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On the construction of set-valued dual processes

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Abstract

Markov intertwining is an important tool in stochastic processes: it enables to prove equalities in law, to assess convergence to equilibrium in a probabilistic way, to relate apparently distinct random models or to make links with wave equations, see Carmona, Petit and Yor [5], Aldous and Diaconis [2], Borodin and Olshanski [4] and Pal and Shkolnikov [18] for examples of applications in these domains. Unfortunately the basic construction the underlying coupling by Diaconis and Fill [7] is not easy to manipulate. A direct approach is proposed here to remedy to this drawback, via random mappings for set-valued dual processes, first in a discrete time and finite state space setting. This construction is related to the evolving sets of Morris and Peres [17] and to the coupling-from-the-past algorithm of Propp and Wilson [22]. Extensions to continuous frameworks enable to recover, via a coalescing stochastic flow due to Le Jan and Raimond [13], the explicit coupling underlying the intertwining relation between the Brownian motion and the Bessel-3 process due to Pitman [20]. To generalize such a coupling to more general one-dimensional diffusions, new coalescing stochastic flows would be needed and the paper ends with challenging conjectures in this direction.

Keywords: Markov intertwining relations, set-valued dual processes, Diaconis-Fill couplings, random mappings, coalescing stochastic flows, Pitman's theorem, one-dimensional diffusions.

MSC2010: primary: 60J10, secondary: 37A25, 60J05, 60J25, 60J60, 60G17, 60J65.

1 Introduction

Consider two Markov processes X and \mathfrak{X} on respective state spaces V and \mathfrak{V} , whose generators are denoted L and \mathfrak{L} . The Markov process \mathfrak{X} is said to be a dual (by intertwining) of X when \mathfrak{L} and L are linked via a weak conjugation relation $\mathfrak{L}\Lambda = \Lambda L$, where Λ is a Markov kernel going from \mathfrak{V} to V (so that at least formally, the previous commutation makes sense). In such circumstances, the processes X and \mathfrak{X} can usually be nicely coupled, so that useful informations on the behavior of X can be deduced from \mathfrak{X} . The construction of the coupling was given by Diaconis and Fill [7] in the case of finite state spaces and it is sometimes possible to extend it to more general situations, up to cumbersome technicalities (see e.g. [16], for one-dimensional diffusions). Here we propose a direct construction in the context of set-valued duals \mathfrak{X} , namely those for which \mathfrak{V} is a set of subsets of V , X admits an invariant measure π and Λ corresponds to the conditioning of π with respect to the elements of \mathfrak{V} . When V is finite and X is irreducible, Chapter 17 of Levin, Peres and Wilmer [15] presents a particular set-valued dual as a Doob transform of the evolving sets of Morris and Peres [17]. We go further in this direction, by constructing \mathfrak{X} directly from X via an approach sharing similarities with the coupling-from-the-past algorithm of Propp and Wilson [22]. Some reverse random mappings play the main role, since after having been conditioned to be compatible with a given trajectory of X , they are used to spread the set-valued dual through the state space. We will reinterpret the classical examples of the discrete Pitman theorem [20] and of the top-to-random shuffle due to Aldous and Diaconis [1] as particular instances of this general construction. To facilitate the exposition, we will mainly consider finite state spaces and discrete time in this introduction, leaving the extensions to continuous time and spaces to the last part of the paper, that will enable us to recover the classical Pitman's theorem [20] (see also Rogers and Pitman [24]). To deal with more general one-dimensional diffusions and even multi-dimensional diffusions (see the working paper [6] for the corresponding definition of \mathfrak{L}), we would need some coalescing flows of a new type, whose theory has not yet been developed, despite the works of Le Jan and Raimond, in particular [12, 13]. Their investigation should lead to direct constructions of stochastic perturbations of mean curvature flows, in the spirit of [6], which is the remote motivation for this first study.

Let V be a finite space, endowed with a Markov kernel $P := (P(x, y))_{x, y \in V}$. A traditionally associated generator L is $P - I$, where I is the $V \times V$ identity matrix, to make a connexion with the above paragraph. Instead, we first consider discrete time and keep working with P instead of L . When we are given a distribution m_0 on V , there exists a Markov chain $X := (X_n)_{n \in \mathbb{Z}_+}$ on V whose initial position X_0 is distributed according to m_0 and whose transition are dictated by P . The law $\mathcal{L}(X)$ of X is uniquely determined by m_0 and P . From now on, we assume that P is irreducible (i.e. $\exp(P)$ has only positive entries), so that it admits a unique invariant probability $\pi := (\pi(x))_{x \in V}$. The entries of π are positive. As usual, measures (respectively functions) are seen as row (resp. column) vectors and the **invariance** of π writes $\pi P = \pi$. This terminology comes from the fact that when the initial distribution is chosen to be π , then for any time $n \in \mathbb{Z}_+$, the law of X_n is equal to π . In this situation, it is possible to consider a stationary Markov chain $X := (X_n)_{n \in \mathbb{Z}}$ defined for all times $n \in \mathbb{Z}$. The time-reversed process $(X_{-n})_{n \in \mathbb{Z}}$ is also a stationary Markov chain, whose transition matrix $P^* := (P^*(x, y))$ is given by

$$\forall x, y \in V, \quad P^*(x, y) := \frac{\pi(y)}{\pi(x)} P(y, x) \quad (1)$$

Let us recall the **evolving set process** of Morris and Peres [17] (see also Chapter 17 of Levin, Peres and Wilmer [15]). It is a Markov chain $\mathcal{X} := (\mathcal{X}_n)_{n \in \mathbb{Z}_+}$ taking values in \mathfrak{V} , where \mathfrak{V} is the set of all subsets of V . To define its transition matrix J , fix $S \in \mathfrak{V}$ and consider a random variable U uniformly distributed on $[0, 1]$. Denote

$$\Phi(S) := \{y \in V : P^*(y, S) \geq U\} \quad (2)$$

and

$$\forall S, S' \in \bar{\mathfrak{V}}, \quad J(S, S') := \mathbb{P}[\Phi(S) = S'] \quad (3)$$

Note that \mathfrak{X} is absorbed at \emptyset and at $V \in \mathfrak{V}$. The mapping $\pi : \bar{\mathfrak{V}} \ni S \mapsto \pi(S)$ is harmonic for J , in the sense that $J\pi = \pi$. It leads to consider the **Doob transform** of J by π , which is the $\mathfrak{V} \times \mathfrak{V}$ transition matrix \mathfrak{P}_J given by

$$\forall S, S' \in \mathfrak{V}, \quad \mathfrak{P}_J(S, S') := \frac{\pi(S')}{\pi(S)} J(S, S') \quad (4)$$

where \mathfrak{V} is the set of all nonempty subsets of V . The matrix \mathfrak{P}_J is the transition kernel of \mathcal{X} conditioned not to be absorbed at \emptyset , cf. Chapter 17 of Levin, Peres and Wilmer [15]. Let $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ be a Markov chain whose transitions are dictated by \mathfrak{P}_J . It is a set-valued dual to X . Indeed, let Λ be the Markov matrix from \mathfrak{V} to V given by

$$\forall S \in \mathfrak{V}, \forall x \in X, \quad \Lambda(S, x) := \frac{\pi(x)}{\pi(S)} \quad (5)$$

It is not difficult to check the intertwining relation $\mathfrak{P}_J \Lambda = \Lambda P$.

More generally, let \mathfrak{P} be a transition kernel on \mathfrak{V} satisfying

$$\mathfrak{P} \Lambda = \Lambda P \quad (6)$$

and let $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ be a corresponding Markov chain. Assuming furthermore

$$\mathcal{L}(\mathfrak{X}_0) \Lambda = \mathcal{L}(X_0) \quad (7)$$

Diaconis and Fill [7] constructed a coupling of X and \mathfrak{X} such that the two following properties hold:

$$\forall n \in \mathbb{Z}_+, \quad \mathcal{L}(\mathfrak{X}_{\llbracket 0, n \rrbracket} | X) = \mathcal{L}(\mathfrak{X}_{\llbracket 0, n \rrbracket} | X_{\llbracket 0, n \rrbracket}) \quad (8)$$

$$\forall n \in \mathbb{Z}_+, \quad \mathcal{L}(X_n | \mathfrak{X}_{\llbracket 0, n \rrbracket}) = \Lambda(\mathfrak{X}_n, \cdot) \quad (9)$$

In these identities, $\mathcal{L}(Y|Z)$ stands for the conditional law of Y knowing Z , for any (here finite valued) random variables Y, Z defined on a same probability space, $\llbracket 0, n \rrbracket := \{0, 1, \dots, n\}$ and $Y_{\llbracket 0, n \rrbracket}$ is the stopped trajectory $(Y_m)_{m \in \llbracket 0, n \rrbracket}$ for any process $Y := (Y_n)_{n \in \mathbb{Z}_+}$.

The first relation (8) requires that \mathfrak{X} can be deduced from X in an adapted way: for any $n \in \mathbb{Z}_+$, only the knowledge of $X_{\llbracket 0, n \rrbracket}$ is needed to construct $\mathfrak{X}_{\llbracket 0, n \rrbracket}$, maybe with the help of independent randomness. The second relation can be seen as a stochastic prolongation of both $\mathcal{L}(\mathfrak{X}_0) \Lambda = \mathcal{L}(X_0)$ and $\mathfrak{P} \Lambda = \Lambda P$.

Unfortunately, the construction of Diaconis and Fill [7] is not easy to manipulate, that is why we propose an alternative approach, valid for a generalization of the evolving set point of view.

Our main ingredient is the following object. A random mapping $\psi : V \rightarrow V$ (defined on some underlying probability space) is said to be **associated to P^*** when

$$\forall x, x' \in V, \quad \mathbb{P}[\psi(x) = x'] = P^*(x, x')$$

It is convenient to have at our disposal a random mapping ψ_S for any given $S \in \bar{\mathfrak{V}}$. Such a family $(\psi_S)_{S \in \bar{\mathfrak{V}}}$ is said to be **weakly associated to P^*** if

$$\forall S \in \bar{\mathfrak{V}}, \forall x \in V, \forall x' \in S, \quad \mathbb{P}[\psi_S(x) = x'] = \frac{P^*(x, x')}{\zeta(S)} \quad (10)$$

where $\zeta : \bar{\mathfrak{V}} \rightarrow (0, +\infty)$ is a given positive mapping on $\bar{\mathfrak{V}}$ (note that necessarily, $\zeta(V) = 1$). From now on, all the families $(\psi_S)_{S \in \bar{\mathfrak{V}}}$ we will consider will be implicitly assumed to be weakly associated to P^* .

Remark 1 A family $(\psi_S)_{S \in \bar{\mathfrak{Y}}}$ is said to be **completely associated** to P^* when all its random mappings ψ_S are associated to P^* . All the examples of Section 3 are completely associated to P^* . The interest of the notion of weak association will only appear in Section 5. It is natural from a theoretical point of view, but will also be important for some applications not treated in the present paper. □

A family $(\psi_S)_{S \in \bar{\mathfrak{Y}}}$ enables to define a random mapping Ψ from $\bar{\mathfrak{Y}}$ to $\bar{\mathfrak{Y}}$ via

$$\forall S \in \bar{\mathfrak{Y}}, \quad \Psi(S) := \{y \in V : \psi_S(y) \in S\} \quad (11)$$

We compute that

$$\begin{aligned} \forall S \in \bar{\mathfrak{Y}}, \forall y \in V, \quad \mathbb{P}[y \in \Psi(S)] &= \mathbb{P}[\psi_S(y) \in S] \\ &= \sum_{y' \in S} \mathbb{P}[\psi_S(y) = y'] \\ &= \sum_{y' \in S} P^*(y, y') / \zeta(S) \\ &= P^*(y, S) / \zeta(S) \end{aligned}$$

Thus when $\zeta \equiv 1$, in particular for completely associated to P^* random mappings, it appears that for any $y \in V$ and $S \in \bar{\mathfrak{Y}}$, $\mathbb{P}[y \in \Psi(S)] = \mathbb{P}[y \in \Phi(S)]$, but in general the law of $\Phi(S)$ and $\Psi(S)$ are not equal, as it can be seen on the examples of Section 3.

As in (3), consider the transition matrix K from $\bar{\mathfrak{Y}}$ to $\bar{\mathfrak{Y}}$ given by

$$\forall S, S' \in \bar{\mathfrak{Y}}, \quad K(S, S') := \mathbb{P}[\Psi(S) = S'] \quad (12)$$

and as in (4), define its modified Doob transform via

$$\forall S, S' \in \bar{\mathfrak{Y}}, \quad \mathfrak{P}(S, S') := \frac{\pi(S')\zeta(S)}{\pi(S)} K(S, S') \quad (13)$$

(in general \mathfrak{P} does not coincide with \mathfrak{P}_J). It will be shown in Corollary 4 below that \mathfrak{P} is a Markov transition kernel. It will also be useful to introduce the following conditioned transition: fix $x, x' \in V$ such that $P(x, x') > 0$ (i.e. $P^*(x', x) > 0$) and denote for any $S \in \bar{\mathfrak{Y}}$ containing x ,

$$\forall S' \in \bar{\mathfrak{Y}}, \quad K_{x, x'}(S, S') := \mathbb{P}[\Psi(S) = S' | \psi_S(x') = x] \quad (14)$$

Note that the conditioning is non-degenerate, since $\mathbb{P}[\psi_S(x') = x] = P^*(x', x) / \zeta(S) > 0$, due to the fact that $x \in S$.

Consider

$$W := \{(x, S) \in V \times \bar{\mathfrak{Y}} : x \in S\} \quad (15)$$

and let \mathcal{A} be the set of probability measures m on W which can be written under the form

$$\forall (x, S) \in W, \quad m(x, S) = \mu(S)\Lambda(S, x) \quad (16)$$

where μ is the marginal of m on $\bar{\mathfrak{Y}}$ (i.e. the image of m by the mapping $W \ni (x, S) \mapsto S \in \bar{\mathfrak{Y}}$). Define a Markov kernel Q on W via

$$\forall (x, S), (x', S') \in W, \quad Q((x, S), (x', S')) := P(x, x')K_{x, x'}(S, S') \quad (17)$$

Remark 2 This expression should be compared with the one given by Levin, Peres and Wilmer in Section 17.7 of [15], where they introduce a Markov kernel \tilde{Q} on W via

$$\forall (x, S), (x', S') \in W, \quad \tilde{Q}((x, S), (x', S')) := P(x, x')J_{x'}(S, S')$$

where, with the notation of (2),

$$\forall S \in \mathfrak{S}, \forall (x', S') \in W, \quad J_{x'}(S, S') := \mathbb{P}[\Phi(S) = S' | x' \in S']$$

In Subsection 2.3, we will check that \tilde{Q} is a particular case of Q , for an appropriate choice of the family of random mappings $(\psi_S)_{S \in \mathfrak{S}}$. The generality of the random mapping point of view leads to easy constructions in practice, even if they are not optimal, with a coupling-from-the-past flavor (see Subsection 2.4). In particular, it facilitates the reinterpretation of classical set-valued dual processes (see Subsections 3.1 and 3.2). Another advantage of the random mappings is that they can be transformed into coalescing stochastic flows in the diffusion setting (see Section 6 for the general approach, Subsection 7.1 for an application to the real Brownian motion and Subsection 7.2 for corresponding conjectures about general one-dimensional diffusions). □

We can now state the first main result of this paper (continuous space and time extensions will be presented in Theorems 14, 16 and 25 of Sections 5 and 6).

Theorem 3 *Let $(X_n, \mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ be a Markov chain on W whose initial distribution $\mathcal{L}(X_0, \mathfrak{X}_0)$ belongs to \mathcal{A} and whose transitions are given by Q . Then $X := (X_n)_{n \in \mathbb{N}}$ and $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{N}}$ are Markov chains whose respective transitions are given by P and \mathfrak{P} . Furthermore the conditions (8) and (9) are fulfilled.*

As a first consequence, we get an extension of properties recalled for the evolving sets:

Corollary 4 *The kernel \mathfrak{P} is Markovian and the intertwining relation (6) is satisfied. Furthermore when $\zeta \equiv 1$, in particular in the case of completely associated to P^* random mappings, π is harmonic for K .*

Proof

The first assertion is an immediate consequence of Theorem 3 and the last one is obtained by summing in (13) with respect to $S' \in \mathfrak{S}$. The intertwining relation can be checked directly, but it also comes from the computation of $\mathbb{P}[X_{n+1} = x | \mathfrak{X}_n = S]$, for $n \in \mathbb{Z}_+$, $x \in V$ and $S \in \mathfrak{S}$, in two different ways:

On one hand, we have

$$\begin{aligned} \mathbb{P}[X_{n+1} = x | \mathfrak{X}_n = S] &= \sum_{S' \in \mathfrak{S}} \mathbb{P}[X_{n+1} = x | \mathfrak{X}_{n+1} = S', \mathfrak{X}_n = S] \mathbb{P}[\mathfrak{X}_{n+1} = S' | \mathfrak{X}_n = S] \\ &= \sum_{S' \in \mathfrak{S}} \Lambda(S', x) \mathfrak{P}(S, S') \\ &= \mathfrak{P} \Lambda(S, x) \end{aligned}$$

and on the other one,

$$\begin{aligned} \mathbb{P}[X_{n+1} = x | \mathfrak{X}_n = S] &= \sum_{x' \in V} \mathbb{P}[X_{n+1} = x | X_n = x', \mathfrak{X}_n = S] \mathbb{P}[X_n = x' | \mathfrak{X}_n = S] \\ &= \sum_{x' \in \mathfrak{S}} \Lambda(S, x') \mathbb{P}[X_{n+1} = x | X_n = x', \mathfrak{X}_n = S] \\ &= \sum_{x' \in \mathfrak{S}} \Lambda(S, x') P(x', x) \\ &= \Lambda P(S, x) \end{aligned}$$

where the last-but-one equality comes by summing over S' in (17). ■

Theorem 3 leads to an algorithm for the construction of \mathfrak{X} , given X , when $\mu_0\Lambda = m_0$, where $\mu_0 := \mathcal{L}(\mathfrak{X}_0)$ and $m_0 := \mathcal{L}(X_0)$ (in practice, one is often only interested in the situation where m_0 is a Dirac mass at some $x_0 \in V$ and μ_0 is the Dirac mass at $\{x_0\}$, trivially satisfying $\Lambda(\{x_0\}, x_0) = 1$). Assume that a trajectory $(x_n)_{n \in \mathbb{Z}_+}$ of X is given. We begin by sampling \mathfrak{X}_0 according to the probability measure $\mathfrak{V} \ni S \mapsto \mu_0(S)\Lambda(S, x_0)/m_0(x_0)$. Next, for $n \in \mathbb{Z}_+$, assume that \mathfrak{X}_n has been constructed. We consider a random mapping $\psi_{\mathfrak{X}_n}$ weakly associated to P^* , whose law may depend on \mathfrak{X}_n (but not directly on $(x_m)_{m \in \llbracket 0, n \rrbracket}$, see Subsection 2.2 for a generalization with a dependence on $\mathfrak{X}_{\llbracket 0, n \rrbracket}$) and whose underlying randomness is independent from all that has been done before (except for the index parameter \mathfrak{X}_n). We condition by the fact that $\psi_{\mathfrak{X}_n}(x_{n+1}) = x_n$ and we sample a corresponding mapping φ (which is no longer associated to P^* , since in general this property is not preserved by conditioning), to construct

$$\mathfrak{X}_{n+1} := \{y \in V : \varphi(y) \in \mathfrak{X}_n\}$$

Due to the conditioning, we are sure that $x_{n+1} \in \mathfrak{X}_{n+1}$, since by iteration, we would have checked that $x_n \in \mathfrak{X}_n$.

This procedure is maybe better illustrated by the explicit constructions of Subsections 2.3, 2.4, 3.1 and 3.2. There, only classical examples are considered, as this paper is primarily concerned with the theoretical aspects of the random mapping point of view. Theorem 3 describes a general method of construction of set-valued dual processes as well as their couplings with the primal processes. The obtained dual processes will be good or bad (with respect to their fast spreading over the primal state space) depending on the underlying random mappings. In practice, a relevant choice of the latter is thus crucial. This is another task, so that more examples will be presented in future works. While in the finite state space setting random mappings are easy to exhibit, this is no longer true in continuous frameworks, as the underlying coalescing stochastic flows should be investigated further, to begin with in dimension one. Nevertheless we also expect applications for elliptic diffusions on manifolds, then even the construction of dual processes is difficult (see the forthcoming paper [6]) and the coupling apparently out-of-reach by traditional approaches. We believe that the coalescing stochastic flows will provide a constructive existence of the dual processes and of their couplings with the primal processes. As illustrated by Theorem 3 and Corollary 4, the coupling should be constructed first and the dual process deduced in a second step. Thus our method is in reverse order in comparison to the original work of Diaconis and Fill [7].

The plan of the paper is as follows. Theorem 3 is proven in the next section and we will see how random mappings can also be used to deduce non-set-valued dual processes. The classical set-valued dual processes of the discrete Pitman theorem and of the top-to-random shuffle are treated in Section 3. In Section 4, we generalize the discrete Pitman theorem to restless birth and death chains, i.e. moving at each time step. In Sections 5 and 6, we provide extensions of the random mapping analysis, respectively to Polish state spaces and to continuous time. From these abstract considerations, Section 7 recovers Pitman's theorem [20] about the explicit and deterministic coupling associated to the intertwining of the Brownian motion with the Bessel-3 process and proposes some conjectures about general one-dimensional diffusion processes. Our hope is that the underlying challenge of the existence of some needed singular coalescing stochastic flows will motive a more thorough investigation in their direction. An appendix ends the paper, showing why in dimension 1 it is sufficient to study diffusions whose variance coefficient is 1, via some traditional transformations of the state space \mathbb{R} .

2 Random mappings

The proof of Theorem 3 follows the lines of that of Theorem 17.23 of Levin, Peres and Wilmer [15], itself in the spirit of Diaconis and Fill [7]. The argument will be extended to auxiliary set-valued processes in Subsection 2.2, obtained by slightly relaxing the notion of random mappings, it leads

to more general dual processes than the set-valued ones. In Subsection 2.3, we justify the assertions of Remark 2 and in Subsection 2.4 we discuss the link with the coupling-from-the-past algorithm.

2.1 Proof of Theorem 3

By definition, we have for any $n \in \mathbb{Z}_+$ and $(x_0, S_0), (x_1, S_1), \dots, (x_n, S_n) \in W$,

$$\begin{aligned} & \mathbb{P}[(X_0, \mathfrak{X}_0) = (x_0, S_0), (X_1, \mathfrak{X}_1) = (x_1, S_1), \dots, (X_n, \mathfrak{X}_n) = (x_n, S_n)] \\ &= \mu_0(S_0)\Lambda(S_0, x_0) \prod_{m \in \llbracket 0, n-1 \rrbracket} P(x_m, x_{m+1})K_{x_m, x_{m+1}}(S_m, S_{m+1}) \end{aligned} \quad (18)$$

where $\mu_0 := \mathcal{L}(\mathfrak{X}_0)$. Summing over all $S_0, S_1, \dots, S_n \in \mathfrak{V}$ (so that $x_0 \in S_0, x_1 \in S_1, \dots, x_n \in S_n$), we get that for any $x_0, x_1, \dots, x_n \in V$,

$$\mathbb{P}[X_0 = x_0, X_1 = x_1, \dots, X_n = x_n] = m_0(x_0) \prod_{m \in \llbracket 0, n-1 \rrbracket} P(x_m, x_{m+1}) \quad (19)$$

where

$$m_0(x_0) := \sum_{S_0 \in \mathfrak{V}} \mu_0(S_0)\Lambda(S_0, x_0)$$

(the r.h.s. sum can be restricted to S_0 containing x_0 , since otherwise $\Lambda(S_0, x_0) = 0$). It follows that $(X_m)_{m \in \llbracket 0, n \rrbracket}$ is Markovian with transitions given by P and initial distribution m_0 . Since this is true for all $n \in \mathbb{Z}_+$, we get that X is Markovian with transitions given by P and initial distribution m_0 .

For any $m \in \llbracket 0, n \rrbracket$, dividing (18) by (19) and summing over S_{m+1}, \dots, S_n , we get

$$\begin{aligned} & \mathbb{P}[\mathfrak{X}_0 = S_0, \mathfrak{X}_1 = S_1, \dots, \mathfrak{X}_m = S_m | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n] \\ &= \frac{\mu_0(S_0)\Lambda(S_0, x_0)}{m_0(x_0)} \prod_{l \in \llbracket 0, m-1 \rrbracket} K_{x_l, x_{l+1}}(S_l, S_{l+1}) \\ &= \mathbb{P}[\mathfrak{X}_0 = S_0, \mathfrak{X}_1 = S_1, \dots, \mathfrak{X}_m = S_m | X_0 = x_0, X_1 = x_1, \dots, X_m = x_m] \end{aligned}$$

Fixing $m \in \mathbb{Z}_+$, letting n go to infinity and using the martingale convergence theorem, we get that

$$\mathbb{P}[\mathfrak{X}_{\llbracket 0, m \rrbracket} = S_{\llbracket 0, m \rrbracket} | X] = \mathbb{P}[\mathfrak{X}_{\llbracket 0, m \rrbracket} = S_{\llbracket 0, m \rrbracket} | X_{\llbracket 0, m \rrbracket}]$$

namely (8).

The Markov property of \mathfrak{X} and (9) are less immediate and the argument is based on an iteration with respect to the following statements, for $n \in \mathbb{Z}_+$:

The finite stochastic chain $(\mathfrak{X}_m)_{m \in \llbracket 0, n \rrbracket}$ is Markovian with transitions given by \mathfrak{P} (A_n)

$$\mathcal{L}(X_n | \mathfrak{X}_{\llbracket 0, n \rrbracket}) = \Lambda(\mathfrak{X}_n, \cdot) \quad (B_n)$$

For $n = 0$, the assertion (A₀) is void and (B₀) is a rewriting of the assumption $\mathcal{L}(X_0, \mathfrak{X}_0) \in \mathcal{A}$. Next assume that (A_n) and (B_n) are true for some $n \in \mathbb{Z}_+$ and let us prove (A_{n+1}) and (B_{n+1}). Let $(x, S) \in W$ be given, we compute that

$$\mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = S | \mathfrak{X}_{\llbracket 0, n \rrbracket}] = \sum_{y \in \mathfrak{X}_n} \mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = S | X_n = y, \mathfrak{X}_{\llbracket 0, n \rrbracket}] \mathbb{P}[X_n = y | \mathfrak{X}_{\llbracket 0, n \rrbracket}]$$

Due to the Markov property of the couple (X, \mathfrak{X}) , we deduce that for any $y \in \mathfrak{X}_n$,

$$\begin{aligned}
\mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = S | X_n = y, \mathfrak{X}_{[0,n]}] &= \mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = S | X_n = y, \mathfrak{X}_n] \\
&= P(y, x) K_{y,x}(\mathfrak{X}_n, S) \\
&= P(y, x) \frac{\mathbb{P}[\Psi(\mathfrak{X}_n) = S, \psi_{\mathfrak{X}_n}(x) = y]}{\mathbb{P}[\psi_{\mathfrak{X}_n}(x) = y]} \\
&= P(y, x) \zeta(\mathfrak{X}_n) \frac{\mathbb{P}[\Psi(\mathfrak{X}_n) = S, \psi_{\mathfrak{X}_n}(x) = y]}{P^*(x, y)} \\
&= \frac{\pi(x)}{\pi(y)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = S, \psi_{\mathfrak{X}_n}(x) = y] \quad (20)
\end{aligned}$$

where the weak association with P^* was used in the fourth equality. On the other hand, (B_n) asserts that

$$\begin{aligned}
\mathbb{P}[X_n = y | \mathfrak{X}_{[0,n]}] &= \Lambda(\mathfrak{X}_n, y) \\
&= \frac{\pi(y)}{\pi(\mathfrak{X}_n)}
\end{aligned}$$

and we get that

$$\begin{aligned}
\mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = S | \mathfrak{X}_{[0,n]}] &= \sum_{y \in \mathfrak{X}_n} \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = S, \psi_{\mathfrak{X}_n}(x) = y] \\
&= \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = S, \psi_{\mathfrak{X}_n}(x) \in \mathfrak{X}_n] \\
&= \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = S, x \in S] \\
&= \frac{\pi(x)}{\pi(\mathfrak{X}_n)} \zeta(\mathfrak{X}_n) \mathbb{P}[\Psi(\mathfrak{X}_n) = S] \\
&= \frac{\pi(x)}{\pi(S)} \mathfrak{P}(\mathfrak{X}_n, S)
\end{aligned}$$

where we used Definitions (12) and (13).

Summing over $x \in S$, we get

$$\mathbb{P}[\mathfrak{X}_{n+1} = S | \mathfrak{X}_{[0,n]}] = \mathfrak{P}(\mathfrak{X}_n, S)$$

whose validity for all $S \in \mathfrak{Y}$ is (A_{n+1}) . Using that for all $(x, S) \in W$,

$$\begin{aligned}
\mathbb{P}[X_{n+1} = x | \mathfrak{X}_{n+1} = S, \mathfrak{X}_{[0,n]}] &= \frac{\mathbb{P}[X_{n+1} = x, \mathfrak{X}_{n+1} = S | \mathfrak{X}_{[0,n]}]}{\mathbb{P}[\mathfrak{X}_{n+1} = S | \mathfrak{X}_{[0,n]}]} \\
&= \frac{\pi(x)}{\pi(S)} \\
&= \Lambda(S, x)
\end{aligned}$$

we conclude to (B_{n+1}) .

