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► **To cite this version:**

Nikolas Tapia, Lorenzo Zambotti. The geometry of the space of branched Rough Paths. 2018. hal-01908559v1

**HAL Id: hal-01908559**

**<https://hal.science/hal-01908559v1>**

Preprint submitted on 30 Oct 2018 (v1), last revised 20 Jan 2020 (v2)

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# THE GEOMETRY OF THE SPACE OF BRANCHED ROUGH PATHS

NIKOLAS TAPIA AND LORENZO ZAMBOTTI

ABSTRACT. We construct an explicit transitive free action of a Banach space of Hölder functions on the space of branched rough paths, which yields in particular a bijection between these two spaces. This endows the space of branched rough paths the structure of a principal homogeneous space over a Banach space and allows to characterize its automorphisms. The construction is based on the Baker-Campbell-Hausdorff formula and on the Hairer-Kelly map, which allows to describe branched rough paths in terms of anisotropic geometric rough paths.

## 1. INTRODUCTION

The theory of Rough Paths has been introduced by Terry Lyons in the '90s with the aim of giving an alternative construction of stochastic integrations and stochastic differential equations. More recently, it has been expanded by Martin Hairer to cover stochastic partial differential equations, with the invention of regularity structures.

A rough path and a *model* of a regularity structure are mathematical objects which must satisfy some algebraic and analytical constraints. For instance, a rough path can be described as a Hölder function defined on an interval and taking values in a non-linear finite-dimensional Lie group; models of regularity structures are a generalisation of this idea. A crucial ingredient of regularity structures is the *renormalisation* procedure: given a family of models depending on a parameter  $\varepsilon > 0$ , which fails to converge in an appropriate topology as  $\varepsilon \rightarrow 0$ , one wants to modify it in a such a way that the algebraic and analytical constraints are still satisfied and the modified version converges. This procedure has been obtained in [5, 7] for a general class of models with a *stationary* character.

The same question could have been asked much earlier about rough paths. Maybe this has not happened because the motivation was less compelling; although one can construct examples of rough paths depending on a parameter  $\varepsilon > 0$  which do not converge as  $\varepsilon \rightarrow 0$ , this phenomenon is the exception rather than the rule. However the problem of characterizing the automorphisms of the space of rough paths is clearly of interest; one example is the transformation from Itô to Stratonovich integration, see e.g. [1, 14, 15]. However our aim is to put this particular example in a much larger context.

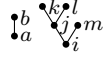
We recall that there are several possible notions of rough paths; in particular we have *geometric* RPs and *branched* RPs, two notions defined respectively by Terry Lyons [28] and Massimiliano Gubinelli [22], see Sections 3 and 4 below. These two notions are intimately related to each other, as shown by Hairer and Kelly [25], see Section 4 below.

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*Date:* 30th October 2018.

We note that regularity structures [24] are a natural and far-reaching generalisation of branched RPs.

In this paper we concentrate on the automorphisms of the space of branched RPs, see below for a discussion of the geometric case. Let  $\mathcal{F}$  be the collection of all non-planar rooted forests with nodes decorated by  $\{1, \dots, d\}$ , see Section 4 below. For instance the following forest



is an element of  $\mathcal{F}$ . We call  $\mathcal{T} \subset \mathcal{F}$  the set of *rooted trees*, namely of *non-empty forests* with a single connected component. Grading elements  $\tau \in \mathcal{F}$  by the number  $|\tau|$  of their nodes we set

$$\mathcal{T}_n := \{\tau \in \mathcal{T} : |\tau| \leq n\}, \quad n \in \mathbb{N}.$$

Let now  $\mathcal{H}$  be the linear span of  $\mathcal{F}$ . It is possible to endow  $\mathcal{H}$  with a product and a coproduct  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  which make it a Hopf algebra, also known as the Butcher-Connes-Kreimer Hopf algebra, see Section 4.2 below. We let  $\mathcal{G}$  denote the set of all *characters* over  $\mathcal{H}$ , that is, elements of  $\mathcal{G}$  are functionals  $X \in \mathcal{H}^*$  that are also multiplicative in the sense that

$$\langle X, \tau\sigma \rangle = \langle X, \tau \rangle \langle X, \sigma \rangle$$

for all forests (and in particular trees)  $\tau, \sigma \in \mathcal{F}$ . Furthermore, the set  $\mathcal{G}$  can be endowed with a product  $*$ , dual to the coproduct, defined pointwise by  $\langle X * Y, \tau \rangle = \langle X \otimes Y, \Delta\tau \rangle$ . We work on the compact interval  $[0, 1]$  for simplicity, and all results can be proved without difficulty on  $[0, T]$  for any  $T \geq 0$ .

**Definition 1.1** (Gubinelli [22]). *Given  $\gamma \in ]0, 1[$ , a branched  $\gamma$ -rough path is a path  $X : [0, 1]^2 \rightarrow \mathcal{G}$  such that  $X_{tt} = \varepsilon$ , it satisfies Chen's rule*

$$X_{su} * X_{ut} = X_{st}, \quad s, u, t \in [0, 1],$$

and the analytical condition

$$|\langle X_{st}, \tau \rangle| \lesssim |t - s|^{\gamma|\tau|}, \quad \tau \in \mathcal{F}.$$

Setting  $x_t^i := \langle X_{0t}, \bullet_i \rangle$ ,  $t \in [0, 1]$ , we say that  $X$  is a branched  $\gamma$ -rough path over the path  $x = (x^1, \dots, x^d)$ . We denote by  $\mathbf{BRP}^\gamma$  the set of all branched  $\gamma$ -rough paths (for a fixed finite alphabet  $\{1, \dots, d\}$ ).

By introducing the *reduced coproduct*  $\Delta' : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$

$$\Delta'\tau := \Delta\tau - \tau \otimes \mathbf{1} - \mathbf{1} \otimes \tau,$$

Chen's rule can be rewritten as follows

$$\delta \langle X, \tau \rangle_{sut} = \langle X_{su} \otimes X_{ut}, \Delta'\tau \rangle, \quad s, u, t \in [0, 1], \quad (1.1)$$

where for  $F : [0, 1]^2 \rightarrow \mathbb{R}$  we set  $\delta F : [0, 1]^3 \rightarrow \mathbb{R}$ ,

$$\delta F_{sut} := F_{st} - F_{su} - F_{ut}, \quad (1.2)$$

which is the second order finite increment considered by Gubinelli [21]. Note that the right-hand side of (1.1) depends on the values of  $X$  on trees with strictly fewer nodes than  $\tau$ ; if we can invert the operator  $\delta$ , then the right-hand side of (1.1) determines the left-hand side. This is however not a trivial result. In fact, a simple (but crucial for us)

remark is the following: if  $\gamma|\tau| \leq 1$ , then for any  $g^\tau : [0, 1] \rightarrow \mathbb{R}$  such that  $g^\tau \in C^{\gamma|\tau|}([0, 1])$ , the classical homogeneous Hölder space on  $[0, 1]$  with Hölder exponent  $\gamma|\tau|$ , the function

$$[0, 1]^2 \ni (s, t) \mapsto F_{st} := \langle X_{st}, \tau \rangle + g_t^\tau - g_s^\tau \quad (1.3)$$

also satisfies

$$\delta F_{sut} = \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle, \quad |F_{st}| \lesssim |t - s|^{\gamma|\tau|}, \quad s, u, t \in [0, 1]. \quad (1.4)$$

Inversely, if  $F : [0, 1]^2 \rightarrow \mathbb{R}$  satisfies (1.4), then  $F$  must satisfy (1.3) with  $g^\tau \in C^{\gamma|\tau|}([0, 1])$ .

If  $\gamma|\tau| > 1$ , then Gubinelli's Sewing Lemma [21] yields that the function  $(s, t) \mapsto \langle X_{st}, \tau \rangle$  is uniquely determined by (1.4) i.e. by the values of  $X$  on trees with at most  $|\tau| - 1$  nodes, and therefore, applying a recursion, on trees with at most  $N := \lfloor \gamma^{-1} \rfloor$  nodes. More explicitly, the Sewing Lemma is an *existence and uniqueness result* for  $[0, 1]^2 \ni (s, t) \mapsto \langle X_{st}, \tau \rangle$  with  $\gamma|\tau| > 1$ , once the right-hand side of (1.1) is known. However, for  $\gamma|\tau| \leq 1$  we have no uniqueness, as we have already seen, and existence is not trivial.

As we have seen in (1.3), the value of  $\langle X, \tau \rangle$  can be modified by adding the increment of a function in  $C^{\gamma|\tau|}([0, 1])$ , as long as  $\gamma|\tau| \leq 1$ . It seems reasonable to think that it is therefore possible to construct an action on the set of branched  $\gamma$ -rough paths of the abelian group (under pointwise addition)

$$\mathcal{C}^\gamma := \{(g^\tau)_{\tau \in \mathcal{T}_N} : g_0^\tau = 0, g^\tau \in C^{\gamma|\tau|}([0, 1]), \forall \tau \in \mathcal{T}, |\tau| \leq N\},$$

namely the set of all collections of functions  $(g^\tau \in C^{\gamma|\tau|}([0, 1]) : \tau \in \mathcal{T}, |\tau| \leq N)$  indexed by rooted trees with fewer than  $N := \lfloor \gamma^{-1} \rfloor$  nodes, such that  $g_0^\tau = 0$  and  $g^\tau \in C^{\gamma|\tau|}([0, 1])$ . This is indeed the content of the following

**Theorem 1.2.** *Let  $\gamma \in ]0, 1[$  such that  $\gamma^{-1} \notin \mathbb{N}$ . There is a transitive free action of  $\mathcal{C}^\gamma$  on  $\mathbf{BRP}^\gamma$ , namely a map  $(g, X) \mapsto gX$  such that*

- (1) *for each  $g, g' \in \mathcal{C}^\gamma$  and  $X \in \mathbf{BRP}^\gamma$  the identity  $g'(gX) = (g + g')X$  holds.*
- (2) *if  $(g^\tau)_{\tau \in \mathcal{T}_N} \in \mathcal{C}^\gamma$  is such that there exists a unique  $\tau \in \mathcal{T}_N$  with  $g^\tau \neq 0$ , then*

$$\langle gX, \tau \rangle = \langle X, \tau \rangle + \delta g^\tau$$

*and  $\langle gX, \sigma \rangle = \langle X, \sigma \rangle$  for all  $\sigma \in \mathcal{T}$  not containing  $\tau$ .*

- (3) *For every pair  $X, X' \in \mathbf{BRP}^\gamma$  there exists a unique  $g \in \mathcal{C}^\gamma$  such that  $gX = X'$ .*

We say that a tree  $\sigma \in \mathcal{T}$  *contains* a tree  $\tau \in \mathcal{T}$  if there exists a subtree  $\tau'$  of  $\sigma$ , not necessarily containing the root of  $\sigma$ , such that  $\tau$  and  $\tau'$  are isomorphic as rooted trees, where the root of  $\tau'$  is its node which is closest to the root of  $\sigma$ . Note that every  $(g^\tau)_{\tau \in \mathcal{T}_N} \in \mathcal{C}^\gamma$  is the sum of finitely many elements of  $\mathcal{C}^\gamma$  having satisfying the property required in point (2) of Theorem 1.2.

If  $\gamma > 1/2$  then the result of Theorem 1.2 is trivial. Indeed, in this case  $N = 1$ ,  $\mathcal{T}_N = \{\bullet i : i = 1, \dots, d\}$ , and  $\mathcal{C}^\gamma = \{g^{\bullet i} \in C^\gamma([0, 1]) : g_0^{\bullet i} = 0, i = 1, \dots, d\}$ . Then the action is

$$(g, X) \mapsto gX, \quad \langle (gX)_{st}, \bullet i \rangle := \langle X_{st}, \bullet i \rangle + g_t^{\bullet i} - g_s^{\bullet i}, \quad (1.5)$$

while the value of  $\langle gX, \tau \rangle$  for  $|\tau| \geq 2$  is uniquely determined by (1.1) via the Sewing Lemma. For example

$$\langle (gX)_{st}, \mathfrak{I}_i^j \rangle := \int_s^t (x_u^j - x_s^j + g_u^j - g_s^j) d(x_u^i + g_u^i), \quad (1.6)$$

where  $x_u^i := \langle X_{0u}, \bullet_i \rangle$  and the integral is well-defined in the Young sense, see [21, section 3].

If  $1/3 < \gamma \leq 1/2$  then  $N = 2$  and  $\mathcal{T}_2 = \mathcal{T}_1 \sqcup \{\mathfrak{I}_i^j : i, j = 1, \dots, d\}$ . Then the action at level  $|\tau| = 1$  is still given by (1.5), while at level  $|\tau| = 2$  we must have by (1.1)

$$\delta \langle gX, \mathfrak{I}_i^j \rangle_{sut} = \langle (gX)_{su} \otimes (gX)_{ut}, \Delta' \tau \rangle = (x_u^j - x_s^j + g_u^j - g_s^j)(x_t^i - x_u^i + g_t^i - g_u^i). \quad (1.7)$$

Although the right-hand side of (1.7) is explicit and simple, in this case there is no canonical choice for  $\langle gX, \mathfrak{I}_i^j \rangle$ . An expression like (1.6) is ill-defined in the Young sense, and the same is true if we try the formulation

$$\langle (gX)_{st}, \mathfrak{I}_i^j \rangle = \langle X_{st}, \mathfrak{I}_i^j \rangle + \int_s^t \left( (x_u^j - x_s^j + g_u^j - g_s^j) dg_u^i + (g_u^j - g_s^j) dx_u^i \right), \quad (1.8)$$

which satisfies formally (1.7), but the Young integrals are ill defined since  $2\gamma \leq 1$ . The construction of  $\langle gX, \mathfrak{I}_i^j \rangle$  is therefore not trivial in this case.

The same argument applies for any  $\gamma \leq 1/2$  and any tree  $\tau$  such that  $2 \leq |\tau| \leq N = \lfloor \gamma^{-1} \rfloor$ , and the fact that the above Young integrals are not well defined shows why existence of the map  $X \rightarrow gX$  is not trivial.

Since Theorem 1.2 yields an action of  $\mathcal{C}^\gamma$  on  $\mathbf{BRP}^\gamma$  which is *regular*, i.e. free and transitive, then  $\mathbf{BRP}^\gamma$  is a *principal  $\mathcal{C}^\gamma$ -homogeneous space* or  *$\mathcal{C}^\gamma$ -torsor*. In particular,  $\mathbf{BRP}^\gamma$  is a copy of  $\mathcal{C}^\gamma$ , but there is no canonical choice of an origin in  $\mathbf{BRP}^\gamma$ .

Therefore, Theorem 1.2 also yields the following

**Corollary 1.3.** *Given a branched  $\gamma$ -rough path  $X$ , the map  $g \rightarrow gX$  yields a bijection between  $\mathcal{C}^\gamma$  and the set of branched  $\gamma$ -rough paths.*

Therefore Corollary 1.3 yields a complete parametrisation of the space of branched rough paths. This result is somewhat surprising, since rough paths form a non-linear space, in particular because of the Chen relation; however Corollary 1.3 yields a natural bijection between the space of branched  $\gamma$ -rough paths and the linear space  $\mathcal{C}^\gamma$ .

Moreover, the fact that the above Young integrals are not well defined shows why existence of the map  $X \rightarrow gX$  is not trivial.

Corollary 1.3 also gives a complete answer to the question of existence and characterization of branched  $\gamma$ -rough paths over a  $\gamma$ -Hölder path  $x$ . Unsurprisingly, for our construction we start from a result of T. Lyons and N. B. Victoir's [29] of 2007, which was the first general theorem of existence of a geometric  $\gamma$ -rough path over a  $\gamma$ -Hölder path  $x$ , see our discussion of Theorem 1.4 below.

An important point to stress is that the action constructed in Theorem 1.2 is neither unique nor canonical. In the proof of Theorem 3.5 below, some parameters have to be

fixed arbitrarily, and the final outcome depends on them, see Remark 3.7. In this respect, the situation is similar to what happens in regularity structures with the reconstruction operator on spaces  $\mathfrak{D}^\gamma$  with a negative exponent  $\gamma < 0$ , see [24, Theorem 3.10].

