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Nested Locally Hamiltonian Graphs and the Oberly-Sumner Conjecture

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Abstract

A graph G is *locally \mathcal{P}* , abbreviated $L\mathcal{P}$, if for every vertex v in G the open neighbourhood $N(v)$ of v is non-empty and induces a graph with property \mathcal{P} . Specifically, a graph G without isolated vertices is *locally connected* (LC) if $N(v)$ induces a connected graph for each $v \in V(G)$, and *locally hamiltonian* (LH) if $N(v)$ induces a hamiltonian graph for each $v \in V(G)$. A graph G is *locally locally \mathcal{P}* (abbreviated $L^2\mathcal{P}$) if $N(v)$ is non-empty and induces a locally \mathcal{P} graph for every $v \in V(G)$. This concept is generalized to an arbitrary degree of nesting, to make it possible to work with $L^k C$ and $L^k H$ graphs for any integer $k \geq 0$ (with $L^0 C$ and $L^0 H$ meaning *connected* and *hamiltonian*, respectively.) We call a graph *locally k -nested hamiltonian* if it is $L^m C$ for $m = 0, 1, \dots, k$ and $L^k H$. The class of locally k -nested-hamiltonian graphs contains important subclasses. For example, Skupien had already observed in 1963 that the class of connected LH graphs (which is the class of locally 1-nested-hamiltonian graphs) contains all triangulations of closed surfaces. We show that for any $k \geq 1$ the class of locally k -nested-hamiltonian graphs contains all simple-clique $(k+2)$ -trees. In 1979 Oberly and Sumner proved that every connected $K_{1,3}$ -free graph that is locally connected is hamiltonian. They conjectured that for every $k \geq 1$, every connected $K_{1,k+3}$ -free graph that is locally $(k+1)$ -connected is hamiltonian. We show that locally k -nested-hamiltonian graphs are locally $(k+1)$ -connected and consider the weaker conjecture that every $K_{1,k+3}$ -free graph that is locally k -nested-hamiltonian is hamiltonian. We show that if our conjecture is true, it would be “best possible” in the sense that for every $k \geq 1$ there exist $K_{1,k+4}$ -free locally k -nested-hamiltonian graphs that are nonhamiltonian. We also attempt to establish the minimum order of nonhamiltonian locally k -nested-hamiltonian graphs and investigate the complexity of the Hamilton Cycle Problem for locally k -nested-hamiltonian graphs with restricted maximum degree.

Keywords: locally traceable, locally hamiltonian, Hamilton Cycle Problem

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1 Introduction and Background

A graph G is *traceable* if it has a *Hamilton path* (a path that visits every vertex in G), and *hamiltonian* if it has a *Hamilton cycle* (a cycle that visits every vertex in G). A graph G is *fully cycle extendable* if every vertex in G lies in a 3-cycle and for every nonhamiltonian cycle C of G there is a cycle C^* in G that contains all the vertices of C plus a single new vertex. The number of components of a graph G is denoted by $\text{comp}(G)$. We call a connected graph G *1-tough* if $\text{comp}(G - S) \leq |S|$ for every vertex cut S of G . We note that full cycle extendability is a stronger property than hamiltonicity, and toughness is a weaker property.

For standard concepts we use the notation and terminology of [2].

Let v be a vertex in a graph G . The *open neighbourhood* $N(v)$ of v consists of all the vertices in $G - v$ that are adjacent to v , and the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $\langle S \rangle$. We use $n(G)$ to denote the *order* of G and $d(v)$ to denote the *degree* of v in G , i.e., $n(G) = |V(G)|$ and $d(v) = |N(v)|$. We also write $N_S(v) = N(v) \cap S$ and $d_S(v) = |N_S(v)|$.

For a given graph property \mathcal{P} , we say a graph G is *locally \mathcal{P}* if $N(v) \neq \emptyset$ and $\langle N(v) \rangle$ has the property \mathcal{P} for each $v \in V(G)$. For example, G is *locally connected* (*LC*), *locally traceable* (*LT*) and *locally hamiltonian* (*LH*) if for each $v \in V(G)$ the set $N(v)$ is nonempty and the graph $\langle N(v) \rangle$ is connected, traceable and hamiltonian, respectively.

A graph G is *locally locally \mathcal{P}* (abbreviated $L^2\mathcal{P}$) if $N(v) \neq \emptyset$ and $\langle N(v) \rangle$ is locally \mathcal{P} for every $v \in V(G)$. We extend this concept inductively to arrive at the following definition.

Definition 1.1. A graph is $L^0\mathcal{P}$ if it has the property \mathcal{P} . For any integer $k \geq 1$, a graph G is $L^k\mathcal{P}$ if $N(v) \neq \emptyset$ and $\langle N(v) \rangle$ is $L^{k-1}\mathcal{P}$ for every $v \in V(G)$. We say G is $L^{\leq k}\mathcal{P}$ if G is $L^m\mathcal{P}$ for $m = 1, \dots, k$.

Throughout this paper k will denote a non-negative integer.

For a given graph H , a graph G is said to be *H -free* if G does not contain H as an induced subgraph. Our interest in local properties that imply hamiltonicity was sparked by the well-known theorem of Oberly and Sumner, stated below.

Theorem 1.2. [13] *If G is a $K_{1,3}$ -free, connected, LC graph of order at least 3, then G is hamiltonian.*

A graph G is *k -connected* if $G - S$ is a connected graph for every subset S of $V(G)$ consisting of fewer than k vertices. Oberly and Sumner conjectured an extension of their theorem.

Conjecture 1.3. [13] *If G is a $K_{1,k+3}$ -free connected, locally $(k+1)$ -connected graph, then G is hamiltonian, for $k \geq 0$.*

It is easy to prove the weaker result that graphs satisfying the requirements of Conjecture 1.3 are 1-tough, although to the best of our knowledge, this is not in the literature, so we present the proof here. We will need the following result due to Chartrand and Kapoor.

Theorem 1.4. [3] *If G is a connected, locally k -connected graph, $k \geq 0$, then G is $(k+1)$ -connected.*

Theorem 1.5. *If G is a $K_{1,k+3}$ -free connected, locally $(k+1)$ -connected graph, $k \geq 0$, then G is 1-tough.*

Proof. Let S be any vertex cut of G . Since G is locally $(k+1)$ -connected, it follows from Theorem 1.4 that G is $(k+2)$ -connected, so each component of $G - S$ has at least $k+2$ neighbours in S . On the other hand, since G is K_{k+3} -free, each vertex in S has neighbours in at most $k+2$ different components of $G - S$. This implies that $\text{comp}(G - S) \leq |S|$, so G is 1-tough. \square

However, $K_{1,k+3}$ -free connected, locally $(k+1)$ -connected graphs have not been proved to be hamiltonian. In fact, it is not even known whether there exists an integer t such that every $K_{1,4}$ -free locally t -connected graph is hamiltonian. We are therefore interested in replacing the local connectivity condition in the Oberly-Sumner Conjecture with a stronger local condition that might guarantee hamiltonicity. For example, local hamiltonicity is stronger than local 2-connectivity, so the following conjecture is weaker than the case $k = 1$ of the Oberly-Sumner Conjecture.

Conjecture 1.6. *If G is a $K_{1,4}$ -free, connected LH graph, then G is hamiltonian.*

This motivated us to consider locally k -nested-hamiltonian graphs, which we define as follows.

Definition 1.7. A graph G is *locally k -nested-hamiltonian* if G is connected, $L^{\leq k-1}C$ and L^kH .

We shall show in Section 2 that every locally k -nested-hamiltonian graph is locally $(k+1)$ -connected. Thus the following conjecture, which extends Conjecture 1.6, is weaker than Conjecture 1.3.

Conjecture 1.8. *If G is a $K_{1,k+3}$ -free graph that is locally k -nested-hamiltonian, $k \geq 1$, then G is hamiltonian.*

We note that an L^kH graph is also L^kC but not necessarily L^mC if $0 \leq m \leq k-1$. For example, Figure 1 depicts a $K_{1,3}$ -free L^3H graph that is L^mC for $m = 0, 2, 3$ but not for $m = 1$, and is obviously not hamiltonian. For this reason we include the requirement that G be L^mC for $m = 0, 1, \dots, k-1$ in Definition 1.7. This is analogous to restricting our investigation to connected graphs when studying the hamiltonicity of LH graphs.

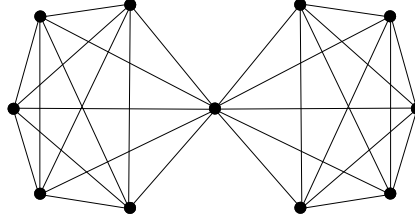


Figure 1: An L^3H graph that is not LC .

We shall show in Section 3 that if Conjecture 1.8 is true, it would be “best possible” in the sense that for each $k \geq 1$ there exists a $K_{1,k+4}$ -free locally k -nested hamiltonian graph that is nonhamiltonian.

It is easily seen that if G is a connected nonhamiltonian LH graph, then $n(G) \geq \Delta(G) + 3$. Suppose an LH graph G contains an induced $K_{1,4}$ with v as its central vertex. Then $\langle N(v) \rangle$ is a hamiltonian graph with independence number 4, so $|N(v)| \geq 8$. Thus, if Conjecture 1.6 is true, it would imply that every connected nonhamiltonian LH graph has maximum degree at least 8 and order at least 11. Indeed, Pareek and Skupień [15] proved that the graph with maximum degree 8 and order 11 depicted in Figure 2 (a) is the smallest connected nonhamiltonian LH graph. (This graph, known as the *Goldner-Harary graph*, was shown by Goldner and Harary [10] to be the smallest nonhamiltonian maximal planar graph.) De Wet [4, 6] showed that there are four nonhamiltonian, connected LH graphs of order 11 and they all have maximum degree 8 (Figure 2).

In Section 3 we prove that the minimum order of a locally 2-nested-hamiltonian graphs is 13. By generalizing the graph of order 11 in Figure 2(a) we obtain for each $k \geq 1$ a locally k -nested-hamiltonian graph of order $9 + 2k$ that is non- L^mH for $m = 0, 1, \dots, k-1$. On the other hand, a generalization of the graph in Figure 2(b) yields for each $k \geq 1$ a nonhamiltonian connected graph that is $L^{\leq k}H$. We show that if Conjecture 1.8 is true, the minimum order of a nonhamiltonian connected $L^{\leq k}H$ graph would be $2k + 9$, as is the case for $k = 1, 2$.