2.2 Auxiliary set-valued processes

In the definition of random mappings weakly associated to P^* given in the introduction, we assumed that we had at our disposal a family $(\psi_S)_{S \in \mathfrak{Y}}$. Suppose now that we rather have a family $(\psi_{S_{[0,n]}})_{S_{[0,n]} \in \mathfrak{W}}$, where

$$\mathfrak{W} := \sqcup_{n \in \mathbb{N}} \mathfrak{Y}^n$$

and whose elements are written under the form $S_{\llbracket 0, n \rrbracket} := (S_0, S_1, \dots, S_n)$, for some $n \in \mathbb{Z}_+$ and $S_0, S_1, \dots, S_n \in \mathfrak{V}$. The weak association with P^* of the family $(\psi_{S_{\llbracket 0, n \rrbracket}})_{S_{\llbracket 0, n \rrbracket} \in \bar{\mathfrak{M}}}$ now means that

$$\forall S_{\llbracket 0, n \rrbracket} \in \bar{\mathfrak{M}}, \forall x \in V, \forall x' \in S_n, \quad \mathbb{P}[\psi_{S_{\llbracket 0, n \rrbracket}}(x) = x'] = \frac{P^*(x, x')}{\zeta(S_{\llbracket 0, n \rrbracket})}$$

where $\zeta : \bar{\mathfrak{M}} \rightarrow (0, +\infty)$ is a positive mapping on $\bar{\mathfrak{M}}$ (note that necessarily $\zeta(S_{\llbracket 0, n \rrbracket}) = 1$ as soon as $S_n = V$). Following (11), we get a random set depending on $S_{\llbracket 0, n \rrbracket} \in \bar{\mathfrak{M}}$ and given by:

$$\forall S_{\llbracket 0, n \rrbracket} \in \bar{\mathfrak{M}}, \quad \Psi(S_{\llbracket 0, n \rrbracket}) := \{y \in V : \psi_{S_{\llbracket 0, n \rrbracket}}(y) \in S_n\}$$

so that the analogue of (12) is a transition kernel from $\bar{\mathfrak{M}}$ to $\bar{\mathfrak{V}}$

$$\forall S_{\llbracket 0, n \rrbracket} \in \bar{\mathfrak{M}}, \forall S' \in \bar{\mathfrak{V}}, \quad K(S_{\llbracket 0, n \rrbracket}, S') := \mathbb{P}[\Psi(S_{\llbracket 0, n \rrbracket}) = S']$$

and (13) must be transformed into a kernel from \mathfrak{W} to \mathfrak{V} :

$$\forall S_{\llbracket 0, n \rrbracket} \in \mathfrak{W}, \forall S' \in \mathfrak{V}, \quad \mathfrak{P}(S_{\llbracket 0, n \rrbracket}, S') := \frac{\pi(S')\zeta(S_{\llbracket 0, n \rrbracket})}{\pi(S_n)} K(S_{\llbracket 0, n \rrbracket}, S') \quad (21)$$

where \mathfrak{W} is the denumerable space

$$\mathfrak{W} := \sqcup_{n \in \mathbb{N}} \mathfrak{V}^n$$

By working similarly to the introduction, define conditioned transition kernels as in (14) via

$$\forall S_{\llbracket 0, n \rrbracket} \in \mathfrak{W}, \forall S' \in \mathfrak{V}, \quad K_{x, x'}(S_{\llbracket 0, n \rrbracket}, S') := \mathbb{P}[\Psi(S_{\llbracket 0, n \rrbracket}) = S' | \psi_{S_{\llbracket 0, n \rrbracket}}(x') = x]$$

for any $x, x' \in V$ with $x \in S_n$ and $P(x, x') > 0$. As a consequence of the dependence on $S_{\llbracket 0, n \rrbracket} \in \mathfrak{W}$ in (21), we cannot expect in Theorem 3 that \mathfrak{X} will still be Markovian.

Nevertheless, the other parts of Theorem 3 are satisfied. Indeed, it is sufficient to replace everywhere in the previous subsection $\psi_{\mathfrak{X}_n}$ by $\psi_{\mathfrak{X}_{\llbracket 0, n \rrbracket}}$, $\Psi(\mathfrak{X}_n)$ by $\Psi(\mathfrak{X}_{\llbracket 0, n \rrbracket})$ and $\mathfrak{P}(\mathfrak{X}_n, S)$ by $\mathfrak{P}(\mathfrak{X}_{\llbracket 0, n \rrbracket}, S)$.

The algorithm explained after the proof of Corollary 4 is straightforwardly adapted to this extended situation. The finite sequence $\mathfrak{X}_{\llbracket 0, n \rrbracket}$ constructed in this way from a given trajectory $x_{\llbracket 0, n \rrbracket}$ is called an auxiliary set-valued process. It can be used to construct strong stationary times for X (see Fill and Diaconis [7]): consider

$$\tau := \inf\{n \in \mathbb{Z}_+ : \mathfrak{X}_n = V\} \quad (22)$$

When τ is (a.s.) finite, we have that τ and X_τ are independent and the distribution of X_τ is the invariant measure π .

The notion of auxiliary set-valued process is an instance of non-set-valued dual process. Indeed, use the traditional trick transforming any process into a Markov process by adding all its history in its present state. More precisely, consider the \mathfrak{W} -valued process $\mathfrak{Y} := (\mathfrak{X}_{\llbracket 0, n \rrbracket})_{n \in \mathbb{Z}_+}$. The process \mathfrak{Y} is clearly Markovian, its transition kernel Ω being given by

$$\forall S_{\llbracket 0, n \rrbracket}, S'_{\llbracket 0, n' \rrbracket} \in \mathfrak{W}, \quad \Omega(S_{\llbracket 0, n \rrbracket}, S'_{\llbracket 0, n' \rrbracket}) := \begin{cases} \mathfrak{P}(S_{\llbracket 0, n \rrbracket}, S'_{n+1}) & , \text{ if } n' = n + 1 \text{ and } S_{\llbracket 0, n \rrbracket} = S'_{\llbracket 0, n \rrbracket} \\ 0 & , \text{ otherwise} \end{cases}$$

Extend Λ into a Markov kernel from \mathfrak{W} to V via

$$\forall S_{\llbracket 0, n \rrbracket} \in \mathfrak{W}, \forall x \in V, \quad \Lambda(S_{\llbracket 0, n \rrbracket}, x) = \frac{\pi(x)}{\pi(S_n)}$$

As in Corollary 4, we deduce the intertwining relation

$$\Omega \Lambda = \Lambda P$$

showing that \mathfrak{Y} is a dual process to X . Thus from the general theory of Fill and Diaconis [7], we know that a Markov chain with transition kernel Ω can be used to construct a strong stationary time, as soon as a.s. it ends up reaching the set $\{Y \in \mathfrak{W} : \Lambda(Y, \cdot) = \pi\}$, which corresponds in the above situation to the fact that τ defined in (22) is a.s. finite.

2.3 The Levin, Peres and Wilmer construction

Let us come back to the construction of Chapter 17 of Levin, Peres and Wilmer [15] and interpret it in the random mapping setting.

In practice, a random mapping ψ is often given in the following way: let $I := (I_{x,y})_{x,y \in V}$ be a family of measurable subsets of $[0, 1)$ such that

$$\forall x \in V, \quad [0, 1) = \bigsqcup_{y \in V} I_{x,y} \quad (23)$$

$$\forall x, y \in V, \quad \lambda(I_{x,y}) = P^*(x, y) \quad (24)$$

where λ is the Lebesgue measure. Let U be a random variable uniformly distributed on $[0, 1)$. A random mapping ψ associated to P^* can be defined in terms of I and U by deciding that

$$\forall x, y \in V, \quad \{\psi(x) = y\} = \{U \in I_{x,y}\}$$

Remark 5 Conversely, any random mapping associated to P^* has the same law as a random mapping constructed as above. This is a consequence of the fact that the set of functions from V to V is finite and that any probability distribution on a finite number of points can be seen as an image of the restriction of the Lebesgue measure on $[0, 1)$. Thus there is no loss of generality in only considering random mappings of the previous form. □

Let $S \in \mathfrak{B}$ be fixed and label the elements of V as $1, 2, \dots, |V|$, where $|V|$ is the cardinal of V , so that $S = \llbracket 1, |S| \rrbracket$. Next define a family $I := (I_{x,y})_{x,y \in V}$ via

$$\forall x, y \in V = \llbracket 1, |V| \rrbracket, \quad I_{x,y} := [P^*(x, \llbracket 1, y-1 \rrbracket), P^*(x, \llbracket 1, y \rrbracket))$$

it is immediate to check both (23) and (24). Let ϕ_S be the completely associated to P^* random mapping constructed in this way. With the notations of the introduction, we have

$$\begin{aligned} \Psi(S) &= \{x \in V : \phi_S(x) \in S\} \\ &= \{x \in V : U \in [0, P^*(x, S))\} \\ &= \{x \in V : U \leq P^*(x, S)\} \\ &= \Phi(S) \end{aligned}$$

It follows that the evolving sets of Morris and Peres [17] are a particular case of the construction via the random mappings. The special random mapping ϕ_S has the tendency to put a maximal number of points inside $\Psi(S)$, when U is small, and a minimal number of points inside $\Psi(S)$, when U is close to 1. So it seems that among all random mappings, ϕ_S induces the maximal possible oscillation for $\pi(\Psi(S))$ (e.g. measured through its variance). By analogy with the result stating that the best way to couple two Brownian motions is the mirror symmetry coupling (see for instance Jacka, Mijatović, and Siraj [10]), a tempting conjecture is that the evolving set construction is the best possible choice for \mathfrak{X} to grow as fast as possible (property which is important in the construction of strong stationary times). We believe it is true when the underlying geometry is simple (as for birth and death processes, see Section 4, this phenomenon was encountered for one-dimensional diffusions in [16]), but maybe not in the general case.

2.4 Independent random iterative mappings

A particularly simple instance of completely associated to P^* random mappings is when they are not allowed to depend on a set $S \in \mathfrak{B}$, corresponding to the current state of \mathfrak{X} . The algorithm of

the introduction then takes the following form. Let be given $n \in \mathbb{Z}_+$ and a trajectory $x_{[0,n]}$ of X . In this whole subsection, we assume for simplicity that $\mathfrak{X}_0 = \{x_0\}$. Consider n independent mappings $\psi_1, \psi_2, \dots, \psi_n$ completely associated to P^* , which may not have the same law. For any $m \in \llbracket 1, n \rrbracket$, condition ψ_m by $\psi_m(x_m) = x_{m-1}$, and to avoid confusion, let us call the new random mapping φ_m (it is no longer associated to P^* , except when $P^*(x_m, x_{m-1}) = 1$). The construction of \mathfrak{X}_n is now:

$$\mathfrak{X}_n = \{y \in V : \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(y) = x_0\} \quad (25)$$

In particular, the strong stationary time defined by (22) is given by the collapsing time

$$\tau = \inf\{n \in \mathbb{Z}_+ : \forall y \in V, \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(y) = x_0\} \quad (26)$$

Formulas (25) and (26) are valid more generally, up to the forward simultaneous definition of the $(\varphi_1, \mathfrak{X}_1), (\varphi_2, \mathfrak{X}_2), \dots, (\varphi_n, \mathfrak{X}_n)$ presented in the introduction. But when $\psi_1, \psi_2, \dots, \psi_n$ are independent, a backward construction can also be envisaged. This possibility will also be satisfied by the non-independent random mappings associated to restless birth and death chains in Section 4. Let us describe an algorithm constructing \mathfrak{X}_n , for any fixed $n \in \mathbb{Z}_+$, in this spirit. Label V as $\{y_0, y_1, \dots, y_{|V|-1}\}$, with $y_0 = x_n$ and define $\mathfrak{X}_n^{(0)} := \{x_n\}$. We look iteratively for the first $m \in \llbracket 1, n \rrbracket$ such that $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_1) = x_{n-m}$. If there is no such m , we know that $y_1 \notin \mathfrak{X}_n$ and we let $\mathfrak{X}_n^{(1)} := \{x_n\}$. Otherwise we define $\mathfrak{X}_n^{(1)} := \{x_n, y_1\}$. Assume that $\mathfrak{X}_n^{(l)}$ has been constructed for some $l \in \llbracket 1, |V| - 2 \rrbracket$. We look iteratively for the first $m \in \llbracket 1, n \rrbracket$ such that $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_{l+1}) \in \mathfrak{X}_n^{(l)}$. If there is no such m , we infer that $y_{l+1} \notin \mathfrak{X}_n$ and let $\mathfrak{X}_n^{(l+1)} = \mathfrak{X}_n^{(l)}$. Otherwise we define $\mathfrak{X}_n^{(l+1)} = \mathfrak{X}_n^{(l)} \sqcup \{y_{l+1}\}$. At the end, we consider $\mathfrak{X}_n := \mathfrak{X}_n^{(|V|-1)}$. This procedure can also be used to test if $\tau \leq n$: this is equivalent to the fact that for all the above steps for $l \in \llbracket 0, |V| - 2 \rrbracket$, there is a $m \in \llbracket 1, n \rrbracket$ such that $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_{l+1}) \in \mathfrak{X}_n^{(l)}$.

This test can be strongly simplified when V is endowed with a partial order admitting a unique minimal element y_\wedge and a unique maximal element y_\vee and when the independent random mappings $\psi_1, \psi_2, \dots, \psi_n$ preserve the partial order (of course this is only possible if P^* is equally preserving the partial order). Then the random mappings $\varphi_1, \varphi_2, \dots, \varphi_n$ equally preserve the partial order and the validity of $\tau \leq n$ is equivalent to the existence of $m \in \llbracket 1, n \rrbracket$ such that

$$\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_\wedge) = \varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(y_\vee)$$

These observations are strongly reminiscent of the coupling-from-the-past algorithm of Propp and Wilson [22] (see also their review in Chapter 22 of Levin, Peres and Wilmer [15]). Recall they consider a family $(\psi_n)_{n \in \mathbb{N}}$ of independent identically distributed random mappings associated to P . For any $N \in \mathbb{N}$, they test if $\psi_{-1} \circ \dots \circ \psi_{-N+1} \circ \psi_{-N}$ send the whole state space V into a single point. When this is true, the single point is distributed according to π . Otherwise they consider another integer number $N' > N$ (usually $N' = 2N$) and start again the above procedure. Their algorithm is equally greatly simplified under the assumptions that V is endowed with a partial order admitting a unique minimal element y_\wedge and a unique maximal element y_\vee and that the independent random mappings $\psi_{-1}, \psi_{-2}, \dots, \psi_{-N}$ preserve the partial order (this is only possible if P is equally preserving the partial order).

Of course there are big differences between the two procedures: our initial point is fixed, the final point of coupling-from-the-past is distributed accordingly to the invariant measure, the preservation of the partial order by P^* and P are not equivalent, we fix a trajectory and allow (except in this subsection) dependence between the random mappings through the already constructed set-valued dual etc. Nevertheless, it would be interesting to investigate further the links between the two algorithms, e.g.:

- Could the convergence of the coupling-from-the-past algorithm be improved by allowing, for $N \in \mathbb{N}$ fixed, choices of ψ_{-n} depending on $\psi_{-n-1} \circ \dots \circ \psi_{-N+1} \circ \psi_{-N}(V)$, for $n \in \llbracket 1, N \rrbracket$?
- What happens to the examples treated by the coupling-from-the-past algorithm if we look at them from the intertwining/strong stationary times point of view?

3 Classical examples

Up to now, the considerations were both abstract and simple. It is time give some examples showing how Theorem 3 works in practice. As it can be guessed, all the difficulty is to find nice random mappings. The families of random mappings considered here will be completely associated to P^* , as mentioned in Remark 1.

3.1 The discrete Pitman theorem

For this example due to Pitman [20], the state space is \mathbb{Z} , but the extension of our construction to this denumerable state space situation is straightforward. The kernel P is the transition “matrix” of the simple random walk on \mathbb{Z} , namely,

$$\forall x, y \in \mathbb{Z}, \quad P(x, y) := \begin{cases} 1/2 & , \text{ if } |y - x| = 1 \\ 0 & , \text{ otherwise} \end{cases}$$

The invariant measure π is the counting measure, up to a factor. The kernel Λ given in (5) is still well-defined, if we take \mathfrak{V} the set of finite non-empty subsets of \mathbb{Z} . The measure π is even reversible for P , in the sense that $P^* = P$, where P^* is defined as in (1).

Let $X := (X_n)_{n \in \mathbb{Z}_+}$ be a random walk with transition kernel P and starting from 0. Introduce the process $X^\vee := (X_n^\vee)_{n \in \mathbb{Z}_+}$ defined by

$$\forall n \in \mathbb{Z}_+, \quad X_n^\vee := 2M_n - X_n$$

where $M_n := \max\{X_m : m \in \llbracket 0, n \rrbracket\}$. Finally consider $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ given by

$$\forall n \in \mathbb{Z}_+, \quad \mathfrak{X}_n := \{X_n^\vee - 2m : m \in \llbracket 0, X_n^\vee \rrbracket\} \quad (27)$$

Pitman [20] has shown that (8) and (9) hold with the above processes X and \mathfrak{X} , it is in fact the first historical instance of such a coupling.

Let us prove that this result is a consequence of Theorem 3. Since (8) is obvious, we concentrate our attention on (9).

Consider the function ψ given by

$$\forall S \in \mathfrak{V}, \forall x \in \mathbb{Z}, \forall b \in \{-1, 1\}, \quad \psi(S, x, b) := \begin{cases} x + b & , \text{ if } x > \max(S) \\ x - b & , \text{ if } x \leq \max(S) \end{cases} \quad (28)$$

Consider a Rademacher variable B , i.e. such that $\mathbb{P}[B = -1] = \mathbb{P}[B = 1] = 1/2$ and for fixed $S \in \mathfrak{V}$, let ψ_S be the random mapping given by

$$\forall x \in \mathbb{Z}, \quad \psi_S(x) := \psi(S, x, B) \quad (29)$$

It is clear that ψ_S is a random mapping associated to $P^* = P$. So let be given a trajectory $x_{\llbracket 0, n \rrbracket}$ of X stopped at time $n \in \mathbb{Z}_+$ and starting with $x_0 = 0$. Construct the finite sequence $\mathfrak{X}_{\llbracket 0, n \rrbracket}$ as in the introduction, starting with $\mathfrak{X}_0 := \{0\}$. Denote by $(\varphi_m)_{m \in \llbracket 1, n \rrbracket}$ the corresponding random mappings used in this construction, conditioned by the compatibility relations $\varphi_m(x_m) = x_{m-1}$ for $m \in \llbracket 1, n \rrbracket$. Given the stopped trajectory $x_{\llbracket 0, n \rrbracket}$, these random mappings are here deterministic:

$$\forall m \in \llbracket 1, n \rrbracket, \quad \varphi_m = \psi(\mathfrak{X}_{m-1}, \cdot, b_m)$$

with

$$b_m := \begin{cases} -1 & , \text{ if } x_{m-1} = \max(\mathfrak{X}_{m-1}) \text{ and } x_m - x_{m-1} = 1 \\ x_m - x_{m-1} & , \text{ otherwise} \end{cases}$$

Since under these mappings the parity of the positions are alternating, we remark that

$$\forall m \in \llbracket 0, n \rrbracket, \quad \mathfrak{X}_m \subset \begin{cases} 2\mathbb{Z} & , \text{ if } m \text{ is even} \\ 2\mathbb{Z} + 1 & , \text{ if } m \text{ is odd} \end{cases}$$

Consider for any $m \in \llbracket 0, n \rrbracket$, $Y_m := \min(\mathfrak{X}_m)$ and $Z_m := \max(\mathfrak{X}_m)$. By a forward iteration on $m \in \llbracket 0, n \rrbracket$, we show that $Z_m = X_n^\vee$, $Y_m = -Z_m$ and that \mathfrak{X}_m contains all the elements in $\llbracket Y_m, Z_m \rrbracket$ with the same parity as Z_m . It proves the validity of (27). This is well-illustrated by Figure 1, where for $m \in \llbracket 0, n - 1 \rrbracket$, the elements of \mathfrak{X}_m are represented by full disks, the elements of \mathfrak{X}_{m+1} by circles, the transition from X_m to X_{m+1} by a double line, the dashed lines are the actions of the random mappings (from the right to the left), the green (respectively red) line is the transition from Z_m to Z_{m+1} (resp. from Y_m to Y_{m+1}).

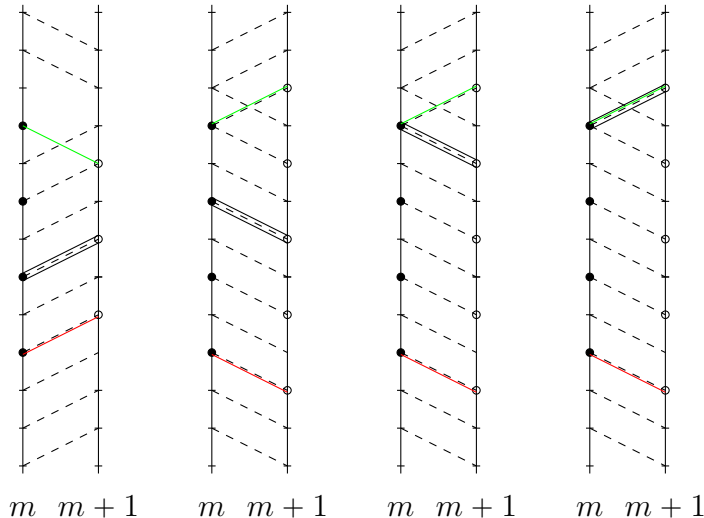


Figure 1: Schematic proof of the discrete Pitman theorem via random mappings

The symmetry with respect 0 leads to another Pitman transformation: rather introduce the process $X^\wedge := (X_n^\wedge)_{n \in \mathbb{Z}_+}$ defined by

$$\forall n \in \mathbb{Z}_+, \quad X_n^\wedge := X_n - 2M_n^\wedge$$

where $M_n^\wedge := \min\{X_m : m \in \llbracket 0, n \rrbracket\}$ and consider $\tilde{\mathfrak{X}} := (\tilde{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$ given by

$$\forall n \in \mathbb{Z}_+, \quad \tilde{\mathfrak{X}}_n := \{X_n^\wedge - 2m : m \in \llbracket 0, X_n^\wedge \rrbracket\} \quad (30)$$

By symmetry in law of X , it is clear that (8) and (9) equally hold for processes X and $\tilde{\mathfrak{X}}$. This can also be obtained by replacing the mapping ψ of (28) by

$$\forall S \in \mathfrak{X}, \forall x \in \mathbb{Z}, \forall b \in \{-1, 1\}, \quad \tilde{\psi}(S, x, b) := \begin{cases} x + b & , \text{ if } x \geq \min(S) \\ x - b & , \text{ if } x < \min(S) \end{cases} \quad (31)$$

More generally, at each time $n \in \mathbb{Z}_+$, either ψ or $\tilde{\psi}$ can be chosen to construct random mappings and this choice itself may depend on the current state \mathfrak{X}_n and on independent noise. Of course the description of the deduced set-valued dual \mathfrak{X} will then be more tricky than (27) or (30).

The dual processes deduced from (28) and (31) (as well as their above variants) are examples of Doob transforms of the evolving sets of Morris and Peres [17]. In the spirit and the notations of Subsection 2.3, the above random mappings can be described via labelings of \mathbb{Z} depending on S . For instance for (29), $\max(S) + 1$ is named 1, $\max(S)$ is named 2, $\max(S) - 1$ is named 3, ...

until $\min(S) - 1$ is named $\max(S) - \min(S) + 3$, then $\max(S) + 2$ is named $\max(S) - \min(S) + 4$, $\min(S) - 2$ is named $\max(S) - \min(S) + 5$, $\max(S) + 3$ is named $\max(S) - \min(S) + 6$, etc. ((31) corresponds to a “mirror” labeling). One can imagine other labelings (where the first labels are given to the elements of $\llbracket \min(S) - 1, \max(S) + 1 \rrbracket$, or only to those of S as in Subsection 2.3, according to the any fancy rules), it will not change the law of \mathfrak{X} (starting from $\{0\}$), only the law of the coupling (X, \mathfrak{X}) will be modified, as in the above cases deduced from (28) and (31).

It is also possible to consider random mappings not depending on a finite subset S of \mathbb{Z} , as mentioned in Subsection 2.4, but they are expected to be worse than the previous Doob transform of the evolving sets of Morris and Peres [17], in the sense of the stochastic domination for the stopping times defined, for any $A \in \mathbb{N}$, by

$$\tau_A := \inf\{n \in \mathbb{Z}_+ : \pi(\mathfrak{X}_n) \geq A\}$$

(see [16], for a partial result in this direction for one-dimensional diffusions). Note that the worst case is to consider with probability 1/2 one of the two mappings

$$\begin{aligned} \forall x \in \mathbb{Z}, \quad \psi_-(x) &:= x - 1 \\ \forall x \in \mathbb{Z}, \quad \psi_+(x) &:= x + 1 \end{aligned}$$

since starting with $\mathfrak{X}_0 = \{X_0\}$, we get that for all $n \in \mathbb{Z}_+$, $\mathfrak{X}_n = \{X_n\}$ (more generally, for any initial law of \mathfrak{X}_0 on \mathfrak{V} satisfying (7), we end up with $\pi(\mathfrak{X}_n) = \pi(\mathfrak{X}_0)$ for all $n \in \mathbb{Z}_+$).

3.2 The top-to-random shuffle

The top-to-random shuffle is a simple model of shuffling cards: at each time, take the top card and put it at a uniform random location in the deck. This stochastic evolution is described mathematically by a Markov chain $X := (X_n)_{n \in \mathbb{Z}_+}$ on the symmetric group $V := \mathcal{S}_N$, with $N \in \mathbb{N} \setminus \{1\}$, whose transition matrix P is given by

$$\begin{aligned} \forall \sigma, \sigma' \in \mathcal{S}_N, \\ P(\sigma, \sigma') &:= \begin{cases} 1/N & , \text{ if there exists } l \in \llbracket 1, N \rrbracket \text{ with } \sigma' = (1 \rightarrow l \rightarrow l - 1 \rightarrow \dots \rightarrow 2) \circ \sigma \\ 0 & , \text{ otherwise} \end{cases} \end{aligned}$$

where $(1 \rightarrow l \rightarrow l - 1 \rightarrow \dots \rightarrow 2)$ is the cyclic permutation, seen as the function from $\llbracket 1, N \rrbracket$ to $\llbracket 1, N \rrbracket$, transferring the card at position 1 to position l , the card at position l to position $l - 1$, ... and the card at position 2 to position 1.

The transition matrix P is irreducible and the corresponding invariant measure π is the uniform probability distribution on \mathcal{S}_N . The Markov chain X admits a famous dual process defined by Aldous and Diaconis [1] in terms of the position of the last card of X_0 in the deck X_n at time $n \in \mathbb{Z}_+$. More precisely, represent a permutation σ by the sequence of its values $(\sigma(1), \sigma(2), \dots, \sigma(N))$. Start X from the identity: $X_0 = (1, 2, \dots, N) =: \text{id}$ and at any time $n \in \mathbb{Z}_+$, let $Y_n \in \llbracket 1, N \rrbracket$ be the position of the last card defined by $X_n(Y_n) = N$. It is not difficult to check that $Y := (Y_n)_{n \in \mathbb{Z}_+}$ is a Markov chain. Define $\tilde{\tau} := \inf\{n \in \mathbb{Z}_+ : Y_n = 1\}$, the first time the last card arrives at the top of the deck. It is well-known that $\tilde{\tau} + 1$ is a strong stationary time, it is even the first historical instance of a strong stationary time in a finite context. We modify Y by imposing that $Y_n = 0$ for any $n > \tau$.

For any $\sigma \in \mathcal{S}_N$ and $y \in \llbracket 0, N \rrbracket$, define

$$A_{\sigma, y} := \{\sigma' \in \mathcal{S}_N : \sigma'(1) = \sigma(1), \dots, \sigma'(y) = \sigma(y)\}$$

with the usual convention that $A_{\sigma, 0} = \mathcal{S}_N$. Aldous and Diaconis [1] considered the set-valued dual $\tilde{\mathfrak{X}} := (\tilde{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$ deduced from (X, Y) by defining

$$\forall n \in \mathbb{Z}_+, \quad \tilde{\mathfrak{X}}_n := A_{X_n, Y_n} \tag{32}$$

Let us construct a better set-valued dual $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ via random mappings. Note that P^* is the transition matrix of the random-to-top shuffle and corresponds to taking a card of the deck at a uniform random location and putting it at the top. Consider for any $x \in \llbracket 1, N \rrbracket$, the mapping $\psi^{(x)} : \mathcal{S}_N \rightarrow \mathcal{S}_N$ which acts on any permutation σ by removing the card x from the deck and putting it at the top. Formally, we have

$$\forall \sigma \in \mathcal{S}_N, \quad \psi^{(x)}(\sigma) = (1 \rightarrow 2 \rightarrow \dots \rightarrow \sigma^{-1}(x)) \circ \sigma \quad (33)$$

(note that $\sigma^{-1}(x)$ is the position of the card x). Let $(U_n)_{n \in \mathbb{N}}$ be a family of independent random variables uniformly distributed on $\llbracket 1, N \rrbracket$ and for any $n \in \mathbb{N}$, denote by ψ_n the random mapping $\psi^{(U_n)}$, which is clearly associated to P^* . There is no dependence on a subset $S \in \mathfrak{V}$ and we are in the context of independent random mappings of Subsection 2.4. Let be given a trajectory $x_{\llbracket 0, n \rrbracket}$, for some fixed $n \in \mathbb{Z}_+$, starting from the identity, $x_0 = \text{id}$. For any $m \in \llbracket 1, n \rrbracket$, let φ_m be the conditioning of ψ_m by $\psi_m(x_m) = x_{m-1}$. Remark that as in the previous subsection, φ_m is deterministic, as we have $\varphi_n = \psi^{(x_{n-1}(1))}$. Starting from $\mathfrak{X}_0 = \{\text{id}\}$, we get from (25) that

$$\mathfrak{X}_n = \{\sigma \in \mathcal{S}_N : \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(\sigma) = \text{id}\}$$

Let us check that \mathfrak{X} is better than the set-valued dual $\tilde{\mathfrak{X}}$ of Aldous and Diaconis [1], in the sense that

$$\forall n \in \mathbb{Z}_+, \quad \tilde{\mathfrak{X}}_n \subset \mathfrak{X}_n \quad (34)$$

It is furthermore strictly better, because $\tilde{\mathfrak{X}}_{\tilde{\tau}} = A_{X_{\tilde{\tau}, 1}}$ is strictly included into $\mathfrak{X}_{\tilde{\tau}} = \mathcal{S}_N$, as we will see below. It implies that $\tau \leq \tilde{\tau} < \tilde{\tau} + 1$, where τ is the strong stationary time associated to \mathfrak{X} as in (26) (recall that $\tilde{\tau} + 1$ is the strong stationary time associated to $\tilde{\mathfrak{X}}$).

