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Multidisciplinary Optimization of a Quad-Rotor by Integrating Multi-level Models

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Abstract—The quad-rotor (or quad copter) is a complex system that has lately submerged the aerospace field. In this article, components are chosen from Modelica’s different block libraries to build the multi-body model. On this model, simplified dynamic equations are implemented according to a certain disposing configuration. Based on disk actuator theory, a multidisciplinary optimization takes place in order to improve some system characteristics such as power, weight throughout a link between ModelCenter and Catia V5 and with a focus only on the propeller.

Keywords—Quad-rotor; hover; modeling; multi body; dynamic; implementation; Modelica; optimization; multidisciplinary; ModelCenter; coupling

I. INTRODUCTION

Nowadays, quad-copters are created in different fields and for multiple purposes. They can be intended for both academic and industrial worlds.

In the research field, the Mesicopter project at Stanford University (1999 -2001) [1], the MARK I and II at the Australia National University (2002-2006) [2][3], the Stanford Testbed of Autonomous Rotorcraft for Multi-Agent Control (STARMACI and II) at Stanford University (2004-2007) [4][5], the Omnidirectional Stationary Flying Outstretched Robot (OS4) at the EPFL (2005-2007) [6] and the Flying Machine Arena (FMA) at the ETH Zurich (since the 1990s) [7] can be cited.

Microdrones MD4-200, Draganfly and Silverlit X-UFO and Parrot AR.Drone I, II and III [8] are the commercially available quad-rotors.

Since their invention, quad-rotors have interested many amateurs and researchers. In fact, they have tried to model them, to construct them, to control them in different flight scenarios and to optimize some of their technical specifications...

In the same context, this paper has as aims: to build, first, a dynamical model of the quad-rotor in Modelica; second, to make an optimization of some parameters to improve its characteristics.

The paper is organized as follows. The first part is about both multi body and dynamic quad-rotor modeling where, at first, the airframe is defined, then, the dynamics are implemented to get the complete system model which is controlled in hover in the next step. In the second part, the optimization process is described in order to get the optimal quad-rotor. Finally, the paper is concluded in the last section.

II. QUADROTOR MODELING

A. Multi Body Modeling

The first step in the modeling process is to have a multi-body model containing almost all the quad-rotor’s necessary components for its flight in hover. Indeed, this model has to be faithful to the real quad-rotors. It is mentioned that the OS4 is the basic model.

In fact, a preliminary model is built as it’s shown in *Fig.1*; then, some important improvements should take place in order to get to the detailed model of the quad-rotor. It is composed of:

- A central mass m which its center coincides with the center of gravity of the quad-rotor; that includes all the electronic devices, in some cases a computer module or even a camera.
- Four symmetrical arms of length l attached to the center, which each one ends with a propulsion group composed of a propeller, a gearbox and a motor supposed as a point mass m_0 .

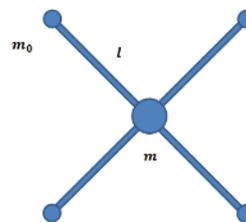


Fig.1. Quad-rotor preliminary model

This model may seem generalized and complete but is not sufficient enough to set up the 3D system architecture on which the flight in hover may be simulated. That’s why, some

ameliorations are required touching the structure of the airframe, the material used for the different components and the manner of disposing the components in the Diagram Layer profiting from Modelica oriented-object character such as connecting components from multiple domains (Mechanical, Electrical...).

B. Dynamic Modeling

1) Complete dynamics

The quad-rotor's dynamic model resumes different aerodynamic and mechanic effects. In fact, there are many sources affecting its behavior like it is mentioned in TABLE I.

