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Bessel SPDEs and renormalized local times

Henri Elad Altman and Lorenzo Zambotti

Sorbonne Université
Laboratoire de Probabilités Statistique et Modélisation
4 Pl. Jussieu, 75005 Paris, France
Email: eladaltman@lpsm.paris, zambotti@lpsm.paris

Abstract

In this article, we prove integration by parts formulae (IbPFs) for the laws of Bessel bridges from 0 to 0 over the interval $[0, 1]$ of dimension smaller than 3. As an application, we construct a weak version of an SPDE having the law of a one-dimensional Bessel bridge (i.e. the law of a reflected Brownian bridge) as reversible measure, the dimension 1 being particularly relevant in view of applications to scaling limits of dynamical critical pinning models. We also exploit the IbPFs to conjecture the structure of the SPDEs associated with Bessel bridges of all dimensions smaller than 3.

1 Introduction

1.1 From Bessel SDEs to Bessel SPDEs

A Bessel process of dimension $\delta \geq 0$ is defined as the unique continuous non-negative process $(X_t)_{t \geq 0}$ such that $Y_t := X_t^2$ is the solution to the SDE

$$Y_t = Y_0 + \int_0^t 2\sqrt{Y_s} dB_s + \delta t, \quad t \geq 0, \quad (\delta \geq 0) \quad (1.1)$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion. These processes play an important role in the study of the fine properties of Brownian motion, see [RY13]. Moreover their fascinating behavior at the boundary point 0 can be studied in great detail, see e.g. [Zam17] for a recent account.

While well-posedness of the SDE (1.1) satisfied by $(Y_t)_{t \geq 0}$ follows from the classical Yamada-Watanabe theorem [RY13, Theorem IX.3.5], the situation for $(X_t)_{t \geq 0}$ is more involved. For $\delta > 1$, X is solution to

$$X_t = X_0 + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} ds + B_t, \quad t \geq 0, \quad (\delta > 1) \quad (1.2)$$

and this equation satisfies pathwise uniqueness and existence of strong solutions since the drift is monotone decreasing. For $\delta = 1$, X is the solution to

$$X_t = X_0 + L_t + B_t, \quad t \geq 0, \quad (\delta = 1)$$

where $(L_t)_{t \geq 0}$ is continuous and monotone non-decreasing, with $L_0 = 0$ and

$$X \geq 0, \quad \int_0^\infty X_s dL_s = 0. \quad (1.3)$$

In other words X is a reflecting Brownian motion, and the above equation has a unique solution by the Skorokhod Lemma [RY13, Lemma VI.2.1].

For $\delta \in (0, 1)$, the situation is substantially more difficult. One can show, see e.g. [Zam17, Proposition 3.12], that X admits *diffusion local times*, namely continuous processes $(\ell_t^a)_{t \geq 0, a \geq 0}$ such that

$$\int_0^t \varphi(X_s) ds = \int_0^\infty \varphi(a) \ell_t^a a^{\delta-1} da,$$

for all Borel $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and that X satisfies

$$X_t = X_0 + \frac{\delta-1}{2} \int_0^\infty \frac{\ell_t^a - \ell_t^0}{a} a^{\delta-1} da + B_t, \quad t \geq 0, \quad (0 < \delta < 1). \quad (1.4)$$

The latter formula is not really an SDE, and to our knowledge one cannot (so far) characterize X as the unique process satisfying this property, unless one manages to prove that X^2 is a solution to (1.1).

In a series of papers [Zam01, Zam02, Zam03, Zam04] the second author of this article studied a class of stochastic partial differential equations (SPDEs) with analogous properties. For a parameter $\delta > 3$ the equation, that we call *Bessel SPDE*, is

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\kappa(\delta)}{2u^3} + \xi \\ u(t, 0) = u(t, 1) = 0, \quad t \geq 0 \\ u(0, x) = u_0(x), \quad x \in [0, 1] \end{cases} \quad (\delta > 3) \quad (1.5)$$

where $u \geq 0$ is continuous and ξ is a space-time white noise on $\mathbb{R}_+ \times [0, 1]$, and we set

$$\kappa(\delta) := \frac{(\delta-3)(\delta-1)}{4}. \quad (1.6)$$

As $\delta \downarrow 3$, the solution to (1.5) converges to the solution of the Nualart-Pardoux equation [NP92]

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \eta + \xi \\ u(0, \cdot) = u_0, \quad u(t, 0) = u(t, 1) = 0 \\ u \geq 0, \quad d\eta \geq 0, \quad \int_{\mathbb{R}_+ \times [0, 1]} u d\eta = 0, \end{cases} \quad (\delta = 3) \quad (1.7)$$

where η is a reflection measure on $]0, \infty[\times]0, 1[$. The unique invariant measure of (1.5) for $\delta > 3$, respectively (1.7), is the Bessel bridge of dimension δ , resp. 3. Equation (1.7) also describes the fluctuations of an effective (1 + 1) interface model near a wall [FO01, Fun05].

While (1.5) for $\delta > 3$ is the analog of (1.2) for $\delta > 1$, (1.7) is the analog of (1.3). The analogy can be justified in terms of scaling invariance: the equations (1.2) and (1.3) are invariant (in law) under the rescaling $X_t \mapsto \lambda^{-1} X_{\lambda^2 t}$ for $\lambda > 0$, while (1.5) and (1.7) are invariant under $u(t, x) \mapsto \lambda^{-1} u(\lambda^4 t, \lambda^2 x)$ (apart from the fact that the space interval changes from $[0, 1]$ to $[0, \lambda^{-2}]$).

It has been an open problem for over 15 years to understand the analog of (1.4) in this context. In other words, the question is: what is a SPDE whose invariant measure is the Bessel bridge of dimension $\delta < 3$?

This problem is particularly interesting for $\delta = 1$, which corresponds to the reflecting Brownian bridge as an invariant measure. Indeed, the reflecting Brownian bridge arises as the scaling limit of critical pinning models, see [DGZ05]. In turn, pinning models can be seen as the reversible measure of some finite-dimensional diffusions: such a construction was addressed in Chapter 15.2 of [Fun05], while in [FGV16] and [GV18], the authors constructed and analyzed these diffusions using mainly Dirichlet form techniques. Moreover, recently [DO18] proposed an easy construction of strong solutions in the case of pinning models with a shrinking strip. These dynamical pinning models are believed to have a scaling limit, which would be an infinite-dimensional diffusion having the law of a reflecting Brownian motion as reversible measure. What kind of SPDE that limit should satisfy has however remained a very open question so far: we refer to Section 5 in [Voß16] for some conjectures based on formulae obtained in [Zam05] and [GV16].

In this paper we provide an answer to these questions through integration by parts formulae on the law of Bessel bridges of dimension $\delta < 3$. These formulae involve the laws of pinned Bessel bridges (or, more precisely, the laws $\Sigma_{0,0}^{\delta,r}(\cdot|b)$, $b \geq 0$, $r \in (0, 1)$, defined below) which should correspond to the local times of the solution u to our would-be SPDEs. Thus, the drift term in these SPDEs should involve the process $(\ell_{t,x}^a)_{a \geq 0}$ defined by

$$\int_0^t \varphi(u(s, x)) ds = \int_0^\infty \varphi(a) \ell_{t,x}^a a^{\delta-1} da, \quad (1.8)$$

for all Borel $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Some explicit computations on the measures $\Sigma_{0,0}^{\delta,r}(\cdot|b)$ suggest that this process should moreover have a vanishing first-order derivative at 0, that is

$$\left. \frac{\partial}{\partial b} \ell_{t,x}^b \right|_{b=0} = 0, \quad t \geq 0, \quad x \in (0, 1). \quad (1.9)$$

Finally, the integration by parts formulae that we find imply that the Bessel SPDE for $1 < \delta < 3$ should have the following form

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\kappa(\delta)}{2} \frac{\partial}{\partial t} \int_0^\infty \frac{1}{a^3} (\ell_{t,x}^a - \ell_{t,x}^0) a^{\delta-1} da + \xi, \quad (1 < \delta < 3). \quad (1.10)$$

Then (1.10) is the SPDE analog of (1.4). For $\delta = 1$ we have

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{8} \frac{\partial}{\partial t} \frac{\partial^2}{\partial b^2} \ell_{t,x}^b \Big|_{b=0} + \xi, \quad (\delta = 1), \quad (1.11)$$

and for $0 < \delta < 1$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \xi & (0 < \delta < 1) \\ &+ \frac{\kappa(\delta)}{2} \frac{\partial}{\partial t} \int_0^\infty \frac{1}{a^3} \left(\ell_{t,x}^a - \ell_{t,x}^0 - \frac{a^2}{2} \frac{\partial^2}{\partial b^2} \ell_{t,x}^b \Big|_{b=0} \right) a^{\delta-1} da. \end{aligned} \quad (1.12)$$

In (1.10), as in (1.4), we have a Taylor expansion at order 0 of the local times functions $a \mapsto \ell^a$. By contrast, equations (1.11) and (1.12) have no analog in the context of one-dimensional Bessel processes. In (1.12) the Taylor expansion is at order 2, while (1.11) is a limit case, like (1.7), which can be explained as follows.

We introduce for $\alpha \in \mathbb{R}$ the following distributions on $[0, \infty)$:

- if $\alpha = -k$ with $k \in \mathbb{N}$, then

$$\langle \mu_\alpha, \varphi \rangle := (-1)^k \varphi^{(k)}(0), \quad \forall \varphi \in C_0^\infty([0, \infty))$$

- else,

$$\langle \mu_\alpha, \varphi \rangle := \int_0^{+\infty} \left(\varphi(a) - \sum_{0 \leq j \leq -\alpha} \frac{a^j}{j!} \varphi^{(j)}(0) \right) \frac{a^{\alpha-1}}{\Gamma(\alpha)} da, \quad \forall \varphi \in C_0^\infty([0, \infty)).$$

Then we have that $\alpha \mapsto \langle \mu_\alpha, \varphi \rangle$ is continuous for all $\varphi \in C_0^\infty([0, \infty))$. Moreover, for $\delta > 3$ the non-linearity in (1.5) can be expressed as

$$\frac{\kappa(\delta)}{2} \int_0^t \frac{1}{(u(s, x))^3} ds = \frac{\kappa(\delta)}{2} \int_0^\infty \frac{1}{a^3} \ell_{t,x}^a a^{\delta-1} da = \frac{\kappa(\delta) \Gamma(\delta-3)}{2} \langle \mu_{\delta-3}, \ell_{t,x} \rangle,$$

which, by (1.6), we can in turn rewrite as

$$\frac{\Gamma(\delta)}{8(\delta-2)} \langle \mu_{\delta-3}, \ell_{t,x} \rangle$$

Note that, at least formally, the latter expression makes sense for any $\delta > 0$ (at the expense of taking a limit in the case $\delta = 2$, and taking into account the cancellation (1.9)). Moreover, this expression precisely gives the non-linearity of (1.7)-(1.10)-(1.11)-(1.12). In particular, for $\delta = 3$, it equals $\frac{1}{4} \ell_{t,x}^0$, which is coherent with the results about the structure of the reflection measure η in (1.7) proved in [Zam04] and showing that a.s. :

$$\eta([0, t] \times dx) = \frac{1}{4} \ell_{t,x}^0 dx.$$

While the formulae (1.10), (1.11) and (1.12) remain conjectural, for the case $\delta = 1$, we do construct, by Dirichlet Forms methods, a Markov process $(u_t)_{t \geq 0}$ with the reflected Brownian bridge as reversible measure, and satisfying a modified version of equation (1.11) above. More precisely, we show that

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{4} \lim_{\epsilon \rightarrow 0} \rho_\epsilon''(u) + \xi, \quad (\delta = 1) \quad (1.13)$$

where $\rho_\epsilon(x) = \frac{1}{\epsilon} \rho(\frac{x}{\epsilon})$ is a smooth approximation of the Dirac measure at 0, see Theorem 5.8 for the precise statements. Similar arguments could allow us to treat the case $\delta = 2$ similarly: this will be done in a forthcoming article.

1.2 Integration by parts formulae for the laws of Bessel bridges

Integration by parts plays a fundamental role in analysis, and most notably in stochastic analysis. For instance, it lies at the core of Malliavin Calculus and the theory of Dirichlet forms, see e.g. [Nua09, FOT10, MR92].

While it is relatively easy in finite dimension, where the standard rules of calculus apply, obtaining integration by parts formulae (IbPFs for short) for measures on infinite-dimensional spaces can be a difficult task, one of the main reasons being the absence of Lebesgue measure in that context. The most celebrated example is the IbPF associated with Brownian motion, or its corresponding bridge, on the interval $[0, 1]$, which reads

$$E[\partial_h \Phi(B)] = -E[\langle h'', B \rangle \Phi(B)],$$

for all Fréchet differentiable $\Phi : L^2(0, 1) \rightarrow \mathbb{R}$ and all $h \in C_c^2(0, 1)$, where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product in $L^2(0, 1)$. This formula follows for instance from the quasi-invariance property of the Wiener measure on $[0, 1]$ along the Cameron-Martin space.

In [Zam02], the second author exploited the relation between the law of the Brownian bridge and the law of a Brownian excursion (also known as 3-dimensional Bessel bridge) on $[0, 1]$ to deduce an IbPF for the latter measure; other proofs were given later, see e.g. [FI07, Zam17]. In [Zam03], exploiting an absolute continuity relation with respect to the 3-dimensional Bessel bridge, the second author obtained IbPFs for the Bessel bridges of dimension larger than 3. Put in a nutshell, these formulae read as follows:

$$E_{a,a}^\delta[\partial_h \Phi(X)] = -E_{a,a}^\delta[\langle h'', X \rangle \Phi(X)] - \kappa(\delta) E_{a,a}^\delta[\langle h, X^{-3} \rangle \Phi(X)] \quad (1.14)$$

for all $\delta > 3$, and

$$\begin{aligned} E_{a,a}^3[\partial_h \Phi(X)] &= -E_{a,a}^3[\langle h'', X \rangle \Phi(X)] \\ &\quad - \int_0^1 dr h_r \gamma(r, a) E_{a,a}^\delta[\Phi(X) | X_r = 0], \end{aligned} \quad (1.15)$$

where Φ and h are as above, and, for all $r \in (0, 1)$, $\gamma(r, a)$ is some explicit positive number. Here, for all $\delta > 0$ and $a \geq 0$, $E_{a,a}^\delta$ denotes expectation with respect to the law $P_{a,a}^\delta$, on the space of continuous real-valued functions on $[0, 1]$, of the δ -dimensional Bessel bridge from a to a over the interval $[0, 1]$, and $\kappa(\delta)$ is defined in (1.6). Note that $\kappa(\delta) > 0$ for $\delta > 3$, and κ vanishes at $\delta = 3$, the dimension corresponding to the Brownian excursion. At the same time, the quantity

$$\langle |h|, X^{-3} \rangle$$

is integrable with respect to $P_{a,a}^\delta$ for $\delta > 3$, but is non-integrable with respect to $P_{a,a}^3$ for h that is not identically 0. Hence, formula (1.15) can be seen as the limit, as $\delta \searrow 3$, of the formula (1.14), with the last term in the right-hand side corresponding to an indeterminate form $0 \times \infty$, converging here to an integral over a measure involving the laws of pinned Bessel bridges $E_{a,a}^3[dX | X_r = 0]$, $r \in (0, 1)$. Actually, formula (1.15) possesses a geometric-measure theory interpretation as a Gauss-Green formula in an infinite-dimensional space, the second term in the right-hand side corresponding to a boundary term (see Chapter 6.1.2 in [Zam17]).

What can we say for Bessel bridges of dimension $\delta < 3$? In such a regime, the techniques used in [Zam03], based on absolute continuity relations with the Brownian excursion as well as monotonicity arguments, fall apart. Indeed, when $\delta \in (1, 3)$, $\kappa(\delta) < 0$, so the required monotonicity properties do not hold anymore, while for $\delta < 2$ the absolute continuity relations fail to exist. Hence, the problem of finding IbPFs for the measures $P_{a,a}^\delta$, when $\delta < 3$, has remained open until now,

excepted for the value $\delta = 1$, corresponding to the reflected Brownian bridge, for which some IbPFs have been obtained, see [Zam05] for the case of the reflected Brownian motion, [GV16] for the case of a genuine bridge, and Remark 4.11 below for a discussion.

1.3 Outline of the results

Here and below, let $C([0, 1]) := C([0, 1], \mathbb{R})$ be the space of continuous real-valued functions on $[0, 1]$. In this article, we obtain IbPFs for the laws $P_{0,0}^\delta$ of Bessel bridges of dimension $\delta \in (0, 3)$ from 0 to 0 over $[0, 1]$. Our formulae hold for a large class of functionals $\Phi : C([0, 1]) \rightarrow \mathbb{R}$. More precisely, we consider linear combinations of functionals of the form

$$\Phi(\zeta) = \exp(-\langle m, \zeta^2 \rangle), \quad \zeta \in C([0, 1]),$$

with m a finite Borel measure on $[0, 1]$, and where $\langle m, \zeta^2 \rangle := \int_0^1 \zeta_t^2 m(dt)$. We prove that these functionals satisfy IbPFs for the laws $P_{0,0}^\delta$, for all $\delta > 0$. Our method is based on deriving semi-explicit expressions for quantities of the form

$$E_{0,0}^\delta[\Phi(X)] \quad \text{and} \quad E_{0,0}^\delta[\Phi(X) | X_r = b], \quad b \geq 0, r \in (0, 1),$$

using solutions to some second-order differential equations, and exploiting the nice computations done in Chapter XI of [RY13]. The fundamental property enabling these computations is the additivity property of the squared Bessel processes, which in particular implies that, when $a = a' = 0$, both of the quantities above factorize in a very specific way, see the expression (3.17) below. As a consequence, for functionals as above, all the IbPFs for $P_{0,0}^\delta$, $\delta \geq 3$ are just multiples of a single differential relation which does not depend on δ (see Lemma 4.6 below), the dependence in δ entering only through the multiplying constant which involves some Γ values. When $\delta \geq 3$, expressing these Γ values as integrals, and performing a change of variable, we retrieve the formulae already obtained in [Zam02] and [Zam03]. On the other hand, when $\delta < 3$, one of the Γ values appearing is negative, so we cannot express it using the usual integral formula, but must rather use *renormalized* integrals.

