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On the complexity of cache analysis for different replacement policies*

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Abstract

Modern processors use cache memory: a memory access that “hits” the cache returns early, while a “miss” takes more time. Given a memory access in a program, cache analysis consists in deciding whether this access is always a hit, always a miss, or is a hit or a miss depending on execution. Such an analysis is of high importance for bounding the worst-case execution time of safety-critical real-time programs.

There exist multiple possible policies for evicting old data from the cache when new data are brought in, and different policies, though apparently similar in goals and performance, may be very different from the analysis point of view. In this paper, we explore these differences from a complexity-theoretical point of view. Specifically, we show that, among the common replacement policies, LRU (Least Recently Used) is the only one whose analysis is NP-complete, whereas the analysis problems for the other policies are PSPACE-complete.

1 Introduction

Most high performance processors implement some form of *caching*: frequently used instructions and data are retained in fast memory close to the processing unit, to avoid costly requests from the main memory. While the intuition is that a cache retains the most recently accessed memory words, up to its size, reality is far more complex: what happens depends on the number of cache levels, the size of each level, the “number of ways” (also known as the *associativity*) of the

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cache and the *cache replacement policy*, that is, the algorithm used for choosing which memory block to evict from the cache to make room for a new block.

Reading a memory block not in cache can take 10 times to 100 times the time needed to access it if it is cache. Thus, static analysis approaches for bounding the worst-case execution time (WCET) of programs have to take into account whether or not data are cached. Such analyses are used, for instance, for proving that the execution time of software in a critical control loop (e.g. in avionics) can never exceed the period of the loop. Not only does caching, or lack of caching, directly influence execution time, it also complicates analysis itself, as different microarchitectural execution paths may be taken inside the processor depending on whether or not data are cached, and analysis has to take all these paths into account.

For these reasons, all static analyses for bounding execution time include a cache analysis, which determines which of the memory accesses made by the program are hits, which are misses, and which cannot be classified. Analyses depend on the cache replacement policy, and, in the literature, there is a clear preference for the LRU (*Least Recently Used*) policy, from the well-known age-based abstract analysis of Ferdinand [3] to recent work proved to be optimally precise in a certain sense [10, 11].

In contrast, other policies such as PLRU (pseudo-LRU), NMRU and FIFO (*First-In, First-Out*) have a reputation for being very hard to analyze [4] and for having poor predictability [9]. A legitimate question is whether these problems are intrinsically difficult, or is it just that research has not so far yielded efficient analyses.

Issues of static analysis of programs under different cache policies are not necessarily correlated with the practical efficiency of cache policies. Static analysis is concerned with worst-case behavior, and policies with approximately equal “average”¹ practical performance may be very different from the analysis point of view. Even though PLRU and NMRU were designed as “cheap” (easier to implement in hardware) alternatives to LRU and have comparable practical efficiency [1], they are very different from the worst-case analysis point of view.

In this paper, we explore these questions as decision problems:

Definition 1 (Exist-Hit). The *exist-hit* problem is, for a given replacement policy:

- | | |
|----------------|---|
| <i>Inputs</i> | a control flow graph with edges adorned with memory block names
a starting node S in the graph
a final location F in the graph
the cache <i>associativity</i> (<i>number of ways</i> N), in unary
a memory block name a |
| <i>Outputs</i> | a Boolean: is there an execution trace from S to F , starting with an empty initial cache and ending with a cache containing a ? |

¹By “average” we do not imply any probabilistic distribution, but rather an informal meaning over industrially relevant workloads, as opposed to examples concocted for exhibiting very good or very bad behavior.

Definition 2 (Exist-Miss). The *exist-miss* problem is defined as above but with an ending state *not* containing a .

We shall also study the variant of this problem where the initial cache contents are arbitrary:

Definition 3. The exist-hit (respectively, exist-miss) problem with arbitrary initial state contents is defined as above, except that the output is “are there a legal initial cache state σ and an execution trace from S to F , starting in σ and ending with a cache containing (respectively, not containing) a ?”.

We shall here prove that

- for policies LRU, FIFO, pseudo-RR, PLRU, and NMRU, the exist-hit and exist-miss problems are NP-complete for acyclic control flow graphs;
- for LRU, these problems are still NP-complete for cyclic control flow graphs;
- for PLRU, FIFO, pseudo-RR, PLRU, and NMRU, these problems are PSPACE-complete for cyclic control flow graphs
- for LRU, FIFO, pseudo-RR, and PLRU, the above results extend to exist-miss and exist-hit problems from an arbitrary starting state

Under the usual conjecture that PSPACE-complete problems are not in NP, this may justify why analyzing properties of FIFO, PLRU and NMRU caches is harder than for LRU.

Real-life CPU cache systems are generally complex (multiple levels of caches) and poorly documented (often, the only information about replacement policies is by reverse engineering). For our complexity-theoretical analyses we need simple models with clear mathematical definitions; thus we consider only one level of cache, and only one “cache set” per cache.²

In this paper, we consider that the control-flow graph carries only identifiers of memory blocks to be accessed, abstracting away the data that are read or written, as well as arithmetic operations and guards. Therefore, we take into account executions that cannot take place on the real system. This is the same setting used by many static analyses for cache properties. Some more precise static analyses attempt to discard some infeasible executions — e.g. an execution with guards $x < 0$ and $x > 0$ with no intervening write to x is infeasible. In general, however, this entails deciding the reachability of program locations, a problem that is undecidable if the program operates over unbounded

²A real cache system is composed of a large number of “cache sets”: a memory block may fit in only one cache set depending on its address, and the replacement policy applies only within a given cache set. For all commonly found cache replacement policies except pseudo-round-robin, and disregarding complex CPU pipelines, this means that the cache sets operate completely independently, each seeing only memory blocks that map to it; each can be analyzed independently. It is therefore very natural to consider the complexity of analysis over one single cache set, as we do in this paper.

integers, and already PSPACE-complete if the program operates on a finite vector of bits. Clearly we cannot use such a setting to isolate the contribution of the cache analysis itself.

2 LRU

The “Least Recently Used” (LRU) replacement policy is simple and intuitive: the data block least recently used is evicted when a cache miss occurs. The cache is thus a queue ordered by age: on a miss, the oldest block is discarded to make room for a new one, which has age 0; the ages of all other blocks are incremented. If a block is already in the cache, it is “rejuvenated”: its age is set to zero, and the ages of the blocks before it in the queue are incremented.

In other words, the state of an LRU cache with associativity N is a word of length at most N over the alphabet of cache blocks, composed of pairwise distinct letters; an empty cache is defined by the empty word. When an access is made to a block a , if it belongs to the word (*hit*), then this letter is removed from the word and appended to the word. If it does not belong to the word (*miss*), and the length of the word is less than N , then a is appended to the word; otherwise, the length of the word is exactly N — the first letter of the word is discarded and a is appended.

LRU has been used in Intel Pentium I, MIPS 24K/34K [8, p.21], among others. Notably, for Kalray processors K1a and K1b, LRU caches are advertised as advancing “timing predictability”.

In this section, we shall extend our recent NP-hardness results [11] to NP-completeness, and also prove NP-hardness for exist-hit on a restricted class of control-flow graphs.

2.1 Motivation and fundamental properties

LRU caches are appreciated by designers of static analysis tools that bound the worst-case execution time of the program, since an analysis based on abstract interpretation by Ferdinand and Wilhelm [3] (basically, an interval for the age of each possible block) has long been known. The analysis classifies each access in the program as “always hit” (all execution traces leading to that access produce a hit there), “always miss” (all execution traces leading to that access produce a miss there), or “unknown”. When it answers “unknown”, it may be that it is in fact “always hit” or “always miss”, but the analysis is too weak to come to a conclusion about it, or that there is at least one execution leading to a hit there and one leading to a miss there (“definitely unknown”).