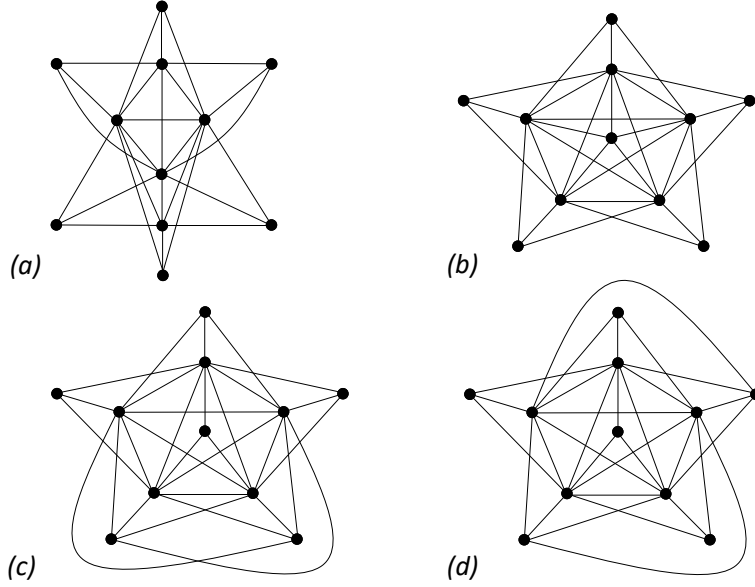


Figure 2: The smallest connected LH graphs that are not hamiltonian.

Pareek [14] claimed that every nonhamiltonian connected LH graph has maximum degree at least 8, but there are flaws in his “proof” that we have not been able to rectify, as discussed in [4, 6]. If Conjecture 1.6 is true, it would immediately prove Pareek’s (as yet unproved) claim. It is shown in [18] that every connected LH graph with maximum degree at most 6 is fully cycle extendable. We showed in [5] that the Hamilton Cycle Problem for LH graphs with maximum degree 9 is NP-complete.

We show in Section 4 that every locally 2-nested-hamiltonian graph with maximum degree at most 7 is fully cycle extendable, while the Hamilton Cycle Problem for locally 2-nested-hamiltonian graphs with maximum degree 12 is NP-complete.

A k -tree is a graph that can be constructed by starting with a K_{k+1} and then recursively performing the following operation. Choose a k -clique in the graph, add a new vertex and add an edge between the new vertex and each vertex in the chosen k -clique. If no k -clique is chosen more than once during the construction, the resulting k -tree is called a *simple-clique k -tree*. In Section 5 we investigate the connection between LH graphs, 3-trees and maximal planar graphs, and we show that for every integer $k \geq 0$, the simple-clique k -trees constitute a subclass of the class of $L^{k-2}H$ graphs.

2 Basic properties and constructions of L^kC graphs and L^kH graphs

The proposition below provides a characterization of $L^k\mathcal{P}$ graphs for $k \geq 1$, which may be used as a convenient alternative definition for these graphs.

Proposition 2.1. *For $k \geq 1$, a graph G is $L^k\mathcal{P}$ if and only if each of the following holds.*

- (1) *If $1 \leq m \leq k$, and X is an m -clique in G , then X is contained in an $(m+1)$ -clique in G .*
- (2) *If X is a k -clique in G , then the neighbourhood intersection $\bigcap_{x \in V(X)} N(x)$ induces a graph with property \mathcal{P} .*

Proof. The proof is by induction on k .

First, suppose G is $L^k\mathcal{P}$. If $k = 1$, then it follows immediately from the definition of an $L\mathcal{P}$ graph that (1) and (2) hold.

Now let $k \geq 2$ and let X be an m -clique in G . If $m = 1$, then X is contained in a 2-clique by Definition 1.1. If $m \geq 2$, let $x \in V(X)$. Then $X - x$ is an $(m - 1)$ -clique in $\langle N(x) \rangle$. But $\langle N(x) \rangle$ is an $L^{k-1}\mathcal{P}$ graph by definition 1.1, so by our induction hypothesis, $X - x$ is contained in an m -clique Y in $\langle N(x) \rangle$. But then $\langle V(Y) \cup \{x\} \rangle$ is an $(m + 1)$ -clique in G that contains X . Thus G satisfies (1).

Now let X be a k -clique in G with $V(X) = \{x_1, \dots, x_k\}$. Then $\{x_1, \dots, x_{k-1}\}$ induces a $(k - 1)$ -clique in $\langle N(x_k) \rangle$. But $\langle N(x_k) \rangle$ is $L^{k-1}\mathcal{P}$ by Definition 1.1, so our induction hypothesis implies that the subgraph of $\langle N(x_k) \rangle$ induced by $\bigcap_{i=1}^{k-1} N_{N(x_k)}(x_i)$ has the property \mathcal{P} . But $\bigcap_{i=1}^{k-1} N_{N(x_k)}(x_i) = \bigcap_{i=1}^k N(x_i)$ (since $N_{N(x_k)}(x_i) = N(x_i) \cap N(x_k)$ for $i = 1, \dots, k - 1$), so G also satisfies (2).

Now suppose (1) and (2) hold. If $k = 1$, then (1) implies that $N(v)$ is nonempty for every $v \in V(G)$, and (2) implies that $\langle N(v) \rangle$ induces a graph with property \mathcal{P} for every $v \in V(G)$, so then G is $L\mathcal{P}$.

Now let $k \geq 2$ and consider any $v \in V(G)$. If $1 \leq m \leq k - 1$ and X is an m -clique in $\langle N(v) \rangle$, then $V(X) \cup \{v\}$ induces an $(m + 1)$ -clique in G , so then (2) implies that X lies in an $(m + 2)$ -clique Y in G . But then $Y - v$ is an $(m + 1)$ -clique in $\langle N(v) \rangle$ that contains X . Thus the graph $\langle N(v) \rangle$ satisfies (1) with k replaced by $k - 1$. Now let X be a $(k - 1)$ -clique in $\langle N(v) \rangle$ with $V(X) = \{x_1, \dots, x_{k-1}\}$. Then $\{x_1, \dots, x_{k-1}, v\}$ induces a k -clique in G , so (2) implies that $\bigcap_{i=1}^{k-1} N(x_i) \cap N(v)$, induces a graph with property \mathcal{P} , i.e., the subgraph of $\langle N(v) \rangle$ induced by $\bigcap_{i=1}^{k-1} N_{N(v)}(x_i)$ has the property \mathcal{P} . Hence $\langle N(v) \rangle$ also satisfies (2) with k replaced by $k - 1$. Hence, by our induction hypothesis, $\langle N(v) \rangle$ is an $L^{k-1}\mathcal{P}$ graph, so by Definition 1.1, G is an $L^k\mathcal{P}$ graph. \square

If v is any vertex in an LH graph, then $\langle N[v] \rangle$ contains a wheel of order $d(v) + 1$, centered at v , as spanning subgraph. The following result is therefore useful when dealing with LH graphs.

Lemma 2.2. *Suppose a graph G contains a wheel W with centre v and rim C . Let C be the cycle $v_0v_1 \dots v_tv_0$ ($t \geq 2$) and let $V(G) - V(W) = \{x_1, \dots, x_r\}$. If $d_C(x_i) \geq 4$ for each $i \in \{1, \dots, r\}$, then the following hold.*

- (a) *If $r = 1$, then G is hamiltonian.*
- (b) *If $r = 2$ and there is a pair of consecutive vertices on C that each have a neighbour in $\{x_1, x_2\}$, then G is hamiltonian.*
- (c) *If $r = 3$ and there are two pairs of consecutive vertices $\{v_i, v_{i+1}\}$ and $\{v_j, v_{j+1}\}$ ($i \neq j$) on C such that $\{v_i, v_{i+1}\} \subseteq N(x_1)$ and $\{v_j, v_{j+1}\} \subseteq N(x_2)$, then G is hamiltonian.*

Proof. Hamilton cycles that illustrate the proof of the result in each of the three cases are shown in Figure 3. Since $d_C(x_2) \geq 4$, we may assume in Case (b) that v_i, v_{i+1}, v_k, v_l are four distinct vertices. In Case (c), we may for example have $v_k = v_i$, but then we can choose v_l to be neither v_{i+1} nor v_{j+1} and we still have a valid Hamilton cycle. \square

Corollary 2.3. *If G is a connected nonhamiltonian LH graph, then $n(G) \geq \Delta(G) + 3$.*

Our next result is implied by Proposition 2.1.

Lemma 2.4. *If G is an L^kH graph, then every vertex of G lies in a $(k + 2)$ -clique and $\delta(G) \geq k + 2$.*

Proof. Let $v \in V(G)$. Repeated application of Proposition 2.1(1) shows that v lies in a k -clique X . By Proposition 2.1, the neighbourhood intersection $\bigcap_{x \in X} N(x)$ is nonempty and induces a hamiltonian graph. But a hamiltonian graph has at least three vertices and contains K_2 's, so it follows that $|N(v)| \geq k + 2$ and v lies in a K_{k+2} . \square

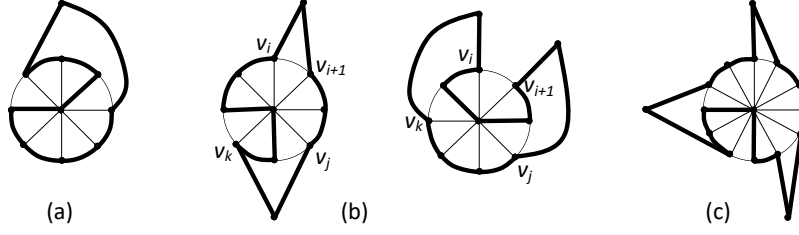


Figure 3: Hamilton cycles illustrating the proof of Lemma 2.2.

Corollary 2.5. *The smallest L^kH graph is K_{k+3} .*

Corollary 2.6. *If G is an L^kH graph and $d(v) = k + 2$ for some $v \in V(G)$, then $\langle N(v) \rangle$ is a $(k + 2)$ -clique in G .*

Lemma 2.7. *If v is a vertex in an L^kH graph G such that $\langle N(v) \rangle$ is contained in a clique of order at least $k + 3$ in G , then $G - v$ is also L^kH .*

Proof. Only the neighbourhoods of vertices adjacent to v are affected by the removal of v from G , so to show that $G - v$ is L^kH , we need only consider the k -cliques that are contained in $\langle N(v) \rangle$. Let X be a k -clique in $\langle N(v) \rangle$. Then the graph $\langle \bigcap_{x \in V(X)} N(x) \rangle$ contains the vertex v plus at least three vertices in $G - v$, so it has a Hamilton cycle C that contains a subpath u_1vu_2 , with $u_1, u_2 \in N(v)$. Replacing the path u_1vu_2 in C with the edge u_1u_2 yields a Hamilton cycle of $\langle \bigcap_{x \in V(X)} N_{G-v}(x) \rangle$. Hence $G - v$ is L^kH . \square

Theorem 1.4 implies the following.

Theorem 2.8. *If G is locally k -nested-hamiltonian graph, then G is $(k + 2)$ -connected and locally $(k + 1)$ -connected.*

Proof. It is easily seen that a connected LH graph is 3-connected and locally 2-connected, so the result holds for $k = 1$. Now let $k \geq 2$, and let $v \in V(G)$. Then by Definitions 1.1 and 1.7, $\langle N(v) \rangle$ is locally $(k - 1)$ -nested-hamiltonian, so by the induction hypothesis, $\langle N(v) \rangle$ is $(k + 1)$ -connected. Hence G is locally $(k + 1)$ -connected and therefore, by Theorem 1.4, G is $(k + 2)$ -connected. \square

De Wet [4, 6] developed a method, called *triangle identification*, for obtaining LH graphs with certain properties by combining suitable LH graphs. A triangle Y in an LH graph G is called *suitable for triangle identification*, or simply a *suitable triangle* if for every $y \in Y$ there is a Hamilton cycle in $\langle N(y) \rangle$ that contains the edge between the two vertices in $Y - y$. The following results are proved in [4, 6].