As a result, when $\delta \in (1, 3)$, the IbPFs contain Taylor remainders at order 0, of the form

$$\Sigma_{0,0}^{\delta,r}(\Phi(X) | b) - \Sigma_{0,0}^{\delta,r}(\Phi(X) | 0), \quad b \geq 0, r \in (0, 1),$$

where $\Sigma_{0,0}^{\delta,r}(dX | b)$ is a measure on $C([0, 1])$ proportional to the law of the Bessel bridge conditioned to hit b at r . When $\delta \in (0, 1)$, this renormalization phenomenon becomes even more acute, with the presence in the IbPFs of Taylor remainders at order 2, with expressions of the form

$$\Sigma_{0,0}^{\delta,r}(\Phi(X) | b) - \Sigma_{0,0}^{\delta,r}(\Phi(X) | 0) - \frac{b^2}{2} \frac{d^2}{da^2} \Sigma_{0,0}^{\delta,r}(\Phi(X) | a) \Big|_{a=0}, \quad b \geq 0, r \in (0, 1),$$

noting that the term of order 1 vanishes:

$$\frac{d}{da} \Sigma_{0,0}^{\delta,r}(\Phi(X) | a) \Big|_{a=0} = 0, \quad r \in (0, 1). \quad (1.16)$$

These formulae are stated in Theorem 4.1 below. One important, expected feature is the transition that occurs at the values $\delta = 3$ and $\delta = 1$. In the latter case, which

corresponds to the reflected Brownian bridge, the IbPF degenerates in an expression similar to (1.15), but with the boundary term replaced with its double derivatives at $b = 0$

$$\frac{d^2}{db^2} \Sigma_{0,0}^{1,r}(\Phi(X) | b) \Big|_{b=0}, \quad r \in (0, 1),$$

see formula (4.4) below. Another important but less expected feature is the absence of transition at $\delta = 2$, as well as the related remarkable fact that the functions $b \mapsto \Sigma_{0,0}^{\delta,r}(\Phi(X) | b)$, are, for all $r \in (0, 1)$, smooth functions in b^2 , so that all their odd-order derivatives vanish at 0. This is the reason why there only ever appear derivatives of even order in our formulae. An objection to this observation might be that the class of functionals we consider is too restrictive. However, in a forthcoming article, we will show that the IbPFs obtained in the present article still hold for a class of very general functionals. In particular, vanishing of first-order derivatives at $b = 0$ can be established for $b \mapsto \Sigma_{0,0}^{\delta,r}(\Phi(X) | b)$ for any $\Phi \in C_b^1(L^2(0, 1))$, which confirms the absence of transition at $\delta = 2$ observed in this article.

The IbPF above should help us with our original SPDE problem. Indeed, assuming that we could construct Markov processes on $L^2(0, 1)$ having $P_{0,0}^\delta$ for $\delta \in (0, 3)$ as their invariant measure, we should be able to prove that these satisfy the Bessel SPDEs presented above, at least in a weak form, using for example Dirichlet Form techniques. Note that, in the case $\delta \geq 3$, the SPDEs had been solved by the second author in [Zam02] and [Zam03] using pathwise techniques, and many fine properties of the solution had been studied, such as their hitting properties (see [DMZ06]), or the existence of occupation densities (see [Zam04]). By contrast, in the case $\delta < 3$, the SPDEs (1.10), (1.11) and (1.12) do not seem to possess any strong notion of solution, and essentially seem to lie outside the scope of any existing theory of SPDEs. However, in this article, for $\delta = 1$, using Dirichlet form techniques, and thanks to the IbPF for the reflecting Brownian bridge, we are able to construct a weak version of the associated SPDE in the stationary regime. Thus, the dynamics for $\delta = 1$ can be described formally by (1.13). A similar construction can be implemented in the case $\delta = 2$, which will be the subject of a forthcoming article. The approach using Dirichlet forms has already been used in [Voß16], where the author constructed the Markov process for $\delta = 1$, but not the SPDE.

The cases corresponding to $\delta \in (0, 3) \setminus \{1\}$ do not seem to have been provided any partial answer yet. However, the IbPFs we obtain below justify our conjecture for the structure of the corresponding would-be SPDEs (1.10), (1.11) and (1.12), containing Taylor remainders of local times, in a way reminiscent of the principal values of local times appearing in (1.4).

The article is organized as follows: in Section 2 we address a toy-model consisting of a family of measures on \mathbb{R}_+ , hence much simpler than the laws of Bessel bridges, but displaying a similar renormalization phenomena at the level of the IbPFs. In Section 3 we recall and prove some useful facts on the laws of squared Bessel processes, Bessel processes, and their bridges. In Section 4, we state and prove the IbPFs for the laws of Bessel bridges. The formula for $P_{0,0}^1$ is used in Section 5 to construct a weak form of the corresponding SPDE, using Dirichlet form techniques. We also show that the absolute value of the solution to the stochastic heat equation does *not* coincide with the Markov process solution of (1.13) (see Thm 5.9 below for the precise statement). Finally, in Section 6, based on the IbPFs we obtained, we justify more in detail our conjectures on the structure of the Bessel SPDEs for $\delta < 3$.

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2 A prelude

In this section we introduce a toy model consisting of a family of measures on \mathbb{R}_+ , for which we derive integration by parts formulae. We will obtain a unified formulation for these upon extending the initial family of measures to a family of *distributions*, and this will be achieved by *renormalization*. The content of this section is well-known and classical, but it will serve as a useful finite-dimensional example for the theory to come.

We consider the following Borel measures on \mathbb{R}_+ . For $\alpha \geq 0$, we set

$$\mu_\alpha(dx) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} dx, \quad \alpha > 0, \quad \mu_0 = \delta_0,$$

where δ_0 denotes the Dirac measure at 0. Anticipating on Section 3 below, we readily mention that, for each $\alpha \geq 0$, μ_α is a reversible measure for the α -dimensional Bessel process.

Remark 2.1. For the sequel it will be useful to compute the Laplace transform of the measures μ_α , $\alpha \geq 0$. A simple change of variable shows that

$$\int \exp(-\lambda x) \mu_\alpha(dx) = \lambda^{-\alpha}, \quad \lambda > 0, \quad \alpha \geq 0. \quad (2.1)$$

Proposition 2.2 (Integration by parts formulae). *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a C^1 function with compact support in $[0, \infty)$. Then, for all $\alpha \geq 1$:*

$$\int \phi'(x) \mu_\alpha(dx) = - \int \phi(x) \mu_{\alpha-1}(dx). \quad (2.2)$$

Proof. For $\alpha > 1$, integrating by parts, we obtain

$$\int_0^\infty \varphi'(x) \frac{x^{\alpha-1}}{\Gamma(\alpha)} dx = \left[\varphi(x) \frac{x^{\alpha-1}}{\Gamma(\alpha)} \right]_0^\infty - \int_0^\infty \varphi(x) \frac{(\alpha-1)x^{\alpha-2}}{\Gamma(\alpha)} dx. \quad (2.3)$$

Since φ has compact support in \mathbb{R}_+ , and since $\alpha > 1$, the bracket term vanishes. On the other hand, recalling that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, we can rewrite the right-hand integral as

$$\int_0^\infty \varphi(x) \mu_{\alpha-1}(dx),$$

and the claim follows. Turning to $\alpha = 1$, we have $\mu_1 = \mathbf{1}_{x>0} dx$. Hence, recalling that φ has compact support in \mathbb{R}_+ , we have

$$\int_0^\infty \varphi'(x) \mu_1(dx) = -\varphi(0) = -\int_0^\infty \varphi(x) \mu_0(dx),$$

and the claim follows as well. \square

We are going to prove that the family of measures $(\mu_\alpha)_{\alpha \geq 0}$ can be extended to a family of *distributions* $(\mu_\alpha)_{\alpha \in \mathbb{R}}$ which satisfies equation (2.2) for all $\alpha \in \mathbb{R}$. To do so, we first define the appropriate space of test functions on $[0, \infty)$.

Definition 2.3. Let $\mathcal{S}([0, \infty))$ be the space of C^∞ functions $\varphi : [0, \infty) \rightarrow \mathbb{R}$ such that, for all $k, l \geq 0$, there exists $C_{k,l} \geq 0$ such that

$$\forall x \geq 0, \quad |\varphi^{(k)}(x)| x^\ell \leq C_{k,\ell}. \quad (2.4)$$

Now, for each $\alpha \geq 0$, the measure μ_α defines a Schwartz distribution (which we still denote by μ_α) as follows: for all test function $\varphi \in \mathcal{S}([0, \infty))$,

$$\langle \mu_\alpha, \varphi \rangle := \int_0^\infty \varphi(x) \mu_\alpha(dx). \quad (2.5)$$

Remark 2.4. The quantity in (2.5) is equal to $\Gamma(\alpha)\mathcal{M}\varphi(\alpha)$, where $\mathcal{M}\varphi(\alpha)$ is the Mellin transform of the function φ computed at $\alpha \geq 0$.

For $\alpha < 0$, we will define μ_α as a distribution, using a *renormalization* procedure based on Taylor polynomials. To do so, for any smooth function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$, for all $n \in \mathbb{Z}$, and all $x \geq 0$, we set

$$\mathcal{T}_x^n \varphi := \varphi(x) - \sum_{0 \leq j \leq n} \frac{x^j}{j!} \varphi^{(j)}(0). \quad (2.6)$$

In words, if $n \geq 0$ then $\mathcal{T}_x^n \varphi$ is the Taylor remainder based at 0, of order $n + 1$, of the function φ , evaluated at x ; if $n < 0$ then $\mathcal{T}_x^n \varphi$ is simply the value of φ at x .

Definition 2.5. For $\alpha < 0$, we define the distribution μ_α as follows:

- if $\alpha = -k$ with $k \in \mathbb{N}$, then

$$\langle \mu_\alpha, \varphi \rangle := (-1)^k \varphi^{(k)}(0), \quad \forall \varphi \in \mathcal{S}([0, \infty)) \quad (2.7)$$

- if $-k - 1 < \alpha < -k$ with $k \in \mathbb{N}$, then

$$\langle \mu_\alpha, \varphi \rangle := \int_0^{+\infty} \mathcal{T}_x^k \varphi \frac{x^{\alpha-1}}{\Gamma(\alpha)} dx, \quad \forall \varphi \in \mathcal{S}([0, \infty)). \quad (2.8)$$

Remark 2.6. Note that formula (2.8) defines a bona fide distribution on $\mathcal{S}([0, \infty))$. Indeed, by Taylor's theorem, the integrand is of order $x^{k+\alpha}$ near 0, therefore integrable there, while it is dominated by $x^{k+\alpha-1}$ near $+\infty$, so is integrable at infinity as well.

Remark 2.7. Definition 2.5 thus provides an extension of the Mellin transform of a function $\varphi \in \mathcal{S}([0, \infty))$ to the whole real line. In particular, equality (2.7) is natural in view of Ramanujan's Master Theorem, which allows to see the successive derivatives at 0 of an analytic function as the values, for non-positive integers, of the analytic extension of its Mellin transform. We refer to [AEG⁺12] for more details on this theorem. We also stress that the renormalization procedure used in equation (2.8) to define μ_α for $\alpha < 0$ is very natural, and can also be used to extend the domain of validity of Ramanujan's Master Theorem, see Theorem 8.1 in [AEG⁺12].

Note that, for $k \in \mathbb{N}$ and α such that $-k - 1 < \alpha < -k$, by performing $k + 1$ successive integration by parts, we can rewrite equality (2.8) as

$$\langle \mu_\alpha, \varphi \rangle := \frac{(-1)^{k+1}}{\Gamma(\alpha)} \int_0^{+\infty} \varphi^{(k+1)}(x) \frac{x^{\alpha+k}}{\alpha(\alpha+1)\dots(\alpha+k)} dx.$$

Hence, recalling that $\Gamma(\alpha)\alpha(\alpha+1)\dots(\alpha+k) = \Gamma(\alpha+k+1)$, we deduce that

$$\langle \mu_\alpha, \varphi \rangle := (-1)^{k+1} \int_0^{+\infty} \varphi^{(k+1)}(x) \mu_{\alpha+k+1}(dx), \quad (2.9)$$

which can be interpreted as a variant of the Caputo differential, at order $-\alpha$, of φ (see e.g. (1.17) in [GM08]). Note that we supposed that $k \geq 0$ so that $\alpha < 0$, but equality (2.9) clearly also holds for $k = -1$ and $\alpha \in (0, 1)$.

Proposition 2.8 (Integration by parts formulae). *For all $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{S}([0, \infty))$:*

$$\langle \mu_\alpha, \varphi' \rangle = -\langle \mu_{\alpha-1}, \varphi \rangle.$$

Proof. The case $\alpha \geq 1$ was treated in Proposition 2.2. The case $\alpha = -k$ with $k \in \mathbb{N}$ follows from the definition (2.7). Finally, the case when $\alpha \in (-k - 1, -k)$ for some integer $k \geq -1$ follows at once from the representation (2.9). \square

Remark 2.9. As a consequence of Proposition 2.8, we deduce that the expression (2.1) for the Laplace transform of μ_α remains true also for negative α . Indeed, for such α , picking $k \in \mathbb{N}$ such that $\alpha + k > 0$, we have, for all $\lambda > 0$

$$\begin{aligned} \langle \mu_\alpha, e^{-\lambda \cdot} \rangle &= (-1)^k \langle \mu_{\alpha+k}, \frac{d^k}{dx^k} e^{-\lambda \cdot} \rangle = \lambda^k \langle \mu_{\alpha+k}, e^{-\lambda \cdot} \rangle \\ &= \lambda^k \lambda^{-\alpha-k} = \lambda^{-\alpha}. \end{aligned}$$

3 Squared Bessel processes, Bessel processes, and associated bridges

In this section we recall and prove some useful facts about squared Bessel processes, Bessel processes, and their corresponding bridges.

We recall that, for all $\alpha, \theta > 0$, $\Gamma(\alpha, \theta)$ denotes the probability law on \mathbb{R}_+

$$\Gamma(\alpha, \theta)(dx) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} \mathbf{1}_{x>0} dx.$$

For $\alpha = 0$, we set $\Gamma(0, \theta) := \delta_0$.

3.1 Squared Bessel processes and Bessel processes

For all $x, \delta \geq 0$, denote by Q_x^δ the law, on $C(\mathbb{R}_+, \mathbb{R}_+)$, of the δ -dimensional squared Bessel process started at x , namely the unique solution to the SDE

$$Y_t = x + 2 \int_0^t \sqrt{Y_s} dB_s + \delta t, \quad t \geq 0, \quad (3.1)$$

where B is a standard Brownian motion, see Chapter XI of [RY13]. We denote by $(X_t)_{t \geq 0}$ the canonical process

$$X_t : C([0, 1]) \rightarrow \mathbb{R}, \quad X_t(\omega) := \omega_t, \quad \omega \in C([0, 1]).$$

Definition 3.1. For any interval $I \subset \mathbb{R}_+$, and any two probability laws μ, ν on $C(I, \mathbb{R}_+)$, let $\mu * \nu$ denote the convolution of μ and ν , i.e. the image of $\mu \otimes \nu$ under the addition map:

$$C(I, \mathbb{R}_+) \times C(I, \mathbb{R}_+) \rightarrow C(I, \mathbb{R}_+), \quad (x, y) \mapsto x + y.$$

The family of probability measures $(Q_x^\delta)_{\delta, x \geq 0}$ satisfies the following well-known additivity property, first observed by Shiga and Watanabe in [SW73].

Proposition 3.2. For all x, x', δ, δ' , we have the following equality of probability laws on $C(\mathbb{R}_+, \mathbb{R}_+)$

$$Q_x^\delta * Q_{x'}^{\delta'} = Q_{x+x}^{\delta+\delta'} \quad (3.2)$$

We recall that squared Bessel processes are homogeneous Markov processes on \mathbb{R}_+ . Exploiting the additivity property (3.2), Revuz and Yor provided, in section XI of [RY13], explicit expressions for their transition densities $(q_t^\delta(x, y))_{t > 0, x, y \geq 0}$. When $\delta > 0$, these are given by

$$q_t^\delta(x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right), \quad t > 0, x > 0 \quad (3.3)$$

and

$$q_t^\delta(0, y) = (2t)^{-\frac{\delta}{2}} \Gamma(\delta/2)^{-1} y^{\delta/2-1} \exp\left(-\frac{y}{2t}\right), \quad t > 0. \quad (3.4)$$

In particular, for $x = 0$, we have

$$q_t^\delta(0, y) dy = \Gamma\left(\frac{\delta}{2}, \frac{1}{2t}\right)(dy).$$

If $(Y_t)_{t \geq 0}$ is a δ -dimensional squared Bessel process started at x , then the process

$$(\sqrt{Y_t})_{t \geq 0}$$

is called a δ -dimensional Bessel process started at a , where $a = \sqrt{x}$. We shall denote by $(p_t^\delta(a, b))_{t > 0, a, b \geq 0}$ the corresponding transition densities. These are given in terms of the densities of the squared Bessel process by the relation

$$\forall t > 0, \quad \forall a, b \geq 0, \quad p_t^\delta(a, b) = 2b q_t^\delta(a^2, b^2). \quad (3.5)$$

In section XI of [RY13], Revuz and Yor provided semi-explicit expressions for the Laplace transforms of squared Bessel processes (and also the corresponding bridges).

Their proof is based on the fact that, for all $\delta, x \geq 0$, and all finite Borel measure m on $[0, 1]$, the measure $\exp(-\langle m, X \rangle) Q_x^\delta$ possesses a nice probabilistic interpretation, where we use the notation:

$$\langle m, f \rangle := \int_0^1 f(r) m(dr)$$

for any Borel function $f : [0, 1] \rightarrow \mathbb{R}_+$. This remarkable fact is used implicitly in [RY13] (see e.g. the proof of Theorem (3.2) of Chap XI.3), where the authors compute the one-dimensional marginal distributions of this measure. By contrast, in the proof of Lemma 3.6 below, we will need to compute higher-dimensional marginals. We thus need a way to interpret the measure $\exp(-\langle m, X \rangle) Q_x^\delta$ probabilistically. To do so, we first introduce some notations.

Let m be a finite, Borel measure on $[0, 1]$. As in Chap. XI of [RY13], we consider the unique solution $\phi = (\phi_r, r \geq 0)$ on \mathbb{R}_+ of the following problem

$$\begin{cases} \phi''(dr) = 2\mathbf{1}_{[0,1]}(r) \phi_r m(dr) \\ \phi_0 = 1, \\ \phi > 0, \phi' \leq 0 \text{ on } \mathbb{R}_+, \end{cases} \quad (3.6)$$

where the first equality is an equality in distribution (see Appendix 8 of [RY13] for existence and uniqueness of solutions to this problem). Note that the above function ϕ coincides with the function ϕ_μ of Chap XI.1 of [RY13], with $\mu := 2\mathbf{1}_{[0,1]} m$.

