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# Cayley graphs of basic algebraic structures

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## Abstract

We present simple graph-theoretic characterizations for the Cayley graphs of monoids, right-cancellative monoids, left-cancellative monoids, and groups.

## 1 Introduction

Arthur Cayley was the first to define in 1854 [3] the notion of a group as well as the table of its operation known as the Cayley table. To describe the structure of a group  $(G, \cdot)$ , Cayley also introduced in 1878 [4] the concept of graph for  $G$  according to a generating subset  $S$ , namely the set of labeled oriented edges  $g \xrightarrow{s} g \cdot s$  for every  $g$  of  $G$  and  $s$  of  $S$ . Such a graph, called Cayley graph, is directed and labeled in  $S$  (or an encoding of  $S$  by symbols called letters or colors). A characterization of unlabeled and undirected Cayley graphs was given by Sabidussi in 1958 [7]: an unlabeled and undirected graph is a Cayley graph if and only if we can find a group with a free and transitive action on the graph. Following a question asked by Hamkins in 2010 [5]: ‘Which graphs are Cayley graphs?’, we present simple graph-theoretic characterizations of Cayley graphs first for the monoids, then for the right-cancellative or/and left-cancellative monoids, and finally for the groups.

Let us present the main structural characterizations starting with the Cayley graphs of monoids. Among many properties of these graphs, we retain only basic ones. First and by definition, any Cayley graph is deterministic: there are no two edges of the same source and label. They are also source-complete: for any label of the graph and from any vertex, there is at least one edge. These first two properties are well known in automata theory. As any Cayley graph of a monoid is according to a generating subset, it is rooted: there is a path from the identity element to any vertex. The identity is also an out-simple vertex: it is not source of two edges with the same target. Finally, the identity is a propagating vertex meaning that if it is source of two paths labeled by  $u$  and  $v$  with the same target then any two paths labeled by  $u$  and  $v$  of the same source have the same target. These properties satisfied by the Cayley graphs of monoids are sufficient to characterize them: they are the deterministic and source-complete graphs with a propagating out-simple root (Theorem 9). In fact for any graph  $\Gamma$  and root  $r$  satisfying these properties, the vertex set of  $\Gamma$  is a monoid whose  $s \cdot t$  is the target of the path from  $s$  labeled by  $u$  if  $u$  labels a path from  $r$  to  $t$ . It follows that  $\Gamma$  is a Cayley graph of this monoid generated by the set of successors of  $r$ .

We then characterize the Cayley graphs of right-cancellative monoids as being the Cayley graphs of monoids which are co-deterministic: there are no two edges of the same target and label (Theorem 13). We also characterize the Cayley graphs of left-cancellative monoids as being the Cayley graphs of monoids which are propagating: any vertex is propagating (Theorem 16). By extending to chains the vertex propagation, we get the Cayley graphs of groups: they are the deterministic and co-deterministic, chain-propagating connected simple graphs (Theorem 20).

## 2 Directed labeled graphs

We present some basic concepts on directed labeled graph, namely the determinism, the source-completeness, and the simplicity, which are basic notions in automata theory.

Let  $A$  be an arbitrary (finite or infinite) set. We denote by  $A^*$  the set of tuples (words) over  $A$  which is for the concatenation the free monoid generated by  $A$ . We denote by  $\varepsilon$  the 0-tuple *i.e.* the identity element called the *empty word*. A directed  $A$ -graph  $(V, G)$  is defined by a set  $V$  of *vertices* and a subset  $G \subseteq V \times A \times V$  of *edges*. Any edge  $(s, a, t) \in G$  is from the *source*  $s$  to the *target*  $t$  with *label*  $a$ , and is also written by the *transition*  $s \xrightarrow{a}_G t$  or directly  $s \xrightarrow{a} t$  if  $G$  is clear from the context. The sources and targets of edges form the set  $V_G$  of *non-isolated vertices* of  $G$  and we denote by  $A_G$  the set of edge labels:

$$V_G = \{ s \mid \exists a, t (s \xrightarrow{a} t \vee t \xrightarrow{a} s) \} \quad \text{and} \quad A_G = \{ a \mid \exists s, t (s \xrightarrow{a} t) \}.$$

The set  $V - V_G$  is the set of *isolated vertices*. From now on, we assume that any graph  $(V, G)$  is without isolated vertex (*i.e.*  $V = V_G$ ), hence the graph can be identified with its edge set  $G$ . We also exclude the empty graph  $\emptyset$ : every graph is a non-empty set of labeled edges. Let  $\rightarrow_G$  be the *edge relation* defined by  $s \rightarrow_G t$  if  $s \xrightarrow{a}_G t$  for some  $a \in A_G$ .

A *path*  $(s_0, a_1, s_1, \dots, a_n, s_n)$  of length  $n \geq 0$  in a graph  $G$  is a sequence  $s_0 \xrightarrow{a_1} s_1 \dots \xrightarrow{a_n} s_n$  of  $n$  consecutive edges, and we write  $s_0 \xrightarrow{a_1 \dots a_n} s_n$  for indicating the source  $s_0$ , the target  $s_n$  and the label word  $a_1 \dots a_n \in A_G^*$  of the path.

Let  $\rightarrow_G^*$  be the *path relation* defined by  $s \rightarrow_G^* t$  if  $s \xrightarrow{u}_G t$  for some  $u \in A_G^*$ . We denote by  $\rightarrow_G(r) = \{ s \mid r \rightarrow_G s \}$  the set of *successors* of a vertex  $r$ . A graph  $G$  is *accessible* from  $P \subseteq V_G$  if for any  $s \in V_G$  there is  $r \in P$  such that  $r \rightarrow_G^* s$ . A vertex  $r$  is a *root* if the graph is accessible from  $\{r\}$ . A graph  $G$  is *strongly connected* if every vertex is a root:  $s \rightarrow_G^* t$  for all  $s, t \in V_G$ .

A graph  $G$  is *deterministic* if there are no two edges of the same source and label *i.e.*

$$(r \xrightarrow{a}_G s \wedge r \xrightarrow{a}_G t) \implies s = t \quad \text{for any } r, s, t \in V_G \text{ and } a \in A_G$$

meaning that for any label  $a \in A_G$ , the relation  $\xrightarrow{a}$  is one to one. This local condition is equivalent to the injectivity of  $\xrightarrow{u}$  for any word  $u \in A_G^*$ .

A vertex  $s$  is a *source-complete vertex* if for any label  $a$  of the graph, there is at least one edge from  $s$  label by  $a$  *i.e.*  $\forall a \in A_G \exists t (s \xrightarrow{a}_G t)$ . A graph is a *source-complete graph* if all its vertices are source-complete. This condition is equivalent for any label word  $u$  to the existence of a path labeled by  $u$  from any vertex:  $\forall u \in A_G^* \forall s \in V_G \exists t (s \xrightarrow{u}_G t)$ . Thus a graph  $G$  is deterministic and source-complete if and only if for any label  $a \in A_G$ ,  $\xrightarrow{a}_G$  is a mapping from  $V_G$  into  $V_G$  or equivalently for any label word  $u \in A_G^*$ ,  $\xrightarrow{u}_G$  is a mapping.

A vertex  $s$  is an *out-simple vertex* if there are no two edges of source  $s$  with the same target:  $(s \xrightarrow{a}_G t \wedge s \xrightarrow{b}_G t) \implies a = b$  for any  $t \in V_G$  and  $a, b \in A_G$ . We also say that  $s$  is an *in-simple vertex* if there are no two edges of target  $s$  with the same source:  $(t \xrightarrow{a}_G s \wedge t \xrightarrow{b}_G s) \implies a = b$  for any  $t \in V_G$  and  $a, b \in A_G$ . A *simple vertex* is a vertex which is both in-simple and out-simple. A *simple graph* is a graph whose any vertex is simple (or in-simple, or out-simple).

## 3 Generalized Cayley graphs

We present graph-theoretic characterizations for the generalized Cayley graphs of magmas with a left identity, and with an identity.

Recall that a *magma* is a set  $M$  equipped with a binary operation  $\cdot : M \times M \rightarrow M$  that sends any two elements  $p, q \in M$  to the element  $p \cdot q$ . Given a subset  $Q \subseteq M$  and an injective mapping  $[\ ] : Q \rightarrow A$ , we define the *generalized Cayley graph*:

$$\mathcal{C}[M, Q] = \{ p \xrightarrow{[q]} p \cdot q \mid p \in M \wedge q \in Q \}$$

of vertex set  $M$  and of label set  $[Q] = \{ [q] \mid q \in Q \}$ .