In recent work [10, 11], we closed that loophole and proposed an analysis that completely decides whether a given access is “always hit”, “always miss”, or “definitely unknown”: the classical abstract interpretation is applied, along with another age-based abstract interpretation capable of concluding, in some cases, that an access is “definitely unknown”; the remaining cases are decided by an exact but expensive (exponential worst-case) analysis. These analyses

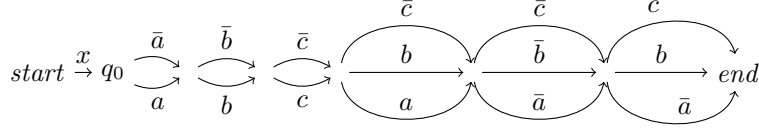


Figure 1: There is a path from q_0 to end with at most 3 different labels if and only if the formula $(\bar{c} \vee b \vee a) \wedge (\bar{c} \vee \bar{b} \vee \bar{a}) \wedge (c \vee b \vee \bar{a})$ has a model. Thus, for a LRU cache with associativity $N = 4$, there is an execution from $start$ to end ending in a cache state containing x if and only if this formula has a model.

solve both the exist-hit and the exist-miss problems; was such an exponential cost unavoidable? This motivated the studies in this paper.

Our analyses, as well as all our results on LRU in this paper, are based on the following easy, but fundamental, property of LRU caches:

Proposition 4. *After an execution path starting from an empty cache, a block a is in the cache if and only if there has been at least one access to a along that path and the number of distinct blocks accessed since the last access to a is at most $N - 1$.*

Example 5. Assume a 4-way cache, initially empty. After the sequence of accesses $bcabdcdb$, a is in the cache because $bdcdb$ contains only 3 distinct blocks b, c, d . In contrast, after the sequence $bcabdcceb$, a is no longer in the cache because $bdceb$ contains 4 distinct blocks b, c, d, e .

Remark 6. In definitions 1 and 2, it does not matter if the associativity is specified in unary or binary. An associativity larger than the number of different edge labels always produces hits, thus the problems become trivial. This also applies to FIFO, PLRU, NMRU caches and, more generally, to any cache analysis problem starting from an empty cache with a replacement policy that never evicts cache blocks as long as there is a free cache line.

2.2 Exist-Hit

Theorem 7. *The exist-hit problem is NP-complete for LRU and acyclic control-flow graphs.*

Proof. Obviously, the problem is in NP: a path may be chosen nondeterministically then checked in polynomial time.

Now consider the following reduction from CNF-SAT (see Figure 1 for an example). Let n_V be the number of variables in the SAT problem. With each variable v in the SAT problem we associate two cache block labels v and \bar{v} . The control-flow graph is a sequence of switches:

- For each variable v in the SAT problem, a switch between two edges labeled with v and \bar{v} respectively.
- For each clause in the SAT problem, a switch between edges labeled with the literals present in the clause.

Each path through the sequence of switches with at most n_V different labels corresponds to a SAT valid assignment, and conversely.

Let x be a fresh name. Prepend to the sequence of switches a read from memory block x and set the associativity N of the cache to $n_V + 1$. Then there exists an execution such that at *end* the cache contains x if and only if there exists a SAT valid assignment. \square

The objection can be made that the reduction in this proof produces control-flow graphs in which the same label occurs an arbitrary number of times — the number of times the corresponding literal occurs in the CNF-SAT problem, plus one. This may be appropriate for a data cache (the same data may be accessed many times within a loop-free program) but is unrealistic for an instruction cache:³ a given cache block may not overlap with arbitrarily many basic blocks in the machine code.⁴ However, we can refine the preceding result to account for this criticism.

Theorem 8. *The exist-hit problem is NP-complete for LRU for acyclic control-flow graphs, even when the same cache block label occurs no more than thrice.*

Proof. We use the same reduction as in Th. 8, but from a CNF-SAT problem where each literal occurs at most twice, as per the following lemma. \square

Lemma 9. *CNF-SAT is NP-hard even when restricted to sets of clauses where the same literal occurs at most twice, the same variable exactly thrice.⁵*

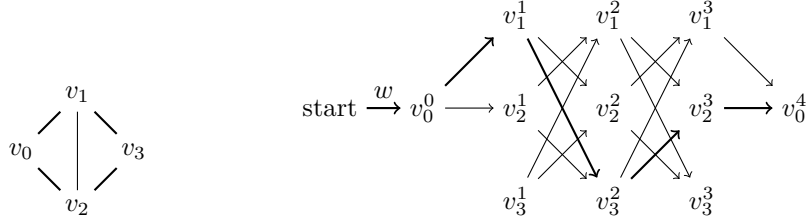
Proof. In the set of clauses, rename each occurrence of the same variable v_i as a different variable name $v_{i,j}$, then add clauses $v_{i,1} \Rightarrow v_{i,2}$, $v_{i,2} \Rightarrow v_{i,3}$, \dots , $v_{i,n-1} \Rightarrow v_{i,n}$, $v_{i,n} \Rightarrow v_{i,1}$ to establish logical equivalence between all renamings. Each literal now occurs once or twice, each variable thrice. Each model of the original formula corresponds to a model of the renamed formula, and conversely. \square

Remark 10. The exist-hit problem is easy when the same cache block occurs only once in the graph. Assume that the aim is to test whether there exists an execution leading to a cache containing x at the final location F . Either there exists one reachable access R to x in the control-flow graph, or there is none (in the latter case, x cannot be in the cache at location F). Then there exists an execution leading to a cache state containing x at location F if and only if there exists a path of length at most $N - 1$ between R and F , which may be tested for instance by breadth-first traversal. The complexity question remains open when cache blocks occur at most twice.

³Unless procedure calls are “inlined” in the graph.

⁴Consider a cache with 64-byte cache lines, as typical in x86 processors. In order for several basic blocks to overlap with that cache line, each, except perhaps the last one, must end with a branch instruction, which, in the shortest case, takes 2 bytes. No more than 32 basic blocks can overlap this cache line, and this upper bound is achieved by highly unrealistic programs.

⁵We thank Pálvölgyi Dömötör for pointing out to us that this restriction is still NP-hard.



(a) Graph with (thick) Hamiltonian cycle (b) Acyclic control-flow graph obtained by the reduction. Edge labels are not shown; the path corresponding to the Hamiltonian cycle is shown in a thick line.

Figure 2: Reduction from Theorem 13 from the Hamiltonian cycle problem to the exist-miss problem for LRU caches.

Definition 11. The *content* of a path π , denoted by $c(\pi)$, is the set of edge labels in this path.

Theorem 12. *The exist-hit problem is still in NP for LRU when the graph may be cyclic.*

Proof. Consider a path π from v_1 to v_2 , such that $|c(\pi)| < K$. By removing all cycles from π we can construct a path π' from v_1 to v_2 such that $c(\pi') \subseteq c(\pi)$ and π' has at most $|V| - 1$ steps. Thus, the nondeterministic search for a witness hit path may be restricted to simple paths of length at most $|V| - 1$, which ensures membership in NP. \square

2.3 Exist-Miss

Theorem 13. *The exist-miss problem is NP-complete for LRU for acyclic control-flow graphs.*

Proof. Obviously, the problem is in NP: a path may be chosen nondeterministically, then checked in polynomial time.

We reduce the Hamiltonian circuit problem to the exist-miss problem (see Figure 2 for an example). Let (V, E) be a graph, let $n = |V|$, $v = \{v_0, \dots, v_{n-1}\}$ (the ordering is arbitrary). Let us construct an acyclic control-flow graph G suitable for cache analysis as follows:

- two copies v_0^0 and v_0^n of v_0
- for each v_i , $i \geq 1$, $|V| - 1$ copies v_i^j , $1 \leq j < n$ (this arranges these vertices in layers indexed by j)
- for each pair $v_i^j, v_{i'}^{j+1}$ of nodes in consecutive layers, an edge, labeled by the address i' , if and only if there is an edge (i, i') in E .