Indeed, to show (34), consider $\sigma \in \tilde{\mathfrak{X}}_n$. By definition, we have

$$\sigma = (\sigma(1), \sigma(2), \dots, \sigma(Y_n - 1), N, \sigma(Y_n + 1), \dots, \sigma(N))$$

Observe that for $l \in \llbracket 1, Y_n - 1 \rrbracket$, $\sigma(l) = X_n(l)$ and for $l \in \llbracket Y_n + 1, N \rrbracket$, the values $\sigma(l)$ belongs to $\{x_m(1) : m \in \llbracket 0, n \rrbracket\}$, since they have had to be at the top of the deck before time n to be sent below the last card N . By iteration on $m \in \llbracket 1, n \rrbracket$, it follows that the Y_{n-m+1} first coordinates of $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(\sigma)$ and $\varphi_{n-m+1} \circ \varphi_{n-m+2} \circ \dots \circ \varphi_n(X_n)$ coincide. In particular for $m = 1$, since $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(X_n) = \text{id}$, we get that $\sigma \in \mathfrak{X}_n$, so that (34) is proven.

To prove that \mathfrak{X} is strictly better than $\tilde{\mathfrak{X}}$, note that $\tilde{\mathfrak{X}}_{\tilde{\tau}}$ is the set of permutations $\sigma \in \mathcal{S}_N$ such that $\sigma(1) = N$, in particular $\tilde{\mathfrak{X}}_{\tilde{\tau}} \neq \mathcal{S}_N$. Applying a reasoning similar to the proof of (34), we get that for any $\sigma \in \mathcal{S}_N$, $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_{\tilde{\tau}}(\sigma) = \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_{\tilde{\tau}}(X_{\tilde{\tau}})$, except maybe for the last coordinate $\sigma(N)$. This is a consequence of the fact that all the values of $\llbracket 1, N - 1 \rrbracket$ will have been placed at a same time $m \in \llbracket 1, \tilde{\tau} \rrbracket$ at the top of the deck by $\varphi_m \circ \varphi_{m+1} \circ \dots \circ \varphi_{\tilde{\tau}}$. But if all the coordinates except the last one coincide for two permutations, it means that the permutations are the same. It follows that $\mathfrak{X}_{\tilde{\tau}} = \mathcal{S}_N$ as announced.

Corollary 6 *The first time $\hat{\tau}$ that the card $N - 1$ comes to the top and is inserted is a strong stationary time. It is a strict improvement over the strong stationary time constructed by Aldous and Diaconis [1], which is the first time $\tilde{\tau} + 1$ that the card N comes to the top and is inserted. But this improvement is negligible: we have $\mathbb{E}[\hat{\tau}] = \mathbb{E}[\tau] + N$, while as N goes to infinity, $\mathbb{E}[\tilde{\tau}] \sim N \ln(N)$.*

Proof

Let us show that $\hat{\tau}$ coincides with the strong stationary time τ defined in (26). Indeed, as a consequence of the above proof that $\mathfrak{X}_{\tilde{\tau}} = \mathcal{S}_N$, we see that τ is smaller than 1 plus the first time when all the cards except N have been at the top, namely $\tau \leq \hat{\tau}$. Conversely, let σ be a permutation where the card N is above the card $N - 1$. For $n < \hat{\tau}$, the same is true for the

permutation $\varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_n(\sigma)$, since neither N nor $N - 1$ have been put at the top. So we get that $\sigma \notin \mathfrak{X}_n$, i.e. $\tau > n$ and it follows that $\hat{\tau} = \tau$. Note that $\hat{\tau}$ is a sum of independent geometric random variables of parameters $2/N, 3/N, \dots, 1$, which correspond respectively to the first time a card goes under $N - 1$, the inter-time until a second card goes behind $N - 1$, etc. Similarly, $\tilde{\tau}$ is a sum of independent geometric random variables of parameters $1/N, 2/N, \dots, 1$. Thus $\mathbb{E}[\tilde{\tau}] - \mathbb{E}[\tau]$ is equal to the expectation of a geometric random variable of parameter $1/N$, namely N . The last assertion of the corollary is a consequence of the equality

$$\mathbb{E}[\tilde{\tau}] = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N}$$

■

One can wonder if the set-valued dual given in (32) has a random mapping representation. It is indeed the case, the subsequent simple construction resorts to random mappings depending on a fix set $S \in \mathfrak{V}$ (we don't know if it is possible to devise a construction via independent random mappings as in Subsection 2.4). The underlying random mappings $\tilde{\psi}_S$ are described as follows.

- Assume S is of the form $A_{\sigma,k}$ for some $\sigma \in \mathcal{S}_N$ and $k \in \llbracket 1, N \rrbracket$ with $\sigma(k) = N$. Let (U, \tilde{U}) be a random variable taking values in $\{0, 1\} \times \llbracket 1, N \rrbracket$ such that $\mathbb{P}[U = 0] = (k - 1)/N$ and knowing that $U = 0$ (respectively $U = 1$), \tilde{U} is uniformly distributed on $\llbracket 1, k - 1 \rrbracket$ (resp. on $\llbracket 1, N - k + 1 \rrbracket$). Let be given a permutation $\sigma' \in \mathcal{S}_N$. When $U = 0$, we take $\tilde{\psi}_S(\sigma') = \hat{\psi}^{(\tilde{U})}(\sigma')$, where for any $x \in \llbracket 1, N \rrbracket$,

$$\forall \sigma' \in \mathcal{S}_N, \quad \hat{\psi}^{(x)}(\sigma') = (1 \rightarrow 2 \rightarrow \dots \rightarrow x) \circ \sigma'$$

In words, a position is chosen among the first $k - 1$ ones and the corresponding card is sent to the top. On the contrary, when $U = 1$, we choose a label of card among those whose position are in $\llbracket k, N \rrbracket$ according to the following procedure. Let $\hat{\sigma}(1) < \hat{\sigma}(2) < \dots < \hat{\sigma}(N - k + 1)$ be the ordering of the set $\{\sigma'(l) : l \in \llbracket k, N \rrbracket\}$. We then take $\tilde{\psi}_S(\sigma') := \psi^{(\hat{\sigma}(U))}(\sigma')$, with the notation introduced in (33). The mappings $\psi^{(x)}$ and $\hat{\psi}^{(x)}$ may look similar at first view, but it is the difference between the choice of a position and a label of card that will result in the distinction between the dual processes \mathfrak{X} and $\tilde{\mathfrak{X}}$. Note that $\tilde{\psi}_S$ depends on subsets S as above only through k .

- Assume S is not of the form $A_{\sigma,k}$ for some $\sigma \in \mathcal{S}_N$ and $k \in \llbracket 1, N \rrbracket$ with $\sigma(k) = N$. This situation is not important, since the algorithm will only construct subsets of the previous form (starting with $\mathfrak{X}_0 = \{\text{id}\}$), and it would be possible to restrict \mathfrak{V} to contain only such sets. Nevertheless, for definiteness, take for instance $\tilde{\psi}_S := \psi^{(U)}$ where U is uniformly distributed on $\llbracket 1, N \rrbracket$.

It is immediate to check that $\tilde{\psi}_S$ is associated to P^* and its interest is encapsulated in the following result.

Lemma 7 *Let S be of the form $A_{\sigma,k}$ for some $\sigma \in \mathcal{S}_N$ and $k \in \llbracket 1, N \rrbracket$ with $\sigma(k) = N$. Fix $x \in \llbracket 1, N \rrbracket$ and denote by $\tilde{\sigma}$ the deck of cards obtained from σ by putting the top card $\sigma(1)$ at position x . Condition $\tilde{\psi}_S$ by the fact that $\tilde{\psi}_S(\tilde{\sigma}) = \sigma$, to obtain a random mapping $\tilde{\varphi}_S$. The mapping $\tilde{\varphi}_S$ is in fact deterministic and defining*

$$B := \{\sigma' \in \mathcal{S}_N : \tilde{\varphi}_S(\sigma') \in S\}$$

we have

$$B = A_{\tilde{\sigma}, \tilde{k}}$$

with $\tilde{k} := (\tilde{\sigma})^{-1}(N)$.

Proof

Let (U, \tilde{U}) be the random variable appearing in the definition of $\tilde{\psi}_S$. The value of U can be deduced by comparing x and k : if $x \in \llbracket 1, k-1 \rrbracket$ then $U = 0$ and otherwise $U = 1$. When $U = 0$, \tilde{U} is determined by the relation $\tilde{\sigma}(\tilde{U}) = \sigma(1)$ and when $U = 1$, \tilde{U} is determined as the rank of $\sigma(1)$ in $\{\tilde{\sigma}(k), \tilde{\sigma}(k+1), \dots, \tilde{\sigma}(N)\}$ (which is also the rank of $\sigma(1)$ in $\{\sigma(1), \sigma(k+1), \sigma(k+2), \dots, \sigma(N)\}$). It follows that $\tilde{\varphi}_S$ is determined, since its randomness only comes from (U, \tilde{U}) .

For the assertion concerning B , let be given $\sigma'' \in S$ and $\sigma' \in \mathcal{S}_N$ such that $\tilde{\varphi}_S(\sigma') = \sigma''$ and consider separately two alternatives.

- The case $U = 0$, which is equivalent to the identity $\tilde{k} = k$. Necessarily we have $\sigma'(l) = \sigma''(l)$ for $l \in \llbracket \tilde{U} + 1, N \rrbracket$ and

$$(\sigma'(\tilde{U}), \sigma'(1), \dots, \sigma'(\tilde{U} - 1)) = (\sigma''(1), \sigma''(2), \dots, \sigma''(\tilde{U}))$$

Furthermore, note that

$$\begin{aligned} (\sigma''(1), \sigma''(2), \dots, \sigma''(\tilde{U})) &= (\sigma(1), \sigma(2), \dots, \sigma(\tilde{U})) \\ &= (\tilde{\sigma}(\tilde{U}), \tilde{\sigma}(1), \dots, \tilde{\sigma}(\tilde{U} - 1)) \end{aligned}$$

It follows that the set of σ' obtained when σ'' runs through S is just $A_{\tilde{\sigma}, \tilde{k}}$, as announced.

- The case $U = 1$, which is equivalent to the identity $\tilde{k} = k - 1$. We get that

$$\begin{aligned} (\sigma'(1), \sigma'(2), \dots, \sigma'(k-1)) &= (\sigma''(2), \sigma''(3), \dots, \sigma''(k)) \\ (\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))_{\sigma''(1)} &= (\sigma''(k+1), \sigma''(k+2), \dots, \sigma''(N)) \end{aligned}$$

where $(\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))_{\sigma''(1)}$ stands for the finite sequence $(\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))$ where $\sigma''(1)$ has been deleted. It appears that contrary to the case $U = 0$, the permutation σ' is not determined by σ'' , as we have $N - k + 1$ possibilities for the insertion of $\sigma''(1)$ in $(\sigma'(k), \sigma'(k+1), \dots, \sigma'(N))_{\sigma''(1)}$. Nevertheless, the set of σ' obtained when σ'' run through S is again $A_{\tilde{\sigma}, \tilde{k}}$. ■

Construct, as described in the introduction, a set-valued dual $\tilde{\mathfrak{X}} := (\tilde{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$ starting from $\tilde{\mathfrak{X}}_0 := \{\text{id}\}$ and using the random mappings modeled after the family $(\tilde{\psi}_S)_{S \in \mathfrak{Y}}$. By applying iteratively Lemma 7, we get that $\tilde{\mathfrak{X}}$ is given by (32), showing in particular that it is taking values in subsets of the form $A_{\sigma, k}$ for some $\sigma \in \mathcal{S}_N$ and $k \in \llbracket 1, N \rrbracket$ with $\sigma(k) = N$, as forecasted.

4 Birth and death chains

The construction of the random mappings used to recover the discrete Pitman theorem is extended here to “restless” birth and death chains. It is a discrete analogue of the results we are looking for in the context of one-dimensional diffusions (see Subsection 7.2). As in the previous section, only completely associated to P^* random mappings will be considered here and *associated* will mean *completely associated*.

Let be given $(p(x))_{x \in \mathbb{Z}}$ a family of elements of $(0, 1)$. We are interested in the irreducible transition kernel P given by

$$\forall x, y \in \mathbb{Z}, \quad P(x, y) := \begin{cases} p(x) & , \text{ if } y = x + 1 \\ 1 - p(x) & , \text{ if } y = x - 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (35)$$

An associated Markov chain $X := (X_n)_{n \in \mathbb{Z}_+}$ is said to be a **restless** birth and death chain, since at each time $n \in \mathbb{Z}_+$ it chooses to go up or down of one unity and cannot stay at the same position. Up to a factor, a corresponding invariant measure π is given by

$$\forall x \in \mathbb{Z}, \quad \pi(x) := \begin{cases} \frac{P(0,1)P(1,2)\dots P(x-1,x)}{P(x,x-1)P(x-1,x-2)\dots P(1,0)} & , \text{ if } x \geq 1 \\ 1 & , \text{ if } x = 0 \\ \frac{P(0,-1)P(-1,-2)\dots P(x+1,x)}{P(x,x+1)P(x+1,x+2)\dots P(-1,0)} & , \text{ if } x \leq -1 \end{cases}$$

Depending on the family of coefficients $(p(x))_{x \in \mathbb{Z}}$, the measure π may be finite or not. Whatever the case, as in Subsection 3.1, the measure π is reversible for P , in the sense that $P^* = P$, where P^* is defined as in (1). The kernel Λ given in (5) is also well-defined, as soon as we restrict \mathfrak{A} to be the set of finite non-empty subsets of \mathbb{Z} .

A random mapping associated to $P^* = P$ can be constructed by mimicking the definition given in Subsection 3.1. Define for $x \in \mathbb{Z}$ and $u \in [0, 1)$,

$$\begin{aligned}\psi_+(x, u) &= \begin{cases} x - 1 & , \text{ if } u \in [0, 1 - p(x)) \\ x + 1 & , \text{ if } u \in [1 - p(x), 1) \end{cases} \\ \psi_-(x, u) &= \psi_+(x, 1 - u)\end{aligned}$$

and introduce the mapping

$$\forall S \in \mathfrak{A}, \forall x \in \mathbb{Z}, \forall u \in [0, 1), \quad \psi(S, x, u) := \begin{cases} \psi_+(x, u) & , \text{ if } x > \max(S) \\ \psi_-(x, u) & , \text{ if } x \leq \max(S) \end{cases} \quad (36)$$

A random mapping ψ_S is obtained by considering $\psi(S, \cdot, U)$, where U is uniformly distributed on $[0, 1)$. It leads to the construction of a (conditioned) set-valued dual \mathfrak{X} (starting from a singleton) as prescribed after the proof of Corollary 4. More precisely, fix a finite trajectory $x_{[0, n]}$ of X with $n \in \mathbb{Z}_+$ and let U_1, U_2, \dots, U_n be independent random variables uniformly distributed on $[0, 1)$, so that \mathfrak{X}_m is constructed recursively on $m \in \llbracket 1, n \rrbracket$ as follows. Condition the random mappings $\psi(\mathfrak{X}_0, \cdot, U_1), \psi(\mathfrak{X}_1, \cdot, U_2), \dots, \psi(\mathfrak{X}_{m-1}, \cdot, U_m)$ by $\psi(\mathfrak{X}_0, x_1, U_1) = x_0, \psi(\mathfrak{X}_1, x_2, U_2) = x_1, \dots, \psi(\mathfrak{X}_{m-1}, x_m, U_m) = x_{m-1}$ and call $\varphi_1, \varphi_2, \dots, \varphi_m$ the induced random mappings. We define $\mathfrak{X}_0 := \{x_0\}$ and iteratively

$$\forall m \in \llbracket 1, n \rrbracket, \quad \mathfrak{X}_m := \{y \in \mathbb{Z} : \varphi_m(y) \in \mathfrak{X}_{m-1}\} \quad (37)$$

Our goal in this section is to simplify the description of $\mathfrak{X}_{[0, n]}$ via a backward construction that will be useful for one-dimensional diffusion processes.

For any $m \in \llbracket 0, n \rrbracket$, consider

$$\begin{aligned}R_m &:= \max(\mathfrak{X}_m) \\ R_m^\wedge &:= \min(\mathfrak{X}_m)\end{aligned}$$

These numbers determine \mathfrak{X}_m :

Lemma 8 *For any $m \in \llbracket 0, n \rrbracket$, R_m and R_m^\wedge have the same parity and \mathfrak{X}_m is the set of integers between R_m and R_m^\wedge with the same parity as R_m .*

Proof

All the mappings $\varphi_1, \varphi_2, \dots, \varphi_m$ change the parity of their argument. Since $\mathfrak{X}_0 = \{x_0\}$, we deduce that for all $m \in \llbracket 0, n \rrbracket$, the parity of all the elements of \mathfrak{X}_m is the same as that of x_0 when m is even and is the other one when m is odd.

The second assertion of the lemma is proven by iteration on $m \in \llbracket 1, n \rrbracket$, based on the fact that if x and $x + 2$ belong to \mathfrak{X}_{m-1} , then necessary $x + 1 \in \mathfrak{X}_m$, by restlessness. ■

Before proceeding toward a simplified presentation, let us be more specific about our conditioning operations. For any $m \in \llbracket 1, n \rrbracket$ and any $r \in \mathbb{Z}$, consider

$$a(x_{m-1}, x_m, r) := \begin{cases} p(x_m) & , \text{ if } x_{m-1} < x_m \leq r \\ 0 & , \text{ otherwise} \end{cases} \quad (38)$$

Define another number $b(x_{m-1}, x_m)$ via

$$b(x_{m-1}, x_m) := \begin{cases} 1 - p(x_m) & , \text{ if } x_{m-1} < x_m \\ p(x_m) & , \text{ if } x_{m-1} > x_m \end{cases} \quad (39)$$

The interest of these numbers is:

Lemma 9 *The conditioning $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = x_{m-1}$ a.s. amounts to the conditioning $U_m \in [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$.*

Proof

We consider several cases:

- When $x_m > x_{m-1}$ and $x_m > R_{m-1}$: then $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = \psi_+(x_m, U_m)$, so for this term to be equal to $x_{m-1} = x_m - 1$, we must have $U_m \in [0, 1 - p(x_m)] = [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$, with $a(x_{m-1}, x_m, R_{m-1}) = 0$ and $b(x_{m-1}, x_m) = 1 - p(x_m)$.
- When $x_m > x_{m-1}$ and $x_m \leq R_{m-1}$: then $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = \psi_-(x_m, U_m) = \psi_+(x_m, 1 - U_m)$ so for this term to be equal to $x_{m-1} = x_m - 1$, we must have $1 - U_m \in [0, 1 - p(x_m)]$, namely $U_m \in (p(x_m), 1]$, which a.s. corresponds to $U_m \in [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$, with $a(x_{m-1}, x_m, R_{m-1}) = p(x_m)$ and $b(x_{m-1}, x_m) = 1 - p(x_m)$.
- When $x_m < x_{m-1}$: since $x_m < x_{m-1} \leq R_{m-1}$, we have $\psi(\mathfrak{X}_{m-1}, x_m, U_m) = \psi_-(x_m, U_m) = \psi_+(x_m, 1 - U_m)$, so for this term to be equal to $x_{m-1} = x_m + 1$, we must have $1 - U_m \in [1 - p(x_m), 1)$, namely $U_m \in (0, p(x_m)]$, which a.s. corresponds to $U_m \in [a(x_{m-1}, x_m, R_{m-1}), a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)]$, with $a(x_{m-1}, x_m, R_{m-1}) = 0$ and $b(x_{m-1}, x_m) = p(x_m)$.

■

It follows from Lemma 9 there exist $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n$ independent random variables uniformly distributed on $[0, 1)$ so that

$$\forall m \in \llbracket 1, n \rrbracket, \quad \varphi_m(\cdot) = \psi(\mathfrak{X}_{m-1}, \cdot, a(x_{m-1}, x_m, R_{m-1}) + b(x_{m-1}, x_m)\tilde{U}_m) \quad (40)$$

where we recall that ψ is defined in (36).

Let us now show how $(R_m)_{m \in \llbracket 0, n \rrbracket}$ can be constructed backwardly in the spirit of Subsection 2.4. Similarly to $\varphi_1, \varphi_2, \dots, \varphi_m$, we would like to consider the random mappings $\phi_1, \phi_2, \dots, \phi_m$ obtained by conditioning the random mappings $\psi(\{x_0, x_1\}, \cdot, U_1), \psi(\{x_1, x_2\}, \cdot, U_2), \dots, \psi(\{x_{m-1}, x_m\}, \cdot, U_m)$ by $\psi(\{x_0, x_1\}, x_1, U_1) = x_0, \psi(\{x_1, x_2\}, x_2, U_2) = x_1, \dots, \psi(\{x_{m-1}, x_m\}, x_m, U_m) = x_{m-1}$. Since we want the construction of $\phi_1, \phi_2, \dots, \phi_m$ to be coupled with $\varphi_1, \varphi_2, \dots, \varphi_m$, we define

$$\begin{aligned} \forall m \in \llbracket 1, n \rrbracket, \quad \phi_m(\cdot) &:= \psi(\{x_{m-1}, x_m\}, \cdot, a(x_{m-1}, x_m, x_{m-1} \vee x_m) + b(x_{m-1}, x_m)\tilde{U}_m) \\ &= \psi(\{x_{m-1}, x_m\}, \cdot, \tilde{a}(x_{m-1}, x_m) + b(x_{m-1}, x_m)\tilde{U}_m) \end{aligned} \quad (41)$$

where

$$\tilde{a}(x_{m-1}, x_m) := \begin{cases} p(x_m) & , \text{ if } x_{m-1} < x_m \\ 0 & , \text{ otherwise} \end{cases}$$

Remark that the above definition of the ϕ_m , for $m \in \llbracket 1, n \rrbracket$, is using the finite trajectory $\mathfrak{X}_{[0, m-1]}$, only through x_m and x_{m-1} , and is not recursive (ϕ_n does not need the knowledge of $\phi_{n-1}, \dots, \phi_1$). More precisely, this definition coincides with (40) if we had $\mathfrak{X}_{m-1} = \{x_{m-1}, x_m\}$. Denote

$$\forall m \in \llbracket 1, n \rrbracket, \quad \phi_{m,n} := \phi_{m+1} \circ \phi_{m+2} \circ \dots \circ \phi_n$$

with the usual convention that $\phi_{n,n}$ is the identity mapping.

Proposition 10 *With the above notations, we have*

$$R_n = \max\{x \geq x_n : \exists m \in \llbracket 0, n \rrbracket \text{ with } \phi_{m,n}(x) = x_m\}$$

Proof

Let us prove by iteration on $m \in \llbracket 1, n \rrbracket$ that

$$R_m = \tilde{R}_m := \max\{x \geq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(x) = x_l\}$$

For $m = 1$, by definition

$$\begin{aligned} \tilde{R}_1 &= \max\{x \geq x_1 : \exists l \in \llbracket 0, 1 \rrbracket \text{ with } \phi_{l,1}(x) = x_l\} \\ &= \max\{x \geq x_1 : x = x_1 \text{ or } \phi_1(x) = x_0\} \\ &= \max\{x \in \mathbb{Z} : \phi_1(x) = x_0\} \\ &= \max\{x \in \mathbb{Z} : \varphi_1(x) = x_0\} \\ &= \max(\mathfrak{X}_1) \\ &= R_1 \end{aligned}$$

where in the third equality we used that $\phi_1(x_1) = x_0$ and in the fourth, that $\varphi_1 = \phi_1$, when $x_1 < x_0$, and that $\varphi_1(x_1) = \phi_1(x_1) = x_0$ (as well as $\varphi_1(x_1 + 2) > x_0$ and $\phi_1(x_1 + 2) > x_0$), when $x_1 > x_0$.

Assume that $R_m = \tilde{R}_m$ for some $m \in \llbracket 1, n - 1 \rrbracket$ and let us show that $R_{m+1} = \tilde{R}_{m+1}$.

We have

$$\begin{aligned} \tilde{R}_{m+1} &= \max\{x \geq x_{m+1} : \exists l \in \llbracket 0, m + 1 \rrbracket \text{ with } \phi_{l,m+1}(x) = x_l\} \\ &= \max\{x \geq x_{m+1} : x = x_{m+1} \text{ or } \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(\phi_{m+1}(x)) = x_l\} \\ &= \max(x_{m+1}, \max\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\}) \end{aligned}$$

where

$$A_m := \{y \in \mathbb{Z} : \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(y) = x_l\}$$

In particular $x_m \in A_m$, so that $x_{m+1} \in \{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\}$, since $\phi_{m+1}(x_{m+1}) = x_m$. We deduce that

$$\tilde{R}_{m+1} = \max\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\} \quad (42)$$

Due to restlessness, the parity of the elements of A_m is the parity of x_m and the parity of the elements of $\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\}$ is the same as the parity of x_{m+1} . Furthermore, on the set of odd (respectively even) integers, the mapping ϕ_{m+1} is non-decreasing, thus

$$\{x \geq x_{m+1} : \phi_{m+1}(x) \in A_m\} = \{x \geq x_{m+1} : \phi_{m+1}(x) \in B_m\}$$

where

$$B_m := \{y \geq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \phi_{l,m}(y) = x_l\}$$

It follows by our iteration assumption that $\max(B_m) = \tilde{R}_m = R_m$.

Note also that B_m is exactly equal to the subset of elements from $\llbracket x_m, R_m \rrbracket$ which have the same parity as x_m . This comes from restlessness, which implies that two trajectories $(\phi_{l,m}(y))_{l \in \llbracket 0, m \rrbracket}$ and $(\phi_{l,m}(y'))_{l \in \llbracket 0, m \rrbracket}$, where y and y' are integers with the same parity, either stay one above the other or end up coalescing. So for any $y \in \llbracket x_m, R_m \rrbracket$ which has the same parity as x_m , the trajectory $(\phi_{l,m}(y))_{l \in \llbracket 0, m \rrbracket}$ is sandwiched between $(\phi_{l,m}(x_m))_{l \in \llbracket 0, m \rrbracket} = (x_l)_{l \in \llbracket 0, m \rrbracket}$ and $(\phi_{l,m}(R_m))_{l \in \llbracket 0, m \rrbracket}$, thus ends up coalescing with $(x_l)_{l \in \llbracket 0, m \rrbracket}$.

It follows from this description of B_m and (42) that $\tilde{R}_{m+1} = R_m - 1$, except if $\phi_{m+1}(R_m + 1) = R_m$, in which case $\tilde{R}_{m+1} = R_m + 1$. Let us show that

$$\phi_{m+1}(R_m + 1) = \varphi_{m+1}(R_m + 1) \quad (43)$$

We consider two cases:

- When $R_m + 1 > x_m \vee x_{m+1}$: we have

$$\phi_{m+1}(R_m + 1) = \psi_+(R_m + 1, \tilde{a}(x_m, x_{m+1}) + b(x_m, x_{m+1})\tilde{U}_{m+1})$$

Since $R_m + 1 > R_m$, we also get

$$\varphi_{m+1}(R_m + 1) = \psi_+(R_m + 1, a(x_m, x_{m+1}, R_m) + b(x_m, x_{m+1})\tilde{U}_{m+1})$$

Thus to deduce (43), it is sufficient to see that $\tilde{a}(x_m, x_{m+1}) = a(x_m, x_{m+1}, R_m)$. This is always true when $x_{m+1} > x_m$ and when $x_{m+1} < x_m$, it requires that $x_{m+1} \leq R_m$, which is implied by $R_m + 1 > x_{m+1}$ here.

- When $R_m + 1 \leq x_m \vee x_{m+1}$: since $x_m \leq R_m$, we get $R_m = x_m < x_{m+1} = R_m + 1$. It follows that $\phi_{m+1}(R_m + 1) = \phi_m(x_{m+1}) = x_m = \varphi_{m+1}(R_m + 1)$.

It follows that

$$\tilde{R}_{m+1} = \begin{cases} R_m + 1 & , \text{ if } \varphi_{m+1}(R_m + 1) = R_m \\ R_m - 1 & , \text{ otherwise} \end{cases}$$

It is easy to check by similar arguments that the r.h.s. is the iteration defining R_{m+1} , showing that $\tilde{R}_{m+1} = R_{m+1}$. ■

A first guess would be that a similar formula holds for the minima of \mathfrak{X}_n , namely that

$$R_n^\wedge = \tilde{R}_n^\wedge := \min\{x \leq x_n : \exists m \in \llbracket 0, n \rrbracket \text{ with } \phi_{m,n}(x) = x_m\}$$

but this is wrong! Indeed, in the discrete Pitman example of Subsection 3.1, we get that $\tilde{R}_n^\wedge = x_n$ for all $n \in \mathbb{Z}_+$, see Picture 2 showing that $x_{n+1} - \tilde{R}_{n+1}^\wedge = x_n - \tilde{R}_n^\wedge$ for all $n \in \mathbb{Z}_+$ (the dotted lines correspond to the action of the mapping ϕ_{n+1} , to be read from the right to the left, remember it is constructed by pretending that $\mathfrak{X}_n = \{x_n, x_{n+1}\}$): whatever the motion from x_n to x_{n+1} , \tilde{R}_n^\wedge follows a parallel motion to go to \tilde{R}_{n+1}^\wedge . Since $\tilde{R}_0^\wedge = x_0$, we deduce that $\tilde{R}_n^\wedge = x_n$ for all $n \in \mathbb{Z}_+$.