► **Fact 1.** Any generalized Cayley graph of a magma is deterministic and source-complete.

The deterministic source-complete graphs are the graphs such that for any label word  $u$  and for any vertex  $s$ , there is a unique path from  $s$  labeled by  $u$ . Among the deterministic and source-complete graphs, we want to determine those that are generalized Cayley graphs.

Given a vertex  $r$  of a graph  $G$ , a binary operation  $\cdot$  on  $V_G$  is an *r-edge operation* of  $G$  if

$$r \xrightarrow{a} t \implies s \xrightarrow{a} s \cdot t \text{ for any } s, t \in V_G \text{ and } a \in A_G$$

Such an operation is illustrated below and will be refined later (see Figures 10,21,24).

$$\begin{array}{ccc} \begin{array}{c} \bullet \\ \xrightarrow{a} \\ s \end{array} & \begin{array}{c} \bullet \\ \xrightarrow{a} \\ s \cdot t \end{array} & \text{for} \quad \begin{array}{c} \bullet \\ \xrightarrow{a} \\ r \end{array} & \begin{array}{c} \bullet \\ \xrightarrow{a} \\ t \end{array} & \text{and} \quad a \in A \end{array}$$

► **Figure 2.** An  $r$ -edge operation on the vertex set of a graph.

In particular if a graph  $G$  has an  $r$ -edge operation with  $r$  source-complete then  $G$  is source-complete. For any deterministic and source-complete graph, the existence of an edge-operation from an out-simple vertex allows to express it as a generalized Cayley graph.

► **Lemma 3.** *Let  $r$  be an out-simple source-complete vertex of a deterministic graph  $G$ . Let  $\cdot$  be an  $r$ -edge operation of  $G$ . So  $G = \mathcal{C}[V_G, \rightarrow_G(r)]$  with  $[s] = a$  for any  $r \xrightarrow{a} s$ .*

**Proof.**

We denote by  $S_r = \rightarrow_G(r) = \{ s \mid \exists a \in A_G (r \xrightarrow{a} s) \}$  the set of successors of  $r$ .

As  $r$  is out-simple,  $[\ ]$  is a mapping from  $S_r$  into  $A_G$ . As  $G$  is deterministic,  $[\ ]$  is injective. As  $r$  is source-complete,  $[\ ]$  is surjective hence  $[\ ]$  is bijective.

Let us check that  $G = \mathcal{C}[V_G, S_r]$ .

⊆ : Let  $s \xrightarrow{a} t$ . As  $r$  is source-complete, there exists  $q$  such that  $r \xrightarrow{a} q$ . So  $[q] = a$ .

As  $\cdot$  is an  $r$ -edge operation, we have  $s \xrightarrow{a} s \cdot q$ .

As  $G$  is deterministic, we get  $t = s \cdot q$  hence  $s \xrightarrow{[q]} s \cdot q = t$ .

⊇ : Let  $s \xrightarrow{a} t$ . There exists (a unique)  $q \in S_r$  such that  $[q] = a$ .

So  $t = s \cdot q$  and  $r \xrightarrow{a} q$ . As  $\cdot$  is an  $r$ -edge operation, we have  $s \xrightarrow{a} s \cdot q = t$ . ◀

Recall that an element  $e$  of  $M$  is a *left identity* (resp. *right identity*) if  $e \cdot p = p$  (resp.  $p \cdot e = p$ ) for any  $p \in M$ . Note that any left identity of a magma is an out-simple vertex of its generalized Cayley graphs. This property with those of Fact 1 suffices to characterize the generalized Cayley graphs of magmas with a left identity.

► **Proposition 4.** *A graph is a generalized Cayley graph of a magma with a left identity if and only if it is deterministic, source-complete with an out-simple vertex.*

**Proof.**

⇒ : Let  $G = \mathcal{C}[M, Q]$  for some magma  $M$  with a left identity  $e$  and some  $Q \subseteq M$ .

By Fact 1, it remains to check that  $e$  is out-simple.

Let  $e \xrightarrow{a} s$  and  $e \xrightarrow{b} s$  with  $a, b \in A_G$ .

As  $A_G = [Q]$ , we have  $a = [p]$  and  $b = [q]$  for some  $p, q \in Q$ .

Thus  $s = e \cdot p = p$  and  $s = e \cdot q = q$  hence  $p = q$  i.e.  $a = b$ .

**XX:4 Cayley graphs**

$\Leftarrow$ : Let  $G$  be a deterministic and source-complete graph and let  $r$  be an out-simple vertex. We consider the magma  $(V_G, \cdot)$  with  $\cdot$  defined for any  $s, t \in V_G$  by

$$s \cdot t = \begin{cases} t & \text{if } s = r \\ x & \text{if } r \xrightarrow{a} t \text{ and } s \xrightarrow{a} x \text{ for some } a \in A_G \\ r & \text{otherwise.} \end{cases}$$

As  $r$  is out-simple and  $G$  is deterministic,  $\cdot$  is well-defined.

So  $r$  is a left identity and  $\cdot$  is an  $r$ -edge operation.

As  $G$  is source-complete and by Lemma 3,  $G$  is a generalized Cayley graph of  $(V_G, \cdot)$ .

Note that we could also defined  $\cdot$  for any  $s, t \in V_G$  by

$$s \cdot t = \begin{cases} x & \text{if } r \xrightarrow{a} t \text{ and } s \xrightarrow{a} x \text{ for some } a \in A_G \\ t & \text{otherwise.} \end{cases} \quad \blacktriangleleft$$

Let us introduce a graph notion. A vertex  $r$  of a graph  $G$  is a *loop-propagating vertex* if

$$r \xrightarrow{a} r \implies s \xrightarrow{a} s \text{ for any } s \in V_G \text{ and } a \in A_G$$

meaning that if  $r$  has a loop labeled by  $a$  then any vertex has a loop labeled by  $a$ . In particular any vertex without loop is loop-propagating.

If a magma  $M$  has a left identity  $e$  and a right identity  $e'$  then  $e = e \cdot e' = e'$  is the *identity* of  $M$  and we say that  $M$  is a *unital magma*. The identity  $e$  of a magma is a loop-propagating vertex for its generalized Cayley graphs. We can strengthen Proposition 4 to magmas with an identity.

**► Proposition 5.** *A graph is a generalized Cayley graph of a unital magma if and only if it is deterministic, source-complete with an out-simple and loop-propagating vertex.*

**Proof.**

$\implies$ : Let  $G = \mathcal{C}[M, Q]$  for some magma  $M$  with an identity  $e$  and some  $Q \subseteq M$ .

In the proof of Proposition 4, we have seen that  $e$  is out-simple. It remains to check that  $e$  is loop-propagating. Let  $e \xrightarrow{a} e$  with  $a \in A_G$  and let  $s \in V_G = M$ .

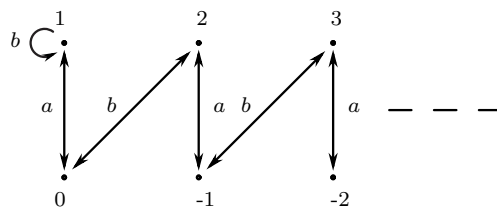
So  $a = [q]$  for some  $q \in Q$  hence  $e = e \cdot q = q$ . Thus  $s \xrightarrow{a} s \cdot q = s \cdot e = s$ .

$\Leftarrow$ : Let  $G$  be a deterministic and source-complete graph and let  $r$  be an out-simple and loop-propagating vertex. We have just to refine the operation  $\cdot$  of the previous proof as follows: for any  $s, t \in V_G$

$$s \cdot t = \begin{cases} t & \text{if } s = r \\ s & \text{if } t = r \\ x & \text{if } r \xrightarrow{a} t \text{ and } s \xrightarrow{a} x \text{ for some } a \in A_G \\ r & \text{otherwise.} \end{cases}$$

As  $r$  is out-simple, loop-propagating and as  $G$  is deterministic,  $\cdot$  is well-defined.  $\blacktriangleleft$

For instance consider the magma  $(\mathbb{Z}, \cdot)$  where  $m \cdot n = -m + n$  for any  $m, n \in \mathbb{Z}$ . There is no right identity and 0 is the unique left identity. The generalized Cayley graph  $G = \mathcal{C}[\mathbb{Z}, \{1, 2\}]$  with  $[1] = a$  and  $[2] = b$  is the following simple graph:



Any vertex other than 1 is loop-propagating. By Proposition 5,  $G$  is a generalized Cayley graph of a unital magma. Precisely by applying the construction given in its proof for  $r = 0$ , we get  $G = \mathcal{C}[\mathbb{Z}, \{1, 2\}]$  with  $[1] = a$  and  $[2] = b$  for the magma  $(\mathbb{Z}, \cdot)$  of identity 0 defined for any  $n$  by  $0 \cdot n = n$  and for any  $m \neq 0$  by

$$m \cdot n = \begin{cases} m & \text{if } n = 0 \\ -m + n & \text{if } n = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

#### 4 Cayley graphs of monoids

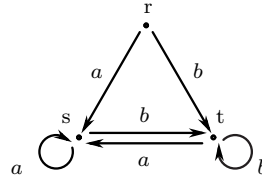
We present a graph theoretic characterization for the Cayley graphs of monoids. Such a characterization is based on a structural property describing a partial symmetry of these graphs: every vertex is the image of the unit element by an endomorphism. We express this general property by an elementary notion of propagation.

