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# BESSEL SPDES AND RENORMALIZED LOCAL TIMES

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Abstract. In this article, we prove integration by parts formulae (IbPFs) for the laws of Bessel bridges from 0 to 0 over the interval $[0, 1]$ of dimension smaller than 3. As an application, we construct a weak version of a stochastic PDE having the law of a one-dimensional Bessel bridge (i.e. the law of a reflected Brownian bridge) as reversible measure, the dimension 1 being particularly relevant in view of applications to scaling limits of dynamical critical pinning models. We also exploit the IbPFs to conjecture the structure of the stochastic PDEs associated with Bessel bridges of all dimensions smaller than 3.

## 1. Introduction

The classical stochastic calculus due to Kiyoshi Itô was originally created as a tool to define and solve stochastic differential equations (SDEs). In classical monographs on the subject, see e.g. [33, 25, 34], Bessel processes play a prominent role as a fundamental example on which the extraordinary power of the theory can be tested.

Stochastic partial differential equations (SPDEs) were invented around fifty years ago as a natural function-valued analog of SDEs, and are by now a well-established field which is increasingly active and lively. SPDEs driven by a space-time white noise have recently received much attention, because they are naturally associated with *ultraviolet divergences* and *renormalization*, phenomena which are now mathematically well-understood in many circumstances using the recent theories of regularity structures [23, 4] and of paracontrolled distributions [21].

In particular, the classical stochastic calculus for semimartingales and SDEs has no analog for space-time-white-noise driven SPDEs, despite some early and more recent attempts [43, 3], because of the divergences created by the white noise. A partial substitute is given by the Fukushima stochastic calculus associated with Dirichlet forms [13, 27], but the formulae that one obtains are often less explicit than one would hope. The marvellous power of the Itô calculus for the study of fine properties of semimartingales remains without proper analog in genuinely infinite-dimensional processes.

In this paper we discuss a particular class of equations which seems a natural analog of Bessel processes in the context of SPDEs driven by a space-time white noise. As we explain below, the standard approach to Bessel processes does not

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work at all for these Bessel SPDEs, and we have to apply a different method, with necessarily weaker results, at least in comparison with the finite-dimensional situation. We will rely on Dirichlet forms methods and on integration by parts formulae on path spaces. These will include distributional terms - rather than $\sigma$-finite measures - as in the theory of white noise calculus [24].

The processes that we consider have interesting path properties, as it is the case for Bessel processes, but with the enhanced richness of infinite-dimensional objects, see e.g. [44] for a recent account. We hope that this work will further motivate the study of infinite-dimensional stochastic calculus, which is still in its infancy.

1.1. **From Bessel SDEs to Bessel SPDEs.** A squared Bessel process of dimension $\delta \geq 0$ is defined as the unique continuous non-negative process $(Y_t)_{t\geq 0}$ solving the SDE

$$Y_t = Y_0 + \int_0^t 2\sqrt{Y_s}\,\mathrm{d}B_s + \delta\, t, \quad t \geq 0, \qquad (\delta \geq 0) \tag{1.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Squared Bessel processes enjoy a remarkable additivity property (see [35] and (3.1) below), and play a prominent in several areas of probability theory. For instance, in population dynamics, they arise as the scaling limit of Galton-Watson processes with immigration. On the other hand, they play an important role in the study of the fine properties of Brownian motion, see e.g. [33]. Moreover their fascinating behavior at the boundary point 0 can be studied in great detail, see e.g. [44] for a recent account.

Well-posedness of the SDE (1.1) satisfied by $(Y_t)_{t\geq 0}$ follows from the classical Yamada-Watanabe theorem [33, Theorem IX.3.5]. If we consider the *Bessel process* $X_t := \sqrt{Y_t}$, $t \geq 0$, the situation is more involved. For $\delta > 1$, by the Itô formula, $X$ is solution to

$$X_t = X_0 + \frac{\delta - 1}{2}\int_0^t \frac{1}{X_s}\,\mathrm{d}s + B_t, \quad t \geq 0, \qquad (\delta > 1) \tag{1.2}$$

and this equation satisfies pathwise uniqueness and existence of strong solutions since the drift is monotone decreasing. For $\delta = 1$, $X$ is the solution to

$$X_t = X_0 + L_t + B_t, \qquad t \geq 0, \qquad (\delta = 1)$$

where $(L_t)_{t\geq 0}$ is continuous and monotone non-decreasing, with $L_0 = 0$ and

$$X \geq 0, \qquad \int_0^\infty X_s\,\mathrm{d}L_s = 0. \tag{1.3}$$

In other words $X$ is a reflecting Brownian motion, and the above equation has a unique solution by the Skorokhod Lemma [33, Lemma VI.2.1].

For $\delta \in (0, 1)$, the situation is substantially more difficult and it turns out that the relation (1.2) is not valid anymore in this regime. One can show, see e.g. [44,

Proposition 3.12], that $X$ admits *diffusion local times*, namely continuous processes $(\ell_t^a)_{t\geq 0, a\geq 0}$ such that

$$\int_0^t \varphi(X_s)\,\mathrm{d}s = \int_0^\infty \varphi(a)\,\ell_t^a\,a^{\delta-1}\,\mathrm{d}a, \tag{1.4}$$

for all Borel $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, and that $X$ satisfies

$$X_t = X_0 + \frac{\delta-1}{2}\int_0^\infty \frac{\ell_t^a - \ell_t^0}{a}\,a^{\delta-1}\,\mathrm{d}a + B_t, \quad t\geq 0, \qquad (0<\delta<1). \tag{1.5}$$

Note that by the occupation time formula (1.4) we have

$$\int_0^\infty \frac{\ell_t^a-\ell_t^0}{a}\,a^{\delta-1}\,\mathrm{d}a = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty \frac{\ell_t^a-\ell_t^0}{a}\,a^{\delta-1}\,\mathrm{d}a =$$
$$= \lim_{\varepsilon\downarrow 0}\left(\int_0^t 1_{(X_s\geq\varepsilon)}\frac{1}{X_s}\,\mathrm{d}s - \ell_t^0\int_\varepsilon^\infty a^{\delta-2}\,\mathrm{d}a\right)$$

and in the latter expression both terms diverge as $\varepsilon\downarrow 0$, while the difference converges since $|\ell_t^a-\ell_t^0| \lesssim a^{1-\frac{\delta}{2}-\kappa}$ with $\kappa>0$: this is why we speak of *renormalised local times*.

The formula (1.5) is not really an SDE, and to our knowledge one cannot (so far) characterize $X$ as the unique process satisfying this property, unless one manages to prove that $X^2$ is a solution to (1.1). We stress again that the relation between (1.1) and (1.2)-(1.3)-(1.5) is based on Itô's stochastic calculus.

In a series of papers [38, 39, 40, 41] the second author of this article studied a class of stochastic partial differential equations (SPDEs) with analogous properties. For a parameter $\delta>3$ the equation, that we call *Bessel SPDE*, is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\kappa(\delta)}{2\,u^3} + \xi \\ \\ u(0,\cdot)=u_0,\ u(t,0)=u(t,1)=0 \end{cases} \qquad (\delta>3) \tag{1.6}$$

where $u\geq 0$ is continuous and $\xi$ is a space-time white noise on $\mathbb{R}_+\times[0,1]$, and

$$\kappa(\delta) := \frac{(\delta-3)(\delta-1)}{4}. \tag{1.7}$$

As $\delta\downarrow 3$, the solution to (1.6) converges to the solution of the Nualart-Pardoux equation [31], namely the random obstacle problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \eta + \xi \\ \\ u(0,\cdot)=u_0,\ u(t,0)=u(t,1)=0 \\ \\ u\geq 0,\ d\eta\geq 0,\ \int_{\mathbb{R}_+\times[0,1]} u\,\mathrm{d}\eta = 0, \end{cases} \qquad (\delta=3) \tag{1.8}$$

where $\eta$ is a Radon measure on $]0,\infty[\times]0,1[$. The unique invariant measure of (1.6) for $\delta > 3$, respectively (1.8), is the Bessel bridge of dimension $\delta$, resp. 3. In other words, the invariant measure has the law of $(X_t)_{t\in[0,1]}$ conditioned to return to 0 at time 1, where $X$ solves (1.2) with $X_0 = 0$ and $\delta > 3$, respectively $\delta = 3$. Equation (1.8) also describes the fluctuations of an effective $(1+1)$ interface model near a wall [16, 14] and also arises as the scaling limit of several weakly asymmetric interface models, see [11].

While (1.6) for $\delta > 3$ is the analog of (1.2) for $\delta > 1$, (1.8) is the analog of (1.3). The analogy can be justified in terms of scaling invariance: the equations (1.2) and (1.3) are invariant (in law) under the rescaling $X_t \mapsto \lambda^{-1}X_{\lambda^2 t}$ for $\lambda > 0$, while (1.6) and (1.8) are invariant under $u(t,x) \mapsto \lambda^{-1}u(\lambda^4 t, \lambda^2 x)$ (apart from the fact that the space interval changes from $[0,1]$ to $[0,\lambda^{-2}]$).

It has been an open problem for over 15 years to complete the above picture. Namely, what is an SPDE whose invariant measure is the Bessel bridge of dimension $\delta < 3$ ? Is it an SPDE analogue of (1.5) ?

We stress that equations (1.6) and (1.8) enjoy nice properties (pathwise uniqueness, continuity with respect to initial data, the Strong Feller property) because of the *dissipative*, namely monotone non-increasing, character of the drift. This is however true only as long as the coefficient $\kappa(\delta)$ is positive, and fails for $\delta \in ]1,3[$. In the regime $\delta < 3$, we shall see that even the notion of solution becomes highly non-trivial, as for Bessel processes in the regime $\delta < 1$. The nice properties mentioned above may still be true but the known techniques become ineffective.

This problem is particularly interesting for $\delta = 1$, which corresponds to the reflecting Brownian bridge as an invariant measure. Indeed, the reflecting Brownian bridge arises as the scaling limit of critical pinning models, see [7], [14, Chapter 15.2] and [12, 20]. Dynamical pinning models are believed to have a scaling limit, which would be an infinite-dimensional diffusion having the law of a reflecting Brownian motion as reversible measure. What kind of SPDE that limit should satisfy has however remained a very open question so far.

Note that the one-dimensional trick of considering a power of $u$, in this case for instance $v := u^4$, in order to find a more tractable SPDE fails because one obtains rather frightening equations of the form

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + 2\,\kappa(\delta) - \frac{3}{8}\,\frac{1}{\sqrt{v}}\,:\left(\frac{\partial v}{\partial x}\right)^2: + 4v^{\frac{3}{4}}\,\xi$$

where the : : notation denotes a KPZ-type renormalization. Even the theory of regularity structures does not cover this kind of equations, due to the non-Lipschitz character of the coefficients. One could hope that a Yamada-Watanabe result could be proved for this class of equations; inspiringly, the exponent $\frac{3}{4}$ in the noise-term is known to be critical for pathwise uniqueness of parabolic SPDEs (without the KPZ-type term), see [29, 28]. This approach is, at present, completely out of reach.

Therefore, in this paper, rather than tackling these difficulties, we answer the above questions by exploiting the specific, very nice structure underlying Bessel processes. More precisely, we derive integration by parts formulae for the law of Bessel bridges of dimension $\delta < 3$. These formulae involve the laws of pinned Bessel bridges (or, more precisely, the measures $\Sigma_r^\delta(\,\cdot\,|\,a)$ defined in (3.12) below) which should correspond to the local times of the solution $u$ to our would-be SPDEs. Thus, the drift term in these SPDEs should involve the process $(\ell^a_{t,x})_{a\geq 0}$ defined by

$$\int_0^t \varphi(u(s,x))\,\mathrm{d}s = \int_0^\infty \varphi(a)\,\ell^a_{t,x}\,a^{\delta-1}\,\mathrm{d}a, \tag{1.9}$$

for all Borel $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Some explicit computations on the measures $\Sigma_r^\delta(\,\cdot\,|\,a)$ suggest that this process should moreover have a vanishing first-order derivative at 0, that is

$$\frac{\partial}{\partial a}\ell^a_{t,x}\bigg|_{a=0} = 0, \qquad t \geq 0, \quad x \in (0,1). \tag{1.10}$$

Finally, the integration by parts formulae that we find enable us to identify the corresponding Bessel SPDEs. For $1 < \delta < 3$, the SPDE should have the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\kappa(\delta)}{2}\frac{\partial}{\partial t}\int_0^\infty \frac{1}{a^3}\left(\ell^a_{t,x} - \ell^0_{t,x}\right) a^{\delta-1}\,\mathrm{d}a + \xi, \qquad (1<\delta<3). \tag{1.11}$$

Then (1.11) is the SPDE analog of (1.5). On the other hand, for $\delta = 1$, we find that the SPDE should be of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{8}\frac{\partial}{\partial t}\frac{\partial^2}{\partial a^2}\,\ell^a_{t,x}\bigg|_{a=0} + \xi, \qquad (\delta = 1), \tag{1.12}$$

while for $0 < \delta < 1$

$$\begin{aligned}\frac{\partial u}{\partial t} = {}& \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \xi \qquad\qquad (0<\delta<1)\\ &+ \frac{\kappa(\delta)}{2}\frac{\partial}{\partial t}\int_0^\infty \frac{1}{a^3}\left(\ell^a_{t,x} - \ell^0_{t,x} - \frac{a^2}{2}\frac{\partial^2}{\partial a^2}\,\ell^a_{t,x}\bigg|_{a=0}\right) a^{\delta-1}\,\mathrm{d}a.\end{aligned} \tag{1.13}$$

In (1.11), as in (1.5), we have a Taylor expansion at order 0 of the local times functions $a \mapsto \ell^a$. By contrast, equations (1.12) and (1.13) have no analog in the context of one-dimensional Bessel processes. In (1.13) the Taylor expansion is at order 2, while (1.12) is a limit case, like (1.8). Although they seem quite different, all the above SPDEs can be written in a unified way as follows. We introduce for $\alpha \in \mathbb{R}$ the following distributions on $[0,\infty)$

- if $\alpha = -k$ with $k \in \mathbb{N}$, then

$$\langle \mu_\alpha, \varphi\rangle := (-1)^k \varphi^{(k)}(0), \qquad \forall\, \varphi \in C_0^\infty([0,\infty))$$

- else,

$$\langle \mu_\alpha, \varphi \rangle := \int_0^{+\infty} \left( \varphi(a) - \sum_{0 \le j \le -\alpha} \frac{a^j}{j!} \varphi^{(j)}(0) \right) \frac{a^{\alpha-1}}{\Gamma(\alpha)} \, \mathrm{d}a, \quad \forall \, \varphi \in C_0^\infty([0,\infty)).$$

Note that, for all $\alpha \in \mathbb{R}$, $\mu_\alpha$ coincides with the distribution $\frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$ considered in Section 3.5 of [17]. Then for all $\varphi \in C_0^\infty([0,\infty))$, the map $\alpha \mapsto \langle \mu_\alpha, \varphi \rangle$ is analytic. Moreover, for $\delta > 3$ the non-linearity in (1.6) can be expressed as

$$\frac{\kappa(\delta)}{2} \int_0^t \frac{1}{(u(s,x))^3} \, \mathrm{d}s = \frac{\kappa(\delta)}{2} \int_0^\infty \frac{1}{a^3} \, \ell_{t,x}^a \, a^{\delta-1} \, \mathrm{d}a = \frac{\kappa(\delta)\,\Gamma(\delta-3)}{2} \langle \mu_{\delta-3}, \ell_{t,x}^{\cdot} \rangle,$$

which, by (1.7), we can in turn rewrite as

$$\frac{\Gamma(\delta)}{8(\delta-2)} \langle \mu_{\delta-3}, \ell_{t,x}^{\cdot} \rangle, \tag{1.14}$$

an expression which, at least formally, makes sense for any $\delta \in (0,\infty) \setminus \{2\}$. Note that the singularity at $\delta = 2$ is just apparent, since in this case, we can rewrite the previous expression simply as:

$$-\frac{1}{8} \int_0^\infty \frac{1}{a^3} \left( \ell_{t,x}^a - \ell_{t,x}^0 \right) a \, \mathrm{d}a,$$

where the integral is indeed convergent due to the cancellation (1.10). Then, the expression (1.14) encapsulates, in a unified way, the non-linearities of (1.8)-(1.11)-(1.12)-(1.13). In particular, for $\delta = 3$, it equals $\frac{1}{4}\ell_{t,x}^0$, which is coherent with the results about the structure of the reflection measure $\eta$ in (1.8) proved in [41] and showing that a.s.

$$\eta([0,t] \times \mathrm{d}x) = \frac{1}{4} \, \ell_{t,x}^0 \, \mathrm{d}x.$$

At least formally, the $\delta$-Bessel SPDEs for $\delta < 3$ correspond to the unique analytic continuation of the $\delta$-Bessel SPDEs for $\delta \ge 3$. This is justified by considering the corresponding integration by parts formulae on a specific set of test functions, where every term depends in an analytic way on $\delta$, see (4.10) below.

While the formulae (1.11), (1.12) and (1.13) remain conjectural, for the case $\delta = 1$, we do construct, by Dirichlet Forms methods, a Markov process $(u_t)_{t \ge 0}$ with the reflected Brownian bridge as reversible measure, and satisfying a modified version of equation (1.12) above. More precisely, we show that

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{4} \lim_{\epsilon \to 0} \rho_\epsilon''(u) + \xi, \qquad (\delta = 1) \tag{1.15}$$

where $\rho_\epsilon(x) = \frac{1}{\epsilon} \rho(\frac{x}{\epsilon})$ is a smooth approximation of the Dirac measure at 0, see Theorem 5.7 for the precise statements. Similar arguments allow to treat the case $\delta = 2$: this will be done in a forthcoming article.

1.2. **Integration by parts formulae for the laws of Bessel bridges.** Integration by parts plays a fundamental role in analysis, and most notably in stochastic analysis. For instance, it lies at the core of Malliavin Calculus and the theory of Dirichlet forms, see e.g. [30, 13, 27].

While it is relatively easy in finite dimension, where the standard rules of calculus apply, obtaining integration by parts formulae (IbPFs for short) for measures on infinite-dimensional spaces can be a difficult task, one of the main reasons being the absence of Lebesgue measure in that context. The most celebrated example is the IbPF associated with Brownian motion, or its corresponding bridge, on the interval $[0,1]$, which reads

$$E\left[\partial_h \Phi(B)\right] = -E\left[\langle h'', B\rangle\, \Phi(B)\right],$$

for all Fréchet differentiable $\Phi : L^2(0,1) \to \mathbb{R}$ and all $h \in C_c^2(0,1)$, where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product in $L^2(0,1)$. This formula follows for instance from the quasi-invariance property of the Wiener measure on $[0,1]$ along the Cameron-Martin space.