Lemma 3.3. *Let m be a finite, Borel measure on $[0, 1]$, and let ϕ be the unique solution of (3.6). Then, for all $x, \delta \geq 0$, the measure R_x^δ defined on $\mathcal{F}_1 := \sigma(\{X_s, s \leq 1\})$ by*

$$R_x^\delta := \exp\left(-\frac{x}{2}\phi'_0\right) \phi_1^{-\frac{\delta}{2}} e^{-\langle m, X \rangle} Q_x^\delta \quad (3.7)$$

is a probability measure, equal to the law of the process

$$(\phi_t^2 Y_{\varrho_t})_{t \in [0,1]},$$

where $Y \stackrel{(d)}{=} Q_x^\delta$ and ϱ is the deterministic time change

$$\varrho_t = \int_0^t \phi_u^{-2} du, \quad t \geq 0. \quad (3.8)$$

Proof. We proceed as in the proofs of Theorem (1.7) and (3.2) in Chapter XI of [RY13]. Let $x, \delta \geq 0$. Under Q_x^δ , $M_t := X_t - \delta t$ is a local martingale, so we can define an exponential local martingale by setting

$$Z_t = \mathcal{E}\left(\frac{1}{2} \int_0^t \frac{\phi'_s}{\phi_s} dM_s\right)_t.$$

As established in the proof of Theorem (1.7) of [RY13], we have

$$\begin{aligned} Z_t &= \exp\left(\frac{1}{2} \left(\frac{\phi'_t}{\phi_t} X_t - \phi'_0 x - \delta \ln \phi_t\right) - \int_0^t X_s dm(s)\right) \\ &= \exp\left(-\frac{x}{2}\phi'_0\right) \phi_t^{-\frac{\delta}{2}} \exp\left(\frac{1}{2} \frac{\phi'_t}{\phi_t} X_t - \int_0^t X_s dm(s)\right), \end{aligned}$$

recalling that the measure μ considered in [RY13] is given in our case by $2 \mathbf{1}_{[0,1]} m$. In particular, we deduce that the measure R_x^δ defined by (3.7) coincides with $Z_1 Q_x^\delta$ (note that $\phi'_1 = 0$ as a consequence of (3.6)). Moreover, by the above expression, $(Z_t)_{t \in [0,1]}$ is uniformly bounded by $\exp(-\frac{1}{2} \phi'_0) \phi_1^{-\frac{\delta}{2}}$, so it is a martingale on $[0, 1]$. Hence, R_x^δ defines a probability measure on \mathcal{F}_1 .

There remains to give a description of R_x^δ . By Girsanov's theorem, under R_x^1 , $(X_t)_{t \in [0,1]}$ solves the following SDE on $[0, 1]$

$$X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + 2 \int_0^t \frac{\phi'_s}{\phi_s} X_s ds + t. \quad (3.9)$$

But a weak solution to this SDE is provided by $(H_t^2)_{t \in [0,1]}$, where

$$H_t := \phi_t \left(x + \int_0^t \phi_s^{-1} dW_s \right),$$

where W is a standard Brownian motion. By strong and therefore weak uniqueness of solutions to equation (3.9), see [RY13, Theorem IX.3.5], we deduce that X is equal in law to the process $(H_t^2)_{t \in [0,1]}$. On the other hand, by Lévy's theorem, we have

$$(H_t)_{t \in [0,1]} \stackrel{(d)}{=} (\phi_t \gamma_{\varrho_t})_{t \in [0,1]},$$

where γ is a standard Brownian motion started at x . Hence we deduce that

$$(H_t^2)_{t \in [0,1]} \stackrel{(d)}{=} (\phi_t^2 Y_{\varrho_t})_{t \in [0,1]},$$

where $Y \stackrel{(d)}{=} Q_x^1$. Therefore, under R_x^1 , we have

$$X \stackrel{(d)}{=} (\phi_t^2 Y_{\varrho_t})_{t \in [0,1]}.$$

The claim is thus proven for $\delta = 1$ and for any $x \geq 0$. Now, by the additivity property (3.2) satisfied by $(Q_x^\delta)_{\delta, x \geq 0}$, there exist $A, B > 0$ such that, for all $x, \delta \geq 0$, and all finite Borel measure ν on $[0, 1]$, we have

$$Q_x^\delta \left[\exp \left(- \int_0^1 \phi_t^2 X_{\varrho_t} d\nu(t) \right) \right] = A^x B^\delta,$$

which can be proved exactly as Corollary 1.3 in Chapter XI of [RY13]. Note now that the family of probability laws $(R_x^\delta)_{\delta, x \geq 0}$ satisfies the same additivity property

$$\forall \delta, \delta', x, x' \geq 0, \quad R_x^\delta * R_{x'}^{\delta'} = R_{x+x'}^{\delta+\delta'}.$$

Hence, there also exist $\tilde{A}, \tilde{B} > 0$ such that, for all $x, \delta \geq 0$, and μ as above:

$$R_x^\delta \left[\exp \left(- \int_0^1 X_t d\nu(t) \right) \right] = \tilde{A}^x \tilde{B}^\delta.$$

By the previous point, evaluating at $\delta = 1$, we obtain

$$\forall x \geq 0, \quad A^x B = \tilde{A}^x \tilde{B}.$$

Hence $A = \tilde{A}$ and $B = \tilde{B}$, whence we deduce that, for all $\delta, x \geq 0$

$$Q_x^\delta \left[\exp \left(- \int_0^1 \phi_t^2 X_{\theta_t} d\nu(t) \right) \right] = R_x^\delta \left[\exp \left(- \int_0^1 X_t d\nu(t) \right) \right].$$

Since this holds for any finite measure μ on $[0, 1]$, by injectivity of the Laplace transform, we deduce that the claim equality in law holds for all values of $\delta, x \geq 0$. \square

3.2 Squared Bessel bridges and Bessel bridges

For all $\delta, x, y \geq 0$, we denote by $Q_{x,y}^\delta$ the law, on $C([0, 1])$, of the δ -dimensional squared Bessel bridge from x to y over the interval $[0, 1]$. In other words, $Q_{x,y}^\delta$ is the law of a δ -dimensional squared Bessel bridge started at x , and conditioned to hitting y at time 1. A rigorous construction of these probability laws is provided in Chap. XI.3 of [RY13] (see also [PY82] for a discussion on the particular case $\delta = y = 0$). Note that, if $y > 0$, and $0 < r < 1$, then, for $X \stackrel{(d)}{=} Q_{x,y}^\delta$, the random variable X_r admits the density $q_{x,y}^{\delta,r}$ on \mathbb{R}_+ , where

$$q_{x,y}^{\delta,r}(z) := \frac{q_r^\delta(x, z) q_{1-r}^\delta(z, y)}{q_1^\delta(x, y)}, \quad z \geq 0 \quad (3.10)$$

see Chap. XI.3 of [RY13]. In the case $x = y = 0$, $X_r \stackrel{(d)}{=} \Gamma(\frac{\delta}{2}, \frac{1}{2r(1-r)})$, so the associated density q_r^δ is given by

$$q_r^\delta(z) := \frac{z^{\delta/2-1}}{(2r(1-r))^{\delta/2} \Gamma(\delta/2)} \exp \left(- \frac{z}{2r(1-r)} \right), \quad z \geq 0, \quad (3.11)$$

In the same way as one constructs the laws of squared Bessel bridges $Q_{x,y}^\delta$, $\delta, x, y \geq 0$, one can also construct the laws of Bessel bridges. In the following, for any $\delta, a, b \geq 0$, we shall denote by $P_{a,b}^\delta$ the law, on $C([0, 1])$, of the δ -dimensional Bessel bridge from a to b over the time interval $[0, 1]$. In other words, $P_{a,b}^\delta$ is the law of a δ -dimensional Bessel process started at a and conditioned to hit b at time 1. We shall denote by $E_{a,b}^\delta$ the associated expectation operator. Note that $P_{a,b}^\delta$ is the image of Q_{a^2, b^2}^δ under the map:

$$C([0, 1]) \ni \omega \mapsto \mathbf{1}_{\omega \geq 0} \sqrt{\omega} \in C([0, 1]). \quad (3.12)$$

In particular, under $P_{0,0}^\delta$, for $r \in (0, 1)$ the canonical process X_r admits the density p_r^δ on \mathbb{R}_+ , where

$$p_r^\delta(b) = 2b q_r^\delta(b^2) = \frac{b^{\delta-1}}{2^{\frac{\delta}{2}-1} \Gamma(\frac{\delta}{2}) (r(1-r))^{\delta/2}} \exp \left(- \frac{b^2}{2r(1-r)} \right), \quad b \geq 0. \quad (3.13)$$

We finally introduce the last family of measures that we shall consider, which are further conditioned versions of the Bessel bridges.

3.3 Pinned bridges

For all $\delta, x, y, z \geq 0$ and $r \in (0, 1)$, we shall denote by $Q_{x,y}^\delta[\cdot | X_r = z]$ the law, on $C([0, 1])$, of a δ -dimensional squared Bessel bridge between x and y , pinned at z at time r . Intuitively, it corresponds to the law of a $Q_{x,y}^\delta$ bridge conditioned to hitting z at time r . Such a probability law can be constructed using the same procedure as for the construction of squared Bessel bridges.

Similarly, one can define, for all $\delta, a, b, c \geq 0$ and $r \in (0, 1)$, the law $P_{a,b}^\delta[\cdot | X_r = c]$ of a δ -dimensional Bessel bridge between a and b , pinned at c at time r . Note that this probability measure is then the image of $Q_{a^2,b^2}^\delta[\cdot | X_r = c^2]$ under the map (3.12).

With these notations at hand, we now define a family of measures which will play an important role in the IbPF for Bessel bridges. Heuristically, they should be related to the local times of the solution $(u(t, x))_{t \geq 0, x \in [0, 1]}$ to an SPDE having the law of a Bessel bridge as reversible measure.

Definition 3.4. For all $b \geq 0$ and $r \in (0, 1)$, we set

$$\Sigma_{0,0}^{\delta,r}(dX | b) := \frac{p_r^\delta(b)}{b^{\delta-1}} P_{0,0}^\delta[dX | X_r = b], \quad (3.14)$$

where p_r^δ is the probability density function of X_r under $P_{0,0}^\delta$, see (3.13).

The measure $\Sigma_{0,0}^{\delta,r}(\cdot | b)$ is meant to be the Revuz measure of the diffusion local time of $(u(t, r))_{t \geq 0}$ at level $b \geq 0$.

Remark 3.5. Note that, for all $r \in (0, 1)$, by (3.13), we have

$$\frac{p_r^\delta(b)}{b^{\delta-1}} = \frac{1}{2^{\frac{\delta}{2}-1} \Gamma(\frac{\delta}{2})(r(1-r))^{\delta/2}} \exp\left(-\frac{b^2}{2r(1-r)}\right), \quad b > 0,$$

and the right-hand side is well-defined also for $b = 0$. It is this quantity that we consider in equality (3.14) above.

In the sequel we will have to compute quantities of the form

$$\Sigma_{0,0}^{\delta,r}[\exp(-\langle m, X^2 \rangle) | b]$$

for m a finite Borel measure on $[0, 1]$. In that perspective, we introduce some further notations. Given such a m , following the notation used in [PY82] (see also Exercise (1.34), Chap. XI, of [RY13]), we denote by ψ the function on $[0, 1]$ given by

$$\psi_r := \phi_r \int_0^r \phi_u^{-2} du = \phi_r \varrho_r, \quad r \in [0, 1], \quad (3.15)$$

where ϱ is as in (3.8). Note that ψ is the unique solution on $[0, 1]$ of the Cauchy problem

$$\begin{cases} \psi''(dr) = 2\psi_r m(dr) \\ \psi_0 = 0, \quad \psi'_0 = 1. \end{cases}$$

Moreover, we denote by $\hat{\psi}$ the function on $[0, 1]$ given by

$$\hat{\psi}_r := \phi_1 \phi_r (\varrho_1 - \varrho_r) = \psi_1 \phi_r - \psi_r \phi_1, \quad r \in [0, 1]. \quad (3.16)$$

Note that $\hat{\psi}$ satisfies the following problem on $[0, 1]$

$$\begin{cases} \hat{\psi}''(dr) = 2\hat{\psi}_r m(dr) \\ \hat{\psi}_1 = 0, \quad \hat{\psi}'_1 = -1. \end{cases}$$

With these notations at hand, we obtain the following result.

Lemma 3.6. *For all $r \in (0, 1)$, $\delta > 0$ and $b \geq 0$, the following holds:*

$$\int \exp(-\langle m, X^2 \rangle) \Sigma_{0,0}^{\delta,r}(dX | b) = \frac{1}{2^{\frac{\delta}{2}-1} \Gamma(\frac{\delta}{2})} \exp\left(-\frac{b^2}{2} C_r\right) D_r^{\delta/2}, \quad (3.17)$$

where

$$C_r = \frac{\psi_1}{\psi_r \hat{\psi}_r}, \quad D_r = \frac{1}{\psi_r \hat{\psi}_r}.$$

Remark 3.7. Note that the functions ϕ , ψ and $\hat{\psi}$ take positive values on $]0, 1[$, so that the above quantities are well-defined for all $r \in]0, 1[$.

Proof. First note that by the relation (3.5), and by the expression (3.14), we have

$$\int \exp(-\langle m, X^2 \rangle) \Sigma_{0,0}^{\delta,r}(dX | b) = 2 \frac{q_r^\delta(b^2)}{b^{\delta-2}} Q_{0,0}^\delta[\exp(-\langle m, X \rangle) | X_r = b^2]. \quad (3.18)$$

To obtain the claim, it therefore suffices to compute

$$Q_{0,0}^\delta[\exp(-\langle m, X \rangle) | X_r = b^2].$$

To do so, consider two Borel functions $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We have

$$\begin{aligned} & \int_0^\infty \int_0^\infty Q_0^\delta[\exp(-\langle m, X \rangle) | X_r = x, X_1 = y] q_r^\delta(a^2, x) q_{1-r}^\delta(x, y) f(x) g(y) dx dy = \\ & = Q_0^\delta[\exp(-\langle m, X \rangle) f(X_r) g(X_1)] = \phi_1^{\delta/2} Q_{a^2}^\delta [f(\phi_r^2 X_{\varrho_r}) g(\phi_1^2 X_{\varrho_1})] \\ & = \phi_1^{\delta/2-2} \phi_r^{-2} \int_0^\infty \int_0^\infty q_{\varrho_r}^\delta\left(a^2, \frac{x}{\phi_r^2}\right) q_{\varrho_1-\varrho_r}^\delta\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right) f(x) g(y) dx dy. \end{aligned}$$

Here, we used Lemma 3.3 to obtain the third line. Since the functions f and g are arbitrary we deduce that:

$$Q_0^\delta[\exp(-\langle m, X \rangle) | X_r = x, X_1 = y] = \phi_1^{\delta/2-2} \phi_r^{-2} \frac{q_{\varrho_r}^\delta\left(0, \frac{x}{\phi_r^2}\right) q_{\varrho_1-\varrho_r}^\delta\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right)}{q_r^\delta(0, x) q_{1-r}^\delta(x, y)}$$

$dx dy$ a.e. on \mathbb{R}_+^{*2} . Since the family of measures $(Q_{x,y}^\delta)_{x,y \geq 0}$ is continuous in $(x, y) \in \mathbb{R}_+^2$ for the weak topology on probability measures (see [RY13], Section XI.3), we deduce that, for all $x \geq 0$

$$Q_{0,0}^\delta[\exp(-\langle m, X \rangle) | X_r = x] = \lim_{\substack{y \rightarrow 0 \\ y > 0}} \phi_1^{\delta/2-2} \phi_r^{-2} \frac{q_{\varrho_r}^\delta\left(0, \frac{x}{\phi_r^2}\right) q_{\varrho_1-\varrho_r}^\delta\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right)}{q_r^\delta(0, x) q_{1-r}^\delta(x, y)}.$$

But, by (3.3) and (3.4), we have

$$\frac{q_{\varrho_r}^\delta \left(0, \frac{x}{\phi_r^2}\right)}{q_r^\delta(0, x)} = \left(\frac{r}{\varrho_r}\right)^{\delta/2} \phi_r^{2-\delta} \exp\left(-\frac{x}{2} \left(\frac{1}{\phi_r^2 \varrho_r} - \frac{1}{r}\right)\right)$$

and

$$\lim_{\substack{y \rightarrow 0 \\ y > 0}} \frac{q_{\varrho_1 - \varrho_r}^\delta \left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right)}{q_{1-r}^\delta(x, y)} = \left(\frac{1-r}{\varrho_1 - \varrho_r}\right)^{\delta/2} \phi_1^{2-\delta} \exp\left(-\frac{x}{2} \left(\frac{1}{\phi_r^2(\varrho_1 - \varrho_r)} - \frac{1}{1-r}\right)\right).$$

We thus obtain

$$\begin{aligned} Q_{0,0}^\delta[\exp(-\langle m, X \rangle) | X_r = x] &= \\ &= \phi_1^{-\delta/2} \phi_r^{-\delta} \left(\frac{r(1-r)}{\varrho_r(\varrho_1 - \varrho_r)}\right)^{\delta/2} \exp\left(-\frac{x}{2} \left(\frac{\varrho_1}{\phi_r^2 \varrho_r(\varrho_1 - \varrho_r)} - \frac{1}{r(1-r)}\right)\right) \quad (3.19) \\ &= \left(\frac{r(1-r)}{\psi_r \hat{\psi}_r}\right)^{\delta/2} \exp\left(-\frac{x}{2} \left(\frac{\psi_1}{\psi_r \hat{\psi}_r} - \frac{1}{r(1-r)}\right)\right), \end{aligned}$$

where the second equality follows from the relations (3.15)-(3.16) defining ψ and $\hat{\psi}$. Applying this equality to $x = b^2$, and replacing in (3.18), we obtain the claim. \square

Remark 3.8. Along the proof of the above Proposition, for $\delta > 0$, $b \geq 0$, $r \in (0, 1)$ and m as above, we also obtained the following, useful expression

$$\begin{aligned} Q_{0,0}^\delta[\exp(-\langle m, X \rangle) | X_r = b^2] &= E_{0,0}^\delta[\exp(-\langle m, X^2 \rangle) | X_r = b] \\ &= \exp\left(-\frac{b^2}{2} \left(\frac{\psi_1}{\psi_r \hat{\psi}_r} - \frac{1}{r(1-r)}\right)\right) \left(\frac{r(1-r)}{\psi_r \hat{\psi}_r}\right)^{\delta/2}, \quad (3.20) \end{aligned}$$

which follows at once from equality (3.19).