We write  $s \xrightarrow{u,v}_G t$  if  $s \xrightarrow{u}_G t$  and  $s \xrightarrow{v}_G t$ . We denote by  $s \xrightarrow{u,v}_G$  if  $s \xrightarrow{u,v}_G t$  for some  $t$ . A vertex  $r$  of a graph  $G$  is a *propagating vertex* if

$$r \xrightarrow{u,v} \implies s \xrightarrow{u,v} \text{ for any } s \in V_G \text{ and } u, v \in A_G^*$$

meaning that if there are two paths from  $r$  of the same target labeled by  $u$  and  $v$  then from any vertex, there are two paths of the same target labeled by  $u$  and  $v$ .

In particular for  $u = v \in A$ , a graph is source-complete if it has a source-complete propagating vertex. Furthermore for  $u = \varepsilon$  and  $v \in A$ , a propagating vertex is loop-propagating. For the following deterministic and source-complete graph:



the vertex  $r$  is propagating but the vertices  $s, t$  are not propagating: we have  $s \xrightarrow{\varepsilon, a}$  (resp.  $t \xrightarrow{\varepsilon, b}$ ) which is not the case for the other vertices.

Let us express differently a propagating vertex for a deterministic graph. Given a vertex  $r$  of a graph  $G$ , we denote by  $G_{\downarrow r}$  the restriction of  $G$  to the vertices accessible from  $r$ :

$$G_{\downarrow r} = \{ (s, a, t) \in G \mid r \xrightarrow{*}_G s \}.$$

A *morphism* from a graph  $G$  into a graph  $H$  is a mapping  $h$  from  $V_G$  into  $V_H$  such that

$$s \xrightarrow{a}_G t \implies h(s) \xrightarrow{a}_H h(t).$$

A propagating vertex  $r$  of a deterministic graph  $G$  is a vertex from which there is for any vertex  $s$  a morphism from  $G_{\downarrow r}$  into  $G$  linking  $r$  to  $s$ .

► **Lemma 6.** For any deterministic graph  $G$  and vertices  $r, s$ , we have

$$r \xrightarrow{u,v}_G \implies s \xrightarrow{u,v}_G \text{ for any } u, v \in A_G^*$$

if and only if there is a morphism  $h$  from  $G_{\downarrow r}$  to  $G_{\downarrow s}$  such that  $h(r) = s$ .

**Proof.**

$\Leftarrow$ : Immediate for any graph.

$\Rightarrow$ : As  $G$  is deterministic, it allows to define the mapping  $h : V_{G_{\downarrow r}} \rightarrow V_{G_{\downarrow s}}$  by

$$h(p) = q \text{ if } r \xrightarrow{u}_G p \text{ and } s \xrightarrow{u}_G q \text{ for some } u \in A^*.$$

Thus  $h(r) = s$ . It remains to check that  $h$  is a morphism.

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Let  $p \xrightarrow{a}_{G \downarrow r} q$ . There exists  $u \in A_G^*$  such that  $r \xrightarrow{u}_G p$ .  
 As  $r \xrightarrow{ua}_G$ , we have  $s \xrightarrow{ua}_G$  i.e.  $s \xrightarrow{u}_G p' \xrightarrow{a}_G q'$  for some vertices  $p', q'$ .  
 As  $G$  is deterministic,  $h(p) = p'$  and  $h(q) = q'$  hence  $h(p) \xrightarrow{a}_{G \downarrow s} h(q)$ . ◀

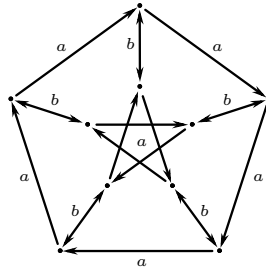
Recall that a magma  $(M, \cdot)$  is a *semigroup* if  $\cdot$  is associative:  $(p \cdot q) \cdot r = p \cdot (q \cdot r)$  for any  $p, q, r \in M$ . A *monoid*  $M$  is a semigroup with an identity 1. In that case, 1 is a propagating vertex of its generalized Cayley graphs.

► **Fact 7.** For any generalized Cayley graph of a monoid, 1 is a propagating vertex.

**Proof.**

Let  $1 \xrightarrow{u,v}_G s$  with  $u, v \in A_G^*$ . We have  $u = [p_1] \dots [p_m]$  and  $v = [q_1] \dots [q_n]$  for some  $m, n \geq 0$  and  $p_1, \dots, p_m, q_1, \dots, q_n \in Q$ .  
 Thus  $(\dots(1 \cdot p_1) \dots) \cdot p_m = s = (\dots(1 \cdot q_1) \dots) \cdot q_n$ . As  $\cdot$  is associative,  $p_1 \dots p_m = q_1 \dots q_n$ .  
 Let  $t$  be any vertex of  $G$ . So  $t \xrightarrow{u}_G (\dots(t \cdot p_1) \dots) \cdot p_m = t \cdot (p_1 \dots p_m)$  and  
 $t \xrightarrow{v}_G t \cdot (q_1 \dots q_n) = t \cdot (p_1 \dots p_m)$ . Hence  $t \xrightarrow{u,v}_G$ . ◀

For instance the following oriented and labeled Petersen graph:



is not a generalized Cayley graph of a monoid since it has no propagating vertex: for  $s$  an inner vertex and  $t$  an outer vertex, we have  $s \xrightarrow{\varepsilon, ababa}$  and  $t \xrightarrow{\varepsilon, baaba}$  while  $t \not\xrightarrow{ababa} t$  and  $s \not\xrightarrow{baaba} s$ .

Recall that the *submonoid generated* by a subset  $Q$  of a monoid  $M$  is the least submonoid  $Q^* = \{ q_1 \dots q_n \mid n \geq 0 \wedge q_1, \dots, q_n \in Q \}$  containing  $Q$ . A *Cayley graph of a monoid*  $M$  is a generalized Cayley graph  $\mathcal{C}[M, Q]$  for  $M$  generated by  $Q$  i.e.  $M = Q^*$ .

► **Fact 8.** For any Cayley graph of a monoid, 1 is a root.

**Proof.**

Let  $G = \mathcal{C}[M, Q]$  for some monoid  $M$  generated by  $Q$ .  
 For any  $m \in M$ , we have  $m = q_1 \dots q_n$  for some  $n \geq 0$  and  $q_1, \dots, q_n \in Q$ , thus  $1 \xrightarrow{v}_G m$  for the label word  $v = [q_1] \dots [q_n]$ . ◀

Thus a generalized Cayley graph of a monoid is a Cayley graph if and only if it is rooted by 1. We can give a simple graph-theoretic characterization for the Cayley graphs of monoids.

► **Theorem 9.** A graph is a Cayley graph of a monoid if and only if it is deterministic, source-complete with a propagating out-simple root.

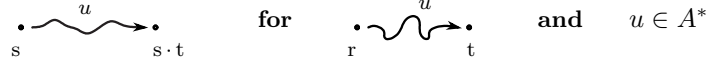
**Proof.**

$\implies$ : Let  $G = \mathcal{C}[M, Q]$  for some monoid  $M$  and some  $Q \subseteq M$  such that  $Q^* = M$ .  
 By Fact 1,  $G$  is deterministic and source-complete. By Proposition 4, 1 is out-simple.  
 By Facts 7 and 8, the vertex 1 is a propagating root.

$\Leftarrow$ : Let  $G$  be a deterministic and source-complete graph and  $r$  be a propagating out-simple root. Let  $\cdot$  be an  $r$ -path operation i.e. a binary operation on  $V_G$  such that

$$r \xrightarrow{u} t \implies s \xrightarrow{u} s \cdot t \text{ for any } s, t \in V_G \text{ and } u \in A_G^*$$

which is illustrated below. Any  $r$ -path operation is an  $r$ -edge operation.