Theorem 2.9. *For $i = 1, 2$ let G_i be an LH graph that contains a suitable triangle Y_i . Let G be the graph obtained from G_1 and G_2 by identifying the triangle Y_1 with the triangle Y_2 . Then the following hold.*

- (1) G is LH .
- (2) If G is hamiltonian, then both G_1 and G_2 are hamiltonian.

A $(k + 2)$ -clique Y in an L^kH graph G is called *suitable for K_{k+2} -identification*, or simply a *suitable $(k + 2)$ -clique* if for each k -clique X in Y , the graph induced by $\bigcap_{v \in V(X)} N(v)$ has a Hamilton cycle

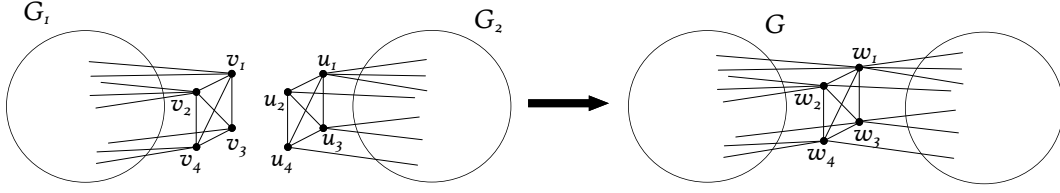


Figure 4: The K_4 -identification procedure.

that contains the edge between the two vertices in $V(Y) - V(X)$. The procedure of combining two L^2H graphs by means of K_4 -identification is illustrated in Figure 4.

Our next result is a straightforward generalization of Theorem 2.9.

Theorem 2.10. *For $i = 1, 2$ suppose G_i is a locally k -nested-hamiltonian graph that contains a suitable $(k+2)$ -clique Y_i . Let G be the graph obtained from G_1 and G_2 by identifying the $(k+2)$ -clique Y_1 with the $(k+2)$ -clique Y_2 . Then the following hold.*

- (1) G is locally k -nested-hamiltonian.
- (2) If G is hamiltonian, then both G_1 and G_2 are hamiltonian.

Proof. (1) For $i = 1, 2$, it follows from Proposition 2.1 that if X_i is an m -clique in G_i , then $\bigcap_{x \in X_i} N_{G_i}(x)$ is nonempty and induces a connected graph if $m \leq k$, which is also hamiltonian if $m = k$.

Let Y be the $(k+2)$ -clique obtained by identifying Y_1 and Y_2 . Let X be an m -clique in G , $1 \leq m \leq k$. If X has a vertex in $V(G_1) - Y$, then since there are no edges between $V(G_2) - Y$ and $V(G_1) - Y$, it follows that X lies in G_1 and that $\bigcap_{x \in X} N_G(x) = \bigcap_{x \in X} N_{G_1}(x)$. Hence $\bigcap_{x \in X} N_G(x)$ induces a connected graph, which is also hamiltonian if $m = k$. The same is true if X has a vertex in $G_2 - Y$.

Now suppose X is a subgraph of Y . Then, for $i = 1, 2$, the subgraph of G_i induced by $\bigcap_{x \in X} N_{G_i}(x)$ is connected (since G_i is L^kC). We note that $\bigcap_{x \in X} N_G(x) = (\bigcap_{x \in X} N_{G_1}(x)) \cup (\bigcap_{x \in X} N_{G_2}(x))$ and that $V(Y) - V(X)$ is contained in both $\bigcap_{x \in X} N_{G_1}(x)$ and $\bigcap_{x \in X} N_{G_2}(x)$. This implies that $\bigcap_{x \in X} N_G(x)$ induces a connected graph. Moreover, if $m = k$, then since Y_i is a suitable $(k+2)$ -clique and X_i is a k -clique in Y_i , the graph $\langle \bigcap_{x \in X_i} N_{G_i}(x) \rangle$ has a Hamilton cycle C_i containing the edge between the two vertices in $Y_i - V(X_i)$, for $i = 1, 2$. It follows that $\langle \bigcap_{x \in X} N_G(x) \rangle$ is a hamiltonian graph. This proves that G is locally k -nested-hamiltonian

- (2) Suppose $v_0 v_1 \dots v_n v_0$ is a Hamilton cycle of G . If $v_i v_{i+1} \dots v_j$ is a path of length at least 2 on C that lies in G_2 and $v_i, v_j \in V(X)$, then we replace the path $v_{i+1} \dots v_{j-1}$ with the edge $v_i v_j$. We do this for every such path in G_1 that has its end-vertices in X . The result is a Hamilton cycle of G_1 . A similar argument shows that G_2 also has a Hamilton cycle. □

Remark 2.11. *If $m < k$, then the K_{k+2} -identification of two locally m -nested hamiltonian graphs does not necessarily result in an L^mH graph. For example, the graph in Figure 6(a) was constructed by starting with a K_5 and then recursively using K_4 -identification with other K_5 's. The result is a graph that is locally 2-nested hamiltonian, but not LH .*

By a slight modification of the proof of Theorem 2.10 we can also prove the following.

Theorem 2.12. *Suppose G is a locally k -nested-hamiltonian graph and G contains two disjoint suitable $(k+2)$ -cliques X_1 and X_2 such that $N(X_1) \cap N(X_2) = \emptyset$. Then the graph obtained from G by identifying X_1 with X_2 is also locally k -nested-hamiltonian.*

A suitable $(k+2)$ -clique in a locally k -nested-hamiltonian graph G_1 may be used only once in K_{k+2} -identification. That is to say, if G was obtained by identifying suitable $(k+2)$ -cliques of two locally k -nested-hamiltonian graphs G_1 and G_2 to a single $(k+2)$ -clique Y , then Y is not a suitable $(k+2)$ -clique of G . This is because, if X is a $(k+2)$ -clique in Y and $V(Y) - V(X) = \{y_1, y_2\}$, then any Hamilton cycle in $\bigcap_{x \in X} N_G(x)$ contains vertices in both G_1 and G_2 , so it does not contain the edge $y_1 y_2$.

However, the following result implies that certain vertices may be used several times in K_{k+2} -identification, each time as a member of a different $(k+2)$ -clique.

Lemma 2.13. *Let G_1 be a locally k -nested-hamiltonian graph that contains a vertex v_1 such that $d(v_1) = k+2$. Then $\langle N_{G_1}(v_1) \rangle \cong K_{k+2}$ and v_1 can be used $k+2$ times in K_{k+2} -identification, once in combination with each of the $k+2$ distinct subsets of $k+1$ of its neighbours.*

Proof. Let $N_{G_1}(v_1) = \{v_2, v_3, \dots, v_{k+3}\}$. Throughout this proof, the vertices $\{v_1, v_2, \dots, v_{k+3}\}$ that are used in K_{k+2} -identification will retain their labels. Since $d(v_1) = k+2$, it follows from Corollary 2.6 that $\langle N_{G_1}(v_1) \rangle \cong K_{k+2}$. Let G_2, G_3, \dots, G_{k+3} be the locally k -nested-hamiltonian graphs that will successively be used in K_{k+2} -identification to form the graphs $G_{1,2}, G_{1,2,3}, \dots, G_{1,2,\dots,k+3}$. Furthermore, without loss of generality, let G_i be combined with $G_{1,2,\dots,i-1}$ using the $(k+2)$ -clique $\langle \{v_1, v_2, \dots, v_{k+3}\} - \{v_i\} \rangle$, $i = 2, 3, \dots, k+3$ to create the graph $G_{1,2,\dots,i}$. In each K_{k+2} -identification step, let $U = \{u_1, u_2, \dots, u_k\} = \{v_1, v_2, \dots, v_{k+3}\} - \{v_i\} - \{v_l, v_m\}$, where $\{l, m\} \subset (\{1, 2, \dots, k+3\} - \{i\})$, $l \neq m$. It suffices to show that the graph $\langle \bigcap_{u \in U} N_{G_{1,2,\dots,i-1}}(u) \rangle$ has a Hamilton cycle that includes the edge $v_l v_m$ for all possible values of l and m (this shows that $\langle \{v_1, v_2, \dots, v_{k+3}\} - \{v_i\} \rangle$, $i = 2, 3, \dots, k+3$ is a suitable $(k+2)$ -clique).

First consider using K_{k+2} -identification to combine G_1 with G_2 to create the graph $G_{1,2}$. Since G_1 is locally k -nested-hamiltonian, if $v_1 \in U$, then $\bigcap_{u \in U} N_{G_1}(u) = \{v_2, v_l, v_m\}$, and $\langle \bigcap_{u \in U} N_{G_1}(u) \rangle \cong K_3$, and it clearly follows that the edge $v_l v_m$ is part of the Hamilton cycle in $\langle \bigcap_{u \in U} N_{G_1}(u) \rangle$. If $v_1 \notin U$ (that is, $l = 1$ or $m = 1$), then v_1 has degree 2 in $\langle \bigcap_{u \in U} N_{G_1}(u) \rangle$ and again it follows that the edge $v_l v_m$ is part of any Hamilton cycle in $\langle \bigcap_{u \in U} N_{G_1}(u) \rangle$. This argument applies to any choice of l and m . It follows that $G_{1,2}$ is locally k -nested-hamiltonian. Note that after the K_{k+2} -identification is done, the edge $v_l v_m$ in a Hamilton cycle in $\langle \bigcap_{u \in U} N_{G_1}(u) \rangle$ is replaced by a path in the corresponding Hamilton cycle in $\langle \bigcap_{u \in U} N_{G_{1,2}}(u) \rangle$, and the internal vertices of this path all originated from G_2 .

We now proceed with the next K_{k+2} -identification, that between $G_{1,2}$ and G_3 to create the graph $G_{1,2,3}$, and continue in this manner. Consider the case in which we combine $G_{1,2,\dots,i-1}$ with G_i to form the graph $G_{1,2,\dots,i}$, $2 \leq i \leq k+3$. This is done by identifying $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+3}\} \subset V(G_{1,2,\dots,i-1})$ with vertices in $V(G_i)$.