**1.1. Outline of our approach.** A key point in Theorem 1.2 is the construction of branched  $\gamma$ -rough paths. In the case of *geometric* rough paths, see Definition 4.1, the *signature* [9, 28] of a smooth path  $x : [0, 1] \rightarrow \mathbb{R}^d$  yields a canonical construction. Other cases where geometric rough paths over non-smooth paths have been constructed are Brownian motion and fractional Brownian motion (see [11] for the case  $H > \frac{1}{4}$  and [32] for the general case) among others. However, until T. Lyons and N. B. Victoir's paper [29] in 2007, this question remained largely open in the general case. The precise result is as follows

**Theorem 1.4** (Lyons–Victoir extension). *If  $p \in [1, \infty) \setminus \mathbb{N}$  and  $\gamma := 1/p$ , a  $\gamma$ -Hölder path  $x : [0, 1] \rightarrow \mathbb{R}^d$  can be lifted to a geometric  $\gamma$ -rough path. For any  $p \geq 1$  and  $\varepsilon \in ]0, \gamma[$ , a  $\gamma$ -Hölder path can be lifted to a geometric  $(\gamma - \varepsilon)$ -rough path.*

Our first result is a version of this theorem which holds for rough paths in a more general algebraic context, see Theorem 3.5 below. We use the Lyons–Victoir approach and an explicit form of the Baker–Campbell–Hausdorff formula by Reutenauer [33], see formula (2.8) below. Whereas Lyons and Victoir used in one passage the axiom of choice, our method is completely constructive.

Using the same idea we extend this construction to the case where the collection  $(x^1, \dots, x^d)$  is allowed to have different regularities in each component, which we call *anisotropic (geometric) rough paths (aGRP)*, see Definition 5.5.

**Theorem 1.5.** *To each collection  $(x^i)_{i=1, \dots, d}$ , with  $x^i \in C^{\gamma_i}([0, 1])$ , we can associate an anisotropic rough path  $\bar{X}$  over  $(x^i)_{i=1, \dots, d}$ . For every collection  $(g^i)_{i=1, \dots, d}$ , with  $g^i \in C^{\gamma_i}([0, 1])$ , denoting by  $g\bar{X}$  the anisotropic geometric rough path over  $(x^i + g^i)_{i=1, \dots, d}$ , we have*

$$g'(g\bar{X}) = (g + g')\bar{X}.$$

This kind of extension to rough paths has already been explored in the papers [2, 23] in the context of isomorphisms between geometric and branched rough paths. It turns out that the additional property obtained by our method enables us to explicitly describe the propagation of suitable modifications from lower to higher degrees.

We then go on to describe the interpretation of the above results in the context of branched rough paths. The main tool is the Hairer–Kelly map [25], that we introduce and describe in Proposition 6.5 a sum over a suitable set of partitions of the given tree, as opposed to the original iterative definition, which we then use to encode branched rough paths via anisotropic geometric rough paths, along the same lines as in [2, Theorem 4.3].

**Theorem 1.6.** *Let  $X$  be a branched  $\gamma$ -rough path. There exists an anisotropic geometric rough path  $\bar{X}$  indexed by words on the alphabet  $\mathcal{T}_N$ , with exponents  $(\gamma_\tau = \gamma|\tau|, \tau \in \mathcal{T}_N)$ , and such that  $\langle X, \tau \rangle = \langle \bar{X}, \psi(\tau) \rangle$ , where  $\psi$  is the Hairer–Kelly map.*

The main difference of this result with [25, Theorem 1.9] is that we obtain an anisotropic geometric rough path instead of a classical geometric rough path. This means that we do not construct unneeded components, i.e. components with regularity larger than 1, and we also obtain the right Hölder estimates in terms of the size of the indexing tree. This addresses two problems mentioned in Hairer and Kelly’s work, namely Remarks 4.14 and 5.9 in [25].

We then use Theorem 1.5 and Theorem 1.6 to construct our action on branched rough paths. Given  $(g, X) \in \mathcal{C}^\gamma \times \mathbf{BRP}^\gamma$ , we construct the anisotropic geometric rough paths  $\bar{X}$  and  $g\bar{X}$  and then define the branched rough path  $gX \in \mathbf{BRP}^\gamma$  as  $\langle gX, \tau \rangle = \langle g\bar{X}, \psi(\tau) \rangle$ , where  $\psi$  is the Hairer–Kelly map.

Our approach also does not make use of Foissy’s and Chapoton’s Hopf-algebra isomorphism [8, 17] between the Butcher–Connes–Kreimer Hopf algebra and the shuffle algebra over a complicated set  $I$  of trees as is done in [2]. This allows us to construct an action of a *larger* group on the set of branched rough paths; indeed, using the above isomorphism one would obtain a transformation group parametrised by  $(g^\tau)_{\tau \in I}$  where  $I$  is the aforementioned set of trees of Foissy’s and Chapoton’s results and  $g^\tau \in C^{\gamma|\tau|}$ ; on the other hand our approach yields a transformation group parametrised by  $(g^\tau)_{\tau \in \mathcal{T}_N}$ . With the smaller set  $I \cap \mathcal{T}_N$ , transitivity of the action  $g \mapsto gX$  would be lost.

Finally we note that we use a special property of the Butcher–Connes–Kreimer Hopf algebra: the fact that it is freely generated as an algebra by the set of trees, so defining characters over it is significantly easier than in the geometric case. To define an element  $X \in G$  it suffices to give the values  $\langle X, \tau \rangle$  for all trees  $\tau \in \mathcal{T}$ ; by freeness there is a unique multiplicative extension to all of  $\mathcal{H}$ . This is not at all the case for geometric rough paths: the tensor algebra  $T^c(A)$  over an alphabet  $A$  is not free over the linear span of words so if one is willing to define a character  $x$  over  $T^c(A)$  there are additional algebraic constraints that the values of  $X$  on words must satisfy.

**Outline.** We start by reviewing all the theoretical concepts needed to make the exposition in this section formal. In Section 3 we state and prove the main result of this chapter, namely we give an explicit construction of a geometric rough path above any given path  $x \in C^\gamma(\mathbb{R}^d)$ . Next, in Section 5 we extend this result to the wider class of anisotropic geometric rough paths. Finally, in Section 4 we connect our construction with M. Gubinelli’s branched rough paths, and we extend M. Hairer and D. Kelly’s work in Section 6.1. We also explore possible connections with renormalisation in Section 7 by studying how our construction behaves under modification of the underlying paths. Then, we connect this approach with a recent work by Bruned, Chevyrev, Friz and Preiß [3] in Section 7.1 who borrowed ideas from the theory of Regularity Structures [5, 24] and proposed a renormalisation procedure for geometric and branched rough paths [3] based on pre-Lie morphisms.

The main difference between our result and the BCFP procedure is that they consider translation only by time-independent factors, whereas –under reasonable hypotheses– we are also able to handle general translations depending on the time parameter. We also mention that some further algebraic aspects of renormalisation in rough paths have been recently developed in [4].

**Acknowledgements.** The first author acknowledges support by the CONICYT/Doctorado Nacional/2013-21130733 doctoral scholarship and *Núcleo Milenio Modelos Estocásticos de Sistemas Complejos y Desordenados*.

## 2. PRELIMINARIES

For the rest of this section we fix a locally finite graded connected Hopf algebra  $\mathcal{H}$ , that is,  $\mathcal{H}$  is a vector space endowed with an associative product  $m : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$  and a coassociative coproduct  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  satisfying certain compatibility assumptions. There is also a unit  $\mathbf{1} \in \mathcal{H}$ , a counit  $\varepsilon \in \mathcal{H}^*$  and an antipode  $S : \mathcal{H} \rightarrow \mathcal{H}$  such that

$$m(\text{id} \otimes S)\Delta x = \varepsilon(x)\mathbf{1} = m(S \otimes \text{id})\Delta x$$

for all  $x \in \mathcal{H}$ . As usual we will denote the image  $m(x \otimes y) = xy$  in order to reduce notation. The fact that  $\mathcal{H}$  is locally finite graded connected means that it can be decomposed as a direct sum

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{(n)}$$

with  $\mathcal{H}_{(0)} = \mathbb{R}\mathbf{1}$ , each  $\mathcal{H}_{(n)}$  is finite-dimensional and

$$m : \mathcal{H}_{(n)} \otimes \mathcal{H}_{(m)} \rightarrow \mathcal{H}_{(n+m)}, \quad \Delta : \mathcal{H}_{(n)} \rightarrow \bigoplus_{p+q=n} \mathcal{H}_{(p)} \otimes \mathcal{H}_{(q)}. \quad (2.1)$$

Each element  $x \in \mathcal{H}$  can thus be decomposed as a sum

$$x = \sum_{n=0}^{\infty} x_n, \quad x_n \in \mathcal{H}_{(n)},$$

where only a finite number of the summands are non-zero. We call each  $x_n$  the *homogeneous part of degree  $n$*  of  $x$ , and elements of  $\mathcal{H}_{(n)}$  are said to be homogeneous of degree  $n$ . In this case we write  $|x_n| = n$ . From now on the grading  $(\mathcal{H}_{(n)} : n \in \mathbb{N})$  will be called the *standard grading*.

The grading property of  $\mathcal{H}$  implies in particular that for homogeneous elements  $x \in \mathcal{H}_{(n)}$  the coproduct can be written as

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \Delta' x$$

where

$$\Delta' x \in \bigoplus_{\substack{p+q=n \\ p, q \geq 1}} \mathcal{H}_{(p)} \otimes \mathcal{H}_{(q)}$$

is known as the *reduced coproduct*. Furthermore, the coassociativity of  $\Delta$ , i.e. the identity  $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ , allows to unambiguously define its iterates  $\Delta_n : \mathcal{H} \rightarrow \mathcal{H}^{\otimes(n+1)}$  by setting

$$\Delta_0 = \text{id}, \quad \Delta_n = (\text{id} \otimes \Delta_{n-1})\Delta$$

and we have, for a homogeneous element  $x \in \mathcal{H}_{(k)}$  of degree  $k$

$$\Delta_n x = x \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes x \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes x + \Delta'_n x$$



where now

$$\Delta'_n x \in \bigoplus_{\substack{p_1 + \dots + p_{n+1} = k \\ p_j \geq 1}} \mathcal{H}_{(p_1)} \otimes \dots \otimes \mathcal{H}_{(p_{n+1})}.$$

*Remark 2.1.* These properties of the iterated coproduct imply that the bialgebra  $(\mathcal{H}, \Delta)$  is *conilpotent*, that is, for each homogeneous  $x \in \mathcal{H}_{(k)}$  there is an integer  $n \geq 1$  such that  $\Delta'_n x = 0$  and we see that in fact  $n = k$ .

*Remark 2.2.* From the above discussion we obtain in particular the inclusion

$$\Delta'_n \mathcal{H}_{(n+1)} \subset \mathcal{H}_{(1)}^{\otimes(n+1)},$$

that is, the  $n$ -fold reduced coproduct of a homogeneous element of degree  $n + 1$  is a sum of  $(n + 1)$ -fold tensor products of homogeneous elements of degree 1.

We recall that in general the dual space  $\mathcal{H}^*$  carries an algebra structure given by the convolution product  $\star$ , dual to the coproduct  $\Delta$ , defined by

$$\langle X \star Y, x \rangle = \langle X \otimes Y, \Delta x \rangle,$$

but in general  $\mathcal{H}^*$  cannot be made into a coalgebra by dualizing the product. In particular,  $\mathcal{H}^*$  is commutative if and only if  $\mathcal{H}$  is cocommutative. For a sequence of maps  $X_1, \dots, X_k \in \mathcal{H}^*$  we have the formula

$$X_1 \star \dots \star X_k = m^{\otimes(k-1)}(X_1 \otimes \dots \otimes X_k) \Delta_{k-1}. \quad (2.2)$$

**Definition 2.3.** A character on  $\mathcal{H}$  is a linear map  $X : \mathcal{H} \rightarrow \mathbb{R}$  such that

$$\langle X, xy \rangle = \langle X, x \rangle \langle X, y \rangle$$

for all  $x, y \in \mathcal{H}$ . We call  $G$  the set of all characters on  $\mathcal{H}$ .

An infinitesimal character (or derivation) on  $\mathcal{H}$  is a linear map  $\alpha : \mathcal{H} \rightarrow \mathbb{R}$  such that

$$\langle \alpha, xy \rangle = \langle \alpha, x \rangle \langle \varepsilon, y \rangle + \langle \varepsilon, x \rangle \langle \alpha, y \rangle.$$

We call  $\mathfrak{g}$  the set of all infinitesimal characters on  $\mathcal{H}$ .

We observe that  $\langle X, \mathbf{1} \rangle = 1$  and  $\langle \alpha, \mathbf{1} \rangle = 0$  for all  $X \in G$  and  $\alpha \in \mathfrak{g}$ .

It is well known that the set  $G$  of characters on  $\mathcal{H}$  is a group with unit  $\varepsilon$  and inverse  $X^{-1} = S^* X = X \circ S$ . The space  $\mathfrak{g}$  of infinitesimal characters on  $\mathcal{H}$  is a Lie algebra under the bracket  $[\alpha, \beta] = \alpha \star \beta - \beta \star \alpha$ . Moreover, there is an exponential map  $\exp : \mathfrak{g} \rightarrow G$

$$\exp(\alpha) := \sum_{n=0}^{\infty} \frac{\alpha^{\star n}}{n!}$$

which is a bijection and its inverse is the map  $\log : G \rightarrow \mathfrak{g}$

$$\log(X) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(X - \varepsilon)^{\star n}}{n}.$$

*Remark 2.4.* The above maps are actually defined over bigger spaces. These definitions make sense for maps  $\alpha \in \hat{\mathfrak{g}}$  and  $X \in \hat{G}$ , where  $\hat{\mathfrak{g}}$  is the Lie algebra of linear maps mapping  $\mathbf{1} \rightarrow 0$  and  $\hat{G}$  is the group of linear maps mapping  $\mathbf{1} \rightarrow 1$ . In fact,  $\mathfrak{g}$  is a sub-Lie algebra of  $\hat{\mathfrak{g}}$  and  $G$  is a subgroup of  $\hat{G}$ .

*Remark 2.5.* The conilpotency of  $(\mathcal{H}, \Delta)$  implies that for each homogeneous element  $x \in \mathcal{H}_{(n)}$  the above series defining  $\exp$  and  $\log$  terminate after a finite number of terms (precisely  $n$ , in fact). Therefore, for a general element  $x \in \mathcal{H}$  and  $\alpha \in \mathfrak{g}$  the value of  $\langle \exp(\alpha), x \rangle$  is made up of a finite number of finite sums, so there are no convergence issues involved whatsoever.

Using the grading of  $\mathcal{H}$  we can inductively define a collection of subspaces

$$W_1 = \mathcal{H}_{(1)}^*, \quad W_{k+1} = [W_1, W_k] \subset \mathcal{H}_{(k+1)}^*. \quad (2.3)$$

The Lie algebra  $\mathfrak{g}$  consists of formal series of the type

$$\alpha = \sum_{k=1}^{\infty} \alpha_{(k)}, \quad \alpha_{(k)} = \alpha|_{\mathcal{H}_{(k)}} \in W_k$$

Formally,  $\mathfrak{g}$  corresponds to the completion

$$\mathfrak{g} = \prod_{k=1}^{\infty} W_k$$

of the graded Lie algebra  $W_1 \oplus W_2 \oplus \dots$  with respect to the natural filtration induced by the grading, and so it also admits a filtration

$$\mathfrak{g}^{(n)} = \prod_{k \geq n} W_k.$$

*Remark 2.6.* The elements of  $\mathfrak{g}$  correspond to infinite formal series and cannot in general be reduced to finite sums. The same goes for elements of the character group  $G$  and more in general for arbitrary elements of the dual space  $\mathcal{H}^*$ . That is, we can think of elements  $f \in \mathcal{H}^*$  as formal series

$$f = \sum_{k=0}^{\infty} f_k$$

with  $f_k = f|_{\mathcal{H}_{(k)}} \in \mathcal{H}_{(k)}^*$ . This is one of the reasons why it is not possible to dualize the product on  $\mathcal{H}$  to induce a coproduct on  $\mathcal{H}^*$ .

The Baker–Campbell–Hausdorff formula below describes the group law on  $G$  in terms of the Lie bracketing on  $\mathfrak{g}$  via the exponential map. See [26] for a proof.

**Theorem 2.7** (Baker–Campbell–Hausdorff). *Let  $\alpha, \beta \in \mathfrak{g}$ , then  $\log(\exp(\alpha) \star \exp(\beta)) \in \mathfrak{g}$ .*

We define the map BCH:  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  by

$$\text{BCH}(\alpha, \beta) := \log(\exp(\alpha) \star \exp(\beta)). \quad (2.4)$$

The main point of Theorem 2.7 is that, even if one can compute explicitly all the terms appearing in the series defining  $\log(\exp(\alpha) \star \exp(\beta))$  it is not immediately clear that each of these terms can be rewritten in terms of iterated commutators as the definition of  $\mathfrak{g}$  requires. Another way to interpret this theorem is to say that there exists an element  $\gamma = \text{BCH}(\alpha, \beta) \in \mathfrak{g}$  such that  $\exp(\alpha) \star \exp(\beta) = \exp(\gamma)$ .