► **Figure 10.** An  $r$ -path operation on the vertex set of a graph.

i) Let  $s, t \in V_G$ . Let us check that  $s \cdot t$  is well-defined.

As  $r$  is a root, there exists  $u \in A_G^*$  such that  $r \xrightarrow{u}_G t$ .

As  $G$  is source-complete, there exists  $x$  such that  $s \xrightarrow{u}_G x$ .

Let  $r \xrightarrow{v}_G t$  for some  $v \in A_G^*$ . So  $r \xrightarrow{u,v}_G$ . As  $r$  is propagating, we have  $s \xrightarrow{u,v}_G$ .

As  $G$  is deterministic, we get  $s \xrightarrow{v}_G x$ . Thus  $s \cdot t = x$  is well-defined.

The path operation  $\cdot$  is an  $r$ -edge operation.

By Lemma 3,  $G = \mathcal{C}[V_G, \rightarrow_G(r)]$  where  $[s] = a$  for any  $r \xrightarrow{a}_G s$ .

It remains to check that  $(V_G, \cdot)$  is a monoid generated by  $\rightarrow_G(r)$ .

ii) Let us check that  $r$  is a left identity.

Let  $s \in V_G$ . As  $r$  is a root, there exists  $u \in A_G^*$  such that  $r \xrightarrow{u}_G s$ .

By definition of  $\cdot$ , we have  $r \xrightarrow{u}_G r \cdot s$ . As  $G$  is deterministic, we get  $r \cdot s = s$ .

iii) Let us check that  $r$  is also a right identity.

Let  $s \in V_G$ . As  $r \xrightarrow{\varepsilon}_G r$  and by definition of  $\cdot$ , we have  $s \xrightarrow{\varepsilon}_G s \cdot r$  i.e.  $s \cdot r = s$ .

iv) Let us check that  $\cdot$  is associative. Let  $x, y, z \in V_G$ .

As  $r$  is a root, there exists  $v, w \in A_G^*$  such that  $r \xrightarrow{v}_G y$  and  $r \xrightarrow{w}_G z$ .

Thus  $x \xrightarrow{v}_G x \cdot y \xrightarrow{w}_G (x \cdot y) \cdot z$  and  $y \xrightarrow{w}_G y \cdot z$ . So  $r \xrightarrow{vw}_G y \cdot z$  hence  $x \xrightarrow{vw}_G x \cdot (y \cdot z)$ .

As  $G$  is deterministic, we get  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ .

v) Let us check that  $\rightarrow_G(r)$  generates  $V_G$ . Let  $s \in V_G$ .

There exists a path  $r = s_0 \xrightarrow{a_1} s_1 \dots s_{n-1} \xrightarrow{a_n} s_n = s$ .

As  $r$  is source-complete, there exists  $r_1, \dots, r_n$  such that  $r \xrightarrow{a_1} r_1, \dots, r \xrightarrow{a_n} r_n$ .

By induction on  $1 \leq i \leq n$ , we have  $s_i = s_{i-1} \cdot r_i$ .

Therefore  $s = (\dots (r \cdot r_1) \dots) \cdot r_n = r_1 \cdot \dots \cdot r_n \in (\rightarrow_G(r))^*$ . ◀

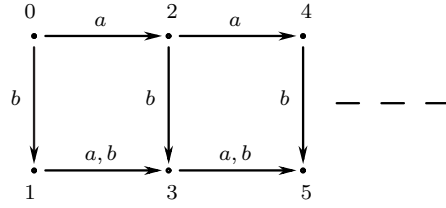
By Theorem 9, the graph of the first figure of this section is a Cayley graph of a monoid: it is  $\mathcal{C}[\{r, s, t\}, \{s, t\}]$  with  $[s] = a$ ,  $[t] = b$  for the monoid  $(\{r, s, t\}, \cdot)$  where  $\cdot$  is the  $r$ -path operation defined by the following Cayley table:

$\cdot$	$r$	$s$	$t$
$r$	$r$	$s$	$t$
$s$	$s$	$s$	$t$
$t$	$t$	$s$	$t$

Another example is given by the following infinite graph:

$G = \{ n \xrightarrow{a} n+2 \mid n \in \mathbb{N} \} \cup \{ 2n \xrightarrow{b} 2n+1 \mid n \in \mathbb{N} \} \cup \{ 2n+1 \xrightarrow{a,b} 2n+3 \mid n \in \mathbb{N} \}$   
represented as follows:





Such a graph is deterministic, source-complete, and the root 0 is out-simple and propagating. By Theorem 9,  $G$  is a Cayley graph of a monoid. Precisely  $G = \mathcal{C}[\mathbb{N}, \{1, 2\}]$  where  $[1] = b$  and  $[2] = a$  for the 0-path operation  $\cdot$  defined for any  $p, q \in \mathbb{N}$  by

$$p \cdot q = \begin{cases} p + q & \text{if } p \text{ or } q \text{ is even,} \\ p + q + 1 & \text{if } p \text{ and } q \text{ are odd} \end{cases}$$

which is indeed associative.

### 5 Cayley graphs of semigroups

We apply the previous characterization for the Cayley graphs of monoids to the Cayley graphs of semigroups.

Recall that a *Cayley graph of a semigroup*  $M$  is a generalized Cayley graph  $\mathcal{C}[M, Q]$  such that  $M = Q^+$  whose  $Q^+ = \{ q_1 \dots q_n \mid n > 0 \wedge q_1, \dots, q_n \in Q \}$  is the *subsemigroup generated by*  $Q$ . Let us extend Theorem 9 into a characterization of these graphs.

► **Theorem 11.** *A graph  $G$  is a Cayley graph of a semigroup if and only if it is deterministic and there is an injection  $i$  from  $A_G$  into  $V_G$  such that*

$$G \text{ is accessible from } i(A_G) \text{ and } i(a) \xrightarrow{u} \xleftarrow{v} i(b) \implies s \xrightarrow{au, bv} \text{ for any } s \in V_G, a, b \in A_G \text{ and } u, v \in A_G^*.$$

**Proof.**

⇒: Let  $G = \mathcal{C}[M, Q]$  for some semigroup  $(M, \cdot)$  generated by  $Q$ .

By Fact 1,  $G$  is deterministic.

If  $M$  is not a monoid *i.e.* it has no identity, we turn  $M$  into a monoid  $M' = M \cup \{1\}$  by just adding an identity 1 *i.e.*  $p \cdot 1 = 1 \cdot p = p$  for any  $p \in M'$ .

Let  $M' = M$  when  $M$  is a monoid.

In both cases,  $M'$  is a monoid of identity 1. Let  $G' = \mathcal{C}[M', Q] = G \cup \{ 1 \xrightarrow{[q]} q \mid q \in Q \}$ .

Let  $i : A_G \rightarrow Q$  defined by  $i([q]) = q$  for any  $q \in Q$ .

As  $M$  is generated by  $Q$ , the graph  $G$  is accessible from  $Q = i(A_G)$ .

Let  $i([p]) \xrightarrow{u} \xleftarrow{v} i([q])$  for some  $p, q \in Q$  and  $u, v \in A_G^*$ .

So  $p \xrightarrow{u} \xleftarrow{v} q$  hence  $1 \xrightarrow{[p]u, [q]v} G'$ . By Fact 7, 1 is a propagating vertex of  $G'$ .

Let  $s \in V_G$ . Therefore  $s \xrightarrow{[p]u, [q]v} G'$  hence  $s \xrightarrow{[p]u, [q]v} G$ .

⇐: Let  $G$  be a deterministic graph.

Let an injection  $i : A_G \rightarrow V_G$  such that  $G$  is accessible from  $Q = i(A_G)$  and such that

$$s \xrightarrow{au, bv} \text{ when } i(a) \xrightarrow{u} \xleftarrow{v} i(b) \text{ for any } s \in V_G, a, b \in A_G \text{ and } u, v \in A_G^*. \quad (1)$$

Note that by (1), we get  $s \xrightarrow{a} G$  for any  $s \in V_G$  and  $a \in A_G$ , hence  $G$  is source-complete.

We take a new vertex  $r$  and we define the graph

$$\widehat{G} = G \cup \{ r \xrightarrow{a} i(a) \mid a \in A_G \}.$$

So  $\widehat{G}$  remains deterministic and source-complete, and  $r$  is out-simple.