Without loss of generality let $U = \{u_1, u_2, \dots, u_k\} = \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+3}\} - \{v_l, v_m\}$, where $\{l, m\} \subset \{1, 2, \dots, i-1, i+1, \dots, k+3\}$, $l \neq m$. Note that no vertex in $N_{G_{1,2,\dots,i-1}}(v_j)$ originated from $V(G_j)$, for all $j = 2, 3, \dots, k+3$. It follows that $\bigcap_{u \in U} N_{G_{1,2,\dots,i-1}}(u) = \{v_l, v_m, v_i\} \cup X \cup Y$, where $X \subset V(G_l)$, $Y \subset V(G_m)$ if $l, m < i$, respectively, and $Y = \emptyset$ and $Z = \emptyset$ if $l, m > i$, respectively (in this instance, when we write $Y \subset V(G_l)$, we mean every vertex in $Y \cap V(G_{1,2,\dots,i-1})$ originated from $V(G_l)$). It follows that there are at most two internally disjoint paths from v_l to v_m in $\langle \bigcap_{u \in U} N_{G_{1,2,\dots,i-1}}(u) \rangle$, and since by inductive hypothesis $\langle \bigcap_{u \in U} N_{G_{1,2,\dots,i-1}}(u) \rangle$ is hamiltonian, there is a Hamilton cycle in $\langle \bigcap_{u \in U} N_{G_{1,2,\dots,i-1}}(u) \rangle$ that includes the edge $v_l v_m$ (see Figure 5). Therefore in $G_{1,2,\dots,i-1}$, the K_{k+2} graph induced by $\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_{k+3}\}$ is suitable for use in K_{k+2} -identification. □

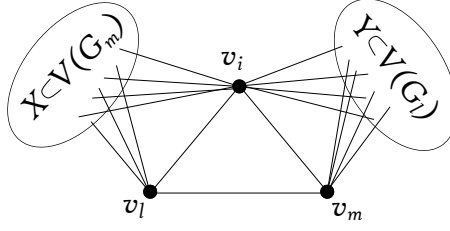


Figure 5: The graph $\langle \bigcap_{u \in U} N_{G_{1,2,\dots,i-1}}(u) \rangle$ used in the proof of Theorem 2.13.

3 Nonhamiltonian and Nontraceable locally k -nested-hamiltonian graphs of small order

As mentioned in Section 1, Pareek and Skupien proved the following.

Theorem 3.1. [15] *The minimum order of a nonhamiltonian connected LH graph is 11.*

De Wet et al. proved the following.

Theorem 3.2. [4, 7] *The minimum order of a nontraceable connected LH graph is 14.*

We shall also use the following result of de Wet.

Theorem 3.3. [4] *If G is a nonhamiltonian connected LH graph of order 12, then $\Delta(G) = 9$.*

We are now ready to prove the main results of this section.

Theorem 3.4. *The minimum order of a nonhamiltonian locally 2-nested-hamiltonian graph is 13.*

Proof. The graph in Figure 6(a), which is redrawn in Figure 10(a), as well as the graph in Figure 14 are examples of locally 2-nested-hamiltonian graphs of order 13.

Now suppose G is a nonhamiltonian locally k -nested-hamiltonian graph of order at most 12.

First, suppose G is not LH. Then there is a vertex $v \in V(G)$ such that $\langle N(v) \rangle$ is nonhamiltonian, so then Theorem 3.1 implies that $|N(v)| = 11$. But then $G = \langle N[v] \rangle$, and $\langle N(v) \rangle$ is traceable by Theorem 3.2, and hence G is hamiltonian.

We therefore assume that G is LH. Now let w be a vertex of degree $\Delta(G)$ in G . Then $\langle N(w) \rangle$ has a Hamilton cycle $C = v_0 v_1 \dots v_t v_0$, where $t = \Delta(G) - 1$. Let $X = \langle V(G) - N[w] \rangle$ with $V(X) = \{x_1, \dots, x_r\}$, $r = n(G) - \Delta(G) - 1$. By Theorems 4.1 and 3.3, $2 \leq r \leq 3$, and by Lemma 2.4, $\delta(G) \geq 4$. There are three cases to consider. Each case has subcases depending on $\text{comp}(X)$, the number of components of X . Since G is nonhamiltonian, $\text{comp}(X) \geq 2$. In each case where $\text{comp}(X) = 2$ and $r = 3$, we assume that $E(X) = \{x_1 x_2\}$. Then x_i has at least 3 neighbours on C for $i = 1, 2$, and x_3 has at least 4 neighbours on C .

Case 1: $n(G) = 11$ and $\Delta = 7$ (so $r = 3$).

If $\text{comp}(X) = 2$, then $d_C(x_i) \geq 3$ for $i = 1, 2$ and $d_C(x_3) \geq 4$, so therefore $N(x_3)$ contains two consecutive vertices v_i, v_{i+1} on C and there are two distinct vertices v_l, v_k in $V(C) - \{v_i, v_{i+1}\}$ such that $x_l \in N(x_1)$ and $x_k \in N(x_2)$ (since G is 4-connected). Thus G has a Hamilton cycle similar to the one illustrated in Figure 3(b).

If $\text{comp}(X) = 3$, then $d_C(x_i) \geq 4$ for $i = 1, 2, 3$ and hence each set $N(x_i)$ contains a pair of consecutive vertices on C . It therefore follows from Lemma 2.2(c) that there is a pair of consecutive vertices v_j, v_{j+1}

on C that is contained in $N(x_1) \cap N(x_2) \cap N(x_3)$. Then $\{w, x_1, x_2, x_3\}$ is an independent set in $N(v_j)$. But $|N(v_j)| \leq \Delta = 7$, so $\langle N(v) \rangle$ is nonhamiltonian, contradicting our assumption that G is LH .

Case 2: $n(G) = 11$ and $\Delta = 8$ (so $r = 2$).

By Lemma 2.2(b), we may assume without loss of generality that $N(x_1) = \{v_1, v_3, v_5, v_7\}$ and that, $N(x_2) = N(x_1)$. Since G is L^2H , it follows that $\langle \{v_1, v_3, v_5, v_7\} \rangle \cong K_4$ and that $\langle N(x_1) \rangle$ is LH , so that $d(v_i) = 8$ for $i = 1, 3, 5, 7$. Since $\Delta(G) = 8$, v_2 is not adjacent to either of v_5 or v_7 . If v_2 is adjacent to v_0 , then $v_1x_1v_7v_6v_5x_2v_3v_4wv_2v_0v_1$ is a Hamilton cycle in G . Hence $|N(v_1) \cap N(v_2)| = 2$ contradicting that $\langle N(v_1) \cap N(v_2) \rangle$ is hamiltonian.

Case 3: $n(G) = 12$.

In this case, $\Delta(G) = 9$ by Theorem 3.3, so $r = 2$.

By Lemma 2.2(b) we may assume without loss of generality that $N(x_1) = \{v_1, v_3, v_5, v_7\}$ and it follows that $N(x_2) = N(x_1)$. Since G is L^2H , $\langle N(x_1) \rangle$ is LH and since $d(x_1) = 4$, we get $\langle \{v_1, v_3, v_5, v_7\} \rangle \cong K_4$. We now show that, with the exception of v_8v_0 there are no edges in G between vertices in $\{v_0, v_2, v_4, v_6, v_8\}$, since otherwise G would be hamiltonian.

If $v_2v_6 \in E(G)$, then $v_1x_1v_3v_4v_5x_2v_7v_6v_2wv_8v_0v_1$ is a Hamilton cycle in G .

If $v_2v_4 \in E(G)$, then $v_1x_1v_3v_4v_2wv_6v_5x_2v_7v_8v_0v_1$ is a Hamilton cycle in G .

Hence $v_4, v_6 \notin N(v_2)$, and by symmetry $v_4 \notin N(v_6)$, so $\{v_2, v_4, v_6\}$ is an independent set in G .

If $v_0v_2 \in E(G)$, then $v_1x_1v_3v_4v_5x_2v_7v_6wv_8v_0v_2v_1$ is a Hamilton cycle in G .

If $v_0v_4 \in E(G)$, then $v_1v_2v_3x_2v_5v_6wv_4v_0v_8v_7x_1v_1$ is a Hamilton cycle in G .

If $v_0v_6 \in E(G)$, then $v_1v_2wv_6v_0v_8v_7x_2v_5v_4v_3x_1v_1$ is a Hamilton cycle in G .

Hence v_0 does not have a neighbour in $\{v_2, v_4, v_6\}$ and by symmetry, neither does v_8 .

Since $\delta(G) \geq 4$, it follows that each of v_2, v_4, v_6 has three neighbours in the set $\{v_1, v_3, v_5, v_7\}$ and each of v_0, v_8 has two neighbours in this set. From the pigeonhole principle it follows that at least one of v_1, v_3, v_5, v_7 has degree at least 10, contradicting our assumption that $\Delta(G) = 9$.

Thus we have proved that $n(G) \geq 13$. □

In view of Theorem 3.2, our next result is somewhat surprising.

Theorem 3.5. *Let G be a nontraceable, locally 2-nested-hamiltonian graph of minimum order. Then $n(G) = 14$. Moreover, if G is not LH , then G has a two-path cover.*

Proof. First note that the graph in Figure 6 (b) is a nontraceable locally 2-nested-hamiltonian graph of order 14. We already know that a connected nontraceable LH graph has order at least 14, so we can assume G is not LH . Then there is a vertex $v \in V(G)$ such that $\langle N(v) \rangle$ is LH but not hamiltonian. It follows that $d(v) \geq 11$. Since all LH graphs of order less than 14 are traceable, $\langle N[v] \rangle$ is hamiltonian, and therefore if $n(G) = 13$, G is traceable, and if $n(G) = 14$, G has a two-path cover. □

Note that the graphs in Figure 6 are L^2H , but not LH . It is therefore not surprising that $\langle N(w) \rangle$, where w is the vertex shown in Figure 6, is the Goldner-Harary graph (Figure 2 (a)), which is the smallest connected nonhamiltonian LH graph [15]. A method to construct a connected nonhamiltonian L^2H graph of order 13 that is also LH can be found as a special case of the graphs constructed in the proof of Theorem 3.8.

Observation 3.6. *If G is any nonhamiltonian LH graph, then according to Corollary 2.3, $\Delta(G) \leq n - 3$. However, if G is a nonhamiltonian, locally 2-nested-hamiltonian graph, then $\Delta(G)$ can be as large as $n - 1$.*

The graph in Figure 7 is an example of a nonhamiltonian locally 2-nested-hamiltonian graph of order 15 for which the maximum degree is 14. To see that 15 is the smallest order for which this is possible,

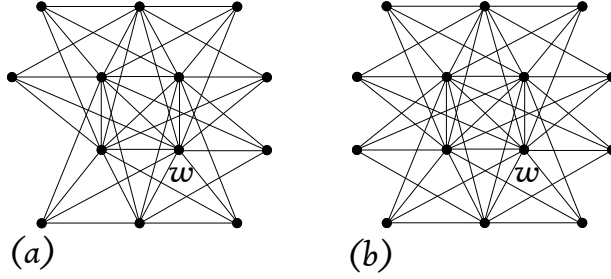


Figure 6: (a) nonhamiltonian and (b) nontraceable locally 2-nested-hamiltonian graphs of orders 13 and 14, respectively.

note that if G is L^2H with $\Delta(G) = n - 1$, there exists a vertex $v \in V(G)$ such that $d(v) = n - 1$ and $\langle N(v) \rangle$ is LH and nontraceable, otherwise G is hamiltonian. Therefore $|N(v)| \geq 14$ and $n(G) \geq 15$.