4 Integration by parts formulae

Here and in the sequel, we denote by \mathcal{S} the linear span of all functionals on $C([0, 1])$ of the form

$$C([0, 1]) \ni X \mapsto \exp(-\langle m, X^2 \rangle) \in \mathbb{R} \quad (4.1)$$

where m is a finite Borel measure on $[0, 1]$. The elements of \mathcal{S} are the functionals for which we will derive our IbPFs wrt the laws of Bessel bridges.

4.1 The statement

To keep the formulae synthetic, for all $\delta > 0$, $r \in (0, 1)$, $b \geq 0$, and $\Phi \in \mathcal{S}$, we shall write with a slight abuse of language

$$\Sigma_{0,0}^{\delta,r}(\Phi(X) | b) := \int \Phi(X) \Sigma_{0,0}^{\delta,r}(dX | b).$$

After recalling the definition (1.6) of $\kappa(\delta) = \frac{(\delta-3)(\delta-1)}{4}$, $\delta \in \mathbb{R}$, we can state one of the main results of this article.

Theorem 4.1. Let $\delta \in (0, \infty) \setminus \{1, 3\}$, and $k := \lfloor \frac{3-\delta}{2} \rfloor \leq 1$. Then, for all $\Phi \in \mathcal{S}$

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) = \\ - \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \left[\mathcal{T}_b^{2k} \Sigma_{0,0}^{\delta,r}(\Phi(X) | \cdot) \right] db dr, \end{aligned} \quad (4.2)$$

where \mathcal{T}_x^n is the Taylor remainder defined in (2.6). On the other hand, when $\delta \in \{1, 3\}$, the following formulae hold for all $\Phi \in \mathcal{S}$

$$E_{0,0}^3(\partial_h \Phi(X)) + E_{0,0}^3(\langle h'', X \rangle \Phi(X)) = -\frac{1}{2} \int_0^1 h_r \Sigma_{0,0}^{3,r}(\Phi(X) | 0) dr, \quad (4.3)$$

$$E_{0,0}^1(\partial_h \Phi(X)) + E_{0,0}^1(\langle h'', X \rangle \Phi(X)) = \frac{1}{4} \int_0^1 h_r \frac{d^2}{db^2} \Sigma_{0,0}^{1,r}(\Phi(X) | b) \Big|_{b=0} dr. \quad (4.4)$$

Remark 4.2. Note that the last integral in (4.5) is indeed convergent. Indeed, recall that $\mathcal{T}_b^{2k} \Sigma_{0,0}^{\delta,r}(\Phi(X) | \cdot)$ is the Taylor remainder of order $2k$ at 0 of a smooth, even, function. Hence, near 0, the integrand is of order $O(b^{\delta+2k-2})$. Since, $\delta+2k-2 > -1$, the integral is convergent at 0. On the other hand, near ∞ , the integrand is of order $O(b^{\delta+2k-4})$. Since $\delta+2k-4 < -1$, integrability also holds at $+\infty$.

Remark 4.3. For all $\delta \in (1, 3)$ the right-hand side in the IbPF (4.13) takes the form

$$- \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \left[\Sigma_{0,0}^{\delta,r}(\Phi(X) | b) - \Sigma_{0,0}^{\delta,r}(\Phi(X) | 0) \right] db dr.$$

Note the absence of transition at the threshold $\delta = 2$. This might seem surprising given the transition that the Bessel bridges undergo at $\delta = 2$, which is the smallest value of δ satisfying

$$P_{0,0}^\delta [\exists r \in]0, 1[: X_r = 0] = 0.$$

This lack of transition is related to the fact that, as a consequence of Lemma 3.6, we have for all $\Phi \in \mathcal{E}$

$$\frac{d}{db} \Sigma_{0,0}^{\delta,r}(\Phi(X) | b) \Big|_{b=0} = 0.$$

Remark 4.4. In the IbPF (4.2), the last term may equivalently be written as

$$- \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{-3} \left[\mathcal{T}_b^{2k} \Sigma_{0,0}^{\delta,r}(\Phi(X) | \cdot) \right] m_\delta(db) dr \quad (4.5)$$

where m_δ is the measure on \mathbb{R}_+ defined by :

$$m_\delta(db) = \mathbf{1}_{b>0} b^{\delta-1} db.$$

Note that m_δ is a reversible measure for the δ -dimensional Bessel process. Actually, if $(X_t)_{t \geq 0}$ is a δ dimensional Bessel process, we can construct a bicontinuous family of *diffusion local times* $(\ell_t^b)_{b, t \geq 0}$, satisfying the occupation times formula

$$\int_0^t f(X_s) ds = \int_0^{+\infty} f(b) \ell_t^b m_\delta(db),$$

for all $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ bounded and Borel. We hope that such a property should hold also for $(u(t, x))_{t \geq 0}$, for all $x \in (0, 1)$ where u is the hypothetical solution of

the dynamics corresponding to $P_{0,0}^\delta$. In that case the term (4.5) should correspond, in the dynamics, to a drift in u^{-3} integrated against renormalized local times. We shall develop this idea more in detail in Section 6 below.

As a consequence of the above theorem, we retrieve the following known results, see Chapter 6 of [Zam17].

Proposition 4.5. *Let $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$. Then, for all $\delta > 3$, the following IbPF holds*

$$E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) = -\kappa(\delta) E_{0,0}^\delta(\langle h, X^{-3} \rangle \Phi(X)). \quad (4.6)$$

Moreover, for $\delta = 3$, the following IbPF holds

$$\begin{aligned} E_{0,0}^3(\partial_h \Phi(X)) + E_{0,0}^3(\langle h'', X \rangle \Phi(X)) = \\ - \int_0^1 dr \frac{h_r}{\sqrt{2\pi r^3(1-r)^3}} E_{0,0}^3[\Phi(X) | X_r = 0]. \end{aligned} \quad (4.7)$$

Proof. For $\delta > 3$ we have $k := \lfloor \frac{3-\delta}{2} \rfloor < 0$, and by (4.13)

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) = \\ = -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \Sigma_{0,0}^{\delta,r}(\Phi(X) | b) db dr \\ = -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{-3} p_r^\delta(b) E_{0,0}^\delta[\Phi(X) | X_r = b] db dr \\ = -\kappa(\delta) E_{0,0}^\delta(\langle h, X^{-3} \rangle \Phi(X)). \end{aligned}$$

For $\delta = 3$, it suffices to note that, for all $r \in (0,1)$

$$\frac{1}{2} \lim_{\epsilon \downarrow 0} \frac{p_r^3(\epsilon)}{\epsilon^2} = \frac{1}{\sqrt{2\pi r^3(1-r)^3}},$$

so that

$$\frac{1}{2} \Sigma_{0,0}^{3,r}(\Phi(X) | 0) = \frac{1}{\sqrt{2\pi r^3(1-r)^3}} E_{0,0}^3[\Phi(X) | X_r = 0],$$

and the proof is complete. \square

4.2 Proof of Theorem 4.1

We first state a differential relation satisfied by the product of the functions ψ and $\hat{\psi}$ associated as above with a finite Borel measure m on $[0,1]$. This relation is the skeleton of all the IbPFs for $P_{0,0}^\delta$, $\delta > 0$: the latter will all be deduced from the former with a simple multiplication by a constant (depending on the parameter δ).

Lemma 4.6. *Let m be a finite Borel measure on $[0,1]$, and consider the functions ψ and $\hat{\psi}$ as in (3.15) and (3.16). Then, for all $h \in C_c^2(0,1)$ and $\delta > 0$, the following equality holds*

$$\int_0^1 \sqrt{\psi_r \hat{\psi}_r} (h_r'' dr - 2h_r m(dr)) = -\frac{1}{4} \psi_1^2 \int_0^1 h_r (\psi_r \hat{\psi}_r)^{-\frac{3}{2}} dr. \quad (4.8)$$

Proof. Performing an integration by parts, we can rewrite the left-hand side as

$$\int_0^1 h_r \left(\frac{d^2}{dr^2} - 2m(dr) \right) (\psi_r \hat{\psi}_r)^{\frac{1}{2}}.$$

Note that here we are integrating wrt the signed measure

$$\left(\frac{d^2}{dr^2} - 2m(dr) \right) (\psi_r \hat{\psi}_r)^{\frac{1}{2}} = \frac{d^2}{dr^2} (\psi_r \hat{\psi}_r)^{\frac{1}{2}} - 2 (\psi_r \hat{\psi}_r)^{\frac{1}{2}} m(dr).$$

Now, we have

$$\frac{d^2}{dr^2} (\psi \hat{\psi})^{\frac{1}{2}} = \frac{1}{2} \frac{\psi'' \hat{\psi} + 2\psi' \hat{\psi}' + \psi \hat{\psi}''}{(\psi \hat{\psi})^{1/2}} - \frac{1}{4} \frac{(\psi' \hat{\psi} + \psi \hat{\psi}')^2}{(\psi \hat{\psi})^{3/2}}.$$

Recalling that $\psi'' = 2\psi m$ and $\hat{\psi}'' = 2\hat{\psi} m$, we obtain

$$\begin{aligned} \left(\frac{d^2}{dr^2} - 2m(dr) \right) (\psi \hat{\psi})^{\frac{1}{2}} &= \frac{\psi' \hat{\psi}' \psi \hat{\psi} - \frac{1}{4} (\psi' \hat{\psi} + \psi \hat{\psi}')^2}{(\psi \hat{\psi})^{3/2}} \\ &= -\frac{1}{4} \frac{(\psi' \hat{\psi} - \psi \hat{\psi}')^2}{(\psi \hat{\psi})^{3/2}}. \end{aligned}$$

Using the expressions (3.15) and (3.16) for ψ and $\hat{\psi}$, we easily see that

$$\psi'_r \hat{\psi}_r - \psi \hat{\psi}'_r = \psi_1, \quad r \in (0, 1). \quad (4.9)$$

Hence, we obtain the following equality of signed measures:

$$\left(\frac{d^2}{dr^2} - 2m \right) (\psi \hat{\psi})^{\frac{1}{2}} = -\frac{1}{4} \frac{\psi_1^2}{(\psi_r \hat{\psi}_r)^{3/2}} dr.$$

Consequently, the left-hand side in (4.8) is equal to

$$-\frac{1}{4} \psi_1^2 \int_0^1 dr h_r (\psi_r \hat{\psi}_r)^{-3/2}.$$

The claim follows. \square

As a consequence, we obtain the following preliminary result.

Lemma 4.7. *Let m be a finite measure on $[0, 1]$, and let $\Phi : C([0, 1]) \rightarrow \mathbb{R}$ be the functional thereto associated as in (4.1). Then, for all $\delta > 0$ and $h \in C_c^2(0, 1)$,*

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) &= \\ -\frac{\Gamma(\frac{\delta+1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\delta}{2})} \psi_1^{-\frac{\delta-3}{2}} \int_0^1 h_r (\psi_r \hat{\psi}_r)^{-\frac{3}{2}} dr, & \quad (4.10) \end{aligned}$$

where ψ and $\hat{\psi}$ are associated with m as in (3.15) and (3.16).

Proof. By the expression (4.1) for $\overline{\Phi}$, we have

$$\partial_h \Phi(X) = -2\langle Xh, m \rangle \Phi(X).$$

Therefore

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) &= \\ Q_{0,0}^\delta \left[\left(\langle h'', \sqrt{X} \rangle - 2\langle h\sqrt{X}, m \rangle \right) \exp(-\langle m, X \rangle) \right] &= \\ \int_0^1 (h_r'' dr - 2h_r m(dr)) \int_0^{+\infty} \Gamma\left(\frac{\delta}{2}, \frac{1}{2r(1-r)}\right) (db) \sqrt{b} Q_{0,0}^\delta[\exp(-\langle m, X \rangle) | X_r = b]. \end{aligned}$$

By (3.20) we obtain:

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) &= \\ = \int_0^1 (h_r'' dr - 2h_r m(dr)) \frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2})} \left(\frac{C_r}{2} \psi_1^\delta\right)^{-\frac{1}{2}} \int_0^{+\infty} \Gamma\left(\frac{\delta+1}{2}, \frac{C_r}{2}\right) (db) &= \\ = \sqrt{2} \frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2})} \psi_1^{-\frac{\delta+1}{2}} \int_0^1 dr (h_r'' dr - 2h_r m(dr)) \sqrt{\psi_r \hat{\psi}_r}. \end{aligned}$$

Finally, by (4.8), the latter expression is equal to

$$-\frac{\Gamma(\frac{\delta+1}{2})}{2^{\frac{3}{2}} \Gamma(\frac{\delta}{2})} \psi_1^{-\frac{\delta-3}{2}} \int_0^1 h_r (\psi_r \hat{\psi}_r)^{-\frac{3}{2}} dr$$

and the proof is complete. \square

Apart from the above lemma, the proof of the IbPF for $P_{0,0}^\delta$, $\delta > 0$, will require integral expressions for negative Gamma values. For all $x \in \mathbb{R}$ we set $\lfloor x \rfloor := \sup\{k \in \mathbb{Z} : k \leq x\}$. We also use the notation $\mathbb{Z}^- := \{n \in \mathbb{Z} : n \leq 0\}$.

Lemma 4.8. *For all $x \in \mathbb{R} \setminus \mathbb{Z}^-$*

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathcal{T}_t^{\lfloor -x \rfloor} (e^{-\cdot}) dt.$$

Proof. By Remark 2.9 we have

$$\int_0^\infty t^{x-1} \mathcal{T}_t^{\lfloor -x \rfloor} (e^{-\cdot}) dt = \Gamma(x) \langle \mu_\alpha, e^{-\cdot} \rangle = \Gamma(x) 1^x = \Gamma(x),$$

and the claim follows. \square

From Lemma 4.8 we obtain for all $C > 0$, $x \in \mathbb{R} \setminus \mathbb{Z}^-$

$$\Gamma(x) C^{-x} = 2^{1-x} \int_0^{+\infty} b^{2x-1} \left(e^{-C\frac{b^2}{2}} - \sum_{0 \leq j \leq \lfloor -x \rfloor} \frac{(-C)^j b^{2j}}{2^j j!} \right) db \quad (4.11)$$

by a simple change of variable $t = Cb^2/2$. Then (4.11) can be rewritten as follows

$$\Gamma(x) C^{-x} = 2^{1-x} \int_0^{+\infty} b^{2x-1} \mathcal{T}_b^{2\lfloor -x \rfloor} \left(e^{-C\frac{(\cdot)^2}{2}} \right) db, \quad x \in \mathbb{R} \setminus \mathbb{Z}^-. \quad (4.12)$$

Theorem 4.9. Let $\delta > 0$, $\delta \notin \{1, 3\}$, and $k := \lfloor \frac{3-\delta}{2} \rfloor \leq 1$. Then

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) &= \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \left[\mathcal{T}_b^{2k} \Sigma_{0,0}^{\delta,r}(\Phi(X) | \cdot) \right] db dr. \end{aligned} \quad (4.13)$$

Proof of Theorem 4.9. Let $\delta > 0$ and $\delta \notin \{1, 3\}$. Then by (4.10)

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) &= \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2} \Gamma(\frac{\delta}{2})} \int_0^1 h_r \left(\frac{\psi_1}{\psi_r \hat{\psi}_r} \right)^{\frac{3-\delta}{2}} (\psi_r \hat{\psi}_r)^{-\frac{\delta}{2}} dr \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2} \Gamma(\frac{\delta}{2})} \int_0^1 h_r C_r^{\frac{3-\delta}{2}} D_r^{\delta/2} dr \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2} \Gamma(\frac{\delta}{2})} \frac{2^{\frac{5-\delta}{2}}}{\Gamma(\frac{\delta-3}{2})} \int_0^1 h_r D_r^{\delta/2} \int_0^\infty b^{\delta-4} \mathcal{T}_b^{2k} e^{-\frac{C_r}{2}(\cdot)^2} db dr, \end{aligned}$$

where we used (4.12) with $C = C_r$ and $x = \frac{\delta-3}{2}$ to obtain the last line. Recalling the expression (3.17) for $\Sigma_{0,0}^{\delta,r}(\Phi(X) | b)$, we thus obtain

$$\begin{aligned} E_{0,0}^\delta(\partial_h \Phi(X)) + E_{0,0}^\delta(\langle h'', X \rangle \Phi(X)) &= \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta-3}{2})} \int_0^1 h_r \int_0^\infty b^{\delta-4} \mathcal{T}_{2b}^k \Sigma_{0,0}^{\delta,r}(\Phi(X) | b) db dr. \end{aligned}$$

Now, since $\delta \notin \{1, 3\}$,

$$\Gamma(\frac{\delta+1}{2}) = \frac{\delta-1}{2} \Gamma(\frac{\delta-1}{2}) = \frac{\delta-1}{2} \frac{\delta-3}{2} \Gamma(\frac{\delta-3}{2}) = \kappa(\delta) \Gamma(\frac{\delta-3}{2}).$$

Therefore $\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta-3}{2})} = \kappa(\delta)$ and we obtain the claim. \square

There remains to treat the critical cases $\delta \in \{1, 3\}$.