As  $G$  is accessible from  $Q$ ,  $r$  is a root of  $\widehat{G}$ . By (1),  $r$  is a propagating vertex of  $\widehat{G}$ .

By the proof of Theorem 9,  $V_{\widehat{G}} = V_G \cup \{r\}$  is a monoid generated by  $Q$  for the  $r$ -path

operation  $\cdot$  of identity  $r$  with  $\widehat{G} = \mathcal{C}[V_{\widehat{G}}, Q] = \mathcal{C}[V_G, Q] \cup \{ r \xrightarrow{[q]} q \mid q \in Q \}$  where  $i([q]) = q$  for every  $q \in Q$ .

As  $r$  is not the target of an edge of  $\widehat{G}$  and by definition,  $\cdot$  remains an internal operation on  $V_G$  i.e.  $p \cdot q \neq r$  for any  $p, q \in V_G$ . Thus  $\cdot$  remains associative on  $V_G$ .

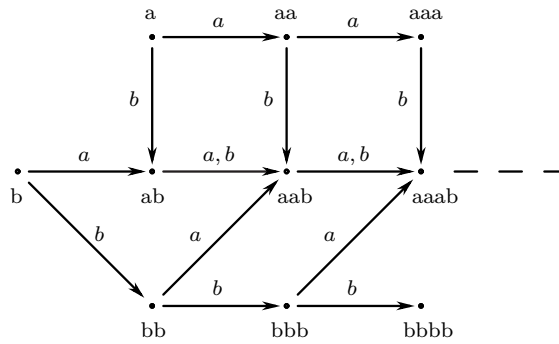
Finally  $G = \mathcal{C}[V_G, Q]$  and  $(V_G, \cdot)$  is a semigroup. ◀

For instance, the two semilines  $G_a = \{ n \xrightarrow{a} n+1 \mid n \in \mathbb{N} - \{0\} \} \cup \{ n \xrightarrow{a} n-1 \mid n \in \mathbb{Z} - \mathbb{N} \}$  is not a Cayley graph of a semigroup since it is not connected and has a unique label. On the other hand,  $G_a \cup G_b$  is a Cayley graph of a semigroup. Precisely  $G_a \cup G_b = \mathcal{C}[\mathbb{Z} - \{0\}, \{1, -1\}]$  where  $[1] = a$  and  $[-1] = b$  for the associative operation  $\cdot$  defined by

$$m \cdot n = m + \text{sign}(m) |n| \text{ for any } m, n \in \mathbb{Z} - \{0\}$$

where  $\text{sign}(m) = 1$  for any  $m > 0$  and  $\text{sign}(m) = -1$  for any  $m < 0$ .

Another example is given by the following graph  $G$ :



whose its vertex set  $V$  is the regular set  $a^*a \cup a^*b \cup b^*bb$ .

By Theorem 11,  $G$  is the Cayley graph  $\mathcal{C}(V, \{a, b\})$  for the associative binary operation  $\cdot$  on  $V$  defined for any  $m, n > 0, i, j \geq 0, p, q > 1$  by the following Cayley table:

$\cdot$	$a^n$	$a^j b$	$b^q$
$a^m$	$a^{m+n}$	$a^{j+m} b$	$a^{m+q-1} b$
$a^i b$	$a^{i+n} b$	$a^{i+j+1} b$	$a^{i+q} b$
$b^p$	$a^{n+p-1} b$	$a^{j+p} b$	$b^{p+q}$

## 6 Cayley graphs of right-cancellative and left-cancellative monoids

We strengthen the characterization for the Cayley graphs of monoids (Theorem 9) to the Cayley graphs of right-cancellative monoids and of left-cancellative monoids.

Recall that a magma  $(M, \cdot)$  is *right-cancellative* if  $p \cdot r = q \cdot r \implies p = q$  for any  $p, q, r \in M$ . Let us give a basic property of their generalized Cayley graphs.

A graph  $G$  is *co-deterministic* if there are no two edges of the same target and label i.e.

$$(s \xrightarrow{a}_G r \wedge t \xrightarrow{a}_G r) \implies s = t \text{ for any } r, s, t \in V_G \text{ and } a \in A_G.$$

► **Fact 12.** Any generalized Cayley graph of a right-cancellative magma is co-deterministic.

By adding the condition of co-determinism in Theorem 9, we get a characterization for the Cayley graphs of right-cancellative monoids.

► **Theorem 13.** A graph is a Cayley graph of a right-cancellative monoid if and only if it is deterministic, co-deterministic, source-complete with a propagating out-simple root.

## XX:10 Cayley graphs

**Proof.**

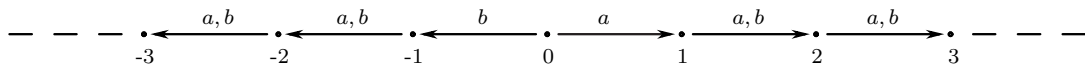
$\implies$ : By Theorem 9 and Fact 12.

$\impliedby$ : Let  $G$  be a deterministic and co-deterministic source-complete graph with a propagating out-simple root  $r$ . It remains to check that the  $r$ -path operation  $\cdot$  defined in the proof of Theorem 9 is right-cancellative. Let  $s, s', t \in V_G$  such that  $s \cdot t = s' \cdot t$ .

There exists  $u \in A_G^*$  such that  $r \xrightarrow{u} t$ . So  $s \xrightarrow{u} s \cdot t$  and  $s' \xrightarrow{u} s' \cdot t = s \cdot t$ .

As  $G$  is co-deterministic, we get  $s = s'$ .  $\blacktriangleleft$

For instance, let us consider the following source-complete, deterministic and co-deterministic graph  $G$ :



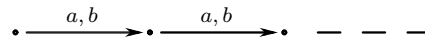
The vertex 0 is the unique out-simple vertex, the unique root and the unique propagating vertex. By Theorem 13,  $G$  is a Cayley graph of a right-cancellative monoid. Precisely  $G = \mathcal{C}[\mathbb{Z}, \{1, -1\}]$  where  $[1] = a$  and  $[-1] = b$  for the right-cancellative monoid  $(\mathbb{Z}, \cdot)$  defined for any  $m, n \in \mathbb{Z}$  by

$$m \cdot n = \begin{cases} n & \text{if } m = 0, \\ m + |n| & \text{if } m > 0, \\ m - |n| & \text{if } m < 0. \end{cases}$$

We will now consider the Cayley graphs of left-cancellative monoids. Recall that a magma  $(M, \cdot)$  is *left-cancellative* if  $r \cdot p = r \cdot q \implies p = q$  for any  $p, q, r \in M$ .

► **Fact 14.** Any generalized Cayley graph of a left-cancellative magma is simple.

Let us strengthen Theorem 9 in order to characterize the Cayley graphs of left-cancellative monoids. We say that a graph is a *propagating graph* when all its vertices are propagating. The previous graph is not propagating since only its root is a propagating vertex. Here is a propagating graph:



Let us express differently the propagating graphs when they are deterministic.

If a morphism  $h$  from a graph  $G$  into a graph  $H$  is bijective and  $h^{-1}$  is a morphism,  $h$  is called an *isomorphism* from  $G$  to  $H$ . We say vertices  $s, t$  of a graph  $G$  are *accessible-isomorphic* if  $t = h(s)$  for some isomorphism  $h$  from  $G_{\downarrow s}$  to  $G_{\downarrow t}$ . By Lemma 6,

*a deterministic graph is propagating if and only if all its vertices are accessible-isomorphic.*

When a monoid is left-cancellative, its generalized Cayley graphs are propagating.

► **Fact 15.** Any generalized Cayley graph of a left-cancellative monoid is propagating.

**Proof.**

Let  $r \xrightarrow{u,v}_G s$  with  $u, v \in A_G^*$ . We have  $u = [p_1] \dots [p_m]$  and  $v = [q_1] \dots [q_n]$  for some  $m, n \geq 0$  and  $p_1, \dots, p_m, q_1, \dots, q_n \in Q$ .

Thus  $(\dots(r \cdot p_1) \dots) \cdot p_m = s = (\dots(r \cdot q_1) \dots) \cdot q_n$ .

As  $\cdot$  is associative and left-cancellative, we get  $p_1 \dots p_m = q_1 \dots q_n$ .

Let  $t$  be any vertex of  $G$ . So  $t \xrightarrow{u,v}_G t \cdot (p_1 \dots p_m) = t \cdot (q_1 \dots q_n)$ .  $\blacktriangleleft$

We get a graph-theoretic characterization for the Cayley graphs of left-cancellative monoids: they are the Cayley graphs of monoids which are propagating.