More can be said about the Lie series

$$\text{BCH}(\alpha, \beta) = \sum_{k \geq 1} \text{BCH}_{(k)}(\alpha, \beta) \in \mathfrak{g} \quad (2.5)$$

given by the previous theorem. In fact, it is possible to show that each homogeneous term  $\text{BCH}_{(k)}$  of this series is formed by iterated Lie brackets of  $\alpha$  and  $\beta$ , where the first are

$$\text{BCH}(\alpha, \beta) = \alpha + \beta + \frac{1}{2}[\alpha, \beta] + \frac{1}{12}[\alpha, [\alpha, \beta]] - \frac{1}{12}[\beta, [\alpha, \beta]] + \dots, \quad (2.6)$$

and the following terms are explicit but difficult to compute. Nevertheless, fully explicit formulas have been known since 1947 by Dynkin [13].

For our purposes, however, Dynkin's formula is too complicated so we rely on a different expression first show by Reutenauer [33]. In order to describe it, let  $\varphi_k : (\mathcal{H}^*)^{\otimes k} \rightarrow \mathcal{H}^*$  be the linear map

$$\varphi_k(\alpha_1 \otimes \dots \otimes \alpha_k) = \sum_{\sigma \in S_k} a_\sigma \alpha_{\sigma(1)} \star \dots \star \alpha_{\sigma(k)} \quad (2.7)$$

where  $S_k$  denotes the symmetric group of order  $k$ , and  $a_\sigma := \frac{(-1)^{d(\sigma)}}{k} \binom{k-1}{d(\sigma)}^{-1}$  is a constant depending only on the *descent number*<sup>1</sup>  $d(\sigma)$  of the permutation  $\sigma \in S_k$ .

**Lemma 2.8** (Reutenauer's formula). *For all  $k \geq 1$ ,  $\varphi_k : \mathfrak{g}^{\otimes k} \rightarrow W_k$ , i.e.  $\varphi_k(\alpha_1, \dots, \alpha_k) \in W_k$  if  $\alpha_1, \dots, \alpha_k \in \mathfrak{g}$ . Moreover, the  $k$ -th homogeneous term in (2.5) is*

$$\text{BCH}_{(k)}(\alpha, \beta) = \sum_{i+j=k} \frac{1}{i!j!} \varphi_k(\alpha^{\otimes i} \otimes \beta^{\otimes j}) \in W_k. \quad (2.8)$$

*Remark 2.9.* *A priori* the homogeneous term (2.8) looks different from the ones appearing in (2.6) which are written in terms of iterated Lie brackets. This can be explained by using a theorem by Dynkin, Specht and Wever [27] relating products of elements in  $\mathfrak{g}$  with their iterated brackets. More concretely, one can define the *Dynkin operator*  $D : \mathcal{H}^* \rightarrow \mathfrak{g}$  by

$$D(x_1 \otimes \dots \otimes x_k) = [x_1, \dots, [x_{k-1}, x_k] \dots]$$

for  $x_1, \dots, x_k \in \mathcal{H}_{(1)}^*$ , and then the Dynkin–Specht–Wever theorem asserts that a homogeneous element  $X \in \mathcal{H}_{(k)}^*$  is in  $\mathfrak{g}$  (i.e.  $X \in W_k$ ) if and only if  $D(X) = kX$ . A nice short proof of this fact in terms of Hopf algebras can be found in [36]. The point is that the coefficients in (2.7)–(2.8) are such that if we replace the products with iterated brackets then we recover the terms in (2.6). For example, we have that  $\varphi_1(\alpha) = \alpha$ ,  $\varphi_2(\alpha \otimes \beta) = \frac{1}{2}(\alpha \star \beta - \beta \star \alpha)$  and (omitting the  $\star$  product)

$$\varphi_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \frac{1}{3}\alpha_1\alpha_2\alpha_3 - \frac{1}{6}(\alpha_2\alpha_1\alpha_3 + \alpha_1\alpha_3\alpha_2 + \alpha_2\alpha_3\alpha_1 + \alpha_3\alpha_1\alpha_2) + \frac{1}{3}\alpha_3\alpha_2\alpha_1.$$

In any case, this is not needed in what follows and we work with formulas (2.7)–(2.8).

From all these considerations we obtain

<sup>1</sup>The number of  $i$  such that  $\sigma(i) > \sigma(i+1)$ .

**Lemma 2.10.** *Let  $x \in \mathcal{H}_{(k)}$  such that*

$$\Delta'_{k-1}x = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(k)} \in \mathcal{H}_{(1)}^{\otimes k}.$$

*Then for all  $\alpha_1, \dots, \alpha_k \in \mathfrak{g}$*

$$\langle \varphi_k(\alpha_1 \otimes \cdots \otimes \alpha_k), x \rangle = \sum_{(x)} \sum_{\sigma \in S_k} a_\sigma \prod_{j=1}^k \langle \alpha_{\sigma(j)}, x_{(j)} \rangle. \quad (2.9)$$

*Proof.* This follows directly from the definition of  $\varphi_k$  in (2.7) together with (2.2) and the fact that since  $\alpha_j(\mathbf{1}) = 0$  we can write

$$\alpha_1 \star \cdots \star \alpha_k = m^{\otimes(k-1)}(\alpha_1 \otimes \cdots \otimes \alpha_k) \Delta'_{k-1}$$

instead (note the reduced coproduct in place of the full coproduct).  $\square$

Combining (2.9) with (2.8) we get

$$\langle \text{BCH}_{(k)}(\alpha, \beta), x \rangle = \sum_{i+j=k} \frac{1}{i!j!} \sum_{\sigma \in S_k} a_\sigma \prod_{p=1}^i \langle \alpha, x_{(i_{\sigma^{-1}(p)})} \rangle \prod_{q=i+1}^k \langle \beta, x_{(i_{\sigma^{-1}(q)})} \rangle \quad (2.10)$$

for all  $x \in \mathcal{H}_{(k)}$ .

**2.1. Nilpotent Lie algebras.** From (2.1) we have

**Lemma 2.11.** *For any  $N \in \mathbb{N}$  the subspace*

$$\mathcal{H}_N := \bigoplus_{k=0}^N \mathcal{H}_{(k)} \subset \mathcal{H}$$

*is a counital subcoalgebra of  $(\mathcal{H}, \Delta, \varepsilon)$ . The canonical projection  $\pi_N : \mathcal{H} \rightarrow \mathcal{H}_N$  is a coalgebra epimorphism.*

Then we can consider the dual algebra  $(\mathcal{H}_N^*, \star, \varepsilon)$  and the corresponding truncated Lie algebra

$$\mathfrak{g}^N = \mathfrak{g}/\mathfrak{g}^{(N)} \cong \bigoplus_{k=1}^N W_k.$$

There is a canonical injection  $\iota_N := \pi_N^* : \mathcal{H}_N^* \rightarrow \mathcal{H}^*$  such that  $\langle \iota_N X, x \rangle = 0$  if  $|x| > N$  so when working with elements of  $\mathcal{H}_N^*$  or  $\mathfrak{g}^N$  we will always assume that they satisfy this property. There is also a restricted exponential map  $\exp_N : \mathfrak{g}^N \rightarrow G^N$  where  $G^N := \exp_N(\mathfrak{g}^N)$  defined by the truncated series

$$\exp_N(\alpha) := \sum_{k=0}^N \frac{\alpha^{\star k}}{k!}. \quad (2.11)$$

The (truncated) BCH formula again shows that  $G^N$  defined in this way is a group. Also, as before,  $\exp_N$  is a bijection with inverse  $\log_N : G^N \rightarrow \mathfrak{g}^N$  given by

$$\log_N(X) := \sum_{k=1}^N (-1)^{k+1} \frac{(X - \varepsilon)^{\star k}}{k}.$$

**Proposition 2.12.** *The orthogonal subspace*

$$\mathcal{K}^N := \mathcal{K}_N^\perp = \{ X \in \mathcal{K}^* : \langle X, x \rangle = 0, \forall x \in \mathcal{K}_N \}$$

is an ideal of the algebra  $(\mathcal{K}^*, \star, \varepsilon)$ . In particular, the quotient algebra  $\mathcal{K}^*/\mathcal{K}^N$  is isomorphic to  $(\mathcal{K}_N^*, \star, \varepsilon)$ .

*Proof.* This is standard, see [30, Proposition I.3.1].  $\square$

*Remark 2.13.* The canonical inclusion  $\iota_N$  is an algebra monomorphism, being the dual map to a coalgebra epimorphism. Moreover, it is such that if  $\varpi_N : \mathcal{K}^* \rightarrow \mathcal{K}_N^*$  is the canonical projection then  $\varpi_N \circ \iota_N = \text{id}_{\mathcal{K}_N^*}$ . We see that, a priori,  $\iota_N$  maps  $G^N$  to  $\hat{G}$  and  $\mathfrak{g}^N$  to  $\hat{\mathfrak{g}}$  defined inside Remark 2.4.

**Proposition 2.14.** *The canonical inclusion  $\iota_N$  maps  $\mathfrak{g}^N$  to  $\mathfrak{g}$ .*

*Proof.* We already know that  $\iota_N : \mathfrak{g}^N \rightarrow \hat{\mathfrak{g}}$  so it only suffices to check that given  $\alpha \in \mathfrak{g}^N$  we have that  $\iota_N \alpha$  is an infinitesimal character. Let  $x, y \in \mathcal{H}$ , then

$$\begin{aligned} \langle \iota_N \alpha, xy \rangle &= \langle \alpha, \pi_N(xy) \rangle = \sum_{n=0}^N \sum_{j=0}^n \langle \alpha, x_j y_{n-j} \rangle \\ &= \sum_{n=0}^N \sum_{j=0}^n (\langle \alpha, x_j \rangle \langle \varepsilon, y_{n-j} \rangle + \langle \varepsilon, x_j \rangle \langle \alpha, y_{n-j} \rangle) \\ &= \sum_{n=0}^N (\langle \alpha, x_n \rangle \langle \varepsilon, y_0 \rangle + \langle \varepsilon, x_0 \rangle \langle \alpha, y_n \rangle) = \langle \alpha, \pi_N x \rangle \langle \varepsilon, y \rangle + \langle \varepsilon, x \rangle \langle \alpha, \pi_N y \rangle, \end{aligned}$$

hence  $\iota_N \alpha \in \mathfrak{g}$ .  $\square$

*Remark 2.15.* A similar statement cannot hold for  $G^N$ . For instance take  $X \in G^N$  and an element  $x \in \mathcal{H}_N \setminus \{1\}$  such that  $\langle X, x \rangle \neq 0$ . Without loss of generality we may suppose that  $x$  is homogeneous. Take  $k$  large enough so that  $k|x| \geq N + 1$ . Then

$$0 = \langle \iota_N X, x^k \rangle \neq \langle X, x \rangle^k.$$

*Remark 2.16.* All the above considerations work, with minor modifications, if linear maps are allowed to take values in an arbitrary commutative algebra instead of the ground field (in this case  $\mathbb{R}$ ). That is, we may consider instead of the dual space  $\mathcal{K}^*$ , the space  $\mathcal{L}(\mathcal{H}, A)$  of linear maps from  $\mathcal{H}$  to a commutative unital algebra  $A$ . Even though this level of generality may seem superfluous, it could be interesting since the added structure may reveal some further connections with renormalisation via the Birkhoff decomposition of characters. Since our aim is to define paths taking values in the group  $G^N$  satisfying some extra properties, for example, we could make them depend on an extra parameter  $\varepsilon > 0$  and  $A$  then could be taken to be the algebra of Laurent series in this extra parameter. See [30] for further details.

**Proposition 2.17.** *The finite-dimensional Lie algebra  $\mathfrak{g}^N$  is step  $N$  nilpotent.*

*Proof.* We recall that nilpotency means that the lower central series  $\mathfrak{g}_1^N \supset \mathfrak{g}_2^N \supset \dots$  defined inductively  $\mathfrak{g}_1^N := \mathfrak{g}^N$ ,  $\mathfrak{g}_{k+1}^N := [\mathfrak{g}^N, \mathfrak{g}_k^N]$  terminates after a finite number of steps, that is, there exists  $k_0 \in \mathbb{N}$  such that  $\mathfrak{g}_k^N = \{0\}$  for all  $k \geq k_0$ . The smallest number

such that this happens is called the nilpotency step of  $\mathfrak{g}^N$ . Thus, we have to show that  $\mathfrak{g}_{N+1}^N = \{0\}$  but  $\mathfrak{g}_N^N \neq \{0\}$ .

We first prove that  $[W_k, W_j] \subset W_{k+j}$ . Take  $\alpha \in W_k, \beta \in W_j$  and an element  $x \in \mathcal{H}$ . Then

$$\langle [\alpha, \beta], x \rangle = \langle \alpha \otimes \beta - \beta \otimes \alpha, \Delta x \rangle = \sum_{(x)} \langle \alpha, x_{(1)} \rangle \langle \beta, x_{(2)} \rangle - \langle \beta, x_{(1)} \rangle \langle \alpha, x_{(2)} \rangle$$

where the only surviving terms are those with  $|x_{(1)}| = k$  and  $|x_{(2)}| = j$  or  $|x_{(1)}| = j$  and  $|x_{(2)}| = k$ . In any case this is only possible if  $x$  is homogeneous of degree  $k + j$ .

Now, observe that using this fact and induction one can show that

$$\mathfrak{g}_k^N \subset \bigoplus_{j=k}^N W_j$$

thus  $\mathfrak{g}_N^N \subset W_N$  and  $\mathfrak{g}_{N+1}^N = \{0\}$ . □

**Corollary 2.18.** *The centre of  $\mathfrak{g}^N$  is  $W_N$ .*

Note that the Baker–Campbell–Hausdorff is also valid in  $\mathfrak{g}^N$  with the additional property that the sum is now finite due to nilpotency. We define an operator  $\text{BCH}_N : \mathfrak{g}^N \times \mathfrak{g}^N \rightarrow \mathfrak{g}^N$  by

$$\text{BCH}_N(\alpha, \beta) := \sum_{k=1}^N \text{BCH}_{(k)}(\alpha, \beta) \quad (2.12)$$

where  $\text{BCH}_{(k)} : \mathfrak{g} \times \mathfrak{g} \rightarrow W_k$  was defined in (2.8).

**2.2. Homogeneous norms.** Let  $\mathfrak{L}$  be a finite-dimensional Lie algebra. A family of dilations on  $\mathfrak{L}$  is a family  $(\Omega_r)_{r>0}$  of automorphisms of  $\mathfrak{L}$  such that  $\Omega_r \Omega_s = \Omega_{rs}$ . A homogeneous group is a connected simply connected Lie group whose Lie algebra is endowed with a family of dilations. If  $G$  is a homogeneous group, the map  $\exp \circ \Omega_r \circ \log$  is a group automorphism of  $G$  and we also call them dilations.

**Definition 2.19.** *The element  $X \in \mathfrak{L}$  is said to be an eigenvector of the dilation  $\Omega$  with eigenvalue  $\alpha \in \mathbb{R}$  if  $\Omega_r X = r^\alpha X$  for all  $r > 0$ . For an eigenvalue  $\alpha \in \mathbb{R}$  the eigenspace  $E_\alpha$  is the subspace of  $\mathfrak{L}$  spanned by all the eigenvectors of  $\Omega$  with eigenvalue  $\alpha$ .*

Since  $\Omega_r$  is a Lie homomorphism we have that  $[E_\alpha, E_\beta] \subset E_{\alpha+\beta}$ .

**Lemma 2.20.** *A family  $(\Omega_r)_{r>0}$  is a dilation if and only if  $\Omega_r = e^{\log(r)A}$  for some matrix  $A$ .*

*Proof.* It suffices to observe that  $f(r) := \Omega_{e^r}$  satisfies  $f(r+s) = f(r) \circ f(s)$ . □

Thus, a dilation can only have a finite number of eigenvalues which correspond to eigenvalues of the matrix  $A$ . In the sequel we order the spectrum of  $\Omega$  (or  $A$ ) increasingly, i.e.  $\alpha_1 \leq \dots \leq \alpha_n$  where  $n = \dim \mathfrak{L}$ . Since if  $\Omega$  is a dilation then  $\tilde{\Omega}_r = \Omega_{r^\alpha}$  is also a dilation, we may and do suppose that  $\alpha_1 \geq 1$ .