Theorem 4.10. Let $\Phi \in \mathcal{S}$ and $h \in C_c^2(0, 1)$. The following IbPFs holds

$$\begin{aligned} E_{0,0}^3(\partial_h \Phi(X)) + E_{0,0}^3(\langle h'', X \rangle \Phi(X)) &= -\frac{1}{2} \int_0^1 dr h_r \Sigma_{0,0}^{3,r}(\Phi(X) | 0), \\ E_{0,0}^1(\partial_h \Phi(X)) + E_{0,0}^1(\langle h'', X \rangle \Phi(X)) &= \frac{1}{4} \int_0^1 dr h_r \frac{d^2}{db^2} \Sigma_{0,0}^{1,r}(\Phi(X) | b) \Big|_{b=0}. \end{aligned} \quad (4.14)$$

Proof. By linearity, we may assume that Φ is of the form (4.1). For $\delta = 3$ we have by (4.10)

$$E_{0,0}^3(\partial_h \Phi(X)) + E_{0,0}^3(\langle h'', X \rangle \Phi(X)) = -\frac{1}{2^{\frac{3}{2}} \Gamma(\frac{3}{2})} \int_0^1 h_r (\psi_r \hat{\psi}_r)^{-\frac{3}{2}} dr.$$

By (3.17) this equals

$$-\frac{1}{2} \int_0^1 dr h_r \Sigma_{0,0}^{3,r}(\Phi(X) | 0)$$

and the proof is complete. For $\delta = 1$, by (4.10), we have

$$E_{0,0}^1(\partial_h \Phi(X) + \langle h'', X \rangle \Phi(X)) = -\frac{1}{2\sqrt{2\pi}} \psi_1 \int_0^1 h_r (\psi_r \hat{\psi}_r)^{-\frac{3}{2}} dr.$$

But by (3.17) we have, for all $r \in (0, 1)$

$$\left. \frac{d^2}{db^2} \Sigma_{0,0}^{1,r}(\Phi(X) | b) \right|_{b=0} = -\frac{C_r D_r^{1/2}}{2^{-\frac{1}{2}} \Gamma(\frac{1}{2})} = -\sqrt{\frac{2}{\pi}} \psi_1 (\psi_r \hat{\psi}_r)^{-\frac{3}{2}}.$$

The claimed IbPF follows. \square

Remark 4.11. In [Zam05] for the reflecting Brownian motion, and then in [GV16] for the Reflecting Brownian bridge, a different formula was proved in the case $\delta = 1$. In our present notations, for $(\beta_r)_{r \in [0,1]}$ a Brownian bridge and $X := |\beta|$, the formula reads

$$\mathbb{E}(\partial_h \Phi(X)) + \mathbb{E}(\langle h'', X \rangle \Phi(X)) = \lim_{\epsilon \rightarrow 0} 2 \mathbb{E} \left(\Phi(X) \int_0^1 h_r \left[(\dot{\beta}_r^\epsilon)^2 - c_r^\epsilon \right] dL_r^0 \right), \quad (4.15)$$

where $\Phi : H \rightarrow \mathbb{R}$ is any Lipschitz function, $h \in C_0^2(0, 1)$, L^0 is the standard local time of β at 0 and for some even smooth mollifier ρ_ϵ we set

$$\beta^\epsilon := \rho_\epsilon * \beta, \quad c_r^\epsilon := \frac{\|\rho\|_{L^2(0,1)}^2}{\epsilon}.$$

The reason why (4.15) is strictly weaker than (4.14), is that the former depends explicitly on β , while the latter is written only in terms of X . This will become crucial when we compute the SPDE satisfied by u for $\delta = 1$ in Theorem 5.8 below.

5 The dynamics via Dirichlet forms

In this section we exploit the IbPF obtained above to construct a weak version of the gradient dynamics associated with $P_{0,0}^1$, using the theory of Dirichlet forms. The reason for considering the particular value $\delta = 1$ is that we can exploit a representation of the Bessel bridge in terms of a Brownian bridge, for which the corresponding gradient dynamics is well-known and corresponds to a linear stochastic heat equation. Moreover the value $\delta = 1$ is of particular interest with regard to scaling limits of dynamical critical pinning models (see e.g. [Voß16] and [DO18]).

For the sake of our analysis, instead of working on the Banach space $C([0, 1])$, it shall actually be more convenient to work on the Hilbert space $H := L^2(0, 1)$ endowed with the L^2 inner product:

$$\langle f, g \rangle = \int_0^1 f_t g_t dt, \quad f, g \in H.$$

We shall denote by $\|\cdot\|$ the corresponding norm on H . Moreover we denote by μ the law of β on H , where β is a Brownian bridge from 0 to 0 over the interval $[0, 1]$. We shall use the shorthand notation $L^2(\mu)$ for the space $L^2(H, \mu)$. We also consider the closed subset $K \subset H$ of nonnegative functions:

$$K := \{z \in H, z \geq 0 \text{ a.e.}\}.$$

Note that K is a Polish space. We further denote by ν the law, on K , of the 1-Bessel bridge from 0 to 0 on $[0, 1]$ (so that $P_{0,0}^1$ is then the restriction of ν to $C([0, 1])$). We shall use the shorthand $L^2(\nu)$ to denote the space $L^2(K, \nu)$. Denoting by $j : H \rightarrow K$ the absolute value map:

$$j(z) := |z|, \quad z \in H, \quad (5.1)$$

we remark that the map:

$$\begin{cases} L^2(\nu) \rightarrow L^2(\mu) \\ \varphi \mapsto \varphi \circ j. \end{cases}$$

is an isometry.

5.1 The one-dimensional random string

Consider the Ornstein-Uhlenbeck semigroup $(\mathbf{Q}_t)_{t \geq 0}$ on H defined, for all $F \in L^2(\mu)$ and $z \in H$, by

$$\mathbf{Q}_t F(z) := \mathbb{E}[F(v_t(z))], \quad t \geq 0,$$

where $(v_t(z))_{t \geq 0}$ is the solution to the stochastic heat equation on $[0, 1]$ with initial condition z , and with homogeneous Dirichlet boundary conditions

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \xi \\ v(0, x) = z(x), & x \in [0, 1] \\ v(t, 0) = v(t, 1) = 0, & t > 0 \end{cases} \quad (5.2)$$

with ξ a space-time white noise on $\mathbb{R}_+ \times [0, 1]$. Recall that v can be written explicitly in terms of the fundamental solution $(g_t(x, x'))_{t \geq 0, x, x' \in (0, 1)}$ of the stochastic heat equation with homogeneous Dirichlet boundary conditions on $[0, 1]$, which by definition is the unique solution to

$$\begin{cases} \frac{\partial g}{\partial t} = \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \\ g_0(x, x') = \delta_x(x') \\ g_t(x, 0) = g_t(x, 1) = 0. \end{cases}$$

Recall further that g can be represented as follows:

$$\forall t > 0, \quad \forall x, x' \geq 0, \quad g_t(x, x') = \sum_{k=1}^{\infty} e^{-\frac{\lambda_k}{2} t} e_k(x) e_k(x'), \quad (5.3)$$

where $(e_k)_{k \geq 1}$ is the complete orthonormal system of H given by

$$e_k(x) := \sqrt{2} \sin(k\pi x), \quad x \in [0, 1], \quad k \geq 1$$

and $\lambda_k := k^2 \pi^2$, $k \geq 1$. We can then represent u as follows:

$$v(t, x) = z(t, x) + \int_0^t \int_0^1 g_{t-s}(x, x') \xi(ds, dx'),$$

where $z(t, x) := \int_0^1 g_t(x, x') z(x') dx'$, and the double integral is a stochastic convolution. In particular, it follows from this formula that v is a Gaussian process. An

important role will be played by its covariance function. Namely, for all $t \geq 0$ and $x, x' \in (0, 1)$, we set

$$q_t(x, x') := \text{Cov}(v(t, x), v(t, x')) = \int_0^t g_{2\tau}(x, x') d\tau.$$

We also set

$$q_\infty(x, x') := \int_0^\infty g_{2\tau}(x, x') d\tau = \mathbb{E}[\beta_x \beta_{x'}] = x \wedge x' - xx'.$$

For all $t \geq 0$, we set moreover

$$q^t(x, x') := q_\infty(x, x') - q_t(x, x') = \int_t^\infty g_{2\tau}(x, x') d\tau.$$

When $x = x'$, we will use the shorthand notations $q_t(x)$, $q_\infty(x)$ and $q^t(x)$ instead of $q_t(x, x)$, $q_\infty(x, x)$ and $q^t(x, x)$ respectively. Finally, we denote by $(\Lambda, D(\Lambda))$ the Dirichlet form associated with $(\mathbf{Q}_t)_{t \geq 0}$ in $L^2(H, \mu)$, and which is given by

$$\Lambda(F, G) = \frac{1}{2} \int_H \langle \nabla F, \nabla G \rangle d\mu, \quad F, G \in D(\Lambda) = W^{1,2}(\mu).$$

Here, for all $F \in W^{1,2}(\mu)$, $\nabla F : H \rightarrow H$ is the gradient of F , see [DPZ02]. The corresponding family of resolvents $(\mathbf{R}_\lambda)_{\lambda > 0}$ is then given by:

$$\mathbf{R}_\lambda F(z) = \int_0^\infty e^{-\lambda t} \mathbf{Q}_t F(z) dt, \quad z \in H, \lambda > 0,$$

for all $F \in L^2(\mu)$.

5.2 Gradient Dirichlet form associated with the reflected Brownian bridge

In this section we construct the Dirichlet form associated with our equation (1.11), and the associated Markov process $(u_t)_{t \geq 0}$.

Let $\mathcal{FC}_b^\infty(H)$ denote the space of all functionals $F : H \rightarrow \mathbb{R}$ of the form

$$F(z) = \psi(\langle l_1, z \rangle, \dots, \langle l_m, z \rangle), \quad z \in H,$$

with $m \in \mathbb{N}$, $\psi \in C_b^\infty(\mathbb{R}^m)$, and $l_1, \dots, l_m \in \text{Span}\{e_k, k \geq 1\}$. Recalling that $K := \{z \in H, z \geq 0\}$, we also define:

$$\mathcal{FC}_b^\infty(K) := \{F|_K, \quad F \in \mathcal{FC}_b^\infty(H)\}.$$

Moreover, for $f \in \mathcal{FC}_b^\infty(K)$ of the form $f = F|_K$, with $F \in \mathcal{FC}_b^\infty(H)$, we define $\nabla f : K \rightarrow H$ by

$$\nabla f(z) = \nabla F(z), \quad z \in K,$$

where this definition does not depend on the choice of F . We denote by \mathcal{E} the bilinear form defined on $\mathcal{FC}_b^\infty(K)$ by

$$\mathcal{E}(f, g) := \frac{1}{2} \int \langle \nabla f, \nabla g \rangle d\nu, \quad f, g \in \mathcal{FC}_b^\infty(K),$$

Proposition 5.1. *The form $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ is closable. Its closure $(\mathcal{E}, D(\mathcal{E}))$ is a local, quasi-regular Dirichlet form on $L^2(\nu)$. Moreover, for all $f \in D(\mathcal{E})$, $f \circ j \in D(\Lambda)$, and we have:*

$$\forall f, g \in D(\mathcal{E}), \quad \mathcal{E}(f, g) = \Lambda(f \circ j, g \circ j) \quad (5.4)$$

Proof. We follow arguments communicated to us by Rongchan Zhu and Xiangchan Zhu. Since $D(\Lambda)$ contains all globally Lipschitz functions on H , for all $f \in \mathcal{FC}_b^\infty(K)$ we have $f \circ j \in D(\Lambda)$. Moreover, writing f as

$$f(z) = \psi(\langle l_1, z \rangle, \dots, \langle l_m, z \rangle), \quad z \in K,$$

with $m \in \mathbb{N}$, $\psi \in C_b^\infty(\mathbb{R}^m)$, and $l_1, \dots, l_m \in \text{Span}\{e_k, k \geq 1\}$, we have, for μ a.e. $z \in H$

$$\nabla(f \circ j)(z) = \sum_{i=1}^m \partial_i \psi(\langle l_1, j(z) \rangle, \dots, \langle l_m, j(z) \rangle) l_i \text{sgn}(z),$$

that is

$$\nabla(f \circ j)(z) = \nabla f(j(z)) \text{sgn}(z). \quad (5.5)$$

Hence, for all $f, g \in \mathcal{FC}_b^\infty(K)$, we have

$$\begin{aligned} \mathcal{E}(f, g) &= \frac{1}{2} \int \langle \nabla f(x), \nabla g(x) \rangle d\nu(x) = \frac{1}{2} \int \langle \nabla f(j(z)), \nabla g(j(z)) \rangle d\mu(z) \\ &= \frac{1}{2} \int \langle \nabla(f \circ j)(z), \nabla(g \circ j)(z) \rangle d\mu(z) = \Lambda(f \circ j, g \circ j), \end{aligned}$$

where the third equality follows from (5.5). This shows that the bilinear symmetric form $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ admits as an extension the image of the Dirichlet form $(\Lambda, D(\Lambda))$ under the map j . Since $\mathcal{FC}_b^\infty(K)$ is dense in $L^2(\nu)$, this extension is a Dirichlet form. In particular, $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ is closable, its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form, and we have the isometry property (5.4).

There remains to prove that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Since it is the closure of $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$, it suffices to show that the associated capacity is tight. Since K is separable, we can find a countable dense subset $\{y_k, k \in \mathbb{N}\} \subset K$ such that $y_k \neq 0$ for all $k \in \mathbb{N}$.

Let now $\varphi \in C_b^\infty(\mathbb{R})$ be an increasing function such that $\varphi(t) = t$ for all $t \in [-1, 1]$ and $\|\varphi'\|_\infty \leq 1$. For all $m \in \mathbb{N}$, we define the function $v_m : K \rightarrow \mathbb{R}$ by

$$v_m(z) := \varphi(\|z - y_m\|), \quad z \in K.$$

Moreover, we set, for all $n \in \mathbb{N}$

$$w_n(z) := \inf_{m \leq n} v_m(z), \quad z \in K.$$

We claim that $w_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, and that $w_n \xrightarrow{n \rightarrow \infty} 0$, \mathcal{E} quasi-uniformly in K . Assuming this claim for the moment, for all $k \geq 1$ we can find a closed subset F_k of K such that $\text{Cap}(K \setminus F_k) < 1/k$, and $w_n \xrightarrow{n \rightarrow \infty} 0$ uniformly on F_k . Hence, for all $\epsilon > 0$, we can find $n \in \mathbb{N}$ such that $w_n < \epsilon$ on F_k . Therefore:

$$F_k \subset \bigcup_{m \leq n} B(y_m, \epsilon)$$

where, for all $y \in K$ and $r > 0$

$$B(y, r) := \{y' \in K, \quad \|y' - y\| < r\}.$$

This shows that F_k is totally bounded. Since it is, moreover, complete as a closed subspace of a complete metric space, it is compact, and the tightness of Cap follows.

We now justify our claim. For all i such that $y_i \neq 0$, we set $l_i = \|y_i\|^{-1} y_i$. Then for all $i \geq 1$, $l_i \in K$, $\|l_i\| = 1$ and, for all $z \in K$:

$$\|z\| = \sup_{i \geq 1} \langle l_i, z \rangle.$$

Let $m \in \mathbb{N}$ be fixed. For all $i \geq 1$, let:

$$u_i(z) := \sup_{j \leq i} \varphi(\langle l_j, z - y_m \rangle), \quad z \in K.$$

We have $u_i \in D(\mathcal{E})$, and, for ν - a.e. $z \in K$

$$\sum_{k=1}^{\infty} \frac{\partial u_i}{\partial e_k}(z)^2 \leq \sup_{j \leq i} \left(\sum_{k=1}^{\infty} \varphi'(\langle l_j, z - y_m \rangle)^2 \langle l_j, e_k \rangle^2 \right) \leq 1,$$

whence

$$\mathcal{E}(u_i, u_i) = \int_K \sum_{k=1}^{\infty} \left(\frac{\partial u_i}{\partial e_k} \right)^2 d\nu \leq 1.$$

By the definition of v_m , as $i \rightarrow \infty$, $u_i \uparrow v_m$ on K , hence in $L^2(K, \nu)$. By [MR92, I.2.12], we deduce that $v_m \in D(\mathcal{E})$, and that

$$\mathcal{E}(v_m, v_m) \leq 1.$$

Therefore, for all $n \in \mathbb{N}$, $w_n \in D(\mathcal{E})$, and

$$\mathcal{E}(w_n, w_n) \leq 1.$$

But, since $\{y_k, k \in \mathbb{N}\}$ is dense in K , as $n \rightarrow \infty$, $w_n \downarrow 0$ on K . Hence $w_n \xrightarrow[n \rightarrow \infty]{} 0$ in $L^2(K, \nu)$. This and the previous bound imply, by [MR92, I.2.12], that the Cesàro means of some subsequence of $(w_n)_{n \geq 0}$ converge to 0 in $D(\mathcal{E})$. By [MR92, III.3.5], some subsequence thereof converges \mathcal{E} quasi-uniformly to 0. But, since $(w_n)_{n \geq 0}$ is non-increasing, we deduce that it converges \mathcal{E} quasi-uniformly to 0. The claimed quasi-regularity follows. There finally remains to check that $(\mathcal{E}, D(\mathcal{E}))$ is local in the sense of Definition [MR92, V.1.1]. Let $u, v \in D(\mathcal{E})$ satisfying $\text{supp}(u) \cap \text{supp}(v) = \emptyset$. Then, $u \circ j$ and $v \circ j$ are two elements of $D(\Lambda) = W^{1,2}(\mu)$ with disjoint supports, and, recalling (5.4), we have :

$$\mathcal{E}(u, v) = \Lambda(u \circ j, v \circ j) = \frac{1}{2} \int_H \nabla(u \circ j) \cdot \nabla(v \circ j) d\mu = 0.$$

The claim follows. \square

Let $(Q_t)_{t \geq 0}$ be the contraction semigroup on $L^2(K, \nu)$ associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, and let $(R_\lambda)_{\lambda > 0}$ be the associated family of resolvents. Let also $\mathcal{B}_b(K)$ denote the set of Borel and bounded functions on K . As a consequence of Prop. 5.1, in virtue of Thm IV.3.5 and Thm V.1.5 in [MR92], we obtain the following:

Corollary 5.2. *There exists a Markov diffusion process $M = \{\Omega, \mathcal{F}, (u_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in K}\}$ properly associated to $(\mathcal{E}, D(\mathcal{E}))$, i.e. for all $\varphi \in L^2(\nu) \cap \mathcal{B}_b(K)$, and for all $t > 0$, $E_x(\varphi(u_t))$, $x \in K$, defines an \mathcal{E} quasi-continuous version of $Q_t\varphi$. Moreover, the process M admits the following continuity property:*

$$\mathbb{P}_x[t \mapsto u_t \text{ is continuous on } \mathbb{R}_+] = 1, \quad \text{for } \mathcal{E}\text{-q.e. } x \in K.$$

The rest of this section will be devoted to show that for \mathcal{E} q.e. $x \in K$, under \mathbb{P}_x , $(u_t)_{t \geq 0}$ solves (1.11), or rather its weaker form (1.13).

In the sequel, we set $\Lambda_1 := \Lambda + (\cdot, \cdot)_{L^2(\mu)}$ and $\mathcal{E}_1 := \mathcal{E} + (\cdot, \cdot)_{L^2(\nu)}$, which are inner products for the Hilbert spaces $D(\Lambda)$ and $D(\mathcal{E})$ respectively. We shall also write in an abusive way, for any $\Phi \in C^1(H)$:

$$\mathcal{E}_1(\Phi, \Phi) := \int_K (\Phi(x))^2 d\nu(x) + \frac{1}{2} \int_K \|\nabla \Phi(x)\|_H^2 d\nu(x) \in [0, +\infty].$$

Since the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, by the transfer method stated in VI.2 of [MR92], we can apply several results of [FOT10] in our setting.