► **Theorem 16.** *A graph is a Cayley graph of a left-cancellative monoid if and only if it is rooted, simple, deterministic and propagating.*

**Proof.**

⇒: By Facts 1, 8, 14, 15.

⇐: Let  $G$  be a simple, deterministic and propagating graph with a root  $r$ .

It remains to check that the  $r$ -path operation  $\cdot$  defined in the proof of Theorem 9 is left-cancellative. Assume that  $s \cdot t = s \cdot t'$ .

As  $r$  is a root, there exists  $u, v \in A_G^*$  such that  $r \xrightarrow{u}_G t$  and  $r \xrightarrow{v}_G t'$ .

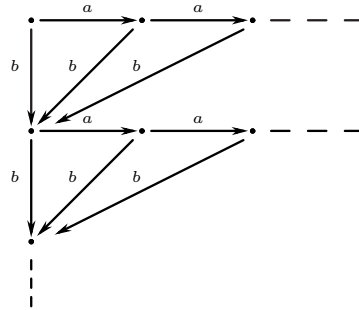
By definition of  $\cdot$  we get  $s \xrightarrow{u}_G s \cdot t$  and  $s \xrightarrow{v}_G s \cdot t' = s \cdot t$ . Thus  $s \xrightarrow{u,v}_G$ .

As  $G$  is propagating,  $r \xrightarrow{u,v}_G$ . As  $G$  is deterministic, it follows that  $t = t'$ . ◀

For instance, let us consider the following graph:

$$G = \{ b^m a^n \xrightarrow{a} b^m a^{n+1} \mid m, n \geq 0 \} \cup \{ b^m a^n \xrightarrow{b} b^{m+1} \mid m, n \geq 0 \}$$

represented below.



It is a *skeleton* of the graph of  $\omega^2$  where  $a$  is the successor and  $b$  goes to the next limit ordinal:  $(V_G, \rightarrow_G^*)$  is isomorphic to  $(\omega^2, \leq)$ . This graph is not co-deterministic hence by Fact 12 is not a generalized Cayley graph of a right-cancellative monoid. By Theorem 16,  $G = \mathcal{C}(V_G, \{a, b\})$  where  $(V_G, \cdot)$  is the left-cancellative monoid defined for any  $m, n, p, q \geq 0$  by

$$b^m a^n \cdot b^p a^q = \begin{cases} b^m a^{n+q} & \text{if } p = 0 \\ b^{m+p} a^q & \text{if } p \neq 0. \end{cases}$$

As the converse of Theorems 13 and 16 define the same monoid, it follows a characterization for the Cayley graphs of monoids which are *cancellative i.e.* both left-cancellative and right-cancellative.

► **Corollary 17.** *A graph is a Cayley graph of a cancellative monoid if and only if it is rooted, simple, deterministic, co-deterministic, propagating.*

Let us summarize the previous characterizations.

	deterministic + source-complete + out-simple vertex $r$ (4)	Generalized Cayley graphs of magmas with a left identity
	-----	
	+ $r$ loop-propagating (5)	of magmas with an identity
	-----	
	+ $r$ propagating root (9)	Cayley graphs of monoids
	-----	
of left-cancellative monoids	+ propagating (16)   (17)   + co-deterministic (13)	of right-cancellative monoids

It remains to extend the path propagation to chains in Corollary 17 to get a characterization for the Cayley graphs of groups.

## 7 Cayley graphs of groups

We present two graph-theoretic characterizations for the Cayley graphs of groups, one in a strong form [1] and the other in a weak form [2]. It follows a characterization for the generalized Cayley graphs of groups.

We start by introducing some basic graph notions. We need to circulate in a graph  $G$  in the direct and inverse direction of the arrows. Let  $\bar{\cdot} : A_G \rightarrow A - A_G$  be an injective mapping whose image  $\overline{A_G} = \{ \bar{a} \mid a \in A_G \}$  is a disjoint copy of  $A_G$ . This allows to define the *inverse* of  $G$  as the graph  $\overline{G} = \{ t \xrightarrow{\bar{a}} s \mid s \xrightarrow{a} t \}$ . So  $V_{\overline{G}} = V_G$  and  $A_{\overline{G}} = \overline{A_G}$ .

A path of  $G \cup \overline{G}$  i.e.  $s \xrightarrow{u}_{G \cup \overline{G}} t$  with  $u \in (A_G \cup \overline{A_G})^*$  is a *chain* of  $G$  also denoted by  $s \xrightarrow{u}_G t$  where  $s \xrightarrow{\bar{a}}_G t$  means that  $t \xrightarrow{a}_G s$  for any  $a \in A_G$ . So the chain  $r \xrightarrow{a}_G s \xrightarrow{\bar{b}}_G t$  corresponds to  $r \xrightarrow{a}_G s \xleftarrow{b}_G t$ . We write  $\longleftrightarrow_G$  for the *adjacency relation*  $\rightarrow_{G \cup \overline{G}}$  i.e.  $s \longleftrightarrow_G t$  for  $s \rightarrow_G t$  or  $t \rightarrow_G s$ . We denote by  $\longleftrightarrow_G^*$  for the *chain relation*  $\rightarrow_{G \cup \overline{G}}^*$ . Recall that a graph  $G$  is *connected* if  $G \cup \overline{G}$  is strongly connected i.e.  $s \longleftrightarrow_G^* t$  for all  $s, t \in V_G$ . So  $G$  is co-deterministic if  $\overline{G}$  is deterministic. We say that  $G$  is *target-complete* if  $\overline{G}$  is source-complete i.e. for any label  $a$  and any vertex  $s$ , there is at least one edge labeled by  $a$  of target  $s$ .

A *chain-propagating vertex*  $r$  of a graph  $G$  is a propagating vertex of  $G \cup \overline{G}$  i.e.

$$r \xrightarrow{u,v}_G \implies s \xrightarrow{u,v}_G \text{ for any } s \in V_G \text{ and } u, v \in (A_G \cup \overline{A_G})^*$$

meaning that if there are two chains from  $r$  of the same target labeled by  $u$  and  $v$  then from any vertex, there are two chains of the same target labeled by  $u$  and  $v$ .

We say that a graph is a *chain-propagating graph* when all its vertices are chain-propagating. Let us recall the vertex-transitivity of a graph. An *automorphism* of  $G$  is an isomorphism from  $G$  to  $G$ . Two vertices  $s, t$  of a graph  $G$  are *isomorphic* and we write  $s \simeq_G t$  if  $t = h(s)$  for some automorphism  $h$  of  $G$ . A graph  $G$  is *vertex-transitive* if its automorphism group acts transitively upon its vertices i.e. if all its vertices are isomorphic:  $s \simeq_G t$  for every  $s, t \in V_G$ . This implies that the graph is chain-propagating. The converse is true when the graph is both deterministic and co-deterministic.

► **Lemma 18.** *A deterministic and co-deterministic graph is vertex-transitive if and only if it is chain-propagating.*

**Proof.**

$\implies$ : Let  $G$  be a vertex-transitive graph and  $s \xrightarrow{u,v}_G$  for some  $u, v \in (A_G \cup \overline{A_G})^*$ .

For any  $t \in V_G$ , we have  $s \simeq_G t$  hence  $t \xrightarrow{u,v}_G$  thus  $s$  is chain-propagating.

$\impliedby$ : Let  $G$  be a deterministic and co-deterministic graph which is chain-propagating.

In particular  $G$  is source and target-complete. Let  $s, t \in V_G$ . Let us check that  $s \simeq_G t$ .

As  $s, t$  are chain-propagating, we have  $s \xrightarrow{u,v}_G \iff t \xrightarrow{u,v}_G$  for any  $u, v \in (A_G \cup \overline{A_G})^*$ .

So we can define the mapping  $f_{s,t} : V_G \rightarrow V_G$  for any  $r \in V_G$  by

$$f_{s,t}(r) = \begin{cases} r' & \text{if } s \xrightarrow{u} r \text{ and } t \xrightarrow{u} r' \text{ for some } u \in (A_G \cup \overline{A_G})^* \\ r' & \text{otherwise and if } s \xrightarrow{u} r' \text{ and } t \xrightarrow{u} r \text{ for some } u \in (A_G \cup \overline{A_G})^* \\ r & \text{otherwise.} \end{cases}$$

Therefore  $f_{s,t}$  is a bijective morphism of inverse  $f_{t,s}$  with  $f_{s,t}(s) = t$ , hence  $s \simeq_G t$ . ◀

A group  $(M, \cdot)$  is a monoid whose each element  $p$  has an *inverse*  $p^{-1}$ :  $p \cdot p^{-1} = p^{-1} \cdot p = 1$ .