There is a Hamiltonian circuit in (V, E) if and only if there is a path in G from v_0^0 to v_0^n such that no edge label is repeated, thus if and only if there exists a

path from v_0^0 to v_0^n with at least n distinct edge labels. Prepend an edge labeled x from the start node to v_0^0 , then there exists a trace such that x is not in the cache at v_0^n if and only if this Hamiltonian circuit exists. \square

The proof of Theorem 12 does not carry over to the exist-miss case: we replace a witness path by one of bounded length with possibly *fewer* distinct labels, whereas we would need *as many or more* distinct labels.

Lemma 14. *Let (V, E) be a graph with at most N edge labelings, $v_1, v_2 \in G$. From any path from v_1 to v_2 we can extract a path from v_1 to v_2 with the same contents and length at most $|V| \cdot N$ (and thus at most $|V| \cdot |E|$).*

Proof. Consider a path π from v_1 to v_2 . π can be segmented into sub-paths π_1, \dots, π_m , each beginning with the first occurrence of a new label not present in previous sub-paths.

Each sub-path π_i consists of an initial edge e_i followed by π'_i . From π'_i one can extract a simple path π''_i — that is, π''_i has no repeated vertex — of length at most $|V| - 1$. The concatenated path $e_1\pi''_1 \dots e_m\pi''_m$ has the same contents as π , starts and ends with the same vertices, and has at most $|V| \cdot N$ edges. \square

Theorem 15. *The exist-miss problem is still in NP for cyclic control-flow graphs.*

Proof. Follows from the preceding lemma: in a graph (V, E) search for a witness path of length at most $|V| \cdot |E|$. \square

2.4 Extensions

Remark 16. The above theorems hold even if the starting cache state is unspecified, because what matters in their proofs is the contents of the path from the initial access to x to the final location, and not the initial contents of the cache.

Remark 17. The proofs of NP-hardness for exist-hit and exist-miss on acyclic graphs for LRU carry over to FIFO (section 4).

3 Boolean register machine

In the next sections, we shall prove that the exist-hit and the exist-miss problems for a variety of replacement policies are NP-hard for acyclic control-flow graphs and PSPACE-hard for general control-flow graphs. All proofs will be by reduction from the reachability problem on a class of very simple machines, which we describe in this section: this problem is NP-complete if the control-flow graph of the machine is assumed to be acyclic, and PSPACE-complete in general.

Definition 18. A *Boolean register machine* is defined by a number r of registers and a directed (multi)graph with an initial vertex and a final vertex, with edges adorned by instructions of the form:

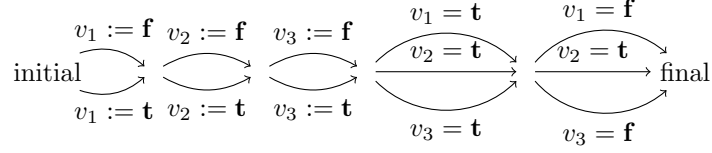


Figure 3: Reduction of CNF-SAT over 3 unknowns with clauses $\{v_1 \vee v_2 \vee v_3, \bar{v}_1 \vee v_2 \vee \bar{v}_3\}$ to a Boolean 3-register machine

Guard $v_i = b$ where $1 \leq i \leq r$ and $b \in \{\mathbf{f}, \mathbf{t}\}$,

Assignment $v_i := b$ where $1 \leq i \leq r$ and $b \in \{\mathbf{f}, \mathbf{t}\}$.

The *register state* is a vector of r Booleans. An edge with a guard $v_i = b$ may be taken only if the i -th register contains b ; the register state is unchanged. The register state after the execution of an edge with an assignment $v_i := b$ is the same as the preceding internal state except that the i -th register now contains b .

The *reachability problem* for such a system is the existence of a valid execution starting in the initial vertex with all registers equal to 0, and leading to the final vertex.

Lemma 19. *The reachability problem for Boolean register machines is PSPACE-complete.*

Proof. Such a machine is easily simulated by a polynomial-space nondeterministic Turing machine; based on Savitch's theorem, the reachability problem is thus in PSPACE.

Any Turing machine using space $P(|x|)$ on input x can be simulated by a Boolean register machine with $O(P(|x|))$ registers, encoding the state of the tape of the Turing machine, and a number of transitions in $O(P(|x|) \cdot |D|)$ where D is the description of the Turing machine. \square

Lemma 20. *The reachability problem for Boolean register machines with acyclic control-flow is NP-complete.*

Proof. A path from initial to final vertices, along with register values, may be guessed nondeterministically, then checked in polynomial time, thus reachability is in NP.

Any CNF-SAT problem with r Boolean unknowns may be encoded as a Boolean r -register machine as follows: a sequence of r disjunctions between $v_i := \mathbf{t}$ and $v_i := \mathbf{f}$ for each variable i , and then for each clause $v_{i_1^+} \vee \dots \vee v_{i_{n^+}^+} \vee v_{i_1^-} \vee \dots \vee v_{i_{n^-}^-}$, a disjunction between edges $v_{i_i^+} = \mathbf{t}$ for all $1 \leq i \leq n^+$ and edges $v_{i_i^-} = \mathbf{f}$ for all $1 \leq i \leq n^-$ (Figure 3). \square

All forthcoming reductions will replace each instruction edge ($v_i = b$ guard edges, $v_i := b$ assignment edge) by a “gadget”, a small acyclic piece of control-flow graph adorned with accesses to memory blocks; they will also add a prologue and an epilogue. The idea is to simulate executions of the Boolean register machine by executions of the cache system.

However, there is one problem: in Boolean register machines, an execution aborts when a guard is not satisfied, whereas in our cache analysis problems, executions never abort except when reaching a control location with no outgoing edge. We overcome this limitation by arranging the cache analysis problems so that the cache states following the simulation of an *invalid* guard (e.g. the simulated state encodes $v_i = \mathbf{t}$ but the guard is $v_i = \mathbf{f}$) are irremediably marked as incorrectly formed; thus the reachability problem for the Boolean register machine is reduced to the reachability of a correctly formed cache state at a certain location, which, in the epilogue, is encoded into the reachability of a cache hit (or a cache miss, respectively).

4 FIFO

FIFO (First-In, First-Out), also known as “round-robin”, caches follow the same mechanism as LRU (a bounded queue ordered by age in the cache), except that a block is not rejuvenated on a hit. They are used in Motorola PowerPC 56x, Intel XScale, ARM9, ARM11 [8, p.21], among others.

4.1 Fundamental properties

The state of a FIFO cache with associativity N is a word of length at most N over the alphabet of cache blocks, composed of pairwise distinct letters; an empty cache is defined by the empty word. When an access is made to a block a , if it belongs to the word (*hit*) then the cache state does not change. If it does not belong to the word (*miss*), and the length of the word is less than N , then a is appended to the word; otherwise, the length of the word is exactly N — the first letter of the word is discarded and a is appended.