TABLE I. Sources affecting quad-rotor dynamics

Effect	Source
Aerodynamic effect	Propeller rotation
	Blades flapping
Inertial counter torques	Change in propeller rotation speed
Gravity effect	Center of mass
Gyroscopic effects	Change in orientation of the rigid body
	Change of the propeller plan
Friction	All quad-copter motion

The dynamic modeling begins with defining both reference frames [9]. The first is an earth-fixed reference frame (O_e, X_e, Y_e, Z_e) fixed to the earth. The second is a body-fixed reference frame (O_b, X_b, Y_b, Z_b) fixed to the quad-rotor. Then, because it is quasi impossible to implement a model conform to reality, some assumptions are required to be made:

- The quad-copter's structure and propellers are rigid
- The quad-copter's structure is symmetrical
- Thrust and drag are proportional to the square of the propeller's angular speed
- The ground effect is neglected
- The center of gravity and the origin of the body-fixed frame are coincident

The momentum theory and the blade element theory [10] [11] permit to calculate the aerodynamic forces and torques acting on a rotor like it's shown in Fig.2.

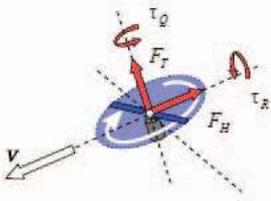


Fig.2. Forces and moments applied on one rotor

The thrust force is the resultant of forces acting on all blade elements perpendicular to the rotor shaft.

$$F_T = C_{F_T} \cdot \rho \cdot A \cdot (\Omega \cdot R)^2 \quad (1)$$

The hub force is defined as the resultant of forces acting on all the elements of the blade in the horizontal plane.

$$F_H = C_{F_H} \cdot \rho \cdot A \cdot (\Omega \cdot R)^2 \quad (2)$$

The drag moment is the resultant moment of all horizontal forces acting on the rotor's center.

$$\tau_Q = C_{\tau_Q} \rho A (\Omega R)^2 R \quad (3)$$

The blades move horizontally through the air, the blade which is advancing will produce more lift than the one which is retracting, and thus, a rolling moment is produced on the rotor.

$$\tau_R = C_{\tau_R} \rho A (\Omega R)^2 R \quad (4)$$

Using the method of Mulder [12] and gathering all the forces and torques coming from different sources, the general dynamics of the whole quad-rotor can be written in the first place, then some simplifications are considered in the second place for implementation purposes and will be detailed in the next paragraph.

There are some additional effects not included in these equations like it was mentioned in the assumptions such as the ground effect [10] and the blade flapping [2] [3].

Added to the rotor dynamics, the motor dynamics are now detailed. Knowing that the used motors are DC motors having a very low inductance, motor dynamics are approximated by this relation:

$$J_{mot} \cdot \frac{d\Omega_{mot}}{dt} = -\frac{k_m^2}{R_{mot}} \cdot \Omega_{mot} - \tau_d + \frac{k_m}{R_{mot}} \cdot u \quad (5)$$

2) Simplified dynamics

In order to make the modeling possible, a simplified model should take place instead of the complete one taking into consideration only the actuator forces and moments and central weight m (Fig.1). This simplification is done because previous simulations of the model [3] have shown that there is no big difference between the complete and simplified models, so the other sources can be neglected.

Consequently, the dynamic equations can be rewritten as below:

$$m \cdot \dot{u} = -m \cdot (g \cdot \sin \theta - q \cdot w + r \cdot v) \quad (6)$$

$$m \cdot \dot{v} = m \cdot (g \cdot \cos \theta \cdot \sin \varphi - r \cdot u + p \cdot w) \quad (7)$$

$$m \cdot \dot{w} = m \cdot (g \cdot \cos \theta \cdot \cos \varphi - p \cdot v + q \cdot u) - U_1 \quad (8)$$

$$I_{xx} \cdot \dot{p} = (I_{yy} - I_{zz}) \cdot q \cdot r + l U_2 \quad (9)$$

$$I_{yy} \cdot \dot{q} = (I_{zz} - I_{xx}) \cdot r \cdot p + l U_3 \quad (10)$$

$$I_{zz} \cdot \dot{r} = (I_{xx} - I_{yy}) \cdot p \cdot q + U_4 \quad (11)$$

Where U_1 , U_2 , U_3 and U_4 are parameters ensuring the establishment of a formula is derived to calculate the motor voltages from the inputs.