In [39], the second author exploited the relation between the law of the Brownian bridge and the law $P^3$ of the 3-dimensional Bessel bridge (also known as the normalised Brownian excursion) on $[0,1]$ to deduce an IbPF for the latter measure; other proofs were given later, see e.g. [15, 44]. In [40], exploiting an absolute continuity relation with respect to the 3-dimensional Bessel bridge, the second author obtained IbPFs for the law $P^\delta$ of Bessel bridges of dimension $\delta > 3$. Put in a nutshell, these formulae read as follows:

$$E^\delta\left[\partial_h \Phi(X)\right] + E^\delta\left[\langle h'', X\rangle\, \Phi(X)\right] = -\kappa(\delta)\, E^\delta\left[\langle h, X^{-3}\rangle\, \Phi(X)\right] \tag{1.16}$$

for all $\delta > 3$, and

$$\begin{aligned} &E^3\left[\partial_h \Phi(X)\right] + E^3\left[\langle h'', X\rangle\, \Phi(X)\right] = \\ &= -\int_0^1 \mathrm{d}r\, \frac{h_r}{\sqrt{2\pi r^3 (1-r)^3}}\, E^\delta\left[\Phi(X) \,|\, X_r = 0\right], \end{aligned} \tag{1.17}$$

where $\Phi$ and $h$ are as above. Here, for all $\delta > 0$, $E^\delta$ denotes the expectation with respect to the law $P^\delta$, on the space of continuous real-valued functions on $[0,1]$, of the $\delta$-dimensional Bessel bridge from 0 to 0 over the interval $[0,1]$, and $\kappa(\delta)$ is defined in (1.7). Note that $\kappa(\delta) > 0$ for $\delta > 3$, and $\kappa$ vanishes at $\delta = 3$, the dimension corresponding to the Brownian excursion. At the same time, the quantity $\langle |h|, X^{-3}\rangle$ is integrable with respect to $P^\delta$ for $\delta > 3$, but is non-integrable with respect to $P^3$ for $h$ that is not identically 0. Hence, (1.17) can be seen as the limit, as $\delta \searrow 3$, of (1.16). Formula (1.17) also possesses a geometric-measure theory interpretation as a Gauss-Green formula in an infinite-dimensional space, the second term in the rigt-hand side corresponding to a boundary term (see Chapter 6.1.2 in [44]).

What can we say for Bessel bridges of dimension $\delta < 3$ ? In such a regime, the techniques used in [40], based on absolute continuity relations with the Brownian excursion as well as monotonicity arguments, fall apart. Indeed, when $\delta \in (1,3)$, $\kappa(\delta) < 0$, so the required monotonicity properties do not hold anymore, while for $\delta < 2$ the absolute continuity relations fail to exist. Hence, the problem of finding IbPFs for the measures $P^\delta$, when $\delta < 3$, has remained open until now, excepted for the value $\delta = 1$, corresponding to the reflected Brownian bridge, for which some (strictly weaker) IbPFs have been obtained, see [42] for the case of the reflected Brownian motion, [19] for the case of a genuine bridge, and Remark 4.11 below for a discussion.

1.3. **Outline of the results.** Here and below, let $C([0,1]) := C([0,1], \mathbb{R})$ be the space of continuous real-valued functions on $[0,1]$. In this article, we obtain IbPFs for the laws $P^\delta$ of Bessel bridges of dimension $\delta \in (0,3)$ from 0 to 0 over $[0,1]$. Our formulae hold for a large class of functionals $\Phi : C([0,1]) \to \mathbb{R}$. More precisely, we consider linear combinations of functionals of the form

$$\Phi(\zeta) = \exp(-\langle m, \zeta^2 \rangle), \quad \zeta \in C([0,1]), \tag{1.18}$$

with $m$ a finite Borel measure on $[0,1]$, and where $\langle m, \zeta^2 \rangle := \int_0^1 \zeta_t^2 \, m(\mathrm{d}t)$. We prove that these functionals satisfy IbPFs for the laws $P^\delta$, for all $\delta > 0$. Our method is based on deriving semi-explicit expressions for quantities of the form

$$E^\delta \left[ \Phi(X) \right] \qquad \text{and} \qquad E^\delta \left[ \Phi(X) \,|\, X_r = a \right], \quad a \geq 0, \, r \in (0,1),$$

using solutions to some second-order differential equations, and exploiting the nice computations done in Chapter XI of [33]. The fundamental property enabling these computations is the additivity property of the squared Bessel processes, which in particular implies that both of the quantities above factorize in a very specific way, see the expression (3.15) below. As a consequence, for functionals as above, all the IbPFs for $P^\delta$, $\delta \geq 3$ are just multiples of a single differential relation which does not depend on $\delta$ (see Lemma 4.6 below), the dependence in $\delta$ entering only through the multiplying constant which involves some $\Gamma$ values. When $\delta \geq 3$, expressing these $\Gamma$ values as integrals, and performing a change of variable, we retrieve the formulae already obtained in [39] and [40]. On the other hand, when $\delta < 3$, one of the $\Gamma$ values appearing is negative, so we cannot express it using the usual integral formula, but must rather use *renormalized* integrals.

As a result, when $\delta \in (1,3)$, the IbPFs can be written

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X \rangle \, \Phi(X)) = \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty a^{\delta-4} \Big[ \Sigma_r^\delta \left( \Phi(X) \,|\, a \right) - \Sigma_r^\delta \left( \Phi(X) \,|\, 0 \right) \Big] \, \mathrm{d}a \, \mathrm{d}r, \end{aligned} \tag{1.19}$$

where, for all $a \geq 0$, $\Sigma_r^\delta \left( \mathrm{d}X \,|\, a \right)$ is a measure on $C([0,1])$ proportional to the law of the Bessel bridge conditioned to hit $a$ at $r$, see (3.12). Thus, the left-hand side is the same as for (1.16) and (1.17), but the right-hand side now contains Taylor

remainders at order 0 of the functions $a \mapsto \Sigma_r^\delta(\Phi(X) \,|\, a)$. When $\delta \in (0,1)$, this renormalization phenomenon becomes even more acute. Indeed, in that case, the IbPFs are similar to (1.19), but the right-hand side is replaced by

$$-\kappa(\delta) \int_0^1 h_r \int_0^\infty a^{\delta-4} \left[\varphi(a) - \varphi(0) - \frac{a^2}{2}\varphi''(0)\right] \mathrm{d}a \, \mathrm{d}r, \tag{1.20}$$

where $\varphi(a) := \Sigma_r^\delta(\Phi(X) \,|\, a)$, and where we see Taylor remainders at order 2 appearing. An important remark is that the terms of order 1 vanish

$$\varphi'(0) = \frac{\mathrm{d}}{\mathrm{d}a}\Sigma_r^\delta(\Phi(X) \,|\, a)\Big|_{a=0} = 0, \quad r \in (0,1), \tag{1.21}$$

so we do not see them in the above Taylor remainders. Finally, in the critical case $\delta = 1$, we obtain the fomula

$$E^1(\partial_h \Phi(X)) + E^1(\langle h'', X\rangle\, \Phi(X)) = \frac{1}{4}\int_0^1 h_r \frac{\mathrm{d}^2}{\mathrm{d}a^2}\Sigma_r^1(\Phi(X) \,|\, a)\Big|_{a=0} \mathrm{d}r. \tag{1.22}$$

The IbPFs are stated in Theorem 4.1 below. One important, expected feature is the transition that occurs at the critical values $\delta = 3$ and $\delta = 1$. Another important but less expected feature is the absence of transition at $\delta = 2$, as well as the related remarkable fact that the functions $a \mapsto \Sigma_r^\delta(\Phi(X) \,|\, a)$ are, for all $r \in (0,1)$, smooth functions in $a^2$, so that all their odd-order derivatives vanish at 0. This is the reason why there only ever appear derivatives of even order in our formulae. An objection to this observation might be that the class of functionals (1.18) is too restrictive. However, in a forthcoming article, we will show that the IbPFs obtained in the present article still hold for a class of very general functionals. In particular, vanishing of first-order derivatives at $a = 0$ can be established for $a \mapsto \Sigma_r^\delta(\Phi(X) \,|\, a)$ for any $\Phi \in C_b^1(L^2(0,1))$, which confirms the absence of transition at $\delta = 2$ observed in this article. Finally, note that all the IbPFs above can be written in a unified way, by re-expressing the last term as

$$-\frac{\Gamma(\delta)}{4(\delta-2)}\int_0^1 \langle \mu_{\delta-3}, \Sigma_r^\delta(\Phi(X) \,|\, \cdot\,)\rangle$$

in analogy with (1.14). The latter formula bears out the idea that the new IbPFs for Bessel bridges of dimension $\delta < 3$ are given by the unique analytic continuation of those for $\delta \geq 3$, at least for suitable test functionals $\Phi$ as in (1.18).

The IbPFs (1.19), (1.20) and (1.22) above suggest that the gradient dynamics associated with the laws of Bessel bridges of dimension $\delta < 3$ should be given by the SPDEs (1.11), (1.13) and (1.12) respectively. Note that, in the case $\delta \geq 3$, the SPDEs had been solved in [31, 40] using pathwise techniques, and many fine properties of the solution had been studied, such as their hitting properties (see [6]), or the existence of occupation densities (see [41]). By contrast, in the case $\delta < 3$, the SPDEs (1.11), (1.12) and (1.13) do not yet seem to possess any strong notion of solution, and essentially lie outside the scope of any existing theory of SPDEs.

However, in this article, for $\delta = 1$, using Dirichlet form techniques, and thanks to the IbPF (1.22) for the reflecting Brownian bridge, we are able to construct a weak version of the associated SPDE in the stationary regime. Thus, the dynamics for $\delta = 1$ can be described by (1.15), which is a weaker version of (1.12). We also prove (see Theorem 5.8 below) that the corresponding Markov process does not coincide with the process associated with the absolute value of the solution to the stochastic heat equation. A similar construction can be implemented in the case $\delta = 2$: this will be done in the forthcoming article [9]. The approach using Dirichlet forms was already used in Robert Voßhall's thesis [37], which provided a construction of the Markov process for $\delta = 1$, but not the SPDE.

The article is organized as follows: in Section 2 we address a toy-model consisting of a family of measures on $\mathbb{R}_+$, hence much simpler than the laws of Bessel bridges, but displaying a similar renormalization phenomenon at the level of the IbPFs. In Section 3 we recall and prove some useful facts on the laws of squared Bessel processes, Bessel processes, and their bridges. In Section 4, we state and prove the IbPFs for the laws of Bessel bridges. The dynamics associated with the law of a reflected Brownian bridge is constructed and studied in Section 5. Finally, in Section 6, we justify our conjectures (1.11) (1.12) and (1.13) for the $\delta$-Bessel SPDEs for $\delta < 3$, and we formulate some additional related conjectures.

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## 2. A prelude

In this section we consider a toy model consisting of a family of Schwartz distributions on $\mathbb{R}_+$ satisfying nice integration by parts formulae. The content of this section is classical (see e.g. Section 3.5 of [17]), but it will serve as a useful finite-dimensional example for the theory to come. For $\alpha \geq 0$, we set

$$\mu_\alpha(\mathrm{d}x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}x, \quad \alpha > 0, \qquad \mu_0 = \delta_0,$$

where $\delta_0$ denotes the Dirac measure at 0. A simple change of variable yields the Laplace transform of the measures $\mu_\alpha$, $\alpha \geq 0$

$$\int \exp(-\lambda x)\, \mu_\alpha(\mathrm{d}x) = \lambda^{-\alpha}, \qquad \lambda > 0,\ \alpha \geq 0. \tag{2.1}$$

It turns out that the family of measures $(\mu_\alpha)_{\alpha\geq 0}$ can be extended in a natural way to a family of *distributions* $(\mu_\alpha)_{\alpha\in\mathbb{R}}$ . We first define the appropriate space of test functions on $[0,\infty)$.

**Definition 2.1.** Let $S([0,\infty))$ be the space of $C^\infty$ functions $\varphi : [0,\infty) \to \mathbb{R}$ such that, for all $k, l \geq 0$, there exists $C_{k,\ell} \geq 0$ such that

$$\forall x \geq 0, \quad |\varphi^{(k)}(x)|\, x^\ell \leq C_{k,\ell}. \tag{2.2}$$

For $\alpha < 0$, we will define $\mu_\alpha$ as a distribution, using a *renormalization* procedure based on Taylor polynomials. To do so, for any smooth function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, for all $n \in \mathbb{Z}$, and all $x \geq 0$, we set

$$\mathcal{T}_x^n \varphi := \varphi(x) - \sum_{0\leq j\leq n} \frac{x^j}{j!}\, \varphi^{(j)}(0). \tag{2.3}$$

In words, if $n \geq 0$ then $\mathcal{T}_x^n\varphi$ is the Taylor remainder based at 0, of order $n+1$, of the function $\varphi$, evaluated at $x$; if $n < 0$ then $\mathcal{T}_x^n\varphi$ is simply the value of $\varphi$ at $x$.

**Definition 2.2.** For $\alpha < 0$, we define the distribution $\mu_\alpha$ as follows

- if $\alpha = -k$ with $k \in \mathbb{N}$, then
$$\langle \mu_\alpha, \varphi\rangle := (-1)^k \varphi^{(k)}(0), \qquad \forall\, \varphi \in \mathcal{S}([0,\infty)) \tag{2.4}$$
- if $-k-1 < \alpha < -k$ with $k \in \mathbb{N}$, then
$$\langle \mu_\alpha, \varphi\rangle := \int_0^{+\infty} \mathcal{T}_x^k \varphi\, \frac{x^{\alpha-1}}{\Gamma(\alpha)}\, \mathrm{d}x, \qquad \forall\, \varphi \in \mathcal{S}([0,\infty)). \tag{2.5}$$

Note that formula (2.5) defines a bona fide distribution on $\mathcal{S}([0,\infty))$. Indeed, by Taylor's theorem, the integrand is of order $x^{k+\alpha}$ near 0, therefore integrable there, while it is dominated by $x^{k+\alpha-1}$ near $+\infty$, so is integrable at infinity as well. We note that $\mu_\alpha$ is equal the generalized function $(\frac{x_+^{\alpha-1}}{\Gamma(\alpha)})_{\alpha\in\mathbb{R}}$ of Section 3.5 of [17].

**Remark 2.3.** Note that for all $\alpha > 0$ and all Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, the integral $\int_0^\infty \varphi(x)\, \mu_\alpha(\,\mathrm{d}x)$ coincides with $\Gamma(\alpha)^{-1}\mathcal{M}\varphi(\alpha)$, where $\mathcal{M}\varphi(\alpha)$ is the value of the Mellin transform of the function $\varphi$ computed at $\alpha$. Definition 2.2 thus provides an extension of the Mellin transform of a function $\varphi \in \mathcal{S}([0,\infty))$ to the whole real line. In particular, equality (2.4) is natural in view of Ramanujan's Master Theorem, which allows to see the successive derivatives at 0 of an analytic function as the values, for non-positive integers, of the analytic extension of its Mellin transform. We refer to [2] for more details on this theorem. We also stress that the renormalization procedure used in equation (2.5) to define $\mu_\alpha$ for $\alpha < 0$ is

very natural, and can also be used to extend the domain of validity of Ramanujan's Master Theorem, see Theorem 8.1 in [2].

**Remark 2.4.** For $k \in \mathbb{N}$ and $\alpha$ such that $-k-1 < \alpha < -k$, and for all $\varphi \in \mathcal{S}([0,\infty))$, we obtain after $k+1$ successive integration by parts the equality:

$$\langle \mu_\alpha, \varphi \rangle := (-1)^{k+1} \int_0^{+\infty} \varphi^{(k+1)}(x)\, \mu_{\alpha+k+1}(\mathrm{d}x), \tag{2.6}$$

which can be interpreted as a variant of the Caputo differential, at order $-\alpha$, of $\varphi$, see e.g. (1.17) in [18].

We recall the following basic fact, which is easily proven (see e.g. (5) in Section 3.5 of [17]). It can be seen as a toy-version of the integration by parts formulae of Theorem 4.1 below.

**Proposition 2.5.** *For all $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{S}([0,\infty))$*

$$\langle \mu_\alpha, \varphi' \rangle = -\langle \mu_{\alpha-1}, \varphi \rangle.$$

In parficular, for $\alpha \in (0,1)$ we have the measure $\mu_\alpha$ in the left-hand side of the IbPF and the distribution $\mu_{\alpha-1}$ in the right-hand side.

**Remark 2.6.** As a consequence of Proposition 2.5, we deduce that the expression (2.1) for the Laplace tranform of $\mu_\alpha$ remains true also for negative $\alpha$. Indeed, for such $\alpha$, picking $k \in \mathbb{N}$ such that $\alpha + k > 0$, we have, for all $\lambda > 0$

$$\langle \mu_\alpha, e^{-\lambda \cdot} \rangle = (-1)^k \langle \mu_{\alpha+k}, \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-\lambda \cdot} \rangle = \lambda^k \langle \mu_{\alpha+k}, e^{-\lambda \cdot} \rangle = \lambda^k \lambda^{-\alpha-k} = \lambda^{-\alpha}.$$

## 3. Bessel processes and associated bridges

In this section we recall and prove some useful facts about squared Bessel processes, Bessel processes, and their corresponding bridges. We recall that, for all $\alpha \geq 0$, $\theta > 0$, $\Gamma(\alpha, \theta)$ denotes the Gamma probability law on $\mathbb{R}_+$

$$\Gamma(\alpha,\theta)(\mathrm{d}x) = \frac{\theta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\theta x}\, \mathbf{1}_{x>0}\, \mathrm{d}x, \qquad \Gamma(0,\theta) := \delta_0.$$

3.1. **Squared Bessel processes and Bessel processes.** For all $x, \delta \geq 0$, denote by $Q_x^\delta$ the law, on $C(\mathbb{R}_+, \mathbb{R}_+)$, of the $\delta$-dimensional squared Bessel process started at $x$, namely the unique solution to the SDE (1.1) with $Y_0 = x$, see Chapter XI of [33]. We denote by $(X_t)_{t\geq 0}$ the canonical process

$$X_t : C([0,1]) \to \mathbb{R}, \qquad X_t(\omega) := \omega_t, \quad \omega \in C([0,1]).$$

**Definition 3.1.** For any interval $I \subset \mathbb{R}_+$, and any two probability laws $\mu, \nu$ on $C(I, \mathbb{R}_+)$, let $\mu * \nu$ denote the convolution of $\mu$ and $\nu$, i.e. the image of $\mu \otimes \nu$ under the addition map:

$$C(I,\mathbb{R}_+) \times C(I,\mathbb{R}_+) \to C(I,\mathbb{R}_+), \quad (x,y) \mapsto x+y.$$

The family of probability measures $\left(Q_x^\delta\right)_{\delta,x\geq 0}$ satisfies the following well-known additivity property, first observed by Shiga and Watanabe in [35].