Lemma 21. *The exist-hit and the exist-miss problems are in NP for acyclic control flow graphs.*

Proof. Guess a path nondeterministically and execute the policy along it. \square

Lemma 22. *The exist-hit and the exist-miss problems are in PSPACE for general graphs.*

Proof. Simulate the execution of the policy using a polynomial-space nondeterministic Turing machine. Based on Savitch’s theorem, both problems are in PSPACE. \square

4.2 Reduction to Exist-Hit

We reduce the reachability problem for the Boolean register machine to the exist-hit problem for the FIFO cache as follows. The associativity of the cache is chosen as $N = 2r - 1$. The alphabet of cache blocks is $\{(a_{i,b})_{1 \leq i \leq r, b \in \{f,t\}}\} \cup \{(e_i)_{1 \leq i \leq r}\} \cup \{(f_i)_{1 \leq i \leq r-1}\} \cup \{(g_i)_{1 \leq i \leq r-1}\}$. The internal state v_1, \dots, v_r of the register machine is to be encoded as the FIFO word

$$a_{1,v_1} e_2 a_{2,v_2} \dots e_r a_{r,v_r}. \quad (1)$$

Definition 23. We say that a FIFO word of that form is *well-formed at shift 1*, or *well-formed* for short if it is of the form

$$a_{1,v_1} e_2 a_{2,v_2} \dots e_r a_{r,v_r} \quad (2)$$

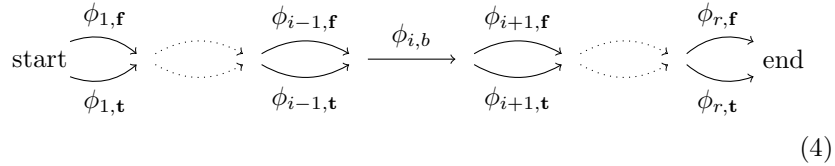
We say that a FIFO word is well-formed at shift i ($2 \leq i \leq r$) if it is of the form

$$a_{i,v_i} e_{i+1} a_{i+1,v_{i+1}} \dots a_{r,v_r} e_1 a_{1,v_1} e_2 \dots a_{i-1,v_{i-1}} \quad (3)$$

In both cases, we say that the FIFO word *corresponds* to the state v_1, \dots, v_r .

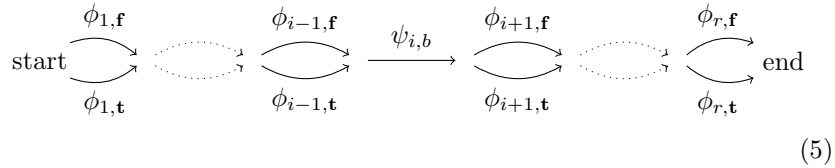
Definition 24. We turn the register machine graph into a cache analysis graph as follows.

- From the cache analysis initial vertex I_f to the register machine former initial vertex I_r there is a prologue, a sequence of accesses $a_{1,f} e_1 \dots a_{r-1,f} e_{r-1} a_{r,f}$.
- Each guard edge $v_i = b$ is replaced by the gadget



where $\phi_{i,b}$ denotes the sequence of accesses $a_{i,b} e_i a_{i,b}$.

- Each assignment edge $v_i := b$ is replaced by the gadget



where $\psi_{i,b}$ denotes the sequence of accesses $e_i a_{i,b} e_i$.

- From the register machine former final vertex F_r to a vertex F_a there is a sequence of accesses $\psi_{1,f} \dots \psi_{r,f}$, constituting the first part of the epilogue.

- From F_a to a vertex F_h there is a sequence of accesses $a_{1,\mathbf{f}}g_1e_2f_2a_{2,\mathbf{f}}g_2 \dots e_{r-1}f_{r-1}a_{r-1,\mathbf{f}}g_{r-1}e_rf_r$, constituting the second part of the epilogue.
- The final vertex is $F_f = F_h$.

The main difficulty in this reduction is that the Boolean register machines may terminate traces if a guard is not satisfied, whereas the cache problem has no guards and no way to terminate traces. Our workaround is that cache states that do not correspond to traces from the Boolean machine are irremediably marked as incorrect (formally: *well-phased but not well-formed*, per the following definition).

Definition 25. We say that a FIFO word of that form is *well-phased at shift 1*, or *well-phased* for short if it is of the form

$$\beta_1\alpha_2\beta_2 \dots \alpha_r\beta_r \quad (6)$$

where, for each i :

- either $\alpha_i = e_i$ and $\beta_i = a_{i,b_i}$ for some b_i ,
- or $\beta_i = e_i$ and $\alpha_i = a_{i,b_i}$ for some b_i .

We say that a FIFO word is well-phased at shift i ($2 \leq i \leq r$) if it is of the form

$$\beta_i\alpha_{i+1}\beta_{i+1} \dots \alpha_r\beta_r\alpha_1\beta_1 \dots \alpha_{i-1}\beta_{i-1} \quad (7)$$

Lemma 26. Assume w is well-formed at shift i , corresponding to state $\sigma = (\sigma_1, \dots, \sigma_r)$. If $\sigma_i = b$, then executing $\phi_{i,b}$ over FIFO state w leads to a state well-formed at shift $i + 1$ (1 if $i = r$), corresponding to σ too. If $\sigma_i = \neg b$, then executing $\phi_{i,b}$ over FIFO state w leads to a state well-phased, but not well-formed, at shift $i + 1$ (1 if $i = r$).

Proof. Without loss of generality we prove this for $i = 1$ and $b = \mathbf{f}$. Assume $w = a_{1,\mathbf{f}}e_2a_{2,v_2} \dots e_ra_{r,v_r}$; then the sequence $\phi_{1,\mathbf{f}} = a_{1,\mathbf{f}}e_1a_{1,\mathbf{f}}$ yields $a_{2,v_2} \dots e_ra_{r,v_r}e_1a_{1,\mathbf{f}}$. Assume now $w = a_{1,\mathbf{t}}e_2a_{2,v_2} \dots e_ra_{r,v_r}$; then $\phi_{1,\mathbf{f}}$ yields $a_{2,v_2} \dots e_ra_{r,v_r}a_{1,\mathbf{f}}e_1$. \square

Lemma 27. Assume w is well-formed at shift i , corresponding to state $\sigma = (\sigma_1, \dots, \sigma_r)$. Executing $\psi_{i,b}$ over FIFO state w leads to a state well-formed at shift $i + 1$ (1 if $i = r$), corresponding to σ where σ_i has been replaced by b .

Proof. Without loss of generality we prove it for $i = 1$ and $b = \mathbf{f}$. Assume $w = a_{1,v_1}e_2a_{2,v_2} \dots e_ra_{r,v_r}$; then the sequence $\psi_{1,\mathbf{f}} = e_1a_{1,\mathbf{f}}e_1$ yields $a_{2,v_2} \dots e_ra_{r,v_r}e_1a_{1,\mathbf{f}}$. \square

Corollary 28. Assume starting in a well-formed FIFO state, corresponding to state σ , then any path through the gadget encoding an assignment or a guard

- either leads to a well-formed FIFO state, corresponding to the state σ' obtained by executing the assignment, or $\sigma' = \sigma$ for a valid guard;
- or leads to a well-phased but not well-formed state.

Lemma 29. *Assume w is well-phased, but not well-formed, at shift i , then executing $\psi_{i,b}$ or $\phi_{i,b}$ over FIFO state w leads to a state well-phased, but not well-formed, at shift $i + 1$.*

Proof. Without loss of generality, we shall prove this for $i = 1$. Let $w = \beta_1\alpha_2\beta_2 \dots \alpha_r\beta_r$.

First case: $\beta_1 = e_1$. $\psi_{1,b} = e_1a_{1,b}e_1$ then leads to $\alpha_2\beta_2 \dots \alpha_r\beta_ra_{1,b}e_1$, which is well-phased, but not well-formed due to the last two letters, at shift 2. $\phi_{1,b} = a_{1,b}e_1a_{1,b}$ also leads to the same state.

Second case: β_1 is either $a_{1,f}$ or $a_{1,t}$; assume the former without loss of generality. Then there exists $j > 1$ such that $\alpha_j = a_{j,v_j}$ and $\beta_j = e_j$. $\psi_{1,b}$ then leads to $\alpha_2\beta_2 \dots \alpha_r\beta_re_1a_{1,b}$, which is well-phased, but not well-formed due to the α_j, β_j , at shift 2.