► **Fact 19.** Any generalized Cayley graph of a group is co-deterministic, simple and vertex-transitive.

**Proof.**

Let  $G$  be a generalized Cayley graph of a group.

The graph  $G$  is co-deterministic: for  $r \xrightarrow{[q]} t$  and  $s \xrightarrow{[q]} t$ ,  $r \cdot q = t = s \cdot q$  thus  $r = s$ .

The graph  $G$  is simple: for  $s \xrightarrow{[p]} t$  and  $s \xrightarrow{[q]} t$ ,  $s \cdot p = t = s \cdot q$  thus  $p = q$ .

It remains to check that  $G$  is vertex-transitive.

Let  $r \xrightarrow{u,v} s$  for  $u, v \in (A_G \cup \overline{A_G})^*$ . Let  $t \in V_G$ .

By Lemma 18, it remains to check that  $t \xrightarrow{u,v}$ . As

$$x \xrightarrow{[q]} y \iff y \xrightarrow{[q]} x \iff x = y \cdot q \iff y = x \cdot q^{-1}$$

we extend  $[\ ]$  on  $Q^{-1} - Q$  by  $[q^{-1}] = \overline{[q]}$  for any  $q \in Q$  such that  $q^{-1} \notin Q$ .

Thus  $u = [p_1] \dots [p_m]$  and  $v = [q_1] \dots [q_n]$  for some  $m, n \geq 0$  and  $p_1, \dots, p_m, q_1, \dots, q_n$  in  $Q \cup Q^{-1}$ . So  $r \cdot p_1 \dots p_m = s = r \cdot q_1 \dots q_n$  i.e.  $p_1 \dots p_m = q_1 \dots q_n$ .

Hence  $t \xrightarrow{u,v} t \cdot p_1 \dots p_m = t \cdot q_1 \dots q_n$ . ◀

Recall that a *Cayley graph of a group*  $M$  is a generalized Cayley graph  $\mathcal{C}[M, Q]$  such that  $M$  is generated by  $Q$  i.e.  $M$  is equal to the least subgroup  $(Q \cup Q^{-1})^*$  containing  $Q$  where  $Q^{-1} = \{ q^{-1} \mid q \in Q \}$  is the set of inverses of the elements in  $Q$ .

We can give a simple characterization for the Cayley graphs of groups.

► **Theorem 20.** For any graph  $G$ , the following properties are equivalent:

- a)  $G$  is a Cayley graph of a group,
- b)  $G$  is connected, deterministic, co-deterministic, simple, vertex-transitive,
- c)  $G$  is connected, deterministic, co-deterministic, with a chain-propagating simple vertex which is source and target-complete.

**Proof.**

(a)  $\implies$  (b): Let  $G = \mathcal{C}[M, Q]$  for some group  $(M, \cdot)$  generated by  $Q$ .

By Facts 1 and 19, it remains to check that  $G$  is connected. Let  $s \in V_G = M$ .

So  $s = q_1^{\epsilon_1} \dots q_n^{\epsilon_n}$  for some  $n \geq 0$ ,  $q_1, \dots, q_n \in Q$  and  $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ .

Thus  $1 \xrightarrow{a_1 \dots a_n} s$  where for any  $1 \leq i \leq n$ ,  $a_i = [q_i]$  if  $\epsilon_i = 1$  and  $a_i = \overline{[q_i]}$  if  $\epsilon_i = -1$ .

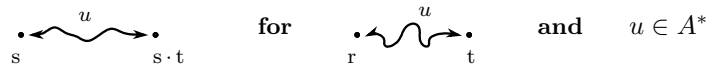
(b)  $\implies$  (c): any vertex-transitive graph is source and target-complete, and by Lemma 18.

(c)  $\implies$  (a): Let  $r$  be a chain-propagating vertex of  $G$  which is also simple, source and target-complete. So  $G$  is source-complete and target-complete.

We consider the magma  $(V_G, \cdot)$  where  $\cdot$  is an  $r$ -chain operation:

$$r \xrightarrow{u} t \implies s \xrightarrow{u} s \cdot t \text{ for any } s, t \in V_G \text{ and } u \in (A_G \cup \overline{A_G})^*$$

which is illustrated below. Any  $r$ -chain operation is an  $r$ -path operation.



► **Figure 21.** An  $r$ -chain operation on the vertex set of a graph.

i) Let  $s, t \in V_G$ . Let us check that  $s \cdot t$  is well-defined.

As  $G$  is connected, there exists  $u \in (A_G \cup \overline{A_G})^*$  such that  $r \xrightarrow{u} t$ .

As  $r$  is chain-propagating, there exists  $x$  such that  $s \xrightarrow{u} x$ .

Let  $r \xrightarrow{v} t$  for some  $v \in (A_G \cup \overline{A_G})^*$ . So  $r \xrightarrow{u,v} x$ .

As  $r$  is chain-propagating, we have  $s \xrightarrow{u,v} x$ .

As  $G$  is deterministic and co-deterministic, we get  $s \xrightarrow{v} x$ . Thus  $s \cdot t = x$  is well-defined.

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The chain operation  $\cdot$  is an  $r$ -edge operation.

By Lemma 3,  $G = \mathcal{C}[V_G, \rightarrow_G(r)]$  where  $[s] = a$  for any  $r \xrightarrow{a} s$ .

It remains to check that  $(V_G, \cdot)$  is a group.

In that case and having  $\mathcal{C}[V_G, \rightarrow_G(r)]$  connected, it is generated by  $\rightarrow_G(r)$ .

ii) Let us check that  $r$  is an identity. Let  $s \in V_G$ .

As  $r \xrightarrow{\varepsilon} r$  and by definition of  $\cdot$ , we have  $s \xrightarrow{\varepsilon} s \cdot r$  i.e.  $s \cdot r = s$ .

Furthermore and as  $G$  is connected, there exists  $u \in (A_G \cup A_{\overline{G}})^*$  such that  $r \xrightarrow{u} s$ .

By definition of  $\cdot$ , we have  $r \xrightarrow{u} r \cdot s$ . As  $G$  is deterministic and co-deterministic,  $r \cdot s = s$ .

iii) Let us check that  $\cdot$  is associative. Let  $x, y, z \in V_G$ .

As  $G$  is connected, there exists  $v, w \in (A_G \cup A_{\overline{G}})^*$  such that  $r \xrightarrow{v} y$  and  $r \xrightarrow{w} z$ .

Thus  $x \xrightarrow{v} x \cdot y \xrightarrow{w} (x \cdot y) \cdot z$  and  $y \xrightarrow{w} y \cdot z$ . So  $r \xrightarrow{vw} y \cdot z$  hence  $x \xrightarrow{vw} x \cdot (y \cdot z)$ .

As  $G$  is deterministic and co-deterministic, we get  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ .

iv) Let  $s \in V_G$ . Let us check that  $s$  has an inverse.

As  $G$  is connected, there exists  $u \in (A_G \cup A_{\overline{G}})^*$  such that  $r \xrightarrow{u} s$ .

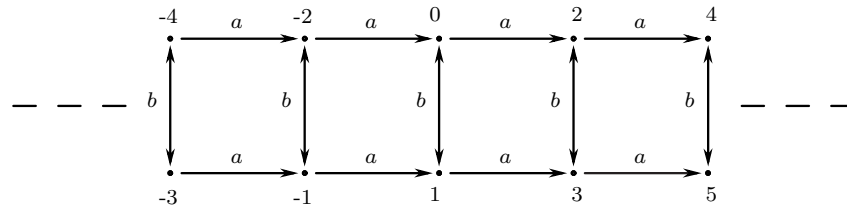
As  $G$  is source and target-complete, there exists  $\bar{s} \in V_G$  such that  $\bar{s} \xrightarrow{u} r$ .

We have  $\bar{s} \xrightarrow{u} \bar{s} \cdot s$ . As  $G$  is deterministic and co-deterministic,  $\bar{s} \cdot s = r$ .

For  $u = a_1 \dots a_n$  with  $n \geq 0$  and  $a_1, \dots, a_n \in A_G \cup A_{\overline{G}}$ , let  $u' = \bar{a}_n \dots \bar{a}_1$  with  $\bar{a} = a$  for any  $a \in A_G$ . Thus  $s \xrightarrow{u'} r$  and  $r \xrightarrow{u'} \bar{s}$ . So  $s \xrightarrow{u'} s \cdot \bar{s}$ .