In the following, we assume the matrix  $A$  to be diagonalizable. In this case, we may fix a basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{L}$  such that  $AX_j = \alpha_j X_j$ . We use this basis to obtain a norm  $\|\cdot\|$  on  $\mathfrak{L}$  by requiring that this basis is orthonormal.

**Definition 2.21.** *A homogeneous norm on a homogeneous group  $G$  is a continuous function  $|\cdot| : G \rightarrow \mathbb{R}_+$  which is of class  $C^\infty$  on  $G \setminus \{\mathbf{1}\}$  and such that  $|X^{-1}| = |X|$ , and  $|\Omega_r X| = r|X|$ . The homogeneous norm  $|\cdot|$  is said to be sub-additive if  $|XY| \leq |X| + |Y|$ .*

In case the homogeneous norm is sub-additive, we can induce a left-invariant metric on  $G$  by setting  $\rho(X, Y) = |Y^{-1}X|$ .

**Lemma 2.22** ([19]). *Suppose  $G$  is a homogeneous group with Lie algebra  $\mathfrak{L}$ . Then there exist constants  $C_1, C_2 > 0$  such that*

$$C_1 \|\log X\| \leq |X| \leq C_2 \|\log X\|^{1/\alpha_n}$$

for all  $X \in G$  with  $|X| \leq 1$ .

A simple consequence of this lemma is the following

**Corollary 2.23.** *All homogeneous norms on  $G$  are equivalent.*

We can build a dilation on  $\mathfrak{g}^N$  as follows: for  $x \in \mathcal{H}_{(k)}$  set  $\Omega_r x = r^k x$  and transpose this map to  $\mathcal{H}^*$  by setting  $\langle \Omega_r X, x \rangle = \langle X, \Omega_r x \rangle$ .

**Proposition 2.24.** *The maps  $\Omega_r$  are algebra automorphisms of  $\mathcal{H}^*$ .*

*Proof.* The map  $\Omega_r$  is a coalgebra morphism of  $\mathcal{H}$ . Indeed, for a homogeneous element  $x \in \mathcal{H}$

$$\Delta(\Omega_r x) = r^{|x|} \Delta x = (\Omega_r \otimes \Omega_r) \Delta x$$

by Equation (2.1). □

In fact, the maps  $\Omega_r$  are bialgebra automorphisms of  $\mathcal{H}$ , hence Hopf algebra automorphisms. Therefore, we obtain a dilation on  $\mathfrak{g}^N$  by simple restriction, and we remark that the spaces  $W_k$  act as eigenspaces for  $\Omega$ , with  $k$  as the associated eigenvalue.

Fix a basis  $B$  of  $\mathcal{H}_N$  and define a norm  $\|\cdot\|$  on this space by requiring that  $B$  is orthonormal. Define the constant

$$D := \max_{v \in B} \sum_{(v)} |c(v_1, v_2; v)| \|v_1\| \|v_2\| < \infty$$

where  $\Delta v = \sum c(v_1, v_2; v) v_1 \otimes v_2$ , and set, for  $f \in \mathcal{H}_N^*$ ,

$$\|f\| := D \|f\|_\infty$$

where  $\|\cdot\|_\infty$  is the usual operator norm with respect to  $B$ . Then, if  $f \in \mathcal{H}_{(k)}^*$  and  $g \in \mathcal{H}_{(k)}^*$  we have that  $f \star g \in \mathcal{H}_{(k+j)}^*$  and for any  $v \in B$

$$|\langle f \star g, v \rangle| \leq \sum_{(v)} |c(v_1, v_2; v)| |\langle f, v_1 \rangle| |\langle g, v_2 \rangle| \leq \frac{1}{D} \|f\| \|g\|,$$

thus  $\|f \star g\| \leq \|f\| \|g\|$ .

We obtain a homogeneous norm on  $G^N$  by setting

$$|X| := \max_{k=1,\dots,N} (k! \|X_k\|)^{1/k} + \max_{k=1,\dots,N} (k! \|X_k^{-1}\|)^{1/k}, \quad (2.13)$$

where for  $X \in G^N$  we write uniquely (see Remark 2.6)

$$X = \varepsilon + \sum_{k=1}^N X_k, \quad X_k \in \mathcal{H}_{(k)}^*.$$

The following formula for the components of the convolution product between two linear maps follows directly from (2.1): given linear maps  $f, g \in \mathcal{H}_N^*$ , writing

$$f = \varepsilon + \sum_{k=1}^N f_k, \quad g = \varepsilon + \sum_{k=1}^N g_k$$

we then have

$$f \star g = \varepsilon + \sum_{k=1}^N \sum_{j=1}^k f_j \star g_{k-j}. \quad (2.14)$$

**Proposition 2.25.** *The group  $(G^N, |\cdot|)$  is homogeneous with  $|\cdot|$  sub-additive.*

*Proof.* We only need to prove that the norm defined in Equation (2.13) is sub-additive, the other properties being clear. By (2.14) we have that

$$\|(X \star Y)_k\| \leq \sum_{j=1}^k \|X_j\| \|Y_{k-j}\| \leq \frac{1}{k!} \sum_{j=1}^k \binom{k}{j} |X|^j |Y|^{k-j} = \frac{1}{k!} (|X| + |Y|)^{1/k}$$

whence the result.  $\square$

In particular we obtain a metric  $\rho_N$  on  $G^N$  which is left-invariant and such that the metric space  $(G^N, \rho_N)$  is complete. This distance may be explicitly computed by Equation (2.13) as

$$\rho_N(X, Y) = \max_{k=1,\dots,N} (k! \|(Y^{-1} \star X)_k\|)^{1/k}$$

*Remark 2.26.* In view of Corollary 2.23 we may obtain bounds over the distance  $\rho_N(X, Y)$  by bounding first  $p(Y^{-1} \star X)$  for any homogeneous norm  $p$  on  $G$ . The importance of  $|\cdot|$  resides in the fact that we know it to be sub-additive by Proposition 2.25 so we obtain a distance. On some concrete cases there might be other sub-additive homogeneous norms defined on  $G^N$  but we choose to work with the one defined in (2.13) since it is closely related to rough paths, see Definition 3.1 and Proposition 4.2

### 3. CONSTRUCTION OF ROUGH PATHS

As in the previous section, we fix a locally-finite graded connected Hopf algebra  $\mathcal{H}$ . We also fix a number  $\gamma \in ]0, 1[$  and let  $N := \lfloor \gamma^{-1} \rfloor$  be the biggest integer such that  $N\gamma \leq 1$ . Without loss of generality we can fix a basis  $B$  of  $\mathcal{H}_N$  consisting only of homogeneous elements and in particular we let  $\{e_1, \dots, e_d\} = B \cap \mathcal{H}_{(1)}$  where  $d := \dim \mathcal{H}_{(1)}$ .



**Definition 3.1.** A  $(\mathcal{H}, \gamma)$ -Hölder path is a path  $\mathbb{X} : [0, 1] \rightarrow G^N$ , with  $N = \lfloor \gamma^{-1} \rfloor$ , which is  $\gamma$ -Hölder with respect to the metric  $\rho_N$  defined by the homogeneous norm in (2.13) and such that  $\mathbb{X}_0 = \varepsilon$ . Setting  $x_t^i = \langle \mathbb{X}_t, e_i \rangle$  we say that  $\mathbb{X}$  is a  $\gamma$ -rough path over  $(x^1, \dots, x^d)$ .

*Remark 3.2.* The classical definitions of rough paths of various types consider functions  $X : [0, 1]^2 \rightarrow G$  with values in an appropriate group, satisfying Chen's rule

$$X_{su} * X_{ut} = X_{st}$$

and an analytical estimate. It turns out that one can reduce  $X$  to a one-variable path  $\mathbb{X} : [0, 1] \rightarrow G$  by noting that the above equation implies that  $X_{st} = (X_{0s})^{-1} * X_{0t}$ . The analytical estimate can be seen to be equivalent to requiring that the resulting path  $\mathbb{X}_t = X_{0t}$  is  $\gamma$ -Hölder with respect to the corresponding homogeneous metric  $\rho_N$ . For a more detailed exposition, albeit in the setting of geometric rough paths under the  $p$ -variation norm, see [20].

*Remark 3.3.* By specializing this definition to different values of  $\mathcal{H}$  we recover both *geometric rough paths* [28] where  $\mathcal{H}$  is the shuffle Hopf algebra over an alphabet and *branched rough paths* [22] where  $\mathcal{H}$  is the Butcher–Connes–Kreimer Hopf algebra on decorated non-planar rooted trees.

**Proposition 3.4.** *The path  $\mathbb{X} : [0, 1] \rightarrow G^N$  is a  $(\mathcal{H}, \gamma)$ -Hölder path if and only if the two-variable function  $X : [0, 1]^2 \rightarrow G^N$  given by  $X_{st} = \mathbb{X}_s^{-1} \star \mathbb{X}_t$  satisfies Chen's rule and for all  $v \in B$  one has*

$$|\langle X_{st}, v \rangle| \lesssim |t - s|^{\gamma|v|}. \quad (3.1)$$

*Proof.* First note that the homogeneous norm in (2.13) is defined with respect a fixed (but arbitrary) basis so we use the basis  $B$  fixed at the beginning of this section. Also, due to the above remark we only have to verify that  $\mathbb{X}$  is  $\gamma$ -Hölder with respect to  $\rho_N$  if and only if  $X$  satisfies (3.1) using the same basis. In one direction, if  $\mathbb{X}$  is  $\gamma$ -Hölder then, by definition

$$|X_{st}| = \rho_N(\mathbb{X}_s, \mathbb{X}_t) \lesssim |t - s|^\gamma$$

and so, for a basis element  $v \in B$  we have

$$|\langle X_{st}, v \rangle| \lesssim \frac{1}{D|v|!} |t - s|^{\gamma|v|}.$$

Conversely, if (3.1) holds then  $|X_{st}| \lesssim |t - s|^\gamma$  and so by definition also  $\rho_N(\mathbb{X}_s, \mathbb{X}_t) \lesssim |t - s|^\gamma$ , i.e.  $\mathbb{X}$  is  $\gamma$ -Hölder with respect to  $\rho_N$ .  $\square$

We now come to the problem of existence. Our construction of a rough path in the sense of Definition 3.1 over an arbitrary collection of  $\gamma$ -Hölder paths  $(x^1, \dots, x^d)$  relies in the following extension theorem. We note that the proof is a reinterpretation of the approach of Lyons-Victoir [29, Theorem 1].

**Theorem 3.5** (Rough path extension). *Let  $1 \leq n \leq N - 1$  and  $\gamma \in ]0, 1[$  such that  $\gamma^{-1} \notin \mathbb{N}$ . Suppose we have a  $(\mathcal{H}_n, \gamma)$ -Hölder path  $\mathbb{X} : [0, 1] \rightarrow (G^n, \rho_n)$ . There is a  $(\mathcal{H}_{n+1}, \gamma)$ -Hölder path  $\tilde{\mathbb{X}} : [0, 1] \rightarrow (G^{n+1}, \rho_{n+1})$  extending  $\mathbb{X}$ , i.e. such that  $\tilde{\mathbb{X}}|_{\mathcal{H}_n} = \mathbb{X}$ .*

A key tool is the following technical lemma whose proof can be found in [29, Lemma 2].

**Lemma 3.6.** *Let  $(E, \rho)$  be a complete metric space and set  $D = \{t_k^m := k2^{-m} : m \geq 0, k = 0, \dots, 2^m - 1\}$ . Suppose  $y : D \rightarrow E$  is a path satisfying the bound  $\rho(y_{t_k^m}, y_{t_{k+1}^m}) \lesssim 2^{-\gamma m}$  for some  $\gamma \in (0, 1)$ . Then, there exists a  $\gamma$ -Hölder path  $x : [0, 1] \rightarrow E$  such that  $x|_D = y$ .*

*Proof of Theorem 3.5.* The construction of  $\tilde{\mathbb{X}}$  is made in two steps.

Step 1. Let  $D = \{t_k^m := 2^{-m}k \mid m \in \mathbb{N}_0, k = 0, \dots, 2^m\}$  be the dyadics in  $[0, 1]$ . Set  $X_{st} = (\mathbb{X}_s^n)^{-1} \star \mathbb{X}_t^n \in G^n$  and  $L_{st} = \log_n(X_{st}) \in \mathfrak{g}^n$  where  $\log_n$  was defined in (2.11). Then, the BCH formula (2.12) and Chen's rule imply that

$$L_{st} = \text{BCH}_n(L_{su}, L_{ut}) = L_{su} + L_{ut} + \text{BCH}'_n(L_{su}, L_{ut}). \quad (3.2)$$

We look for  $Z_{st} \in W_{n+1}$  such that the exponential  $X_{st}^{n+1} = \exp_{n+1}(\iota_{n+1}L_{st} + Z_{st})$  still satisfies Chen's rule. To this effect, we first define  $Z$  on the dyadics, starting from the initial condition  $Z_{0,1} = 0$  and

$$Z_{t_{2k}^m, t_{2k+1}^m} = Z_{t_{2k+1}^m, t_{2k+2}^m} := \frac{1}{2} Z_{t_k^{m-1}, t_{k+1}^{m-1}} - \frac{1}{2} \text{BCH}_{(n+1)}(\iota_{n+1}L_{t_{2k}^m, t_{2k+1}^m}, \iota_{n+1}L_{t_{2k+1}^m, t_{2k+2}^m}) \quad (3.3)$$

where  $\iota_{n+1}$  is the canonical inclusion and we have used Proposition 2.14 so that the right-hand side is well defined. To ease notation in the following we identify  $L$  with  $\iota_{n+1}L$  where appropriate. Note that with this  $Z_{st} \in W_{n+1}$  for each pair of consecutive dyadics.

We now look to extend this definition to more general pairs of dyadics  $s, t \in D$ . Set  $Y_{0,1} := \exp_{n+1}(L_{0,1} + w)$ . If  $s = t_{2k}^m$ ,  $u = t_{2k+1}^m$  and  $t_{2k+2}^m$  are consecutive dyadics then we define

$$Y_{su} := \exp_{n+1}(L_{su} + Z_{su}), \quad Y_{ut} := \exp_{n+1}(L_{ut} + Z_{ut})$$

and note that by (3.3) we have

$$\begin{aligned} \log_{n+1}(Y_{su} \star Y_{ut}) &= L_{su} + L_{ut} + \text{BCH}'_{n+1}(L_{su}, L_{ut}) + Z_{su} + Z_{ut} \\ &= L_{su} + L_{ut} + \text{BCH}'_n(L_{su}, L_{ut}) + Z_{st} = L_{st} + Z_{st} \end{aligned}$$

by (3.2), so that

$$Y_{su} \star Y_{ut} = Y_{st}.$$

We have also used the fact that  $\iota_{n+1}$  is an algebra morphism for the  $\star$  product and that  $W_{n+1}$  is in the centre of  $\mathfrak{g}^{n+1}$  by Corollary 2.18. Therefore, we may set

$$Y_{t_k^m, t_j^m} := Y_{t_k^m, t_{k+1}^m} \star Y_{t_{k+1}^m, t_{k+2}^m} \star \dots \star Y_{t_{j-1}^m, t_j^m}$$

so that the identity  $Y_{t_i^m, t_j^m} \star Y_{t_j^m, t_k^m} = Y_{t_i^m, t_k^m}$  is valid for any  $0 \leq i < j < k \leq 2^m$ .