The thrust force (1) and the moment (3) can be rewritten for the flight in hover as below:

$$F_{T_i} \approx b \cdot \Omega_i^2 \quad (12)$$

$$\tau_{Q_i} \approx d \cdot \Omega_i^2 \quad (13)$$

Where: Ω is the angular speed for each rotor according to the disposition previously given in Modelica, b and d are respectively the thrust and drag factors in hover:

$$b = C_{F_T} \cdot \rho \cdot A \cdot R^2 \quad (14)$$

$$d = C_{\tau_Q} \cdot \rho \cdot A \cdot R^3 \quad (15)$$

Thus, the inputs defined previously are written as:

$$U_1 = \sum_{i=1}^4 F_{T_i} = b \cdot (\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \quad (16)$$

$$U_2 = (-F_{T_2} + F_{T_4}) = b \cdot (-\Omega_2^2 + \Omega_4^2) \quad (17)$$

$$U_3 = (F_{T_1} - F_{T_3}) = b \cdot (\Omega_1^2 - \Omega_3^2) \quad (18)$$

$$U_4 = (-1)^i \cdot \sum_{i=1}^4 \tau_{Q_i} = d \cdot (-\Omega_1^2 + \Omega_2^2 - \Omega_3^2 + \Omega_4^2) \quad (19)$$

III. IMPLEMENTATION IN MODELICA

A. Multi Body Model

Since a faithful model to reality is required, these are the improvements made in Modelica:

- Airframe structure: The four point masses representing the propulsion groups can be ameliorated. Thanks to the different libraries especially Mechanics-Multi Body-Parts, motors and propellers can be detailed.
- Used materials: An improvement in the field of the used material can be made because all the Parts are assumed to be initially of steel and it's obvious that the qua-rotor's components are not necessarily of the same material and it's known that materials play a big role in control and stability.
- Components disposing configuration: Another helpful tool is about how to dispose components in the Diagram Layer. Thus, numerous classes can be created.

Responding to all the previous criteria and after many trials, the optimal configuration is composed of three classes: a main class, a propeller class shown in Fig.3 and a motor class detailed below.

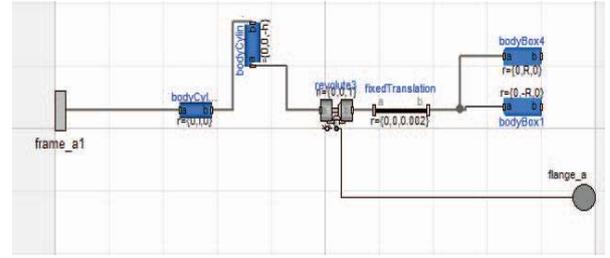


Fig.3. Propeller class

The main class presents the general airframe but the propeller class details the geometry of the propeller thanks to a Body Cylinder of length l representing the arm of Polystyrene related to the gravity center of the quad-rotor, another Body Cylinder of h length of aluminum representing the distance between the propeller plan and the center of gravity plan, two Body Boxes of length R , the radius of the propeller, of Nylon6.6 representing both blades and a Revolute component ensuring the rotation of the propeller around the third axis.

B. Dynamic Model

For the motor, its dynamics can be implemented in a specified class like it's shown in Fig.4 thanks to mechanical and electrical components and some connectors.

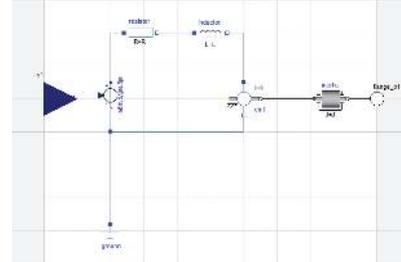


Fig.4. Motor Class

All the forces and moments are applied in the center of gravity of the quad-copter [13]. For the implementation in Modelica, there are two points that have to be respected:

- The speed sensor measuring the angular velocity should be positioned for each rotor.
- All the torques' expressions are multiplied by the arm length (distance between the quad-rotor center of gravity and the rotor) since the choice was made earlier to apply the torques on the quad-rotor center of gravity.