An important technical point is the density of the space \mathcal{S} introduced in Section 4 above in the Dirichlet space $D(\mathcal{E})$. To state this precisely, we consider \mathcal{S} to be the vector space generated by functionals $F : H \rightarrow \mathbb{R}$ of the form:

$$F(\zeta) = \exp(-\langle \theta, \zeta^2 \rangle), \quad \zeta \in H,$$

for some $\theta : [0, 1] \rightarrow \mathbb{R}_+$ Borel and bounded. Note that \mathcal{S} may be seen as a subspace of the space \mathcal{S} of Section 4 in the following sense: for any $F \in \mathcal{S}$, $F|_{C([0,1])} \in \mathcal{S}$. We also set:

$$\mathcal{S}_K := \{F|_K, \quad F \in \mathcal{S}\}.$$

Lemma 5.3. *\mathcal{S}_K is dense in $D(\mathcal{E})$.*

Proof. Recall that $D(\mathcal{E})$ is the closure under the bilinear form \mathcal{E}_1 of the space $\mathcal{FC}_b^\infty(K)$ of functionals of the form $F = \Phi|_K$, where $\Phi \in \mathcal{FC}_b^\infty(H)$. Therefore, to prove the claim, it suffices to show that for any functional $\Phi \in \mathcal{FC}_b^\infty(H)$ and all $\epsilon > 0$, there exists $\Psi \in \mathcal{S}$ such that $\mathcal{E}_1(\Phi - \Psi, \Phi - \Psi) < \epsilon$.

We proceed in two steps. In a first step, we prove that a functional Φ as before can be approximated in \mathcal{E}_1 norm by a sequence $(\Phi_\epsilon^d)_{\epsilon > 0, d \geq 1}$ of functionals $\Phi_\epsilon^d : H \rightarrow \mathbb{R}$ of the form:

$$\Phi_\epsilon^d(\zeta) = f_\epsilon^d(\langle \zeta_{d,1}, \zeta^2 \rangle, \dots, \langle \zeta_{d,d}, \zeta^2 \rangle), \quad \zeta \in H, \quad (5.6)$$

where $(\zeta_{d,i})_{i=1, \dots, d}$ is a family of elements of K , and $f_\epsilon^d \in C_b^1(\mathbb{R}_+^d)$. Then, in a second step, invoking a differentiable type of Stone-Weierstrass argument, we will show that the functions f_ϵ^d can be approximated on \mathbb{R}_+^d by linear combinations of functions of the form $e^{-\lambda \cdot}$, $\lambda \in \mathbb{R}_+^d$. This will yield the claim.

We proceed to the first step. Given $\Phi \in \mathcal{FC}_b^\infty(H)$, we set, for all $\epsilon > 0$:

$$\Phi_\epsilon(\zeta) := \Phi(\sqrt{\zeta^2 + \epsilon}), \quad \zeta \in H.$$

Then $\Phi_\epsilon \xrightarrow{\epsilon \rightarrow 0} \Phi$ pointwise, and $\|\Phi_\epsilon\|_\infty \leq \|\Phi\|_\infty$. Furthermore Φ_ϵ is C^1 and :

$$\nabla \Phi_\epsilon(\zeta) = \nabla \Phi \left(\sqrt{\zeta^2 + \epsilon} \right) \frac{\zeta}{\sqrt{\zeta^2 + \epsilon}},$$

so that $\nabla\Phi_\epsilon \xrightarrow{\epsilon \rightarrow 0} \nabla\Phi$ pointwise with the domination $\|\nabla\Phi_\epsilon\|_\infty \leq \|\nabla\Phi\|_\infty$. Hence, by dominated convergence, $\mathcal{E}_1(\Phi_\epsilon - \Phi, \Phi_\epsilon - \Phi) \xrightarrow{\epsilon \rightarrow 0} 0$. Then, introducing for all $d \geq 1$ $(\zeta_i^d)_{1 \leq i \leq d}$ the orthonormal family in $L^2(0, 1)$ given by:

$$\zeta_i^d := \sqrt{d} \mathbf{1}_{[\frac{i-1}{d}, \frac{i}{d}]}, \quad i = 1, \dots, d, \quad (5.7)$$

and setting

$$\Phi_\epsilon^d(\zeta) := \Phi_\epsilon \left(\left(\sum_{i=1}^d \langle \zeta_{d,i}, \zeta^2 \rangle \right)^{1/2} \right) = \Phi \left(\left(\sum_{i=1}^d \langle \zeta_{d,i}, \zeta^2 \rangle + \epsilon \right)^{1/2} \right), \quad \zeta \in H,$$

it follows that Φ_ϵ^d is indeed of the form (5.6), with:

$$f_\epsilon^d(x) := \Phi \left(\left(\sum_{i=1}^d x_i + \epsilon \right)^{1/2} \right), \quad x \in \mathbb{R}_+^d.$$

Note that we indeed have $f_\epsilon^d \in C_b^1(\mathbb{R}_+^d)$ due to the regularity of Φ , and to the fact that the function $x \rightarrow \sqrt{x + \epsilon}$ is in $C_b^1(\mathbb{R}_+)$. Moreover, the convergence $\mathcal{E}_1(\Phi_\epsilon^d - \Phi_\epsilon, \Phi_\epsilon^d - \Phi_\epsilon) \xrightarrow{d \rightarrow \infty} 0$ holds, again by dominated convergence. This concludes the first step.

In the second step, there remains to show that any fixed functional of the form :

$$\Phi(\zeta) = f(\langle \zeta_{d,1}, \zeta^2 \rangle, \dots, \langle \zeta_{d,d}, \zeta^2 \rangle), \quad \zeta \in H$$

with $d \geq 1$, $f \in C_b^1(\mathbb{R}_+^d)$, and $(\zeta_i^d)_{i=1, \dots, d}$ as above can be approximated by elements of \mathcal{S} . To do so, let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with values in $[0, 1]$, such that $\chi = 1$ on $(-\infty, -1]$ and $\chi = 0$ on $[0, +\infty)$. Set $\chi_n(\cdot) := \chi(\cdot - n)$ and let :

$$f_n(x) := f(x) \prod_{i=1}^d \chi_n(x_i), \quad x \in \mathbb{R}_+^d.$$

Note that f_n is in $C_b^1(\mathbb{R}_+^d)$ and is moreover compactly supported in $[0, n]^d$. Then, setting:

$$\Phi_n(\zeta) = f_n(\langle \zeta_{d,1}, \zeta^2 \rangle, \dots, \langle \zeta_{d,d}, \zeta^2 \rangle), \quad \zeta \in H,$$

we have $\mathcal{E}_1(\Phi - \Phi_n, \Phi - \Phi_n) \xrightarrow{n \rightarrow \infty} 0$ by dominated convergence. Therefore, there remains to show that, for any n fixed, Φ_n can be approximated by elements of \mathcal{S} . First consider $g_n \in C_b^1([0, 1]^d)$, the map defined by:

$$g_n(y) := f_n(-\ln(y_1), \dots, -\ln(y_d)), \quad y \in [0, 1]^d.$$

By convention, if there is some i such that $y_i = 0$, we set $g_n(y) = 0$. Note that, since f_n has compact support in $[0, n]^d$, g_n has compact support in $[1/n, 1]^d$, and is C^1 . By a differentiable version of the Weierstrass Approximation Theorem (see Theorem 1.1.2 in [Lla86]), there exists a sequence $(p_{n,k})_{k \geq 1}$ of polynomial functions converging to g_n for the C^1 topology on $[0, 1]^d$. Defining for all $k \geq 1$ the function $f_{n,k} : \mathbb{R}_+^d \rightarrow \mathbb{R}$ by:

$$f_{n,k}(x) = p_{n,k}(e^{-x_1}, \dots, e^{-x_d}), \quad x \in \mathbb{R}_+^d,$$

we define $\Phi_{n,k} \in \mathcal{S}$ by:

$$\Phi_{n,k}(\zeta) = f_{n,k}(\langle \zeta_{d,1}, \zeta^2 \rangle, \dots, \langle \zeta_{d,d}, \zeta^2 \rangle), \quad \zeta \in H.$$

Since $p_{n,k} \xrightarrow[k \rightarrow \infty]{} g_n$ for the C^1 topology on $[0, 1]^d$, $f_{n,k} \xrightarrow[k \rightarrow \infty]{} f_n$ uniformly on \mathbb{R}_+^d together with its first order derivatives. Hence, it follows that $\Phi_{n,k} \xrightarrow[k \rightarrow \infty]{} \Phi_n$ pointwise on K together with its gradient. It also follows that there is some $C_n > 0$ (that may depend on n) such that for all $k \geq 1$:

$$\forall \zeta \in K, \quad |\Phi_{n,k}(\zeta)|^2 + \|\nabla \Phi_{n,k}(\zeta)\|^2 \leq C_n(1 + \|\zeta\|^2).$$

Since the quantity in the right-hand side is ν integrable in ζ , it follows by dominated convergence that $\mathcal{E}_1(\Phi_{n,k} - \Phi_n, \Phi_{n,k} - \Phi_n) \xrightarrow[k \rightarrow \infty]{} 0$. This yields the claim. \square

5.3 Convergence of one-potentials

Let ρ be a smooth function supported on $[-1, 1]$ such that

$$\rho \geq 0, \quad \int_{-1}^1 \rho = 1, \quad \rho(y) = \rho(-y), \quad y \in \mathbb{R}.$$

For all $\epsilon > 0$, let:

$$\rho_\epsilon(y) := \frac{1}{\epsilon} \rho\left(\frac{y}{\epsilon}\right), \quad y \in \mathbb{R}. \quad (5.8)$$

Then, for all $\Phi \in \mathcal{S}$ and $h \in C_c^2(0, 1)$, the right-hand side of the IbPF (4.14) can be rewritten as follows:

$$\frac{1}{4} \int_0^1 h_r \frac{d^2}{db^2} \Sigma_{0,0}^{1,r}(\Phi(X) | b) \Big|_{b=0} dr = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\Phi(|\beta|) \int_0^1 h_r \rho_\epsilon''(\beta_r) dr \right]. \quad (5.9)$$

Indeed, starting from the right-hand side, by conditioning on the value of $|\beta_r|$, and recalling that $|\beta| \stackrel{(d)}{=} \nu$, the equality follows at once.

Henceforth, we fix a function $h \in C_c^2(0, 1)$. Then there exists $\delta \in (0, 1)$ such that h is supported in $[\delta, 1 - \delta]$. For all $\epsilon > 0$, let G_ϵ denote the functional defined on H by

$$G_\epsilon(z) := \frac{1}{2} \int_0^1 h_r \rho_\epsilon''(z_r) dr, \quad z \in H. \quad (5.10)$$

For all $t > 0$ and $z \in H$, we have:

$$\mathbf{Q}_t G_\epsilon(z) = \int_0^1 \frac{h_r}{2\sqrt{2\pi q_t(r)}} \int_{\mathbb{R}} \rho_\epsilon''(b) \exp\left(-\frac{(b - z(t, r))^2}{2q_t(r)}\right) db dr,$$

where, for all $x \in (0, 1)$, $z(t, x) := \int_0^1 g_t(x, x') z(x') dx'$. For all $\epsilon > 0$, we define the functional $U_\epsilon : H \rightarrow \mathbb{R}$ by

$$U_\epsilon(z) = \int_0^\infty e^{-t} \mathbf{Q}_t G_\epsilon(z) dt, \quad z \in H.$$

Note that U_ϵ is the one-potential of the additive functional

$$\int_0^t G_\epsilon(v(s, \cdot)) ds, \quad t \geq 0,$$

associated with the Markov process $(v(t, \cdot))_{t \geq 0}$ in H defined in (5.2) (see Section 5 of [FOT10] for this terminology). In particular, $U_\epsilon \in D(\Lambda)$. We will show that U_ϵ converges in $D(\Lambda)$ as $\epsilon \rightarrow 0$.

Proposition 5.4. *As $\epsilon \downarrow 0$, $U_\epsilon \rightarrow U$ in $D(\Lambda)$, where for all $z \in H$*

$$U(z) := \int_0^\infty dt e^{-t} \int_0^1 dr \frac{h_r}{2\sqrt{2\pi q_t(r)^3}} \left[\frac{z(t, r)^2}{q_t(r)} - 1 \right] \exp\left(-\frac{z(t, r)^2}{2q_t(r)}\right). \quad (5.11)$$

Proof. First note that U is well-defined. Indeed, the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi(y) = (y^2 - 1)e^{-y^2/2}, \quad y \in \mathbb{R}$$

is globally bounded on \mathbb{R} . Note also that, by the bound (22) in [Zam04], there exists $c_\delta > 0$ such that :

$$\forall t > 0, \quad \forall r \in [\delta, 1 - \delta], \quad q_t(r) \geq c_\delta \sqrt{t} \quad (5.12)$$

Hence, the integrand in (5.11) is bounded by

$$e^{-t} \frac{|h_r|}{2\sqrt{2\pi q_t(r)^3}} \|\varphi\|_{L^\infty(\mathbb{R})} \leq e^{-t} t^{-3/4} \frac{|h_r|}{2\sqrt{2\pi c_\delta^3}} \|\varphi\|_{L^\infty(\mathbb{R})},$$

which is integrable on $\mathbb{R}_+ \times [0, 1]$. Note now that $U \in L^2(\mu)$. Indeed, if z is an H valued random variable with law μ , then, for all $r \in (0, 1)$, $z(t, r) = \int_0^1 g_t(r, x) z(x) dx$ is a centered Gaussian random variable with variance $\int_0^1 \int_0^1 g_t(r, x) g_t(r, y) q_\infty(x, y) dx dy = q^t(r)$. Therefore:

$$\begin{aligned} \|U\|_{L^2(\mu)} &\leq \int_0^\infty dt e^{-t} \int_0^1 dr \frac{|h_r|}{2\sqrt{2\pi q_t(r)^3}} \left(\int_{\mathbb{R}} \varphi\left(x/\sqrt{q_t(r)}\right)^2 \mathcal{N}(0, q^t(r))(dx) \right)^{1/2} \\ &\leq \int_0^\infty dt e^{-t} \int_0^1 dr \frac{|h_r|}{2\sqrt{2\pi q_t(r)^3}} \left\{ \|\varphi\|_{L^\infty(\mathbb{R})} \wedge \left(\frac{q_t(r)}{2\pi q^t(r)} \right)^{1/4} \|\varphi\|_{L^2(\mathbb{R})} \right\} \\ &\leq \int_0^1 dt \int_0^1 dr \frac{|h_r| \|\varphi\|_{L^2(\mathbb{R})}}{2(2\pi)^{3/4} q_t(r)^{5/4} q^1(r)} + \int_1^\infty dt e^{-t} \int_0^1 dr \frac{|h_r| \|\varphi\|_{L^\infty(\mathbb{R})}}{2\sqrt{2\pi q_1(r)^3}}. \end{aligned}$$

By (5.12), the first integral in the last line is convergent, and so is the second one, whence $U \in L^2(\mu)$ as claimed. Now, differentiating under the integrals in (5.11), we see that U is differentiable on H , and that for all $z \in H$

$$\nabla U(z) = \int_0^\infty dt e^{-t} \int_0^1 dr \frac{h_r g_t(r, \cdot)}{2\sqrt{2\pi q_t(r)^4}} \psi\left(z(t, r)/\sqrt{q_t(r)}\right), \quad (5.13)$$

where

$$\psi(x) := (3x - x^3)e^{-x^2/2}, \quad x \in \mathbb{R}.$$

Note that ψ is globally bounded and square integrable on \mathbb{R} . In particular, the right-hand side of (5.13) is well-defined. Indeed, by (22) in [Zam04], there exists $c > 0$ such that, for all $t > 0$ and $r \in (0, 1)$

$$\frac{1}{q_t(r)} \leq \frac{c}{r(1-r)}. \quad (5.14)$$

Since, moreover, for all $r, s \in (0, 1)$

$$\int_0^\infty g_t(r, s) dt \leq \sum_{k=1}^\infty \frac{1}{\lambda_k} < \infty,$$

the integrand in (5.13) is indeed integrable on $\mathbb{R}_+ \times (0, 1)$. We now show that $\nabla U \in L^2(H, \mu; H)$. Noting that, by (27) in [Zam04],

$$\|g_t(r, \cdot)\| \leq \frac{1}{t^{1/4}}, \quad t > 0,$$

we have

$$\|\nabla U\|_{L^2} \leq \int_0^\infty dt e^{-t} \int_0^1 dr \frac{|h_r| t^{-1/4}}{2\sqrt{2\pi q_t(r)^4}} \left(\int_{\mathbb{R}} \psi \left(x/\sqrt{q_t(r)} \right)^2 \mathcal{N}(0, q^t(r))(dx) \right)^{1/2},$$

where we use the shorthand notation $\|\cdot\|_{L^2}$ for $\|\cdot\|_{L^2(H, \mu; H)}$. Hence

$$\begin{aligned} \|\nabla U\|_{L^2} &\leq \int_0^\infty dt e^{-t} \int_0^1 dr \frac{|h_r| t^{-1/4}}{2\sqrt{2\pi q_t(r)^4}} \left\{ \|\psi\|_{L^\infty(\mathbb{R})} \wedge \left(\frac{q_t(r)}{2\pi q^t(r)} \right)^{1/4} \|\psi\|_{L^2(\mathbb{R})} \right\} \\ &\leq \int_0^1 dt \int_0^1 dr \frac{|h_r| \|\psi\|_{L^2(\mathbb{R})} t^{-1/4}}{2(2\pi)^{3/4} (q^1(r))^{1/4} q_t(r)^{7/4}} + \int_1^\infty dt e^{-t} \int_0^1 dr \frac{|h_r| \|\psi\|_{L^\infty(\mathbb{R})}}{2\sqrt{2\pi q_1(r)^4}}. \end{aligned}$$

Since, for all $t > 0$ and $r \in (0, 1)$

$$\frac{t^{-1/4}}{q_t(r)^{7/4}} \leq \left(\frac{c}{r(1-r)} \right)^{7/4} t^{-1/4},$$

and since h is compactly supported in $(0, 1)$, the first integral in the last line above is convergent. Since the second integral is also convergent, we deduce that $\nabla U \in L^2(H, \mu; H)$ as claimed. We have thus proved that $U \in W^{1,2}(H, \mu) = D(\Lambda)$.

