not enjoy full cut[9] are quite rare in the history of logic, this is a second important common point between Bolzano and Tennant.

Let us move to the third point. It concerns disjunctive syllogism (see principle 1.); this inference is indeed considered as valid by Tennant, as well as by Bolzano, while it is not valid in Anderson and Belnap's logic. Therefore this is another aspect that unites the two approaches and distinguishes them from Anderson and Belnap's.

For the sake of clarity, let us remark that there is a notion of exact derivability introduced by Bolzano in the *Wissenschaftslehre* §155 (we mentioned it above), for which disjunctive syllogism is not a valid inference. This notion, that from now on we call ED2, is basically a special case of the notion of exact derivability introduced in Definition 2.2. Indeed it is obtained by adding to Definition 2.2 the following clause: *no part* of any premiss of an exact derivability relation can be deleted without losing derivability itself. Now, if one considers the premisses of disjunctive syllogism, namely $\neg A$ and $A \vee B$, one quickly realizes that $A\vee$ can be deleted by these premisses whilst preserving derivability.

[8]In Tennant (2015) the reader can find a very accurate analysis of the relevance of the derivability relation. This analysis is carried out in the so-called *core logic*, a refined version of the logic **CR** mentioned in this article.

[9]On this point see also Tennant (2016).

Figure 3: Summary

	Relation/ Connective	Non-monotonicity	No Lewis's paradoxes	Cut	Disjunctive syllogism	Insistance on normality/ subformula property
Anderson and Belnap	Connective	✓	✓	✓	x	x
Bolzano	Relation	✓	✓	x	✓	✓
Tennant	Relation	✓	✓	x	✓	✓

From B and $\neg A$ one can indeed still derive B. Thus disjunctive syllogism is not an exact inference according to ED2.[10]

As remarked by Rusnock and George (2015) and as already emphasized before, Bolzano preferred the notion of exact derivability as defined in Definition 2.2 over the ED2 notion. Let us illustrate whether, beyond Bolzano's preferences, there are good reasons for choosing one notion over the other. First of all, let us note that ED2 is, just like the relation of exact derivability introduced in Definition 2.2, a relation that does not enjoy cut. But then, if ED2 does not enjoy cut, it makes little sense for it not to enjoy disjunctive syllogism as well: Lewis's first paradox is already blocked by the lack of transitivity and thus there is no need to add a further restriction. Moreover, a derivability relation that enjoys neither cut nor disjunctive syllogism is a quite narrow relation and several doubts can be cast against its conceptual utility. Finally, and perhaps most importantly, disjunctive syllogism *is* to be counted as a relevant inference: indeed both its premisses $\neg A$ and $A \vee B$ are relevant to derive B.[11] Thus, it is the notion of exact derivability introduced in Definition 2.2 the one that should be taken into account.

Let us pass to the fourth common point between Bolzano and Tennant. As we have seen in the previous section, Tennant insists on and defends the normality of derivations. In an analogous way, Bolzano (see (Rusnock and George, 2015, §378 and §609)) states principles that are very close to the subformula property.[12] But the subformula property is precisely the property enjoyed by normal derivations (see Poggiolesi (2010)). Hence, once more, the two philosophers seem to share the same point of view.

We have summed up in Figure 3 analogies and differences between the notion of exact derivability, the logic **CR** and the logic **R**. The relevant logic **R** and the notion of exact derivability share some important features: they both are non-monotonic and they both block Lewis's paradoxes. This is the reason why – we believe - several scholars have drawn a parallel between the two. However, as soon as other properties are taken into account such as disjunctive syllogism, cut or normality (subformula property), it is **CR** the logic that best matches the notion of exact derivability.

[10] For a precise analysis of this point see (Rusnock and George, 2015, p. xxxvi, II volume).

[11] This, we think, is Tennant's central insight.

[12] Note that many scholars have investigated the importance of the subformula property for Bolzano's grounding proofs, see in particular Rumberg (2013) (Roski, 2014, Ch. 5.2.3) and the chapter of A. Tatzel contained in this volume.

5 Some reflections about grounding rules for implication

As already emphasized, in the last ten years the notion of grounding has become a vibrant area of research. Amongst the several different approaches to this concept, one of the most interesting concerns the logics of grounding: several different logics of grounding have indeed been proposed, e.g. see Correia (2014); Fine (2012); Poggiolesi (2017); Schnieder (2011). Most of them are characterized by grounding rules for the classical connectives: these rules give us the grounds of classical conjunction, disjunction and negation. Perhaps unsurprisingly, little has been said about the connective of implication. More precisely, Schnieder (2011) is the only author who have explicitly formulated the grounding rules for implication, which are the following ones:

$$\frac{B}{A \to B \textit{ because } B} \to_1 \qquad \frac{\neg A}{A \to B \textit{ because } \neg A} \to_2$$

These rules should be read in the following way. Suppose that B is true, then B is the ground of $A \to B$; suppose that $\neg A$ is true, then $\neg A$ is the ground of $A \to B$. Thus the grounds of $A \to B$ are either $\neg A$ or B.