**Proposition 3.2.** *For all $x, x', \delta, \delta' \geq 0$, we have the following equality of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$*

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x'}^{\delta+\delta'} \tag{3.1}$$

We recall that squared Bessel processes are homogeneous Markov processes on $\mathbb{R}_+$. Exploiting the additivity property (3.1), Revuz and Yor provided, in section XI of [33], explicit expressions for their transition densities $\left(q_t^\delta(x,y)\right)_{t>0,x,y\geq 0}$. When $\delta > 0$, these are given by

$$q_t^\delta(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right), \quad t>0,\ x>0 \tag{3.2}$$

and

$$q_t^\delta(0,y) = (2t)^{-\frac{\delta}{2}}\,\Gamma\left(\delta/2\right)^{-1} y^{\delta/2-1} \exp\left(-\frac{y}{2t}\right), \quad t>0. \tag{3.3}$$

In particular, for $x=0$, we have

$$q_t^\delta(0,y)\,\mathrm{d}y = \Gamma\left(\frac{\delta}{2}, \frac{1}{2t}\right)(\mathrm{d}y).$$

We also denote by $P_x^\delta$ the law of the $\delta$-Bessel process, image of $Q_{x^2}^\delta$ under the map

$$C(\mathbb{R}_+,\mathbb{R}_+) \ni \omega \mapsto \sqrt{\omega} \in C(\mathbb{R}_+,\mathbb{R}_+). \tag{3.4}$$

We shall denote by $\left(p_t^\delta(a,b)\right)_{t>0,\,a,b\geq 0}$ the transition densities of a $\delta$-Bessel process. They are given in terms of the densities of the squared Bessel process by the relation

$$\forall t>0, \quad \forall a,b\geq 0, \quad p_t^\delta(a,b) = 2\,b\,q_t^\delta(a^2,b^2). \tag{3.5}$$

In section XI of [33], Revuz and Yor provided semi-explicit expressions for the Laplace transforms of squared Bessel processes (and also the corresponding bridges). Their proof is based on the fact that, for all $\delta, x \geq 0$, and all finite Borel measure $m$ on $[0,1]$, the measure $\exp\left(-\langle m, X\rangle\right) Q_x^\delta$ possesses a nice probabilistic interpretation, where we use the notation

$$\langle m, f\rangle := \int_0^1 f(r)\, m(\mathrm{d}r)$$

for any Borel function $f : [0,1] \to \mathbb{R}_+$. This remarkable fact is used implicitly in [33] (see e.g. the proof of Theorem (3.2) of Chap XI.3), where the authors compute the one-dimensional marginal distributions of this measure. By contrast, in the proof of Lemma 3.6 below, we will need to compute higher-dimensional marginals. We thus need a way to interpret the measure $\exp\left(-\langle m, X\rangle\right) Q_x^\delta$ probabilistically. To do so, we first introduce some notations.

Let $m$ be a finite, Borel measure on $[0,1]$. As in Chap. XI of [33], we consider the unique solution $\phi = (\phi_r, r \geq 0)$ on $\mathbb{R}_+$ of the following problem

$$\begin{cases} \phi''(\mathrm{d}r) = 2\mathbf{1}_{[0,1]}(r)\,\phi_r\, m(\mathrm{d}r) \\ \phi_0 = 1,\ \phi > 0,\ \phi' \leq 0 \text{ on } \mathbb{R}_+, \end{cases} \tag{3.6}$$

where the first is an equality of measures (see Appendix 8 of [33] for existence and uniqueness of solutions to this problem). Note that the above function $\phi$ coincides with the function $\phi_\mu$ of Chap XI.1 of [33], with $\mu := 2\mathbf{1}_{[0,1]}\, m$.

**Lemma 3.3.** *Let $m$ be a finite, Borel measure on $[0,1]$, and let $\phi$ be the unique solution of (3.6). Then, for all $x, \delta \geq 0$, the measure $R^\delta_x$ on $C([0,1])$ defined by*

$$R^\delta_x := \exp\left(-\frac{x}{2}\phi'_0\right) \phi_1^{-\frac{\delta}{2}}\ e^{-\langle m, X\rangle}\ Q^\delta_x \tag{3.7}$$

*is a probability measure, equal to the law of the process*

$$\left(\phi_t^2\ Y_{\varrho_t}\right)_{t\in[0,1]},$$

*where $Y \overset{(d)}{=} Q^\delta_x$ and $\varrho$ is the deterministic time change*

$$\varrho_t = \int_0^t \phi_u^{-2}\,\mathrm{d}u, \quad t \geq 0. \tag{3.8}$$

*Proof.* We proceed as in the proofs of Theorem (1.7) and (3.2) in Chapter XI of [33]. Let $x, \delta \geq 0$. Under $Q^\delta_x$, $M_t := X_t - \delta t$ is a local martingale, so we can define an exponential local martingale by setting

$$Z_t = \mathscr{E}\left(\frac{1}{2}\int_0^\cdot \frac{\phi'_s}{\phi_s}\,\mathrm{d}M_s\right)_t.$$

As established in the proof of Theorem (1.7) of [33], we have

$$\begin{aligned} Z_t &= \exp\left(\frac{1}{2}\left(\frac{\phi'_t}{\phi_t}X_t - \phi'_0 x - \delta\ln\phi_t\right) - \int_0^t X_s\, m(\mathrm{d}s)\right) = \\ &= \exp\left(-\frac{x}{2}\phi'_0\right)\phi_t^{-\frac{\delta}{2}} \exp\left(\frac{1}{2}\frac{\phi'_t}{\phi_t}X_t - \int_0^t X_s\, m(\mathrm{d}s)\right), \end{aligned}$$

recalling that the measure $\mu$ considered in [33] is given in our case by $2\,\mathbf{1}_{[0,1]}\, m$. In particular, we deduce that the measure $R^\delta_x$ defined by (3.7) coincides with $Z_1 Q^\delta_x$ (note that $\phi'_1 = 0$ as a consequence of (3.6)). Moreover, by the above expression, $(Z_t)_{t\in[0,1]}$ is uniformly bounded by $\exp\left(-\frac{1}{2}\phi'_0\right)\phi_1^{-\frac{\delta}{2}}$, so it is a martingale on $[0,1]$. Hence, $R^\delta_x$ defines a probability measure.

There remains to give a description of $R^\delta_x$. By Girsanov's theorem, under $R^1_x$, $(X_t)_{t\in[0,1]}$ solves the following SDE on $[0,1]$

$$X_t = x + 2\int_0^t \sqrt{X_s}\,\mathrm{d}B_s + 2\int_0^t \frac{\phi'_s}{\phi_s}X_s\,\mathrm{d}s + t. \tag{3.9}$$

But a weak solution to this SDE is provided by $(H_t^2)_{t\in[0,1]}$, where

$$H_t := \phi_t \left( x + \int_0^t \phi_s^{-1} \, \mathrm{d}W_s \right),$$

where $W$ is a standard Brownian motion. By strong and therefore weak uniqueness of solutions to equation (3.9), see [33, Theorem IX.3.5], we deduce that $X$ is equal in law to the process $(H_t^2)_{t\in[0,1]}$. On the other hand, by Lévy's theorem, we have

$$(H_t)_{t\in[0,1]} \overset{(d)}{=} (\phi_t \, \gamma_{\varrho_t})_{t\in[0,1]},$$

where $\gamma$ is a standard Brownian motion started at $x$. Hence we deduce that

$$(H_t^2)_{t\in[0,1]} \overset{(d)}{=} \left(\phi_t^2 \, Y_{\varrho_t}\right)_{t\in[0,1]},$$

where $Y \overset{(d)}{=} Q_x^1$. Therefore, under $R_x^1$, we have

$$X \overset{(d)}{=} \left(\phi_t^2 \, Y_{\varrho_t}\right)_{t\in[0,1]}.$$

The claim is thus proven for $\delta = 1$ and for any $x \geq 0$. Now, by the additivity property (3.1) satisfied by $\left(Q_x^\delta\right)_{\delta, x\geq 0}$, there exist $A, B > 0$ such that, for all $x, \delta \geq 0$, and all finite Borel measure $\nu$ on $[0,1]$, we have

$$Q_x^\delta \left[ \exp\left( - \int_0^1 \phi_t^2 \, X_{\varrho_t} \, \nu(\mathrm{d}t) \right) \right] = A^x B^\delta,$$

which can be proved exactly as Corollary 1.3 in Chapter XI of [33]. Note now that the family of probability laws $\left(R_x^\delta\right)_{\delta, x\geq 0}$ satisfies the same additivity property

$$\forall \ \delta, \delta', x, x' \geq 0, \quad R_x^\delta * R_{x'}^{\delta'} = R_{x+x'}^{\delta+\delta'}.$$

Hence, there also exist $\tilde{A}, \tilde{B} > 0$ such that, for all $x, \delta \geq 0$, and $\mu$ as above:

$$R_x^\delta \left[ \exp\left( - \int_0^1 X_t \, \nu(\mathrm{d}t) \right) \right] = \tilde{A}^x \tilde{B}^\delta.$$

By the previous point, evaluating at $\delta = 1$, we obtain

$$\forall x \geq 0, \quad A^x B = \tilde{A}^x \tilde{B}.$$

Hence $A = \tilde{A}$ and $B = \tilde{B}$, whence we deduce that, for all $\delta, x \geq 0$

$$Q_x^\delta \left[ \exp\left( - \int_0^1 \phi_t^2 \, X_{\varrho_t} \, \nu(\mathrm{d}t) \right) \right] = R_x^\delta \left[ \exp\left( - \int_0^1 X_t \, \nu(\mathrm{d}t) \right) \right].$$

Since this holds for any finite measure $\nu$ on $[0,1]$, by injectivity of the Laplace transform, the claimed equality in law holds for all $\delta, x \geq 0$. □

3.2. **Squared Bessel bridges and Bessel bridges.** For all $\delta, x, y \geq 0$, we denote by $Q^\delta_{x,y}$ the law, on $C([0,1])$, of the $\delta$-dimensional squared Bessel bridge from $x$ to $y$ over the interval $[0,1]$. In other words, $Q^\delta_{x,y}$ is the law of of a $\delta$-dimensional squared Bessel bridge started at $x$, and conditioned to hitting $y$ at time 1. A rigourous construction of these probability laws is provided in Chap. XI.3 of [33] (see also [32] for a discussion on the particular case $\delta = y = 0$).

In the sequel we shall chiefly consider the case $x = y = 0$, and write $Q^\delta$ instead of $Q^\delta_{0,0}$. We recall that if $X \overset{(d)}{=} Q^\delta$, then, for all $r \in (0,1)$, the distribution of the random variable $X_r$ is given by $\Gamma(\frac{\delta}{2}, \frac{1}{2r(1-r)})$, so it admits the density $q^\delta_r$ given by:

$$q^\delta_r(z) := \frac{z^{\delta/2-1}}{(2r(1-r))^{\frac{\delta}{2}}\Gamma(\delta/2)} \exp\left(-\frac{z}{2r(1-r)}\right), \quad z \geq 0, \tag{3.10}$$

see Chap. XI.3 of [33].

In the same way as one constructs the laws of squared Bessel bridges $Q^\delta_{x,y}$, $\delta, x, y \geq 0$, one can also construct the laws of Bessel bridges. In the following, for any $\delta, a, b \geq 0$, we shall denote by $P^\delta_{a,b}$ the law, on $C([0,1])$, of the $\delta$-dimensional Bessel bridge from $a$ to $b$ over the time interval $[0,1]$ (that is, the law of a $\delta$-dimensional Bessel process started at $a$ and conditioned to hit $b$ at time 1). We shall denote by $E^\delta_{a,b}$ the expectation operator for $P^\delta_{a,b}$. Morever, when $a = b = 0$, we shall drop the subindices and use the compact notations $P^\delta$ and $E^\delta$. Note that, for all $a, b \geq 0$, $P^\delta_{a,b}$ is the image of $Q^\delta_{a^2,b^2}$ under the map $\omega \mapsto \sqrt{\omega}$. In particular, under the measure $P^\delta$, for all $r \in (0,1)$, $X_r$ admits the density $p^\delta_r$ on $\mathbb{R}_+$, where by (3.10)

$$p^\delta_r(a) = 2a\, q^\delta_r(a^2) = \frac{a^{\delta-1}}{2^{\frac{\delta}{2}-1}\,\Gamma(\frac{\delta}{2})(r(1-r))^{\delta/2}} \exp\left(-\frac{a^2}{2r(1-r)}\right), \quad a \geq 0. \tag{3.11}$$

3.3. **Pinned bridges.** Let $\delta \geq 0$. For all $x \geq 0$ and $r \in (0,1)$, we denote by $Q^\delta[\,\cdot\,|\,X_r = x]$ the law, on $C([0,1])$, of a $\delta$-dimensional squared Bessel bridge between 0 and 0, pinned at $x$ at time $r$ (that is, conditioned to hit $x$ at time $r$). Such a probability law can be constructed using the same procedure as for the construction of squared Bessel bridges. One similarly defines, for all $a \geq 0$ and $r \in (0,1)$, the law $P^\delta[\,\cdot\,|\,X_r = a]$ of a $\delta$-dimensional Bessel bridge between 0 and 0 pinned at $a$ at time $r$. Note that the latter probability measure is the image of $Q^\delta[\,\cdot\,|\,X_r = a^2]$ under the map (3.4).

With these notations at hand, we now define a family of measures which will play an important role in the IbPF for Bessel bridges. Heuristically, they should be related to the local times of the solution $(u(t,x))_{t\geq 0,\, x\in[0,1]}$ to an SPDE having the law of a Bessel bridge as reversible measure.

**Definition 3.4.** For all $a \geq 0$ and $r \in (0,1)$, we set

$$\Sigma_r^\delta(\mathrm{d}X \,|\, a) := \frac{p_r^\delta(a)}{a^{\delta-1}}\, P^\delta[\mathrm{d}X \,|\, X_r = a], \tag{3.12}$$

where $p_r^\delta$ is the probability density function of $X_r$ under $P^\delta := P^\delta_{0,0}$, see (3.11).

The measure $\Sigma_r^\delta(\,\cdot\,|\,a)$ is meant to be the Revuz measure of the diffusion local time of $(u(t,r))_{t\geq 0}$ at level $a \geq 0$.

**Remark 3.5.** Note that, for all $r \in (0,1)$, by (3.11), we have

$$\frac{p_r^\delta(a)}{a^{\delta-1}} = \frac{1}{2^{\frac{\delta}{2}-1}\,\Gamma(\frac{\delta}{2})(r(1-r))^{\delta/2}}\, \exp\left(-\frac{a^2}{2r(1-r)}\right), \quad a > 0,$$

and the right-hand side is well-defined also for $a = 0$. It is this quantity that we consider in equality (3.12) above.

To keep the formulae synthetic, for all $r \in (0,1)$ and $a \geq 0$, and all Borel function $\Phi : C([0,1]) \to \mathbb{R}_+$, we shall write with a slight abuse of language

$$\Sigma_r^\delta(\Phi(X) \,|\, a) := \int \Phi(X)\ \Sigma_r^\delta(\mathrm{d}X \,|\, a).$$

In the sequel we will have to compute quantities of the form

$$\Sigma_r^\delta\left(\exp(-\langle m, X^2\rangle) \,|\, a\right)$$

for $m$ a finite Borel measure on $[0,1]$. In that perspective, we introduce some further notations. Given such a $m$, following the notation used in [32] (see also Exercise (1.34), Chap. XI, of [33]), we denote by $\psi$ the function on $[0,1]$ given by

$$\psi_r := \phi_r \int_0^r \phi_u^{-2}\,\mathrm{d}u = \phi_r \varrho_r, \qquad r \in [0,1], \tag{3.13}$$

where $\varrho$ is as in (3.8). Note that $\psi$ is the unique solution on $[0,1]$ of the Cauchy problem

$$\begin{cases} \psi''(\mathrm{d}r) = 2\,\psi_r\, m(\mathrm{d}r) \\ \psi_0 = 0, \quad \psi_0' = 1. \end{cases}$$

Moreover, we denote by $\hat{\psi}$ the function on $[0,1]$ given by

$$\hat{\psi}_r := \phi_1\phi_r(\varrho_1 - \varrho_r) = \psi_1\phi_r - \psi_r\phi_1, \quad r \in [0,1]. \tag{3.14}$$

Note that $\hat{\psi}$ satisfies the following problem on $[0,1]$

$$\begin{cases} \hat{\psi}''(\mathrm{d}r) = 2\,\hat{\psi}_r\, m(\mathrm{d}r) \\ \hat{\psi}_1 = 0, \quad \hat{\psi}_1' = -1. \end{cases}$$

Note that the functions $\phi$, $\psi$ and $\hat{\psi}$ take positive values on $]0,1[$.