$\phi_{1,f}$ leads to $\alpha_2\beta_2 \dots \alpha_r\beta_re_1a_{1,f}$, which is well-phased, but not well-formed due to the α_j, β_j , at shift 2.

$\phi_{1,t}$ leads to $\alpha_2\beta_2 \dots \alpha_r\beta_ra_{1,t}e_1$, which is well-phased, but not well-formed due to the last two letters, at shift 2. \square

Corollary 30. *Assume starting in a well-phased but not well-formed FIFO state, then any path through the gadget encoding an assignment or a guard leads to a well-phased but not well-formed FIFO state.*

Corollary 31. *Any path from a well-formed FIFO state in I_r to F_r in the FIFO graph from Theorem 24*

- either corresponds to a valid sequence of assignments and guards from the register machine from I_r to F_r , and leads to a well-formed FIFO state corresponding to the final state of that sequence
- or corresponds to an invalid sequence of assignments and guards from the register machine, and leads to a well-phased but not well-formed FIFO state.

Conversely, any valid sequence of assignments and guards from the register machine maps from I_r to F_r transforms a well-formed FIFO state into a well-formed FIFO state, corresponding respectively to the initial and final states of that sequence.

Corollary 32. *The path from F_r to F_a (as in Definition 24):*

- transforms a well-phased but not well-formed FIFO state into a well-phased but not well-formed FIFO state
- transforms any well-formed FIFO state into a well-formed FIFO state w_0 corresponding to the initial register state (all registers zero).

Lemma 33. *The path $a_{1,f}g_1e_2f_2a_{2,f}g_2 \dots e_{r-1}f_{r-1}a_{r-1,f}g_{r-1}e_rf_r$ (from F_a to F_h in Definition 24):*

- *transforms w_0 into $a_{r,f}g_1f_2g_2 \dots f_{r-1}g_{r-1}f_r$*
- *transforms any other word w consisting of a 's and e 's into a word not containing $a_{r,f}$.*

Proof. The first item is trivial. We shall now prove that it is necessary for the input word to be exactly w_0 in order for the final word to contain $a_{r,f}$. In order for that, there must have been at most $2r - 2$ misses along the path. The accesses to $g_1, f_2, g_2, \dots, f_{r-1}, g_{r-1}, f_r$ are always misses. As there are $2r - 2$ of them, there must have been exactly those misses and no others. This implies that $a_{r,f}$ was in the last position in w .

When e_r is processed, similarly there were exactly $2r - 3$ misses, and e_r must be a hit. This implies that e_r was in the last or the penultimate position in w , but since the last position was occupied by $a_{r,f}$, e_r must have been in the penultimate position.

The same reasoning holds for all preceding locations, down to the first one, and thus the lemma holds. \square

From all these lemmas, the main result follows:

Corollary 34. *There is an execution of the FIFO cache from I_f to F_f such that $a_{r,f}$ is in the final cache state if and only if there is an execution of the Boolean register machine from I_r to F_r .*

Theorem 35. *The exist-hit problem for FIFO caches is NP-complete for acyclic graphs and PSPACE-complete for general graphs.*

Proof. As seen above, a register machine reachability problem can be reduced in polynomial time to a exist-hit FIFO problem, preserving acyclicity if applicable. \square

Remark 36. We have described a reduction from a r -register machine to a FIFO cache problem with an odd $2r - 1$ number of ways. This reduction may be altered to yield an even number of ways as follows. Two special padding letters p and p' are added. A well-formed state is now $pa_{1,b_1}e_2a_{2,b_2} \dots a_{r,b_r}$; the definition of well-phased states is similarly modified. Each gadget G for assignment or guard is replaced by $p'GpG$. The first p' turns padding p into p' , G is applied. The second p' turns p' into p and G is applied again.

This remark also applies to the exist-miss problem.

4.3 Reduction to Exist-Miss

We modify the reduction for exist-hit in order to exhibit a miss on $a_{r,f}$ later on if and only if it is in the cache at the end of the graph defined above.

Definition 37. We transform the register machine graph into a cache analysis graph as in Theorem 24, with the following modification: in between F_h and F_f we insert a sequence $a_{r,f}e_{1,f} \dots e_{r-1,f}e_r$, constituting the third part of the epilogue.

Lemma 38. *The path from F_h to F_f transforms $a_{r,f}g_1f_2g_2 \dots f_{r-1}g_{r-1}f_r$ into a word not containing $a_{r,f}$. It transforms any word composed of f 's and g 's only into a word containing $a_{r,f}$.*

Theorem 39. *The exist-miss problem for FIFO caches is NP-complete for acyclic graphs and PSPACE-complete for general graphs.*

4.4 Extension to arbitrary starting cache

Lemma 40. *The exist-hit and exist-miss problems for an empty starting FIFO cache state are reduced, in linear time, to the same kind of problem for an arbitrary starting cache state, with the same associativity.*

Proof. Let Σ be the alphabet of blocks in the problem and N its associativity. Let e_1, \dots, e_{2N-1} be new blocks not in Σ ; after accessing them in sequence, the cache contains only elements from these accesses [9, Th. 1]. Prepend this sequence as a prologue to the cache problem; then the rest of the execution of the cache problem will behave as though it started from an empty cache. \square

Corollary 41. *The exist-hit and exist-miss problems for FIFO caches with arbitrary starting state is NP-complete for acyclic graphs and PSPACE-complete for general graphs.*

4.5 Extension to Pseudo-RR caches

Recall how a FIFO cache with multiple cache sets — the usual approach in hardware caches — operates. A memory block of address x is stored in the cache set number $H(x)$ where H is a suitable function, normally a simple combination of the bits of x . In typical situations, this is as though the address x were specified as a pair (s, a) where s is the number of the cache set and a is the block name to be used by the FIFO in cache set number s .

In a FIFO cache, each cache set, being a FIFO, can be implemented as a circular buffer: an array of cache blocks and a “next to be evicted” index. In contrast, in a pseudo-RR cache, the “next to be evicted” index is global to all cache sets.

A FIFO cache exist-hit or exist-miss problem with cache block labels a_1, \dots, a_n can be turned into an equivalent pseudo-RR problem simply by using $(s, a_1), \dots, (s, a_n)$ as addresses for a constant distinguished cache set s . Thus, both exist-hit and exist-miss are NP-hard for acyclic control-flow graphs on pseudo-RR caches, and PSPACE-hard for general control-flow graphs.

The same simulation arguments used for FIFO (subsection 4.1) hold for establishing membership in NP and PSPACE respectively.

5 PLRU

Because LRU caches were considered too difficult to implement efficiently in hardware, various schemes for heuristically approximating the behavior of a LRU cache (keeping the most recently used data) have been proposed. By “heuristically approximating” we mean that these schemes are assumed, on “typical” workloads, to perform close to LRU, even though worst-case performance may be different.⁶ Some authors lump all such schemes as “pseudo-LRU” or “PLRU”, and call the scheme in the present section “tree-based PLRU” or “PLRU-t” [1], while some others [8, p. 26] call “PLRU” only the scheme discussed here.

PLRU has been used in i486 [5] (4-way), Intel Pentium II-IV, PowerPC 75x [8, p. 21]; an 8-way PLRU is used in NXP/Freescale MPC745x [7, p. 3-41], e6500, MPC8540.

5.1 PLRU caches

The cache lines of a PLRU cache, which may contain cached blocks, are arranged as the leaves of a full binary tree — thus the number of ways N is a power of 2, often 4 or 8. Two lines may not contain the same block. Each internal node of the tree has a tag bit, which is represented as an arrow pointing to the left or right branch. The internal state of the cache is thus the content of the lines and the $N - 1$ tag bits.