Our aim in this section is to further discuss the case of the implication connective in the framework of grounding, which in our opinion deserves further reflection. It will turn out that the reflections about grounds for implication are linked to the previous investigations on exact derivability.

Let us start by considering ordinary-language conditionals and the grounding intuitions concerning them. Let us for example consider the following three sentences:

1. "If it rains, then the road will be wet"
2. "If the ball is thrown, then it rolls"
3. "If the snow is white, then it is not black"

The question that we aim at answering concerns the grounds of 1-3. What are the reasons for the truth of these sentences that all have the form "if A, then B"? For each of the conditionals 1-3, the answer seems to be the same: the ground is a sentence C such that from A and C, B can be derived. Thus, more precisely

- Ground For 1: "Water makes things wet"
- Ground For 2: "The ball is a sphere"
- Ground For 3: "Snow only has one color"

In each case from the ground and the antecedent, one can derive the consequent. Consider for example the case 2. Why is it true that "if the ball is thrown, then it rolls"? Because "the ball is a sphere." Indeed if the ball is a sphere and spheres if thrown, roll, then if the ball is thrown, then it rolls.

Let us make three remarks about what has just been said. First of all, these intuitions formulated in a completely non-formal way seem to be both natural and reasonable. Secondly, they only apply to a certain type of conditional, namely indicative conditionals characterized by a connection between antecedent and consequent. This connection is very important since it precisely represents that which is grounded. Thirdly, these intuitions seem to be quite distant from those that are behind Schnieder's rules: according to them, the grounds of an implication are basically its truth-conditions, while, according to the intuitions presented above, the grounds of an implication is a sentence C such that from the antecedent (of the implication) and C one can derive the consequent.

In front of these two divergent approaches, it seems worth deepening our analysis to understand whether there is a way to conciliate them; we will do this by moving to the formal level and thus situating the intuitions behind Schnieder's rules as well as the intuitions described above in the wide and long debate concerning the connective of implication. In this broader and more technical context they both will be clarified and better evaluated.

Let us thus remind the reader the main points of this debate. First of all there is the classical or material implication: a material implication is true if, and only if, the antecedent is false or the consequent is true. On the one hand the material implication has the great advantage of being easy and truth-functional; on the other hand, it does not take into account the connection between antecedent and consequent that we assume conditionals to have when we use them in natural language. Thus the conditional "if 2+2 =3, then the moon is yellow" intuitively seems to be false, while, when formalized by the material implication, it is taken as true. An important defense of the material implication is that of Grice (e.g. see Grice (1989); Thomson (1990)). According to Grice, when conditionals are evaluated, a distinction should be drawn between semantics and pragmatics: on the one hand, lies the question of when we can say that a conditional is true and, on the other, lies the question of when a conditional can be reasonably asserted. These two levels often differ: for example the conditional "if 2+2 =3, then the moon is yellow" is true but it is not reasonable to assert it since it violates some basic rules on conversational cooperation.

This defense of material implication is a milestone in the history of the subject and has attracted the interest and approval of many philosophers. Nevertheless it is not free from defects (e.g. see Gibbard (1981)) and several scholars have preferred different formalizations of conditionals. Let us mention those accounts that are, in our opinion, the most famous ones.

The first attempt to fix the inadequacies of the material implication was made by the aforementioned C. I. Lewis (1918) with the definition of *strict implication*, intended to block the paradoxes of material implication, and go beyond the truth-functional analysis. Strict implication is defined as a material implication under the scope of the necessity operator, i.e. the famous $A \multimap B := \Box(A \rightarrow B)$. An improved version of the strict implication is the *variably strict implication* introduced by Stalnaker (1968) and D. Lewis (1973). While a strict implication says: "if A, C" is true provided C is true in all the worlds where A is true, the variably strict implication says "if A, C" is true provided C is true in all the closest worlds to the actual world where A is true (where closeness depends on the world of evaluation).

Alongside the work on (variably) strict implication, there is also the trivalent approach whose starting point is the proviso that a material implication is true whenever its antecedent is false. A common consideration is that an indicative conditional whose antecedent is not true cannot be evaluated as true or false, and so remains indeterminate in truth value. Trivalent logics implications are basically built to satisfy this criterion, e.g. see De Finetti (1936); Belnap (1970); McDermott (1996).