**Lemma 3.6.** *For all $r \in (0,1)$, $\delta > 0$ and $a \geq 0$, the following holds:*

$$\int \exp(-\langle m, X^2\rangle)\ \Sigma_r^\delta(\mathrm{d}X \,|\, a) = \frac{1}{2^{\frac{\delta}{2}-1}\Gamma(\frac{\delta}{2})}\ \exp\left(-\frac{a^2}{2}C_r\right) D_r^{\delta/2}, \tag{3.15}$$

*where*

$$C_r = \frac{\psi_1}{\psi_r\hat{\psi}_r}, \qquad D_r = \frac{1}{\psi_r\hat{\psi}_r}.$$

*Proof.* First note that by (3.5) and (3.12), we have

$$\int \exp(-\langle m, X^2\rangle)\ \Sigma_r^\delta(\mathrm{d}X \,|\, a) = 2\frac{q_r^\delta(a^2)}{a^{\delta-2}} Q^\delta[\exp(-\langle m, X\rangle) \,|\, X_r = a^2]. \tag{3.16}$$

To obtain the claim, it therefore suffices to compute

$$Q^\delta[\exp(-\langle m, X\rangle) \,|\, X_r = a^2], \qquad Q^\delta := Q^\delta_{0,0}.$$

To do so, consider two Borel functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$. We have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty Q_0^\delta[\exp(-\langle m, X\rangle) \,|\, X_r = x, X_1 = y]\, q_r^\delta(a^2, x) q_{1-r}^\delta(x, y) f(x) g(y)\, \mathrm{d}x\, \mathrm{d}y = \\
&= Q_0^\delta\left[\exp(-\langle m, X\rangle) f(X_r) g(X_1)\right] = \phi_1^{\frac{\delta}{2}} Q_0^\delta\left[f\left(\phi_r^2 X_{\varrho_r}\right) g\left(\phi_1^2 X_{\varrho_1}\right)\right] = \\
&= \phi_1^{\delta/2-2}\phi_r^{-2} \int_0^\infty \int_0^\infty q_{\varrho_r}^\delta\left(0, \frac{x}{\phi_r^2}\right) q_{\varrho_1-\varrho_r}^\delta\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right) f(x) g(y)\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

Here, we used Lemma 3.3 to obtain the third line. Since the functions $f$ and $g$ are arbitrary we deduce that:

$$Q_0^\delta[\exp(-\langle m, X\rangle) \,|\, X_r = x, X_1 = y] \;=\; \phi_1^{\delta/2-2}\phi_r^{-2} \frac{q_{\varrho_r}^\delta\left(0, \frac{x}{\phi_r^2}\right) q_{\varrho_1-\varrho_r}^\delta\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right)}{q_r^\delta(0, x)\, q_{1-r}^\delta(x, y)}$$

$\mathrm{d}x\,\mathrm{d}y$ a.e. on ${\mathbb{R}_+^*}^2$. Since the family of measures $\left(Q_{x,y}^\delta\right)_{x,y\geq 0}$ is continuous in $(x, y) \in \mathbb{R}_+^2$ for the weak topology on probability measures (see [33], Section XI.3), we deduce that, for all $x \geq 0$

$$Q^\delta[\exp(-\langle m, X\rangle) \,|\, X_r = x] = \lim_{\substack{y\to 0 \\ y>0}} \phi_1^{\delta/2-2}\phi_r^{-2} \frac{q_{\varrho_r}^\delta\left(0, \frac{x}{\phi_r^2}\right) q_{\varrho_1-\varrho_r}^\delta\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right)}{q_r^\delta(0, x)\, q_{1-r}^\delta(x, y)}.$$

But, by (3.2) and (3.3), we have

$$\frac{q_{\varrho_r}^\delta\left(0, \frac{x}{\phi_r^2}\right)}{q_r^\delta(0, x)} = \left(\frac{r}{\varrho_r}\right)^{\frac{\delta}{2}} \phi_r^{2-\delta} \exp\left(-\frac{x}{2}\left(\frac{1}{\phi_r^2\varrho_r} - \frac{1}{r}\right)\right)$$

and

$$\lim_{\substack{y\to 0\\ y>0}} \frac{q^{\delta}_{\varrho_1-\varrho_r}\left(\frac{x}{\phi_r^2},\frac{y}{\phi_1^2}\right)}{q^{\delta}_{1-r}(x,y)} = \left(\frac{1-r}{\varrho_1-\varrho_r}\right)^{\frac{\delta}{2}} \phi_1^{2-\delta} \exp\left(-\frac{x}{2}\left(\frac{1}{\phi_r^2(\varrho_1-\varrho_r)}-\frac{1}{1-r}\right)\right).$$

We thus obtain

$$\begin{aligned} &Q^{\delta}[\exp(-\langle m, X\rangle)\,|\,X_r = x] = \\ &= \phi_1^{-\delta/2}\phi_r^{-\delta}\left(\frac{r(1-r)}{\varrho_r(\varrho_1-\varrho_r)}\right)^{\frac{\delta}{2}} \exp\left(-\frac{x}{2}\left(\frac{\varrho_1}{\phi_r^2\varrho_r(\varrho_1-\varrho_r)}-\frac{1}{r(1-r)}\right)\right) = \\ &= \left(\frac{r(1-r)}{\psi_r\hat{\psi}_r}\right)^{\frac{\delta}{2}} \exp\left(-\frac{x}{2}\left(\frac{\psi_1}{\psi_r\hat{\psi}_r}-\frac{1}{r(1-r)}\right)\right), \end{aligned} \tag{3.17}$$

where the second equality follows from the relations (3.13)-(3.14) defining $\psi$ and $\hat{\psi}$. Applying this equality to $x = a^2$, and replacing in (3.16), we obtain the claim. □

**Remark 3.7.** Along the proof of the above Proposition, for $\delta > 0$, $a \geq 0$, $r \in (0,1)$ and $m$ as above, we also obtained from equality (3.17) the following, useful expression

$$\begin{aligned} &Q^{\delta}\left[\exp(-\langle m, X\rangle)\,|\,X_r = a^2\right] = E^{\delta}[\exp(-\langle m, X^2\rangle)\,|\,X_r = a] \\ &= \exp\left(-\frac{a^2}{2}\left(\frac{\psi_1}{\psi_r\hat{\psi}_r}-\frac{1}{r(1-r)}\right)\right)\left(\frac{r(1-r)}{\psi_r\hat{\psi}_r}\right)^{\delta/2}. \end{aligned} \tag{3.18}$$

## 4. Integration by parts formulae

Here and in the sequel, we denote by $\mathcal{S}$ the linear span of all functionals on $C([0,1])$ of the form

$$C([0,1]) \ni X \mapsto \exp\left(-\langle m, X^2\rangle\right) \in \mathbb{R} \tag{4.1}$$

where $m$ is a finite Borel measure on $[0,1]$. The elements of $\mathcal{S}$ are the functionals for which we will derive our IbPFs wrt the laws of Bessel bridges.

### 4.1. The statement.

After recalling the definition (1.7) of $\kappa(\delta) = \frac{(\delta-3)(\delta-1)}{4}$, $\delta \in \mathbb{R}$, we can now state one of the main results of this article.

**Theorem 4.1.** *Let $\delta \in (0,\infty)\setminus\{1,3\}$, and $k := \lfloor \frac{3-\delta}{2} \rfloor \leq 1$. Then, for all $\Phi \in \mathcal{S}$*

$$\begin{aligned} &E^{\delta}(\partial_h \Phi(X)) + E^{\delta}(\langle h'', X\rangle\,\Phi(X)) = \\ &= -\kappa(\delta)\int_0^1 h_r \int_0^{\infty} a^{\delta-4}\Big[\mathcal{T}_a^{2k}\,\Sigma_r^{\delta}(\Phi(X)\,|\,\cdot\,)\Big]\,\mathrm{d}a\,\mathrm{d}r, \end{aligned} \tag{4.2}$$

*where $\mathcal{T}_x^n$ is the Taylor remainder defined in* (2.3)*. On the other hand, when $\delta \in \{1, 3\}$, the following formulae hold for all $\Phi \in \mathcal{S}$*

$$E^3(\partial_h \Phi(X)) + E^3(\langle h'', X\rangle\, \Phi(X)) = -\frac{1}{2}\int_0^1 h_r\, \Sigma_r^3(\Phi(X)\,|\,0)\,\mathrm{d}r, \tag{4.3}$$

$$E^1(\partial_h \Phi(X)) + E^1(\langle h'', X\rangle\, \Phi(X)) = \frac{1}{4}\int_0^1 h_r\, \frac{\mathrm{d}^2}{\mathrm{d}a^2}\, \Sigma_r^1(\Phi(X)\,|\,a)\,\Big|_{a=0}\, \mathrm{d}r. \tag{4.4}$$

**Remark 4.2.** Note that the last integral in (4.5) is indeed convergent. Indeed, by Lemma 3.6 $\mathcal{T}_a^{2k}\, \Sigma_r^\delta(\Phi(X)\,|\,\cdot\,)$ is the Taylor remainder of order $2k$ at 0 of a smooth, even, function. Hence, near 0, the integrand is of order $O(a^{\delta+2k-2})$. Since, $\delta + 2k - 2 > -1$, the integral is convergent at 0. On the other hand, near $\infty$, the integrand is of order $O(a^{\delta+2k-4})$. Since $\delta + 2k - 4 < -1$, integrability also holds at $+\infty$.

**Remark 4.3.** For all $\delta \in (1, 3)$ the right-hand side in the IbPF (4.13) takes the form

$$-\kappa(\delta)\int_0^1 h_r \int_0^\infty a^{\delta-4}\Big[\Sigma_r^\delta(\Phi(X)\,|\,a) - \Sigma_r^\delta(\Phi(X)\,|\,0)\Big]\,\mathrm{d}a\,\mathrm{d}r.$$

Note the absence of transition at the threshold $\delta = 2$. This might seem surprising given the transition that the Bessel bridges undergo at $\delta = 2$, which is the smallest value of $\delta$ satisfying

$$P^\delta\,[\exists r \in\, ]0, 1[\,:\, X_r = 0] = 0.$$

This lack of transition is related to the fact that, as a consequence of Lemma 3.6, we have for all $\Phi \in \mathcal{E}$:

$$\frac{\mathrm{d}}{\mathrm{d}a}\, \Sigma_r^\delta(\Phi(X)\,|\,a)\Big|_{a=0} = 0.$$

**Remark 4.4.** In the IbPF (4.2), the last term may equivalently be written as

$$-\,\kappa(\delta)\int_0^1 h_r \int_0^\infty a^{-3}\Big[\mathcal{T}_a^{2k}\, \Sigma_r^\delta(\Phi(X)\,|\,\cdot\,)\Big] m_\delta(\mathrm{d}a)\,\mathrm{d}r \tag{4.5}$$

where $m_\delta$ is the measure on $\mathbb{R}_+$ defined by

$$m_\delta(\mathrm{d}a) = \mathbf{1}_{a>0}\, a^{\delta-1}\,\mathrm{d}a.$$

Note that $m_\delta$ is a reversible measure for the $\delta$-dimensional Bessel process. Actually, if $(X_t)_{t\geq 0}$ is a $\delta$ dimensional Bessel process, we can construct a bicontinuous family of *diffusion local times* $(\ell_t^a)_{a,t\geq 0}$, satisfying the occupation times formula

$$\int_0^t f\,(X_s)\,\mathrm{d}s = \int_0^{+\infty} f(a)\,\ell_t^a\, m_\delta(\mathrm{d}a),$$

for all $f : \mathbb{R}_+ \to \mathbb{R}_+$ bounded and Borel. We hope that such a property should hold also for $(u(t, x))_{t\geq 0}$, for all $x \in (0, 1)$ where $u$ is the hypothetical solution of the dynamics corresponding to $P^\delta$. In that case the term (4.5) should correspond,

in the dynamics, to a drift in $u^{-3}$ integrated against renormalized local times. We shall develop this idea more in detail in Section 6 below.

As a consequence of the above theorem, we retrieve the following known results, see Chapter 6 of [44] and (1.16)-(1.17) above.

**Proposition 4.5.** *Let $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$. Then, for all $\delta > 3$, the following IbPF holds*

$$E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = -\kappa(\delta)\, E^\delta(\langle h, X^{-3}\rangle\, \Phi(X)). \tag{4.6}$$

*Moreover, for $\delta = 3$, the following IbPF holds*

$$\begin{aligned} &E^3(\partial_h \Phi(X)) + E^3(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\int_0^1 \mathrm{d}r\, \frac{h_r}{\sqrt{2\pi r^3(1-r)^3}}\, E^3[\Phi(X)\,|\,X_r = 0]. \end{aligned} \tag{4.7}$$

*Proof.* For $\delta > 3$ we have $k := \lfloor \frac{3-\delta}{2} \rfloor < 0$, and by (4.13)

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty a^{\delta-4}\, \Sigma_r^\delta(\Phi(X)\,|\,a)\, \mathrm{d}a\, \mathrm{d}r \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty a^{-3}\, p_r^\delta(a)\, E^\delta[\Phi(X)\,|\,X_r = a]\, \mathrm{d}a\, \mathrm{d}r \\ &= -\kappa(\delta)\, E^\delta(\langle h, X^{-3}\rangle\, \Phi(X)). \end{aligned}$$

For $\delta = 3$, it suffices to note that, for all $r \in (0,1)$

$$\frac{1}{2} \lim_{\epsilon \downarrow 0} \frac{p_r^3(\epsilon)}{\epsilon^2} = \frac{1}{\sqrt{2\pi r^3(1-r)^3}},$$

so that

$$\frac{1}{2}\, \Sigma_r^3(\Phi(X)\,|\,0\,) = \frac{1}{\sqrt{2\pi r^3(1-r)^3}} E^3[\Phi(X)\,|\,X_r = 0],$$

and the proof is complete. □

4.2. **Proof of Theorem 4.1.** We first state a differential relation satisfied by the product of the functions $\psi$ and $\hat{\psi}$ associated as above with a finite Borel measure $m$ on $[0,1]$. This relation is the skeleton of all the IbPFs for $P^\delta$, $\delta > 0$ : the latter will all be deduced from the former with a simple multiplication by a constant (depending on the parameter $\delta$).

**Lemma 4.6.** *Let $m$ be a finite Borel measure on $[0,1]$, and consider the functions $\psi$ and $\hat{\psi}$ as in (3.13) and (3.14). Then, for all $h \in C_c^2(0,1)$ and $\delta > 0$, the following equality holds*

$$\int_0^1 \sqrt{\psi_r \hat{\psi}_r}\, (h_r''\, \mathrm{d}r - 2h_r\, m(\mathrm{d}r)) = -\frac{1}{4}\psi_1^2 \int_0^1 h_r (\psi_r \hat{\psi}_r)^{-\frac{3}{2}}\, \mathrm{d}r. \tag{4.8}$$

*Proof.* Performing an integration by parts, we can rewrite the left-hand side as

$$\int_0^1 h_r \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\,m(\mathrm{d}r)\right)\left(\psi_r\hat{\psi}_r\right)^{\frac{1}{2}}.$$

Note that here we are integrating wrt the signed measure

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\,m(\mathrm{d}r)\right)\left(\psi_r\hat{\psi}_r\right)^{\frac{1}{2}} = \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\psi_r\hat{\psi}_r\right)^{\frac{1}{2}} - 2\left(\psi_r\hat{\psi}_r\right)^{\frac{1}{2}}\,m(\mathrm{d}r).$$

Now, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\psi\hat{\psi}\right)^{\frac{1}{2}} = \frac{1}{2}\frac{\psi''\hat{\psi} + 2\psi'\hat{\psi}' + \psi\hat{\psi}''}{(\psi\hat{\psi})^{\frac{1}{2}}} - \frac{1}{4}\frac{(\psi'\hat{\psi} + \psi\hat{\psi}')^2}{(\psi\hat{\psi})^{3/2}}.$$

Recalling that $\psi'' = 2\psi\,m$ and $\hat{\psi}'' = 2\hat{\psi}\,m$, we obtain

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\,m(\mathrm{d}r)\right)\left(\psi\hat{\psi}\right)^{\frac{1}{2}} =& \frac{\psi'\hat{\psi}'\psi\hat{\psi} - \frac{1}{4}(\psi'\hat{\psi} + \psi\hat{\psi}')^2}{(\psi\hat{\psi})^{3/2}}\\ =& -\frac{1}{4}\frac{(\psi'\hat{\psi} - \psi\hat{\psi}')^2}{(\psi\hat{\psi})^{3/2}}.\end{aligned}$$

Using the expressions (3.13) and (3.14) for $\psi$ and $\hat{\psi}$, we easily see that

$$\psi_r'\hat{\psi}_r - \psi\hat{\psi}_r' = \psi_1, \qquad r \in (0,1). \tag{4.9}$$

Hence, we obtain the following equality of signed measures:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\,m\right)\left(\psi\hat{\psi}\right)^{\frac{1}{2}} = -\frac{1}{4}\frac{\psi_1^2}{(\psi_r\hat{\psi}_r)^{3/2}}\,\mathrm{d}r.$$

Consequently, the left-hand side in (4.8) is equal to

$$-\frac{1}{4}\psi_1^2\int_0^1 \mathrm{d}r\; h_r\left(\psi_r\hat{\psi}_r\right)^{-3/2}.$$

The claim follows. □

As a consequence, we obtain the following preliminary result.

**Lemma 4.7.** *Let $m$ be a finite measure on $[0,1]$, and let $\Phi : C([0,1]) \to \mathbb{R}$ be the functional thereto associated as in (4.1). Then, for all $\delta > 0$ and $h \in C_c^2(0,1)$,*

$$\begin{aligned}&E^\delta(\partial_h\Phi(X)) + E^\delta(\langle h'', X\rangle\,\Phi(X)) =\\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{\frac{3}{2}}\,\Gamma(\frac{\delta}{2})}\,\psi_1^{-\frac{\delta-3}{2}}\int_0^1 h_r\left(\psi_r\hat{\psi}_r\right)^{-\frac{3}{2}}\,\mathrm{d}r,\end{aligned} \tag{4.10}$$

*where $\psi$ and $\hat{\psi}$ are associated with $m$ as in (3.13) and (3.14).*

*Proof.* By the expression (4.1) for $\Phi$, we have

$$\partial_h \Phi(X) = -2\langle Xh, m\rangle\, \Phi(X).$$

Therefore

$$E^\delta(\partial_h\Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = Q^\delta\left[\left(\langle h'', \sqrt{X}\rangle - 2\langle h\sqrt{X}, m\rangle\right) e^{-\langle m, X\rangle}\right] =$$

$$= \int_0^1 (h_r''\,\mathrm{d}r - 2h_r\, m(\mathrm{d}r)) \int_0^{+\infty} \Gamma\left(\frac{\delta}{2}, \frac{1}{2r(1-r)}\right)(\mathrm{d}a)\sqrt{a}\, Q^\delta\left[e^{-\langle m, X\rangle} \,\middle|\, X_r = a\right].$$

By (3.18) we obtain:

$$\begin{aligned} &E^\delta(\partial_h\Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= \int_0^1 (h_r''\,\mathrm{d}r - 2h_r\, m(\mathrm{d}r))\, \frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2})} \left(\frac{C_r}{2}\,\psi_1^\delta\right)^{-\frac{1}{2}} \int_0^{+\infty} \Gamma\left(\frac{\delta+1}{2}, \frac{C_r}{2}\right)(\mathrm{d}a) \\ &= \sqrt{2}\,\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2})}\psi_1^{-\frac{\delta+1}{2}} \int_0^1 \mathrm{d}r\ (h_r''\,\mathrm{d}r - 2h_r\, m(\mathrm{d}r))\, \sqrt{\psi_r\hat{\psi}_r}. \end{aligned}$$

Finally, by (4.8), the latter expression is equal to

$$-\frac{\Gamma(\frac{\delta+1}{2})}{2^{\frac{3}{2}}\,\Gamma(\frac{\delta}{2})}\,\psi_1^{-\frac{\delta-3}{2}} \int_0^1 h_r \left(\psi_r\hat{\psi}_r\right)^{-\frac{3}{2}}\,\mathrm{d}r$$

and the proof is complete. □

Apart from the above lemma, the proof of the IbPF for $P^\delta$, $\delta > 0$, will require integral expressions for negative Gamma values. For all $x \in \mathbb{R}$ we set $\lfloor x \rfloor := \sup\{k \in \mathbb{Z} : k \le x\}$. We also use the notation $\mathbb{Z}^- := \{n \in \mathbb{Z} : n \le 0\}$.