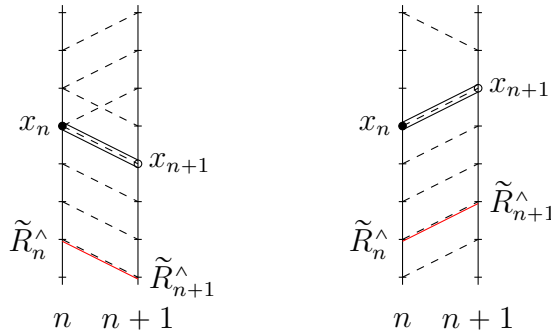


Figure 2: Action of ϕ_{n+1}

To get a correct backward formula, we must replace ϕ_m , for $m \in \llbracket 1, n \rrbracket$, by the random mapping φ_m , using the acquired knowledge of $R_{\llbracket 1, n \rrbracket}$. Indeed, from (40), the mapping φ_m is completely determined by x_{m-1} , x_m , R_{m-1} and the random variable \tilde{U}_m . Denote

$$\forall m \in \llbracket 1, n \rrbracket, \quad \varphi_{m,n} := \varphi_{m+1} \circ \varphi_{m+2} \circ \cdots \circ \varphi_n$$

Proposition 11 *With the above notations, we have*

$$R_n^\wedge = \min\{x \leq x_n : \exists m \in \llbracket 0, n \rrbracket \text{ with } \varphi_{m,n}(x) = x_m\}$$

Proof

The arguments are similar to those of the proof of Proposition 10, but simpler since we just play with the family of random mappings $(\varphi_m)_{m \in \llbracket 1, n \rrbracket}$. So let us prove by iteration on $m \in \llbracket 0, n \rrbracket$ that

$$R_m^\wedge = \tilde{R}_m^\wedge := \min\{x \leq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(x) = x_l\}$$

For $m = 0$, by definition

$$\begin{aligned} \tilde{R}_0^\wedge &= \min\{x \leq x_0 : \varphi_{0,0}(x) = x_0\} \\ &= \min\{x \leq x_0 : x = x_0\} \\ &= x_0 \\ &= R_0^\wedge \end{aligned}$$

Assume that $R_m^\wedge = \tilde{R}_m^\wedge$ for some $m \in \llbracket 0, n-1 \rrbracket$ and let us show that $R_{m+1}^\wedge = \tilde{R}_{m+1}^\wedge$. We have

$$\begin{aligned} \tilde{R}_{m+1}^\wedge &= \min\{x \leq x_{m+1} : \exists l \in \llbracket 0, m+1 \rrbracket \text{ with } \varphi_{l,m+1}(x) = x_l\} \\ &= \min\{x \leq x_{m+1} : x = x_{m+1} \text{ or } \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(\varphi_{m+1}(x)) = x_l\} \\ &= \min(x_{m+1}, \min\{x \leq x_{m+1} : \varphi_{m+1}(x) \in A_m^\wedge\}) \end{aligned}$$

where

$$A_m^\wedge := \{y \in \mathbb{Z} : \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(y) = x_l\}$$

In particular $x_m \in A_m^\wedge$, so that $x_{m+1} \in \{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\}$, since $\varphi_m(x_{m+1}) = x_m$. We deduce that

$$\tilde{R}_{m+1}^\wedge = \min\{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\} \tag{44}$$

Due to restlessness, the parity of the elements of A_m^\wedge is the parity of x_m and the parity of the elements of $\{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\}$ is the same as the parity of x_{m+1} . Furthermore, on the set of odd (respectively even) integers, the mapping φ_m is non-decreasing, thus

$$\{x \leq x_{m+1} : \varphi_m(x) \in A_m^\wedge\} = \{x \leq x_{m+1} : \varphi_m(x) \in B_m^\wedge\}$$

where

$$B_m^\wedge := \{y \leq x_m : \exists l \in \llbracket 0, m \rrbracket \text{ with } \varphi_{l,m}(y) = x_l\}$$

It follows by our iteration assumption that $\min(B_m^\wedge) = \tilde{R}_m^\wedge = R_m^\wedge$.

As in proof of Proposition 10, remark that B_m^\wedge is equal to the subset of elements from $\llbracket R_m^\wedge, x_m \rrbracket$ which have the same parity as x_m . It follows from (44) that $\tilde{R}_{m+1}^\wedge = R_m^\wedge + 1$, except if $\varphi_m(R_m^\wedge - 1) = R_m^\wedge$, in which case $\tilde{R}_{m+1}^\wedge = R_m^\wedge - 1$. Thus we have

$$\tilde{R}_{m+1}^\wedge = \begin{cases} R_m^\wedge - 1 & , \text{ if } \varphi_m(R_m^\wedge - 1) = R_m^\wedge \\ R_m^\wedge + 1 & , \text{ otherwise} \end{cases}$$

It is easy to check by similar arguments that the r.h.s. is the iteration defining R_{m+1}^\wedge , showing that $\tilde{R}_{m+1}^\wedge = R_{m+1}^\wedge$. ■

To facilitate the analogy with the last section, let us summarize the procedure followed here to construct \mathfrak{X}_n for any fixed $n \in \mathbb{Z}_+$, given $x_{\llbracket 0, n \rrbracket}$. First we sample $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n$ independent random variables uniformly distributed on $[0, 1)$. Next we construct the family $(\phi_m)_{m \in \llbracket 1, n \rrbracket}$ via

(41). Proposition 10 enables to deduce the family $R_{\llbracket 0, n \rrbracket}$. From the latter, we construct the family $(\varphi_m)_{m \in \llbracket 1, n \rrbracket}$ via (40). Then Proposition 11 enables to deduce the family $R_{\llbracket 0, n \rrbracket}^\wedge$. Finally, Lemma 8 leads to the construction of \mathfrak{X}_n .

Remark 12 In the discrete Pitman example of Subsection 3.1, the random variables $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_n$ are not needed. Here this extra randomness is necessary to construct the family $R_{\llbracket 0, n \rrbracket}$. But the two families $\tilde{U}_{\llbracket 0, n \rrbracket}$ and $R_{\llbracket 0, n \rrbracket}$ are sufficient to deduce $R_{\llbracket 0, n \rrbracket}^\wedge$, namely no additional randomness is required. In the context of one-dimensional diffusions presented in Subsection 7.2, we conjecture that the extra randomness disappears altogether. \square

5 Markov chains on Polish spaces

The construction of set-valued intertwining dual processes presented in the introduction for finite state spaces can be extended in several directions. Here, while keeping the time discrete, we let the state space be a general Polish space.

Let V be a Polish space endowed with a Markov kernel P . We assume that P admits an invariant probability π , so that P can be extended as a bounded operator on $\mathbb{L}^2(\pi)$. Let P^* be its adjoint operator. It is an abstract Markov operator: P^* preserves non-negativity as well as $\mathbb{1}_V$, the indicator function of V . The probability measure π is invariant for P^* in the sense that $\pi[P^*[f]] = \pi[f]$, for any $f \in \mathcal{B}(V)$, the space of bounded and measurable functions defined on V . The motivation for the Polish assumption on V is that P^* can also be seen as a Markov kernel. More precisely, consider on $V \times V$ the coordinate mappings X_0 and X_1 and the probability measure $\pi(dx_0)P(x_0, dx_1)$. Note that by the invariance of π , the law of X_1 is π . Using that $V \times V$ and V are Polish spaces we get that the conditional law of X_0 given X_1 can be described by a Markov kernel M from V to V (see for instance Section V.8 of Parthasarathy [19]). It follows that $\pi(dx_0)P(x_0, dx_1) = \pi(dx_1)M(x_1, dx_0)$. It is now easy to see that P^* is the extension of M as an operator on $\mathbb{L}^2(\pi)$. From now on, M will be denoted P^* . Let us extend the definitions of the introduction to the present setting.

A random mapping ψ from V to V is a measurable mapping

$$\psi : \Omega \times V \rightarrow V$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is an auxiliary probability space. Since we want to insure the existence of conditional distributions, let us furthermore impose that (Ω, \mathcal{F}) is the Borel σ -field associated to a Polish space. The random mapping ψ is said to be **associated with P^*** , when for any $x \in V$, the law of $\psi(x)$ is $P^*(x, \cdot)$. As in the finite state space case, we want to allow for the dependence of ψ on some subsets $S \subset V$. Let \mathfrak{V} be a set of measurable subsets $S \subset V$ such that $\pi(V) > 0$. We assume that \mathfrak{V} is endowed with a Polish space topology and that the mapping $V \times \mathfrak{V} \ni (x, S) \mapsto \mathbb{1}_S(x)$ is measurable with respect to the corresponding product Borelian σ -field. This hypothesis will be called **(H1)**. Define Λ the mapping associating to any $S \in \mathfrak{V}$ the probability measure $\Lambda(S, \cdot)$ which is the conditioning of π on S . This is an elementary conditioning, since $\pi(V) > 0$. It is straightforward to deduce from the measurability part of Fubini's theorem and from the above assumption on \mathfrak{V} that for any $f \in \mathcal{B}(V)$, the mapping $\mathfrak{V} \ni S \mapsto \Lambda(S, f)$ is measurable. It follows that Λ can be seen as a Markov kernel from \mathfrak{V} to V .

A measurable mapping

$$\psi : \Omega \times \mathfrak{V} \times V \ni (\omega, S, x) \mapsto \psi_S(\omega, x)$$

is called a **\mathfrak{V} -random mapping** and is said to be **weakly associated to P^*** when for any fixed $S \in \mathfrak{V}$, any measurable $A \subset S$ and any $x \in V$, we have

$$\mathbb{P}[\psi_S(x) \in A] = \frac{P^*(x, A)}{\zeta(S)} \quad (45)$$

where $\zeta : \mathfrak{V} \rightarrow (0, +\infty)$ is a measurable and positive mapping on \mathfrak{V} (as in the finite case, we must have $\zeta(V) = 1$). As it is customary, the dependence on $\omega \in \Omega$ will often not be written explicitly. When for any fixed $S \in \mathfrak{V}$, $\psi_S(\cdot)$ is a random mapping associated to P^* , ψ is said to be **completely associated to P^*** .

As in (11), to a \mathfrak{V} -random mapping ψ , we associate

$$\forall \omega \in \Omega, \forall S \in \mathfrak{V}, \quad \Psi(\omega, S) := \{y \in V : \psi_S(\omega, y) \in S\}$$

A priori $\Psi(S)$ is a measurable subset of V for any $S \in \mathfrak{V}$. We furthermore make the assumption, subsequently called **(H2)**, that \mathfrak{V} , its topology and ψ have been chosen so that Ψ is a random mapping from \mathfrak{V} to \mathfrak{V} . We would like to extend the definition given in (14), but the conditioning by $\psi_S(x') = x$ is no longer an elementary one. Nevertheless, our topological hypotheses make it possible, for any given $x' \in V$ and $S \in \mathfrak{V}$, to find a Markov kernel $K_{x', S}$ from V to \mathfrak{V} such that for any $(x, x', S) \in V \times V \times \mathfrak{V}$, $K_{x', S}(x, \cdot)$ is a regular version of the conditional law $\mathbb{P}[\Psi(S) \in \cdot | \psi_S(x') = x]$. Introduce Assumption **(H3)**, asserting that $K(x, x', S, \cdot) := K_{x', S}(x, \cdot)$ is a Markov kernel from $V \times V \times \mathfrak{V}$ to \mathfrak{V} , i.e. we are furthermore requiring that for any measurable $A \subset \mathfrak{V}$, the mapping $V^2 \times \mathfrak{V} \ni (x, x', S) \mapsto K_{x', S}(x, A)$ is measurable. When the measurability assumptions (H1) (H2) and (H3) are satisfied, we say that the \mathfrak{V} -random mapping ψ weakly associated to P^* is **standard**.

Remarks 13

(i) Let $\mathcal{M}_b(V)$ be the set of signed measures on V with finite total absolute weight. From (H1) and the measurability part of Fubini's theorem, we have that for any $\mu \in \mathcal{M}_b(V)$, the mapping

$$\forall S \in \mathfrak{V}, \quad F_\mu(S) := \int_S d\mu$$

is measurable.

If we have that the Borel σ -field on \mathfrak{V} is generated by these mappings, then (H2) follows from (H1). Indeed, it is then sufficient to check that for any $\mu \in \mathcal{M}_b(V)$, the mapping $\Omega \times \mathfrak{V} \ni (\omega, S) \mapsto F_\mu(\Psi(\omega, S))$ is measurable. This is again a consequence of (H1) and of the measurability part of Fubini's theorem, since the mapping $\Omega \times \mathfrak{V} \times V \ni (\omega, S, x) \mapsto \mathbb{1}_S(\psi(\omega, S, x))$ is measurable.

We believe (H3) should equally follow from (H1), under the same assumption on the Borel σ -field on \mathfrak{V} , but we prefer not entering such measurability questions here.

(ii) A priori (H2) and (H3) depend on \mathfrak{V} and ψ , but we rather see these conditions as relative to the topology of \mathfrak{V} , in the sense they should be satisfied for all \mathfrak{V} -random mappings ψ . In this situation, \mathfrak{V} is said to be a **standard subset topological space**. □

Similarly, (17) is replaced by

$$\forall (x, S) \in V \times \mathfrak{V}, \quad Q((x, S), d(x', S')) := P(x, dx')K(x, x', S, dS') \quad (46)$$

As in (16), we are interested in the set \mathcal{A} of probability measures m on $V \times \mathfrak{V}$ which can be decomposed under the form

$$m(dx, dS) = \mu(dS)\Lambda(S, dx) \quad (47)$$

where μ is the marginal law of m on \mathfrak{V} . When considering Markov chains starting from initial distributions in \mathcal{A} and evolving according to Q , it is possible to reduce the state space $V \times \mathfrak{V}$ to

$$W := \{(x, S) \in V \times \mathfrak{V} : x \in S\} \quad (48)$$

as in (15), since for $x \in S$, $K(x, x', S, \cdot)$ should be supported by $\{S' \in \mathfrak{S} : x' \in S'\}$. But in the definition of the regular version of a conditional expectation, one has to be careful with negligible subsets, that is why the justification of this restriction will only be given below, in the proof of Theorem 14. Note that W is a Borelian subset, according to (H1).

Finally, extend (12) and (13) by defining the kernels K and \mathfrak{P} on \mathfrak{S} via:

$$\forall S \in \mathfrak{S}, \quad \begin{cases} K(S, dS') & := \mathbb{P}[\Psi(S) \in dS'] \\ \mathfrak{P}(S, dS') & := \frac{\pi(S')\zeta(S)}{\pi(S)}K(S, dS') \end{cases} \quad (49)$$

Both K and \mathfrak{P} are Markovian: it is obvious for K and it is a consequence of the following generalization of Theorem 3 to the present framework:

Theorem 14 *Assume that we are given a standard \mathfrak{S} -random mapping ψ weakly associated to P^* . Let $(X_n, \mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ be a Markov chain on $V \times \mathfrak{S}$ whose initial distribution $\mathcal{L}(X_0, \mathfrak{X}_0)$ belongs to \mathcal{A} and whose transitions are given by Q , constructed as in (46). Then $X := (X_n)_{n \in \mathbb{N}}$ and $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{N}}$ are Markov chains whose respective transitions are given by P and \mathfrak{P} . Furthermore the conditions (8) and (9) are fulfilled and a.s. for all $n \in \mathbb{Z}_+$, $(X_n, \mathfrak{X}_n) \in W$.*

Proof

The arguments are essentially the same as those of the proof of Theorem 3, to make them rigorous we just have to resort to conditional expectations.

The first part of the proof, namely that $X := (X_n)_{n \in \mathbb{N}}$ is a Markov chain whose transitions are given by P and the validity of (8), is very simple, as well as checking (A_0) and (B_0) , with the notation from the proof of Theorem 3. Thus we concentrate our attention to the deduction of (A_{n+1}) and (B_{n+1}) from (A_n) and (B_n) , for some given $n \in \mathbb{Z}_+$.

Let $G : V \times \mathfrak{S} \rightarrow \mathbb{R}$ be a bounded and measurable test function. We have

$$\begin{aligned} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | \mathfrak{X}_{[0, n]}] &= \int_V \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0, n]}] \mathbb{P}[X_n \in dy | \mathfrak{X}_{[0, n]}] \\ &= \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0, n]}] \mathbb{P}[X_n \in dy | \mathfrak{X}_{[0, n]}] \end{aligned}$$

since due to (B_n) , $\mathbb{P}[X_n \in \cdot | \mathfrak{X}_{[0, n]}]$ is supported by \mathfrak{X}_n . The reformulation of (20) in the present context is that for any bounded measurable test function $h : V \rightarrow \mathbb{R}$, we have

$$\begin{aligned} &\int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0, n]}] h(y) \pi(dy) \\ &= \zeta(\mathfrak{X}_n) \int_V \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) h(\psi_{\mathfrak{X}_n}(x))] \pi(dx) \end{aligned} \quad (50)$$

where the exponent of the expectation of the r.h.s. indicates that the integration is only with respect to the randomness of the random mapping $\psi_{\mathfrak{X}_n}$, while \mathfrak{X}_n is fixed, as told by its presence as an index. In this equality, h can depend on \mathfrak{X}_n (even on $\mathfrak{X}_{[0, n]}$), it will be written $h_{\mathfrak{X}_n}$ in the following computations. Let us prove (50). By the Markov property, we have that the l.h.s. is equal to

$$\begin{aligned} &\int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_n] h_{\mathfrak{X}_n}(y) \pi(dy) \\ &= \int_{\mathfrak{X}_n \times V \times \mathfrak{S}} \pi(dy) P(y, dx) K(y, x, \mathfrak{X}_n, dS') G(x, S') h_{\mathfrak{X}_n}(y) \\ &= \int_{\mathfrak{X}_n \times V} \pi(dy) P(y, dx) \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x) = y] h_{\mathfrak{X}_n}(y) \\ &= \int_{V \times \mathfrak{X}_n} \pi(dx) P^*(x, dy) \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x) = y] h_{\mathfrak{X}_n}(y) \end{aligned} \quad (51)$$

Recall that by weak association, $P^*(x, \cdot)/\zeta(\mathfrak{X}_n)$ and the law of $\psi_{\mathfrak{X}_n}(x)$ coincide when they are restricted on \mathfrak{X}_n , so that

$$\begin{aligned}
& \frac{1}{\zeta(\mathfrak{X}_n)} \int_{\mathfrak{X}_n} P^*(x, dy) \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x) = y] h_{\mathfrak{X}_n}(y) \\
&= \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [\mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) | \psi_{\mathfrak{X}_n}(x)] h_{\mathfrak{X}_n}(\psi_{\mathfrak{X}_n}(x))] \\
&= \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) h_{\mathfrak{X}_n}(\psi_{\mathfrak{X}_n}(x))]
\end{aligned} \tag{52}$$

The announced equality (50) follows. Taking into account (B_n) , asserting that

$$\begin{aligned}
\mathbb{P}[X_n \in dy | \mathfrak{X}_{[0,n]}] &= \Lambda(\mathfrak{X}_n, dy) \\
&= \mathbb{1}_{\mathfrak{X}_n}(y) \frac{\pi(dy)}{\pi(\mathfrak{X}_n)}
\end{aligned}$$

we deduce from (50), with $h = h_{\mathfrak{X}_n} := \mathbb{1}_{\mathfrak{X}_n}/\pi(\mathfrak{X}_n)$,

$$\begin{aligned}
\mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | \mathfrak{X}_{[0,n]}] &= \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0,n]}] \frac{\pi(dy)}{\pi(\mathfrak{X}_n)} \\
&= \int_{\mathfrak{X}_n} \mathbb{E}[G(X_{n+1}, \mathfrak{X}_{n+1}) | X_n = y, \mathfrak{X}_{[0,n]}] \frac{\mathbb{1}_{\mathfrak{X}_n}(y)}{\pi(\mathfrak{X}_n)} \pi(dy) \\
&= \zeta(\mathfrak{X}_n) \int_V \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) \mathbb{1}_{\mathfrak{X}_n}(\psi_{\mathfrak{X}_n}(x))] / \pi(\mathfrak{X}_n) \pi(dx) \\
&= \zeta(\mathfrak{X}_n) \int_V \mathbb{E}_{\mathfrak{X}_n}^{\psi_{\mathfrak{X}_n}} [G(x, \Psi(\mathfrak{X}_n)) \mathbb{1}_{\Psi(\mathfrak{X}_n)}(x)] / \pi(\mathfrak{X}_n) \pi(dx) \\
&= \zeta(\mathfrak{X}_n) \int_{V \times \mathfrak{Y}} \pi(dx) K(\mathfrak{X}_n, dS) G(x, S) \mathbb{1}_S(x) / \pi(\mathfrak{X}_n) \\
&= \int_{V \times \mathfrak{Y}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) G(x, S) \mathbb{1}_S(x) / \pi(S)
\end{aligned} \tag{53}$$

where we used the definitions from (49). When G does not depend on the first variable, i.e. is of the form

$$\forall (x, S) \in V \times \mathfrak{Y}, \quad G(x, S) := g(S)$$

for a bounded and measurable test function $g : V \rightarrow \mathbb{R}$, we get

$$\mathbb{E}[g(\mathfrak{X}_{n+1}) | \mathfrak{X}_{[0,n]}] = \int_{\mathfrak{Y}} \mathfrak{P}(\mathfrak{X}_n, dS) g(S)$$

which amounts to (A_{n+1}) .

Next consider G of product form:

$$\forall (x, S) \in V \times \mathfrak{Y}, \quad G(x, S) := h(x)g(S)$$

where $g : V \rightarrow \mathbb{R}$ and $h : \mathfrak{Y} \rightarrow \mathbb{R}$ are bounded and measurable test functions. We compute that

$$\int_{V \times \mathfrak{Y}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) G(x, S) \mathbb{1}_S(x) / \pi(S) = \int_{\mathfrak{Y}} \mathfrak{P}(\mathfrak{X}_n, dS) g(S) \Lambda[h](S)$$

Let $F : \mathfrak{Y}^{[0,n]} \rightarrow \mathbb{R}$ be another bounded and measurable test function. From the above consideration, we get

$$\begin{aligned}
\mathbb{E}[F(\mathfrak{X}_{[0,n]}) g(\mathfrak{X}_{n+1}) h(X_{n+1})] &= \mathbb{E}[F(\mathfrak{X}_{[0,n]}) \mathbb{E}[g(\mathfrak{X}_{n+1}) h(X_{n+1}) | \mathfrak{X}_{[0,n]}]] \\
&= \mathbb{E}[F(\mathfrak{X}_{[0,n]}) \mathfrak{P}[g \Lambda[h]](\mathfrak{X}_n)] \\
&= \mathbb{E}[F(\mathfrak{X}_{[0,n]}) g(\mathfrak{X}_{n+1}) \Lambda[h](\mathfrak{X}_{n+1})]
\end{aligned}$$

Since F and g are arbitrary bounded and measurable functions, it follows that

$$\mathbb{E}[h(X_{n+1})|\mathfrak{X}_{[0,n+1]}] = \Lambda[h](\mathfrak{X}_{n+1})$$

namely (B_{n+1}) , due to the fact that h is equally an arbitrary bounded and measurable function.

It remains to show that a.s. for all $n \in \mathbb{Z}_+$, $(X_n, \mathfrak{X}_n) \in W$. For $n = 0$, this is an immediate consequence of the belonging of the initial distribution to \mathcal{A} . For $n \in \mathbb{Z}_+$, successively apply (53) with the mappings $G : V \times \mathfrak{V} \ni (x, S) \mapsto \mathbb{1}_S(x)$ and $G := \mathbb{1}_{V \times \mathfrak{V}}$ to get

$$\begin{aligned} \mathbb{P}[X_{n+1} \in \mathfrak{X}_{n+1} | \mathfrak{X}_{[0,n]}] &= \int_{V \times \mathfrak{V}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) \mathbb{1}_S(x) \mathbb{1}_S(x) / \pi(S) \\ &= \int_{V \times \mathfrak{V}} \pi(dx) \mathfrak{P}(\mathfrak{X}_n, dS) \mathbb{1}_S(x) / \pi(S) \\ &= \mathbb{E}[\mathbb{1}_{V \times \mathfrak{V}}(X_{n+1}, \mathfrak{X}_{n+1}) | \mathfrak{X}_{[0,n]}] \\ &= 1 \end{aligned}$$

■

It is straightforward to generalize Corollary 4 to the present framework. In the statement of this result, π is seen as the measurable mapping $\mathfrak{V} \ni S \mapsto \pi(S) \in (0, 1]$.

In the applications we have in mind, the state space \mathfrak{V} is too small since we would like that it contains the singletons, which in practice are often negligible with respect to π when V is not denumerable. Consider for instance $\tilde{X} := (\tilde{X}_t)_{t \geq 0}$ an elliptic diffusion on a compact Riemannian manifold V . For fixed $\epsilon > 0$, we are interested in the time-skeleton Markov chain $X := (\tilde{X}_{\epsilon n})_{n \in \mathbb{Z}_+}$. Its invariant probability measure π is also the invariant probability measure of \tilde{X} , which gives zero mass to every singleton, since it admits a density with respect to the Riemannian measure. Nevertheless, we are looking for a set-valued dual $\mathfrak{X} := (\mathfrak{X}_n)_{n \in \mathbb{Z}_+}$ starting from $\mathfrak{X}_0 := \{X_0\}$, constructed via some random mappings (e.g. to escape the difficulties encountered in [16] and [6] to get singletons as starting points). In discrete time, the problem is only in the initial step, since for $n \in \mathbb{N}$, \mathfrak{X}_n should be a “nice” compact subset of V with $\pi(\mathfrak{X}_n) > 0$. So to end this section, we show a way to enlarge \mathfrak{V} to include the singletons. Again, random mappings are very helpful in this respect, but they will no longer be weakly associated to P^* and we need a slight generalization of this notion, see Remark 15(a) below. Unfortunately, this approach will not receive here the full treatment it deserves and this lack of development will be prevent it from being useful in the next section.

We come back to the general setting of this section and begin by presenting some straightforward modifications of the definitions. Consider

$$\bar{\mathfrak{V}} := \mathfrak{V} \cup \{\{x\} : x \in V\} \tag{54}$$

(here the notation differs substantially from that of the introduction, where $\bar{\mathfrak{V}}$ was the set of all subsets of V and $\mathfrak{V} := \bar{\mathfrak{V}} \setminus \{\emptyset\}$). Assume that $\bar{\mathfrak{V}}$ is endowed with a Polish topology such that \mathfrak{V} is a measurable subset of $\bar{\mathfrak{V}}$ and such that the mapping $V \times \bar{\mathfrak{V}} \ni (x, S) \mapsto \mathbb{1}_S(x)$ is measurable.

Extend Λ into a Markov kernel $\bar{\Lambda}$ from $\bar{\mathfrak{V}}$ to V , via the convention that when S is the singleton $\{x\}$, then $\Lambda(\{x\}, \cdot) := \delta_x$, the Dirac mass at x (this definition is coherent with the conditioning when $\pi(\{x\}) > 0$). The state space W has to be enlarged into

$$\begin{aligned} \bar{W} &:= \{(x, S) \in V \times \bar{\mathfrak{V}} : x \in S\} \\ &= W \cup \{(x, \{x\}) : x \in S\} \end{aligned}$$

The set of initial distribution we are interested in is $\bar{\mathcal{A}}$, the set of probability measure on \bar{W} which can be decomposed as in (47).

Consider a measurable mapping

$$\bar{\psi} : \Omega \times \bar{\mathfrak{Y}} \times V \ni (\omega, S, x) \mapsto \bar{\psi}_S(\omega, x)$$

Assume that the restriction ψ of $\bar{\psi}$ to $\Omega \times \mathfrak{Y} \times V$ is a standard \mathfrak{Y} -random mapping weakly associated with P^* . Replacing \mathfrak{Y} by $\bar{\mathfrak{Y}}$, we could define similarly a notion of a $\bar{\mathfrak{Y}}$ -random mapping $\bar{\psi}$ weakly associated with P^* , nevertheless the condition (45) on singletons $\{x_0\} \in \bar{\mathfrak{Y}}$ would just mean

$$\mathbb{P}[\bar{\psi}_{\{x_0\}}(x) = x_0] = P^*(x, \{x_0\})/\zeta(\{x_0\}) \quad (55)$$

where ζ has been extended on $\bar{\mathfrak{Y}}$ as a measurable and positive function. In our present context, the r.h.s. often vanishes and the requirement that $\mathbb{P}[\bar{\psi}_{\{x_0\}}(x) = x_0] = 0$ is not appropriate for our purposes. In some sense, we need a density equivalent of (55), which leads us to strengthen our assumption on the Markov kernel P . So introduce Hypothesis **(H4)** asking for the existence of a measurable function $p : V \times V \rightarrow \mathbb{R}_+$ such that

$$\forall x, y \in V, \quad P(x, dy) = p(x, y)\pi(dy)$$

This density assumption implies immediately that P^* is given by

$$\forall x, y \in V, \quad P^*(x, dy) = p(y, x)\pi(dy)$$

Condition (45) has to be amended with

$$\forall x_0, x \in V, \quad \mathbb{P}[\bar{\psi}_{\{x_0\}}(x) = x_0] = p(x_0, x)/\zeta(\{x_0\}) \quad (56)$$

where $\zeta : \bar{\mathfrak{Y}} \rightarrow (0, +\infty)$ is a measurable function. When (56) is satisfied (in addition to ψ being a standard \mathfrak{Y} -random mapping weakly associated with P^*), we say that $\bar{\psi}$ is **$\bar{\mathfrak{Y}}$ -random mapping weakly associated with P^*** .

Remarks 15

(a) As observed above and strictly speaking, $\bar{\psi}_{\{x_0\}}$ is not weakly associated to P^* , since it does not satisfy (55) in general. Nevertheless, Equation (56) can be seen as a limit of (45), when S converges to $\{x_0\}$ and $\zeta(S)/\pi(S)$ converges to $\zeta(\{x_0\})$. This point of view inspired the notion of weak association to P^* . It also explains the expression obtained in (58) below.

(b) The measurability of ζ in (56) plays no role when we start with $X_0 = x_0$ and $\mathfrak{X}_0 = \{x_0\}$, for some fixed $x_0 \in V$. Anyway, it seems quite natural to assume that $V \ni x \mapsto \{x\} \in \bar{\mathfrak{Y}}$ is a measurable bijection, as well as its inverse (i.e. V can be identified as a measurable space to $\{\{x\} : x \in V\}$), so that one can go from the measurability of $\mathfrak{Y} \ni S \mapsto \zeta(S)$ to the measurability of $\bar{\mathfrak{Y}} \ni S \mapsto \zeta(S)$ via the additional condition that $V \ni x \mapsto \zeta(\{x\})$ is measurable. □

The mapping $\bar{\psi}$ will be said to be **standard**, when the following extensions $(\bar{\text{H2}})$ and $(\bar{\text{H3}})$ of Assumptions (H2) and (H3) hold:

- **($\bar{\text{H2}}$)** requires that the mapping

$$\forall S \in \bar{\mathfrak{Y}}, \quad \bar{\Psi}(S) := \{y \in V : \bar{\psi}_S(y) \in S\}$$

is measurable from $\bar{\mathfrak{Y}}$ to \mathfrak{Y} . In particular, $\bar{\Psi}(\{x_0\})$ must have positive mass for any $x_0 \in V$.