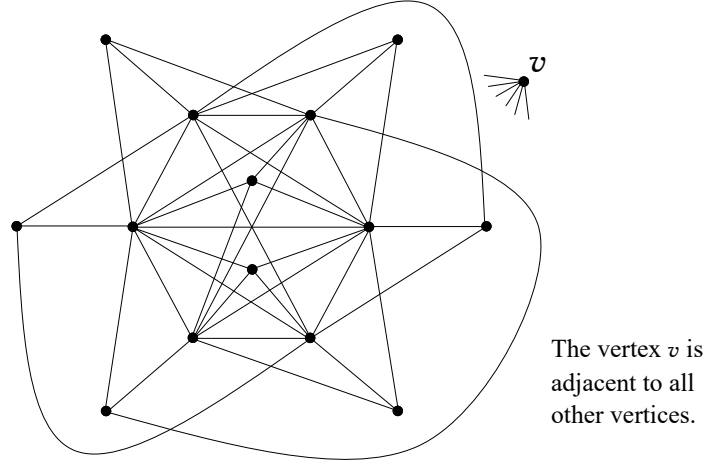


Figure 7: A nonhamiltonian locally 2-nested-hamiltonian graph of order 15 with maximum degree 14.

Lemma 2.13 can be used to construct nonhamiltonian and nontraceable locally 2-nested-hamiltonian graphs, such as the two in Figure 6. These graphs were constructed by combining two copies of K_5 and then repeated combinations using the resulting two vertices of degree four and multiple copies of K_5 .

We now turn our attention to nonhamiltonian and nontraceable locally k -nested-hamiltonian graphs of small order, for higher values of k . We first construct such graphs that are non- L^mH for each $m \in \{0, 1, \dots, k - 1\}$.

Theorem 3.7. *For each $k \geq 1$ there exists a locally k -nested-hamiltonian graph of order $9 + 2k$ that is non- L^mH for $m = 0, 1, \dots, k - 1$, and also a nontraceable locally k -nested-hamiltonian graph of order $10 + 2k$ that is non- L^mH for $l = 0, 1, \dots, k - 1$.*

Proof. By Theorems 3.1 and 3.4, the result holds for $k = 1, 2$. Now let $k \geq 3$.

We first Combine two copies of K_{k+3} using K_{k+2} -identification. This results in a locally k -nested-hamiltonian graph H_k of order $k + 4$, that contains two nonadjacent vertices u and v of degree $k + 2$ each. By Lemma 2.13, we can use $N[u]$ to combine H_k with $k + 2$ copies of K_{k+3} , and we can use $N[v]$ to add another three copies of K_{k+3} to create the locally k -nested-hamiltonian graph G_k , where

$n(G_k) = 9 + 2k$. Let the induced subgraph of G_k whose vertices originated from H_k also be known as H_k . Note that G_k has a vertex cut $V(H_k)$ of order $k + 4$, the removal of which breaks G_k into $k + 5$ components, meaning that G_k is not hamiltonian.

To create a nontraceable graph, use $N[v]$ to combine G_k with another copy of K_{k+3} to create the locally k -nested-hamiltonian graph G'_k . Note that $n(G'_k) = 10 + 2k$ and that G'_k contains a vertex cut of order $k + 4$, the removal of which results in $k + 6$ components, meaning that G'_k is not traceable. Figure 6 illustrates these graphs for $k = 2$.

We still have to show G_k and G'_k are not $L^m H$, where $1 \leq m < k$. The two graphs G_2 and G'_2 are the graphs in Figure 6. In G_2 and G'_2 , $\langle N(w) \rangle$ (where w is the vertex indicated in the figure) is the Goldner-Harary graph, which is the smallest nonhamiltonian LH graph. It follows that for $k = 2$, G_k and G'_k are $L^k H$ but not $L^m H$ for $1 \leq m < k$. Using induction on k , assume that G_k and G'_k are locally k -nested-hamiltonian graphs that are not $L^m H$, where $1 \leq m < k$. In the subgraph H_k in both G_k and G'_k , let the graph induced by the $k + 2$ vertices of degree $k + 3$ be W_k . Add vertices u_1 and w_1 to G_k and G'_k to create the graphs F_{k+1} and F'_{k+1} . In F_{k+1} and F'_{k+1} , u_1 is adjacent to the vertices u and $V(W_k)$, in F_{k+1} , w_1 is adjacent to all the vertices in $V(G_k)$ and in F'_{k+1} , w_1 is adjacent to all the vertices in $V(G'_k)$. Now F_{k+1} and F'_{k+1} are the graphs G_{k+1} and G'_{k+1} constructed above. Note $\langle V(W_k) + \{w_1\} \rangle$ is the subgraph in F_{k+1} and F'_{k+1} equivalent to W_k in G_k and G'_k , respectively. Also, $\langle N_{F_{k+1}}(w_1) \rangle$ is the graph G_k and $\langle N_{F'_{k+1}}(w_1) \rangle$ is the graph G'_k . See Figure 8 for an illustration of the technique.

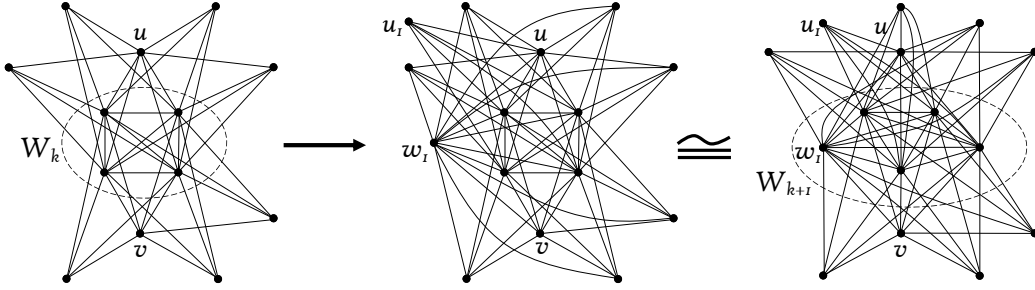


Figure 8: Converting an $L^{\leq 2}H$ graph to an $L^{\leq 3}H$ graph.

This completes the proof. \square

Note that the graphs constructed in the proof of Theorem 3.7 are $(k+2)$ -trees and are a generalization of the Goldner-Harary graph (Figure 2 (a)).

Next we construct nonhamiltonian $L^{\leq k}H$ graphs of order $9 + 2k$ by generalizing the graph in Figure 2(b).

Theorem 3.8. *For each $k \geq 1$ there exists a connected nonhamiltonian $L^{\leq k}H$ graph of order $9 + 2k$.*

Proof. A connected nonhamiltonian $L^{\leq k}H$ graph G_k of order $9 + 2k$ can be constructed in the following way. Start with a K_{k+4} graph W with $V(W) = \{w_0, w_1, \dots, w_{k+3}\}$ and add a vertex u that is adjacent to all vertices in $V(W)$. Then add $k + 4$ vertices v_i , $i = 0, 1, \dots, k + 3$, where $N(v_i) = \{w_i, w_{i+1}, \dots, w_{i+k+1}\}$, where subscripts are taken modulo $k + 4$. The graphs G_1 and G_2 are shown in Figures 2 (b) and 14.

To see that G_k is nonhamiltonian, note that $V(W)$ is a vertex cut, $|V(W)| < V(G)/2$ and $V(G) - V(W)$ is an independent set of vertices. It remains to be shown that G_k is $L^{\leq k}H$. The induced graphs on the neighbourhoods of each of $u, v_0, v_1, \dots, v_{k+3}$ are complete graphs, and it follows that $\langle N(x_0) \cap \dots \cap N(x_j) \rangle$ is hamiltonian, where $x_0 \in \{u, v_0, v_1, \dots, v_{k+3}\}$ and $\{x_2, \dots, x_j\} \subset N(x_0)$ and $j \leq k - 1$.

To prove the result for the intersection of neighbourhoods of vertices in $V(W)$, we will use induction on k . It is easy to see that G_1 and G_2 meet the requirements of the theorem. Now assume that G_k is $L^{\leq k}H$. By inspection of G_{k+1} , we find that $\langle N_{G_{k+1}}(w_1) \rangle \cong G_k - v_1$. It follows from Lemma 2.7 that $\langle N_{G_{k+1}}(w_1) \rangle$ is $L^{\leq k}H$. Also, $\langle N_{G_{k+1}}(w_1) \rangle$ is hamiltonian: $w_2v_0w_3v_{k+4}w_4v_3w_5v_4 \dots w_{k+4}v_{k+3}w_0uw_2$ is a Hamilton cycle.

Note that $\langle N(w_i) \rangle \cong \langle N(w_j) \rangle$, $i, j \in \{0, 1, \dots, k+3\}$, so the result follows. \square

Now suppose G is a nonhamiltonian $L^{<k}H$ graph that contains an induced $K_{1,k+3}$, with v as its central vertex. Then $\alpha(\langle N(v) \rangle) \geq k+3$, so $|N(v)| \geq 2k+6$ since G is LH . Hence, by Corollary 2.3, $n(G) \geq 2k+9$. Thus, if Conjecture 1.8 is true, Theorem 3.7 would imply that the minimum order of a nonhamiltonian $L^{\leq k}H$ graph is $9+2k$. By Theorems 3.1, 3.4 and 3.7, this is indeed the case for $k=1, 2$.

It should be pointed out that the graphs constructed in the proof of Theorem 3.8 were first constructed in [5], where they were described as LH graphs that are $k+2$ -connected. The fact that they are also L^mH for every $m=2, 3, \dots, k$ was not addressed there. In the light of Conjecture 1.8 it is interesting to note that these graphs are locally $(k+1)$ -connected and contain an induced $K_{1,k+3}$. We shall now show that they do not contain an induced $K_{1,4}$.

Corollary 3.9. *For any $k \geq 1$ there exists a $K_{1,k+4}$ -free connected nonhamiltonian $L^{\leq k}H$ graph of order $9+2k$.*

Proof. Consider the graph G_k that is $L^{\leq k}H$ constructed in the proof of Theorem 3.8. We use the same nomenclature as in the proof of Theorem 3.8. The vertex in a $K_{1,q}$ star that has degree greater than one is referred to as the *centre* vertex of the star. Since the neighbourhoods of the vertices $u, v_1, v_2, \dots, v_{k+4}$ all induce complete graphs, it is clear that none of these vertices can be the centre vertex of an induced $K_{1,k+4}$. Since $\langle N(w_i) \rangle \cong \langle N(w_j) \rangle$ for $\{i, j\} \subseteq \{0, 1, \dots, k+3\}$, we need only consider $\langle N(w_{k+3}) \rangle$. $N(w_{k+3}) = \{w_0, w_1, \dots, w_{k+2}, u, v_2, v_3, \dots, v_{k+3}\}$. Since $\langle \{w_0, w_1, \dots, w_{k+2}\} \rangle$ induces a complete graph, say W_{k+3} , and $w_i \sim u$, $i = 0, 1, \dots, k+3$, and v_i , $i = 0, 1, \dots, k+3$, only has neighbours in $V(W)$, it follows that $\alpha(\langle N(w_{k+3}) \rangle) = k+3$, where α is the independence number. \square

Thus Conjecture 1.8, if true, would be a best possible result.