There is always a unique line such that there is a sequence of arrows from the root of the tree to the line; this is the line *pointed at by the tags*. Tags are said to be *adjusted away* from a line as follows: on the path from the root of the tree to the line, tag bits are adjusted so that the arrows all point away from that path.

When a block a is accessed:

- If the block is already in the cache, tags are adjusted away from this line.
- If the block is not already in the cache and one or more cache lines are empty, the leftmost empty line is filled with a , and tags are adjusted away from this block.
- If the block is not already in the cache and no cache line is empty, the block pointed at by the tags is evicted and replaced with a , and tags are adjusted away from this block.

⁶Experimentally, on typical workloads, the tree-based PLRU scheme described in this section is said to produce 5% more misses on a level-1 data cache compared to LRU [1]. However, that scheme may, under specific concocted workloads, indefinitely keep data that are actually never used except once— a misperformance that cannot occur with LRU [4]. This can produce *domino effects*: the cache behavior of a loop body may be indefinitely affected by the cache contents before the loop [2].

Because of the difficulties in obtaining justifiable bounds on the worst-case execution times of programs running on a PLRU cache, some designers of safety-critical real-time systems lock all cache ways except for two, exploiting the fact that a 2-way PLRU cache is the same as a 2-way LRU cache and thus recovering predictability [2, §3].

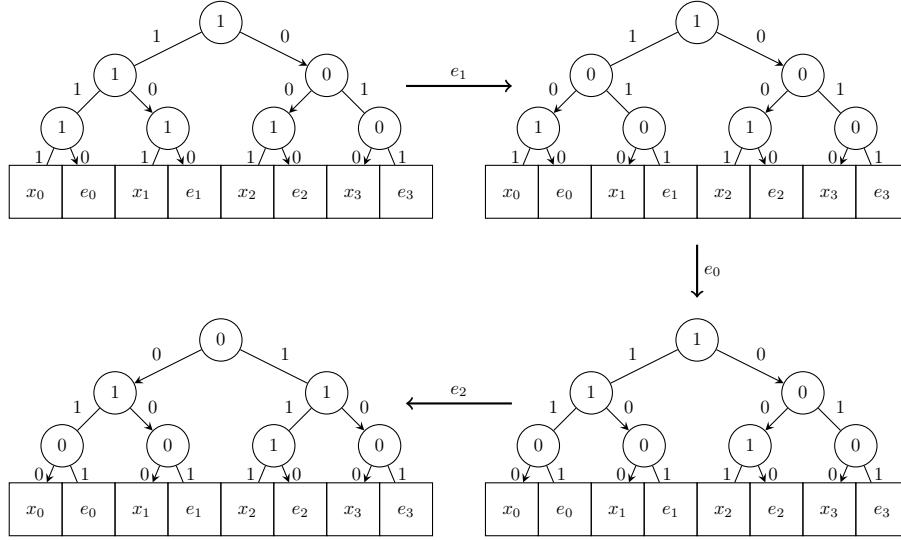


Figure 4: Sequence $\pi_1 = e_1 e_0 e_2$ makes tags point at x_1 without changing cache content

5.2 Exist-Hit Problem

We reduce the reachability problem of a Boolean r -register machine to the PLRU exist-hit problem for a $(2r + 2)$ -way cache — without loss of generality, we can always add useless registers so that $2r + 2$ is a power of two. The alphabet of cache blocks is $\{(a_{i,b})_{1 \leq i \leq r, b \in \{\mathbf{f}, \mathbf{t}\}}\} \cup \{(e_i)_{0 \leq i \leq r}\} \cup \{c\}$.

Definition 42. We say that a PLRU cache state is *well-formed* and *corresponds* to a Boolean state $(b_i)_{1 \leq i \leq r}$ if its leaves are, from left to right: $c, e_0, a_{1,b_1}, e_1, \dots, a_{r,b_r}, e_r$.

Definition 43. We say that a PLRU cache state is *well-phased* if its leaves are, from left to right: $x_0, e_0, a_{1,b_1}, e_1, \dots, a_{r,b_r}, e_r$ where x_0 can be c or any $a_{i,b}$ letter.

Lemma 44. Let $0 \leq i \leq r$, there exists a sequence π_i of accesses, of length logarithmic in r , such that, when run on a well-phased cache state $x_0, e_0, \dots, x_r, e_r$, that sequence makes tags point at x_i without changing the contents of the cache lines.

Proof. Moving from x_i to the root of the tree, at every node one e_i block is accessed from the other branch at that node (see Figure 4 for an example of sequence π_1). \square

Let $1 \leq i \leq r$, $b \in \{\mathbf{t}, \mathbf{f}\}$. Let $\phi_{i,b}$ be the sequence $\pi_0 a_{i,b}$, and $\psi_{i,b}$ the sequence $\pi_i a_{i,b}$.

Definition 45. We turn the register machine graph into a cache analysis graph as follows.

- From the cache analysis initial vertex I_p to the register machine former initial vertex I_r there is a sequence of accesses $c, e_0, a_{1,f}, e_1, \dots, a_{r,f}, e_r$.
- Each guard edge $v_i = b$ is replaced by the sequence $\phi_{i,b}$, and each assignment edge $v_i := b$ by the sequence $\psi_{i,b}$.
- The cache final vertex F_p is the same as the register machine final vertex F_r .

The idea is that any missed guard irremediably removes c from the cache. The following lemmas are easily proved by symbolically simulating the execution of the gadgets over the cache states:

Lemma 46. $\phi_{i,b}$ and $\psi_{i,b}$ map any well-phased but not well-formed state to a well-phased but not well-formed state.

Lemma 47. $\psi_{i,b}$ maps a well-formed state to a well-formed state corresponding to the same Boolean state where register i has been replaced by b .

Lemma 48. $\phi_{i,b}$ maps a well-formed state corresponding to a Boolean state $(b_i)_{1 \leq i \leq r}$ to

- if $b_i = b$, a well-formed state corresponding to the same Boolean state;
- otherwise, a well-phased but not well-formed state.

Corollary 49. There is an execution of the PLRU cache from I_p to F_p such that c is in the final cache state if and only if there is an execution of the Boolean register machine from I_r to F_r .

Proof. A well-phased state is well-formed if and only if it contains c . □

Theorem 50. The exist-hit problem for PLRU caches is NP-complete for acyclic graphs and PSPACE-complete for general graphs.

5.3 Exist-Miss Problem

We use two extra letters d and f .

Definition 51. Let Z be the sequence $\pi_r d \pi_r c \pi_0 f$.

Lemma 52. Z turns any well-formed state into a state not containing c . Z turns any well-phased but not well-formed state into a state containing c .

Proof. Consider a well-formed state $ce_0 a_{1,b_1} e_1 \dots a_{r,b_r} e_r$. Z replaces a_{r,b_r} by d ; then the access to c does not change the line contents since c is in the cache, and c is replaced by f .

Consider a well-phased but not well-formed state $x_0 e_0 a_{1,b_1} e_1 \dots a_{r,b_r} e_r$ where $x_0 \neq c$. Z replaces a_{r,b_r} by d ; then the access to c replaces d by c , and x_0 is replaced by f . □

Definition 53. We turn the register machine graph into a cache analysis graph in the same manner as in Definition 45, but between F_r and F_p we insert Z as epilogue.