Step 2. In order to have a  $(\mathcal{H}_{n+1}, \gamma)$ -Hölder path, Definition 3.1 requires us to construct a  $\gamma$ -Hölder path with values in  $G^{m+1}$ , and for this we will use Lemma 3.6. Set

$$a_m := 2^{m(n+1)\gamma} \max_{k=0, \dots, 2^m-1} \left\| Z_{t_k^m, t_{k+1}^m} \right\|_{n+1}.$$

Then, if  $v$  is a basis element in  $\mathcal{H}_{(n+1)}$  we have by Equation (2.10), for  $s = t_k^m$ ,  $u = t_{k+1}^m$  and  $t = t_{k+2}^m$ , that

$$|\langle \text{BCH}_{(n+1)}(L_{su}, L_{ut}), v \rangle| \leq \sum_{(v)} \sum_{i+j=n+1} \frac{1}{i!j!} \sum_{\sigma \in S_{n+1}} |a_\sigma| \prod_{p=1}^i |\langle L_{su}, v_{\sigma^{-1}(p)} \rangle| \prod_{q=i+1}^{n+1} |\langle L_{ut}, v_{\sigma^{-1}(q)} \rangle|.$$

Now, since  $v_{(j)} \in \mathcal{H}_{(1)}$  for all  $j = 1, \dots, n+1$  we actually have that

$$|\langle L_{su}, v_{(j)} \rangle| \leq \sum_{k=1}^d |x_u^k - x_s^k| |v_{(j)}^k| \leq 2^{-m\gamma} \sum_{k=1}^d |v_{(j)}^k|$$

for some coefficients  $v_{(j)}^k \in \mathbb{R}$  such that  $v_{(j)} = \sum_{k=1}^d v_{(j)}^k e_k$ , and we have a similar estimate for  $L_{ut}$  instead of  $L_{su}$ . Therefore we obtain that

$$\left\| \text{BCH}_{(n+1)}(L_{su}, L_{ut}) \right\|_{n+1} \leq C 2^{-m(n+1)\gamma},$$

where

$$C = \max_v \sum_{(v)} \sum_{i+j=n+1} \frac{1}{i!j!} \sum_{\sigma \in S_{n+1}} |a_\sigma| \sum_{k_1, \dots, k_{n+1}=1}^{n+1} \prod_{\ell=1}^{n+1} |v_{(\ell)}^{k_\ell}|.$$

Therefore, from (3.3) we get

$$2^{-(m+1)(n+1)\gamma} a_{m+1} \leq 2^{-m(n+1)\gamma} + C 2^{-m(n+1)\gamma},$$

hence there is another constant  $C > 0$  such that

$$a_m \leq C \sum_{j=0}^{m-1} 2^{-j(1-(n+1)\gamma)}.$$

Since we are in the regime where  $(n+1)\gamma < 1$  we obtain that the rhs is bounded in  $m$  (here we use that  $\gamma^{-1} \notin \mathbb{N}$ ) and

$$\sup_{m \geq 0} a_m \leq \frac{C}{2 - 2^{(n+1)\gamma}}.$$

To conclude, we observe that

$$L \mapsto \sum_{k=1}^n \|L\|_k^{1/k}$$

defines a norm on  $\mathcal{H}_n^*$ , thus by Lemma 2.22 there's a constant  $C_1$  such that

$$\|L_{st}\|_k \leq \|L_{st}\|^k \leq |(\mathbb{X}_s^n)^{-1} \star \mathbb{X}_t^n|^k = \rho_n(\mathbb{X}_s^n, \mathbb{X}_t^n)^k \leq C_1 2^{-mk\gamma}$$

for all  $k = 1, \dots, n$  and  $s = t_j^m, t = t_{j+1}^m$  since  $\mathbb{X}^n$  is  $\gamma$ -Hölder with respect to  $\rho_n$ . This and the previous estimate provide the bound

$$|Y_{t_j^m, t_{j+1}^m}| \lesssim 2^{-m\gamma}. \quad (3.4)$$

By Chen's rule, the path  $\mathbb{Y} : D \rightarrow G^{n+1}$  defined by  $\mathbb{Y}_{t_j^m} := Y_{0, t_j^m}$  satisfies

$$\rho_{n+1}(\mathbb{Y}_{t_j^m}, \mathbb{Y}_{t_{j+1}^m}) \lesssim 2^{-m\gamma},$$

thus by Lemma 3.6 we obtain a  $\gamma$ -Hölder path  $\tilde{\mathbb{X}} : [0, 1] \rightarrow G^{n+1}$ .  $\square$

*Remark 3.7.* Our construction depends on a finite number of choices, namely we set  $Z_{0,1} = 0$  to start the recursion in (3.3), and this for each level; moreover in (3.3) we make the choice  $Z_{t_{2k}^m, t_{2k+1}^m} = Z_{t_{2k+1}^m, t_{2k+2}^m}$ . These choices are the same as in [29, Proof of Theorem 1] and are indeed the most natural ones, but one could change them and the final outcome would be different.

*Remark 3.8.* While in [29, Proof of Proposition 6] Lyons and Victoir use the axiom of choice, our proof is completely constructive. In particular, we use the explicit map  $\exp_{k+1} \circ \log_k : G^k(\mathcal{T}_n) \rightarrow G^{k+1}(\mathcal{T}_n)$  instead the role of the injection  $i_{G/K,G} : G/K \rightarrow G$  in [29, Proposition 6]. The fact that this map has good continuity estimates is based on the explicit expression (2.10) for the Baker-Campbell-Hausdorff formula.

**Corollary 3.9.** *Given  $\gamma \in ]0, 1[$  with  $\gamma^{-1} \notin \mathbb{N}$  and a collection of  $\gamma$ -Hölder paths  $x^i : [0, 1] \rightarrow \mathbb{R}$ , there exists a  $(\mathcal{H}, \gamma)$ -Hölder path  $\mathbb{X}$  over  $(x^1, \dots, x^d)$  in the sense of Definition 3.1.*

*Proof.* We start with the following observation: for  $n = 1$ , the group  $G^1 \subset \mathcal{H}_{(1)}^*$  is abelian, and isomorphic to the additive group  $\mathcal{H}_{(1)}^*$ . Indeed, let  $X, Y \in G^1$  and  $x \in \mathcal{H}_{(1)}$ . Then, as  $\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x$  by the grading, we have that

$$\langle X \star Y, x \rangle = \langle X, x \rangle + \langle Y, x \rangle,$$

that is,  $X \star Y = X + Y$ . Moreover, in  $\mathcal{H}_1$  the product  $xy = 0$ . Therefore, we may set  $\langle \mathbb{X}_t^1, e_i \rangle := x_t^i$  where  $\{e_1, \dots, e_d\}$  is a basis of  $\mathcal{H}_{(1)}$  and this path is  $\gamma$ -Hölder with respect to  $\rho_1$ .

By Theorem 3.5 there is a  $\gamma$ -Hölder path  $\mathbb{X}^2 : [0, 1] \rightarrow (G^2, \rho_2)$  extending  $\mathbb{X}^1$  so in particular  $\langle \mathbb{X}_t^2, e_i \rangle = x_t^i$  also. Continuing in this way we obtain successive  $\gamma$ -Hölder extensions  $\mathbb{X}^3, \dots, \mathbb{X}^N$  and we set  $\mathbb{X} := \mathbb{X}^N$ .  $\square$

The following result has already been proved in the case where the underlying Hopf algebra  $\mathcal{H}$  is *combinatorial* by Curry, Ebrahimi-Fard, Manchon and Munthe-Kaas in [12]. We remark that their proof works without modifications in our context so we have

**Theorem 3.10.** *Let  $\mathbb{X}$  be a  $(\mathcal{H}, \gamma)$ -Hölder path. There exists a path  $\hat{\mathbb{X}} : [0, 1] \rightarrow G$  such that  $|\langle \hat{\mathbb{X}}_s^{-1} \star \hat{\mathbb{X}}_t, x \rangle| \lesssim |t - s|^{\gamma|x|}$  for all homogeneous  $x \in \mathcal{H}$ .*

*Remark 3.11.* In view of Theorem 3.10 we can replace the truncated group in Definition 3.1 by the full group of characters  $G$ . What this means is that  $\gamma$ -rough paths are uniquely defined once we fix the first  $N$  levels and since  $\mathcal{H}$  is locally finite, this amounts to a finite number of choices. This is of course a generalization of the extension theorem of [28], see also [22, Theorem 7.3] for the branched case.

## 4. APPLICATIONS

We now apply Theorem 3.5 to various kinds of Hopf algebras in order to link this result with the contexts already existing in the literature.

**4.1. Geometric rough paths.** In this settings we fix a finite alphabet  $A := \{1, \dots, d\}$ . As a vector space  $\mathcal{H} := T^c(A)$  is the linear span of the *free monoid*  $M(A)$  generated by  $A$ . The product on  $\mathcal{H}$  is the shuffle product  $\sqcup : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$  defined recursively by  $\mathbf{1} \sqcup v = v \sqcup \mathbf{1} = v$  for all  $v \in \mathcal{H}$ , where  $\mathbf{1} \in M(A)$  is the unit for the monoid operation, and

$$(au \sqcup bv) = a(u \sqcup bv) + b(au \sqcup v)$$

for all  $u, v \in \mathcal{H}$  and  $a, b \in A$ , where  $au$  and  $bv$  denote the product of the letters  $a, b$  with the words  $u, v$  in  $M(A)$ .

The coproduct  $\bar{\Delta} : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  is obtained by *deconcatenation* of words,

$$\bar{\Delta}(a_1 \cdots a_n) = a_1 \cdots a_n \otimes \mathbf{1} + \mathbf{1} \otimes a_1 \cdots a_n + \sum_{k=1}^{n-1} a_1 \cdots a_k \otimes a_{k+1} \cdots a_n.$$

It turns out that  $(\mathcal{H}, \cdot, \bar{\Delta})$  is a commutative unital Hopf algebra, and  $(\mathcal{H}, \bar{\Delta})$  is the cofree coalgebra over the linear span of  $A$ . The antipode is the linear map  $S : \mathcal{H} \rightarrow \mathcal{H}$  given by

$$S(a_1 \cdots a_n) = (-1)^n a_n \cdots a_1.$$

Finally, we recall that  $\mathcal{H}$  is graded by the length  $\ell(a_1 \cdots a_n) = n$  and it is also connected. The homogeneous components  $\mathcal{H}_{(n)}$  are spanned by the sets  $\{a_1 \cdots a_n : a_i \in A\}$ .

Definition 3.1 specialises in this case to *geometric rough paths* (GRP) as defined in [25] (see just below for the precise definition) and Theorem 3.5 coincides with [29, Theorem 6].

**Definition 4.1.** *Let  $\gamma \in ]0, 1[$  and set  $N := \lfloor \gamma^{-1} \rfloor$ . A geometric  $\gamma$ -rough path is a map  $X : [0, 1]^2 \rightarrow G^N$  such that  $X_{tt} = \varepsilon$ , it satisfies Chen's rule*

$$X_{st} = X_{su} \star X_{ut}$$

for all  $s, u, t \in [0, 1]$  and the analytic bound  $|\langle X_{st}, v \rangle| \lesssim |t - s|^{\gamma \ell(v)}$  for all  $v \in \mathcal{H}_N$ .

We observe that Proposition 3.4 connects Definition 3.1 and Definition 4.1.

**Proposition 4.2.** *A path  $X : [0, 1]^2 \rightarrow G^N$  is a geometric rough path if and only if  $\mathbb{X}_t := X_{0t}$  is a  $(\mathcal{H}, \gamma)$ -Hölder path in the sense of Definition 3.1.*

**4.2. Branched rough paths.** Let  $\mathcal{T}$  be the collection of all non-planar non-empty rooted trees with nodes decorated by  $\{1, \dots, d\}$ . Elements of  $\mathcal{T}$  are written as 2-tuples  $\tau = (T, c)$  where  $T$  is a non-planar tree with node set  $N_T$  and edge set  $E_T$ , and  $c : N_T \rightarrow \{1, \dots, d\}$  is a function. Edges in  $E_T$  are oriented away from the root, but this is not reflected in our graphical representation. Examples of elements of  $\mathcal{T}$  include the following

$$\bullet_i, \quad \begin{array}{c} \bullet_j \\ | \\ \bullet_i \end{array}, \quad \begin{array}{c} \bullet_j \bullet_k \\ | \\ \bullet_i \end{array}, \quad \begin{array}{c} \bullet_j \bullet_l \\ | \\ \bullet_j \bullet_m \\ | \\ \bullet_i \end{array}.$$

For  $\tau \in \mathcal{T}$  write  $|\tau| = \#N_T$  for its number of nodes. Also, given an edge  $e = (x, y) \in E_T$  we set  $s(e) = x$  and  $t(e) = y$ . There is a natural partial order relation on  $N_T$  where  $x \leq y$  if and only if there is a path in  $T$  from the root to  $y$  containing  $x$ .

We denote by  $\mathcal{F}$  the collection of decorated rooted forests and we let  $\mathcal{H} := \mathcal{H}_{\text{BCK}}$  denote the vector space spanned by  $\mathcal{F}$ . There is a natural commutative and associative product on  $\mathcal{F}$  given by disjoint union of forests, where the empty forest  $\mathbf{1}$  acts as the unit. Then,  $\mathcal{H}$  is the free commutative algebra over  $\mathcal{T}$ , with grading  $|\tau_1 \cdots \tau_k| = |\tau_1| + \cdots + |\tau_k|$ . Given  $i \in \{1, \dots, d\}$  and a forest  $\tau = \tau_1 \cdots \tau_k$  we denote by  $[\tau_1 \cdots \tau_k]_i$  the tree obtained by grafting each of the trees  $\tau_1, \dots, \tau_k$  to a new root decorated by  $i$ , e.g.

$$[\bullet_j]_i = \begin{array}{c} \bullet_j \\ | \\ \bullet_i \end{array}, \quad [\bullet_j \bullet_k]_i = \begin{array}{c} \bullet_j \bullet_k \\ | \\ \bullet_i \end{array}.$$

The decorated Butcher–Connes–Kreimer coproduct [10, 22] is the unique algebra morphism  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  such that

$$\Delta[\tau]_i = [\tau]_i \otimes \mathbf{1} + (\text{id} \otimes [\cdot]_i)\Delta\tau.$$

This coproduct admits a representation in terms of *cuts*. An *admissible cut*  $C$  of a tree  $T$  is a non-empty subset of  $E_T$  such that any path from any vertex of the tree to the root contains at most one edge from  $C$ ; we denote by  $\mathfrak{A}(T)$  the set of all admissible cuts of the tree  $T$ . Any admissible cut  $C$  containing  $k$  edges maps a tree  $T$  to a forest  $C(T) = T_1 \cdots T_{k+1}$  obtained by removing each of the edges in  $C$ . Observe that only one of the remaining trees  $T_1, \dots, T_{k+1}$  contains the root of  $T$ , which we denote by  $R^C(T)$ ; the forest formed by the other  $k$  factors is denoted by  $P^C(T)$ . This naturally induces a map on decorated trees by considering cuts of the underlying tree, and restriction of the decoration map to each of the rooted subtrees  $T_1, \dots, T_{k+1}$ . Then,

$$\Delta\tau = \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau + \sum_{C \in \mathfrak{A}(\tau)} P^C(\tau) \otimes R^C(\tau).$$

This, together with the counit map  $\varepsilon : \mathcal{F} \rightarrow \mathbb{R}$  such that  $\varepsilon(\tau) = 1$  if and only if  $\tau = \mathbf{1}$  endows  $\mathcal{F}$  with a connected graded commutative non-cocommutative bialgebra structure, hence a Hopf algebra structure [30].

As before we denote by  $\mathcal{H}^*$  the linear dual of  $\mathcal{H}$  which is an algebra via the convolution product  $\langle X \star Y, \tau \rangle = \langle X \otimes Y, \Delta\tau \rangle$  and we denote by  $G$  the set of characters on  $\mathcal{H}$ , that is, linear functionals  $X \in \mathcal{H}^*$  such that  $\langle X, \sigma \cdot \tau \rangle = \langle X, \sigma \rangle \langle X, \tau \rangle$ . For each  $n \in \mathbb{N}$  the finite-dimensional vector space  $\mathcal{H}_n$  spanned by the set  $\mathcal{F}_n$  of forests with at most  $n$  nodes is a subcoalgebra of  $\mathcal{H}$ , hence its dual is an algebra under the convolution product, and we let  $G_n$  be the set of characters on  $\mathcal{H}_n$ .

We have already defined branched rough paths in Definition 1.1. As in Proposition 4.2 for the geometric case, we have the following

**Proposition 4.3.** *A path  $X : [0, 1]^2 \rightarrow G^N$  is a branched rough path if and only if  $\mathbb{X}_t := X_{0t}$  is a rough path in the sense of Definition 3.1.*

*Proof.* See the proof of Proposition 3.4. □

Directly applying Theorem 3.5 to the Butcher–Connes–Kreimer Hopf algebra  $\mathcal{H}$  we obtain

**Corollary 4.4.** *Given  $\gamma \in ]0, 1[$  with  $\gamma^{-1} \notin \mathbb{N}$  and a family of  $\gamma$ -Hölder paths  $(x^i : i = 1, \dots, d)$ , there exists a branched rough path  $X$  above  $(x^i : i = 1, \dots, d)$ , i.e.  $X : [0, 1]^2 \rightarrow G^N$  is such that  $\langle X_{st}, \bullet^i \rangle = x_t^i - x_s^i$  for all  $i = 1, \dots, d$ .*

We end this section with a final remark

*Remark 4.5.* Given the level of generality in which Theorem 3.5 is developed, our results also apply to the case when  $\mathcal{H}$  is a combinatorial Hopf algebra as defined in [12]. In particular, we also have a construction theorem for *planarly branched rough paths* [12] which are characters over Munthe-Kaas and Wright’s Hopf algebra of Lie group integrators [31].