Similar constructions for connected nontraceable graphs that are $L^{\leq k}H$ do not yield graphs of order $10+2k$, as is the case for nontraceable L^kH graphs that are $L^{\leq k-1}C$, but rather graphs of order $12+2k$. This is because it is not possible to add another vertex of degree $k+2$ to the nonhamiltonian graph in such a way that the resulting graph is still $L^{\leq k}H$. Figure 9 is an example of a nontraceable graph that is $L^{\leq 2}H$ of order 16. It is not known at this stage whether it is possible to improve on this result. It is speculated that this is due to these graphs being LH , since for connected LH graphs, the smallest nonhamiltonian graph has order 11 ($= 9+2k$), but the smallest nontraceable graph has order 14 ($= 12+2k$).

4 Hamiltonicity of L^kH graphs with restricted maximum degree

The following result is proved in [18].

Theorem 4.1. [18] *Let G be a connected LH graph with $n(G) \geq 3$ and $\Delta(G) \leq 6$. Then G is fully cycle extendable.*

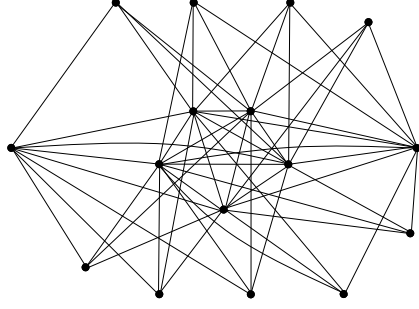


Figure 9: A nontraceable $L^{\leq 2}H$ graph of order 16.

As mentioned in Section 1, it is known that there exist nonhamiltonian connected LH graphs with maximum degree 8, but it remains an open question whether all connected LH graphs with maximum degree 7 are hamiltonian. We suspect that the Hamilton cycle Problem for LH graphs with maximum degree at most 8 is solvable in polynomial time. In [5] we proved the following.

Theorem 4.2. *The Hamilton Cycle Problem for LH graphs with maximum degree 9 is NP-complete.*

We now consider the hamiltonicity of L^2H graphs with small maximum degree. We note that it follows from Corollary 2.5 that any L^2H graph with maximum degree at most 7 is LC . We prove the following.

Theorem 4.3. *If G is a connected L^2H graph with maximum degree at most 7, then G is fully cycle extendable.*

Proof. Suppose G contains a nonextendable nonhamiltonian cycle $C = v_0v_1 \dots v_{t-1}v_0$. Then some vertex on C , say v_0 , has a neighbour x in $V(G) - V(C)$. We consider two cases.

Case 1. $v_{t-1}v_1 \notin E(G)$.

In this case $\{v_{t-1}, v_1, x\}$ is an independent set in $\langle N(v_0) \rangle$, and since $\langle N(v_0) \rangle$ is LH , it follows from Lemma 2.4 that each of the vertices in $\{v_{t-1}, v_1, x\}$ has at least three neighbours in the set $N(v_0) - \{v_{t-1}, v_1, x\}$. But the latter set has at most 4 vertices (since $|N(v_0)| \leq 7$), so there is a vertex v_i in C (with $2 \leq i \leq t-2$) that is a common neighbour of v_1, v_{t-1}, x and v_0 . We note that $N(v_0) \cap N(v_i) \subseteq N(v_0) - \{v_i\}$, so $|N(v_0) \cap N(v_i)| \leq 6$. But $\langle N(v_0) \cap N(v_i) \rangle$ is hamiltonian by Proposition 2.1 and contains the independent set $\{v_1, x, v_{t-1}\}$, so it follows that $|N(v_0) \cap N(v_i)| = 6$ and each of the three vertices in $N(v_0) \cap N(v_i) - \{v_1, x, v_{t-1}\}$ has at least two neighbours in $\{v_1, x, v_{t-1}\}$. Since C is nonextendable, v_1 is not a neighbour of v_{i+1} , and v_{t-1} is not a neighbour of v_{i-1} , and neither v_{i-1} nor v_{i+1} is adjacent to x . But then $d(v_i) \geq 8$, contradicting that $\Delta(G) \leq 7$.

Case 2. $v_{t-1}v_1 \in E(G)$.

In this case x has at least three neighbours in the set $N(v_0) - \{v_1, x, v_{t-1}\}$ and v_1 has at least two neighbours in that set, so v_1 and x have a common neighbour v_i in $N(v_0)$, with $2 \leq i \leq t-2$. If $v_{i-1}v_{i+1} \in E(G)$, then the cycle $v_{i-1}v_{i+1} \xrightarrow{C} v_0xv_1 \dots v_{i+1}$ is an extension of C . Hence $v_{i-1}v_{i+1} \notin E(G)$, but then we have Case 1. \square

In Section 3 we presented nonhamiltonian locally 2-nested-hamiltonian graphs with $\Delta = 10$. We do not know whether nonhamiltonian locally 2-nested-hamiltonian graphs with maximum degree 8 or 9 exist. We now prove the following.

Theorem 4.4. *The Hamilton Cycle Problem for locally 2-nested-hamiltonian graphs with maximum degree 12 is NP-complete.*

Proof. We start with a cubic graph G' and construct a locally 2-nested-hamiltonian graph G that is hamiltonian if and only if G' is hamiltonian.

Each vertex in G' is represented by a copy of K_5 in G , and will be referred to as a *node* in G .

Each edge in G' is represented by a more complex structure, that is based on the graph H in Figure 10, which is the graph in Figure 6 (a) redrawn. We use K_4 -identification to combine H with two copies of graph D in Figure 10 in the following way: using the first copy of D we identify u_j and x_j , $j = 1, 2, 3, 4$, and using the second copy of D we identify v_j and x_j , $j = 1, 2, 3, 4$. This creates the graph B_i shown in Figure 11.

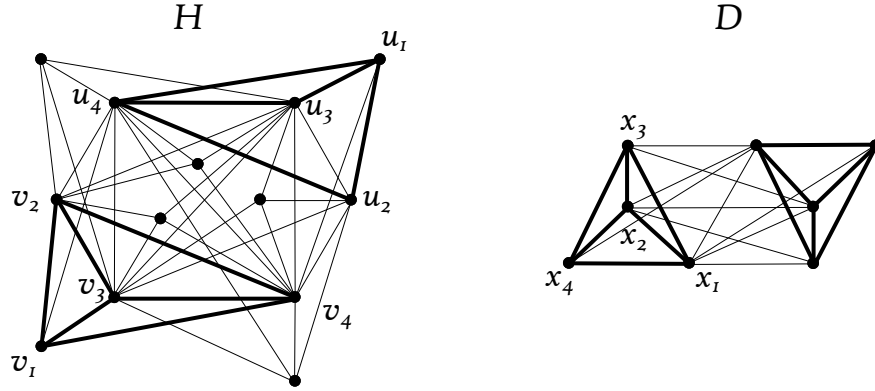


Figure 10: The graphs H and D used in the proof of Theorem 4.4.

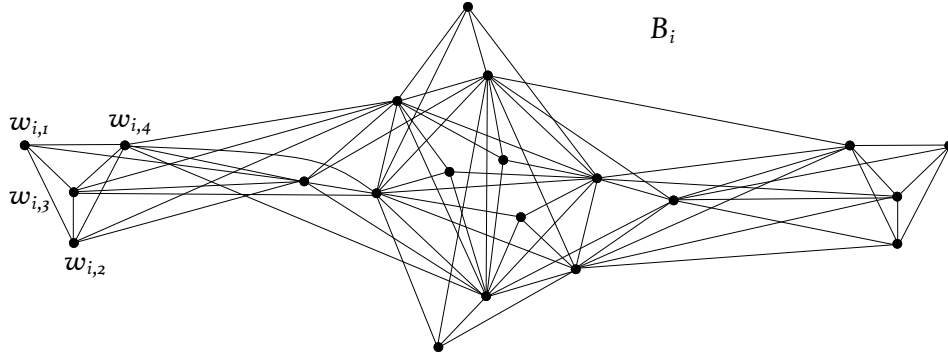


Figure 11: The graph B_i used in the proof of Theorem 4.4.

The edges in G' are represented by copies of B_i in G , and will be referred to as *borders*. The borders are connected to the nodes by means of K_4 -identification. Let the vertices in a node in G be y_1, y_2, y_3, y_4, y_5 and let the vertices in B_i be labeled as shown in Figure 11. Since each vertex in G' has degree three, each node in G is attached to three copies of B_i . We identify the vertices as shown in Table 1 (after each vertex identification, the resulting vertex retains the y -label). We use the graphs B_1, B_2 and B_3 for illustrative purposes. See Figure 12 (the heavy lines in G represent edges belonging to the nodes).

Checking the degrees of the vertices that have been identified shows that $\Delta(G) = 12$ and by Theorems 2.10 and 2.12 and Lemma 2.13, G is connected, LC and L^2H .

Figure 13 shows how a Hamilton cycle in G' can be translated to a Hamilton cycle in G (the heavy

Vertex in node	Vertex in B_i
y_1	$w_{1,2}$
y_2	$w_{1,1}$
y_4	$w_{1,4}$
y_5	$w_{1,3}$
y_1	$w_{2,3}$
y_2	$w_{2,2}$
y_3	$w_{2,1}$
y_5	$w_{2,4}$
y_1	$w_{3,1}$
y_2	$w_{3,2}$
y_3	$w_{3,3}$
y_4	$w_{3,4}$

Table 1: Vertices identified in the proof of Theorem 4.4.

lines represent edges that are in the Hamilton cycles). To see that if G is hamiltonian, then G' is also hamiltonian, consider the graph H in Figure 10 that forms the core of the connection between two nodes in G . Note that $u_2, u_3, u_4, v_2, v_3, v_4$ are the only neighbours of the five unlabeled vertices in Figure 10. Therefore any path cover of H contains at most one path that has one end vertex in u_1, u_2, u_3, u_4 and one end vertex in v_1, v_2, v_3, v_4 . Thus every Hamilton cycle in G has at most one path from node Z_i to node Z_j that passes through the border between them. Since each node has three borders incident to it, the result follows.