Relevant implication together with relevant logic, which have already been introduced in the previous sections, are the last account that we present. Relevant logic is an umbrella term for denoting several different logics whose main (but not only) characteristic is that of having a relevant conditional, i.e. a conditional where the antecedent is relevant for the consequent, or, in other terms, a conditional where antecedent and consequent are on the same topic. Note that there is formal principle that relevant logicians apply to force theorems and inferences to be on the same topic. This is the *variable sharing principle*. The variable sharing principle says that no formula of the form $A \rightarrow B$ can be proven in a relevant logic if A and B do not have at least one propositional variable (sometimes called a proposition letter) in common (and that no inference can be shown valid if the premises and conclusion do not share at least one propositional variable).

We have thus finished our brief panorama on the discussion concerning material implication, which involved an excursus through other types of implication. Let us now go back to our main concern namely the grounding rules for implication. Recall that we had identified two different ways of treating grounding of conditional sentences. On the one hand, according to Schnieder's approach, the grounds of a conditional are its truth-values, and on the other hand, our intuitions suggest that the grounds of a conditional are strictly related to the connection between antecedent and consequent of that conditional. Let us now link these insights with what has just been said about the implication connective. Two possible scenarios emerge.

Scenario number 1. We focus on the material implication adopting Grice's defense. In this context a conditional sentence like "if 2+2 =3, then the moon is yellow" is true. But why is it true? The only plausible answer seems to be the following: because the antecedent is false or the consequent is true. This answer completely matches with Schnieder's rules which identify the grounds of an implication with its truth-conditions. Thus in the framework depicted by material implication Schnieder's rules are the appropriate grounding rules for implication.

Scenario number 2. We do not adopt the material implication and we are thus faced to the other types of implication connective. In this second scenario Schnieder's intuitions are of course no longer appropriate and thus the challenge is to combine our intuitions on grounding rules for conditionals with the several connectives of the market. More specifically, the question which naturally arises seems to be the following: what kind of implication best allows us to formalise the intuitions on grounds for conditionals presented above? The trivalent analysis of conditionals fairs poorly since it mainly focuses on the situation where an implication has a false antecedent. Then we remain with the (variably) strict implication accounts and the relevant one. Both approaches appear to create a tighter link between antecedents and consequents of implication, so at the first glance they both seem a plausible choice. However relevant logic is definitely closer to our insights since it explicitly requires a connection between antecedent and consequent and that is exactly what is needed to find the grounds of a conditional. To see this in more detail, let us consider the case of a conditional where antecedent and consequent are tautologies with no atomic sentence in common, such as "if it is not the case that it rains and it does not rain, then the moon is yellow or the moon is not yellow". This conditional formalized in both Lewis and Stalnaker-Lewis accounts is true because both antecedent and consequent are true in any world; on the contrary if formalized as a relevant implication is not provable since it does not enjoy the variable sharing principle. The analysis offered by relevant logics is thus compatible with our perspective: according to our insights the lack of connection between antecedent and consequent in the sentence "if it is not the case that it rains and it does not rain, then the moon is yellow or the moon is not yellow" does not make it possible to formulate the grounds for such a sentence.

So relevant logic seems to be the best option for formulating grounding rules for implication that reflect our intuitions. But since relevant logic is an umbrella term that gathers together several different formal systems, it would be interesting to identify the most appropriate. There are two good reasons for preferring the system **CR**, which has been introduced in Section 4, over the others. Let us explain these two reasons in detail. The first is simplicity: compared to the other relevant logics, **CR** has both an elegant and simple semantics and and elegant and simple proof theory. The proof theory has been introduced in Section 4 and it is striking for its clarity. The semantics has not been presented for brevity but we refer to Tennant (1984). Grounding is a complicated and problematic object to treat formally; choosing a simple framework in which to formulate grounding rules is thus a primary requirement if we do not want the complexity of the topic to explode. Thus, **CR** appears to be the best relevant logic for this purpose.

A second reason for choosing **CR** is its peculiar link with normality of derivations: as already said (see Section 4) in **CR** all derivations are normal. There exists a viable tradition that starts with Aristotle, passes trough Bolzano, but is also present in the contemporary literature (e.g. see Casari (1987); Rumberg (2013)) that requires ground-revealing proofs to be normal in some sense. This aspect has been deeply discussed in the recent debate. Hence, from this perspective, **CR** seems to be an ideal framework where to develop grounding rules for implication since it ensures normality right from the start. This certainly is a second significant reason for preferring it over the other existing relevant logics.

We have thus terminated our reflections on grounding rules for implication. We have shown that depending on which type of implication is adopted the grounding rules for implication vary. While Schnieder's intuitions match very well with the material implication, our intuitions are captured very well by relevant logic and in particular by the system **CR**. Schnieder's intuitions have already been formalized by means of the rules $\rightarrow 1$ and $\rightarrow 2$. As for our intuitions (or analogous ones) no result has so far been presented. This is certainly an interesting and compelling line of work for future research.

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