Lemma 54. *There is an execution of the PLRU cache from I_p to F_p such that c is not in the final cache state if and only if there is an execution of the Boolean register machine from I_r to F_r .*

Theorem 55. *The exist-miss problem for PLRU caches is NP-complete for acyclic graphs and PSPACE-complete for general graphs.*

5.4 Extension to an arbitrary starting cache

Lemma 56. *The exist-hit and exist-miss problems for an empty starting PLRU cache state are reduced, in linear time, to the same kind of problem for an arbitrary starting cache state, with the same associativity.*

Proof. Same proof as Theorem 40, except we need a sequence of $\frac{N}{2} \log_2 N + 1$ new blocks [9, Th. 12]. \square

Corollary 57. *The exist-hit and exist-miss problems for FIFO caches with arbitrary starting state is NP-complete for acyclic graphs and PSPACE-complete for general graphs.*

6 NMRU

Other forms of “pseudo-LRU” schemes have been proposed than the one discussed in section 5. One of them, due to Malamy, Patel, and Hayes [6] is based on the use of “most recently used” bits. It is thus sometimes referred to as the “not most recently used” (NMRU) policy, or “PLRU-m” [1]. Confusingly, some literature [8] also refers to this policy as “MRU” despite the fact that in this policy, it is *not* the most recently used data block that is evicted first.

NMRU is used in the Intel Nehalem architecture, among others.

6.1 NMRU caches

Definition 58. The internal state of an N -way NMRU cache is a sequence of at most N memory blocks α_i , each tagged by a 0/1 “MRU-bit” r_i saying whether the associated block is to be considered not recently used (0) or recently used (1), denoted by $\alpha_1^{r_1} \dots \alpha_N^{r_N}$.

An access to a block in the cache, a *hit*, results in the associated MRU-bit being set to 1. If there were already $N - 1$ MRU-bits equal to 1, then all the other MRU-bits are set to 0.

An access to a block a not in the cache, a *miss*, results in:

- if the cache is not full (number of blocks less than N), then a^1 is appended to the sequence

- if the cache is full (number of blocks equal to N), then the leftmost (least index i) block with associated MRU-bit 0 is replaced by a^1 . If there were already $N - 1$ MRU-bits equal to 1, then all the other MRU-bits are set to 0.

Remark 59. This definition is correct because the following invariant is maintained: either the cache is not full, or it is full but at least one MRU-bit is zero.

Example 60. Assume $N = 4$. If the cache contains $a^0b^0c^0$, then an access to d yields $a^0b^0c^0d^1$ since the cache was not full. If a is then accessed, the state becomes $a^1b^0c^0d^1$. If e is then accessed, the state becomes $a^1e^1c^0d^1$ since b was the leftmost block with a zero MRU-bit. If f is then accessed, then the state becomes $a^0e^0f^1d^0$.

6.2 Reduction to Exist-Hit

We reduce the reachability problem for the register machine to the exist-hit problem for the NMRU cache as follows. The associativity of the cache is chosen as $N = 2r + 3$. The alphabet of the cache blocks is $\{(a_{i,b})_{1 \leq i \leq r, b \in \{f,t\}}\} \cup \{(e_i)_{1 \leq i \leq r}\} \cup \{(c_i)_{1 \leq i \leq r}\} \cup \{d\} \cup \{g_0, g_1\}$.

The internal state $w = v_1, \dots, v_r$ of the register machine is to be encoded as the NMRU state

$$e_1^0 \dots e_r^0 d^0 a_{1,v_1} \dots a_{r,v_r} g_0^0 g_1^1 \quad (8)$$

where the exponent (0 or 1) is the MRU-bit associated with the block.

Definition 61. We turn the register machine graph into a cache analysis graph as follows.

- From the cache analysis initial vertex I_f to the register machine former initial vertex I_r there is the prologue: the sequence of accesses $e_1 \dots e_r d a_{1,f} \dots a_{r,f} g_0 g_1$.
- Each guard edge $v_i = b$ is replaced by the gadget $\phi_{i,b} g_0 \phi_{i,b} g_1$, where $\phi_{i,b}$ is

$$\begin{array}{ccccccc} \text{start} & \xrightarrow{d} & \begin{array}{c} \xrightarrow{a_{1,f}} \\ \xleftarrow{a_{1,t}} \end{array} & \cdots & \begin{array}{c} \xrightarrow{a_{i-1,f}} \\ \xleftarrow{a_{i-1,t}} \end{array} & \xrightarrow{a_{i,b}} & \begin{array}{c} \xrightarrow{a_{i+1,f}} \\ \xleftarrow{a_{i+1,t}} \end{array} & \cdots & \begin{array}{c} \xrightarrow{a_{r,f}} \\ \xleftarrow{a_{r,t}} \end{array} & \xrightarrow{e_1} & \cdots & \xrightarrow{e_r} & \text{end} \end{array} \quad (9)$$

- Each assignment edge $v_i := b$ is replaced by the gadget $\psi_{i,b} g_0 \psi_{i,b} g_1$, where $\psi_{i,b}$ is

$$\begin{array}{ccccccc} \text{start} & \xrightarrow{d} & \begin{array}{c} \xrightarrow{a_{1,f}} \\ \xleftarrow{a_{1,t}} \end{array} & \cdots & \begin{array}{c} \xrightarrow{a_{i-1,f}} \\ \xleftarrow{a_{i-1,t}} \end{array} & \xrightarrow{a_{i+1,f}} & \begin{array}{c} \xrightarrow{a_{i+1,t}} \\ \xleftarrow{a_{i+1,f}} \end{array} & \cdots & \begin{array}{c} \xrightarrow{a_{r,f}} \\ \xleftarrow{a_{r,t}} \end{array} & \xrightarrow{e_1} & \cdots & \xrightarrow{e_r} & \xrightarrow{a_{i,b}} & \text{end} \end{array} \quad (10)$$

- From the register machine former final vertex F_r to a vertex F_a there is a sequence of gadgets for the assignments $v_1 := 0 \dots v_r := 0$, the first part of the epilogue.
- From F_a to a vertex F_h there is a sequence of accesses $a_{1,\mathbf{f}} \dots a_{r,\mathbf{f}} c_1 \dots c_r$, the second part of the epilogue.
- The final vertex is $F_f = F_h$.

Definition 62. We say that an NMRU state is well-formed at step $s \in \{0, 1\}$ if it is of the form

$$\beta_1^0 \dots \beta_r^0 d^0 \alpha_1^0 \dots \alpha_r^0 g_0^s g_1^{1-s} \quad (11)$$

where $\forall i, 1 \leq i \leq r, \alpha_i \in \{a_{\sigma(i),\mathbf{f}}, a_{\sigma(i),\mathbf{t}}\}, \beta_i = e_{\sigma'(i)}$ and σ and σ' are two permutations of $[1, r]$. In other words, a well-formed state contains r distinct blocks e_i placed before d , and r blocks $a_{i,b}$, with distinct i 's, placed between d and g_0 . We say “well-formed” for short if $s = 0$.

Definition 63. We say that an NMRU state is well-phased at step $s \in \{0, 1\}$ if it is of the form

$$\gamma_{\sigma(1)}^0 \dots \gamma_{\sigma(r)}^0 d^0 \gamma_{\sigma(r+1)}^0 \dots \gamma_{\sigma(2r)}^0 g_0^s g_1^{1-s} \quad (12)$$

where $\gamma_1 = e_1, \dots, \gamma_r = e_r, \gamma_{r+1} \in \{a_{1,\mathbf{f}}, a_{1,\mathbf{t}}\}, \dots, \gamma_{2r} \in \{a_{r,\mathbf{f}}, a_{r,\mathbf{t}}\}$ and σ is a permutation of $[1, 2r]$. We say “well-phased” for short if $s = 0$.

Lemma 64. *Executing a path through $\phi_{i,b} g_s$ over an NMRU state w well-phased at step s always leads to a state well-phased at step $1 - s$. Furthermore that state*

- *either is not well-formed at step $1 - s$*
- *or is identical to w except for the g_0 and g_1 blocks, and this may occur only if $a_{i,b}$ belongs to w .*

Proof. The input state w is $x_1^0, \dots, x_r^0, d^0, x_{r+1}^0, \dots, x_{2r}^0, g_0^0, g_1^1$ where the $(x_i)_{1 \leq i \leq 2r}$ are a permutation of $\{e_i \mid 1 \leq i \leq r\} \cup \{a_{i,\beta_i} \mid 1 \leq i \leq r\}$ for some sequence of Booleans $(\beta_i)_{1 \leq i \leq r}$.