**Lemma 4.8.** *For all $x \in \mathbb{R} \setminus \mathbb{Z}^-$*

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathcal{T}_t^{\lfloor -x \rfloor}(e^{-\cdot})\,\mathrm{d}t.$$

*Proof.* By Remark 2.6 we have

$$\int_0^\infty t^{x-1} \mathcal{T}_t^{\lfloor -x \rfloor}(e^{-\cdot})\,\mathrm{d}t = \Gamma(x)\,\langle \mu_\alpha, e^{-\cdot}\rangle = \Gamma(x)\, 1^x = \Gamma(x),$$

and the claim follows. □

From Lemma 4.8 we obtain for all $C > 0$, $x \in \mathbb{R} \setminus \mathbb{Z}^-$

$$\Gamma(x)\, C^{-x} = 2^{1-x} \int_0^{+\infty} a^{2x-1} \left(e^{-C\frac{a^2}{2}} - \sum_{0 \le j \le \lfloor -x \rfloor} \frac{(-C)^j a^{2j}}{2^j j!}\right) \mathrm{d}a \tag{4.11}$$

by a simple change of variable $t = Cb^2/2$. Then (4.11) can be rewritten as follows

$$\Gamma(x)\, C^{-x} = 2^{1-x} \int_0^{+\infty} a^{2x-1}\, \mathcal{T}_a^{2\lfloor -x \rfloor} \left(e^{-C\frac{(\cdot)^2}{2}}\right) \mathrm{d}a, \quad x \in \mathbb{R} \setminus \mathbb{Z}^-. \tag{4.12}$$

**Theorem 4.9.** *Let $\delta > 0$, $\delta \notin \{1,3\}$, and $k := \lfloor \frac{3-\delta}{2} \rfloor \leq 1$. Then*

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty a^{\delta-4} \Big[ \mathcal{T}_a^{2k}\, \Sigma_r^\delta(\Phi(X)\,|\,\cdot\,) \Big]\, \mathrm{d}a\, \mathrm{d}r. \end{aligned} \tag{4.13}$$

*Proof of Theorem 4.9.* Let $\delta > 0$ and $\delta \notin \{1,3\}$. Then by (4.10)

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2}\,\Gamma(\frac{\delta}{2})} \int_0^1 h_r \left( \frac{\psi_1}{\psi_r \hat{\psi}_r} \right)^{\frac{3-\delta}{2}} \left( \psi_r \hat{\psi}_r \right)^{-\frac{\delta}{2}} \mathrm{d}r \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2}\,\Gamma(\frac{\delta}{2})} \int_0^1 h_r\, C_r^{\frac{3-\delta}{2}}\, D_r^{\delta/2}\, \mathrm{d}r \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2}\Gamma(\frac{\delta}{2})} \frac{2^{\frac{5-\delta}{2}}}{\Gamma\left(\frac{\delta-3}{2}\right)} \int_0^1 h_r\, D_r^{\delta/2} \int_0^\infty a^{\delta-4}\, \mathcal{T}_a^{2k} e^{-\frac{C_r}{2}(\cdot)^2}\, \mathrm{d}a\, \mathrm{d}r, \end{aligned}$$

where we used (4.12) with $C = C_r$ and $x = \frac{\delta-3}{2}$ to obtain the last line. Recalling the expression (3.15) for $\Sigma_r^\delta(\Phi(X)\,|\,a)$, we thus obtain

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta-3}{2})} \int_0^1 h_r \int_0^\infty a^{\delta-4}\, \mathcal{T}_{2a}^k\, \Sigma_r^\delta(\Phi(X)\,|\,a)\, \mathrm{d}a\, \mathrm{d}r. \end{aligned}$$

Now, since $\delta \notin \{1,3\}$,

$$\Gamma(\tfrac{\delta+1}{2}) = \tfrac{\delta-1}{2}\, \Gamma(\tfrac{\delta-1}{2}) = \tfrac{\delta-1}{2}\, \tfrac{\delta-3}{2}\, \Gamma(\tfrac{\delta-3}{2}) = \kappa(\delta)\, \Gamma(\tfrac{\delta-3}{2}).$$

Therefore $\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta-3}{2})} = \kappa(\delta)$ and we obtain the claim. □

There remains to treat the critical cases $\delta \in \{1,3\}$.

**Theorem 4.10.** *Let $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$. The following IbPFs holds*

$$\begin{aligned} E^3(\partial_h \Phi(X)) + E^3(\langle h'', X\rangle\, \Phi(X)) &= -\frac{1}{2} \int_0^1 \mathrm{d}r\, h_r\, \Sigma_r^3(\Phi(X)\,|\,0), \\ E^1(\partial_h \Phi(X)) + E^1(\langle h'', X\rangle\, \Phi(X)) &= \frac{1}{4} \int_0^1 \mathrm{d}r\, h_r\, \frac{\mathrm{d}^2}{\mathrm{d}a^2}\, \Sigma_r^1(\Phi(X)\,|\,a) \Big|_{a=0}. \end{aligned} \tag{4.14}$$

*Proof.* By linearity, we may assume that $\Phi$ is of the form (4.1). For $\delta = 3$ we have by (4.10)

$$E^3(\partial_h \Phi(X)) + E^3(\langle h'', X\rangle\, \Phi(X)) = -\frac{1}{2^{\frac{3}{2}}\,\Gamma(\frac{3}{2})} \int_0^1 h_r \left( \psi_r \hat{\psi}_r \right)^{-\frac{3}{2}} \mathrm{d}r.$$

By (3.15) this equals

$$-\frac{1}{2}\int_0^1 \mathrm{d}r\, h_r\, \Sigma^3_r(\Phi(X)\,|\,0)$$

and the proof is complete. For $\delta = 1$, by (4.10), we have

$$E^1(\partial_h\Phi(X) + \langle h'', X\rangle\, \Phi(X)) = -\frac{1}{2\sqrt{2\pi}}\,\psi_1 \int_0^1 h_r \left(\psi_r\hat{\psi}_r\right)^{-\frac{3}{2}} \mathrm{d}r.$$

But by (3.15) we have, for all $r \in (0,1)$

$$\frac{\mathrm{d}^2}{\mathrm{d}a^2}\,\Sigma^1_r(\Phi(X)\,|\,a)\,\Big|_{a=0} = -\frac{C_r\, D_r^{\frac{1}{2}}}{2^{-\frac{1}{2}}\,\Gamma(\frac{1}{2})} = -\sqrt{\frac{2}{\pi}}\,\psi_1\left(\psi_r\hat{\psi}_r\right)^{-\frac{3}{2}}.$$

The claimed IbPF follows. □

**Remark 4.11.** In [42] for the reflecting Brownian motion, and then in [19] for the Reflecting Brownian bridge, a different formula was proved in the case $\delta = 1$. In our present notations, for $(\beta_r)_{r\in[0,1]}$ a Brownian bridge and $X := |\beta|$, the formula reads

$$\mathbb{E}(\partial_h\Phi(X)) + \mathbb{E}(\langle h'', X\rangle\, \Phi(X)) = \lim_{\epsilon\to 0} 2\,\mathbb{E}\left(\Phi(X)\int_0^1 h_r\left[\left(\dot{\beta}^\epsilon_r\right)^2 - c^\epsilon_r\right]\mathrm{d}L^0_r\right), \tag{4.15}$$

where $\Phi : H \to \mathbb{R}$ is any Lipschitz function, $h \in C^2_0(0,1)$, $L^0$ is the standard local time of $\beta$ at 0 and for some even smooth mollifier $\rho_\epsilon$ we set

$$\beta^\epsilon := \rho_\epsilon * \beta, \qquad c^\epsilon_r := \frac{\|\rho\|^2_{L^2(0,1)}}{\epsilon}.$$

The reason why (4.15) is strictly weaker than (4.14), is that the former depends explicitly on $\beta$, while the latter is written only in terms of $X$. This will become crucial when we compute the SPDE satisfied by $u$ for $\delta = 1$ in Theorem 5.7 below.

## 5. The dynamics via Dirichlet forms

In this section we exploit the IbPF obtained above to construct a weak version of the gradient dynamics associated with $P^1$, using the theory of Dirichlet forms. The reason for considering the particular value $\delta = 1$ is that we can exploit a representation of the Bessel bridge in terms of a Brownian bridge, for which the corresponding gradient dynamics is well-known and corresponds to a linear stochastic heat equation. This representation allows us to construct a quasi-regular Dirichlet form associated with $P^1$, a construction which does not follow from the IbPF (4.10) due to the distributional character of its last term. The IbPF (4.10) is then exploited to prove that the associated Markov process, at equilibrium, satisfies (1.15). The treatment of the particular value $\delta = 1$ is also motivated by potential applications to scaling limits of dynamical critical pinning models, see e.g. [37] and [8].

For the sake of our analysis, instead of working on the Banach space $C([0,1])$, it shall actually be more convenient to work on the Hilbert space $H := L^2(0,1)$ endowed with the $L^2$ inner product

$$\langle f, g\rangle = \int_0^1 f_r\, g_r \,\mathrm{d}r, \quad f, g \in H.$$

We shall denote by $\|\cdot\|$ the corresponding norm on $H$. Moreover we denote by $\mu$ the law of $\beta$ on $H$, where $\beta$ is a Brownian bridge from 0 to 0 over the interval $[0,1]$. We shall use the shorthand notation $L^2(\mu)$ for the space $L^2(H,\mu)$. We also consider the closed subset $K \subset H$ of nonnegative functions

$$K := \{z \in H,\ z \geq 0 \text{ a.e.}\}.$$

Note that $K$ is a Polish space. We further denote by $\nu$ the law, on $K$, of the 1-Bessel bridge from 0 to 0 on $[0,1]$ (so that $P^1$ is then the restriction of $\nu$ to $C([0,1])$). We shall use the shorthand $L^2(\nu)$ to denote the space $L^2(K,\nu)$. Denoting by $j : H \to K$ the absolute value map

$$j(z) := |z|, \quad z \in H, \tag{5.1}$$

we remark that the map $L^2(\nu) \ni \varphi \mapsto \varphi \circ j \in L^2(\mu)$ is an isometry.

5.1. **The one-dimensional random string.** Consider the Ornstein-Uhlenbeck semigroup $(\mathbf{Q}_t)_{t\geq 0}$ on $H$ defined, for all $F \in L^2(\mu)$ and $z \in H$, by

$$\mathbf{Q}_t F(z) := \mathbb{E}\left[F(v_t(z))\right], \quad t \geq 0,$$

where $(v_t(z))_{t\geq 0}$ is the solution to the stochastic heat equation on $[0,1]$ with initial condition $z$, and with homogeneous Dirichlet boundary conditions

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \xi & \\ v(0,x) = z(x), & x \in [0,1] \\ v(t,0) = v(t,1) = 0, & t > 0 \end{cases} \tag{5.2}$$

with $\xi$ a space-time white noise on $\mathbb{R}_+ \times [0,1]$. Recall that $v$ can be written explicitly in terms of the fundamental solution $(g_t(x,x'))_{t\geq 0,\, x,x'\in(0,1)}$ of the stochastic heat equation with homogeneous Dirichlet boundary conditions on $[0,1]$, which by definition is the unique solution to

$$\begin{cases} \frac{\partial g}{\partial t} = \frac{1}{2}\frac{\partial^2 g}{\partial x^2} & \\ g_0(x,x') = \delta_x(x') & \\ g_t(x,0) = g_t(x,1) = 0. & \end{cases}$$

Recall further that $g$ can be represented as follows:

$$\forall t > 0, \quad \forall x, x' \geq 0, \quad g_t(x,x') = \sum_{k=1}^{\infty} e^{-\frac{\lambda_k}{2}t} e_k(x) e_k(x'), \tag{5.3}$$

where $(e_k)_{k\geq 1}$ is the complete orthornormal system of $H$ given by

$$e_k(x) := \sqrt{2}\sin(k\pi x), \quad x \in [0,1], \quad k \geq 1$$

and $\lambda_k := k^2\pi^2$, $k \geq 1$. We can then represent $u$ as follows:

$$v(t,x) = z(t,x) + \int_0^t \int_0^1 g_{t-s}(x,x')\,\xi(\mathrm{d}s,\,\mathrm{d}x'),$$

where $z(t,x) := \int_0^1 g_t(x,x')z(x')\,\mathrm{d}x'$, and the double integral is a stochastic convolution. In particular, it follows from this fomula that $v$ is a Gaussian process. An important role will be played by its covariance function. Namely, for all $t \geq 0$ and $x, x' \in (0,1)$, we set

$$q_t(x,x') := \mathrm{Cov}(v(t,x), v(t,x')) = \int_0^t g_{2\tau}(x,x')\,\mathrm{d}\tau.$$

We also set

$$q_\infty(x,x') := \int_0^\infty g_{2\tau}(x,x')\,\mathrm{d}\tau = \mathbb{E}[\beta_x\beta_{x'}] = x\wedge x' - xx'.$$

For all $t \geq 0$, we set moreover

$$q^t(x,x') := q_\infty(x,x') - q_t(x,x') = \int_t^\infty g_{2\tau}(x,x')\,\mathrm{d}\tau.$$

When $x = x'$, we will use the shorthand notations $q_t(x), q_\infty(x)$ and $q^t(x)$ instead of $q_t(x,x), q_\infty(x,x)$ and $q^t(x,x)$ respectively. Finally, we denote by $(\Lambda, D(\Lambda))$ the Dirichlet form associated with $(\mathbf{Q}_t)_{t\geq 0}$ in $L^2(H,\mu)$, and which is given by

$$\Lambda(F,G) = \frac{1}{2}\int_H \langle \nabla F, \nabla G\rangle\,\mathrm{d}\mu, \quad F, G \in D(\Lambda) = W^{1,2}(\mu).$$

Here, for all $F \in W^{1,2}(\mu)$, $\nabla F : H \to H$ is the gradient of $F$, see [5]. The corresponding family of resolvents $(\mathbf{R}_\lambda)_{\lambda>0}$ is then given by

$$\mathbf{R}_\lambda F(z) = \int_0^\infty e^{-\lambda t}\mathbf{Q}_t F(z)\,\mathrm{d}t, \quad z \in H,\, \lambda > 0, \qquad F \in L^2(\mu).$$

5.2. **Dirichlet form.** In this section we construct the Dirichlet form associated with our equation (1.12), and the associated Markov process $(u_t)_{t\geq 0}$.

Let $\mathcal{FC}_b^\infty(H)$ denote the space of all functionals $F : H \to \mathbb{R}$ of the form

$$F(z) = \psi(\langle l_1, z\rangle, \ldots, \langle l_m, z\rangle), \quad z \in H, \tag{5.4}$$

with $m \in \mathbb{N}$, $\psi \in C_b^\infty(\mathbb{R}^m)$, and $l_1, \ldots, l_m \in \mathrm{Span}\{e_k, k \geq 1\}$. Recalling that $K := \{z \in H, z \geq 0\}$, we also define:

$$\mathcal{FC}_b^\infty(K) := \left\{F\big|_K,\ F \in \mathcal{FC}_b^\infty(H)\right\}.$$

Moreover, for $f \in \mathcal{FC}_b^\infty(K)$ of the form $f = F\big|_K$, with $F \in \mathcal{FC}_b^\infty(H)$, we define $\nabla f : K \to H$ by

$$\nabla f(z) = \nabla F(z), \quad z \in K,$$

where this definition does not depend on the choice of $F$. We denote by $\mathcal{E}$ the bilinear form defined on $\mathcal{FC}_b^\infty(K)$ by

$$\mathcal{E}(f,g) := \frac{1}{2}\int \langle \nabla f, \nabla g\rangle \,\mathrm{d}\nu, \qquad f,g \in \mathcal{FC}_b^\infty(K),$$

**Proposition 5.1.** *The form $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ is closable. Its closure $(\mathcal{E}, D(\mathcal{E}))$ is a local, quasi-regular Dirichlet form on $L^2(\nu)$. Moreover, for all $f \in D(\mathcal{E})$, $f\circ j \in D(\Lambda)$, and we have*

$$\forall f,g \in D(\mathcal{E}), \quad \mathcal{E}(f,g) = \Lambda(f\circ j, g\circ j) \tag{5.5}$$

The proof of Proposition 5.1 is postponed to Appendix A.

Let $(Q_t)_{t\geq 0}$ be the contraction semigroup on $L^2(K,\nu)$ associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, and let $(R_\lambda)_{\lambda>0}$ be the associated family of resolvents. Let also $\mathcal{B}_b(K)$ denote the set of Borel and bounded functions on $K$. As a consequence of Prop. 5.1, in virtue of Thm IV.3.5 and Thm V.1.5 in [27], we obtain the following result.

**Corollary 5.2.** *There exists a diffusion process $M = \{\Omega, \mathcal{F}, (u_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in K}\}$ properly associated to $(\mathcal{E}, D(\mathcal{E}))$, i.e. for all $\varphi \in L^2(\nu)\cap \mathcal{B}_b(K)$, and for all $t>0$, $E_x(\varphi(u_t))$, $x\in K$, defines an $\mathcal{E}$ quasi-continuous version of $Q_t\varphi$. Moreover, the process $M$ admits the following continuity property*

$$\mathbb{P}_x[t\mapsto u_t \text{ is continuous on } \mathbb{R}_+] = 1, \quad \text{for } \mathcal{E}-\text{q.e. } x\in K.$$

The rest of this section will be devoted to show that for $\mathcal{E}$-q.e. $x\in K$, under $\mathbb{P}_x$, $(u_t)_{t\geq 0}$ solves (1.12), or rather its weaker form (1.15).