- **($\bar{\text{H3}}$)** provides us with a Markov kernel \bar{K} from $V \times V \times \bar{\mathfrak{Y}}$ to \mathfrak{Y} , which, as in (46), enables us to define a Markov kernel \bar{Q} from $V \times \bar{\mathfrak{Y}}$ to $V \times \mathfrak{Y}$ via

$$\forall (x, S) \in V \times \bar{\mathfrak{Y}}, \quad \bar{Q}((x, S), d(x', S')) := P(x, dx')\bar{K}(x, x', S, dS') \quad (57)$$

Next let us come to the analogue of (49). The kernel \bar{K} is defined similarly to K :

$$\forall S \in \bar{\mathfrak{V}}, \quad \bar{K}(S, dS') := \mathbb{P}[\bar{\Psi}(S) \in dS']$$

Due to $(\bar{H}2)$, \bar{K} is a Markov kernel from $\bar{\mathfrak{V}}$ to \mathfrak{V} . One has to be more careful with the definition of $\bar{\mathfrak{P}}$:

$$\forall S \in \bar{\mathfrak{V}}, \quad \bar{\mathfrak{P}}(S, dS') := \begin{cases} \frac{\pi(S')\zeta(S)}{\pi(S)} \bar{K}(S, dS') & , \text{ if } \pi(S) > 0 \\ \pi(S')\zeta(\{x_0\}) \bar{K}(S, dS') & , \text{ if } S = \{x_0\} \text{ is a singleton} \end{cases} \quad (58)$$

Now we have all the ingredients necessary for stating the extension of Theorem 14:

Theorem 16 *Assume that we are given a standard $\bar{\mathfrak{V}}$ -random mapping $\bar{\psi}$ weakly associated to P^* . Let $(\bar{X}_n, \bar{\mathfrak{X}}_n)_{n \in \mathbb{Z}_+}$ be a Markov chain on $V \times \bar{\mathfrak{V}}$ whose initial distribution $\mathcal{L}(\bar{X}_0, \bar{\mathfrak{X}}_0)$ belongs to \bar{A} and whose transitions are given by \bar{Q} , constructed in (57). Then $\bar{X} := (\bar{X}_n)_{n \in \mathbb{N}}$ and $\bar{\mathfrak{X}} := (\bar{\mathfrak{X}}_n)_{n \in \mathbb{N}}$ are Markov chains whose respective transitions are given by P and $\bar{\mathfrak{P}}$. Furthermore the conditions (8) and (9) are fulfilled and a.s. for all $n \in \mathbb{Z}_+$, $(\bar{X}_n, \bar{\mathfrak{X}}_n) \in \bar{W}$.*

As in Corollary 4, we deduce the following consequences.

Corollary 17 *The kernel $\bar{\mathfrak{P}}$ is Markovian and the intertwining relation (6) is satisfied. When $\zeta \equiv 1$ (in particular for completely associated to P^* random mappings), π is harmonic for \bar{K} . Furthermore, $\bar{\mathfrak{V}} \setminus \mathfrak{V}$ is an entrance boundary for $\bar{\mathfrak{X}}$, in the sense that for any $n \geq 1$, $\bar{\mathfrak{X}}_n \in \mathfrak{V}$, namely after time 1, $(\bar{X}_n, \bar{\mathfrak{X}}_n)_{n \in \mathbb{N}}$ is a Markov chain on $V \times \mathfrak{V}$ of the same type as those considered in Theorem 14.*

Proof of Theorem 16

Let us come back to the proof of Theorem 14 and review the changes to be made. They correspond to the situation where $\bar{\mathfrak{X}}_n$ is a singleton and it is sufficient to consider the time $n = 0$. Fix some $x_0 \in V$ and assume that $\bar{\mathfrak{X}}_0 = \{x_0\}$. Equation (50) is to be replaced by

$$\mathbb{E}[G(\bar{X}_1, \bar{\mathfrak{X}}_1) | \bar{X}_0 = x_0, \bar{\mathfrak{X}}_0 = \{x_0\}] = \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\psi}_{\{x_0\}}(x) = x_0}] \pi(dx)$$

where $G : V \times \bar{\mathfrak{V}} \rightarrow \mathbb{R}$ is a bounded and measurable test function. Its proof, justifying Condition (H4), is the following modification of (51):

$$\begin{aligned} \mathbb{E}[G(\bar{X}_1, \bar{\mathfrak{X}}_1) | \bar{X}_0 = x_0, \bar{\mathfrak{X}}_0 = \{x_0\}] &= P(x_0, dx) \bar{K}(x_0, x, \{x_0\}, dS') G(x, S') \\ &= \int_V P(x_0, dx) \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) | \bar{\psi}_{\{x_0\}}(x) = x_0] \\ &= \int_V p(x_0, x) \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) | \bar{\psi}_{\{x_0\}}(x) = x_0] \pi(dx) \\ &= \zeta(\{x_0\}) \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\psi}_{\{x_0\}}(x) = x_0}] \pi(dx) \\ &= \zeta(\{x_0\}) \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\Psi}_{\{x_0\}}(x)}] \pi(dx) \end{aligned}$$

The definition of $\bar{\mathfrak{P}}$ is dictated by the analogue of (53), which now writes:

$$\begin{aligned} \mathbb{E}[G(\bar{X}_1, \bar{\mathfrak{X}}_1) | \bar{\mathfrak{X}}_0] &= \mathbb{E}[G(\bar{X}_1, \bar{\mathfrak{X}}_1) | \bar{X}_0 = x_0, \bar{\mathfrak{X}}_0 = \{x_0\}] \\ &= \zeta(\{x_0\}) \int_V \mathbb{E}^{\bar{\psi}_{\{x_0\}}} [G(x, \bar{\Psi}(\{x_0\})) \mathbb{1}_{\bar{\Psi}_{\{x_0\}}(x_0)}] \pi(dx) \\ &= \zeta(\{x_0\}) \int_{V \times \mathfrak{V}} \pi(dx) \bar{K}(\{x_0\}, dS) G(x, S) \mathbb{1}_S(x) \\ &= \int_{V \times \mathfrak{V}} \pi(dx) \bar{\mathfrak{P}}(\{x_0\}, dS) G(x, S) \mathbb{1}_S(x) / \pi(S) \end{aligned}$$

The end of the proof readily follows the arguments given in the proof of Theorem 14. ■

Remark 18 As in the finite situation, the law of the random mapping enabling to construct $\bar{\mathfrak{X}}_{n+1}$ (or \mathfrak{X}_{n+1} in Theorem 14) from $(X_n, \bar{\mathfrak{X}}_n, X_{n+1})$ may depend on the time $n \in \mathbb{Z}_+$. Indeed the proofs of Theorems 14 and 16 are only concerned with a transition from n to $n + 1$. One can even go further, by considering different state spaces V_n at each time $n \in \mathbb{Z}_+$. The invariant probability π has then to be replaced by a family $(\pi_n)_{n \in \mathbb{Z}_+}$ of probability measures which are related by the underlying Markov kernels: $\pi_n P_n = \pi_{n+1}$, for any $n \in \mathbb{Z}_+$, where P_n is the transition kernel between times n and $n + 1$. Corresponding Markov kernels $(\Lambda_n)_{n \in \mathbb{N}}$ have to be considered. □

6 Markov processes

Here we leave the discrete-time setting for a continuous-time framework, with the purpose of extending the construction of set-valued intertwining duals given by Theorem 14 to diffusion processes. The full development of this theory is out of the scope of the present paper and we hope to provide more details in future works. Nevertheless, below we outline the principles underlying such extensions. An illustration will be given in the next section, where we will also discuss further applications, up to the availability of convenient stochastic flows.

Let V be a Polish space endowed with a probability distribution π . Let \mathfrak{V} be a set of certain measurable subsets of V which are given a positive weight by π . For any $S \in \mathfrak{V}$, let $\Lambda(S, \cdot)$ be the elementary conditioning of π on S . Assume that \mathfrak{V} is endowed with a Polish topology such that the mapping $V \times \mathfrak{V} \ni (x, S) \mapsto \mathbb{1}_S(x)$ is measurable. It follows that Λ , the mapping associating to any $S \in \mathfrak{V}$ the probability $\Lambda(S, \cdot)$, is a Markov kernel from \mathfrak{V} to V .

Let $X := (X_t)_{t \geq 0}$ be a time-homogeneous V -valued **diffusion** (i.e. a Markov process with continuous paths), whose semigroup $P := (P_t)_{t \geq 0}$ admits π as an invariant probability measure. Denote $P^* := (P_t^*)_{t \geq 0}$ the adjoint Markov semi-group in $\mathbb{L}^2(\pi)$. As in Section 5, our topological assumptions insure that the semi-group is given by Markov kernels. By our hypotheses below, the Markov processes associated to P^* , $X^* := (X_t^*)_{t \geq 0}$, will admit versions that are diffusions. For any $x \in V$, $X^*(x)$ will stand for such a process starting from x .

We want to consider stochastic flows on V extending the random mappings of the previous sections. We will need a notion of stochastic flow more general than that considered in a series of papers by Le Jan and Raimond [11, 12, 13, 14] (see also Tsirelson [25]), since typically, due to the possible dependence on subsets of \mathfrak{V} , we would like the increments of the flow to be non-stationary. It is even worse, since once the time has been returned, the subset on which the construction depends is in the future of the flow, fortunately there is an important independence property helping us, see Lemma 21 below. In some sense, there is the same difference between the stochastic flows of Le Jan and Raimond and those we would like to construct as the coupling-from-the-past technique of Propp and Wilson [22] and the random mappings considered in the introduction. So a lot remains to be investigated in this direction.

It is convenient to be quite explicit about the underlying probability space, so we are led to the following definitions.

We assume that we are given $\mathcal{R}(V)$ a vector space of measurable functions from V to V endowed with a Polish topology such that the mapping $V \times \mathcal{R}(V) \ni (x, \psi) \mapsto \psi(x)$ is measurable (all product spaces are endowed with the product measurable structure). Let Δ stand for $\{(s, t) \in \mathbb{R}_+^2 : s \leq t\}$ and more generally for any $I \subset \mathbb{R}_+$, we define $\Delta_I := \{(u, v) \in I^2 : u \leq v\}$. Define the space $\Omega^{(5)}$ as the set of all measurable mappings

$$\psi : \Delta \times V \ni (s, t, x) \mapsto \psi_{s,t}(x) \in V \tag{59}$$

such that for any fixed $(s, t) \in \Delta$, the restricted mapping $\psi_{s,t}$ belongs to $\mathcal{R}(V)$. The notation $\psi = (\psi_{s,t}(x))_{0 \leq s \leq t, x \in V}$ will designate the canonical coordinates on $\Omega^{(5)}$. The space $\Omega^{(5)}$ is endowed with the sigma-field generated by the canonical coordinates.

Remark 19 The space $\Omega^{(5)}$ is too large to be endowed with a Polish structure. It would be nicer to define $\Omega^{(5)}$ as the space of mappings of the form (59) such that

$$\Delta \ni (s, t) \mapsto \psi_{s,t} \in \mathcal{R}(V)$$

is continuous. Unfortunately this assumption is for the moment too strong and would not enable us to rely on the results of Le Jan and Raimond [14] in Subsection 7.1 below, where $V = \mathbb{R}$ and $\mathcal{R}(\mathbb{R})$ will be the space of non-decreasing càdlàg mappings from \mathbb{R} to \mathbb{R} , endowed with the Skorohod topology.

The above presentation differs from that of Le Jan and Raimond [12] by the introduction of the space $\mathcal{R}(V)$ (in the same spirit as that of \mathfrak{V}) to avoid the handling of measurable representations. \square

A **stochastic flow** on the state space V is a probability distribution $\mathbb{P}^{(5)}$ on $\Omega^{(5)}$ such that for all $0 \leq s \leq t \leq u$, a.s. $\psi_{s,s}$ is the identity operator and $\psi_{s,t} \circ \psi_{t,u} = \psi_{s,u}$. Given such a probability $\mathbb{P}^{(5)}$, the sigma-field of $\Omega^{(5)}$ is completed with all its negligible subsets. By a slight abuse of terminology, we will also say that the canonical ψ on $\Omega^{(5)}$ is a **stochastic flow** (implicitly under $\mathbb{P}^{(5)}$). The stochastic flow is said to be **associated to** P^* if for any fixed $t \geq 0$ and $x \in V$, $(\psi_{t-s,t}(x))_{s \in [0,t]}$ has the same finite-dimensional marginal laws, over the time domain $[0, t]$, as a diffusion associated to P^* and starting from x . Namely ψ (or $\mathbb{P}^{(5)}$) provides a coupling of the $X_{[0,t]}^*(x)$, for any $x \in V$ and $t \geq 0$. Let us ask more, since we are rather interested in the notion of complete association with P^* , where there is an underlying dependence on subsets of \mathfrak{V} . To proceed toward its definition, let us extend (H2) into **(H5)**: for any $0 \leq s \leq t$, the mapping

$$\Omega^{(5)} \times \mathfrak{V} \ni (\psi, S) \mapsto \psi_{s,t}^{-1}(S)$$

takes values in $\mathfrak{V} \sqcup \{\emptyset\}$ and is measurable (where $\mathfrak{V} \sqcup \{\emptyset\}$ is endowed with the sigma-field generated by that of \mathfrak{V} and $\{\emptyset\}$).

Enlarge $\Omega^{(5)}$ into $\Omega^{(4)} := \mathfrak{V} \times \Omega^{(5)}$, the canonical coordinate on \mathfrak{V} will be denoted \mathfrak{X}_0 . A probability distribution $\mathbb{P}^{(4)}$ on $\Omega^{(4)}$ will be said to be a **stochastic flow** when the coordinates \mathfrak{X}_0 and ψ are independent and the distribution of ψ is a stochastic flow. On $(\Omega^{(4)}, \mathbb{P}^{(4)})$, we define

$$\forall t \geq 0, \quad \mathfrak{X}_t := \psi_{0,t}^{-1}(\mathfrak{X}_0) \tag{60}$$

It follows from (H5) that $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$ is a \mathfrak{V} -valued stochastic process whose initial variable \mathfrak{X}_0 is independent from the stochastic flow ψ . Note that no regularity is assumed with respect to the time (to go into this direction, one should improve the time regularity of the flow, for instance by considering the condition mentioned in Remark 19), \mathfrak{X} is only a collection of random variables indexed by the time.

For any $A \subset \Delta \times V$, denote $\psi_A := (\psi_{u,v}(x))_{(u,v,x) \in A}$ and for any $t \geq 0$, let \mathcal{G}_t be the sigma-field generated by $\psi_{\Delta_{[0,t]} \times V}$ and \mathfrak{X}_0 . The stochastic flow ψ is said to be a **\mathfrak{V} -stochastic flow** when

$$\forall t \geq 0, \forall x \in V, \quad \mathcal{L}[\psi_{\Delta_{[t,+\infty)} \times V} | \mathcal{G}_t] = \mathcal{L}[\psi_{\Delta_{[t,+\infty)} \times V} | \mathfrak{X}_t] \tag{61}$$

Since $\Omega^{(5)}$ is not endowed with a Polish topology, we have to be more careful about the meaning of the above identity: the signification is that any corresponding conditional expectations are a.s. equal. With the same convention, a \mathfrak{V} -stochastic flow is said to be **completely associated to** P^* , when

$$\forall 0 \leq s \leq t, \forall x \in V, \quad \mathbb{P}^{(4)}[\psi_{s,t}(x) \in dy | \mathcal{G}_s] = P_{t-s}^*(x, dy) \tag{62}$$

In particular such a flow ψ is associated to P^* .

Remark 20 By analogy with the definitions of the previous sections, a \mathfrak{V} -stochastic flow ψ is said to be **weakly associated to P^*** , when for any $0 \leq s \leq t$, any $x \in V$ and any measurable $A \subset \mathfrak{X}_s$, we have

$$\mathbb{P}^{(4)}[\psi_{s,t}(x) \in A | \mathcal{G}_s] = P_{t-s}^*(x, A) / \zeta(s, t, \mathfrak{X}_s)$$

where $\zeta : \Delta \times (\mathfrak{V} \sqcup \{\emptyset\}) \rightarrow (0, +\infty)$ is a measurable mapping. But to avoid technicalities and since we will not need it in the next section, we will not investigate this promising notion in the time-continuous setting. □

Enlarge the probability space $(\Omega^{(4)}, \mathbb{P}^{(4)})$ into $(\Omega^{(3)}, \mathbb{P}^{(3)})$, with

$$\begin{aligned} \Omega^{(3)} &:= \Omega^{(4)} \times V \\ \mathbb{P}^{(3)} &:= \mathbb{P}^{(4)} \otimes \pi \end{aligned} \tag{63}$$

and denote Z for the canonical coordinate on V .

An important consequence of complete association to P^* is:

Lemma 21 *For any $(s, t) \in \Delta$, \mathfrak{X}_s and $\psi_{s,t}(Z)$ are independent under $\mathbb{P}^{(3)}$ as soon as $\mathbb{P}^{(4)}$ is a \mathfrak{V} -stochastic flow completely associated to P^* . It follows that $\psi_{\Delta_{[0,s]} \times V}$ and $\psi_{s,t}(Z)$ are independent.*

Proof

Let F and G be bounded and measurable functions defined respectively on \mathfrak{V} and V . By definition, we compute that

$$\begin{aligned} \mathbb{E}^{(3)}[F(\mathfrak{X}_s)G(\psi_{s,t}(Z))] &= \int_V \pi(dz) \mathbb{E}^{(4)}[F(\mathfrak{X}_s)G(\psi_{s,t}(z))] \\ &= \int_V \pi(dz) \mathbb{E}^{(4)}[F(\mathfrak{X}_s) \mathbb{E}^{(4)}[G(\psi_{s,t}(z)) | \mathfrak{X}_s]] \\ &= \int_V \pi(dz) \mathbb{E}^{(4)}[F(\mathfrak{X}_s) P_{t-s}^*[G](z)] \\ &= \mathbb{E}^{(4)}[F(\mathfrak{X}_s)] \int_V \pi(dz) P_{t-s}^*[G](z) \\ &= \mathbb{E}^{(3)}[F(\mathfrak{X}_s)] \pi[G] \end{aligned}$$

where we used (62) in the third equality and the invariance of π for the semi-group P^* in the last equality. Considering $F \equiv 1$ in the above computation, we also get that $\pi[G] = \mathbb{E}^{(3)}[G(\psi_{s,t}(Z))]$, so that finally

$$\mathbb{E}^{(3)}[F(\mathfrak{X}_s)G(\psi_{s,t}(Z))] = \mathbb{E}^{(3)}[F(\mathfrak{X}_s)] \mathbb{E}^{(3)}[G(\psi_{s,t}(Z))]$$

and the wanted independence of \mathfrak{X}_s and $\psi_{s,t}(Z)$.

Note that the construction of $\psi_{s,t}(Z)$ depends only on Z and $\psi_{\Delta_{[s,+\infty)} \times V}$, so it follows from (61) that $\psi_{s,t}(Z)$ is in fact also independent from \mathcal{G}_s . ■

A more interesting enlargement of $\Omega^{(4)}$ is

$$\Omega'' := \Omega^{(4)} \times V^{[0,+\infty)}$$

The canonical coordinates on $\mathcal{C}([0, +\infty), V)$ are denoted by the process $Y := (Y_t)_{t \geq 0}$. For fixed $t \geq 0$, let \mathcal{H}_t be the sigma-field generated by \mathcal{G}_t and $Y_{[0,t]}$. We consider the probability \mathbb{P}''_t on $(\Omega'', \mathcal{H}_t)$ which is the image of $\mathbb{P}^{(3)}$ by the mapping

$$\Omega^{(3)} \ni (\mathfrak{X}_0, \psi, Z) \mapsto (\mathfrak{X}_0, \psi, (\psi_{s,t}(Z))_{s \in [0,t]})$$

Lemma 21 insures that the probability spaces $(\Omega'', \mathcal{H}_t, \mathbb{P}''_t)$, for $t \geq 0$, satisfy the Kolmogorov compatibility criterion. We get there exists a probability \mathbb{P}'' on Ω'' endowed with its natural sigma-field, so that \mathbb{P}'' coincides with \mathbb{P}''_t on \mathcal{H}_t , for any $t \geq 0$. Note furthermore that under \mathbb{P}'' , the process Y is a stationary diffusion associated to the semi-group P starting with Y_0 distributed according to π and that we have

$$\forall (s, t) \in \Delta, \quad \mathbb{P}''[Y_s = \psi_{s,t}(Y_t)] = 1 \quad (64)$$

For any measurable functional F defined on Ω'' , which is either bounded or non-negative, we are interested in the conditional expectation, $\mathbb{E}''[F | \mathfrak{X}_s, Y_{[s,t]}]$, of F knowing the sigma-field generated by \mathfrak{X}_s and the Y_u , for $u \in [s, t]$. We denote $\mathbb{E}''_{S, x_{[s,t]}}[F] := \mathbb{E}''[F | \mathfrak{X}_s = S, Y_{[s,t]} = x_{[s,t]}]$, keeping in mind that it is only defined a.s. with respect to a set $S \in \mathfrak{Y}$ and a trajectory $x_{[s,t]} \in V^{[s,t]}$, distributed according to the law of $(\mathfrak{X}_s, Y_{[s,t]})$. Recall from Lemma 21 that \mathfrak{X}_s and $Y_{[s,t]}$ are independent and that the law of the latter is that of a stationary diffusion. Since neither Ω'' nor $V^{[s,t]}$ are Polish spaces, we cannot represent the above conditional expectation via integration w.r.t. a Markov kernel from $\mathfrak{Y} \times V^{[s,t]}$ to Ω'' .

Here is another compatibility consequence of complete association that will be important in the sequel.

Lemma 22 *Assume the \mathfrak{Y} -stochastic flow ψ is completely associated to P^* . Then for any $0 \leq s < t$, any $u \geq 0$ and any bounded and \mathcal{G}_t -measurable functional F , we have*

$$\mathbb{E}''_{S, x_{[s,t+u]}}[F] = \mathbb{E}''_{S, x_{[s,t]}}[F]$$

where the equality holds a.s. with respect to $S \in \mathfrak{Y}$ and $x_{[s,t+u]} \in V^{[s,t+u]}$ independently distributed according to the law of \mathfrak{X}_s and to a stationary $X_{[s,t+u]}$.

Proof

To get the above a.s. identity, it is sufficient to show that for any bounded and \mathcal{G}_t -measurable functional F , for any bounded and measurable functions $G_1 : \mathfrak{Y} \rightarrow \mathbb{R}$, $G_2 : V^{[s,t]} \rightarrow \mathbb{R}$ and $G_3 : V^{[t,t+u]} \rightarrow \mathbb{R}$, we have

$$\mathbb{E}''[FG_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] = \mathbb{E}''[\mathbb{E}''_{\mathfrak{X}_s, Y_{[s,t]}}[F]G_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] \quad (65)$$

So let us start with the l.h.s. and condition it by \mathcal{G}_t :

$$\begin{aligned} \mathbb{E}''[FG_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] &= \mathbb{E}''[FG_1(\mathfrak{X}_s)\mathbb{E}''[G_2(Y_{[s,t]})G_3(Y_{[t,t+u]}) | \mathcal{G}_t]] \\ &= \mathbb{E}''[FG_1(\mathfrak{X}_s)\mathbb{E}''[G_2((\psi_{v,t}(Y_t))_{v \in [s,t]})G_3(Y_{[t,t+u]}) | \mathcal{G}_t]] \end{aligned}$$

Note that under the conditioning by \mathcal{G}_t , the mappings $(\psi_{v,t}(\cdot))_{v \in [s,t]}$ are fixed and that $(Y_{t+v})_{v \in [0,u]}$ is a stationary diffusion associated to P . It follows that

$$\mathbb{E}''[G_2((\psi_{v,t}(Y_t))_{v \in [s,t]})G_3(Y_{[t,t+u]}) | \mathcal{G}_t] = \int G_2((\psi_{v,t}(x))_{v \in [s,t]})\mathbb{E}[G_3((X_{v-t}(x))_{v \in [t,t+u]})] \pi(dx)$$

where $X(x)$ is a diffusion associated to P starting from x .

It leads us to introduce the measurable mapping

$$\begin{aligned} H : V^{[s,t]} &\rightarrow \mathbb{R} \\ y_{[s,t]} &\mapsto G_2(y_{[s,t]})\mathbb{E}[G_3((X_{v-t}(y_t))_{v \in [t,t+u]})] \end{aligned}$$

since we can write, once again taking into account the independence property of Lemma 21,

$$\begin{aligned} \mathbb{E}''[FG_1(\mathfrak{X}_s)G_2(Y_{[s,t]})G_3(Y_{[t,t+u]})] &= \int \mathbb{E}''[FG_1(\mathfrak{X}_s)H((\psi_{v,t}(x))_{v \in [s,t]})] \pi(dx) \\ &= \mathbb{E}''[FG_1(\mathfrak{X}_s)H((\psi_{v,t}(Y_t))_{v \in [s,t]})] \\ &= \mathbb{E}''[FG_1(\mathfrak{X}_s)H(Y_{[s,t]})] \\ &= \mathbb{E}''[\mathbb{E}''_{\mathfrak{X}_s, Y_{[s,t]}}[F]G_1(\mathfrak{X}_s)H(Y_{[s,t]})] \end{aligned}$$

To get (65), it remains to reverse the above computations, or more precisely, to apply them with $F \equiv 1$ and the mapping $\mathfrak{V} \times V^{[s,t]} \ni (S, y_{[s,t]}) \mapsto G_1(S)G_2(y_{[s,t]})$ replaced by

$$\mathfrak{V} \times V^{[s,t]} \ni (S, y_{[s,t]}) \mapsto G_1(S)G_2(y_{[s,t]})\mathbb{E}''_{S, y_{[s,t]}}[F]$$

■

Given a full trajectory $x_{[0,+\infty)} \in V^{[0,+\infty)}$ and a set $S \in \mathfrak{V}$, we can consider conditional expectations $\mathbb{E}_{S, x_{[0,+\infty)}}[\cdot]$ as before Lemma 22, corresponding to the conditioning by $\mathfrak{X}_0 = S$ and $Y = x_{[0,+\infty)}$. The conditional expectations $\mathbb{E}_{S, x_{[0,+\infty)}}[\cdot]$ are only defined a.s. with respect to $(S, x_{[0,+\infty)})$ distributed according to the tensor product of the law of \mathfrak{X}_0 and the stationary law of X . These conditionings are the time-continuous version of the conditioning of the random mappings by the Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ encountered in the previous sections. As we have done before, when working under $\mathbb{E}_{S, x_{[0,+\infty)}}[\cdot]$ or under its integrated version \mathbb{P} defined in the next paragraph, the flow $\psi := (\psi_{s,t})_{0 \leq s \leq t}$ will be denoted $\varphi := (\varphi_{s,t})_{0 \leq s \leq t}$ (to avoid the error of thinking that φ is completely associated to P^*).

Assume that \mathfrak{X}_0 is deterministic and let $X := (X_t)_{t \geq 0}$ be a diffusion process associated to P starting from $\Lambda(\mathfrak{X}_0, \cdot)$, say defined on a probability space (Ω', \mathbb{P}') . We endow $\Omega := \Omega' \times \Omega''$ with the probability measure \mathbb{P} whose marginal distribution on Ω' is \mathbb{P}' and whose conditional distribution on Ω'' knowing the coordinate X on Ω' is $\mathbb{P}_{\mathfrak{X}_0, X}[\cdot]$. This is well-defined through expectations, since the law of X is then absolutely continuous with respect to the stationary law of a diffusion associated to P (the Radon-Nikodym density being $\mathbb{1}_{\mathfrak{X}_0}(X_0)/\pi(\mathfrak{X}_0)$). More generally, this construction has to be integrated with respect to the law of \mathfrak{X}_0 . This is possible when the initial law of X has the form $\int_{\mathfrak{V}} \Lambda(S, \cdot) \mu(dS)$, where μ is a probability measure on \mathfrak{V} . In this case the law of (X_0, \mathfrak{X}_0) belongs to \mathcal{A} , namely is of the form described in (47), or equivalently, we have

$$\mathcal{L}(X_0 | \mathfrak{X}_0) = \Lambda(\mathfrak{X}_0, \cdot) \quad (66)$$

When \mathbb{P} is constructed as above starting with $\mathbb{P}^{(5)}$, a stochastic flow completely associated to P^* , we say that \mathbb{P} is a **(P^*, \mathfrak{V}) -conditioned stochastic flow**. We deduce from Lemma 22 that under such a probability, the analogue of (8) is satisfied:

Lemma 23 *Under a (P^*, \mathfrak{V}) -conditioned stochastic flow \mathbb{P} , we have*

$$\forall t \geq 0, \quad \mathcal{L}(\mathfrak{X}_{[0,t]} | X) = \mathcal{L}(\mathfrak{X}_{[0,t]} | X_{[0,t]})$$

Proof

Let us first compute the conditional expectation under \mathbb{P} knowing X . Consider a bounded and \mathcal{G}_∞ -measurable functional F (i.e. F is measurable with respect to \mathfrak{X}_0 and φ), as well as a bounded and measurable mapping $G : V^{[0,+\infty)} \rightarrow \mathbb{R}$. Denote μ the law of \mathfrak{X}_0 . By definition, we have

$$\mathbb{E}[FG(X)] = \int_{\mathfrak{V} \times V} \mu(dS) \Lambda(S, dx) \mathbb{E}'_x[G(X) \mathbb{E}''_{S, X}[F]]$$

where under \mathbb{P}'_x , X starts from $x \in V$ and is associated to the semi-group P . The previous r.h.s. can be written under the following form, with $\mathfrak{A}_x := \{S \in \mathfrak{A} : x \in S\}$

$$\int_V \pi(dx) \int_{\mathfrak{A}_x} \mu(dS) \frac{\mathbb{E}'_x[G(X)\mathbb{E}''_{S,X}[F]]}{\pi(S)}$$

Since the distribution of X_0 admits

$$V \ni x \mapsto \int_{\mathfrak{A}_x} \mu(dS) \frac{1}{\pi(S)}$$

as density with respect to π , we get that the conditional expectation of F knowing $X = x_{[0,+\infty)} \in V^{[0,+\infty)}$ is given by

$$\mathbb{E}[F|X = x_{[0,+\infty)}] = \frac{\int_{\mathfrak{A}_{x_0}} \mu(dS) \mathbb{E}''_{S,x_{[0,+\infty)}}[F]}{\int_{\mathfrak{A}_{x_0}} \mu(dS) \frac{1}{\pi(S)}} \quad (67)$$

Note that by the martingale convergence theorem, we have

$$\lim_{s \rightarrow +\infty} \mathbb{E}''_{S,X_{[0,s]}}[F] = \mathbb{E}''_{S,X_{[0,+\infty)}}[F]$$

(a priori a.s. with respect to a stationary X , but equally under \mathbb{P} , by absolute continuity). Fix $t \geq 0$ and assume now that F is furthermore \mathcal{G}_t -measurable. We deduce from Lemma 22 that

$$\mathbb{E}''_{S,X_{[0,+\infty)}}[F] = \mathbb{E}''_{S,X_{[0,t]}}[F]$$

so that

$$\mathbb{E}[F|X = x_{[0,+\infty)}] = \frac{\int_{\mathfrak{A}_{x_0}} \mu(dS) \mathbb{E}''_{S,x_{[0,t]}}[F]}{\int_{\mathfrak{A}_{x_0}} \mu(dS) \frac{1}{\pi(S)}}$$

In particular, the l.h.s. only depends on $x_{[0,t]}$. By the tower property of conditional expectation, we get that

$$\mathbb{E}[F|X_{[0,\infty)}] = \mathbb{E}[F|X_{[0,t]}]$$

Since this is true for any \mathcal{G}_t -measurable F , we deduce the wanted equality. ■

To go further in the description of \mathbb{P} , especially to show that the analogue of (9) equally holds, we try to come back to the setting of the previous section by considering discrete-time skeletons.