Consider a path through $\phi_{i,b}$: it consists of d , followed by a sequence of r a 's, then r e 's. Each of these accesses either freshens, or overwrites, one of the x positions. After the sequence of a 's, there are either no a 's to the left of d , or at least one. The former case is possible only if all a 's are hits, freshening positions to the right of d — this means all these positions are left untouched except that their MRU bits are flipped to 1. Then the sequence of e 's just flips to 1 the MRU-bits of the e 's, all located to the left of d . The resulting state is thus identical to w except that all MRU-bits to the left of g_0 have been flipped to 1; thus after accessing g_0 , the state is identical to the initial state except that it ends with $g_0^{1-s} g_1^s$ instead of $g_0^0 g_1^1$.

Now consider the latter case: after the sequence of a 's there is at least one position of the form $a_{i,\beta}^1$ to the left of d . This position cannot be overwritten by the e 's. After the path through $\phi_{i,b}$, the state is thus of the form

$x_1^1, \dots, x_r^1, d^1, x_{r+1}^1, \dots, x_{2r}^1, g_0^0, g_1^1$, and one of the x_i for $1 \leq i \leq r$ is an a . The access to g_0 yields $x_1^0, \dots, x_r^0, d^0, x_{r+1}^0, \dots, x_{2r}^0, g_0^{1-s}, g_1^s$. This state is well-phased but not well-formed. \square

Lemma 65. *Executing a path through $\psi_{i,b}g_s$ over an NMRU state well-phased at step s always leads to a state well-phased at step $1 - s$. Furthermore that state*

- *either is not well-formed at step $1 - s$*
- *or is identical to the initial state except for the g_0 and g_1 blocks, and, possibly, the a_{i,β_i} block replaced by $a_{i,b}$.*

Proof. Again, the initial state is $x_1^0, \dots, x_r^0, d^0, x_{r+1}^0, \dots, x_{2r}^0, g_0^0, g_1^1$ where the $(x_i)_{1 \leq i \leq 2r}$ are a permutation of $\{e_i \mid 1 \leq i \leq r\} \cup \{a_{i,\beta_i} \mid 1 \leq i \leq r\}$ for some sequence of Booleans $(\beta_i)_{1 \leq i \leq r}$.

Consider a path through $\psi_{i,b}$: it consists of d , followed by a sequence of $r - 1$ a 's, then r e 's, then $a_{i,b}$. Each of these accesses either freshens, either overwrites, one of the x positions. After the sequence of a 's, there are either no a 's to the left of d , or at least one. The former case is possible only if all these a 's are hits, freshening positions to the right of d . Then the sequence of e 's freshens the e 's to the left of d . There is one remaining x position with a zero MRU-bit: it is to the right of d and carries a block a_{i,β_i} . This block is then updated or freshened by the $a_{i,b}$ access. Then the access to g_0 flips all MRU-bits to 0 except the one for g_0 , which is flipped to 1. Since all of the accesses before the $a_{i,b}$ access were hits, the permutation of the positions has not changed: the state is the same as the initial state except that a_{i,β_i}^0 is replaced by $a_{i,b}^0$ and g_0^{1-s} is replaced by $g_0^{1-s}g_1^s$.

Now consider the latter case: after the sequence of a 's there is at least one position of the form $a_{i,\beta}^1$ to the left of d . Then, as in the proof of the previous lemma, there is still $a_{i,\beta}^0$ to the left of d at the end of the path through $\psi_{i,b}g_s$. Thus the final state cannot be well-formed. \square

Corollary 66. *Assume starting in a well-formed NMRU state, corresponding to Boolean state σ , then any path through the gadget encoding an assignment or a guard*

- *either leads to a well-formed NMRU state, corresponding to the state σ' obtained by executing the assignment, or $\sigma' = \sigma$ for a valid guard;*
- *or leads to a well-phased but not well-formed state.*

Lemma 67. *Executing the sequence $a_{1,\mathbf{f}} \dots a_{r,\mathbf{f}} c_1 \dots c_r$ from F_a to F_b over a well-formed NMRU state corresponding to a zero Boolean state leads to a state containing d — more specifically, a state of the form $c_1^1, \dots, c_r^1, d^0, a_{\pi(1),\mathbf{f}}^1, \dots, a_{\pi(r),\mathbf{f}}^1, g_0^0 g_1^1$ where π is a permutation.*

Proof. The $a_{1,\mathbf{f}} \dots a_{r,\mathbf{f}}$ just freshen the corresponding blocks (MRU-bit set to 1), and then the $c_1 \dots c_r$ overwrite the e 's. \square

Lemma 68. *Executing the sequence $a_{1,f} \dots a_{r,f} c_1 \dots c_r$ from F_a to F_h over a well-phased but not well-formed NMRU state leads to a state not containing d — where the $2r$ first MRU bits are set to 1, the next one to 0, and then $g_0^0 g_1^1$.*

Proof. The well-phased but not well-formed NMRU state contains at least one b to the right of d . When applying $a_{1,f} \dots a_{r,f}$, at least one of the a 's must thus freshen or replace a letter to the left of d . Then when applying $c_1 \dots c_r$, d gets erased. \square

Corollary 69. *There is an execution sequence from I_f , with empty cache, to F_f , such that the final cache contains d if and only if there is an execution trace from I_r to F_r .*

Theorem 70. *The exist-hit problem for NMRU caches is NP-complete for acyclic graphs and PSPACE-complete for general graphs.*

6.3 Reduction to Exist-Miss

Definition 71. We modify the reduction of Theorem 61 as follows. Between F_h and F_f we insert the sequence $dg_0 c_1 \dots c_r a_{1,t}$, as the third part of the epilogue.

Lemma 72. *Executing $dg_0 c_1 \dots c_r a_{1,t}$ over a state of the form $c_1^1, \dots, c_r^1, d^0, a_{\pi(1),f}^1, \dots, a_{\pi(r),f}^1 g_0^0 g_1^1$, where π is a permutation, leads to a state without d .*

Proof. d gets freshened, then g_0 is the sole block with a zero MRU-bit. Thus, when it is freshened, all other MRU bits are set to zero. Then $c_1 \dots c_r$ freshen the first r blocks, and $a_{1,t}$ erases d . \square

Lemma 73. *Executing $dg_0 c_1 \dots c_r a_{1,t}$ over a state not containing d , where the $2r$ first MRU bits are set to 1, the next one to 0, and then $g_0^0 g_1^1$ leads to a state containing d .*

Proof. d overwrites the $2r + 1$ -th position, then g_0 is the sole block with a zero MRU-bit. Thus, when it is freshened, all other MRU bits are set to zero. Then possibly some blocks get overwritten among the $r + 1$ first blocks, and d is still in the cache. \square

Corollary 74. *There is an execution sequence from I_f , with empty cache, to F_f , such that the final cache does not contain d if and only if there is an execution of the Boolean register machine from I_r to F_r .*

Theorem 75. *The exist-miss problem for NMRU caches is NP-complete for acyclic graphs and PSPACE-complete for general graphs.*

We have no results for reductions to cache analysis problems with arbitrary starting state. The proof method that we used for FIFO and PLRU — prepend a sufficiently long sequence of accesses that will bring the cache to a sufficiently known state — does not seem to easily carry over to NMRU caches. Even though it is known that $2N - 2$ pairwise distinct accesses are sufficient to remove all previous content from an NMRU cache [9, Th. 4], it can be shown that there is no sequence guaranteed to yield a completely known cache state [9, Th. 5].

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