We show now that U_ϵ converges in $D(\Lambda)$ as $\epsilon \rightarrow 0$. To do so, remark that, for all $z \in H$, we have

$$U_\epsilon(z) = \int_0^\infty dt e^{-t} \int_0^1 dr \frac{h_r}{2\sqrt{2\pi q_t(r)}} \int_{\mathbb{R}} \rho_\epsilon''(b) \exp\left(-\frac{(b-z(t,r))^2}{2q_t(r)}\right) db,$$

which, upon integrating by parts twice in the last integral, we can rewrite

$$U_\epsilon(z) = \int_0^\infty \int_0^1 h_r \zeta(dt, dr) \int_{\mathbb{R}} \rho_\epsilon(b) \left[\frac{(b-z(t,r))^2}{q_t(r)} - 1 \right] \exp\left(-\frac{(b-z(t,r))^2}{2q_t(r)}\right) db,$$

where we have introduced the compact notation

$$\zeta(dt, dr) := \frac{e^{-t}}{2\sqrt{2\pi q_t(r)^3}} dt dr.$$

For all $z \in H$, we have

$$U_\epsilon(z) - U(z) = \int_0^\infty \int_0^1 h_r \zeta(dt, dr) \int_{\mathbb{R}} dx \rho(x) \left[\varphi\left(\frac{z(t,r) - \epsilon x}{\sqrt{q_t(r)}}\right) - \varphi\left(\frac{z(t,r)}{\sqrt{q_t(r)}}\right) \right].$$

Hence

$$\begin{aligned} \|U_\epsilon - U\|_{L^2(\mu)} &\leq \int_0^\infty \int_0^1 |h_r| \zeta(dt, dr) \int_{\mathbb{R}} dx \rho(x) \cdot \\ &\cdot \left[\int_{\mathbb{R}} \left(\varphi \left(\frac{y - \epsilon x}{\sqrt{q_t(r)}} \right) - \varphi \left(\frac{y}{\sqrt{q_t(r)}} \right) \right)^2 \mathcal{N}(0, q^t(r))(dy) \right]^{1/2}, \end{aligned}$$

whence, reasoning as in the previous computations

$$\begin{aligned} \|U_\epsilon - U\|_{L^2(\mu)} &\leq \int_0^\infty \int_0^1 |h_r| \zeta(dt, dr) \int_{\mathbb{R}} dx \rho(x) \cdot \\ &\cdot \left\{ \sqrt{2} \|\varphi\|_{L^\infty(\mathbb{R})} \wedge \left(\frac{q_t(r)}{2\pi q^t(r)} \right)^{1/4} \left\| \varphi \left(\cdot - \epsilon x / \sqrt{q_t(r)} \right) - \varphi \right\|_{L^2(\mathbb{R})} \right\}. \end{aligned}$$

By the Dominated Convergence Theorem, for all $t > 0$, $r \in (0, 1)$ and $x \in \mathbb{R}$

$$\left\| \varphi \left(\cdot - \epsilon x / \sqrt{q_t(r)} \right) - \varphi \right\|_{L^2(\mathbb{R})} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Hence, again by the Dominated Convergence Theorem, we deduce that

$$\|U_\epsilon - U\|_{L^2(\mu)} \xrightarrow{\epsilon \rightarrow 0} 0.$$

Looking now at the level of the gradients, we have:

$$\begin{aligned} \|\nabla U_\epsilon - \nabla U\|_{L^2} &\leq \int_0^\infty dt e^{-t} \int_0^1 dr \frac{|h_r| t^{-1/4}}{2\sqrt{2\pi q_t(r)^4}} \cdot \\ &\cdot \left\{ \|\psi\|_{L^\infty(\mathbb{R})} \wedge \left(\frac{q_t(r)}{2\pi q^t(r)} \right)^{1/4} \left\| \psi \left(\cdot - \epsilon x / \sqrt{q_t(r)} \right) - \psi \right\|_{L^2(\mathbb{R})} \right\}, \end{aligned}$$

so we obtain, with similar arguments, the convergence

$$\|\nabla U_\epsilon - \nabla U\|_{L^2} \xrightarrow{\epsilon \rightarrow 0} 0.$$

We have thus shown that $U_\epsilon \xrightarrow{\epsilon \rightarrow 0} U$ in $D(\Lambda)$. □

5.4 A projection principle

An important tool will be the following projection principle, which was first used in [Zam04] for the case of a 3-Bessel bridge (see Lemma 2.2 therein).

Lemma 5.5. *There exists a unique bounded linear operator $\Pi : D(\Lambda) \rightarrow D(\mathcal{E})$ such that, for all $F, G \in D(\Lambda)$ and $f \in D(\mathcal{E})$*

$$\Lambda_1(F, f \circ j) = \mathcal{E}_1(\Pi F, f),$$

where j is as in (5.1). Moreover, we have:

$$\mathcal{E}_1(\Pi F, \Pi F) \leq \Lambda_1(F, F).$$

Proof. We use the same arguments as in the proof of Lemma 2 in [Zam04]. Let $\mathcal{D} := \{\varphi \circ j, \varphi \in D(\mathcal{E})\}$. By Proposition 5.1, \mathcal{D} is a linear subspace of $D(\Lambda)$ which is isometric to $D(\mathcal{E})$. In particular, it is a closed subspace of the Hilbert space $D(\Lambda)$. Hence, we may consider the orthogonal projection operator $\hat{\Pi}$ onto \mathcal{D} . Then, for all $F \in D(\Lambda)$, let ΠF be the unique element of $D(\mathcal{E})$ such that $\hat{\Pi} F = (\Pi F) \circ j$. It then follows that Π possesses the required properties. \square

As a consequence, we obtain the following formula, which may be seen as a refinement of the IbPF (4.4) for $= P_{0,0}^1$.

Corollary 5.6. *For all $f \in D(\mathcal{E})$ and $h \in C_c^2(0, 1)$, we have*

$$\mathcal{E} \left(\langle h, \cdot \rangle - \frac{1}{2} \Pi U, f \right) = -\frac{1}{2} \int_K (\langle h'', \zeta \rangle - \Pi U(\zeta)) f(\zeta) d\nu(\zeta), \quad (5.15)$$

where U is as in (5.11).

Proof. First note that, for all $f \in \mathcal{S}_K$, we have, by (5.9):

$$\begin{aligned} \frac{1}{4} \int_0^1 dr h_r \frac{d^2}{db^2} \Sigma_{0,0}^{1,r} [f(X) | b] \Big|_{b=0} &= \frac{1}{2} \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[f(|\beta|) \int_0^1 h_r \rho_\epsilon''(\beta_r) dr \right] \\ &= \lim_{\epsilon \rightarrow 0} \int (f \circ j) G_\epsilon d\mu = \lim_{\epsilon \rightarrow 0} \Lambda_1(f \circ j, U_\epsilon) \\ &= \Lambda_1(f \circ j, U) = \mathcal{E}_1(f, \Pi U). \end{aligned}$$

Therefore, for all $f \in \mathcal{S}_K$, the IbPF (4.14) can be rewritten:

$$2\mathcal{E}(\langle h, \cdot \rangle, f) = - \int_K \langle h'', \zeta \rangle f(\zeta) d\nu(\zeta) + \mathcal{E}_1(f, \Pi U),$$

that is

$$\mathcal{E} \left(\langle h, \cdot \rangle - \frac{1}{2} \Pi U, f \right) = -\frac{1}{2} \int_K (\langle h'', \zeta \rangle - \Pi U(\zeta)) f(\zeta) d\nu(\zeta).$$

We have thus shown that the statement holds for all $f \in \mathcal{S}_K$. Since the latter is dense in $D(\mathcal{E})$, the claim follows. \square

Recall that $M = \{\Omega, \mathcal{F}, (u_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in K}\}$ denotes the Markov process properly associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ constructed above. In the sequel we shall consider the process $(u_t)_{t \geq 0}$ at equilibrium, i.e. seen under the measure $\mathbb{P}_\nu = \int d\nu(x) \mathbb{P}_x$. Note that, by Theorem 5.2.2 in [FOT10], for all $F \in D(\mathcal{E})$, we can write in a unique way

$$F(u_t) - F(u_0) = M_t^{[F]} + N_t^{[F]}, \quad t \geq 0, \quad (5.16)$$

\mathbb{P}_ν a.s., where $M^{[F]}$ is a martingale additive functional, and $N^{[F]}$ is an additive functional of zero energy. Using this fact we can thus write u as the weak solution to some SPDE, but with coefficients that are not explicit. However the formula (5.15) above will allow us to identify these coefficients. We shall need the following technical lemma.

Lemma 5.7. *Let $(F_n)_{n \in \mathbb{N}} \subset D(\mathcal{E})$, and $F \in D(\mathcal{E})$ be quasi-continuous functions such that $F_n \xrightarrow[n \rightarrow \infty]{} F$ in $D(\mathcal{E})$. Then it holds that*

$$M^{[F_n]} \xrightarrow[n \rightarrow \infty]{} M^{[F]}, \quad N^{[F_n]} \xrightarrow[n \rightarrow \infty]{} N^{[F]},$$

under \mathbb{P}_ν , in probability, for the topology of uniform convergence on finite intervals of $t \in \mathbb{R}_+$.

Proof. Since $F_n \xrightarrow[n \rightarrow \infty]{} F$ in $D(\mathcal{E})$, for all $T > 0$ and $\alpha > 0$, by (5.1.1) in [FOT10], we have

$$\mathbb{P}_\nu \left(\sup_{0 \leq s \leq T} |F_n(u_s) - F(u_s)| > \alpha \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

Moreover, by (5.2.22) and (5.2.25) in [FOT10], we have

$$\mathbb{P}_\nu \left(\sup_{0 \leq s \leq T} |M_s^{[F_n]} - M_s^{[F]}| > \alpha \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Hence, we have the convergence of processes

$$F_n(u_t) - F_n(u_0) \xrightarrow[n \rightarrow \infty]{} F(u_t) - F(u_0), \quad M_t^{[F_n]} \xrightarrow[n \rightarrow \infty]{} M_t^{[F]},$$

in \mathbb{P}_ν -probability, for the topology of uniform convergence on finite intervals of $t \in \mathbb{R}_+$. Hence, we also have the convergence

$$N_t^{[F_n]} \xrightarrow[n \rightarrow \infty]{} N_t^{[F]}$$

for the same topology, and the claim follows. \square

Theorem 5.8. *For all $h \in C_c^2(0, 1)$, we have*

$$\langle u_t, h \rangle - \langle u_0, h \rangle = M_t + N_t, \quad \mathbb{P}_{u_0} - a.s., \quad q.e. \ u_0 \in K.$$

Here $(N_t)_{t \geq 0}$ is a continuous additive functional of zero energy satisfying

$$N_t - \frac{1}{2} \int_0^t \langle h'', u_s \rangle ds = \lim_{\epsilon \rightarrow 0} N_t^\epsilon, \quad N_t^\epsilon := -\frac{1}{4} \int_0^t \langle \rho_\epsilon''(u_s), h \rangle ds,$$

in \mathbb{P}_ν -probability, uniformly in t on finite intervals. Moreover, $(M_t)_{t \geq 0}$ is a martingale additive functional whose sharp bracket has the Revuz measure $\|h\|_H^2 \nu$. Finally we also have

$$N_t - \frac{1}{2} \int_0^t \langle h'', u_s \rangle ds = \lim_{k \rightarrow \infty} N_t^{\epsilon_k}$$

along a subsequence $\epsilon_k \rightarrow 0$ in \mathbb{P}_{u_0} -probability, for $q.e. \ u_0 \in K$.

Proof. On the one hand, by (5.15), we can write

$$\langle u_t, h \rangle - \frac{1}{2} \Pi U(u_t) - \left(\langle u_0, h \rangle - \frac{1}{2} \Pi U(u_0) \right) = N_t^{(1)} + M_t^{(1)}, \quad (5.17)$$

where $N^{(1)}$ is the continuous additive functional of zero energy given by

$$N_t^{(1)} = \frac{1}{2} \int_0^t (\langle h'', u_s \rangle - \Pi U(u_s)) ds, \quad t \geq 0$$

and $M^{(1)}$ is defined by (5.17). On the other hand, for all $\epsilon > 0$, by definition of U_ϵ , we have for G_ϵ as in (5.10)

$$\Lambda_1(U_\epsilon, \Phi) = \int_H G_\epsilon \Phi d\mu, \quad \Phi \in D(\Lambda).$$

Hence, remarking that $G_\epsilon = g_\epsilon \circ j$, where $g_\epsilon : K \rightarrow \mathbb{R}$ is the functional defined by

$$g_\epsilon(z) := \frac{1}{2} \int_0^1 h_r \rho_\epsilon''(z_r) dr = \frac{1}{2} \langle \rho_\epsilon''(z), h \rangle,$$

by Lemma 5.5, we obtain

$$\mathcal{E}_1(\Pi U_\epsilon, f) = \int_K f(z) g_\epsilon(z) d\nu(z), \quad f \in D(\mathcal{E}), \quad (5.18)$$

that is

$$\mathcal{E}(\Pi U_\epsilon, f) = - \int_K f(z) (\Pi U_\epsilon(z) - g_\epsilon(z)) d\nu(z), \quad f \in D(\mathcal{E}).$$

As a consequence, we have the decomposition

$$\frac{1}{2} \Pi U_\epsilon(u_t) - \frac{1}{2} \Pi U_\epsilon(u_0) = N_t^{(2,\epsilon)} + M_t^{(2,\epsilon)}, \quad (5.19)$$

where $N^{(2,\epsilon)}$ is the continuous additive functional of zero energy given by

$$N_t^{(2,\epsilon)} = \frac{1}{2} \int_0^t (\Pi U_\epsilon(u_s) - g_\epsilon(u_s)) ds, \quad t \geq 0$$

and $M^{(2,\epsilon)}$ is defined by (5.19). Since $U_\epsilon \xrightarrow{\epsilon \rightarrow 0} U$ in $D(\Lambda)$ by Proposition 5.4, by the continuity of $\Pi : D(\Lambda) \rightarrow D(\mathcal{E})$, we have the convergence $\Pi U_\epsilon \xrightarrow{\epsilon \rightarrow 0} \Pi U$ in $D(\mathcal{E})$.

Therefore, setting:

$$M_t^{(2)} = M_t^{[\Pi U]}, \quad N_t^{(2)} := N_t^{[\Pi U]},$$

then, by Lemma 5.7, we have :

$$\Pi U_\epsilon(u_t) - \Pi U_\epsilon(u_0) \xrightarrow{\epsilon \rightarrow 0} \Pi U(u_t) - \Pi U(u_0), \quad M_t^{(2,\epsilon)} \xrightarrow{\epsilon \rightarrow 0} M_t^{(2)}, \quad N_t^{(2,\epsilon)} \xrightarrow{\epsilon \rightarrow \infty} N_t^{(2)}$$

in \mathbb{P}_ν -probability, for the topology of uniform convergence on finite intervals of $t \in \mathbb{R}_+$. Adding equality (5.19) to (5.17) yields

$$\langle u_t, h \rangle - \langle u_0, h \rangle = M_t + N_t,$$

with $M_t = M_t^1 + M_t^2$ and

$$\begin{aligned} N_t &= N_t^1 + N_t^2 \\ &= \frac{1}{2} \int_0^t (\langle h'', u_s \rangle - \Pi U(u_s)) ds + \lim_{\epsilon \rightarrow 0} \frac{1}{2} \int_0^t (\Pi U_\epsilon(u_s) - g_\epsilon(u_s)) ds \\ &= \frac{1}{2} \int_0^t \langle h'', u_s \rangle ds - \lim_{\epsilon \rightarrow 0} \frac{1}{2} \int_0^t g_\epsilon(u_s) ds, \end{aligned}$$

Moreover, note that $M = M^{[F_h]}$, where $F_h \in D(\mathcal{E})$ is given by

$$F_h(z) := \langle z, h \rangle, \quad z \in K.$$

Hence, by Theorem 5.2.3 in [FOT10], $\mu_{\langle M \rangle}$ is given by $\|h\|_{L^2(0,1)}^2 \cdot \nu$. For the last statement, we apply [FOT10, Corollary 5.2.1]. The Theorem is proved. \square

5.5 A distinction result

As a consequence of our IbPFs and the above constructions, we can now state a result which shows that the Markov process considered above is not equivalent (in a sense precised below) to the process corresponding to the modulus of the solution to the stochastic heat equation.

Let $K^{\mathbb{R}^+}$ denote the space of functions from \mathbb{R}_+ to K , endowed with the product σ -algebra. For all $x \in K$, let P_x be the law, on $K^{\mathbb{R}^+}$, of the Markov process $(u_t)_{t \geq 0}$ associated with \mathcal{E} , started from x , that is P_x is the image of \mathbb{P}_x under the measurable map:

$$\begin{cases} \Omega \rightarrow K^{\mathbb{R}^+} \\ \omega \mapsto (u_t(\omega))_{t \geq 0}. \end{cases}$$

Similarly, for all $z \in H$, let \mathbf{P}_z be the law, on $K^{\mathbb{R}^+}$, of $(|v_t|)_{t \geq 0}$, where $(v_t)_{t \geq 0}$ is the solution of the stochastic heat equation (5.2), with $v_0 = z$.