More precisely, for $\epsilon > 0$, consider the Markov kernel P_ϵ from V to V . Associated Markov chains are the ϵ -skeleton $X^{(\epsilon)} := (X_{\epsilon n})_{n \in \mathbb{Z}_+}$. For $n \in \mathbb{Z}_+$, let $\psi_{n,\mathfrak{X}_n}^{(\epsilon)}$ be $\psi_{\epsilon n, \epsilon(n+1)}$ conditioned by $\mathcal{G}_{\epsilon n}$. The family $(\psi_{n,S}^{(\epsilon)})_{S \in \mathfrak{A}}$ has to be understood in the following sense: let F be a bounded or positive measurable function on \mathfrak{A} , by definition, the expectation of $F(\psi_{n,S}^{(\epsilon)})$ is equal to $\mathbb{E}''[F(\psi_{\epsilon n, \epsilon(n+1)}) | \mathfrak{X}_{\epsilon n} = S]$, a.s. in S distributed according to the law of $\mathfrak{X}_{\epsilon n}$. Here our notations may be slightly confusing, so let us review the construction considered in Section 5, taking into account Remark 18, since the family $(\psi_{n,S}^{(\epsilon)})_{n \in \mathbb{Z}_+, S \in \mathfrak{A}}$ is inhomogeneous with respect to the time $n \in \mathbb{Z}_+$. Starting from $\mathfrak{X}_0^{(\epsilon)} := \mathfrak{X}_0$, we define

$$\begin{aligned} \mathfrak{X}_1^{(\epsilon)} &:= (\psi_{0,\mathfrak{X}_0}^{(\epsilon)})^{-1}(\mathfrak{X}_0) \\ &= (\psi_{0,\epsilon})^{-1}(\mathfrak{X}_0) \\ &= \mathfrak{X}_\epsilon \end{aligned}$$

What is important is that the law of $\mathfrak{X}_1^{(\epsilon)}$ knowing $\mathfrak{X}_0^{(\epsilon)}$ is exactly the law of \mathfrak{X}_ϵ knowing \mathfrak{X}_0 , so that the law of $(\mathfrak{X}_0^{(\epsilon)}, \mathfrak{X}_1^{(\epsilon)})$ is equal to the law of $(\mathfrak{X}_0, \mathfrak{X}_\epsilon)$. Construct $\mathfrak{X}^{(\epsilon)} := (\mathfrak{X}_n^{(\epsilon)})_{n \in \mathbb{Z}_+}$ via the induction

$$\forall n \in \mathbb{Z}_+, \quad \mathfrak{X}_{n+1}^{(\epsilon)} := (\psi_{n, \mathfrak{X}_n}^{(\epsilon)})^{-1}(\mathfrak{X}_n^{(\epsilon)})$$

By iteration of the above argument and taking into account (61), we end up with $\mathfrak{X}^{(\epsilon)}$ having the same law as $(\mathfrak{X}_{\epsilon n})_{n \in \mathbb{Z}_+}$. We will identify $\mathfrak{X}^{(\epsilon)}$ with $(\mathfrak{X}_{\epsilon n})_{n \in \mathbb{Z}_+}$, since above all we are interested in their law, to be conditioned below. Besides, our previous assumptions on ψ insure that for any $n \in \mathbb{Z}_+$, the random mapping $\psi_{n, S}^{(\epsilon)}$ is, a.s. in S with respect to the law of $\mathfrak{X}_{\epsilon n}$, a \mathfrak{V} -random mapping completely associated to P_ϵ^* . Assumption (H1) was assumed at the beginning of this section and (H5) implies (H2) for the discrete-time random mappings $\psi_{n, S}^{(\epsilon)}$, for $n \in \mathbb{Z}_+$ and $S \in \mathfrak{V}$. To go in the direction of (H3), let us define, for any $0 \leq s < t$, $x, x' \in V$ and $S \in \mathfrak{V}$, $\mathbb{E}_{S; s, x; t, x'}''[\cdot]$ the conditional expectation under \mathbb{P}'' knowing that $\mathfrak{X}_s = S$, $Y_s = x$ and $Y_t = x'$. Next, let $K((S; s, x; t, x'), \cdot)$ be the image of $\mathbb{P}_{S; s, x; t, x'}''[\cdot]$ by the mapping $\psi \mapsto \psi_{s, t}^{-1}(S)$. For fixed $0 \leq s < t$, we can see it as a Markov kernel from $\mathfrak{V} \times V \times V$ to \mathfrak{V} , since it corresponds to the conditioning of \mathfrak{X}_t by $(\mathfrak{X}_s, X_s, X_t)$ and we can work on the Polish space $\mathfrak{V} \times V \times V \times \mathfrak{V}$ endowed with the law of $(\mathfrak{X}_s, X_s, X_t, \mathfrak{X}_t)$. When $s = \epsilon n$ and $t = \epsilon(n+1)$, $K((S; \epsilon n, x; \epsilon(n+1), x'), \cdot)$ can almost play the role of $K(x, x', S, \cdot)$ in Condition (H3) for the discrete-time random mapping $\psi_{n, S}^{(\epsilon)}$, except that $K((S; \epsilon n, x; \epsilon(n+1), x'), \cdot)$ is maybe not a Markov kernel corresponding to the conditional distribution of $(\psi_{n, S}^{(\epsilon)})^{-1}(S)$ knowing $\psi_{n, S}^{(\epsilon)}(x') = x$, for all fixed $S \in \mathfrak{V}$ and $x' \in S$ (and a.s. in x with respect to the law of $\psi_{n, S}^{(\epsilon)}(x')$), but *only* a.s. for S distributed as $\mathfrak{X}_{\epsilon n}$ and for (x, x') distributed (independently) according to $\pi(dx)P_\epsilon(x, dx') = \pi(dx')P_\epsilon(x', dx)$. Nevertheless, this extension of Condition (H3) is sufficient for the validity of Theorem 14, as it is checked by a direct examination of its proof. Indeed, (51) and (52) have now to be understood a.s., respectively w.r.t. the law of $\mathfrak{X}_{\epsilon n}$ and in x w.r.t. π .

Remark 24 The above construction of the process $(Y_t)_{t \geq 0}$ could also be performed in the discrete-time setting of Section 5 for completely associated to P^* random mappings to get a stationary chain $(Y_n)_{n \in \mathbb{Z}_+}$ associated to the transition kernel P and satisfying the analogue of Property (64). In the restricted setting of complete association, it leads to a variant of Theorem 14 where Condition (H3) has been removed and replaced by the hypothesis that the random mappings belongs to some Polish functional space \mathfrak{R} . It would be interesting to get similar constructions in the context of weak association. □

These considerations lead to the following continuous-time extension of (9):

Theorem 25 *Under a (P^*, \mathfrak{V}) -conditioned stochastic flow \mathbb{P} , we have*

$$\forall t \geq 0, \quad \mathcal{L}(X_t | \mathfrak{X}_{[0, t]}) = \Lambda(\mathfrak{X}_t, \cdot) \quad (68)$$

Proof

According to the above observations, we are in position to apply Theorem 14 to the time-inhomogeneous random mappings induced by the family $(\psi_{n, S}^{(\epsilon)})_{n \in \mathbb{Z}_+, S \in \mathfrak{V}}$. More precisely, let us come back to the kernel described by $K((S; \epsilon n, x; \epsilon(n+1), x'), \cdot)$, for $S \in \mathfrak{V}$, $n \in \mathbb{Z}_+$ and $x, x' \in V$. The tower property of conditional expectation implies that

$$\begin{aligned} & \mathbb{E}_{S; \epsilon n, x; \epsilon(n+1), x'}''[\cdot] \\ &= \int_{\mathcal{C}([\epsilon n, \epsilon(n+1)], V)} \mathbb{E}_{S, x_{[\epsilon n, \epsilon(n+1)]}}''[\cdot] \mathbb{P}^{\epsilon n, x; \epsilon(n+1), x'}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned} \quad (69)$$

where $\mathbb{P}^{\epsilon n, x; \epsilon(n+1), x'}$ is the law of the bridge $X_{[\epsilon n, \epsilon(n+1)]}$ associated to P and conditioned by $X_{\epsilon n} = x$ and $X_{\epsilon(n+1)} = x'$. Here we don't need the results of Fitzsimmons, Pitman and Yor [9] for the

existence for such bridge laws, since we just require their existence $\pi(dx)P_\epsilon(x, dx')$ -a.s. and not for every $x, x' \in V$.

Let $\tilde{X}^{(\epsilon)} := (\tilde{X}_n^{(\epsilon)})_{n \in \mathbb{Z}_+}$ be a Markov chain with transition kernel P_ϵ and starting from $\Lambda(\mathfrak{X}_0, \cdot)$, first assuming that \mathfrak{X}_0 is deterministic. As in Section 5, let $\tilde{\mathfrak{X}}^{(\epsilon)} := (\tilde{\mathfrak{X}}_n^{(\epsilon)})_{n \in \mathbb{Z}_+}$ be obtained from $(\psi_{n,S}^{(\epsilon)})_{n \in \mathbb{Z}_+, S \in \mathfrak{W}}$ through its iterative conditionings:

$$\forall n \in \mathbb{Z}_+, \quad \psi_{n, \tilde{\mathfrak{X}}_n^{(\epsilon)}}^{(\epsilon)}(\tilde{X}_{n+1}^{(\epsilon)}) = \tilde{X}_n^{(\epsilon)}$$

The law of $(\tilde{X}^{(\epsilon)}, \tilde{\mathfrak{X}}^{(\epsilon)})$ coincides with the law of $(X^{(\epsilon)}, \mathfrak{X}^{(\epsilon)})$. Indeed, by iteration, it is sufficient to see that for any $n \in \mathbb{Z}_+$, the conditional law of $(\tilde{X}_{n+1}^{(\epsilon)}, \tilde{\mathfrak{X}}_{n+1}^{(\epsilon)})$ knowing $(\tilde{X}_m^{(\epsilon)}, \tilde{\mathfrak{X}}_m^{(\epsilon)})_{m \in [0, n]}$ is equal to the law of $(X_{\epsilon(n+1)}, \mathfrak{X}_{\epsilon(n+1)})$ knowing $(X_{\epsilon m}, \mathfrak{X}_{\epsilon m})_{m \in [0, n]}$. On one hand, by definition, by taking into account (69) and with the kernel K defined above Remark 24, we have

$$\begin{aligned} & \mathbb{P}[(\tilde{X}_{n+1}^{(\epsilon)}, \tilde{\mathfrak{X}}_{n+1}^{(\epsilon)}) \in (dx, dS) | (\tilde{X}_m^{(\epsilon)}, \tilde{\mathfrak{X}}_m^{(\epsilon)})_{m \in [0, n]}] \\ &= P_\epsilon(\tilde{X}_n^{(\epsilon)}, dx) K(\tilde{\mathfrak{X}}_n^{(\epsilon)}; \epsilon n, \tilde{X}_n^{(\epsilon)}; \epsilon(n+1), x), dS) \\ &= P_\epsilon(\tilde{X}_n^{(\epsilon)}, dx) \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)]], V)} K((\tilde{X}_n^{(\epsilon)}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \mathbb{P}_{\epsilon n, \tilde{\mathfrak{X}}_n^{(\epsilon)}; \epsilon(n+1), x}(dx_{[\epsilon n, \epsilon(n+1)]}) \\ &= \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)]], V)} K((\tilde{X}_n^{(\epsilon)}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \delta_{x_{\epsilon(n+1)}}(dx) \mathbb{P}_{\epsilon n, \tilde{\mathfrak{X}}_n^{(\epsilon)}}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned}$$

where

- $K((S; x_{[\epsilon n, \epsilon(n+1)]}), \cdot)$ is the image of $\mathbb{P}_{S; x_{[\epsilon n, \epsilon(n+1)]}}''[\cdot]$ by the mapping $\psi \mapsto \psi_{\epsilon n, \epsilon(n+1)}^{-1}(S)$,
- $\mathbb{P}_{\epsilon n, x'}(\cdot)$ is the law of $X_{[\epsilon n, \epsilon(n+1)]}$ associated to P , starting at time ϵn from $x' \in V$.

On the other hand, with arguments similar to those of Lemmas 22 and 23, we check that

$$\begin{aligned} & \mathbb{P}[(X_{\epsilon(n+1)}, \mathfrak{X}_{\epsilon(n+1)}) \in (dx, dS) | X_{[0, \epsilon n]}, \mathfrak{X}_{[0, \epsilon n]}] \\ &= \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)]], V)} K((X_{[0, \epsilon n]}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \delta_{x_{\epsilon(n+1)}}(dx) \mathbb{P}_{\epsilon n, X_{\epsilon n}}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned} \tag{70}$$

It follows that

$$\begin{aligned} & \mathbb{P}[(X_{\epsilon(n+1)}, \mathfrak{X}_{\epsilon(n+1)}) \in (dx, dS) | (X_{\epsilon m}, \mathfrak{X}_{\epsilon m})_{m \in [0, n]}] \\ &= \int_{\mathcal{C}([[\epsilon n, \epsilon(n+1)]], V)} K((X_{[0, \epsilon n]}, x_{[\epsilon n, \epsilon(n+1)]}), dS) \delta_{x_{\epsilon(n+1)}}(dx) \mathbb{P}_{\epsilon n, X_{\epsilon n}}(dx_{[\epsilon n, \epsilon(n+1)]}) \end{aligned}$$

and this ends the proof of the equality in law of $(\tilde{X}^{(\epsilon)}, \tilde{\mathfrak{X}}^{(\epsilon)})$ and $(X^{(\epsilon)}, \mathfrak{X}^{(\epsilon)})$. At least when \mathfrak{X}_0 is deterministic, but this identity in law is next extended by integration with respect to $\mathcal{L}(\mathfrak{X}_0)$.

Taking into account Remark 18, Theorem 14 now implies that

$$\forall n \in \mathbb{Z}_+, \quad \mathcal{L}(X_{\epsilon n} | \mathfrak{X}_0, \mathfrak{X}_\epsilon, \dots, \mathfrak{X}_{\epsilon n}) = \Lambda(\mathfrak{X}_{\epsilon n}, \cdot) \tag{71}$$

If we had some time regularity for the process \mathfrak{X} , the announced result would follow by usual approximations. To go further, let us remark that in the above arguments, the time mesh $0, \epsilon, 2\epsilon, \dots, n\epsilon$ can be replaced by any finite sequence $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n$. The Markov chain $(X_{t_0}, X_{t_1}, X_{t_2}, \dots, X_{t_n})$ is no longer time homogeneous, but as pointed out in Remark 18, this is not crucial in the deduction that

$$\mathcal{L}(X_{t_n} | \mathfrak{X}_{t_0}, \mathfrak{X}_{t_1}, \mathfrak{X}_{t_2}, \dots, \mathfrak{X}_{t_n}) = \Lambda(\mathfrak{X}_{t_n}, \cdot) \tag{72}$$

Now fix $t \geq 0$ and consider two bounded and measurable functions $F : V \rightarrow \mathbb{R}$ and $G : \mathfrak{X}^{[0,t]} \rightarrow \mathbb{R}$. We want to show that

$$\mathbb{E}[F(X_t)G(\mathfrak{X}_{[0,t]})] = \mathbb{E}[\Lambda[F](\mathfrak{X}_t)G(\mathfrak{X}_{[0,t]})] \quad (73)$$

Due to the product measurable structure of $\mathfrak{X}^{[0,t]}$, there exists a sequence $(s_n)_{n \in \mathbb{Z}_+}$ of distinct elements from $[0, t]$ such that $G(\mathfrak{X}_{[0,t]})$ only depends on the values $(\mathfrak{X}_{s_n})_{n \in \mathbb{Z}_+}$.

For $N \in \mathbb{N}$, consider

$$G_N((\mathfrak{X}_{s_n})_{n \in [0, N]}) := \mathbb{E}[G(\mathfrak{X}_{[0,t]}) | (\mathfrak{X}_{s_n})_{n \in [0, N]}]$$

The martingale convergence theorem implies that $G_N((\mathfrak{X}_{s_n})_{n \in [0, N]})$ converges for large N toward $G(\mathfrak{X}_{[0,t]})$ in $\mathbb{L}^2(\mathbb{P})$, so to prove (73), it is sufficient to see that

$$\mathbb{E}[F(X_t)G_N((\mathfrak{X}_{s_n})_{n \in [0, N]})] = \mathbb{E}[\Lambda[F](\mathfrak{X}_t)G_N((\mathfrak{X}_{s_n})_{n \in [0, N]})] \quad (74)$$

Let $t_0 \leq t_1 \leq t_2 \leq t_3 \leq \dots \leq t_{N+1}$ be the ordering of the elements $s_0, s_1, s_2, \dots, s_N, t$, (72) implies (74) via the conditioning by the sigma-field generated by $(\mathfrak{X}_{s_n})_{n \in [0, N]}$. ■

Other parts of Theorem 14 can be extended to the continuous-time framework: of course under \mathbb{P} , X remains a diffusion process associated to the semi-group P . The process $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$ is Markovian, but in general it will no longer be time-homogenous. The associated semi-group $\mathfrak{P} := (\mathfrak{P}_{s,t})_{0 \leq s \leq t}$ is given by

$$\forall 0 \leq s \leq t, \forall S \in \mathfrak{X}, \quad \begin{cases} \mathfrak{P}_{s,t}(S, dS') & := \frac{\pi(S')}{\pi(S)} K_{s,t}(S, dS') \\ K_{s,t}(S, dS') & := \mathbb{P}^{(4)}[\mathfrak{X}_t \in dS' | \mathfrak{X}_s = S] \end{cases}$$

Furthermore, the process $(X_t, \mathfrak{X}_t)_{t \geq 0}$ is Markovian, indeed, a version of this property was used in (70). Concerning the set W defined in (48), we get that for any sequence $(t_n)_{n \in \mathbb{Z}_+}$ of non-negative times, we have

$$\mathbb{P}[\forall n \in \mathbb{Z}_+, (X_{t_n}, \mathfrak{X}_{t_n}) \in W] = 1 \quad (75)$$

To go further and deduce that W can be taken as state space of (X, \mathfrak{X}) under \mathbb{P} , we would need further regularity conditions, e.g. that there is version of \mathfrak{X} which is continuous and W is closed.

As in Theorem 14, the initial \mathfrak{X}_0 in Theorem 25 have positive weights with respect to π . We cannot go around this drawback by following the approach leading to Theorem 16, since we did not investigate the weak association in the continuous time setting. Instead of trying to develop an alternative general approach to get a set-valued dual process starting from a singleton, we refer to the particular case presented in Subsection 7.1.

7 One-dimensional diffusion processes

As already alluded to, we would like to apply the analysis of the previous section to diffusion processes. Unfortunately and despite the works of Le Jan and Raimond [11, 12, 13, 14], the theory of stochastic flows has not been developed in the direction needed by our purposes. The next subsection presents a treatment of the Brownian case starting from 0 based on a stochastic flow due to Le Jan and Raimond [13], it leads to a segment-valued dual process directly coupled with the primal Brownian motion through its local time at 0. It turns out to be equivalent to the classical Pitman's theorem [20], which is thus recovered in this way. In the second subsection, we show how to extend this analysis to simple one-dimensional diffusions, if we had at our disposal nice stochastic flows. Subsection 7.2 ends by a conjecture about the existence of the strange stochastic flows we would like to use, as a first step for one-dimensional processes. In Appendix A, we will take advantage of classical transformations of the state space to transfer the considerations of Subsection 7.2 to more general elliptic one-dimensional diffusions.

7.1 The Brownian motion case

Here we give a first illustration of how to put in practice the abstract considerations of the previous section.

Let $X := (X_s)_{s \geq 0}$ be a Brownian motion and let $P := (P_s)_{s \geq 0}$ be the associated semi-group, whose generator is half the Laplacian $\partial^2/2$. Note that the Lebesgue measure λ is reversible for P , we also interpret P as a self-adjoint semi-group on $L^2(\lambda)$, so that $P^* = P$.

A priori we are not in the framework of Section 6, since λ cannot be renormalized into a probability measure. Nevertheless Theorem 25 can be extended to this situation when the elements of \mathfrak{X} are furthermore assumed to have a finite weight under λ . Indeed, in (63), $\mathbb{P}^{(3)}$ must be replaced by the sigma-finite measure $\mathbb{P}^{(4)} \otimes \lambda$ and the following expectations have to be changed into integrations with respect to the corresponding measures (the test functions have to be assumed to be integrable or non-negative). We end up with a probability measure \mathbb{P} , since in its construction, \mathbb{P}'' is conditioned by the event $\{X_0 \in \mathfrak{X}_0\}$, which has the finite weight $\lambda(\mathfrak{X}_0) \in (0, +\infty)$.

Let $B := (B_s)_{s \geq 0}$ be another Brownian motion, it is not important to specify its initial condition, since only the associated white noise will be needed. For fixed $t \geq 0$, we consider the following system of equations, for all $y \in \mathbb{R}$,

$$\begin{cases} dY_s^{(t)}(y) &= -\text{sgn}(Y_s^{(t)}(y))dB_s^{(t)}, & \forall s \in [0, t] \\ Y_0^{(t)}(y) &= y \end{cases} \quad (76)$$

where sgn is the sign function on \mathbb{R} taking the value -1 on $(-\infty, 0]$ and 1 on $(0, +\infty)$ and where $B^{(t)} := (B_s^{(t)})_{s \in [0, t]} := (B_{t-s})_{s \in [0, t]}$ is the time-reversed process associated to B at time $t \geq 0$.

Le Jan and Raimond [13] provide a coalescing stochastic flow solution to (76), but it is non-Wiener, meaning that extra-randomness, in addition to the Brownian motion B , is necessary to its construction. For the general meaning of a (Wiener) coalescing stochastic flow solution, we refer to Le Jan and Raimond [12]. Define $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$ via

$$\forall x \in \mathbb{R}, \forall 0 \leq s \leq t, \quad \psi_{s,t}(y) := Y_{t-s}^{(t)}(y) \quad (77)$$

(be careful that our convention for the direction of the time is reversed with respect to that of Le Jan and Raimond [12]: $\psi_{s,t}$ should be seen as acting on the state space at time t toward the state space at time s). The stochastic flow ψ is associated to P , as an immediate consequence of (76) and of Lévy's characterization of the Brownian motion. Since the state space is the real line, the regularity of ψ can be made more precise, bringing us back to the assumptions of Section 6. Let $\mathcal{R}(\mathbb{R})$ stands for the set of mapping from \mathbb{R} to \mathbb{R} which are non-decreasing and càdlàg.

Lemma 26 *There exists a version of ψ which is such that a.s., for any $(s, t) \in \Delta$, $\psi_{s,t} \in \mathcal{R}(\mathbb{R})$.*

Proof

Due to the temporal continuity and coalescing property of the solution $Y := (Y_s^{(t)}(x))_{(s,t) \in \Delta, x \in \mathbb{R}}$ of (76), we get that for any given $x \leq y \in \mathbb{R}$, we have $Y_s^{(t)}(x) \leq Y_s^{(t)}(y)$ a.s. for any $(s, t) \in \Delta$. So we can extend $(Y_s^{(t)}(x))_{(s,t) \in \Delta, x \in \mathbb{Q}}$ into $\tilde{Y} := (\tilde{Y}_s^{(t)}(x))_{(s,t) \in \Delta, x \in \mathbb{R}}$ by

$$\forall (s, t) \in \Delta, \forall x \in \mathbb{R}, \quad \tilde{Y}_s^{(t)}(x) := \lim_{y \rightarrow x, y > x, y \in \mathbb{Q}} Y_s^{(t)}(y)$$

Let $\tilde{\psi}$ be obtained from \tilde{Y} as in (77). By construction, \tilde{Y} is such that a.s. for all $(s, t) \in \Delta$, $\tilde{\psi}_{s,t} \in \mathcal{R}(\mathbb{R})$. Furthermore, according to the point (e) in the Definition 1.6 of Le Jan and Raimond [12], we have that for any $(s, t) \in \Delta$ and any $x \in \mathbb{R}$, a.s. $\tilde{Y}_s^{(t)}(x) = Y_s^{(t)}(x)$. It follows that \tilde{Y} is also a solution of (76) and thus $\tilde{\psi}$ provides the wanted version of ψ . ■

Remark 27 The validity of Lemma 26 is not related to the choice of $\text{sgn}(0)$. With a similar proof, one could deduce a version of ψ which is a.s. càglàd. In fact the definition of $\text{sgn}(0)$ is irrelevant, as $\mathbb{E}[\int_0^t \mathbf{1}_{\{0\}}(Y_s^{(t)}(y)) ds] = 0$, for all $y \in \mathbb{R}$. □

From now on, we will only consider a version of ψ as in Lemma 26. As announced in Remark 19, $\mathcal{R}(\mathbb{R})$ is endowed with the Skorohod topology, which insures the measurability of the mapping $\mathbb{R} \times \mathcal{R}(\mathbb{R}) \ni (x, \psi) \mapsto \psi(x)$. To apply the results of the previous section, we must equally specify the space of nice subdomains \mathfrak{V} . As it will become apparent later, it is convenient to consider:

$$\mathfrak{V} := \{[a, b] : a < b \in \mathbb{R}\} \quad (78)$$

This set \mathfrak{V} is endowed with the topology inherited from \mathbb{R}^2 and satisfies the properties required in Section 6. Indeed, let us check the following property, relating Lemma 26 to our choice of \mathfrak{V} .

Lemma 28 *For any $(s, t) \in \Delta$ the mapping*

$$\mathfrak{V} \ni S \mapsto \psi_{s,t}^{-1}(S) \in \mathfrak{V} \sqcup \{\emptyset\} \quad (79)$$

is measurable.

Proof

First, for given $(s, t) \in \Delta$, we verify that for $[a, b] \in \mathfrak{V}$, the set $\psi_{s,t}^{-1}([a, b])$ belongs to $\mathfrak{V} \sqcup \{\emptyset\}$. This is a consequence of $\psi_{s,t} \in \mathcal{R}(\mathbb{R})$: to see that $\psi_{s,t}^{-1}([a, b])$ is a segment, let $x \leq y$ belong to $[a, b]$. For $z \in [x, y]$, we have $a \leq \psi_{s,t}(x) \leq \psi_{s,t}(z) \leq \psi_{s,t}(y) < b$, so that $z \in \psi_{s,t}^{-1}([a, b])$. Next let $(x_n)_{n \in \mathbb{Z}_+}$ be a decreasing family of elements from $\psi_{s,t}^{-1}([a, b])$ converging toward some $x \in \mathbb{R}$. Then we have $\lim_{n \rightarrow \infty} \psi_{s,t}(x_n) = \psi_{s,t}(x)$ and since the l.h.s. belongs to $[a, b]$, we deduce that $x \in \psi_{s,t}^{-1}([a, b])$ and that the segment $\psi_{s,t}^{-1}([a, b])$ is closed on the left side. To see that it is open on the right side, consider $x \in \psi_{s,t}^{-1}([a, b])$. As $y > x$ decreases toward x , we have that $\psi_{s,t}(y)$ converges toward $\psi_{s,t}(x)$, so that for y sufficiently close to x , we have $\psi_{s,t}(x) \leq \psi_{s,t}(y) < a$, i.e. $y \in \psi_{s,t}^{-1}([a, b])$.

It follows from these observations that for any $x < y \in \mathbb{R}$, we have $\psi_{s,t}^{-1}([x, y]) = [a, b]$, with

$$\begin{aligned} a &:= \inf\{u \in \mathbb{R} : \psi_{s,t}(u) \geq x\} \\ b &:= \inf\{u \in \mathbb{R} : \psi_{s,t}(u) \geq y\} \end{aligned}$$

Fix $u < v \in \mathbb{R}$ and consider $A := \{[a, b] : a \leq u, b > v\} \subset \mathfrak{V}$. We compute that

$$\{S \in \mathfrak{V} : \psi_{s,t}^{-1}(S) \in A\} = \{[x, y] \in \mathfrak{V} : x \leq \psi_{s,t}(u) \text{ and } y > \psi_{s,t}(v)\}$$

It follows without difficulty that the mapping defined in (79) is measurable. ■

Note that the Lebesgue measure λ , invariant for the Brownian semi-group P , takes positive and finite values on \mathfrak{V} . As in Subsection 3.1, we get that the corresponding conditioning kernel Λ can be seen as a Markov kernel from \mathfrak{V} to \mathbb{R} .