## 5. ANISOTROPIC ROUGH PATHS

We now apply our results to another class of rough paths which we call *anisotropic geometric rough paths* (aGRPs for short). L. Gyurkó introduced a similar concept in [23], which he called  $\Pi$ -rough paths; unlike us, he uses a “primal” presentation, i.e. paths taking values in the tensor algebra  $T(\mathbb{R}^d)$ , and  $p$ -variation norms rather than Hölder norms. Geometric rough paths over a inhomogeneous (or anisotropic) set of paths can be traced back to Lyons’ original paper [28].

As in the geometric case, see Section 4.1, fix a finite alphabet  $A = \{1, \dots, d\}$  and denote by  $M(A)$  the free monoid generated by  $A$ . We denote again by  $T^c(A)$  the shuffle Hopf algebra over the alphabet  $A$ .

Let  $(\gamma_a : a \in A)$  be a sequence of real numbers such that  $0 < \gamma_a < 1$  for all  $a$ , and let  $\hat{\gamma} = \min_{a \in A} \gamma_a$ . For a word  $v = a_1 \cdots a_k$  of length  $k$  define

$$\omega(v) = \frac{\gamma_{a_1} + \cdots + \gamma_{a_k}}{\hat{\gamma}} = \frac{1}{\hat{\gamma}} \sum_{a \in A} n_a(v) \gamma_a \quad (5.1)$$

where  $n_a(v) = |\{j : v_j = a\}|$ , and observe that  $\omega$  is additive in the sense that  $\omega(uv) = \omega(u) + \omega(v)$  for each pair of words  $u, v \in M(A)$ . The set

$$\mathfrak{L} := \{v \in M(A) : \omega(v) \leq \hat{\gamma}^{-1}\}$$

is finite; if  $N_a := \lfloor \hat{\gamma}^{-1} \rfloor$  then  $\mathfrak{L} \subset T^c(A)_{N_a}$  and

$$\#\mathfrak{L} \leq \frac{d^{N_a} - 1}{d - 1}.$$

In analogy with Lemma 2.11, the additivity of  $\omega$  implies

**Lemma 5.1.** *The subspace  $\mathcal{H}_a(A) \subset T^c(A)$  spanned by  $\mathfrak{L}$  is a subcoalgebra of  $(\mathcal{H}, \bar{\Delta}, \varepsilon)$ .*

*Remark 5.2.* If the alphabet  $A$  is clear from the context we will omit it from the notation and write  $\mathcal{H}_a$  instead.

Consequently, we will consider the dual algebra  $(\mathcal{H}_a^*, \star, \varepsilon)$ . In this case, we define  $\mathfrak{g}_a$  to be the space of infinitesimal characters on  $\mathcal{H}_a$  and let  $G_a = \exp(\mathfrak{g}_a)$ . As before, there is a canonical injection  $\iota_a : \mathcal{H}_a^* \rightarrow \mathcal{H}^*$  so we suppose that  $\langle X, v \rangle = 0$  for all  $X \in \mathcal{H}^*$  and  $v \notin \mathfrak{L}$ . We grade  $\mathcal{H}_a$  by word length and we observe that since  $\omega(v) \geq \ell(v)$  we have that  $\mathcal{H}_a \subset T^c(A)_{N_a}$ .

Since  $\mathcal{H}$  is the conilpotent cofree coalgebra over the span of  $A$ , for each  $\lambda > 0$  there is a unique coalgebra automorphism  $\Omega_\lambda : \mathcal{H} \rightarrow \mathcal{H}$  such that  $\Omega_\lambda a = \lambda^{\gamma_a/\hat{\gamma}} a$  for all  $a \in A$ . In a similar way as before we have that  $(\Omega_\lambda)_{\lambda > 0}$  is a one-parameter family of automorphisms of  $\mathcal{H}$ . There are also homogeneous norms

$$|X| := \max_{v \in \mathfrak{L}} (\ell(v)! |\langle X, v \rangle|)^{1/\omega(v)} \quad (5.2)$$

and

$$\|X\| = \max_{v \in \mathfrak{L}} |\langle \log X, v \rangle|^{1/\omega(v)}. \quad (5.3)$$

These homogeneous norms are symmetric, but neither is sub-additive thus they do not generate a metric on  $G_a$ .

**5.1. Signatures.** In order to have a useful metric on  $G_a$  we consider *signatures* of smooth paths. We observe that  $A \subset \mathfrak{L}$ . Let  $x = (x^a : a \in A)$  be a collection of (piecewise) smooth paths, and define a map  $S(x) : [0, 1]^2 \rightarrow \mathcal{H}^*$  by

$$\langle S(x)_{st}, v \rangle := \int_s^t dx_{s_k}^{v_k} \int_s^{s_k} dx_{s_{k-1}}^{v_{k-1}} \cdots \int_s^{s_2} dx_{s_1}^{v_1}.$$

In his seminal work [9], K. T. Chen showed that  $S(x)$  is a multiplicative functional, that is,  $S(x)_{st} \in G_a$ . In particular  $\log S(x)_{st} \in \mathfrak{g}$  thus its restriction to  $\mathfrak{L}$  is in  $\mathfrak{g}_a$  and so we can consider  $S(x)$  also as an element of  $G_a$ .

Consider the metric  $d_a(X, Y) = \sum_{a \in A} |\langle X - Y, a \rangle|^{\hat{\gamma}/\gamma_a}$  on  $\mathcal{H}_{(1)}^*$ , where we recall that  $\mathcal{H}_{(1)}^*$  is the vector space spanned by  $A$ . The *anisotropic length* of a smooth curve  $\theta : [0, 1] \rightarrow \mathcal{H}_1^*$  is defined to be its length with respect to this metric and will be denoted by  $L_a(\theta)$ . Observe that since  $d_a(\Omega_\lambda X, \Omega_\lambda Y) = \lambda d_a(X, Y)$  we have that  $L_a(\Omega_\lambda \theta) = \lambda L_a(\theta)$ . We now define another homogeneous norm  $\|\cdot\| : G_a \rightarrow \mathbb{R}_+$ , called the *anisotropic Carnot–Carathéodory norm*, by setting

$$\|X\| := \inf\{L_a(x) : x^a \in C^\infty, S(x)_{0,1} = X\}.$$

Since curve length is invariant under reparametrization in any metric space we obtain, as in [20, Section 7.5.4].

**Proposition 5.3.** *The infimum defining the anisotropic Carnot–Carathéodory norm is finite and attained at some minimizing path  $\hat{x}$ .*

**Proposition 5.4.** *The anisotropic Carnot–Carathéodory norm is homogeneous, that is,  $\|\Omega_\lambda X\| = \lambda \|X\|$ .*

*Proof.* Let  $\hat{x}$  be the curve such that  $\|X\| = L_a(\hat{x})$ . For any  $\lambda > 0$  and word  $v \in \mathfrak{L}$  we have

$$\langle S(\Omega_\lambda \hat{x})_{0,1}, v \rangle = \lambda^{\omega(v)} \langle S(\hat{x})_{0,1}, v \rangle = \langle \Omega_\lambda S(\hat{x})_{0,1}, v \rangle = \langle \Omega_\lambda X, v \rangle,$$

thus  $\|\Omega_\lambda X\| \leq L_a(\Omega_\lambda \hat{x}) = \lambda L_a(\hat{x}) = \|X\|$ . The reverse inequality is obtained by noting that  $X = (\Omega_{\lambda^{-1}} \circ \Omega_\lambda) X$ .  $\square$

The anisotropic Carnot–Carathéodory norm can also be seen to be symmetric and sub-additive, hence it induces a left-invariant metric  $\nu$  on  $G_a$ .

**Definition 5.5.** *An anisotropic geometric  $\gamma$ -rough path, with  $\gamma = (\gamma_a, a \in A)$ , is a map  $X : [0, 1]^2 \rightarrow G_a$  such that*

- (1)  $X_{tt} = \varepsilon$ ,
- (2) *it satisfies Chen's rule  $X_{su} \star X_{ut} = X_{st}$  for all  $(s, u, t) \in [0, 1]^3$ ,*
- (3)  $|\langle X, v \rangle| \lesssim |t - s|^{\hat{\gamma}\omega(v)}$  *for all  $v \in \mathfrak{L}$ .*

**Proposition 5.6.** *Anisotropic geometric  $\gamma$ -rough paths are in one-to-one correspondence with  $\hat{\gamma}$ -Hölder paths  $\mathbb{X} : [0, 1] \rightarrow (G_a, \nu)$ .*

*Proof.* Let  $X$  be an anisotropic geometric  $\gamma$ -rough path and  $v$  a word. By definition we have that  $|\langle X_{st}, v \rangle| \lesssim |t - s|^{\hat{\gamma}\omega(v)}$ , hence  $|X_{st}| \lesssim |t - s|^{\hat{\gamma}}$ . The equivalence between  $|\cdot|$  and  $\|\cdot\|$  implies that  $\nu(\mathbb{X}_s, \mathbb{X}_t) = \|X_{st}\| \lesssim |t - s|^{\hat{\gamma}}$ , hence  $t \mapsto \mathbb{X}_t$  is  $\hat{\gamma}$ -Hölder with respect to  $\nu$ . The other direction follows in a similar manner.  $\square$



**5.2. Constructing anisotropic rough paths.** Proceeding as in the proof of Theorem 3.5 we can show

**Theorem 5.7.** *Let  $(\gamma_a : a \in A)$  be real numbers such that  $\gamma_a \in (0, 1)$  and  $1 \notin \sum_{a \in A} \gamma_a \mathbb{N}$ . Let  $(x^a : a \in A)$  be a collection of real-valued paths such that  $x^a$  is  $\gamma_a$ -Hölder. Then*

- (1) *There exists an anisotropic rough path  $X$  such that  $\langle X_{st}, a \rangle = x_t^a - x_s^a$  for all  $a \in A$ .*
- (2) *Given a collection of functions  $(g^a : a \in A)$  with  $g^a \in C^{\gamma_a}$ , let  $\bar{x}_t^a = x_t^a + g_t^a$  and denote by  $gX$  the anisotropic geometric  $\gamma$ -rough path constructed in point (1) above the path*

$$\bar{x}_t = \sum_{a \in A} \bar{x}_t^a a \in \mathcal{H}_{(1)}, \quad t \in [0, 1].$$

*Then, for any two such functions  $g$  and  $g'$  we have that  $g'(gX) = (g + g')X$ .*

*Proof.* We set  $\hat{\gamma} = \min_{a \in A} \gamma_a$  as above. The only difference in the proof of this theorem is in the analytic step, because now we have to show that

$$\nu(Y_{t_k^m, t_{k+1}^m}) \lesssim 2^{-m\hat{\gamma}}$$

in order to apply Lemma 2 of [29]. We recall that the metric space  $(G_a, \nu)$  only considers the regularity of the components  $\langle X, v \rangle$  where the weight  $\omega(v) < 1$  by the very definition of the homogeneous norms in (5.2)-(5.3). Looking at the proof of Theorem 3.5 we see that this bound comes from the bound on  $\text{BCH}_{(n+1)}(L_{su}^n, L_{ut}^n)$  provided by Lemma 2.10. In this case we have, for a word  $v \in \mathfrak{L}$  of length  $n+1$  and  $s = t_{2k}^{m+1}$ ,  $u = t_{2k+1}^{m+1}$  and  $t = t_{2k+2}^{m+1}$ ,

$$\begin{aligned} |\langle \text{BCH}_{(n+1)}(L_{su}^n, L_{ut}^n), e_I \rangle| &\leq \sum_{p+q=n+1} \frac{1}{p!q!} \sum_{\sigma \in S_{n+1}} |a_\sigma| \prod_{r_1=1}^p |\langle L_{su}^n, v_{i_{\sigma^{-1}(r_1)}} \rangle| \prod_{r_2=p+1}^{n+1} |\langle L_{ut}^n, v_{i_{\sigma^{-1}(r_2)}} \rangle| \\ &= \sum_{p+q=n+1} \frac{1}{p!q!} \sum_{\sigma \in S_{n+1}} |a_\sigma| \prod_{r_1=1}^p |x_u^{v_{\sigma^{-1}(r_1)}} - x_s^{v_{\sigma^{-1}(r_1)}}| \prod_{r_2=p+1}^{n+1} |x_t^{v_{\sigma^{-1}(r_2)}} - x_u^{v_{\sigma^{-1}(r_2)}}| \\ &\leq \sum_{p+q=n+1} \frac{1}{p!q!} \sum_{\sigma \in S_{n+1}} |a_\sigma| \prod_{r_1=1}^p |u - s|^{\gamma_{v_{\sigma^{-1}(r_1)}}} \prod_{r_2=p+1}^{n+1} |t - u|^{\gamma_{v_{\sigma^{-1}(r_2)}}} \\ &= \sum_{p+q=n+1} \frac{1}{p!q!} \sum_{\sigma \in S_{n+1}} |a_\sigma| 2^{-m(\gamma_{v_{\sigma^{-1}(1)}} + \dots + \gamma_{v_{\sigma^{-1}(n+1)}})} \lesssim 2^{-m\hat{\gamma}\omega(v)}. \end{aligned}$$

This implies that at each stage we have  $\|Y_{t_k^m, t_{k+1}^m}\| \lesssim 2^{-m\hat{\gamma}}$ , which is stronger than (3.4) above, and the equivalence of norms implies the desired bound. The rest of the proof follows through.

Now let  $g, g'$  be two collections of functions as in the statement of the theorem. We have the identity

$$\langle [g'(gX)]_t, a \rangle = \langle (gX)_t, a \rangle + (g')_t^a = x_t^a + g_t^a + (g')_t^a = \langle [(g' + g)X]_t, a \rangle.$$

Since both  $g'(gX)$  and  $(g' + g)X$  are constructed iteratively by adding at each step a function  $Z$  satisfying (3.3) on the dyadics, if we let  $L^n$  and  $\bar{L}^n$  denote the logarithms corresponding to  $g'(gX)$  and  $(g' + g)X$ , Lemma 2.10 and the previous identity imply that

$$\text{BCH}_{n+1}(L_{su}^n, L_{ut}^n) = \text{BCH}_{n+1}(\bar{L}_{su}^n, \bar{L}_{ut}^n)$$

and so  $g'(gX) = (g' + g)X$ .  $\square$

**Corollary 5.8.** *Let  $(x^a : a \in A)$  be a collection of real-valued paths such that  $x^a \in C^{\gamma_a}$ , and let  $X$  be the anisotropic geometric  $\gamma$ -rough path on  $G_a(A)$  given by Theorem 5.7. If  $b \notin A$  is a new letter let  $A^0 = A \cup \{b\}$ . Given  $0 < \gamma_b < 1$  such that  $1 \notin \sum_{a \in A^0} \gamma_a \mathbb{N}$  and  $x^b \in C^{\gamma_b}$ , let  $X^0$  be the anisotropic geometric rough path on  $G_a(A^0)$  over  $(x^a : a \in A^0)$  given by Theorem 5.7. Then, the restriction of  $X^0$  to words from  $M(A)$  coincides with  $X$ .*

*Proof.* If  $L^n$  and  $\bar{L}^n$  denote the logarithms used in the construction of  $X$  and  $X^0$  respectively, and  $v \in M(A)$  is a word not containing  $b$ , the BCH formula gives

$$\begin{aligned} \langle \text{BCH}_{(n+1)}(\bar{L}_{su}^n, \bar{L}_{ut}^n), v \rangle &= \sum_{p+q=n+1} \frac{1}{p!q!} \sum_{\sigma \in S_{n+1}} a_\sigma \prod_{r_1=1}^p (x_u^{v_{\sigma(r_1)}} - x_s^{v_{\sigma(r_1)}}) \prod_{r_2=p+1}^{n+1} (x_t^{v_{\sigma(r_2)}} - x_u^{v_{\sigma(r_2)}}) \\ &= \langle \text{BCH}_{(n+1)}(L_{su}^n, L_{ut}^n), v \rangle. \end{aligned}$$

Therefore, at each step in the proof of Theorem 5.7 the paths  $X^n$  and  $(X^0)^n$  are such that for words  $v \in M(A)$  one has  $\langle (X^0)_{st}^n, v \rangle = \langle X_{st}^n, v \rangle$  for all  $s, t \in [0, 1]$ .  $\square$

## 6. BRANCHED ROUGH PATHS

In this section we develop further results specifically for branched rough paths as introduced in Section 4.2 by using our general results from Section 3. We analyse in detail the Hairer-Kelly map introduced in [25], which plays a very important role in our construction, and we use it to prove Theorem 1.2 and Corollary 1.3.