In hovering flight, the applied forces are identical for all the rotors, whereas the moments are distinguished from a propeller to another.

$$F_{R_i} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}_{R_i} = \begin{Bmatrix} 0 \\ 0 \\ F_{T_{R_i}} \end{Bmatrix}$$

$$\tau_{R_1} = \begin{Bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{Bmatrix}_{R_1} = \begin{Bmatrix} 0 \\ l.F_{T_{R_1}} \\ -\tau_{Q_{R_1}} \end{Bmatrix} \quad \tau_{R_2} = \begin{Bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{Bmatrix}_{R_2} = \begin{Bmatrix} -l.F_{T_{R_2}} \\ 0 \\ \tau_{Q_{R_2}} \end{Bmatrix}$$

$$\tau_{R_3} = \begin{Bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{Bmatrix}_{R_3} = \begin{Bmatrix} 0 \\ -l.F_{T_{R_3}} \\ -\tau_{Q_{R_3}} \end{Bmatrix} \quad \tau_{R_4} = \begin{Bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{Bmatrix}_{R_4} = \begin{Bmatrix} l.F_{T_{R_4}} \\ 0 \\ \tau_{Q_{R_4}} \end{Bmatrix}$$

Indeed, Fig.5 illustrates the dynamic changes made in the Propeller1 class in Modelica.

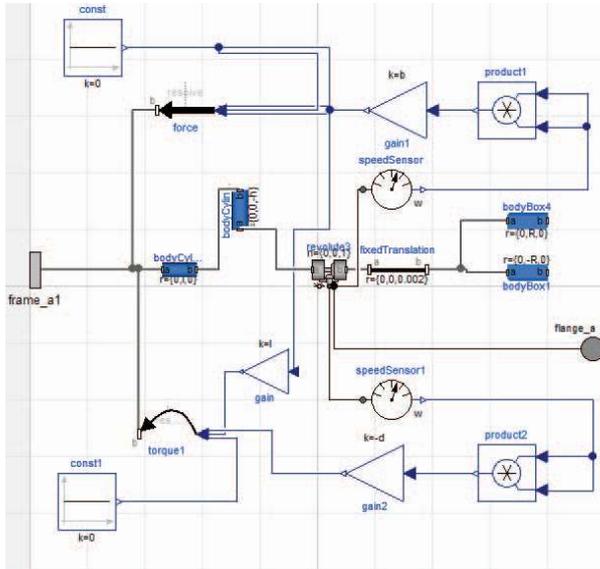


Fig.5. Propeller 1 model implemented in Modelica

Added to that, a new component ensures the six degrees of freedom for the whole system. Then, in order to ensure a stabilized vertical position Z of the quad-rotor in hover, an absolute position sensor is related to its center of gravity and to four PID controllers as it's illustrated in Fig.6.

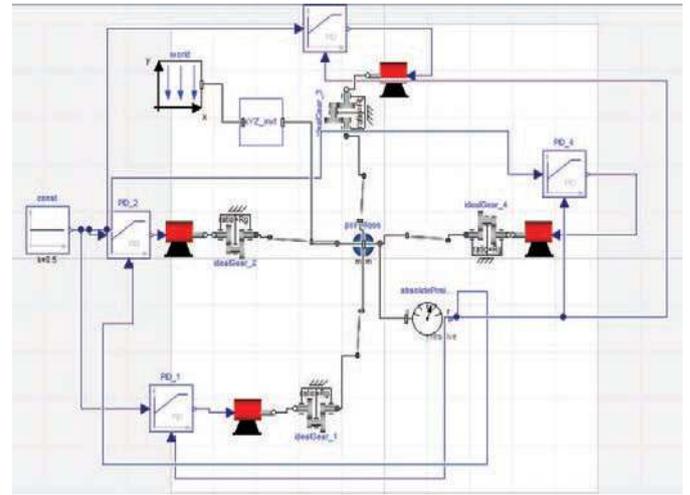


Fig.6. Vertical position control model

The implemented PID controllers are the same for all the four motors and their parameters are listed in TABLE II :

TABLE II. PID parameters

Position	K	T _i	T _d
Z	8	0.5	10

Those parameters are used to simulate the quad-rotor in hover. The result of this simulation is shown in Fig.7.