Fix $t \geq 0$ and a Brownian trajectory $X_{[0,t]}$. Conditioning ψ by the event

$$\forall s \in [0, t], \quad \psi_{s,t}(X_t) = X_s \quad (80)$$

implies in particular that

$$\forall s \in [0, t], \quad dX_s^{(t)} = -\text{sgn}(X_s^{(t)})dB_s^{(t)} \quad (81)$$

but it is not clear what happens to the extra-randomness, since this Tanaka's stochastic differential equation does not admit a strong solution either (see for instance Exercise 1.19 of Chapter 9 from Revuz and Yor [23]). Nevertheless we deduce that

$$\forall s \in [0, t], \quad dB_s^{(t)} = -\text{sgn}(X_s^{(t)})dX_s^{(t)} \quad (82)$$

and it follows that the conditioned flow, denoted $\varphi := (\varphi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$ in Section 6, is given by

$$\forall 0 \leq s \leq t, \forall z \in \mathbb{R}, \quad \varphi_{s,t}(z) := Z_{t-s}^{(t)}(z) \quad (83)$$

where

$$\begin{cases} dZ_s^{(t)}(z) = \text{sgn}(Z_s^{(t)}(z))\text{sgn}(X_s^{(t)})dX_s^{(t)} \\ Z_0^{(t)}(z) = z \end{cases} \quad (84)$$

This system is the same as (76), once we have replaced $B^{(t)}$ by $(-\int_0^s \text{sgn}(X_v^{(t)})dX_v^{(t)})_{s \in [0,t]}$ which is a standard Brownian motion. In particular the resolution of (84) also requires some extra-randomness, but this is not a shortcoming for our present purpose since we just need a solution. Indeed, for any given $r > 0$, assume that the initial law of X is ν_r , the uniform distribution on $[-r, r)$. Define for any $t \geq 0$,

$$\begin{cases} R_t^{\vee,r} := \inf \{z \in \mathbb{R} : Z_0^{(t)}(z) \leq r\} = \sup \{z \in \mathbb{R} : Z_0^{(t)}(z) < r\} \\ R_t^{\wedge,r} := \inf \{z \in \mathbb{R} : Z_0^{(t)}(z) \geq -r\} \\ \mathfrak{X}_t^{(r)} := [R^{\wedge,r}, R^{\vee,r}) = \varphi_{0,t}^{-1}([-r, r)) \end{cases} \quad (85)$$

where for the last equality we took into account the proof of Lemma 28. Remark that $\mathfrak{X}^{(r)} := (\mathfrak{X}_t^{(r)})_{t \geq 0}$ remains non-trivial, namely that it never collapses to a singleton or to the empty set: this is a consequence of the fact that $X_t \in \mathfrak{X}_t$ and of the structure of the elements of \mathfrak{V} . We are now in position to apply Theorem 25 to deduce:

Theorem 29 *For any $r > 0$, the process $\mathfrak{X}^{(r)}$ is a set-valued dual for the Brownian motion X starting from ν_r .*

We would like to let r go to zero to be able to get X starting from 0. It will also provide a more explicit set-valued dual. Indeed, assume that X starts from 0 and consider the coalescing flow φ defined by (83) and (84). By analogy with (85), define the process $\mathfrak{X} := (\mathfrak{X}(t))_{t \geq 0}$ via

$$\forall t \geq 0, \quad \mathfrak{X}(t) := \varphi_{0,t}^{-1}(\{0\}) \quad (86)$$

We still have that for any $t \geq 0$, $\mathfrak{X}(t)$ is a segment closed on the left. We will see in the proof of Proposition 30 that for any $t > 0$, the right boundary of $\mathfrak{X}(t)$ is open (but it is closed at time 0, since $\mathfrak{X}(0) = \{0\}$). Anyway, the closure of $\mathfrak{X}(t)$ is $[R^{\wedge}, R^{\vee}]$ with

$$\begin{aligned} R_t^{\vee} &:= \sup\{z \in \mathbb{R} : Z_0^{(t)}(z) = 0\} \\ R_t^{\wedge} &:= \inf\{z \in \mathbb{R} : Z_0^{(t)}(z) = 0\} \end{aligned}$$

and these quantities can be described explicitly:

Proposition 30 *We have for any $t \geq 0$,*

$$\begin{aligned} R_t^{\vee} &= L_t^0(X) + |X_t| \\ R_t^{\wedge} &= -(L_t^0(X) + |X_t|) \end{aligned}$$

where $L^0(X) := (L_t^0(X))_{t \geq 0}$ is the local time of X at 0.

Proof

Due to the fact that φ is a coalescing flow and that $X_0 = 0$ we have

$$\begin{aligned} R_t^\vee &:= \sup\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = X_s^{(t)}\} \\ R_t^\wedge &:= \inf\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = X_s^{(t)}\} \end{aligned}$$

From (84), we have for any $z \in \mathbb{R}$,

$$\forall s \in [0, t], \quad \text{sgn}(Z_s^{(t)}(z))dZ_s^{(t)}(z) = \text{sgn}(X_s^{(t)})dX_s^{(t)} \quad (87)$$

and Tanaka's formula (see e.g. Chapter 6 of Revuz and Yor [23]) implies that

$$\forall s \in [0, t], \quad d|Z_s^{(t)}(z)| - dL_s^0(Z^{(t)}(z)) = d|X_s^{(t)}| - dL_s^0(X^{(t)}) \quad (88)$$

where $L^0(Z^{(t)}(z)) := (L_s^0(Z^{(t)}(z)))_{s \in [0, t]}$ and $L^0(X^{(t)}) := (L_s^0(X^{(t)}))_{s \in [0, t]}$ are respectively the local times of $Z^{(t)}(z)$ and $X^{(t)}$ at 0.

Fix $z > x := X_0^{(t)} = X_t$, assume that $x \geq 0$ and define

$$\tau_z := \inf\{s \in [0, t] : Z_s^{(t)}(z) = 0\}$$

(with the convention that $\inf(\emptyset) = +\infty$). Consider the case $\tau_z < +\infty$ and let us show that $X_{\tau_z}^{(t)} = 0$. The argument is by contradiction, assuming that $X_{\tau_z}^{(t)} \neq 0$. Define

$$\gamma := \sup\{s \in [0, \tau_z] : X_s^{(t)} = 0\}$$

with the convention that $\gamma = -\infty$ if the set in the r.h.s. is empty. For $s \in [0 \vee \gamma, \tau_z)$, we have $Z_s^{(t)}(z) > 0$, $L_s^0(Z^{(t)}(z)) = 0$ and $L_s^0(X^{(t)}) = L_{0 \vee \gamma}^0(X^{(t)})$. It follows from (88) that

$$\begin{aligned} Z_{0 \vee \gamma}^{(t)}(z) - |X_{0 \vee \gamma}^{(t)}| &= Z_{\tau_z}^{(t)}(z) - |X_{\tau_z}^{(t)}| \\ &= -|X_{\tau_z}^{(t)}| \\ &< 0 \end{aligned}$$

i.e. $Z_{0 \vee \gamma}^{(t)}(z) < |X_{0 \vee \gamma}^{(t)}|$. Note that depending on $\gamma \in [0, t]$ or $\gamma = -\infty$, we have $X_{0 \vee \gamma}^{(t)} = 0$ or $X_{0 \vee \gamma}^{(t)} > 0$ (since when $\gamma = -\infty$, $X^{(t)}$ keeps the same sign on $[0, \tau_z]$, which is the sign of x). In any case, we end up with $Z_{0 \vee \gamma}^{(t)}(z) < X_{0 \vee \gamma}^{(t)}$, which is in contradiction with the fact that $Z^{(t)}(x)$ remains above $X^{(t)}$ by the coalescing property.

Define

$$\sigma_z := \inf\{s \in [0, t] : Z_s^{(t)} = X_s^{(t)}\}$$

Due to the fact that $X_{\tau_z}^{(t)} = 0$ when $\tau_z < +\infty$, we deduce that $\sigma_z \leq \tau_z$. Integrating (88) between the times 0 and σ_z , we thus get

$$\begin{aligned} \sigma_z < +\infty &\Rightarrow |Z_{\sigma_z}^{(t)}| - z = |X_{\sigma_z}^{(t)}| - x - L_{\sigma_z}^0(X^{(t)}) \\ &\Rightarrow z = x + L_{\sigma_z}^0(X^{(t)}) \end{aligned} \quad (89)$$

Since

$$R_t^\vee = \max\{z \geq x : \sigma_z < +\infty\}$$

we get that

$$R_t^\vee = x + L_{\sigma_{R_t^\vee}}^0(X^{(t)}) \quad (90)$$

Let us show that $\sigma := \sigma_{R_t^\vee}$ is equal to t . Again the argument is by contradiction: assume that $\sigma < t$. Define

$$\tau := \inf\{s \in [\sigma, t] : X_s^{(t)} = 0\}$$

Since 0 is an accumulation point of $\{s \in [0, t] : X_s = 0\}$, we get that $\tau < t$. Let us show that $\mathfrak{X}_{t-\tau}$ contains a right neighborhood of 0, namely that for z sufficiently close to 0_+ , we have that $Z^{(t-\tau)}(z)$ coalesces with $X^{(t-\tau)}$. Indeed, if this was not true, according to the first part of the proof above, we would have that $Z^{(t-\tau)}(z)$ does not touch 0 for any $z > 0$. From (84), we deduce that

$$\forall z \in (0, +\infty), \forall s \in [0, \tau], \quad Z_s^{(t-\tau)}(z) = z + \beta_s$$

where $\beta := (\beta_s)_{s \in [0, \tau]}$ is the Brownian motion defined by

$$\forall s \in [0, \tau], \quad \beta_s := \int_0^s \text{sign}(X_u^{(t-\tau)}) dX_u^{(t-\tau)}$$

This shows that $Z^{(t-\tau)}(z)$ touches 0 for any $z \in (0, -\inf_{s \in [0, \tau]} \beta_s]$, a contradiction.

Using the right continuity of our stochastic flows, we get

$$\begin{aligned} \lim_{z \rightarrow R_t^\vee} Z_\tau^{(t)}(z) &= Z_\tau^{(t)}(R_t^\vee) \\ &= X_\tau^{(t)} \\ &= 0 \end{aligned}$$

Thus for z in a right neighborhood of R_t^\vee , we get that $Z_\tau^{(t)}(z)$ belongs to $\mathfrak{X}_{t-\tau}$ and by consequence that $z \in \mathfrak{X}_t$. This is in contradiction with the definition of R_t^\vee and we finally conclude that $\sigma = t$.

Coming back to (90), we get that

$$\begin{aligned} R_t^\vee &= x + L_t^0(X^{(t)}) \\ &= X_t + L_t^0(X) \end{aligned} \tag{91}$$

under the previous assumption that $X_t = x \geq 0$.

When $X_t < 0$, consider

$$\varsigma := \inf\{s \in [0, t] : X_s^{(t)} = 0\}$$

We deduce from (87) that

$$\forall s \in [0, \varsigma], \quad Z_s^{(t)}(-x) = -X_s^{(t)}(x)$$

In particular, we get

$$R_t^\vee = \sup\{z \geq -x : Z_\varsigma^{(t)} \in \mathfrak{X}_{t-\varsigma}\}$$

Note that for all $z \geq -x$, we have

$$\forall s \in [0, \varsigma], \quad Z_s^{(t)}(z) - z = Z_s^{(t)}(-x) - (-x)$$

so that

$$\begin{aligned} R_t^\vee &= -x + R_{t-\varsigma}^\vee \\ &= |X_t| + R_{t-\varsigma}^\vee \end{aligned} \tag{92}$$

Using (91) with t replaced by $t - \varsigma$, we get

$$\begin{aligned} R_{t-\varsigma}^\vee &= X_{t-\varsigma} + L_{t-\varsigma}^0(X) \\ &= 0 + L_t^0(X) \end{aligned}$$

Recalling (92), we conclude to the validity, in all cases, of

$$R_t^\vee = |X_t| + L_t^0(X)$$

By symmetry, the expression for R_t^\wedge is easily deduced:

$$\begin{aligned} R_t^\wedge &:= -\sup\{-z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = X_s^{(t)}\} \\ &= -\sup\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(-z) = X_s^{(t)}\} \\ &= -\sup\{z \in \mathbb{R} : \exists s \in [0, t] \text{ with } Z_s^{(t)}(z) = -X_s^{(t)}\} \\ &= -(|-X_t| + L_t^0(-X)) \\ &= -(|X_t| + L_t^0(X)) \end{aligned}$$

■

In particular, we get that for any $t > 0$, $\mathfrak{X}_t \in \mathfrak{V}$ and is not reduced to a singleton. This property is in fact sufficient to deduce the following variant of Pitman's theorem.

Theorem 31 *The process $\mathfrak{X} = ([-(L_t^0(X) + |X_t|), L_t^0(X) + |X_t|])_{t \geq 0}$ is a set-valued dual for the Brownian motion X starting from 0.*

Proof

The property

$$\forall t \geq 0, \quad \mathcal{L}(\mathfrak{X}_{[0,t]}|X) = \mathcal{L}(\mathfrak{X}_{[0,t]}|X_{[0,t]})$$

is obvious from the explicit expression for \mathfrak{X} .

The important point is to prove that

$$\forall t \geq 0, \quad \mathcal{L}(X_t|\mathfrak{X}_{[0,t]}) = \Lambda(\mathfrak{X}_t, \cdot) \quad (93)$$

To take advantage from Theorem 29, enlarge the underlying probability space for X so that it contains a random variable U independent from X and uniformly distributed on $[0, 1]$. For any $n \in \mathbb{Z}_+$, let $t_n := 1/(n+1)$ and define the random variable $U_n \in (|X_{t_n}|, +\infty)$ by

$$\int_{|X_{t_n}|}^{U_n} u \exp\left(\frac{X_{t_n}^2 - u^2}{2t_n}\right) \frac{du}{t_n} = U \quad (94)$$

The interest of U_n is that, denoting $\mathfrak{X}_{t_n}^{(n)} := [-U_n, U_n] \in \mathfrak{V}$, the law of $(X_{t_n}, \mathfrak{X}_{t_n}^{(n)})$ on $\mathbb{R} \times \mathfrak{V}$ is given by

$$\mathbb{P}[X_{t_n} \in dx, \mathfrak{X}_{t_n}^{(n)} \in dS] = \nu^{(n)}(dS)\Lambda(S, dx) \quad (95)$$

where $\nu^{(n)}$ is the probability distribution on $\mathfrak{V}_{\text{sym}} := \{-u, u : u \geq 0\} \subset \mathfrak{V}$ described by

$$\nu^{(n)}(d[-u, u]) := 2u^2 \exp(-u^2/(2t_n)) du / \sqrt{2\pi t_n^3}$$

Indeed, we compute that

$$\begin{aligned} \int_{\mathfrak{V}_{\text{sym}}} \nu^{(n)}(dS)\Lambda(S, dx) &= \exp(-x^2/(2t_n)) dx / \sqrt{2\pi t_n} \\ &= \mathbb{P}[X_{t_n} \in dx] \end{aligned}$$

and (95) is a consequence of Bayes' formula.

Shifting the origin of time to t_n , we apply the considerations preceding Theorem 29 to the trajectory $X_{[t_n, +\infty)}$ and to the initial set-valued variable $\mathfrak{X}_{t_n}^{(n)}$ to construct

$$\begin{aligned} \forall t \geq t_n, \quad \mathfrak{X}_t^{(n)} &:= \varphi_{t_n, t}^{-1}(\mathfrak{X}_{t_n}^{(n)}) \\ &= \{z \in \mathbb{R} : \varphi_{t_n, t}(z) \in [-U_n, U_n]\} \end{aligned}$$

According to Theorem 29, we have

$$\forall t \geq t_n, \quad \mathcal{L}(X_t | \mathfrak{X}_{[t_n, t]}^{(n)}) = \Lambda(\mathfrak{X}_{t_n}^{(n)}, \cdot) \quad (96)$$

For any $t > 0$ and $n \in \mathbb{Z}_+$, denote

$$[-U_-^{(n)}(t), U_+^{(n)}(t)] := \mathfrak{X}_t^{(n)}$$

According to Lemma 32 below, we have a.s.

$$\begin{cases} \lim_{n \rightarrow \infty} U_-^{(n)}(t) = R_t^\vee \\ \lim_{n \rightarrow \infty} U_+^{(n)}(t) = R_t^\vee \end{cases} \quad (97)$$

Consider $p \in \mathbb{N}$, $0 < s_1 < s_2 < \dots < s_p = t$ and g_1, g_2, \dots, g_p some continuous and bounded functions from \mathbb{R}^2 to \mathbb{R} . We associate to these ingredients the mapping

$$G : \mathfrak{Y}^{(0, t]} \ni ([a_s, b_s])_{s \in (0, t]} \mapsto g_1(a_{s_1}, b_{s_1}) g_2(a_{s_2}, b_{s_2}) \cdots g_p(a_{s_p}, b_{s_p})$$

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable and bounded function. From (97), considered with t replaced by the s_1, s_2, \dots, s_p , we get the a.s. convergence of $G(\mathfrak{X}_{[t_n, t]}^{(n)})$ toward $G(\mathfrak{X}_{[0, t]})$ for n large. We deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}[h(X_t) G(\mathfrak{X}_{[t_n, t]}^{(n)})] = \mathbb{E}[h(X_t) G(\mathfrak{X}_{[0, t]})]$$

Next, taking into account that $\mathfrak{X}_t \in \mathfrak{Y}$, we also deduce from (97) the a.s. convergence of $\Lambda[h](\mathfrak{X}_t^{(n)})$ toward $\Lambda[h](\mathfrak{X}_t)$ and consequently

$$\lim_{n \rightarrow \infty} \mathbb{E}[\Lambda[h](\mathfrak{X}_t^{(n)}) G(\mathfrak{X}_{[t_n, t]}^{(n)})] = \mathbb{E}[\Lambda[h](X_t) G(\mathfrak{X}_{[0, t]})]$$

It follows from (96) that

$$\mathbb{E}[h(X_t) G(\mathfrak{X}_{[0, t]})] = \mathbb{E}[\Lambda[h](X_t) G(\mathfrak{X}_{[0, t]})]$$

Since this is true for all h and G as above and that $\mathfrak{X}_0 = \{0\}$ is deterministic, we get (93). ■

Lemma 32 *For any $t > 0$, the convergences (97) are satisfied a.s.*

Proof

Since $\mathfrak{X}_t^{(n)} = [-U_-^{(n)}(t), U_+^{(n)}(t)]$ and $\mathfrak{X} = [-R_t^\vee, R_t^\vee]$, it is sufficient to see that for any $z \in \mathbb{R}$

$$z \notin \mathfrak{X}_t \Rightarrow \lim_{n \rightarrow \infty} \mathbf{1}_{\mathfrak{X}_t^{(n)}}(z) = 0 \quad (98)$$

$$z \in (-R_t^\vee, R_t^\vee) \Rightarrow \lim_{n \rightarrow \infty} \mathbf{1}_{\mathfrak{X}_t^{(n)}}(z) = 1 \quad (99)$$

Let us come back to (94), which is equivalent to

$$\exp(-U_n^2/(2t_n)) = (1 - U) \exp(-(X_{t_n})^2/(2t_n))$$

namely

$$U_n^2 = (X_{t_n})^2 - 2t_n \ln(1 - U)$$

where we see that $\lim_{n \rightarrow \infty} U_n = 0$.

We begin by showing (98): when $z \notin \mathfrak{X}_t$, we have $\varphi_{0,t}(z) \neq 0$. So since $\lim_{n \rightarrow \infty} \varphi_{t_n,t}(z) = \varphi_{0,t}(z)$, it appears that for $n \in \mathbb{Z}_+$ large enough, we cannot have $\varphi_{t_n,t}(z) \in [-U_n, U_n]$, i.e. z ends up not belonging to $\mathfrak{X}_t^{(n)}$.

We now come to (99). From (89) we deduce that for z in the open set $(-R_t^\vee, R_t^\vee)$, we have $\sigma_z < t$, meaning that the trajectory $[0, t] \ni s \mapsto \varphi_{s,t}(z)$ is equal to X_s for s small enough. Thus for n large enough, we have

$$\begin{aligned} |\varphi_{t_n,t}(z)| &= |X_{t_n}| \\ &< \sqrt{(X_{t_n})^2 - 2t_n \ln(1 - U)} \\ &= U_n \end{aligned}$$

so that z belongs to $\mathfrak{X}_t^{(n)}$. ■

Remark 33 Define

$$\forall t \geq 0, \quad M_t := \max\{X_s : s \in [0, t]\}$$

The classical Pitman's theorem [20] states that the process $\tilde{\mathfrak{X}} = ([-(2M_t - X_t), 2M_t - X_t])_{t \geq 0}$ is a set-valued dual for the Brownian motion X starting from 0 and that $(2M_t - X_t)_{t \geq 0}$ is a Bessel-3 process. The process $(L_t^0(X) + |X_t|)_{t \geq 0}$ is also a Bessel-3 process, as a consequence of Lévy's theorem (see e.g. Theorem 2.3 of Chapter 6 of Revuz and Yor [23]).

A segment-valued process $(\mathfrak{Y}_t)_{t \geq 0}$ is said to be more λ -**expansive** than another segment-valued process $(\hat{\mathfrak{Y}}_t)_{t \geq 0}$, when for any fixed $t \geq 0$, the volume $\lambda(\hat{\mathfrak{Y}}_t)$ is stochastically dominated by $\lambda(\mathfrak{Y}_t)$. Thus the processes \mathfrak{X} and $\tilde{\mathfrak{X}}$ are iso- λ -expansive. It can be proven, by extending the arguments of the Remark 2.39 of Diaconis and Fill [7] (see also Fill and Lyzinski [8] or [16]), that if $(\mathfrak{Y}_t)_{t \geq 0}$ is a set-valued dual for the Brownian motion X starting from 0, then it is less expansive than \mathfrak{X} , i.e. \mathfrak{X} and $\tilde{\mathfrak{X}}$ are two examples of the most expansive set-valued duals for X . □

Let us deduce more precisely the classical Pitman's theorem [20] from Theorem 31. With the notations of the above remark, Lévy's theorem asserts the identity in law

$$(M_t - X_t, M_t)_{t \geq 0} \stackrel{\mathcal{L}}{=} (|X_t|, L_t^0(X))_{t \geq 0}$$

We infer for any $t \geq 0$ the identity in law

$$\begin{aligned} ((L^0(X) + |X|)_{[0,t]}, |X_t|) &\stackrel{\mathcal{L}}{=} ((2M - X)_{[0,t]}, M_t - X_t) \\ &= ((2M - X)_{[0,t]}, (2M_t - X_t)/2 - X_t/2) \end{aligned}$$

From Theorem 31, the distribution of $|X_t|$ knowing $(L^0(X) + |X|)_{[0,t]}$ is the uniform distribution over $[0, L_t^0(X) + |X_t|]$. It follows that the distribution of $(2M_t - X_t)/2 - X_t/2$ knowing $(2M - X)_{[0,t]}$ is the uniform distribution over $[0, 2M_t - X_t]$. To get Pitman's theorem, it remains to note that

for any constant $r \geq 0$, when U is a random variable such that $r/2 - U/2$ is uniformly distributed on $[0, r]$, then U is uniformly distributed on $[-r, r]$.

These arguments can be reversed to conversely deduce Theorem 31 from Pitman's theorem. This implication is succinctly mentioned by Yor [27] (at the end of page 4).

Remark 34 Despite the set-valued dual processes $\mathfrak{X} := ([-(L_t^0(X) + |X_t|), L_t^0(X) + |X_t|])_{t \geq 0}$ and $\tilde{\mathfrak{X}} := ([-(2M_t - X_t), 2M_t - X_t])_{t \geq 0}$ are iso- λ -expansive, some of their features are quite different: the Brownian motion X never hits the boundary of \mathfrak{X} except at time 0 (since for $t > 0$, we have $L_t^0(X) > 0$), while X recurrently hits the upper boundary of $\tilde{\mathfrak{X}}$. \square

An potential advantage of the stochastic flow approach to set-valued dual processes is its expected flexibility in the choice of the stochastic flow. In the next section, we will present a conjectural but promising stochastic flow associated to the Brownian motion, which should enable to directly recover the classical Pitman theorem by mimicking the random mapping proof considered in the finite setting. But one can imagine a lot of other examples, here is another one, which is a Wiener solution to a system of coalescing stochastic flow equations.

Let $B := (B_s)_{s \geq 0}$ and $W := (W_s)_{s \geq 0}$ be two independent Brownian motions. For fixed $t \geq 0$, we consider the following system of equations, for any $y \in \mathbb{R}$,

$$\begin{cases} dY_s^{(t)}(y) &= -\text{sgn}(Y_s^{(t)}(y) + W_s^{(t)})dB_s^{(t)}, & \forall s \in [0, t] \\ Y_0^{(t)}(y) &= y \end{cases} \quad (100)$$

where $W^{(t)} := (W_s^{(t)})_{s \in [0, t]} := (W_{t-s})_{s \in [0, t]}$ and $B^{(t)} := (B_s^{(t)})_{s \in [0, t]} := (B_{t-s})_{s \in [0, t]}$ are time-reversed processes. Again define $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$ via

$$\forall x \in \mathbb{R}, \forall 0 \leq s \leq t, \quad \psi_{s,t}(y) := Y_{t-s}^{(t)}(y) \quad (101)$$

With the help of Le Jan and Raimond [14], we get

Proposition 35 *There exists a Wiener solution of (100) such that ψ is coalescing stochastic flow associated to P and such that a.s., for any $(s, t) \in \Delta$, $\psi_{s,t} \in \mathcal{R}(\mathbb{R})$.*

To be a **Wiener** solution of (100) means that the filtration generated by the stochastic flow is included into the filtration generated by the white noises associated to B and W .

Proof

Consider $Z_- := (Z_-(s))_{s \geq 0}$ and $Z_+ := (Z_+(s))_{s \geq 0}$ two independent Brownian motions (not assumed to be standard). Le Jan and Raimond [14] show there is coalescing stochastic flow $\xi := (\xi_{u,v}(x))_{(u,v,x) \in \Delta \times \mathbb{R}}$ solution to the following system of equations, for any $x \in \mathbb{R}$ and $0 \leq u \leq v$,

$$\begin{cases} d_v \xi_{u,v}(x) &= \mathbb{1}_{\xi_{u,v} < 0} dZ_-(v) + \mathbb{1}_{\xi_{u,v} \geq 0} dZ_+(v) \\ \xi_{u,u}(x) &= x \end{cases} \quad (102)$$

Furthermore the coalescing stochastic flow ξ is a.s. unique and a Wiener solution: it is constructed without resorting to extra-randomness outside Z_- and Z_+ . Fix $t \geq 0$ and consider the two independent Brownian motions $Z_-^{(t)}$ and $Z_+^{(t)}$ over the time interval $[0, t]$ given by

$$\forall s \in [0, t], \quad \begin{cases} Z_+^{(t)}(s) &= \frac{W_s^{(t)} - B_s^{(t)}}{\sqrt{2}} \\ Z_-^{(t)}(s) &= \frac{W_s^{(t)} + B_s^{(t)}}{\sqrt{2}} \end{cases} \quad (103)$$

Let $(\xi_{u,v}^{(t)}(x))_{(u,v,x) \in \Delta_{[0,t]} \times \mathbb{R}}$ be the corresponding solution of (102) and define

$$\forall y \in \mathbb{R}, \forall s \in [0, t], \quad Y_s^{(t)}(y) := \sqrt{2} \xi_{0,s}^{(t)} \left(\frac{y + W_t}{\sqrt{2}} \right) - W_s^{(t)}$$

Via immediate substitution, we check that $(Y_s^{(t)}(y))_{s \in [0,t], y \in \mathbb{R}}$ is a solution of (100).

Let $t \geq 0$ be a free variable again. From (101), we deduce a family of random variables $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$. To check that it is a stochastic flow, let us first remark that

$$\forall x \in \mathbb{R}, \forall t \geq 0, \forall s \in [0, t], \forall v \in [0, s], \quad \xi_{0,s-v}^{(s)}(\xi_{0,t-s}^{(t)}(x)) = \xi_{0,t-v}^{(t)}(x) \quad (104)$$

Indeed, consider for any fixed $x \in \mathbb{R}$, $t \geq 0$ and $s \in [0, t]$,

$$\forall u \in [0, s], \quad \begin{cases} \chi_u & := \xi_{0,u}^{(s)}(\xi_{0,t-s}^{(t)}(x)) \\ \zeta_u & := \xi_{0,t-s+u}^{(t)}(x) \end{cases}$$

By definition, we have

$$\begin{aligned} d\chi_u &= \mathbf{1}_{\chi_u < 0} dZ_-^{(s)}(u) + \mathbf{1}_{\chi_u \geq 0} dZ_+^{(s)}(u) \\ &= \mathbf{1}_{\chi_u < 0} dZ_-(s-u) + \mathbf{1}_{\chi_u \geq 0} dZ_+(s-u) \\ &= \mathbf{1}_{\chi_u < 0} dZ_-^{(t)}(t-s+u) + \mathbf{1}_{\chi_u \geq 0} dZ_+^{(t)}(t-s+u) \end{aligned}$$

where Z_{\pm} are defined as in (103), with the exponents (t) removed. Thus $(\chi_u)_{u \in [0,s]}$ and $(\zeta_u)_{u \in [0,s]}$ satisfy the same evolution equation. Since we also have $\chi_0 = \xi_{0,t-s}^{(t)}(x) = \zeta_0$, we get (104) from the uniqueness result of Le Jan and Raimond [14] (see also Prokaj [21]).