**6.1. The Hairer–Kelly map.** We denote by  $\mathfrak{B}$  the vector space spanned by  $\mathcal{T}$  and for each  $n \in \mathbb{N}$  we denote by  $\mathfrak{B}_n$  the finite-dimensional vector space spanned by the set  $\mathcal{T}_n$  of trees with at most  $n$  edges. Recall that given an alphabet  $A$  we denote by  $T^c(A)$  the shuffle Hopf algebra generated by  $A$ , defined in Section 4.1.

We recall the following result from [25, Lemma 4.9].

**Lemma 6.1.** *There exists a graded morphism of Hopf algebras  $\psi : \mathcal{H} \rightarrow T^c(\mathcal{T})$  satisfying  $\psi(\tau) = \tau + \psi_{n-1}(\tau)$  for all  $\tau \in \mathcal{T}_n$ , where  $\psi_{n-1}$  denotes the projection of  $\psi$  onto  $T^c(\mathcal{T}_{n-1})$ .*

We call  $\psi$  the Hairer-Kelly map. Observe that since  $\psi$  is a Hopf algebra morphism, in particular a coalgebra morphism, then

$$(\psi \otimes \psi)\Delta'\tau = \bar{\Delta}'\psi(\tau) = \bar{\Delta}'\psi_{n-1}(\tau), \quad \tau \in \mathfrak{B}_n,$$

since trees are primitive elements in  $T^c(\mathcal{T})$ . In fact, from the proof of [25, Lemma 4.9] we are able to see that in fact  $\psi_{n-1}$  is given by the recursion  $\psi_{n-1} = (\psi \otimes \text{id})\Delta'$  on  $\mathfrak{B}_n$ .

*Example 6.2.* Here are some examples of the action of  $\psi$  on some trees:

$$\begin{aligned}
\psi(\bullet i) &= \bullet i \\
\psi(\bullet a \bullet b) &= \psi(\bullet a) \sqcup \psi(\bullet b) = \bullet a \otimes \bullet b + \bullet b \otimes \bullet a \\
\psi(\uparrow_a^b) &= \uparrow_a^b + \bullet b \otimes \bullet a \\
\psi\left(\uparrow_a^d \left( \begin{array}{c} \bullet \\ \downarrow \\ \uparrow_a^c \bullet b \end{array} \right)\right) &= \uparrow_a^d \bullet b + \bullet b \otimes \uparrow_a^c + \bullet d \otimes \uparrow_a^c \bullet b + \uparrow_c^d \otimes \uparrow_a^b + \bullet d \otimes \bullet c \otimes \uparrow_a^b + \bullet d \otimes \bullet b \otimes \uparrow_a^c + \bullet b \otimes \bullet d \otimes \uparrow_a^c \\
&\quad + \uparrow_c^d \otimes \bullet b \otimes \bullet a + \bullet b \otimes \uparrow_c^d \otimes \bullet a + \bullet d \otimes \bullet c \otimes \bullet b \otimes \bullet a + \bullet d \otimes \bullet b \otimes \bullet c \otimes \bullet d \\
&\quad + \bullet b \otimes \bullet d \otimes \bullet c \otimes \bullet a.
\end{aligned}$$

In order to describe the image of  $\psi$  we introduce the following extended decorations on the nodes. Recall that a decorated forest is a pair  $\tau = (F, c)$  where  $F$  is a finite non-planar rooted forest and  $c : N_F \rightarrow \{1, \dots, d\}$ . We call a triple  $(F, c, \mathfrak{o})$  an *extended decorated forest* if  $(F, c)$  is a decorated forest and  $\mathfrak{o} : N_F \rightarrow \mathbb{N}$ ; we call  $\mathfrak{o}$  an *extended decoration* and we also use the notation  $\tau^\circ = (F, c, \mathfrak{o})$  where  $\tau := (F, c)$ .

**Definition 6.3.** *An extended decoration on a forest is said to be admissible if*

- (1)  $\mathfrak{o}(N_F)$  is an interval containing 1, we let  $m = m_{\mathfrak{o}} := \max \mathfrak{o}(N_F) \leq |\tau|$ ;
- (2) the function  $\mathfrak{o}$  is increasing, that is,  $\mathfrak{o}(x) \leq \mathfrak{o}(y)$  whenever  $x \leq y$ , and
- (3) for each  $1 \leq j \leq m$  the set  $O_j = \{x \in N_F : \mathfrak{o}(x) = j\}$  spans a subtree  $\tau_j^\circ \subset \tau^\circ$ .

It is clear that the extended decoration  $\mathfrak{o}$  is completely independent of the first decoration  $c$  and only depends on the forest  $F$  so we denote by  $\mathfrak{D}(F)$  the collection of all the admissible extended decorations on the forest  $\tau = (F, c)$ .

Note that condition (2) implies that  $\mathfrak{o}$  must be increasing in each factor since nodes from different trees are not comparable, and condition (3) implies that decorations must not appear twice in different factors since each set  $O_j$  must span a *subtree* and not a subforest. It is fairly clear that for each admissible extended decoration the trees  $\tau_j$  form a partition of  $\tau$  into  $m_{\mathfrak{o}}$  disjoint subtrees.

Let  $F$  be a forest. For  $A \subset F$  a subforest we denote the *boundary* of  $A$  in  $F$  by

$$\partial_F A = \{e \in E_F : s(e) \in N_A, t(e) \notin N_A\}.$$

**Lemma 6.4.** *Let  $F$  be a forest,  $\mathfrak{o} \in \mathfrak{D}(F)$  an admissible extended decoration on  $F$  and let  $T_j$  be the subtree spanned by  $O_j$  as in Definition 6.3. Then  $\partial_F T_j$  is an admissible cut.*

*Proof.* Denote by  $F_{>}$  the subforest of  $F$  spanned by all  $x \in N_F \setminus N_{T_j}$  such that  $y < x$  for some  $y \in T_j$ . If  $x \notin F_{>}$  then the unique path from the root to  $x$  does not contain edges in  $\partial_F T_j$ . If  $x \in F_{>}$ , suppose that the unique path from the root to  $x$  contains two or more edges from  $\partial_F T_j$ . Pick any two distinct such edges  $e_1, e_2 \in E_{T_j}$  and let  $y_1 = s(e_1), y_2 = s(e_2) \in N_{T_j}$ . Then, the paths going from the root to  $x$  through  $y_1$  and through  $y_2$  form a cycle in  $F$ , which is a contradiction.  $\square$

**Proposition 6.5.** *We have the following representation:*

$$\psi(\tau) = \sum_{\mathfrak{o} \in \mathfrak{D}(F)} \tau_m \otimes \cdots \otimes \tau_1 \quad (6.1)$$

*In particular, each term in this expansion satisfies  $|\tau_i| > 0$  and  $|\tau_1| + \cdots + |\tau_m| = |\tau|$ .*

As an example, observe that the following extended decorations are admissible:

$$\bullet(a, 1)\bullet(b, 2), \quad \begin{array}{c} \bullet(d, 2) \\ | \\ \bullet(c, 1) \bullet(b, 3) \\ \diagdown \quad / \\ \bullet(a, 1) \end{array}, \quad \begin{array}{c} \bullet(d, 3) \\ | \\ \bullet(c, 1) \bullet(b, 2) \\ \diagdown \quad / \\ \bullet(a, 1) \end{array},$$

whereas the following are not:

$$\bullet(a, 1)\bullet(b, 1), \quad \begin{array}{c} \bullet(d, 5) \\ | \\ \bullet(c, 1) \bullet(b, 3) \\ \diagdown \quad / \\ \bullet(a, 1) \end{array}, \quad \begin{array}{c} \bullet(d, 2) \\ | \\ \bullet(c, 1) \bullet(b, 2) \\ \diagdown \quad / \\ \bullet(a, 1) \end{array}, \quad \begin{array}{c} \bullet(d, 2) \\ | \\ \bullet(c, 3) \bullet(b, 1) \\ \diagdown \quad / \\ \bullet(a, 1) \end{array}.$$

Observe that the two terms in the first example give exactly the terms

$$\bullet b \otimes \bullet d \otimes \mathfrak{!}_a^c, \quad \text{and} \quad \bullet d \otimes \bullet b \otimes \mathfrak{!}_a^c$$

in Example 6.2.

*Proof of Proposition 6.5.* The proof is by induction on the number of edges of  $\tau$ , the base case being trivial (see Example 6.2).

Suppose identity (6.1) is true for all forests with at most  $k$  edges, and let  $\tau$  be a tree with  $k + 1$  edges. Let  $\mathfrak{A}^*(\tau) = \mathfrak{A}(\tau) \cup \{\emptyset\}$  where  $\mathfrak{A}$  is the set of admissible cuts of  $\tau$ , and set  $R^\emptyset(\tau) = \tau$ ,  $P^\emptyset(\tau) = \mathbf{1}$ . By definition

$$\psi(\tau) = \sum_{C \in \mathfrak{A}^*(\tau)} \psi(P^C(\tau)) \otimes R^C(\tau)$$

and by the induction hypothesis

$$\psi(\tau) = \sum_{C \in \mathfrak{A}^*(\tau)} \sum_{\mathfrak{o} \in \mathfrak{D}(P^C(\tau))} \tau_m \otimes \cdots \otimes \tau_1 \otimes R^C(\tau)$$

Given  $\sigma \in \mathfrak{A}^*(\tau)$  and  $\mathfrak{o} \in \mathfrak{D}(P^C(\tau))$  there is a unique extended decoration  $\tilde{\mathfrak{o}} \in \mathfrak{D}(\tau)$  such that  $\tau_1 = R^C(\tau)$ . This extended decoration is defined by  $\tilde{\mathfrak{o}}(x) = 1$  if  $x \in N_{R^C(\tau)}$  and  $\tilde{\mathfrak{o}}(x) = \mathfrak{o}(x) + 1$  if  $x \in N_\tau \setminus N_{R^C(\tau)}$ . Conversely, given  $\tilde{\mathfrak{o}} \in \mathfrak{D}(\tau)$  we have that  $\partial\tau_1 \in \mathfrak{A}(\tau)$  and the extended decoration such that  $\mathfrak{o}(x) = \tilde{\mathfrak{o}}(x) - 1$  for  $x \in N_\tau \setminus N_{\tau_1}$  belongs to  $\mathfrak{D}(P^C(\tau))$ . Therefore we have the identity

$$\psi(\tau) = \sum_{\tilde{\mathfrak{o}} \in \mathfrak{D}(\tau)} \tau_1 \otimes \cdots \otimes \tau_{m_{\tilde{\mathfrak{o}}}}.$$

This concludes the proof. □

**6.2. Branched rough paths are anisotropic geometric rough paths.** The next theorem is almost the same statement as Theorem 4.10 in [25], the only difference being that we construct an *anisotropic* geometric rough path  $\bar{X}$  while Hairer-Kelly need only that  $\bar{X}$  is geometric in the usual sense (see also [25, Remark 4.14]).

**Theorem 6.6.** *Let  $\gamma \in ]0, 1[$  with  $\gamma^{-1} \notin \mathbb{N}$ , and let  $X$  be a branched  $\gamma$ -rough path. There exists an anisotropic geometric rough path  $\bar{X} : [0, 1]^2 \rightarrow G_a(\mathcal{T}_N)$  with exponents  $\gamma = (\gamma_\tau = \gamma|\tau|, \tau \in \mathcal{T}_N)$ , and such that*

$$\langle X, \tau \rangle = \langle \bar{X}, \psi(\tau) \rangle, \quad \forall \tau \in \mathcal{F}.$$

*Proof.* We construct  $\bar{X}$  iteratively as follows. Let  $\bar{X}^{(1)}$  be the anisotropic geometric rough path indexed by  $\mathcal{T}_1 = \{\bullet, \dots, \bullet\}$  over the paths  $(x_t^i := \langle X_t, \bullet \rangle : i = 1, \dots, d)$  with exponents  $(\gamma_{\bullet i} = \gamma)$  given by Theorem 5.7 (alternatively we could have used Theorem 3.5 since all the exponents are equal). This will give us an anisotropic rough path  $X : [0, 1]^2 \rightarrow G_a(\mathcal{T}_1)$  with exponents  $(\gamma_\tau = \gamma, \tau \in \mathcal{T}_1)$ .

Suppose we have constructed anisotropic geometric rough paths  $\bar{X}^{(k)} : [0, 1]^2 \rightarrow G_a(\mathcal{T}_k)$  over the paths  $(x^\tau : \tau \in \mathcal{T}_k)$  such that  $x_t^\tau - x_s^\tau = \langle X_{st}, \tau \rangle - \langle \bar{X}_{st}^{(k-1)}, \psi_{k-1}(\tau) \rangle$  for  $k = 1, \dots, n$ . This is true for  $n = 1$  by the previous paragraph, since  $\psi(\bullet i) = \bullet i$  for all  $i = 1, \dots, d$ .

If we let  $F_{st}^\tau = \langle X_{st}, \tau \rangle$  and  $G_{st}^\tau = \langle \bar{X}_{st}^{(n)}, \psi_n(\tau) \rangle$  for  $\tau \in \mathcal{T}_{n+1}$  we have, by Chen's rule, that

$$\delta F_{sut}^\tau = \langle X_{su} \otimes X_{ut}, \Delta' \tau \rangle = \langle \bar{X}_{su}^{(n)} \circ \psi \otimes \bar{X}_{ut}^{(n)} \circ \psi, \Delta' \tau \rangle.$$

Since  $\psi$  is in particular a coalgebra morphism between  $(\mathcal{H}, \Delta)$  and  $(T^c(\mathcal{T}), \bar{\Delta})$  we obtain the identity  $\delta F_{sut}^\tau = \langle \bar{X}_{su}^{(n)} \otimes \bar{X}_{ut}^{(n)}, \bar{\Delta}' \psi(\tau) \rangle$ , which then, by Lemma 6.1 becomes

$$\delta F_{sut}^\tau = \langle \bar{X}_{su}^{(n)} \otimes \bar{X}_{ut}^{(n)}, \bar{\Delta}' \psi_n(\tau) \rangle = \delta G_{sut}^\tau. \quad (6.2)$$

since every  $\tau \in \mathcal{T}$  is primitive in  $(T^c(\mathcal{T}), \bar{\Delta})$  being a single-letter word.

The finite increment operator  $\delta$  has the following property: if  $J : [0, 1]^2 \rightarrow \mathbb{R}$  is such that  $\delta J = 0$  then there exists  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $J_{st} = f_t - f_s$ , and the function  $f$  is unique up to an additive constant shift, see also [22, formula (5)]. Thus, by this fundamental property, for each  $\tau \in \mathcal{T}_{(n+1)}$  there exists a function  $x^\tau : [0, 1] \rightarrow \mathbb{R}$  such that  $x_t^\tau - x_s^\tau = F_{st}^\tau - G_{st}^\tau$  and then

$$|x_t^\tau - x_s^\tau| \leq |\langle X_{st}, \tau \rangle| + |\langle \bar{X}_{st}^{(n)}, \psi_n(\tau) \rangle| \lesssim |t - s|^{\gamma|\tau|}$$

since  $\psi_n(\tau)$  preserves the number of nodes by Proposition 6.5.

Repeatedly using Corollary 5.8 we obtain an anisotropic geometric rough path  $\bar{X}^{(n+1)} : [0, 1] \rightarrow G_a(\mathcal{T}_{n+1})$  over  $(x^\tau : \tau \in \mathcal{T}_{n+1})$  whose restriction to  $T^c(\mathcal{T}_n)$  coincides with  $\bar{X}^{(n)}$ .