Theorem 5.9.

$$\mu(\{z \in H : P_{|z|} \neq \mathbf{P}_z\}) > 0.$$

Proof. Assume by contradiction that $P_{|z|} = \mathbf{P}_z$ for μ a.e. $z \in H$. Then, recalling that $(\mathbf{Q}_t)_{t \geq 0}$ denotes the semigroup associated with Λ , and $(Q_t)_{t \geq 0}$ the semigroup associated with \mathcal{E} , we would have:

$$\mathbf{Q}_t(f \circ j) = (Q_t f) \circ j, \quad \mu \text{ a.e.},$$

for all $t \geq 0$ and $f \in L^2(\nu)$. Therefore, the corresponding families of resolvents $(\mathbf{R}_\lambda)_{\lambda > 0}$ and $(R_\lambda)_{\lambda > 0}$ would satisfy, for all $f \in L^2(\nu)$:

$$\mathbf{R}_1(f \circ j) = (R_1 f) \circ j,$$

where the equality holds in $L^2(\mu)$. In particular, this shows that $(R_1 f) \circ j \in D(\Lambda)$ for any f as above. We then claim that, for all $F \in D(\Lambda)$, $\Pi F = \mathbb{E}[F(\beta) \mid |\beta|]$ μ a.e. Indeed, by the previous observations, for all $f \in L^2(\nu)$, it holds:

$$\begin{aligned} \int_H (f \circ j)(z) F(z) d\mu(z) &= \Lambda_1(\mathbf{R}_1(f \circ j), F) \\ &= \Lambda_1((R_1 f) \circ j, F) \\ &= \mathcal{E}_1(R_1 f, \Pi F) \\ &= \int_K f(x) (\Pi F)(x) d\nu(x). \end{aligned}$$

We thus have :

$$\forall f \in L^2(\nu), \quad \int_H f(|z|) F(z) d\mu(z) = \int_K f(x) (\Pi F)(x) d\nu(x), \quad (5.20)$$

i.e. $\Pi F = \mathbb{E}[F(\beta) \mid |\beta|]$ μ a.e., as claimed. By (5.20) and the first equality in Lemma 5.5, we deduce that, for all $f \in D(\mathcal{E})$ and $F \in D(\Lambda)$:

$$\Lambda(F, f \circ j) = \mathcal{E}(\Pi F, f).$$

Consider now the process $(v_t)_{t \geq 0}$ associated with Λ and started from $v_0 = \beta$, where β is a Brownian bridge on $[0, 1]$. Consider also the process $(u_t)_{t \geq 0}$ associated with

\mathcal{E} under the law \mathbb{P}_ν (so that, in particular, $u_0 \stackrel{(d)}{=} |\beta|$). Thus the processes v and u are stationary, and $|v| \stackrel{(d)}{=} u$ by our assumption. Let us set:

$$A_t := \langle |v_t|, h \rangle - \langle |v_0|, h \rangle - \frac{1}{2} \int_0^t \langle |v_s|, h'' \rangle ds,$$

$$C_t := \langle u_t, h \rangle - \langle u_0, h \rangle - \frac{1}{2} \int_0^t \langle u_s, h'' \rangle ds.$$

Let further $k \in C^2([0, 1])$ with $k(0) = k(1) = 0$, and consider the functionals $\Psi_k : H \rightarrow \mathbb{R}$ and $\tilde{\Psi}_k : K \rightarrow \mathbb{R}$ given by:

$$\Psi_k(z) := \exp(\langle k, z \rangle), \quad \tilde{\Psi}_k(y) := \mathbb{E}[\Psi_k(\beta) \mid |\beta| = y], \quad y \in K.$$

Note that $\Psi_k \in D(\Lambda)$, and recall that, by the above remarks, $\tilde{\Psi}_k = \Pi \Psi_k \mu$ a.e., so in particular $\tilde{\Psi}_k \in D(\mathcal{E})$. We then have:

$$\begin{aligned} -\frac{d}{dt} \mathbb{E} \left[A_t \tilde{\Psi}_k(|v_0|) \right] &= -\frac{d}{dt} \mathbb{E} \left[(\langle u_t, h \rangle - \langle u_0, h \rangle) \tilde{\Psi}_k(u_0) \right] + \frac{1}{2} \frac{d}{dt} \mathbb{E} \left[\int_0^t \langle h'', |v_s| \rangle ds \Psi_k(\beta) \right] \\ &= \mathcal{E}(\langle \cdot, h \rangle, \tilde{\Psi}_k) + \frac{1}{2} \mathbb{E}[\langle h'', |\beta| \rangle \Psi_k(\beta)] \\ &= \Lambda(\langle \cdot, | \cdot |, h \rangle, \Psi_k) + \frac{1}{2} \mathbb{E}[\langle h'', |\beta| \rangle \Psi_k(\beta)] \\ &= \frac{1}{2} \mathbb{E}[\langle \nabla \Psi_k(\beta), \text{sign}(\beta) h \rangle + \langle h'', |\beta| \rangle \Psi_k(\beta)] \\ &= \mathbb{E} \left[\Psi_k(\beta) \int_0^1 h : \dot{\beta}^2 : dL^0 \right] \end{aligned}$$

by (3.10) in [Zam05], or rather its analogue for the Brownian bridge as stated in Remark 1.3 of [GV16]. But, by [Zam05, Corollary 3.4] and [GV16, Theorem 3.2], the last quantity equals:

$$\sqrt{\frac{1}{2\pi}} e^{\frac{1}{2} \langle Qk, k \rangle} \int_0^1 \frac{h_r}{\sqrt{r(1-r)}} \exp\left(-\frac{K_r^2}{2r(1-r)}\right) \lambda(K'_r, -K_r, r) dr,$$

where $K = Qk$, with Q the covariance operator of β ,

$$(Qk)_r = \int_0^1 (r \wedge \sigma - r\sigma) k_\sigma d\sigma, \quad r \in [0, 1],$$

and $\lambda : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}$ is defined by:

$$\lambda(x, y, r) := x^2 + xy \frac{1-2r}{r(1-r)} + y^2 \frac{(1-2r)^2}{4r^2(1-r)^2} - \frac{1}{4r(1-r)}, \quad x, y \in \mathbb{R}, \quad r \in [0, 1].$$

Hence,

$$-\frac{d}{dt} \mathbb{E} \left[A_t \tilde{\Psi}_k(|v_0|) \right] = \sqrt{\frac{2}{\pi}} e^{\frac{1}{2} \langle Qk, k \rangle} \int_0^1 \frac{h_r}{\sqrt{r(1-r)}} \exp\left(-\frac{K_r^2}{2r(1-r)}\right) \lambda(K'_r, -K_r, r) dr. \quad (5.21)$$

On the other hand

$$\begin{aligned}
-\frac{d}{dt} \mathbb{E} \left[C_t \tilde{\Psi}_k(|v_0|) \right] \Big|_{t=0} &= \mathcal{E}(\Pi\Psi_k, \langle \cdot, h \rangle) + \frac{1}{2} \mathbb{E}[\langle h'', |\beta| \rangle \Pi\Psi_k(|\beta|)] \\
&= \frac{1}{2} \mathcal{E}(\Pi U, \Pi\Psi_k) \\
&= \frac{1}{4} \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^1 h_r \rho_\epsilon''(|\beta_r|) dr \Pi\Psi_k(|\beta|) \right],
\end{aligned}$$

where we used (5.15) to obtain the second equality, and the fact that $U = \lim_{\epsilon \rightarrow 0} U_\epsilon$ in $D(\mathcal{E})$, combined with (5.18), to obtain the third one. Therefore, recalling that $\Pi\Psi_k = \mathbb{E}(\Psi_k | |\beta|)$ μ a.e.:

$$\begin{aligned}
-\frac{d}{dt} \mathbb{E} \left[C_t \tilde{\Psi}_k(|v_0|) \right] \Big|_{t=0} &= \frac{1}{4} \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^1 h_r \rho_\epsilon''(|\beta_r|) dr \Psi_k(\beta) \right]. \\
&= \frac{1}{4} \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^1 h_r \rho_\epsilon''(\beta_r) dr e^{\langle k, \beta \rangle} \right].
\end{aligned}$$

By the Cameron-Martin formula, for all $\epsilon > 0$:

$$\begin{aligned}
&\frac{1}{4} \mathbb{E} \left[\int_0^1 h_r \rho_\epsilon''(\beta_r) dr e^{\langle k, \beta \rangle} \right] = \\
&= \frac{1}{4} e^{\frac{1}{2} \langle Qk, k \rangle} \int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}} \int_{\mathbb{R}} \rho_\epsilon''(b) \exp\left(-\frac{(b-K_r)^2}{2r(1-r)}\right) db dr \\
&\xrightarrow{\epsilon \rightarrow 0} \frac{1}{4} e^{\frac{1}{2} \langle Qk, k \rangle} \int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}} \left[\frac{K_r^2 - r(1-r)}{r^2(1-r)^2} \right] \exp\left(-\frac{K_r^2}{2r(1-r)}\right) dr.
\end{aligned}$$

Hence we obtain :

$$\begin{aligned}
-\frac{d}{dt} \mathbb{E} \left[C_t \tilde{\Psi}_k(|v_0|) \right] \Big|_{t=0} &= \\
\frac{1}{4} e^{\frac{1}{2} \langle Qk, k \rangle} \int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}} \left[\frac{K_r^2 - r(1-r)}{r^2(1-r)^2} \right] \exp\left(-\frac{K_r^2}{2r(1-r)}\right) dr. & \quad (5.22)
\end{aligned}$$

Since $|v|$ and u have the same law, the left-hand sides (5.21) and (5.22) above coincide, so the right-hand sides as well. This being true for any $h \in C_c^2(0, 1)$, we deduce that

$$\frac{K_r^2 - r(1-r)}{4r^2(1-r)^2} = \lambda(K_r', -K_r, r),$$

for a.e. $r \in (0, 1)$, hence for all r by continuity. We thus deduce that for all $r \in (0, 1)$:

$$(K_r')^2 - \frac{1-2r}{r(1-r)} K_r K_r' - \frac{1}{r(1-r)} K_r^2 = 0.$$

Since we can choose $k \in C_c^2(0, 1)$ such that $K = Qk$ does not satisfy the above equation, we obtain a contradiction. \square

6 Conjectures and open problems

Theorems 4.9 and 4.10 above enable us to conjecture the structure of the Bessel SPDEs for $\delta < 3$. The idea is that the right-hand side of the IbPFs (4.2) (respectively (4.4)) corresponds to the logarithmic derivative of the measure $P_{0,0}^\delta$ for $\delta \in (0, 3) \setminus \{1\}$ (resp. $\delta = 1$), which should yield the drift in the SPDEs we are looking for. More precisely, considering for instance the case $\delta \notin \{1, 3\}$, for all $\Phi \in \mathcal{S}$, we may rewrite the last term in the IbPF (4.2) as follows:

$$\begin{aligned} & -\kappa(\delta) \int_0^1 h_r \int_0^\infty db b^{\delta-4} \left(\Sigma_{0,0}^{\delta,r}(\Phi(X) | b) - \Sigma_{0,0}^{\delta,r}(\Phi(X) | 0) \right) dr \\ & = -\kappa(\delta) \lim_{\epsilon \rightarrow 0} \lim_{\eta \rightarrow 0} \mathbb{E} \left[\Phi(X) \int_0^1 h_r \left(\frac{\mathbf{1}_{X_r \geq \epsilon}}{X_r^3} - 2 \frac{\epsilon^{\delta-3}}{3-\delta} \frac{\rho_\eta(X_r)}{X_r^{\delta-1}} \right) dr \right], \end{aligned} \quad (6.1)$$

where the mollifying functions $\rho_\eta, \eta > 0$ are as in (5.8). To prove this equality, first note that, for all $r \in (0, 1)$, the innermost integral in the left-hand side can be written as the limit:

$$\lim_{\epsilon \rightarrow 0} \int_\epsilon^\infty db b^{\delta-4} \left(\Sigma_{0,0}^{\delta,r}(\Phi(X) | b) - \Sigma_{0,0}^{\delta,r}(\Phi(X) | 0) \right).$$

Now, for all $\epsilon > 0$, recalling the definition (3.14) of $\Sigma_{0,0}^{\delta,r}(\cdot | b)$, $b \geq 0$, we have:

$$\begin{aligned} & \int_\epsilon^\infty db b^{\delta-4} \left(\Sigma_{0,0}^{\delta,r}(\Phi(X) | b) - \Sigma_{0,0}^{\delta,r}(\Phi(X) | 0) \right) = \\ & E_{0,0}^\delta \left[\Phi(X) \mathbf{1}_{X_r \geq \epsilon} X_r^{-3} \right] - \frac{\epsilon^{\delta-3}}{3-\delta} \Sigma_{0,0}^{\delta,r}(\Phi(X) | 0) = \\ & \lim_{\eta \rightarrow 0} E_{0,0}^\delta \left[\Phi(X) \left(\mathbf{1}_{X_r \geq \epsilon} X_r^{-3} - 2 \frac{\epsilon^{\delta-3}}{3-\delta} \frac{\rho_\eta(X_r)}{X_r^{\delta-1}} \right) \right], \end{aligned}$$

and the equality (6.1) follows. As a consequence of that equality, we may write formally the gradient dynamics corresponding to $P_{0,0}^\delta$, $\delta \in (0, 3)$, as follows:

$$\partial_t u = \frac{1}{2} \partial_x^2 u + \xi + \frac{\kappa(\delta)}{2} \lim_{\epsilon \rightarrow 0} \lim_{\eta \rightarrow 0} \left(\frac{\mathbf{1}_{u \geq \epsilon}}{u^3} - 2 \frac{\epsilon^{\delta-3}}{3-\delta} \frac{\rho_\eta(u)}{u^{\delta-1}} \right),$$

where ξ denotes space-time white noise on $\mathbb{R}_+ \times (0, 1)$. Assuming now the existence of a local time process $(\ell_{t,x}^b)_{x \in (0,1), t, b \geq 0}$ satisfying the occupation times formula:

$$\forall f : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \int_0^t f(u(s, x)) ds = \int_0^{+\infty} f(b) \ell_t^b(x) b^{\delta-1} db. \quad (6.2)$$

and possessing sufficient regularity at $b = 0$, we could in turn write:

$$\lim_{\epsilon \rightarrow 0} \lim_{\eta \rightarrow 0} \left(\frac{\mathbf{1}_{u \geq \epsilon}}{u^3} - 2 \frac{\epsilon^{\delta-3}}{3-\delta} \frac{\rho_\eta(u)}{u^{\delta-1}} \right) = \int_0^{+\infty} b^{\delta-4} (\ell_{t,x}^b - \ell_{t,x}^0) db,$$

so the SPDE could be written:

$$\partial_t u = \frac{1}{2} \partial_x^2 u + \xi + \frac{\kappa(\delta)}{2} \frac{\partial}{\partial t} \int_0^{+\infty} b^{\delta-4} (\ell_{t,x}^b - \ell_{t,x}^0) db.$$

The same reasoning can be done for $\delta \in (0, 1)$, yielding for that case:

$$\partial_t u = \frac{1}{2} \partial_x^2 u + \xi + \frac{\kappa(\delta)}{2} \frac{\partial}{\partial t} \int_0^{+\infty} b^{\delta-4} \mathcal{T}_{0,b}^2 \ell_{t,x}^{(\cdot)} db.$$

As for the critical case $\delta = 1$, similar arguments would allow to write the SPDE as follows:

$$\partial_t u = \frac{1}{2} \partial_x^2 u + \xi - \frac{1}{8} \frac{d^2}{db^2} d\ell_t^b(x) \Big|_{b=0}. \quad (6.3)$$

In all the SPDEs above, the unknown would be the couple (u, ℓ) , where u is a continuous nonnegative function on $\mathbb{R}_+ \times (0, 1)$, and, for all $x \in (0, 1)$, $(\ell_t^b(x))_{b,t \geq 0}$ is a family of occupation times satisfying (6.2).

These conjectures raise several problems. Indeed, assuming that the process u could be constructed (as was actually done above for the case $\delta = 1$), it is at present unknown whether a family of occupation times ℓ satisfying (6.2) should exist; if it does, it is not clear whether this process should have any reasonable differentiability property. Finally, well-posedness of such SPDEs containing renormalized local times seems to be out of the reach of existing techniques. For instance, due to the lack of monotonicity, the techniques used in [NP92] to define a solution to the stochastic heat equation with reflection would not be of any help. Finally, we stress that the analogous SDE case of Bessel processes of dimension $\delta \in (0, 1)$ is also a problem of interest in itself; these processes are not semi-martingales, but nonetheless satisfy the stochastic equation:

$$\rho_t = x + \frac{\delta - 1}{2} \int_0^{+\infty} a^{\delta-2} (\ell_t^a - \ell_t^0) da + B_t,$$

where $(\ell_t^a)_{a,t \geq 0}$ is the diffusion local times process of the Bessel process $(\rho_t)_{t \geq 0}$ (see [RY13], Chapter XI, ex. 1.26). Even in this one-dimensional context, such an equation does not seem to fall under the scope of any existing method for solving SDEs. A better understanding of such an equation would be a first step in understanding the more complicated SPDE case.

The Dirichlet form techniques used in Section 5 above to construct u in the case $\delta = 1$ can also be applied successfully to treat the case $\delta = 2$, see the forthcoming paper [EA]. However, for $\delta \in]0, 3[\setminus \{1, 2\}$, it is not even known whether the form which naturally generalizes $(\mathcal{E}, \mathcal{F}C_b^\infty(K))$ in Proposition 5.1 is closable and whether its closure is a quasi-regular Dirichlet form.

We recall the main result of [DMZ06]: for all $\delta \geq 3$, we set

$$\zeta(\delta) := \sup\{k \geq \mathbb{N} : \exists t > 0, 0 < x_1 < \dots < x_k < 1, u(t, x_i) = 0 \quad i = 1, \dots, k\},$$

where u is the solution to the δ -Bessel SPDE (1.5)-(1.7). Then we have

$$\mathbb{P} \left(\zeta(\delta) > \frac{k}{2-\delta} \right) = 0. \quad (6.4)$$

In other words, a.s. u hits the obstacle 0 in at most $\lceil \frac{k}{2-\delta} \rceil$ space points simultaneously in time. It is very tempting to conjecture that (6.4) holds for all $\delta > 2$: in other words, the δ -Bessel SPDE would hit 0 at finitely many space points simultaneously in time for any $\delta > 2$, but the number of such hitting points would tend to

$+\infty$ as $\delta \downarrow 2$. The fact that $\delta = 2$ is the critical value for this behaviour is clearly related to the fact that $\delta = 2$ is also the critical dimension for the probability that the δ -Bessel process or bridge hit 0.

The transition between $\delta \geq 3$ and $\delta < 3$ is visible at the level of the invariant measure, namely the δ -Bessel bridge, since in the former case the measure is log-concave, while this property is lost in the latter case. Therefore the techniques of [ASZ09] based on optimal transport and gradient flows in metric spaces fail for $\delta < 3$. In the same vein, the Strong Feller property holds easily for $\delta \geq 3$, while it is an open problem for $\delta < 3$, again because the drift of the SPDE becomes highly non-dissipative. Still, the recent paper [EA18] of the first author shows that Bessel processes of dimension $\delta < 1$ are Strong Feller even if their drift contains a renormalized local time. Moreover Tsatsoulis and Weber [TW18] have proved that the 2-dimensional stochastic quantization equation satisfies a Strong Feller property, although it is an equation which needs renormalisation. All this suggests that there may be hope that this technically very useful property holds also for δ -SPDEs with $\delta < 3$.

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