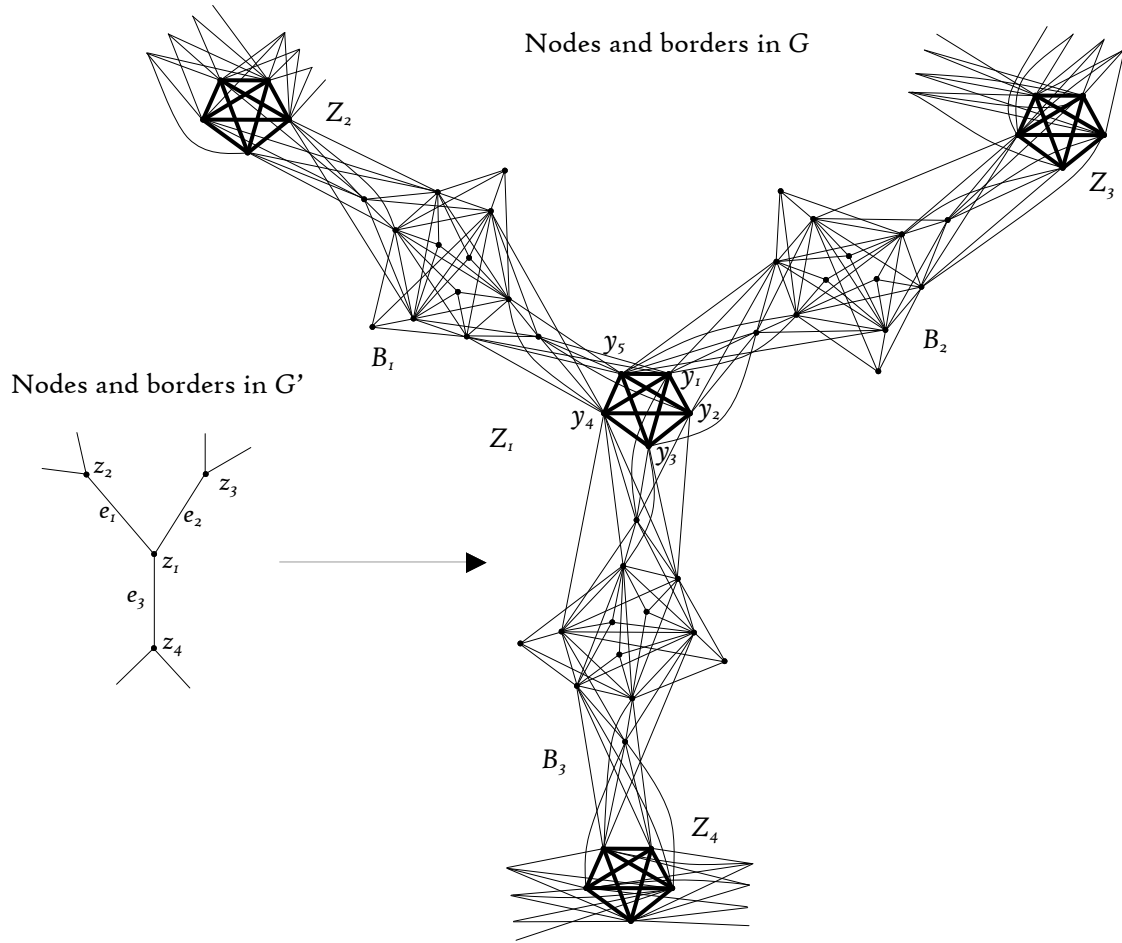


Figure 12: Converting the graph G' to the graph G in Theorem 4.4.

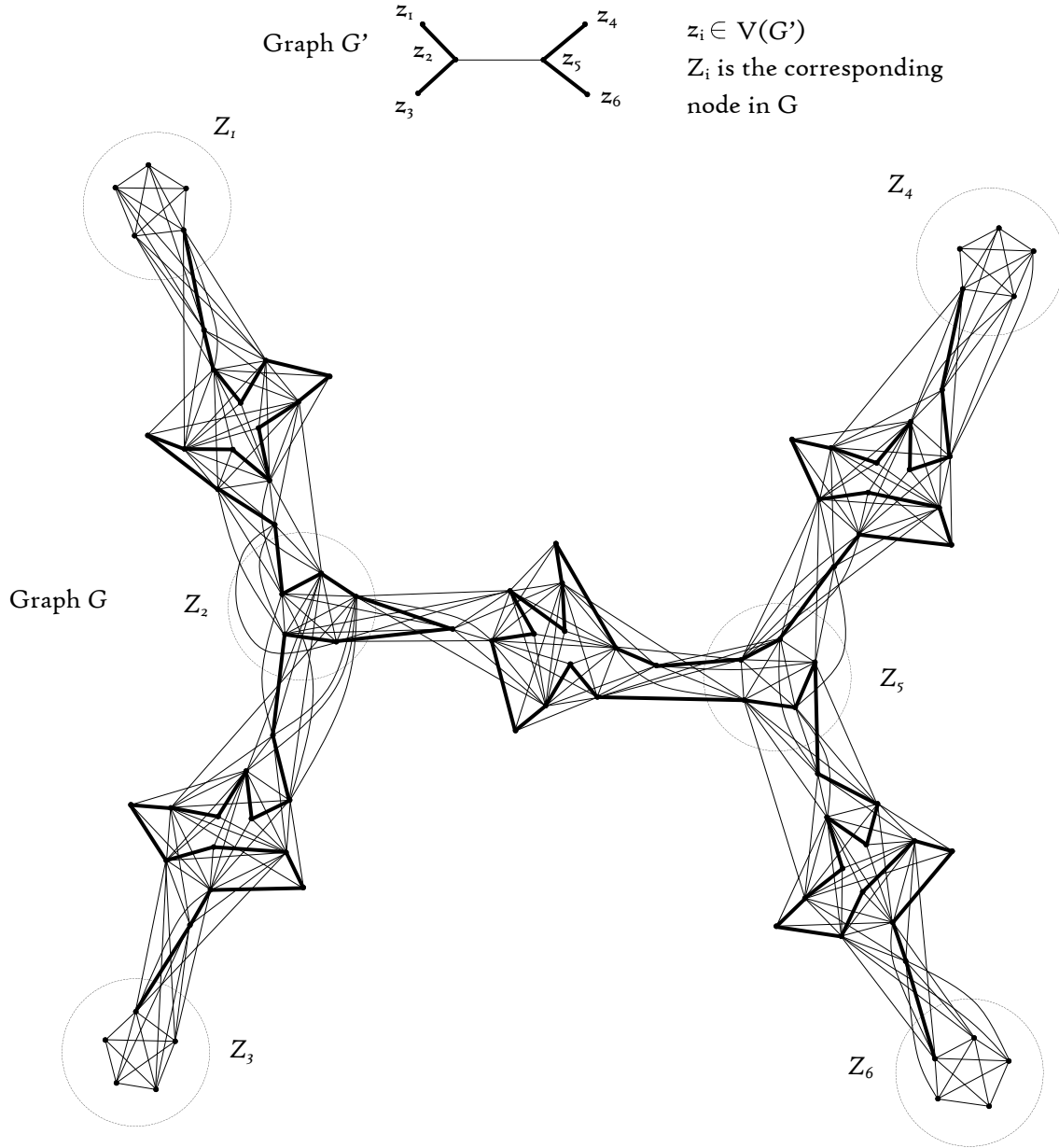


Figure 13: Translating a Hamilton cycle from G' to G in Theorem 4.4.

□

The proof of Theorem 4.4 relies on the fact that the graph H in Figure 10 is locally 2-nested-hamiltonian and nonhamiltonian, has order 13, contains 7 independent vertices of degree 4 each and is traceable between any two vertices of degree 4. In Section 3 we constructed, for each $k \geq 2$, a nonhamiltonian locally k -nested-hamiltonian graph G_k of order $2k+9$ that has $k+5$ vertices of degree $k+2$ each, such that G_k is traceable between any two vertices of degree $k+2$. We conclude that NP-completeness theorems for locally k -nested-hamiltonian graphs with restricted maximum degree are possible for all $k \geq 3$. The smallest value of the maximum degree that these theorems yield

depends on the choice of neighbours for the vertices of degree $k + 2$ in the graphs of order $9 + 2k$. As k increases, there is increasing flexibility in the choice of neighbours for the vertices of degree $k + 2$. Detailed calculations show that for $k = 3, 4, 5, 6, 7, 8$ the Hamilton Cycle Problem for locally k -nested-hamiltonian graphs with maximum degree $3k + 6$ is NP-complete. Since the constructions follow a regular pattern, we expect that this is the case for all $k \geq 1$.

When investigating the possible NP-completeness of the Hamilton Cycle Problem for graphs that are $L^{\leq k}H$, we don't have the advantage of a theorem equivalent to Theorem 2.10 (see Remark 2.11). This means that any construction has to be checked in detail to confirm that the resulting graph is $L^{\leq k}H$. We begin with $k = 2$.

Theorem 4.5. *The Hamilton Cycle Problem for $L^{\leq 2}H$ graphs with maximum degree 13 is NP-complete.*

Proof. We use the same construction as in the proof of Theorem 4.4, except that now the graph H is the graph shown in Figure 14.

We combine H with two copies of the graph D to create the graph shown in Figure 15. When connecting borders to nodes to construct the graph G , we take care to limit the degree of vertices in the nodes to 10, as shown in Figure 16. Since the smallest connected nonhamiltonian LH graph has order 11, this ensures that in G , for any vertex v that lies in a node, $\langle N(v) \rangle$ is a hamiltonian graph. We still have to confirm that for any vertex u that is in a border and adjacent to a node, $\langle N(u) \rangle$ is hamiltonian. This is easily done, since there are only 8 such vertices in any border, and by symmetry, only one border has to be checked (see Figure 16). It follows that G is both LH and L^2H .

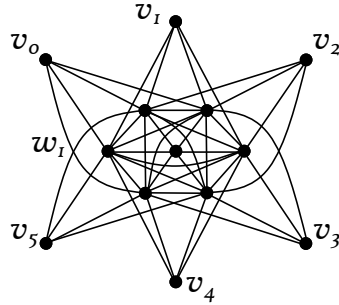


Figure 14: A connected nonhamiltonian $L^{\leq 2}H$ graph of order 13.

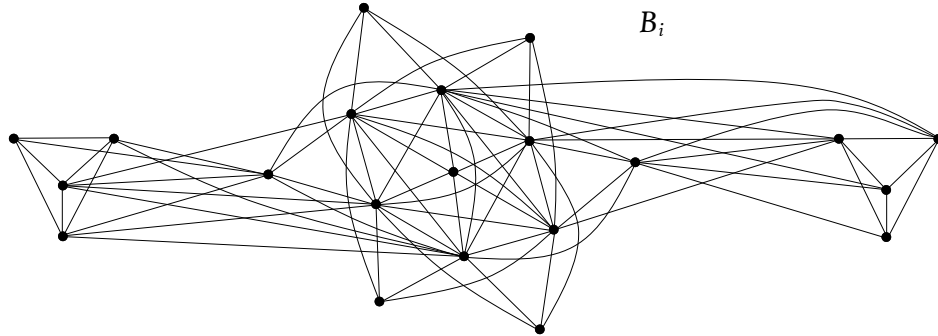


Figure 15: A border used in the construction of the graph G in Theorem 4.5.