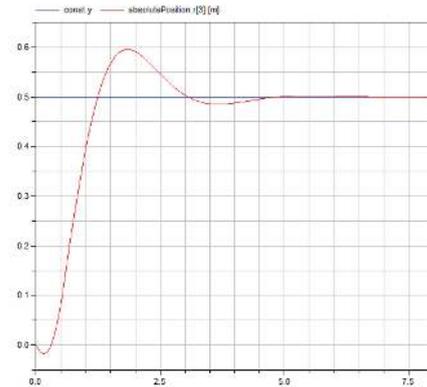


Fig.7. Stabilized vertical position

At last, a 3D window can visualize clearly the 3D quad-rotor integrated in Modelica like it's illustrated by Fig.8.

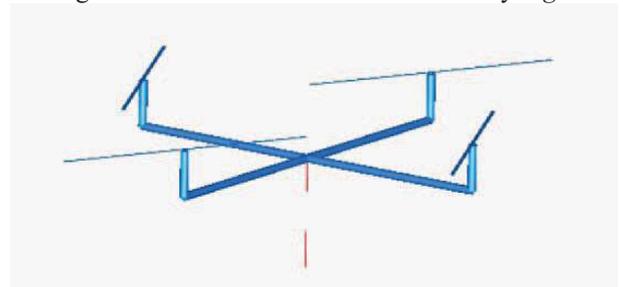


Fig.8. 3D quad-rotor

IV. OPTIMIZATION

This optimization has taken place because of a necessity to study the difference between the first quad-rotor and the redesigned one in terms of characteristics and control parameters such as thrust, setting time; given by Modelica. A

comparison will be made between the two quad-rotors on different levels.

A. Related works

The quad-rotor is a complex system where different disciplines are involved and coupled. This makes of it an excellent environment for multidisciplinary optimization.

The previous works which dealt with rotorcraft optimization are numerous. Some of them were interested in just one discipline such as the works of E. Fresk who tried to optimize the power consumption and the force derivative separately [14] or the works of D. Poinot who tried to get the optimal trajectory in his thesis [15]. Other works dealt with different disciplines such as the works done by Ian D. Cowling et al. which have as aims making a LQR control and optimizing the trajectory with obstacle avoidance at the same time [16]. A. DiCesare tried to ensure an autonomous flight while minimizing lateral drift and reducing 10% of the total weight [17]. Among these works, the works done by A.S. Khalid [18] and S. Ashok et al. [19][20] considered as the most relevant ones because they made a whole Systems Engineering study and linked it to a Product Lifecycle Management approach. This integration of different disciplines through a common platform permitted the optimization and the resolution of multi-disciplinary problems including power, weight, cost, etc...via ModelCenter which was related to many other tools such as Catia V5 and Matlab.

B. Problem formulation

If the quad-rotor makes a flight in hover and stays in a stabilized position, the required thrust has a constant value in order to carry the total of the quad-rotor and its payload. The ideal power per propeller P_p needed to produce this constant thrust per propeller T_p depends only, in this case, on the total area A swept out from the rotor and this is shown by (20).

$$P_p = \frac{T_p^{3/2}}{\sqrt{2 \cdot \rho \cdot A}} \quad (20)$$

Equation (20) is mentioned in the disk actuator theory which is a whole mathematical model of an ideal actuator disk like a propeller done by W.J.M. Rankine [21], A.G. Greenhill and R.E. Froude [22].

From one side, in order to minimize the consumption of power required to produce the constant thrust, the rotor radius has to be increased. From the other side, a higher propeller radius leads to a rise in the weight and the size of the quad-rotor especially in presence of payload coming from the added masses of different accessories such as cameras. All these factors result in a reduction of the endurance. Consequently, an optimal rotor radius exists that solves this multidisciplinary problem. This case is, therefore, a multidisciplinary optimization taking into account rotor radius, weight, payload and endurance with a specific focus on just one part of the quad-copter which is the propeller. The aim of this work is to obtain better characteristics of the existing OS4 by rising the

percentage of payload in the total mass carried by the four rotors in hover, all in reducing the consumed power, minimizing the total weight of the quad-rotor and increasing his endurance. The optimization problem is written below:

$$\begin{cases} \min_{R \in A} f(R); A = [R_{\min}, R_{\max}] = [110, 180] \\ f = \{P_p, M_p\} \end{cases} \quad (21)$$

In order to realize this multidisciplinary optimization, (20) is used in a ModelCenter/ Catia V5 environment in order to get the optimal rotor radius for the given thrust. Then, all the components are chosen based on this result with a necessity to minimize their masses and their costs. Therefore, a redesigned quad-rotor takes its place with better optimized characteristics.