Let us deduce from (104) the flow property of ψ . Indeed, we have for $y \in \mathbb{R}$, $t \geq 0$, $s \in [0, t]$ and $v \in [0, s]$,

$$\begin{aligned} \psi_{v,s}(\psi_{s,t}(y)) &= Y_{s-v}^{(s)}(Y_{t-s}^{(t)}(y)) \\ &= \sqrt{2} \xi_{0,s-v}^{(s)} \left(\frac{Y_{t-s}^{(t)}(y) + W_s}{\sqrt{2}} \right) - W_{s-v}^{(s)} \\ &= \sqrt{2} \xi_{0,s-v}^{(s)} \left(\frac{\sqrt{2} \xi_{0,t-s}^{(t)}((y + W_t)/\sqrt{2}) - W_{t-s}^{(t)} + W_s}{\sqrt{2}} \right) - W_v \\ &= \sqrt{2} \xi_{0,s-v}^{(s)} \left(\xi_{0,t-s}^{(t)} \left(\frac{y + W_t}{\sqrt{2}} \right) \right) - W_v \\ &= \sqrt{2} \xi_{0,t-v}^{(t)} \left(\frac{y + W_t}{\sqrt{2}} \right) - W_{t-v}^{(t)} \\ &= Y_{t-v}^{(t)}(y) \\ &= \psi_{v,t}(y) \end{aligned}$$

Finally, the association of ψ to P is an immediate consequence of (100) and of Lévy's characterization of the Brownian motion and the last assertion of the lemma is proven as in Lemma 26. \blacksquare

Most of the previous arguments for the stochastic flow defined in (77) can now be extended to the stochastic flow considered in (101). E.g. fix $t \geq 0$ and a Brownian trajectory $X_{[0,t]}$. Conditioning ψ by the event described in (80) amounts to asking that $B_{[0,t]}$ satisfies

$$\forall s \in [0, t], \quad dX_s^{(t)} = -\text{sgn}(X_s^{(t)} + W_s^{(t)}) dB_s^{(t)}$$

Indeed, Prokaj [21] has shown (taking into account a transformation similar to that used in the proof of Proposition 35) that this equation in $X^{(t)}$ admits a unique strong solution. We get that

$$\forall s \in [0, t], \quad dB_s^{(t)} = -\text{sgn}(X_s^{(t)} + W_s^{(t)})dX_s^{(t)}$$

and it follows that the conditioned flow $\varphi := (\varphi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$ is given by

$$\forall 0 \leq s \leq t, \forall z \in \mathbb{R}, \quad \varphi_{s,t}(z) := Z_{t-s}^{(t)}(z)$$

where

$$\begin{cases} dZ_s^{(t)}(z) &= \text{sgn}(Z_s^{(t)}(z) + W_s^{(t)})\text{sgn}(X_s^{(t)} + W_s^{(t)})dX_s^{(t)} \\ Z_0^{(t)}(z) &= z \end{cases}$$

Then Theorem 29 is still valid, with $\mathfrak{X}^{(r)}$ defined as in (85), for any $r > 0$. The troubles begin with the process $\hat{\mathfrak{X}}$ defined as in (86), for which we did not find an explicit expression. Nevertheless we believe that this process $\hat{\mathfrak{X}}$ is still a set-valued dual for the Brownian motion X starting from 0 and not reduced to a singleton, for all positive times. Furthermore, we think that this $\hat{\mathfrak{X}}$ is strictly less λ -expansive (in the sense of Remark 33) than the process \mathfrak{X} described in Theorem 31.

More generally, for any $\eta > 0$ it should be possible to solve the coalescing stochastic flow equation

$$\forall y \in \mathbb{R}, \quad \begin{cases} dY_s^{(t)}(y) &= -\text{sgn}(Y_s^{(t)}(y) + \eta W_s^{(t)})dW_s^{(t)} \\ Y_0^{(t)}(y) &= y \end{cases} \quad (105)$$

and to show that the corresponding segment-valued are less and less λ -expansive as η increases (see [16] for certain families of segment-valued dual processes satisfying this monotonicity property, with λ replaced by the underlying invariant probability). Unfortunately and despite the work of Prokaj [21] giving us a strong solution $Y^{(t)}(y)$ for any fixed $y \in \mathbb{R}$, we did not find a reference insuring the existence of a coalescing stochastic flow solution for (105), for $\eta \in (0, +\infty) \setminus \{1\}$.

7.2 Conjectures about one-dimensional diffusions processes

Here we propose an extension of Pitman's theorem [20] (see also Rogers and Pitman [24]) to simple one-dimensional diffusion processes. The considerations of this subsection remain hypothetical since they assume the existence of convenient coalescing stochastic flows, see in particular Conjecture 39.

Let be given a smooth and bounded mapping $b : \mathbb{R} \rightarrow \mathbb{R}$. We consider the stochastic differential equation

$$dX_t = dW_t + b(X_t)dt \quad (106)$$

where $W := (W_t)_{t \geq 0}$ is a standard Brownian motion.

The process X is reversible with respect to the measure π , whose density, still denoted π , with respect to the Lebesgue measure on \mathbb{R} is given by

$$\forall x \in \mathbb{R}, \quad \pi(x) := \exp\left(2 \int_0^x b(y) dy\right) \quad (107)$$

Again the measure π gives an infinite weight to \mathbb{R} and we must take the same precautions as in the previous subsection.

Let $P := (P_t)_{t \geq 0}$ be the semi-group associated to X , it is self-adjoint in $\mathbb{L}^2(\pi)$, so that $P^* = P$. We renew Definition (78) for the set of nice subsets. The kernel Λ is still Markovian from \mathfrak{V} to \mathbb{R} , corresponding to the elementary conditioning operation under π .

Let $B := (B_s)_{s \geq 0}$ be another Brownian motion. For fixed $x_0 \in \mathbb{R}$ and $t \geq 0$, we consider the following system of equations, for all $y \in \mathbb{R}$,

$$\begin{cases} dY_s^{(t)}(y) &= -\text{sgn}(Y_s^{(t)}(y) - x_0)dB_s^{(t)} + b(Y_s^{(t)}(y))ds, & \forall s \in [0, t] \\ Y_0^{(t)}(y) &= y \end{cases} \quad (108)$$

with the same conventions as in Subsection 7.1, in particular for the sign. Let us assume we have at our disposal a coalescing flow solution to (108). Lemma 26 enables us to get a version which is such that for any $s \in [0, t]$, the mapping $\mathbb{R} \ni y \mapsto Y_s^{(t)}(y)$ is non-decreasing and càdlàg. Again to such a version we associate $\psi := (\psi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$ via

$$\forall 0 \leq s \leq t, \forall y \in \mathbb{R}, \quad \psi_{s,t}(y) := Y_{t-s}^{(t)}(y) \quad (109)$$

Subsequently, we can partially adapt the strategy of the previous subsection. Fix $t \geq 0$ and a trajectory $X_{[0,t]}$ associated to P and whose initial distribution is $\Lambda(\mathfrak{X}_0, \cdot)$. Conditioning ψ by the event (80) implies that we have

$$\forall s \in [0, t], \quad dX_s^{(t)} = -\text{sgn}(X_s^{(t)} - x_0)dB_s^{(t)} + b(X_s^{(t)})ds$$

We deduce that

$$\forall s \in [0, t], \quad dB_s^{(t)} = -\text{sgn}(X_s^{(t)} - x_0)(dX_s^{(t)} - b(X_s^{(t)})ds) \quad (110)$$

and it follows that the conditioned flow, denoted $\varphi := (\varphi_{s,t}(y))_{(s,t,y) \in \Delta \times \mathbb{R}}$ in Section 6, is given by

$$\forall 0 \leq s \leq t, \forall z \in \mathbb{R}, \quad \varphi_{s,t}(z) := Z_{t-s}^{(t)}(z) \quad (111)$$

where

$$\begin{cases} dZ_s^{(t)}(z) &= \text{sgn}(Z_s^{(t)}(z) - x_0)\text{sgn}(X_s^{(t)} - x_0)(dX_s^{(t)} - b(X_s^{(t)})ds) + b(Z_s^{(t)}(z))ds \\ Z_0^{(t)}(z) &= z \end{cases} \quad (112)$$

The observations mentioned after (84) are still valid. For given $r^\wedge, r^\vee > 0$, assume that the initial law of X is $\Lambda([r^\wedge, r^\vee], \cdot)$. Define for any $t \geq 0$,

$$\begin{cases} R_t^\vee &:= \inf \left\{ z \in \mathbb{R} : Z_0^{(t)}(z) \leq r^\vee \right\} = \sup \left\{ z \in \mathbb{R} : Z_0^{(t)}(z) < r^\vee \right\} \\ R_t^\wedge &:= \inf \left\{ z \in \mathbb{R} : Z_0^{(t)}(z) \geq r^\wedge \right\} \\ \mathfrak{X}_t &:= [R^\wedge, R^\vee] = \varphi_{0,t}^{-1}([r^\wedge, r^\vee]) \end{cases}$$

where for the last equality we took into account the proof of Lemma 28, which can also be applied here. Thus the process $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$ remains non-trivial and as in the previous subsection, Theorem 25 would lead to the following result, if we had at our disposal a solution to (108):

Conjecture 36 The process \mathfrak{X} is a set-valued dual for the diffusion process X starting from the distribution $\Lambda([r^\wedge, r^\vee], \cdot)$. □

The above conjecture should hold for any $x_0 \in \mathbb{R}$, but we would like the dual process \mathfrak{X} to be quite large and this requires a good choice of x_0 . In Subsection 7, we could have replaced $\text{sgn}(\cdot)$ by $\text{sgn}(\cdot - x_0)$, but the corresponding dual process \mathfrak{X} would have stayed trivial until the first time τ that X , the Brownian motion starting from 0, hits x_0 :

$$\forall t \in [0, \tau], \quad \mathfrak{X}_t = \{X_t\}$$

When, as in Theorem 29, the Brownian motion starts from the uniform distribution on $[-r, r)$ for some $r > 0$, the set valued dual is a translation of $\mathfrak{X}_0 = [-r, r)$ until the closure of \mathfrak{X}_t contain x_0 , in particular the volume of \mathfrak{X}_t remains constant until the occurrence of this event.

In the setting of this subsection, we expect similar behaviors. In particular if we want \mathfrak{X}_t to be quite large for small times $t \geq 0$, it seems preferable to take $x_0 \in \mathfrak{X}_0$. Conjecture 39 below can be seen as an attempt to keep x_0 in the closure of \mathfrak{X} , by letting x_0 evolve and be the supremum of \mathfrak{X} .

We are equally wondering about letting X start from a deterministic point. From the above observation, it seems wise to take x_0 equal to this initial point. So let us assume that (108) admits a solution when $\mathfrak{X}_0 = \{x_0\}$. As in Subsection 7.1, we are led to introduce

$$\begin{aligned} R_t^\vee &:= \sup\{z \in \mathbb{R} : Z_0^{(t)}(z) = x_0\} \\ R_t^\wedge &:= \inf\{z \in \mathbb{R} : Z_0^{(t)}(z) = x_0\} \\ \mathfrak{X}_t &:= [R_t^\vee, R_t^\wedge) \end{aligned}$$

Unfortunately, we did not find a nice explicit expression of \mathfrak{X} in terms of X , as in Proposition 30, this prevents us from to conclude, as in Proposition 31 that \mathfrak{X} is a non-trivial set-dual associated to X starting from x_0 . Nevertheless, here is an indication going in this direction.

Lemma 37 *For any $t > 0$, \mathfrak{X}_t is a right neighborhood of X_t .*

Proof

Fix $t > 0$ as well as $\epsilon \in (\min\{X_s : s \in [0, t]\}, x_0)$. Define

$$\begin{aligned} \tau &:= \inf\{s > 0 : X_s = \epsilon\} \in (0, t) \\ \gamma &:= \sup\{s \in [0, \tau] : X_s = x_0\} \end{aligned}$$

Consider the flow $(F_s^{(t)}(z))_{z \in \mathbb{R}, s \in [0, t]}$ solution of the system

$$\begin{cases} dF_s^{(t)}(z) &= -(dX_s^{(t)} - b(X_s^{(t)})ds) + b(F_s^{(t)}(z))ds \\ F_0^{(t)}(z) &= z \end{cases} \quad (113)$$

It is a regular flow: for any $s \in [0, t]$, the mapping $\mathbb{R} \ni x \mapsto F_s^{(t)}(x)$ is a diffeomorphism of \mathbb{R} . This is also true for the random times τ and γ : the mapping $\mathbb{R} \ni x \mapsto F_{\tau-\gamma}^{(\tau)}(x)$ is a diffeomorphism. Consider the unique $\xi \in \mathbb{R}$ such that $F_{\tau-\gamma}^{(\tau)}(\xi) = x_0$.

Let us show that for $s > 0$ sufficiently small, $F_{\tau-\gamma-s}^{(\tau)}(\xi) > x_0$. Indeed, we have

$$\begin{aligned} F_{\tau-\gamma-s}^{(\tau)}(\xi) - x_0 &= -(F_{\tau-\gamma}^{(\tau)}(\xi) - F_{\tau-\gamma-s}^{(\tau)}(\xi)) \\ &= \int_{\tau-\gamma-s}^{\tau-\gamma} dX_u^{(t)} - \int_{\tau-\gamma-s}^{\tau-\gamma} b(X_u^{(t)}) + b(F_u^{(t)}(z)) du \\ &= x_0 - X_{\tau-\gamma-s}^{(\tau)} - \int_{\tau-\gamma-s}^{\tau-\gamma} b(X_u^{(t)}) + b(F_u^{(t)}(z)) du \end{aligned} \quad (114)$$

If $X_{[0, t]}$ was a Brownian motion, the process $x_0 - X_{[\gamma, \tau]}$ would be a Bessel-3 process starting from 0 and stopped when it reaches $-\epsilon$, according to Williams' Brownian path decomposition (see e.g. Theorem 4.9 of Chapter 7 of Revuz and Yor [23]). From Wichura [26], we would then be able to get that a.s.

$$\liminf_{s \rightarrow 0^+} \frac{x_0 - X_{\gamma+s}}{\sqrt{s/\ln(\ln(1/s))}} > 0$$

This behavior is shared by the diffusion process $X_{[0,t]}$, since Girsanov theorem insures that its law is equivalent to that of the Brownian motion on the time interval $[0, t]$, see also Bass and Erickson [3]. Furthermore, we have that

$$\left| \int_{\tau-\gamma-s}^{\tau-\gamma} b(X_u^{(t)}) + b(F_u^{(t)}(z)) du \right| \leq 2 \|b\|_\infty s$$

thus (114) implies that for $s > 0$ small,

$$\begin{aligned} F_{\tau-\gamma-s}^{(\tau)}(\xi) - x_0 &\sim x_0 - X_{\gamma+s} \\ &> 0 \end{aligned}$$

As a consequence, consider $\sigma \in (0, \tau - \gamma)$ such that

$$\forall s \in (0, \sigma], \quad \begin{cases} X_{\tau-\gamma-s}^{(\tau)}(\xi) < x_0 \\ F_{\tau-\gamma-s}^{(\tau)}(\xi) > x_0 \end{cases}$$

From (112), we deduce that

$$\forall s \in (0, \sigma), \quad F_{\tau-\gamma-s}^{(\tau)}(\xi) = Z_{\tau-\gamma-s}^{(\tau)}(\xi)$$

Define

$$\zeta := F_{\tau-\gamma-\sigma}^{(\tau)}(\xi)$$

so that

$$\begin{aligned} Z_\sigma^{(\gamma+\sigma)}(\zeta) &= F_\sigma^{(\gamma+\sigma)}(\zeta) \\ &= x_0 \\ &= X_\sigma^{(\gamma+\sigma)} \end{aligned}$$

We deduce that $\zeta \in \mathfrak{X}_{\gamma+\sigma}$. Note that $\zeta > x_0 > X_{\gamma+\sigma}$, it follows that $\mathfrak{X}_{\gamma+\sigma}$ is a right neighborhood of $X_{\gamma+\sigma}$. Finally, taking into account the right continuity of $\mathbb{R} \ni z \mapsto Z_{t-\gamma-\sigma}^{(t)}(z)$, we obtain that \mathfrak{X}_t is a right neighborhood of X_t . ■

Remarks 38

(a) The above proof can be adapted to show that for any $t > 0$, there exists $s \in (0, t)$ such that \mathfrak{X}_s is a neighborhood of X_s . But this is not sufficient to get that \mathfrak{X}_t is a neighborhood of X_t . Nevertheless, we believe it is true. This is specific to the set-valued duals to be constructed with flows of the form (108), it is not true for the classical Pitman's dual, recall Remark 34, and cannot be expected for set-valued duals to be constructed with flows of the form (115) below.

(b) Proposition 30 is still valid in the particular case where b is given by the non-regular drift

$$\forall x \in \mathbb{R}, \quad b(x) := c \operatorname{sign}(x)$$

where $c \in \mathbb{R}$ is a fixed constant. Indeed, in general (87) should be replaced by

$$\forall s \in [0, t], \quad \operatorname{sgn}(Z_s^{(t)}(z))(dZ_s^{(t)}(z) - b(Z_s^{(t)}(z))ds) = \operatorname{sgn}(X_s^{(t)})(dX_s^{(t)} - b(X_s^{(t)})ds)$$

and one exactly recovers (87) with b of the above form. □

The existence of a convenient coalescent flow solution to the system (108) seems a reasonable conjecture and hopefully could be worked out using the techniques of Le Jan and Raimond [12, 13, 14], but is out of the scope of this paper. What we really would like to do is to solve the following more challenging one. It will probably require a fixed point approach and thus a complete topological structure on an adequate set of flows, in the spirit of Remark 19.

Conjecture 39 As at the beginning of this subsection, let $B := (B_s)_{s \geq 0}$ be a Brownian motion and fix some $\mathfrak{X}_0 \in \mathfrak{X}$. We are interested in the following system of equations, for all $t \geq 0$ and $y \in \mathbb{R}$,

$$\begin{cases} dY_s^{(t)}(y) &= \operatorname{sgn}(R_{t-s}^\vee - Y_s^{(t)}(y))dB_s^{(t)} + b(Y_s^{(t)}(y))ds, & \forall s \in [0, t] \\ Y_0^{(t)}(y) &= y \\ R_{t-s}^\vee &:= \sup\{y \in \mathbb{R} : Y_{t-s}^{(t-s)}(y) \leq \max(\mathfrak{X}_0)\} \end{cases} \quad (115)$$

Obviously there is a measurability problem in (115), since R_{t-s}^\vee belongs to the future at time s of the filtration generated by $B^{(t)}$. Nevertheless, we believe that for any $s \geq 0$, R_s will be independent from the white noise of B after time s , namely from the sigma-field generated by the $B_v - B_u$, for $v \geq u \geq s$. In particular, the solution will be such that the process $(\int_0^s \operatorname{sgn}(Y_u^{(t)}(y) - R_{t-u}^\vee)dB_u^{(t)})_{s \in [0, t]}$ is a Brownian motion in the filtration generated by $B^{(t)}$, so that for any $y \in \mathbb{R}$, $Y^{(t)}(y) := (Y_s^{(t)}(y))_{s \in [0, t]}$ is a diffusion associated to P . This solution should be sufficiently regular and in particular the process $R^\vee := (R_t^\vee)_{t \geq 0}$ should be a continuous semi-martingale. \square

The interest of this conjecture is that it would enable to construct set-valued dual processes for one-dimensional diffusions in analogy to the treatment presented in Section 4 for birth and death chains.

A Reduction of one-dimensional diffusion processes

The purpose of this appendix is to show it is not very restrictive to only consider diffusions of the form (106).

Consider the second order operator $L := a\partial^2 + b\partial$ on $C^\infty(I)$, where $a > 0$ and b are smooth functions on the open interval $I := (\iota^-, \iota^+) \subset \mathbb{R}$. Up to performing a shift, assume that $0 \in I$. The corresponding **scale and speed functions** ν and μ are defined by

$$\forall x \in I, \quad \begin{cases} \nu(x) &:= \exp\left(-\int_0^x \frac{b}{a}(y) dy\right) \\ \mu(x) &:= \frac{1}{2a(x)\nu(x)} \end{cases} \quad (116)$$

By a usual abuse of notation, ν and μ will also stand for the the **scale and speed measures** which admit respectively the scale and speed functions as densities with respect to the Lebesgue measure restricted to I .

Let $(X_t)_{t \in [0, \tau)}$ be a diffusion whose generator is L (in the sense of martingale problems), where $\tau > 0$ is its (random) explosion time, which is such that

$$\lim_{t \rightarrow \tau^-} X_t = \iota^- \quad \text{or} \quad \lim_{t \rightarrow \tau^-} X_t = \iota^+$$

To simplify the notation, we extend the above diffusion to all times via

$$\forall t \in [\tau, +\infty), \quad X_t := \lim_{s \rightarrow \tau^-} X_s$$

so that $X := (X_t)_{t \geq 0}$ is V -valued with $V := [\iota^-, \iota^+] \subset \overline{\mathbb{R}}$. The process X is absorbed at ι^- and ι^+ (if it reaches them). It is sometimes called a minimal Markov process associated to L and its law is determined by its initial law $\mathcal{L}(X_0)$.

Denote

$$\begin{aligned}\mathfrak{V} &:= \{[z^-, z^+] : z^-, z^+ \in V \text{ and } z^- < z^+\} \\ \bar{\mathfrak{V}} &:= \mathfrak{V} \sqcup \{\{z\} : z \in V \cap \mathbb{R}\}\end{aligned}$$

and as usual, define the Markov kernel Λ from $\bar{\mathfrak{V}}$ to V via

$$\forall z := [z^-, z^+] \in \bar{\mathfrak{V}}, \quad \Lambda([z^-, z^+], \cdot) := \begin{cases} \delta_x & , \text{ if } z^+ = z^- \\ \frac{\mu([z^-, z^+] \cap \cdot)}{\mu(z)} & , \text{ otherwise} \end{cases}$$

We are looking for a Λ -**spreading** for X , namely a set-valued dual process $\mathfrak{X} := (\mathfrak{X}_t)_{t \geq 0}$ such that for any $t > 0$, we have $\mathfrak{X}_t \in \bar{\mathfrak{V}}$. We show below how to come back to the setting of Subsection 7.2.

First, let us check that the problem of finding a Λ -spreading is invariant by diffeomorphisms. More precisely, let ϕ be a smooth function from I to \mathbb{R} , whose derivative is positive (in particular it never vanishes). There will be no loss of generality in assuming that $\phi(0) = 0$. Let $\tilde{I} := (\tilde{t}_-, \tilde{t}_+) := \phi(I)$ and also interpret ϕ as an operator Φ from $\mathcal{C}^\infty(\tilde{I})$ to $\mathcal{C}^\infty(I)$ via

$$\forall f \in \mathcal{C}^\infty(\tilde{I}), \quad \Phi[f] := f \circ \phi \tag{117}$$

Consider the operator $\tilde{L} := \Phi^{-1} \circ L \circ \Phi$ on $\mathcal{C}^\infty(\tilde{I})$. It is not difficult to check that if X is a diffusion associated to L , then $\tilde{X} := (\phi(X_t))_{t \geq 0}$ is a diffusion associated to \tilde{L} (where ϕ has been extended to \tilde{V} by $\phi(t^-) := \tilde{t}^-$ and $\phi(t^+) := \tilde{t}^+$).

Lemma 40 *Assume that \mathfrak{X} is a Λ -spreading for X . Then $\tilde{\mathfrak{X}} := (\phi(\mathfrak{X}_t))_{t \geq 0}$ is a $\tilde{\Lambda}$ -spreading for \tilde{X} .*

Proof

Since ϕ is an homeomorphism between I and \tilde{I} , all the defining properties of a Λ -spreading are immediate to obtain, except (68), which requires the knowledge of $\tilde{\Lambda}$. So let us compute \tilde{L} . Recall that we have

$$\forall f \in \mathcal{C}^\infty(I), \forall F \in \mathcal{C}^\infty(f(I)), \quad L[F \circ f] = F'[f]L[f] + F''[f]\Gamma[f]$$

(this property is equivalent to the fact that L generates processes with continuous trajectories), where the carré du champ operator Γ is defined by

$$\begin{aligned}\forall f \in \mathcal{C}^\infty(I), \quad \Gamma[f] &:= \frac{1}{2}(L[f^2] - 2fL[f]) \\ &= a(\partial f)^2\end{aligned}$$

It follows that for any $F \in \mathcal{C}^\infty(\tilde{I})$, we have the **change of coordinate formula**:

$$L[F \circ \phi] = \Gamma[\phi]F''[\phi] + L[\phi]F'[\phi]$$

so that

$$\tilde{L}[F] = (\Gamma[\phi] \circ \phi^{-1})F'' + (L[\phi] \circ \phi^{-1})F'$$

namely

$$\tilde{a} = \Gamma[\phi] \circ \phi^{-1} = (a(\phi')^2) \circ \phi^{-1} \tag{118}$$

$$\tilde{b} = L[\phi] \circ \phi^{-1} = (a\phi'' + b\phi') \circ \phi^{-1} \tag{119}$$

We deduce that

$$\begin{aligned}
\forall x \in \tilde{I}, \quad \tilde{\nu}(x) &= \exp \left(- \int_0^x \frac{\tilde{b}(y)}{\tilde{a}(y)} dy \right) \\
&= \exp \left(- \int_0^{\phi^{-1}(x)} \frac{\tilde{b}(\phi(y))}{\tilde{a}(\phi(y))} \phi'(y) dy \right) \\
&= \exp \left(- \int_0^{\phi^{-1}(x)} \left(\frac{\phi''}{(\phi')^2}(y) + \frac{b}{a\phi'}(y) \right) \phi'(y) dy \right) \\
&= \exp \left(- \int_0^{\phi^{-1}(x)} \frac{\phi''}{\phi'}(y) + \frac{b(y)}{a(y)} dy \right) \\
&= \frac{\phi'(0)}{\phi'(\phi^{-1}(x))} \nu(\phi^{-1}(x)) \\
&= \phi'(0) \nu(\phi^{-1}(x)) (\phi^{-1})'(x)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\forall x \in \tilde{I}, \quad \tilde{\mu}(x) &= \frac{1}{2\tilde{a}(x)\tilde{\nu}(x)} \\
&= \frac{1}{2(a(\phi')^2) \circ \phi^{-1} \phi'(0) \nu(\phi^{-1}(x)) (\phi^{-1})'(x)} \\
&= \mu(\phi^{-1}(x)) (\phi^{-1})'(x) / \phi'(0)
\end{aligned}$$

It just means that $\tilde{\nu}$ and $\tilde{\mu}$ are the images of ν and μ by ϕ , up to the factors $\phi'(0)$ and $1/\phi'(0)$. It follows that

$$\forall z := [z^-, z^+] \in \tilde{\mathfrak{X}}, \quad \tilde{\Lambda}([z^-, z^+], \cdot) = \Lambda((\phi^{-1}(z^-), \phi^{-1}(z^+)), \phi^{-1}(\cdot)) \quad (120)$$

Denote also by ϕ the mapping $\tilde{\mathfrak{X}} \ni [z^-, z^+] \mapsto (\phi(z^-), \phi(z^+)) \in \tilde{\mathfrak{X}}$ and by Φ the corresponding functional operator, as in (117), transforming measurable mappings on $\tilde{\mathfrak{X}}$ into measurable mappings on $\tilde{\mathfrak{X}}$. At the operator level, the relation (120) translates into the intertwining relation

$$\Phi \tilde{\Lambda} = \Lambda \Phi \quad (121)$$

From these invariance relations and from the fact that the σ -field generated by $\tilde{\mathfrak{X}}_{[0,t]}$ contains the same events as the one generated by $\mathfrak{X}_{[0,t]}$, we deduce that for any $t \geq 0$,

$$\begin{aligned}
\mathcal{L}(\tilde{X}_t | \tilde{\mathfrak{X}}_{[0,t]}) &= \mathcal{L}(\phi(X_t) | \mathfrak{X}_{[0,t]}) \\
&= \Phi(\mathcal{L}(X_t) | \mathfrak{X}_{[0,t]}) \\
&= \Phi(\Lambda(X_t, \cdot)) \\
&= \tilde{\Lambda}(\phi(X_t), \cdot) \\
&= \tilde{\Lambda}(\tilde{X}_t, \cdot)
\end{aligned}$$

where in the second and third lines, Φ stands for the natural action induced on measures by the mapping ϕ , obtained by duality from the action of Φ on the functions, and the fourth equality corresponds to (121). ■

As a consequence, we get:

Corollary 41 *To know how to find a Λ -spreading for all initial distributions $\mathcal{L}(X_0)$ and for all generators L as above is equivalent to know how to solve this problem when $a \equiv 1/2$.*

Proof

Note that conversely in Lemma 40, the diffeomorphism ϕ^{-1} from \tilde{I} to I enables to go from $(\tilde{X}, \tilde{\mathfrak{X}})$ to (X, \mathfrak{X}) . Thus, taking into account (118), the proof of the above corollary is reduced to the finding of increasing diffeomorphism ϕ on I such that $\tilde{a} \equiv 1/2$, namely $\phi' = 1/\sqrt{2a}$. This is solved by considering

$$\forall x \in I, \quad \phi(x) := \int_0^x \frac{1}{\sqrt{2a(y)}} dy$$

From a geometric point of view, this amounts to changing the usual metric on I so that $2a\partial^2$ corresponds to the second order terms of the Laplacian for the new metric. ■

Similarly, we can also removed the drift:

Corollary 42 *To know how to find a Λ -spreading for all initial distributions $\mathcal{L}(X_0)$ and for all generators L as above is equivalent to know how to solve this problem when $b \equiv 0$.*

Proof

By the same reasoning as in the proof of Corollary 41, it is sufficient to find an increasing diffeomorphism ϕ on I such that $\tilde{b} \equiv 0$. From (119), this amounts to $L[\phi] = 0$. This is solved by considering

$$\forall x \in I, \quad \phi(x) := \nu([0, x])$$

Except when $L[1/\sqrt{a}] = 0$, it is not possible to perform the two operations of the proofs of Corollary 41 and 42 simultaneously, to end up with the generator of the Brownian motion on I (absorbed at the boundary when it is reached). Namely, not every search for Λ -spreadings can be reduced to Pitman's theorem [20]. Nevertheless, by Corollary 41, the one-dimensional processes considered at the beginning of this appendix can be reduced to the case of Subsection 7.2, up to relaxing the hypotheses on the drift b there and to assume here that ι^- and ι^+ are natural boundaries. ■

Remark 43 In Subsection 7.2 we preferred to work with the reduction of Corollary 41. We also tried the reduction of Corollary 42 but it did not led to more explicit set-valued dual processes, that is why the computations are not presented in this paper. The reductions of one-dimensional diffusions deduced in Corollaries 41 and 42 are the simplest ones, but other ones can be imagined and maybe among them there is one leading to more natural couplings. □

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