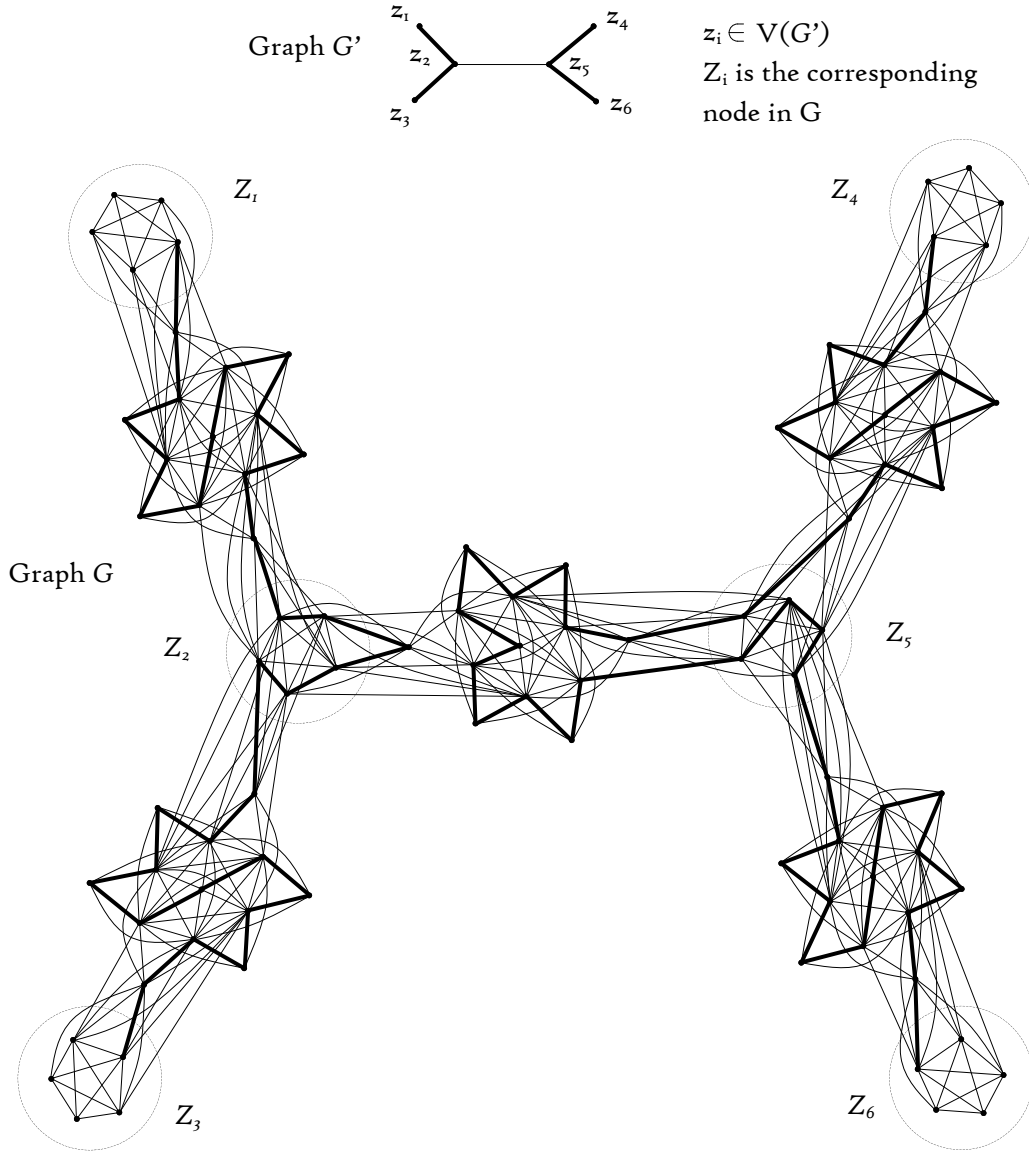


Figure 16: Translating a Hamilton cycle from G' to G in Theorem 4.5.

An argument similar to the one used in Theorem 4.4 can be used to show that if G is hamiltonian then G' is hamiltonian. To see that G is hamiltonian if G' is hamiltonian, the reader is referred to Figure 16, where the heavy lines represent edges that are in a Hamilton cycle. \square

Detailed calculations for the cases $k = 3$ and $k = 4$ show that the Hamilton Cycle Problem is NP-complete for $L^{\leq k}H$ graphs that have maximum degree 16 for $k = 3$ and maximum degree 19 for $k = 4$. There appears to be a pattern according to which the Hamilton Cycle Problem is NP-complete $L^{\leq k}H$ graphs that have maximum degree $3k + 7$, for $k \geq 2$. Again there is reason to expect that the relationship will hold for all values of $k \geq 2$, since the pattern of the construction is quite regular. It is an interesting question whether these results would be best possible, particularly since for $k = 1$ we know the Hamilton Cycle Problem is NP-complete for maximum degree $3k + 6$. It should be noted

that constructions very similar to the ones used in Theorem 4.5 and the discussion in this paragraph appeared in [5]. However, in [5] we used them to prove the Hamilton Cycle Problem is NP-complete for LH graphs that are $(k+2)$ -connected. The fact that these graphs are also $L^{\leq k}H$ was not addressed there.

5 The connection between k -trees and $L^{k-2}H$ graphs

We begin this section by stating some basic properties of k -trees. The first follows directly from the definition of a k -tree.

Proposition 5.1. *If G is a k -tree, then the neighbourhood of every vertex of degree k in G induces a k -clique, and vertices of degree k may be recursively removed until only a k -clique remains.*

A graph G of order n has a *perfect elimination ordering* if the vertices in G may be labelled v_1, v_2, \dots, v_n such that $\langle N[v_i] \rangle$ is a clique in $G - \{v_1, \dots, v_{i-1}\}$ for $i = 1, \dots, n-1$. It is well-known that a graph has a perfect elimination ordering if and only if it is a *chordal graph* (a graph in which every cycle of length greater than 3 has a chord). Thus Proposition 5.1 implies the following.

Corollary 5.2. *Every k -tree is a chordal graph.*

From the construction procedure of k -trees we also observe the following.

Observation 5.3. *If a given k -clique X is used r times ($r \geq 0$) in the construction of a k -tree G , then $G - V(X)$ has $r + 1$ components, each of which contains one vertex of $\bigcap_{x \in V(X)} N(x)$.*

The next result is due to Rose [16].

Lemma 5.4. [16] *Let G be a k -tree and let u and v be any pair of nonadjacent vertices in G . Then there are exactly k vertex disjoint $u - v$ paths in G .*

The smallest nonhamiltonian LH graph (depicted in Figure 2 (a)) happens to be a maximal planar graph as well as a 3-tree. (Note that we can recursively delete a vertex of degree 3 whose neighbourhood is a K_3 , until only a K_4 remains). This prompted us to have a closer look at the connection between LH graphs, 3-trees and maximal planar graphs.

It is well known that every maximal planar graph of order n has exactly $3n - 6$ edges, and an easy calculation shows that the same is true for 3-trees. Markenzon, Justel and Paciorek [12] found a relationship between maximal planar graphs and simple-clique 3-trees.

Theorem 5.5. [12] *A graph G of order $n \geq 3$ is a simple-clique 3-tree if and only if it is a chordal maximal planar graph.*

Skupień [17] found a relationship between LH graphs and maximal planar graphs.

Theorem 5.6. [17] *A connected LH graph G of order $n \geq 3$ is a maximal planar graph if and only if $|E(G)| = 3n - 6$.*

We therefore have the following relationship between LH graphs and simple-clique 3-trees.

Theorem 5.7. *A connected graph G of order $n \geq 3$ is a simple-clique 3-tree if and only if G is a chordal LH graph with $|E(G)| = 3n - 6$.*

Markenzon, Justel and Paciorek [12] also found a relationship between 2-trees and maximal outer-planar graphs.

Theorem 5.8. *A graph G of order $n \geq 3$ is a simple-clique 2-tree if and only if G is a maximal outerplanar graph.*

A maximal outerplanar graph of order at least 3 is obviously hamiltonian, so it follows from Theorems 5.7 and 5.8 that for $k = 2, 3$, the class of $L^{k-2}H$ graphs contains all simple-clique k -trees.

We note that the nonhamiltonian locally k -nested hamiltonian graphs of order $2k + 9$ constructed in the proof of Theorem 3.7 are $(k + 2)$ -trees. We now prove the main result of this section, which establishes a relationship between simple-clique k -trees and $L^{k-2}H$ graphs.

Theorem 5.9. *For $k \geq 3$ a k -tree is locally $(k - 2)$ -nested hamiltonian if and only if it is a simple-clique k -tree.*

Proof. First, suppose G is a k -tree that is not a simple clique k -tree. Then some k -clique X was used more than once in the k -tree construction of G . By Observation 5.3, there are three independent vertices u_1, u_2, u_3 in $\bigcap_{x \in V(X)} N_G(x)$. Now let Y be any $(k - 2)$ -clique in X and let $\{v_1, v_2\} = V(X) - V(Y)$. By Theorem 5.4, there are exactly k internally disjoint paths between any two vertices in $\{u_1, u_2, u_3\}$. Each such path contains exactly one vertex of X . Since $\{v_1, v_2\}$ are the only vertices of X in $\bigcap_{y \in V(Y)} N_G(y)$, any cycle in $\langle \bigcap_{y \in V(Y)} N_G(y) \rangle$ misses at least one of the vertices in $\{u_1, u_2, u_3\}$. Thus $\langle \bigcap_{y \in V(Y)} N_G(y) \rangle$ is not hamiltonian and hence G is not $L^{k-2}H$.

Now let G be a simple clique k -tree of order n . We prove by induction on n that G is locally 2-nested-hamiltonian. If $n = k + 1$, then $G = K_{k+1}$, which is obviously $L^{k-2}H$. Now assume $n \geq k + 2$. Let z be the last vertex added in the k -tree construction of G . Then $G - z$ is a simple clique k -tree of order $n - 1$ and $\langle N(z) \rangle$ is a k -clique in $G - z$ that has not been used in the k -clique construction of $G - z$. Let $N(z) = \{v_1, \dots, v_k\}$. By Observation 5.3, $\langle \bigcap_{v \in N(z)} N_{G-z}(v) \rangle$ consists of a single vertex, say v_{k+1} . By our induction hypothesis, $G - z$ is $L^{k-2}H$. Thus, to prove that G is $L^{k-2}H$, we only need to show that the k -clique $\langle N(z) \rangle$ is suitable for k -clique identification.

Now consider any $(k - 2)$ -clique Y in $\langle N(z) \rangle$. Then $\langle \bigcap_{y \in V(Y)} N_{G-z}(y) \rangle$ has a Hamilton cycle C , since $G - z$ is $L^{k-2}H$. We may assume that $V(Y) = \{v_1, \dots, v_{k-2}\}$. Then $\{v_{k-1}, v_k, v_{k+1}\} \subseteq \bigcap_{y \in Y} N_{G-z}(y)$ and v_{k+1} is the only common neighbour of v_{k-1} and v_k in $\bigcap_{y \in Y} N_{G-z}(y)$. Suppose C does not contain the edge $v_{k-1}v_k$. Then $\bigcap_{y \in Y} N_{G-z}(y)$ contains a $v_{k-1} - v_k$ path that contains neither the edge $v_{k-1}v_k$ nor the vertex v_{k+1} . Let P be a shortest such path. We note that v_{k-1} and v_k do not have a common neighbour on P , so P has at least four vertices and, by the minimality of P , the cycle $v_k v_{k-1} P v_k$ is chordless, contradicting Corollary 5.2. Hence C contains the edge $v_{k-1}v_k$, so $\langle N(z) \rangle$ is suitable for k -clique identification. This proves that G is locally 2-nested-hamiltonian. \square

From Theorems 5.8, 5.7 and 5.9, we conclude the following.

Corollary 5.10. *For each integer $k \geq 1$, the class of locally k -nested-hamiltonian graphs contains the class of simple-clique $(k + 2)$ -trees.*

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