C. Implementation in ModelCenter

Starting from the original design of OS4, some calculations are made in order to increase the possible payload of the quad-rotor with respect to all the parameters cited previously starting from a total mass M_t equal to 1.25 kg.

As the optimization of the propeller is the first step in this process. Its model is elaborated in Catia V5 and all its geometry variables are parameterized and its mass is calculated and related to ModelCenter where a model is created. Indeed, this model makes a link between Catia and a mathematical script via an optimization tool like it's shown in Fig.9.



Fig.9. Multidisciplinary optimization model in ModelCenter

The Catia model has as significant input the rotor radius and as output the propeller mass. For the script, it has as input the radius and as output the ideal power per propeller.

Consequently, the radius is an intermediate variable ensuring the coupling between both objective functions: power and mass.

Throughout the optimization tool, the objectives are defined: minimizing both mass and power under radius constraints. The design variable which is the radius is declared and DAKOTA Multiobjective Genetic Algorithm (MOGA) is chosen.

Next, the link between Catia and Modelcenter is dynamic and the geometry variables are updated while the optimizer changes the design variables. All the geometries have to be parameterized in Catia, otherwise, ModelCenter is not able to call them and the information can't pass between both environments. Consequently, this system is totally parametered and tightly coupled enabling an automated update process by ModelCenter, reducing calculation errors and improving design cycle efficiency.

Finally, the results are shown in this Pareto diagram shown in Fig.10. The mass propeller and the ideal power calculated for each radius according to the described process are gathered in this Pareto diagram.

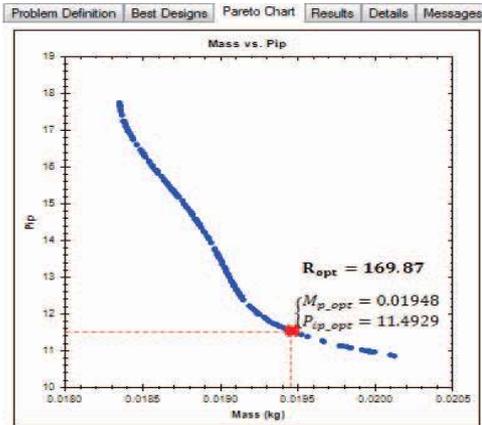


Fig.10. Optimization results

As it's illustrated, there is a whole interval for which both mass and power are acceptable. Out of it, the mass is very big in comparison with the power and the vice versa. That's why, the propeller radius chosen is 169.87 mm.

Obtaining the optimal values of radius, power and propeller mass is the first step to redesign the quad-rotor by choosing new components conform to this multi-disciplinary optimization.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, we have shown interest in a submerging system which is the quad-rotor. A complete study of its geometric and dynamic modeling has taken place before integrating it in Modelica throughout its different blocks according to a very well chosen bodies disposition.

Furthermore, a parametric geometry of the propeller has been elaborated in Catia and linked to ModelCenter via a tightly coupled equation gathering power, mass and rotor radius in order to make a multidisciplinary optimization of the whole system in hover.

This methodology contributes to improve all the quad-rotor characteristics especially weight, payload, required power in hover and flight time.

Our main contributions were : the implementation of a quad-rotor process in Modelica, the establishment of a package of components proper for drones such as propellers. It's an approach for modeling quad-copters. For the optimization, we have integrated many models of different levels starting from the modeling and simulation level and reaching the component level which is the propeller in this paper.

Future works will continue the redesign of the quad rotor. After obtaining the optimal radius and power, the different new components will be chosen from merchant sites with respect to weight and cost. Finally, the system will be validated and total cost and mass reduction will be calculated. Added to that, the new optimized quad-copter can be integrated in the Modelica model to deal with control issues and simulate flight in order to show the achieved improvements.

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