In the sequel, we set $\Lambda_1 := \Lambda + (\cdot,\cdot)_{L^2(\mu)}$ and $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_{L^2(\nu)}$, which are inner products for the Hilbert spaces $D(\Lambda)$ and $D(\mathcal{E})$ respectively. We shall also write in an abusive way, for any $\Phi \in C^1(H)$

$$\mathcal{E}_1(\Phi,\Phi) := \int_K \Phi^2 \,\mathrm{d}\nu + \frac{1}{2}\int_K \|\nabla\Phi\|_H^2.$$

Since the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, by the transfer method stated in VI.2 of [27], we can apply several results of [13] in our setting.

An important technical point is the density of the space $\mathcal{S}$ introduced in Section 4 above in the Dirichlet space $D(\mathcal{E})$. To state this precisely, we consider $\mathscr{S}$ to be the vector space generated by functionals $F : H\to\mathbb{R}$ of the form

$$F(\zeta) = \exp(-\langle\theta, \zeta^2\rangle), \quad \zeta\in H,$$

for some $\theta : [0,1]\to\mathbb{R}_+$ Borel and bounded. Note that $\mathscr{S}$ may be seen as a subspace of the space $\mathcal{S}$ of Section 4 in the following sense: for any $F\in\mathscr{S}$, $F|_{C([0,1])}\in\mathcal{S}$. We also set:

$$\mathscr{S}_K := \{F\big|_K,\ F\in\mathscr{S}\}.$$

**Lemma 5.3.** *$\mathscr{S}_K$ is dense in $D(\mathcal{E})$.*

The proof of Lemma 5.3 is postponed to Appendix A.

5.3. **Convergence of one-potentials.** The key tool in showing that the Markov process constructed above defines a solution of (1.15) is the IBPF (4.10). The rule of thumb is that the last term in the IbPF yields the expression of the drift in the SPDE. Recall however that, for any fixed $h \in C_c^2(0,1)$, the last term in (4.10) is given:

$$\frac{1}{4}\int_0^1 \mathrm{d}r\, h_r \,\frac{\mathrm{d}^2}{\mathrm{d}a^2}\,\Sigma_r^1(\Phi(X)\,|\,a)\Big|_{a=0}, \quad \Phi \in \mathcal{S},$$

which defines a generalized functional in the sense of Schwartz, rather than a genuine measure, on $C([0,1])$. It is therefore not immediate to translate the IbPF in terms of the corresponding dynamics. The strategy we follow to handle this difficulty consists in approximating the above generalized functional by a sequence of measures admitting a smooth density w.r.t. the law of the reflecting Brownian bridge, and showing that the corresponding one-potentials converge in the Dirichlet space $D(\mathcal{E})$. This will imply that the associated additive functionals converge to the functional describing the drift in the SPDE.

More precisely, let $\rho$ be a smooth function supported on $[-1,1]$ such that

$$\rho \geq 0, \quad \int_{-1}^1 \rho = 1, \quad \rho(y) = \rho(-y), \quad y \in \mathbb{R}.$$

For all $\epsilon > 0$, let

$$\rho_\epsilon(y) := \frac{1}{\epsilon}\,\rho\left(\frac{y}{\epsilon}\right), \quad y \in \mathbb{R}. \tag{5.6}$$

Then, for all $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$, the right-hand side of the IbPF (4.14) can be rewritten as follows

$$\frac{1}{4}\int_0^1 h_r\,\frac{\mathrm{d}^2}{\mathrm{d}a^2}\,\Sigma_r^1(\Phi(X)\,|\,a)\Big|_{a=0}\mathrm{d}r = \frac{1}{2}\,\lim_{\epsilon\to 0}\mathbb{E}\left[\Phi(|\beta|)\int_0^1 h_r\,\rho_\epsilon''(\beta_r)\,\mathrm{d}r\right]. \tag{5.7}$$

Indeed, starting from the right-hand side, by conditioning on the value of $|\beta_r|$, and recalling that $|\beta| \overset{(d)}{=} \nu$, the equality follows at once.

We will now show that the convergence of measures (5.7) can be enhanced to a convergence in the Dirichlet space $D(\Lambda)$ of the associated one-potentials. We henceforth fix a function $h \in C_c^2(0,1)$. Then there exists $\delta \in (0,1)$ such that $h$ is supported in $[\delta, 1-\delta]$. For all $\epsilon > 0$, let $G_\epsilon : H \to \mathbb{R}$ be defined by

$$G_\epsilon(z) := \frac{1}{2}\int_0^1 h_r\,\rho_\epsilon''(z_r)\,\mathrm{d}r, \quad z \in H. \tag{5.8}$$

For all $t > 0$ and $z \in H$, we have

$$\mathbf{Q}_t G_\epsilon(z) = \int_0^1 \frac{h_r}{2\sqrt{2\pi q_t(r)}}\int_{\mathbb{R}} \rho_\epsilon''(a)\exp\left(-\frac{(a-z(t,r))^2}{2q_t(r)}\right)\mathrm{d}a\,\mathrm{d}r,$$

where, for all $x \in (0,1)$, $z(t,x) := \int_0^1 g_t(x,x')z(x')\,\mathrm{d}x'$. For all $\epsilon > 0$, we define the functional $U_\epsilon : H \to \mathbb{R}$ by

$$U_\epsilon(z) = \int_0^\infty e^{-t}\mathbf{Q}_t G_\epsilon(z)\,\mathrm{d}t, \quad z \in H.$$

Note that $U_\epsilon$ is the one-potential of the additive functional

$$\int_0^t G_\epsilon(v(s,\cdot))\,\mathrm{d}s, \qquad t \geq 0,$$

associated with the Markov process $(v(t,\cdot))_{t\geq 0}$ in $H$ defined in (5.2) (see Section 5 of [13] for this terminology). In particular, $U_\epsilon \in D(\Lambda)$.

**Proposition 5.4.** *As $\epsilon \downarrow 0$, $U_\epsilon \to U$ in $D(\Lambda)$, where for all $z \in H$*

$$U(z) := \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{h_r}{2\sqrt{2\pi q_t(r)^3}} \left[\frac{z(t,r)^2}{q_t(r)} - 1\right] \exp\left(-\frac{z(t,r)^2}{2q_t(r)}\right). \tag{5.9}$$

*Proof.* First note that $U$ is well-defined. Indeed, the function $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(y) = (y^2 - 1)\, e^{-y^2/2}, \quad y \in \mathbb{R}$$

is globally bounded on $\mathbb{R}$. Note also that, by the bound (22) in [41], there exists $c_\delta > 0$ such that

$$\forall t > 0, \quad \forall r \in [\delta, 1-\delta], \qquad q_t(r) \geq c_\delta \sqrt{t}. \tag{5.10}$$

Hence, the integrand in (5.9) is bounded by

$$e^{-t}\frac{|h_r|}{2\sqrt{2\pi q_t(r)^3}}\|\varphi\|_{L^\infty(\mathbb{R})} \leq e^{-t}t^{-3/4}\frac{|h_r|}{2\sqrt{2\pi}c_\delta^3}\|\varphi\|_{L^\infty(\mathbb{R})},$$

which is integrable on $\mathbb{R}_+ \times [0,1]$. Note now that $U \in L^2(\mu)$. Indeed, if $z$ is an $H$ valued random variable with law $\mu$, then, for all $r \in (0,1)$, $z(t,r) = \int_0^1 g_t(r,x)z(x)\,\mathrm{d}x$ is a centered Gaussian random variable with variance

$$\int_0^1 \int_0^1 g_t(r,x)g_t(r,y)q_\infty(x,y)\,\mathrm{d}x\,\mathrm{d}y = q^t(r).$$

Therefore

$$\begin{aligned}
\|U\|_{L^2(\mu)} &\leq \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{|h_r|}{2\sqrt{2\pi q_t(r)^3}} \left(\int_{\mathbb{R}} \varphi\left(x/\sqrt{q_t(r)}\right)^2 \mathcal{N}(0, q^t(r))(\mathrm{d}x)\right)^{\frac{1}{2}} \\
&\leq \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{|h_r|}{2\sqrt{2\pi q_t(r)^3}} \left\{\|\varphi\|_{L^\infty(\mathbb{R})} \wedge \left(\frac{q_t(r)}{2\pi q^t(r)}\right)^{\frac{1}{4}} \|\varphi\|_{L^2(\mathbb{R})}\right\} \\
&\leq \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}r\, \frac{|h_r|\|\varphi\|_{L^2(\mathbb{R})}}{2(2\pi)^{3/4} q_t(r)^{5/4} q^1(r)} + \int_1^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{|h_r|\|\varphi\|_{L^\infty(\mathbb{R})}}{2\sqrt{2\pi q_1(r)^3}}.
\end{aligned}$$

By (5.10), the first integral in the last line is convergent, and so is the second one, whence $U \in L^2(\mu)$ as claimed. Now, differentiating under the integrals in (5.9), we see that $U$ is differentiable on $H$, and that for all $z \in H$

$$\nabla U(z) = \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{h_r\, g_t(r,\cdot)}{2\sqrt{2\pi q_t(r)^4}}\, \psi\left(z(t,r)/\sqrt{q_t(r)}\right), \tag{5.11}$$

where

$$\psi(x) := (3x - x^3)\, e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Note that $\psi$ is globally bounded and square integrable on $\mathbb{R}$. In particular, the right-hand side of (5.11) is well-defined. Indeed, by (22) in [41], there exists $c > 0$ such that, for all $t > 0$ and $r \in (0,1)$

$$\frac{1}{q_t(r)} \le \frac{c}{r(1-r)}. \tag{5.12}$$

Since, moreover, for all $r, s \in (0,1)$

$$\int_0^\infty g_t(r,s)\, \mathrm{d}t \le \sum_{k=1}^\infty \frac{1}{\lambda_k} < \infty,$$

the integrand in (5.11) is indeed integrable on $\mathbb{R}_+ \times (0,1)$. We now show that $\nabla U \in L^2(H, \mu\,; H)$. Noting that, by (27) in [41],

$$\|g_t(r,\cdot)\| \le \frac{1}{t^{\frac{1}{4}}}, \quad t > 0,$$

we have

$$\|\nabla U\|_{L^2} \le \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{|h_r|\, t^{-1/4}}{2\sqrt{2\pi q_t(r)^4}} \left( \int_{\mathbb{R}} \psi\left(x/\sqrt{q_t(r)}\right)^2 \mathcal{N}(0, q^t(r))(\mathrm{d}x) \right)^{\frac{1}{2}},$$

where we use the shorthand notation $\|\cdot\|_{L^2}$ for $\|\cdot\|_{L^2(H,\mu\,;H)}$. Hence

$$\begin{aligned}
\|\nabla U\|_{L^2} &\le \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{|h_r|\, t^{-1/4}}{2\sqrt{2\pi q_t(r)^4}} \left\{ \|\psi\|_{L^\infty(\mathbb{R})} \wedge \left( \frac{q_t(r)}{2\pi q^t(r)} \right)^{\frac{1}{4}} \|\psi\|_{L^2(\mathbb{R})} \right\} \\
&\le \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}r\, \frac{|h_r|\, \|\psi\|_{L^2(\mathbb{R})}\, t^{-1/4}}{2(2\pi)^{3/4} (q^1(r))^{\frac{1}{4}} q_t(r)^{7/4}} + \int_1^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{|h_r|\, \|\psi\|_{L^\infty(\mathbb{R})}}{2\sqrt{2\pi q_1(r)^4}}.
\end{aligned}$$

Since, for all $t > 0$ and $r \in (0,1)$

$$\frac{t^{-1/4}}{q_t(r)^{7/4}} \le \left( \frac{c}{r(1-r)} \right)^{7/4} t^{-1/4},$$

and since $h$ is compactly supported in $(0,1)$, the first integral in the last line above is convergent. Since the second integral is also convergent, we deduce that $\nabla U \in L^2(H, \mu\,; H)$ as claimed. We have thus proved that $U \in W^{1,2}(H, \mu) = D(\Lambda)$.

We show now that $U_\epsilon$ converges in $D(\Lambda)$ as $\epsilon \to 0$. To do so, remark that, for all $z \in H$, we have

$$U_\epsilon(z) = \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{h_r}{2\sqrt{2\pi q_t(r)}} \int_{\mathbb{R}} \rho_\epsilon''(a) \exp\left(-\frac{(a - z(t,r))^2}{2q_t(r)}\right) \mathrm{d}a,$$

which, upon integrating by parts twice in the last integral, we can rewrite

$$U_\epsilon(z) = \int_0^\infty \int_0^1 h_r \zeta(\mathrm{d}t, \mathrm{d}r) \int_{\mathbb{R}} \rho_\epsilon(a) \left[\frac{(a - z(t,r))^2}{q_t(r)} - 1\right] \exp\left(-\frac{(a - z(t,r))^2}{2q_t(r)}\right) \mathrm{d}a,$$

where we have introduced the compact notation

$$\zeta(\mathrm{d}t, \mathrm{d}r) := \frac{e^{-t}}{2\sqrt{2\pi q_t(r)^3}}\, \mathrm{d}t\, \mathrm{d}r.$$

Hence, for all $z \in H$, $U_\epsilon(z) - U(z)$ is equal to

$$\int_0^\infty \int_0^1 h_r\, \zeta(\mathrm{d}t, \mathrm{d}r) \int_{\mathbb{R}} \mathrm{d}x\, \rho(x) \left[\varphi\left(\frac{z(t,r) - \epsilon x}{\sqrt{q_t(r)}}\right) - \varphi\left(\frac{z(t,r)}{\sqrt{q_t(r)}}\right)\right].$$

Hence, reasoning as in the previous computations,

$$\begin{aligned}\|U_\epsilon - U\|_{L^2(\mu)} \leq \int_0^\infty \int_0^1 |h_r|\, \zeta(\mathrm{d}t, \mathrm{d}r) \int_{\mathbb{R}} \mathrm{d}x\, \rho(x) \cdot \\ \cdot \left\{\sqrt{2}\|\varphi\|_{L^\infty(\mathbb{R})} \wedge \left(\frac{q_t(r)}{2\pi q^t(r)}\right)^{\frac{1}{4}} \left|\left|\varphi\left(\cdot - \epsilon x/\sqrt{q_t(r)}\right) - \varphi\right|\right|_{L^2(\mathbb{R})}\right\}.\end{aligned}$$

By the Dominated Convergence Theorem, for all $t > 0$, $r \in (0,1)$ and $x \in \mathbb{R}$

$$\left|\left|\varphi\left(\cdot - \epsilon x/\sqrt{q_t(r)}\right) - \varphi\right|\right|_{L^2(\mathbb{R})} \underset{\epsilon \to 0}{\longrightarrow} 0.$$

Hence, again by the Dominated Convergence Theorem, we deduce that $\|U_\epsilon - U\|_{L^2(\mu)} \underset{\epsilon \to 0}{\longrightarrow} 0$. Looking now at the level of the gradients, we have

$$\begin{aligned}\|\nabla U_\epsilon - \nabla U\|_{L^2} \leq \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \mathrm{d}r\, \frac{|h_r|\, t^{-1/4}}{2\sqrt{2\pi q_t(r)^4}} \cdot \\ \cdot \left\{\|\psi\|_{L^\infty(\mathbb{R})} \wedge \left(\frac{q_t(r)}{2\pi q^t(r)}\right)^{\frac{1}{4}} \|\psi(\cdot - \epsilon x/\sqrt{q_t(r)}) - \psi\|_{L^2(\mathbb{R})}\right\},\end{aligned}$$

so we obtain, with similar arguments, the convergence $\|\nabla U_\epsilon - \nabla U\|_{L^2} \underset{\epsilon \to 0}{\longrightarrow} 0$. $\square$

5.4. **A projection principle.** Note that in the above section we worked in the Gaussian Dirichlet space $D(\Lambda)$. For our dynamical problem, we shall however need to transfer the above results to the Dirichlet space $D(\mathcal{E})$. To do so, we invoke the following projection principle, which was first used in [41] for the case of a 3-Bessel bridge (see Lemma 2.2 therein).

**Lemma 5.5.** *There exists a unique bounded linear operator* $\Pi : D(\Lambda) \to D(\mathcal{E})$ *such that, for all* $F, G \in D(\Lambda)$ *and* $f \in D(\mathcal{E})$

$$\Lambda_1(F, f \circ j) = \mathcal{E}_1(\Pi F, f),$$

*where* $j$ *is as in* (5.1)*. Moreover, we have*

$$\mathcal{E}_1(\Pi F, \Pi F) \leq \Lambda_1(F, F).$$

*Proof.* We use the same arguments as in the proof of Lemma 2 in [41]. Let $\mathcal{D} := \{\varphi \circ j, \quad \varphi \in D(\mathcal{E})\}$. By Proposition 5.1, $\mathcal{D}$ is a linear subspace of $D(\Lambda)$ which is isometric to $D(\mathcal{E})$. In particular, it is a closed subspace of the Hilbert space $D(\Lambda)$. Hence, we may consider the orthogonal projection operator $\hat{\Pi}$ onto $\mathcal{D}$. Then, for all $F \in D(\Lambda)$, let $\Pi F$ be the unique element of $D(\mathcal{E})$ such that $\hat{\Pi} F = (\Pi F) \circ j$. It then follows that $\Pi$ possesses the required properties. □

We obtain the following refinement of the IbPF (4.4) for $P^1$.

**Corollary 5.6.** *Let* $U$ *be as in* (5.9)*. For all* $f \in D(\mathcal{E})$ *and* $h \in C_c^2(0,1)$*, we have*

$$\mathcal{E}\left(\langle h, \cdot\rangle - \frac{1}{2}\Pi U\,,\, f\right) = -\frac{1}{2}\int_K \left(\langle h'', \zeta\rangle - \Pi U(\zeta)\right) f(\zeta)\,\mathrm{d}\nu(\zeta). \tag{5.13}$$

*Proof.* By the density of $\mathscr{S}_K$ in $D(\mathcal{E})$ proved in Lemma 5.3, it is enough to consider $f \in \mathscr{S}_K$. By (5.7)

$$\begin{aligned}\frac{1}{4}\int_0^1 \mathrm{d}r\, h_r \frac{\mathrm{d}^2}{\mathrm{d}a^2}\Sigma_r^1\left(f(X) \mid a\right)\Big|_{a=0} &= \frac{1}{2}\lim_{\epsilon\to 0}\mathbb{E}\left[f(|\beta|)\int_0^1 h_r\,\rho_\epsilon''(\beta_r)\,\mathrm{d}r\right]\\ = \lim_{\epsilon\to 0}\int (f\circ j)\,G_\epsilon\,\mathrm{d}\mu &= \lim_{\epsilon\to 0}\Lambda_1(f\circ j,\, U_\epsilon) = \Lambda_1(f\circ j,\, U) = \mathcal{E}_1(f,\, \Pi U).\end{aligned}$$

Therefore, for all $f \in \mathscr{S}_K$, the IbPF (4.14) can be rewritten

$$2\mathcal{E}(\langle h, \cdot\rangle, f) = -\int_K \langle h'', \zeta\rangle\, f(\zeta)\,\mathrm{d}\nu(\zeta) + \mathcal{E}_1(f,\, \Pi U),$$

that is

$$\mathcal{E}\left(\langle h, \cdot\rangle - \frac{1}{2}\Pi U, f\right) = -\frac{1}{2}\int_K \left(\langle h'', \zeta\rangle - \Pi U(\zeta)\right) f(\zeta)\,\mathrm{d}\nu(\zeta).$$

The proof is complete. □

Recall that $M = (\Omega, \mathcal{F}, (u_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in K})$ denotes the Markov process properly associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ constructed above. Note that, by Theorem 5.2.2 in [13], for all $F \in D(\mathcal{E})$, we can write in a unique way

$$F(u_t) - F(u_0) = M_t^{[F]} + N_t^{[F]}, \quad t \geq 0, \tag{5.14}$$

$\mathbb{P}_\nu$ a.s., where $M^{[F]}$ is a martingale additive functional, and $N^{[F]}$ is an additive functional of zero energy. Using this fact we can thus write $u$ as the weak solution to some SPDE, but with coefficients that are not explicit. However the formula (5.13) above will allow us to identify these coefficients.