As  $G$  is deterministic and co-deterministic,  $s \cdot \bar{s} = r$ . ◀

Let us consider the following graph:



This graph satisfies the properties of Theorem 20 (b) or (c) hence is the Cayley graph  $\mathcal{C}[\mathbb{Z}, \{1, 2\}]$  with  $[1] = b$  and  $[2] = a$ , where  $\cdot$  is the 0-chain operation defined for any  $p, q \in \mathbb{Z}$  by

$$p \cdot q = \begin{cases} p + q & \text{if } p \text{ or } q \text{ is even,} \\ p + q - 2 & \text{if } p \text{ and } q \text{ are odd} \end{cases}$$

and makes that  $(\mathbb{Z}, \cdot)$  is a group.

We can now obtain a characterization for the generalized Cayley graphs of groups. First, let us start with a basic fact on their *connected components* which are their maximal connected subsets.

► **Fact 22.** The connected components of the generalized Cayley graph of groups are the Cayley graphs of groups.

**Proof.**

⇒: By Fact 19 and Theorem 20.

⇐: Any Cayley graph of a group is a connected generalized Cayley graph. ◀

To characterize the generalized Cayley graphs of groups, we need the axiom of choice that is equivalent in ZF set theory to the property that any non-empty set has a group structure [6].

► **Theorem 23.** *In ZFC set theory, a graph is a generalized Cayley graph of a group if and only if it is simple, deterministic, co-deterministic, vertex-transitive.*

**Proof.**

⇒: By Facts 1 and 19.

⇐: Let  $G$  be a simple, deterministic, co-deterministic, vertex-transitive graph.

Let  $\text{Comp}(G)$  be the set of connected components of  $G$ . Let us recall that a *representative set* (or a *transversal*)  $P$  of  $\text{Comp}(G)$  is a subset of  $V_G$  that has exactly one vertex in each connected component:  $|P \cap V_C| = 1$  for any  $C \in \text{Comp}(G)$ . Its *canonical mapping*  $\pi_P : V_G \rightarrow P$  is the mapping associating to each vertex  $s$  the vertex of  $P$  in the same connected component:  $s \xrightarrow{*}_G \pi_P(s)$  for any  $s \in V_G$ . So  $\pi_P(p) = p$  for any  $p \in P$ .

Using ZFC set theory, there exists a representative set  $P$  and a binary operation  $\circ$  such that  $(P, \circ)$  is a group. Let  $r$  be its identity and  $C \in \text{Comp}(G)$  having  $r$  as vertex.

By (the proof of) Theorem 20,  $(V_C, \cdot_r)$  is a group for the  $r$ -chain-operation  $\cdot_r$  and of identity  $r$ . We take the group product  $(P \times V_C, \cdot)$  with

$$(p, x) \cdot (q, y) = (p \circ q, x \cdot_r y) \text{ for any } p, q \in P \text{ and } x, y \in V_C.$$

This group is of identity  $(r, r)$ .

As  $G$  is vertex-transitive, deterministic and co-deterministic, we can define the mapping

$$f : P \times V_C \rightarrow V_G \text{ such that } r \xrightarrow{u}_G x \implies p \xrightarrow{u}_G f(p, x) \text{ for (any) } u \in (A_G \cup \overline{A_G})^*.$$

Thus  $f$  is a bijection hence  $(V_G, \cdot)$  is a group where

$$f(p, x) \cdot f(q, y) = f(p \circ q, x \cdot_r y) \text{ for any } p, q \in P \text{ and } x, y \in V_C.$$

This group  $(V_G, \cdot)$  is of identity  $f(r, r) = r$ .

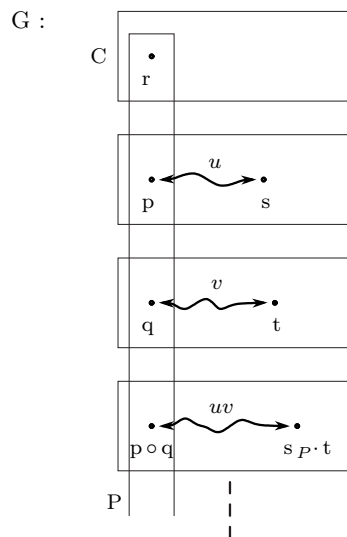
Let us show that this operation  $\cdot$  is an  $r$ -chain operation hence by Lemma 3,

$$G = \mathcal{C}[V_G, \xrightarrow{r}_G] \text{ with } [s] = a \text{ for any } r \xrightarrow{a} s.$$

We say that a binary operation  $P \cdot$  on  $V_G$  is an  $P$ -chain operation if for any  $s, t \in V_G$ ,

$$(\pi_P(s) \xrightarrow{u}_G s \wedge \pi_P(t) \xrightarrow{v}_G t) \implies \pi_P(s) \circ \pi_P(t) \xrightarrow{uv}_G s P \cdot t \text{ for any } u, v \in (A_G \cup \overline{A_G})^*$$

which is illustrated below. Any  $P$ -chain operation is an  $r$ -chain operation.



► **Figure 24.** An  $P$ -chain operation on the vertex set of a graph.

As  $G$  is vertex-transitive, deterministic and co-deterministic,  $P \cdot$  exists and is unique.

Let us check that  $\cdot$  and  $P \cdot$  are equal.



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Let  $p, q \in P$  and  $x, y \in V_G$ . We have to check that  $f(p, x) \cdot_P f(q, y) = f(p, x) \cdot f(q, y)$ .

Let  $u, v \in (A_G \cup \overline{A_G})^*$  such that  $r \xrightarrow{u}_G x$  and  $r \xrightarrow{v}_G y$ .

By definition of  $\cdot_P$ , we have  $x \xrightarrow{v}_G x \cdot_P y$  hence  $r \xrightarrow{uv}_G x \cdot_P y$ .

By definition of  $f$ , we get  $p \circ q \xrightarrow{uv}_G f(p \circ q, x \cdot_P y)$ ,  $p \xrightarrow{u}_G f(p, x)$  and  $q \xrightarrow{v}_G f(q, y)$ .

By definition of  $\cdot_P$ , we have  $p \circ q \xrightarrow{uv}_G f(p, x) \cdot_P f(q, y)$ .

As  $G$  is deterministic, we get  $f(p, x) \cdot_P f(q, y) = f(p \circ q, x \cdot_P y) = f(p, x) \cdot f(q, y)$ . ◀

For instance let us consider the following graph:

$$G = \{ (m, i, p) \xrightarrow{a} (m+1, i, p) \mid m, p \in \mathbb{Z} \wedge i \in \{0, 1\} \} \\ \cup \{ (m, i, p) \xrightarrow{b} (m, 1-i, p) \mid m, p \in \mathbb{Z} \wedge i \in \{0, 1\} \}$$

of representation the countable repetition of the preceding one.

By Theorem 23,  $G = \mathcal{C}[\mathbb{Z} \times \{0, 1\} \times \mathbb{Z}, \{(1, 0, 0), (0, 1, 0)\}]$  with  $[(1, 0, 0)] = a$ ,  $[(0, 1, 0)] = b$  and for the group operation  $(m, i, p) \cdot (n, j, q) = (m+n, i+j \pmod{2}, p+q)$ .

Note that the graph  $G = \{ s \xrightarrow{q} s+q \mid s \in \mathbb{R} \wedge q \in \mathbb{Q} \}$  is the generalized Cayley graph  $\mathcal{C}(\mathbb{R}, \mathbb{Q})$  for the group  $(\mathbb{R}, +)$ . Its unlabeled edge relation  $\xrightarrow{G}$  is the Vitali equivalence (1905) for which we need the axiom of choice to select an element in each class (and the selected elements form a set which is not Lebesgue measurable).

Similarly the graph  $\{ P \xrightarrow{Q} P \Delta Q \mid P, Q \subseteq \mathbb{N} \wedge Q \text{ finite} \}$  is the generalized Cayley graph  $\mathcal{C}(2^{\mathbb{N}}, 2_f^{\mathbb{N}})$  for the group  $2^{\mathbb{N}}$  of the subsets of  $\mathbb{N}$  with the symmetric difference operation  $\Delta$ , and for the set  $2_f^{\mathbb{N}}$  of finite subsets of  $\mathbb{N}$ . Its unlabeled edge relation is the equivalence  $E_0$  for which we also need the axiom of choice to have a selector.

## 8 Conclusion

We obtained simple graph-theoretic characterizations for the Cayley graphs of elementary algebraic structures. This is a first step in a graph description of algebraic structures.

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