We can now finally state the result justifying that that the Markov process constructed above satisfies the SPDE (1.15) above.

**Theorem 5.7.** *For all $h \in C_c^2(0,1)$, we have*

$$\langle u_t, h\rangle - \langle u_0, h\rangle = M_t + N_t, \qquad \mathbb{P}_{u_0} - a.s., \quad q.e.\ u_0 \in K.$$

*Here $(N_t)_{t\geq 0}$ is a continuous additive functional of zero energy satisfying*

$$N_t - \frac{1}{2}\int_0^t \langle h'', u_s\rangle \,\mathrm{d}s = \lim_{\epsilon\to 0} N_t^\epsilon, \qquad N_t^\epsilon := -\frac{1}{4}\int_0^t \langle \rho_\epsilon''(u_s), h\rangle \,\mathrm{d}s,$$

*in $\mathbb{P}_\nu$-probability, uniformly in $t$ on finite intervals. Moreover, $(M_t)_{t\geq 0}$ is a martingale additive functional whose sharp bracket has the Revuz measure $\|h\|_H^2\,\nu$. Finally we also have*

$$N_t - \frac{1}{2}\int_0^t \langle h'', u_s\rangle \,\mathrm{d}s = \lim_{k\to\infty} N_t^{\epsilon_k}$$

*along a subsequence $\epsilon_k \to 0$ in $\mathbb{P}_{u_0}$-probability, for q.e. $u_0 \in K$.*

*Proof.* On the one hand, by (5.13), we can write

$$\langle u_t, h\rangle - \frac{1}{2}\Pi U(u_t) - \left(\langle u_0, h\rangle - \frac{1}{2}\Pi U(u_0)\right) = N_t^{(1)} + M_t^{(1)}, \tag{5.15}$$

where $N^{(1)}$ is the continuous additive functional of zero energy given by

$$N_t^{(1)} = \frac{1}{2}\int_0^t \left(\langle h'', u_s\rangle - \Pi U(u_s)\right) \mathrm{d}s, \quad t \geq 0$$

and $M^{(1)}$ is defined by (5.15). On the other hand, for all $\epsilon > 0$, by definition of $U_\epsilon$, we have for $G_\epsilon$ as in (5.8)

$$\Lambda_1(U_\epsilon, \Phi) = \int_H G_\epsilon\, \Phi \,\mathrm{d}\mu, \quad \Phi \in D(\Lambda).$$

Hence, remarking that $G_\epsilon = g_\epsilon \circ j$, where $g_\epsilon : K \to \mathbb{R}$ is the functional defined by

$$g_\epsilon(z) := \frac{1}{2}\int_0^1 h_r\, \rho_\epsilon''(z_r)\,\mathrm{d}r = \frac{1}{2}\langle \rho_\epsilon''(z), h\rangle,$$

by Lemma 5.5, we obtain for all $f \in D(\mathcal{E})$

$$\mathcal{E}_1(\Pi U_\epsilon, f) = \int_K f(z)\, g_\epsilon(z)\, \mathrm{d}\nu(z) = -\int_K f(z)(\Pi U_\epsilon(z) - g_\epsilon(z))\, \mathrm{d}\nu(z). \tag{5.16}$$

As a consequence, we have the decomposition

$$\frac{1}{2}\Pi U_\epsilon(u_t) - \frac{1}{2}\Pi U_\epsilon(u_0) = N_t^{(2,\epsilon)} + M_t^{(2,\epsilon)}, \tag{5.17}$$

where $N^{(2,\epsilon)}$ is the continuous additive functional of zero energy given by

$$N_t^{(2,\epsilon)} = \frac{1}{2}\int_0^t (\Pi U_\epsilon(u_s) - g_\epsilon(u_s))\ \mathrm{d}s, \quad t \geq 0$$

and $M^{(2,\epsilon)}$ is defined by (5.17). Since $U_\epsilon \underset{\epsilon\to 0}{\longrightarrow} U$ in $D(\Lambda)$ by Proposition 5.4, by the continuity of $\Pi : D(\Lambda) \to D(\mathcal{E})$, we have the convergence $\Pi U_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \Pi U$ in $D(\mathcal{E})$. Therefore, setting

$$M_t^{(2)} = M_t^{[\Pi U]}, \qquad N_t^{(2)} := N_t^{[\Pi U]},$$

then, by (5.1.1), (5.2.22) and (5.2.25) in [13], we have

$$\Pi U_\epsilon(u_t) - \Pi U_\epsilon(u_0) \underset{\epsilon\to 0}{\longrightarrow} \Pi U(u_t) - \Pi U(u_0), \quad M_t^{(2,\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} M_t^{(2)}, \quad N_t^{(2,\epsilon)} \underset{\epsilon\to\infty}{\longrightarrow} N_t^{(2)}$$

in $\mathbb{P}_\nu$-probability, for the topology of uniform convergence on finite intervals of $t \in \mathbb{R}_+$. Adding equality (5.17) to (5.15) yields

$$\langle u_t, h\rangle - \langle u_0, h\rangle = M_t + N_t,$$

with $M_t = M_t^1 + M_t^2$ and

$$\begin{aligned} N_t = N_t^1 + N_t^2 &= \frac{1}{2}\int_0^t (\langle h'', u_s\rangle - \Pi U(u_s))\ \mathrm{d}s + \lim_{\epsilon\to 0}\frac{1}{2}\int_0^t (\Pi U_\epsilon(u_s) - g_\epsilon(u_s))\ \mathrm{d}s \\ &= \frac{1}{2}\int_0^t \langle h'', u_s\rangle\, \mathrm{d}s - \lim_{\epsilon\to 0}\frac{1}{2}\int_0^t g_\epsilon(u_s)\, \mathrm{d}s, \end{aligned}$$

Moreover, note that $M = M^{[F_h]}$, where $F_h \in D(\mathcal{E})$ is given by

$$F_h(z) := \langle z, h\rangle, \quad z \in K.$$

Hence, by Theorem 5.2.3 in [13], $\mu_{<M>}$ is given by $\|h\|^2_{L^2(0,1)} \cdot \nu$. For the last statement, we apply [13, Corollary 5.2.1]. □

5.5. **A distinction result.** As a consequence of our IbPFs and the above constructions, we can prove that the Markov process $(u_t)_{t\geq 0}$ constructed above is not identically equal in law to the process corresponding to the modulus of the solution $(v_t)_{t\geq 0}$ to the stochastic heat equation, as it might otherwise have been guessed by the analogous relation between the invariant measures $\mu$ and $\nu$.

Let $K^{\mathbb{R}_+}$ denote the space of functions from $\mathbb{R}_+$ to $K$, endowed with the product $\sigma$-algebra. For all $x \in K$, let $P_x$ be the law, on $K^{\mathbb{R}_+}$, of the Markox process $(u_t)_{t\geq 0}$ associated with $\mathcal{E}$, started from $x$. Similarly, for all $z \in H$, let $\mathbf{P}_z$ be the law, on

$K^{\mathbb{R}_+}$, of $(|v_t|)_{t\geq 0}$, where $(v_t)_{t\geq 0}$ is the solution of the stochastic heat equation (5.2), with $v_0 = z$.

**Theorem 5.8.**

$$\mu\left(\{z \in H : P_{|z|} \neq \mathbf{P}_z\}\right) > 0.$$

*Proof.* Assume by contradiction that $P_{|z|} = \mathbf{P}_z$ for $\mu$-a.e. $z \in H$. Then, recalling that $(\mathbf{Q}_t)_{t\geq 0}$ denotes the semigroup associated with $\Lambda$, and $(Q_t)_{t\geq 0}$ the semigroup associated with $\mathcal{E}$, we would have

$$\mathbf{Q}_t(f \circ j) = (Q_t f) \circ j, \quad \mu - \text{a.e.},$$

for all $t \geq 0$ and $f \in L^2(\nu)$. Therefore, the corresponding families of resolvents $(\mathbf{R}_\lambda)_{\lambda>0}$ and $(R_\lambda)_{\lambda>0}$ would satisfy, for all $f \in L^2(\nu)$

$$\mathbf{R}_1(f \circ j) = (R_1 f) \circ j,$$

where the equality holds in $L^2(\mu)$. In particular, this shows that $(R_1 f) \circ j \in D(\Lambda)$ for any $f$ as above. We then claim that, for all $F \in D(\Lambda)$, $\Pi F = \mathbb{E}[F(\beta)\,|\,|\beta|\,]$ $\mu$-a.e. Indeed, by the previous observations, for all $f \in L^2(\nu)$, it holds

$$\begin{aligned}\int_H (f \circ j)(z)F(z)\,\mathrm{d}\mu(z) &= \Lambda_1(\mathbf{R}_1(f \circ j), F) = \Lambda_1((R_1 f) \circ j, F)\\ &= \mathcal{E}_1(R_1 f, \Pi F) = \int_K f(x)(\Pi F)(x)\,\mathrm{d}\nu(x),\end{aligned} \tag{5.18}$$

i.e. $\Pi F = \mathbb{E}[F(\beta)\,|\,|\beta|\,]$ $\mu$-a.e., as claimed. By (5.18) and the first equality in Lemma 5.5, we deduce that, for all $f \in D(\mathcal{E})$ and $F \in D(\Lambda)$

$$\Lambda(F, f \circ j) = \mathcal{E}(\Pi F, f).$$

Consider now the process $(v_t)_{t\geq 0}$ associated with $\Lambda$ and started from $v_0 = \beta$, where $\beta$ is a Brownian bridge on $[0, 1]$. Consider also the process $(u_t)_{t\geq 0}$ associated with $\mathcal{E}$ under the law $\mathbb{P}_\nu$ (so that, in particular, $u_0 \overset{(d)}{=} |\beta|$). Thus the processes $v$ and $u$ are stationary, and $|v| \overset{(d)}{=} u$ by our assumption. Let us set

$$A_t := \langle |v_t|, h\rangle - \langle |v_0|, h\rangle - \frac{1}{2}\int_0^t \langle |v_s|, h''\rangle\,\mathrm{d}s,$$

$$C_t := \langle u_t, h\rangle - \langle u_0, h\rangle - \frac{1}{2}\int_0^t \langle u_s, h''\rangle\,\mathrm{d}s.$$

Let further $k \in C^2([0, 1])$ with $k(0) = k(1) = 0$, and consider the functionals $\Psi_k : H \to \mathbb{R}$ and $\tilde{\Psi}_k : K \to \mathbb{R}$ given by

$$\Psi_k(z) := \exp(\langle k, z\rangle), \qquad \tilde{\Psi}_k(y) := \mathbb{E}\left[\Psi_k(\beta)\,|\,|\beta| = y\,\right], \qquad y \in K.$$

Note that $\Psi_k \in D(\Lambda)$, and recall that, by the above remarks, $\tilde{\Psi}_k = \Pi\Psi_k$ $\mu$-a.e., so in particular $\tilde{\Psi}_k \in D(\mathcal{E})$. We then have

$$\begin{aligned}
J(t) &:= -\frac{d}{dt}\mathbb{E}\left[A_t\,\tilde{\Psi}_k(|v_0|)\right] = \\
&= -\frac{d}{dt}\mathbb{E}\left[\left(\langle u_t, h\rangle - \langle u_0, h\rangle\right)\tilde{\Psi}_k(u_0)\right] + \frac{1}{2}\frac{d}{dt}\mathbb{E}\left[\int_0^t \langle h'', |v_s|\rangle\,\mathrm{d}s\,\Psi_k(\beta)\right] \\
&= \mathcal{E}(\langle\cdot, h\rangle\,,\,\tilde{\Psi}_k) + \frac{1}{2}\mathbb{E}[\langle h'', |\beta|\rangle\Psi_k(\beta)] = \Lambda(\langle|\cdot|, h\rangle\,,\,\Psi_k) + \frac{1}{2}\mathbb{E}[\langle h'', |\beta|\rangle\Psi_k(\beta)] \\
&= \frac{1}{2}\mathbb{E}[\langle\nabla\Psi_k(\beta), \mathrm{sign}(\beta)\,h\rangle + \langle h'', |\beta|\rangle\Psi_k(\beta)] = \mathbb{E}\left[\Psi_k(\beta)\int_0^1 h\,:\dot{\beta}^2:\,\mathrm{d}L^0\right]
\end{aligned}$$

by (3.10) in [Zam05], or rather its analogue for the Brownian bridge as stated in Remark 1.3 of [19]. But, by [42, Corollary 3.4] and [19, Theorem 3.2], the last quantity equals

$$\sqrt{\frac{1}{2\pi}}\,e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{r(1-r)}}\exp\left(-\frac{K_r^2}{2r(1-r)}\right)\lambda(K_r', -K_r, r)\,\mathrm{d}r,$$

where $K = Qk$, with $Q$ the covariance operator of $\beta$,

$$(Qk)_r = \int_0^1 (r\wedge\sigma - r\sigma)\,k_\sigma\,\mathrm{d}\sigma, \qquad r\in[0,1],$$

and $\lambda : \mathbb{R}^2\times[0,1]\to\mathbb{R}$ is defined by

$$\lambda(x,y,r) := x^2 + xy\frac{1-2r}{r(1-r)} + y^2\frac{(1-2r)^2}{4r^2(1-r)^2} - \frac{1}{4r(1-r)}, \quad x,y\in\mathbb{R},\ r\in[0,1].$$

Hence,

$$J(t) = \sqrt{\frac{2}{\pi}}e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{r(1-r)}}\exp\left(-\frac{K_r^2}{2r(1-r)}\right)\lambda(K_r', -K_r, r)\,\mathrm{d}r. \quad (5.19)$$

On the other hand

$$\begin{aligned}
L(t) &:= -\frac{d}{dt}\mathbb{E}\left[C_t\,\tilde{\Psi}_k(|v_0|)\right]\Big|_{t=0} = \mathcal{E}(\Pi\Psi_k, \langle\cdot, h\rangle) + \frac{1}{2}\mathbb{E}[\langle h'', |\beta|\rangle\,\Pi\Psi_k(|\beta|)] \\
&= \frac{1}{2}\mathcal{E}(\Pi U, \Pi\Psi_k) = \frac{1}{4}\lim_{\epsilon\to0}\mathbb{E}\left[\int_0^1 h_r\,\rho_\epsilon''(|\beta_r|)\,\mathrm{d}r\,\Pi\Psi_k(|\beta|)\right],
\end{aligned}$$

where we used (5.13) to obtain the second equality, and the fact that $U = \lim_{\epsilon\to0} U_\epsilon$ in $D(\mathcal{E})$, combined with (5.16), to obtain the third one. Therefore, recalling that $\Pi\Psi_k = \mathbb{E}(\Psi_k\,|\,|\beta|)$ $\mu$-a.e., we have

$$L(t) = \frac{1}{4}\lim_{\epsilon\to0}\mathbb{E}\left[\int_0^1 h_r\,\rho_\epsilon''(|\beta_r|)\,\mathrm{d}r\,\Psi_k(\beta)\right] = \frac{1}{4}\lim_{\epsilon\to0}\mathbb{E}\left[\int_0^1 h_r\,\rho_\epsilon''(\beta_r)\,\mathrm{d}r\,e^{\langle k,\beta\rangle}\right].$$

By the Cameron-Martin formula, for all $\epsilon > 0$

$$\begin{aligned}
&\frac{1}{4}\,\mathbb{E}\left[\int_0^1 h_r\,\rho_\epsilon''(\beta_r)\,\mathrm{d}r\,e^{\langle k,\beta\rangle}\right] = \\
&= \frac{1}{4}e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}}\int_{\mathbb{R}}\rho_\epsilon''(a)\exp\left(-\frac{(a-K_r)^2}{2r(1-r)}\right)\mathrm{d}a\,\mathrm{d}r \\
&\underset{\epsilon\to 0}{\to} \frac{1}{4}e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}}\left[\frac{K_r^2-r(1-r)}{r^2(1-r)^2}\right]\exp\left(-\frac{K_r^2}{2r(1-r)}\right)\mathrm{d}r.
\end{aligned}$$

Hence we obtain

$$L(t) = \frac{1}{4}e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}}\left[\frac{K_r^2-r(1-r)}{r^2(1-r)^2}\right]\exp\left(-\frac{K_r^2}{2r(1-r)}\right)\mathrm{d}r. \quad (5.20)$$

Since $|v|$ and $u$ have the same law, $J(t) = L(t)$ and therefore the right-hand sides of (5.19) and (5.20) above are equal. This being true for any $h \in C_c^2(0,1)$, we deduce that

$$\frac{K_r^2-r(1-r)}{4r^2(1-r)^2} = \lambda(K_r', -K_r, r),$$

for a.e. $r \in (0,1)$, hence for all $r$ by continuity. We thus deduce that

$$(K_r')^2 - \frac{1-2r}{r(1-r)}K_rK_r' - \frac{1}{r(1-r)}K_r^2 = 0, \qquad \forall\, r \in (0,1).$$

Since we can choose $k \in C_c^2(0,1)$ such that $K = Qk$ does not satisfy the above equation, we obtain a contradiction. □

## 6. Conjectures and open problems

Theorems 4.9 and 4.10 above enable us to conjecture the structure of the Bessel SPDEs for $\delta < 3$. The idea is that the right-hand side of the IbPFs (4.2) (respectively (4.4)) corresponds to the logarithmic derivative of the measure $P^\delta$ for $\delta \in (0,3)\setminus\{1\}$ (resp. $\delta = 1$), which should yield the drift in the SPDEs we are looking for. More precisely, considering for instance the case $\delta \in (1,3)$, for all $\Phi \in \mathcal{S}$, we may rewrite the last term in the IbPF (4.2) as follows

$$\begin{aligned}
&-\kappa(\delta)\int_0^1 h_r\int_0^\infty a^{\delta-4}\left(\Sigma_r^\delta\left(\Phi(X)\,|\,a\right) - \Sigma_r^\delta\left(\Phi(X)\,|\,0\right)\right)\mathrm{d}a\;\mathrm{d}r = \\
&= -\kappa(\delta)\lim_{\epsilon\to 0}\lim_{\eta\to 0}\mathbb{E}\left[\Phi(X)\int_0^1 h_r\left(\frac{\mathbf{1}_{X_r\geq\epsilon}}{X_r^3} - 2\frac{\epsilon^{\delta-3}}{3-\delta}\frac{\rho_\eta(X_r)}{X_r^{\delta-1}}\right)\mathrm{d}r\right],
\end{aligned} \quad (6.1)$$

where the mollifying functions $\rho_\eta, \eta > 0$ are as in (5.6). As a consequence of this equality, we may write formally the gradient dynamics corresponding to $P^\delta$, $\delta \in (1,3)$, as follows

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \xi + \frac{\kappa(\delta)}{2}\lim_{\epsilon\to 0}\lim_{\eta\to 0}\left(\frac{\mathbf{1}_{u\geq\epsilon}}{u^3} - 2\frac{\epsilon^{\delta-3}}{3-\delta}\frac{\rho_\eta(u)}{u^{\delta-1}}\right),$$

where $\xi$ denotes space-time white noise on $\mathbb{R}_+ \times (0,1)$. Assuming now the existence of a local time process $(\ell^a_{t,x})_{x\in(0,1),t,a\geq 0}$ satisfying the occupation times formula

$$\forall f : \mathbb{R}_+ \to \mathbb{R}_+, \quad \int_0^t f\left(u(s,x)\right) ds = \int_0^{+\infty} f(a)\, \ell^a_t(x)\, a^{\delta-1}\, \mathrm{d}a, \tag{6.2}$$

and possessing sufficient regularity at $a = 0$, we could in turn write

$$\lim_{\epsilon\to 0}\lim_{\eta\to 0}\left(\frac{\mathbf{1}_{u\geq\epsilon}}{u^3} - 2\frac{\epsilon^{\delta-3}}{3-\delta}\frac{\rho_\eta(u)}{u^{\delta-1}}\right) = \int_0^{+\infty} a^{\delta-4}(\ell^a_{t,x} - \ell^0_{t,x})\,\mathrm{d}a,$$

so the SPDE could be written:

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \xi + \frac{\kappa(\delta)}{2}\frac{\partial}{\partial t}\int_0^{+\infty} a^{\delta-4}(\ell^a_{t,x} - \ell^0_{t,x})\,\mathrm{d}a.$$

The same reasoning can be done for $\delta \in (0,1)$, yielding for that case

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \xi + \frac{\kappa(\delta)}{2}\frac{\partial}{\partial t}\int_0^{+\infty} a^{\delta-4}\,\mathcal{T}_a{}^2\ell^{(\cdot)}_{t,x}\,\mathrm{d}a.$$

As for the critical case $\delta = 1$, as shown in Section 5, the dynamics is formally given by (1.15), which we can rewrite using the local times as follows:

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \xi - \frac{1}{8}\frac{\partial}{\partial t}\frac{\mathrm{d}^2}{\mathrm{d}a^2}\ell^a_t(x)\Big|_{b=0}. \tag{6.3}$$

In all the SPDEs above, the unknown would be the couple $(u,\ell)$, where $u$ is a continuous nonnegative function on $\mathbb{R}_+ \times (0,1)$, and, for all $x \in (0,1)$, $(\ell^a_t(x))_{a,t\geq 0}$ is a family of occupation times satisfying (6.2).

These conjectures raise several problems. Indeed, assuming that the process $u$ can be constructed - as done above for the case $\delta = 1$ - it is at present unknown whether a family of occupation times $\ell$ satisfying (6.2) should exist and, if it does, whether it has the requested differentiability property. Moreover, pathwise uniqueness for such equations is at present an open problem. For instance, due to the lack of monotonicity, the techniques used in [31] to define a solution to the stochastic heat equation with reflection would not be of any help. We stress that the analogous SDE case of Bessel processes of dimension $\delta \in (0,1)$ is also a problem of interest in itself; these processes are not semi-martingales, but nonetheless satisfy the stochastic equation

$$X_t = x + \frac{\delta-1}{2}\int_0^{+\infty} a^{\delta-2}(\ell^a_t - \ell^0_t)\,\mathrm{d}a + B_t,$$

where $(\ell^a_t)_{a,t\geq 0}$ is the diffusion local times process of the Bessel process $(X_t)_{t\geq 0}$ (see [33], Chapter XI, ex. 1.26). Even in this one-dimensional context, the only known method for solving this equation is to consider $Y_t := X_t^2$ and show pathwise uniqueness for $Y$; this method breaks down for SPDEs since the Itô formula produces very complicated terms, see the discussion in the Introduction.

The Dirichlet form techniques used in Section 5 above to construct $u$ in the case $\delta = 1$ can also be applied successfully to treat the case $\delta = 2$, see the forthcoming paper [9]. However, for $\delta \in ]0, 3[ \setminus \{1, 2\}$, it is not even known whether the form which naturally generalizes $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ in Proposition 5.1 is closable and whether its closure is a quasi-regular Dirichlet form .

We recall the main result of [6]: for all $\delta \geq 3$, we set

$$\zeta(\delta) := \sup\{k \geq \mathbb{N} : \exists t > 0,\, 0 < x_1 < \ldots < x_k < 1,\, u(t, x_i) = 0 \quad i = 1, \ldots, k\},$$

where $u$ is the solution to the $\delta$-Bessel SPDE (1.6)-(1.8). Then we have

$$\mathbb{P}\left(\zeta(\delta) > \frac{k}{2-\delta}\right) = 0. \tag{6.4}$$

In other words, a.s. $u$ hits the obstacle 0 in at most $\lceil \frac{k}{2-\delta} \rceil$ space points simultaneously in time. It is very tempting to conjecture that (6.4) holds for all $\delta > 2$ in other words, the $\delta$-Bessel SPDE would hit 0 at finitely many space points simultaneously in time for any $\delta > 2$, but the number of such hitting points would tend to $+\infty$ as $\delta \downarrow 2$. The fact that $\delta = 2$ is the critical value for this behaviour is clearly related to the fact that $\delta = 2$ is also the critical dimension for the probability that the $\delta$-Bessel process or bridge hit 0.

The transition between $\delta \geq 3$ and $\delta < 3$ is visible at the level of the invariant measure, namely the $\delta$-Bessel bridge, since in the former case the measure is log-concave, while this property is lost in the latter case. Therefore the techniques of [1] based on optimal transport and gradient flows in metric spaces fail for $\delta < 3$. In the same vein, the Strong Feller property holds easily for $\delta \geq 3$, while it is an open problem for $\delta < 3$, again because the drift of the SPDE becomes highly non-dissipative. Still, the recent paper [10] of the first author shows that Bessel processes of dimension $\delta < 1$ are Strong Feller even if their drift contains a renormalized local time. Moreover Tsatsoulis and Weber [36] have proved that the 2-dimensional stochastic quantization equation satisfies a Strong Feller property, although it is an equation which needs renormalisation; also Hairer and Mattingly [22] have proved the S-F property for a large class of equations with renormalised drifts. All this suggests that there may be hope that this technically very useful property holds also for $\delta$-Bessel SPDEs with $\delta < 3$.

## Appendix A. Proofs of two technical results

*Proof of Proposition 5.1.* Since $D(\Lambda)$ contains all globally Lipschitz functions on $H$, for all $f \in \mathcal{FC}_b^\infty(K)$ we have $f \circ j \in D(\Lambda)$. A simple calculation shows that for any $f \in \mathcal{FC}_b^\infty(K)$ of the form (5.4) we have

$$\nabla(f \circ j)(z) = \nabla f(j(z))\,\mathrm{sgn}(z). \tag{A.1}$$

Hence, for all $f, g \in \mathcal{FC}_b^\infty(K)$, we have

$$\mathcal{E}(f,g) = \frac{1}{2}\int \langle \nabla f(x), \nabla g(x)\rangle \, \mathrm{d}\nu(x) = \frac{1}{2}\int \langle \nabla f(j(z)), \nabla g(j(z))\rangle \, \mathrm{d}\mu(z)$$
$$= \frac{1}{2}\int \langle \nabla (f\circ j)(z), \nabla (g\circ j)(z)\rangle \, \mathrm{d}\mu(z) = \Lambda(f\circ j, g\circ j),$$

where the third equality follows from (A.1). This shows that the bilinear symmetric form $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ admits as an extension the image of the Dirichlet form $(\Lambda, D(\Lambda))$ under the map $j$. Since $\mathcal{FC}_b^\infty(K)$ is dense in $L^2(\nu)$, this extension is a Dirichlet form. In particular, $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ is closable, its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form, and we have the isometry property (5.5).

There remains to prove that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Since it is the closure of $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$, it suffices to show that the associated capacity is tight. Since $K$ is separable, we can find a countable dense subset $\{y_k,\, k \in \mathbb{N}\} \subset K$ such that $y_k \neq 0$ for all $k \in \mathbb{N}$.

Let now $\varphi \in C_b^\infty(\mathbb{R})$ be an increasing function such that $\varphi(t) = t$ for all $t \in [-1,1]$ and $\|\varphi'\|_\infty \leq 1$. For all $m \in \mathbb{N}$, we define the function $v_m : K \to \mathbb{R}$ by

$$v_m(z) := \varphi(\|z - y_m\|), \quad z \in K.$$

Moreover, we set, for all $n \in \mathbb{N}$

$$w_n(z) := \inf_{m\leq n} v_m(z), \quad z \in K.$$

We claim that $w_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, and that $w_n \underset{n\to\infty}{\longrightarrow} 0$, $\mathcal{E}$ quasi-uniformly in $K$. Assuming this claim for the moment, for all $k \geq 1$ we can find a closed subset $F_k$ of $K$ such that $\mathrm{Cap}(K \setminus F_k) < 1/k$, and $w_n \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $F_k$. Hence, for all $\epsilon > 0$, we can find $n \in \mathbb{N}$ such that $w_n < \epsilon$ on $F_k$. Therefore

$$F_k \subset \bigcup_{m\leq n} B(y_m, \epsilon)$$

where $B(y,r)$ is the open ball in $K$ centered at $y \in K$ with radius $r > 0$. This shows that $F_k$ is totally bounded. Since it is, moreover, complete as a closed subspace of a complete metric space, it is compact, and the tightness of Cap follows.

We now justify our claim. For all $i \in \mathbb{N}$, we set $l_i := \|y_i\|^{-1} y_i$. Then for all $i \geq 1$, $l_i \in K$, $\|l_i\| = 1$ and, for all $z \in K$

$$\|z\| = \sup_{i\geq 0} \langle l_i, z\rangle.$$

Let $m \in \mathbb{N}$ be fixed. For all $i \geq 0$, let $u_i(z) := \sup_{j\leq i} \varphi(\, \langle l_j, z - y_m\rangle\,)$, $z \in K$. We have $u_i \in D(\mathcal{E})$, and, for $\nu$ - a.e. $z \in K$

$$\sum_{k=1}^\infty \frac{\partial u_i}{\partial e_k}(z)^2 \leq \sup_{j\leq i}\left(\sum_{k=1}^\infty \varphi'(\langle l_j, z - y_m\rangle)^2 \langle l_j, e_k\rangle^2\right) \leq 1,$$

whence $\mathcal{E}(u_i, u_i) \leq 1$. By the definition of $v_m$, as $i \to \infty$, $u_i \uparrow v_m$ on $K$, hence in $L^2(K, \nu)$. By [27, I.2.12], we deduce that $v_m \in D(\mathcal{E})$, and that $\mathcal{E}(v_m, v_m) \leq 1$. Therefore, for all $n \in \mathbb{N}$, $w_n \in D(\mathcal{E})$, and $\mathcal{E}(w_n, w_n) \leq 1$. But, since $\{y_k,\, k \in \mathbb{N}\}$ is dense in $K$, as $n \to \infty$, $w_n \downarrow 0$ on $K$. Hence $w_n \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(K, \nu)$. This and the previous bound imply, by [27, I.2.12], that the Cesàro means of some subsequence of $(w_n)_{n\geq 0}$ converge to 0 in $D(\mathcal{E})$. By [27, III.3.5], some subsequence thereof converges $\mathcal{E}$ quasi-uniformly to 0. But, since $(w_n)_{n\geq 0}$ is non-increasing, we deduce that it converges $\mathcal{E}$-quasi-uniformly to 0. The claimed quasi-regularity follows. There finally remains to check that $(\mathcal{E}, D(\mathcal{E}))$ is local in the sense of Definition [27, V.1.1]. Let $u, v \in D(\mathcal{E})$ satisfying $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$. Then, $u \circ j$ and $v \circ j$ are two elements of $D(\Lambda) = W^{1,2}(\mu)$ with disjoint supports, and, recalling (5.5), we have

$$\mathcal{E}(u, v) = \Lambda(u \circ j, v \circ j) = \frac{1}{2} \int_H \nabla(u \circ j) \cdot \nabla(v \circ j)\, \mathrm{d}\mu = 0.$$

The claim follows. □

*Proof of Lemma 5.3.* Recall that $D(\mathcal{E})$ is the closure under the bilinear form $\mathcal{E}_1$ of the space $\mathcal{FC}_b^\infty(K)$ of functionals of the form $F = \Phi\big|_K$, where $\Phi \in \mathcal{FC}_b^\infty(H)$. Therefore, to prove the claim, it suffices to show that for any functional $\Phi \in \mathcal{FC}_b^\infty(H)$ and all $\epsilon > 0$, there exists $\Psi \in \mathscr{S}$ such that $\mathcal{E}_1(\Phi - \Psi, \Phi - \Psi) < \epsilon$.

Let $\Phi \in \mathcal{FC}_b^\infty(H)$. We set for all $\epsilon > 0$

$$\Phi_\epsilon(\zeta) := \Phi(\sqrt{\zeta^2 + \epsilon}), \quad \zeta \in H.$$

A simple calculation shows that $\Phi_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \Phi$ and $\nabla\Phi_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \nabla\Phi$ pointwise, with uniform bounds $\|\Phi_\epsilon\|_\infty \leq \|\Phi\|_\infty$ and $\|\nabla\Phi_\epsilon\|_\infty \leq \|\nabla\Phi\|_\infty$. Hence, by dominated convergence, $\mathcal{E}_1(\Phi_\epsilon - \Phi, \Phi_\epsilon - \Phi) \underset{\epsilon\to 0}{\longrightarrow} 0$. Then, introducing for all $d \geq 1$ $(\zeta_i^d)_{1\leq i\leq d}$ the orthonormal family in $L^2(0, 1)$ given by

$$\zeta_i^d := \sqrt{d}\, \mathbf{1}_{[\frac{i-1}{d}, \frac{i}{d}[}, \quad i = 1, \ldots, d,$$

and setting

$$\Phi_\epsilon^d(\zeta) := \Phi_\epsilon\left(\left(\sum_{i=1}^d \langle \zeta_{d,i}, \zeta^2\rangle\right)^{\frac{1}{2}}\right) = \Phi\left(\left(\sum_{i=1}^d \langle \zeta_{d,i}, \zeta^2\rangle + \epsilon\right)^{\frac{1}{2}}\right), \quad \zeta \in H,$$

again we obtain the convergence $\mathcal{E}_1(\Phi_\epsilon^d - \Phi_\epsilon, \Phi_\epsilon^d - \Phi_\epsilon) \underset{d\to\infty}{\longrightarrow} 0$.

There remains to show that any fixed functional of the form

$$\Phi(\zeta) = f\left(\langle \zeta_1, \zeta^2\rangle, \ldots, \langle \zeta_d, \zeta^2\rangle\right), \quad \zeta \in H$$

with $d \geq 1$, $f \in C_b^1(\mathbb{R}_+^d)$, and $(\zeta_i)_{i=1,\ldots,d}$ a family of elements of $K$, can be approximated by elements of $\mathscr{S}$. Again by dominated convergence, we can suppose that

$f$ has compact support in $\mathbb{R}^d_+$. We define $g \in C^1_b([0,1]^d)$,

$$g(y) := f(-\ln(y_1), \cdots, -\ln(y_d)), \quad y \in ]0,1]^d,$$

and $g(y) := 0$ if $y_i = 0$ for any $i = 1, \ldots, d$. By a differentiable version of the Weierstrass Approximation Theorem (see Theorem 1.1.2 in [26]), there exists a sequence $(p_k)_{k\geq 1}$ of polynomial functions converging to $g$ for the $C^1$ topology on $[0,1]^d$. Defining for all $k \geq 1$ the function $f_k : \mathbb{R}^d_+ \to \mathbb{R}$ by

$$f_k(x) = p_k(e^{-x_1}, \cdots, e^{-x_d}), \quad x \in \mathbb{R}^d_+,$$

we define $\Phi_k \in \mathscr{S}$ by

$$\Phi_k(\zeta) = f_k\left(\langle \zeta_1, \zeta^2\rangle, \ldots, \langle \zeta_d, \zeta^2\rangle\right), \quad \zeta \in H.$$

Since $p_k \underset{k\to\infty}{\longrightarrow} g$ for the $C^1$ topology on $[0,1]^d$, $f_k \underset{k\to\infty}{\longrightarrow} f$ uniformly on $\mathbb{R}^d_+$ together with its first order derivatives. Hence, it follows that $\Phi_k \underset{k\to\infty}{\longrightarrow} \Phi$ pointwise on $K$ together with its gradient. It also follows that there is some $C > 0$ such that for all $k \geq 1$

$$\forall \zeta \in K, \quad |\Phi_k(\zeta)|^2 + \|\nabla \Phi_k(\zeta)\|^2 \leq C(1 + \|\zeta\|^2).$$

Since the quantity in the right-hand side is $\nu$ integrable in $\zeta$, it follows by dominated convergence that $\mathcal{E}_1(\Phi_k - \Phi, \Phi_k - \Phi) \underset{k\to\infty}{\longrightarrow} 0$. This yields the claim. □

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