Finally notice that if  $\tau \in \mathcal{T}_{n+1}$  is a tree then

$$\begin{aligned} \langle \bar{X}_{st}^{(n+1)}, \psi(\tau) \rangle &= \langle \bar{X}_{st}^{(|\tau|)}, \tau \rangle + \langle \bar{X}_{st}^{(|\tau|)}, \psi_{|\tau|-1}(\tau) \rangle \\ &= x_t^\tau - x_s^\tau + \langle X_{st}, \tau \rangle - (x_t^\tau - x_s^\tau) = \langle X_{st}, \tau \rangle \end{aligned}$$

and the corresponding identity for arbitrary forests follows by multiplicativity. The anisotropic geometric rough path sought for is  $\bar{X} = \bar{X}^{(N)}$ .  $\square$

We note that our proof is shorter and simpler than that of [25, Theorem 4.10], so we will now dedicate a few paragraphs to highlight the differences between our approach and that of Hairer and Kelly. They define first

$$\hat{\mathbb{X}}_t^1 = \exp_N \left( \sum_{a \in A} x_t^a \bullet a \right) \in G^N(\mathcal{T}_1)$$

then they note that this is not  $\gamma$ -Hölder with values in  $G^N(\mathcal{T}_1)$ , but it is  $\gamma$ -Hölder with values in  $G^N(\mathcal{T}_1)/K_1$ , where  $K_1 := \exp_N(W_2 + \dots + W_N)$ , see (2.3). By the Lyons-Victoir extension theorem there exists a  $\gamma$ -Hölder path  $\bar{\mathbb{X}}_t^1 \rightarrow G^N(\mathcal{T}_1)$  such that  $\pi_{G^N(\mathcal{T}_1) \rightarrow G^N(\mathcal{T}_1)/K_1}(\bar{\mathbb{X}}^1) = \hat{\mathbb{X}}^1$ . Then, in order to add a new tree  $\tau$  with  $|\tau| = 2$ , they define

$$(\delta \bar{X}^\tau)_{st} = \langle X_{st}, \tau \rangle - \langle \bar{X}_{st}^{(1)}, \psi_1(\tau) \rangle$$

and this defines the new function  $t \mapsto \langle \hat{\mathbb{X}}_t, \tau \rangle$ . Then they define

$$\hat{\mathbb{X}}_t^2 = \exp_N \left( \sum_{a \in A} x_t^a \bullet a + \sum_{|\tau|=2} \langle \hat{\mathbb{X}}_t, \tau \rangle \tau \right) \in G^N(\mathcal{T}_2)$$

and again they note that this path is not  $\gamma$ -Hölder with values in  $G^N(\mathcal{T}_2)$ , but it is with values in  $G^N(\mathcal{T}_2)/K_2$ , where  $K_2 := \exp_N(W_3 + \dots + W_N)$ , and again the Lyons-Victoir extension theorem yields a  $\gamma$ -Hölder path  $\bar{\mathbb{X}}_t^2 \rightarrow G^N(\mathcal{T}_2)$  such that  $\pi_{G^N(\mathcal{T}_2) \rightarrow G^N(\mathcal{T}_2)/K_2}(\bar{\mathbb{X}}^2) = \hat{\mathbb{X}}^2$ . Finally they construct recursively in this way  $\hat{\mathbb{X}}^k$  and  $\bar{\mathbb{X}}^k$  for all  $k \leq N$ .

At this point we see the difference with our approach. We do not define  $\hat{\mathbb{X}}_t^2$  nor  $\bar{\mathbb{X}}^k$  but rather we construct  $\bar{\mathbb{X}}$  step by step, namely on all  $G^k(\mathcal{T}_n)$  with  $1 \leq k, n \leq N$ , first by recursion on  $k$  for fixed  $n$  and then by recursion on  $n$ ; at each step we enforce the Hölder continuity on  $G^k(\mathcal{T}_n)$  and the compatibility with the previous levels. This is done using the Lyons-Victoir technique, but in a very explicit and constructive way, in particular without ever using the axiom of choice, since we have the explicit map  $\exp_{k+1} \circ \log_k : G^k(\mathcal{T}_n) \rightarrow G^{k+1}(\mathcal{T}_n)$  which plays the role of the injection  $i_{G/K, G} : G/K \rightarrow G$  in [29, Proposition 6].

## 7. AN ACTION ON BRANCHED ROUGH PATHS

In this section we prove Theorem 1.2.

Given  $\gamma \in ]0, 1[$ , let  $N = \lfloor \gamma^{-1} \rfloor$  and denote by  $\mathcal{C}^\gamma$  the set of collections of functions  $(g^\tau)_{\tau \in \mathcal{T}_N}$  such that  $g^\tau \in C^{\gamma|\tau|}$  and  $g_0^\tau = 0$  for all  $\tau \in \mathcal{T}_N$ . It is easy to see that  $\mathcal{C}^\gamma$  is a group under pointwise addition in  $t$ , that is,

$$(g + h)^\tau := g^\tau + h^\tau.$$

As a consequence of Theorem 5.7,  $(g, \bar{X}) \mapsto g\bar{X}$  is an action of  $\mathcal{C}^\gamma$  on the space of anisotropic geometric rough paths.

We use the Hairer-Kelly map  $\psi$  of Lemma 6.1 to induce an action of  $\mathcal{C}^\gamma$  on branched rough paths. Given a branched rough path  $X$  and  $g \in \mathcal{C}^\gamma$  we let  $gX$  be the branched rough path defined by

$$\langle gX_{st}, \tau \rangle = \langle g\bar{X}_{st}, \psi(\tau) \rangle,$$

where  $\bar{X}$  is the anisotropic geometric rough path given by Theorem 6.6. As a simple consequence of Theorem 5.7 we obtain

**Proposition 7.1.** *Let  $X \in \mathbf{BRP}^\gamma$ .*

- (1) *We have  $g'(gX) = (g' + g)X$  for all  $g, g' \in \mathcal{C}^\gamma$ .*
- (2) *If  $(g^\tau)_{\tau \in \mathcal{T}_N} \in \mathcal{C}^\gamma$  is such that there exists a unique  $\tau \in \mathcal{T}_N$  with  $g^\tau \neq 0$ , then*

$$\langle (gX)_{st}, \tau \rangle = \langle X_{st}, \tau \rangle + g_t^\tau - g_s^\tau$$

*and  $\langle gX, \sigma \rangle = \langle X, \sigma \rangle$  for all  $\sigma \in \mathcal{T}$  not containing  $\tau$  as a subtree.*

*Proof.* The first claim follows from point (1) in Theorem 5.7. In order to prove the second claim, let  $g = (g^\tau)_{\tau \in \mathcal{T}_N} \in \mathcal{C}^\gamma$  be such that there exists a unique  $\tau \in \mathcal{T}_N$  with  $g^\tau \neq 0$ . Then by the property of  $g$  we have

$$\begin{aligned} \langle gX, \tau \rangle &= \langle g\bar{X}, \psi(\tau) \rangle \\ &= \langle g\bar{X}, \tau + \psi_{|\tau|-1}(\tau) \rangle \\ &= \langle \bar{X}, \tau \rangle + \delta g^\tau + \langle g\bar{X}, \psi_{|\tau|-1}(\tau) \rangle \end{aligned}$$

where  $\delta g_{st}^\tau := g_t^\tau - g_s^\tau$ . By Proposition 6.5 the tree  $\tau$  does not appear as a factor in any of the tensor products appearing in  $\psi_{|\tau|-1}(\tau)$ , hence one can recursively show that  $\langle g\bar{X}, \psi_{|\tau|-1}(\tau) \rangle = \langle \bar{X}, \psi_{|\tau|-1}(\tau) \rangle$  so that the above expression becomes

$$\begin{aligned} \langle gX, \tau \rangle &= \langle \bar{X}, \tau + \psi_{|\tau|-1}(\tau) \rangle + \delta g^\tau \\ &= \langle X, \tau \rangle + \delta g^\tau. \end{aligned}$$

For the last assertion, it is enough to note that  $\sigma \in \mathcal{T}$  contains  $\tau \in \mathcal{T}$  if and only if  $\tau$  appears in the expression for  $\psi(\tau)$ ; this can be expressed more precisely by saying that  $\sigma \notin T^c(\mathcal{T}_N \setminus \{\tau\})$ . But if  $\sigma \in T^c(\mathcal{T}_N \setminus \{\tau\})$ , then  $\langle g\bar{X}, \psi(\tau) \rangle = \langle \bar{X}, \psi(\tau) \rangle$ .  $\square$

**Proposition 7.2.** *The action of  $\mathcal{C}^\gamma$  on branched  $\gamma$ -rough paths is transitive: for every pair of branched  $\gamma$ -rough paths  $X$  and  $X'$  there exists  $g \in \mathcal{C}^\gamma$  such that  $gX = X'$ .*

*Proof.* We define  $g \in \mathcal{C}^\gamma$  inductively by imposing the desired identity. For trees  $\tau \in \mathcal{T}_1 = \{\bullet, \dots, \bullet\}$  we set  $g_t^\tau = \langle X'_{0t}, \tau \rangle - \langle X_{0t}, \tau \rangle \in \mathcal{C}^\gamma$  so that

$$\langle gX, \tau \rangle = \langle g\bar{X}, \psi(\tau) \rangle = \langle g\bar{X}, \tau \rangle = \langle \bar{X}, \tau \rangle + \delta g^\tau = \langle X', \tau \rangle$$

where  $\delta g_{st}^\tau := g_t^\tau - g_s^\tau$ . Suppose we have already defined  $g^\tau$  for all  $\tau \in \mathcal{T}_n$  for some  $n \geq 1$ , satisfying the constraints in the definition of  $\mathcal{C}^\gamma$ . For a tree  $\tau$  with  $|\tau| = n + 1$  we define

$$F_{st}^\tau = \langle X'_{st}, \tau \rangle - \langle \bar{X}_{st}, \tau \rangle - \langle g\bar{X}_{st}, \psi_n(\tau) \rangle.$$

Then

$$\begin{aligned}
\delta F_{sut}^\tau &= \langle X'_{su} \otimes X'_{ut}, \Delta' \tau \rangle - \langle g\bar{X}_{su} \otimes g\bar{X}_{ut}, \bar{\Delta}' \psi_n(\tau) \rangle \\
&= \langle X'_{su} \otimes X'_{ut}, \Delta' \tau \rangle - \langle g\bar{X}_{su} \otimes g\bar{X}_{ut}, \bar{\Delta}' \psi(\tau) \rangle \\
&= \langle X'_{su} \otimes X'_{ut}, \Delta' \tau \rangle - \langle g\bar{X}_{su} \circ \psi \otimes g\bar{X}_{ut} \circ \psi, \bar{\Delta}' \tau \rangle \\
&= \langle X'_{su} \otimes X'_{ut}, \Delta' \tau \rangle - \langle gX_{su} \otimes gX_{ut}, \bar{\Delta}' \tau \rangle = 0
\end{aligned}$$

by the induction hypothesis. Hence there is  $g^\tau : [0, 1] \rightarrow \mathbb{R}$  such that  $g_0^\tau = 0$  and

$$g_t^\tau - g_s^\tau = \langle X'_{st}, \tau \rangle - \langle \bar{X}_{st}, \tau \rangle - \langle g\bar{X}_{st}, \psi_n(\tau) \rangle \quad (7.1)$$

whence  $g \in C^{|\gamma|}$ ; by construction

$$\begin{aligned}
\langle gX, \tau \rangle &= \langle g\bar{X}, \psi(\tau) \rangle = \langle g\bar{X}, \tau \rangle + \langle g\bar{X}, \psi_n(\tau) \rangle \\
&= \langle \bar{X}, \tau \rangle + \delta g^\tau + \langle g\bar{X}, \psi_n(\tau) \rangle = \langle X', \tau \rangle,
\end{aligned}$$

where  $\delta g_{st}^\tau = g_t^\tau - g_s^\tau$ . This concludes the proof.  $\square$

**Proposition 7.3.** *The action of  $\mathcal{C}^\gamma$  on branched  $\gamma$ -rough paths is free, namely if  $gX = g'X$  then  $g = g'$ .*

*Proof.* This follows from the fact that by (7.1) the function  $g^\tau$  is defined up to a constant shift. Therefore, the condition  $g_0^\tau = 0$  determines  $g^\tau$  uniquely.  $\square$

Together, Proposition 7.1, Proposition 7.2 and Proposition 7.3 imply Theorem 1.2.

**7.1. The BCFP renormalisation.** In [3] a different kind of modification is proposed. There, a new decoration 0 is considered so rough paths –branched and geometric– are over paths taking values in  $\mathbb{R}^{d+1}$ . Recall that since branched rough paths are seen as Hölder paths taking values in the character group of the Butcher-Connes-Kreimer Hopf algebra, we may think of them as an infinite forest series of the form

$$X_{st} = \sum_{\tau \in \mathcal{F}} \langle X_{st}, \tau \rangle \tau \quad (7.2)$$

where we regard  $\tau$  as a linear functional on  $\mathcal{H}$ , such that  $\langle \tau, \sigma \rangle = 1$  if  $\sigma = \tau$  and zero else. The aforementioned modification procedure then acts as a translation of the series (7.2). Specifically, for each collection  $v = (v_0, \dots, v_d) \in (\mathcal{B}^*)^{d+1}$  an operator  $M_v : \mathcal{H}^* \rightarrow \mathcal{H}^*$  is defined, such that for a  $\gamma$ -branched rough path,  $(M_v X)_{st} := M_v(X_{st})$  is a  $\gamma/N$ -branched rough path.

In the particular case where  $v_j = 0$  except for  $v_0$ , the action of this operator can be described in terms of an extraction/contraction map<sup>2</sup>  $\Psi : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ . This map acts on a tree  $\tau$  by extracting subforests and placing them in the left factor; the right factor

<sup>2</sup>In [3] this map is named  $\delta$  but we choose to call it  $\Psi$  in order to avoid confusion with the operator defined here.



is obtained by contracting the extracted forest and decorating the resulting node with 0. As an example, consider

$$\begin{aligned} \Psi(\mathbb{V}_i^{j,k}) &= \mathbf{1} \otimes \mathbb{V}_i^{j,k} + \bullet_i \otimes \mathbb{V}_0^{j,k} + \bullet_j \otimes \mathbb{V}_i^{0,k} + \bullet_k \otimes \mathbb{V}_i^{j,0} + \mathbb{I}_i^j \otimes \mathbb{I}_0^k + \mathbb{I}_i^k \otimes \mathbb{I}_0^j \\ &\quad + \bullet_{i,j} \otimes \mathbb{V}_0^{0,k} + \bullet_{i,k} \otimes \mathbb{V}_0^{j,0} + \bullet_{j,k} \otimes \mathbb{V}_i^{0,0} + \mathbb{I}_i^j \bullet_k \otimes \mathbb{I}_0^0 + \mathbb{I}_i^k \bullet_j \otimes \mathbb{I}_0^0 + \mathbb{V}_i^{j,k} \otimes \bullet_0. \end{aligned}$$

Extending  $v = v_0 \in \mathcal{B}^*$  to all of  $\mathcal{H}^*$  as an algebra morphism it is shown that

$$\langle (M_v X)_{st}, \tau \rangle = \langle X_{st}, (v \otimes \text{id})\Psi(\tau) \rangle. \quad (7.3)$$

Furthermore, in this case  $M_v X$  is a  $\gamma$ -branched rough path if coefficients corresponding to trees with decoration zero are required to satisfy the stronger analytical condition

$$\sup_{0 \leq s, t \leq 1} \frac{|\langle X_{st}, \tau \rangle|}{|t - s|^{(1-\gamma)|\tau|_0 + \gamma|\tau|}} < \infty, \quad (7.4)$$

where  $|\tau|_0$  counts the times the decoration 0 appears in  $\tau$ . Essentially, this condition imposes that the components corresponding to the zero decoration be Lipschitz on the diagonal  $s = t$ .

We now show how this setting can be recovered from the results of Section 7. Let  $X$  be a  $\gamma$ -branched rough path on  $\mathbb{R}^{d+1}$  satisfying (7.4). Since  $M_v X$  is again a  $\gamma$ -branched rough path, by Proposition 7.2 there exists a collection of functions  $g \in \mathcal{C}^\gamma$  such that  $gX = M_v X$ . Moreover, this collection is the unique one satisfying

$$g_t^\tau - g_s^\tau = \langle X_{st}, (v \otimes \text{id})\Psi(\tau) \rangle - \langle X_{st}, \tau \rangle - \langle g \bar{X}_{st}, \psi_{|\tau|-1}(\tau) \rangle \quad (7.5)$$

for all  $\tau \in \mathcal{T}(\mathbb{R}^{d+1})$  where we have used (7.3) in order to express  $M_v X$  in terms of  $\Psi$ . Theorem 28 in [3] ensures that the first term on the right-hand side is in  $C_2^{\gamma|\tau|}$  hence  $g$  is actually in  $C^{\gamma|\tau|}$  as required.

The approach of [3] is based on pre-Lie morphisms and crucially on a *cointeraction property*, which has been explored by [6], see in particular [3, Lemma 18]. The cointeraction property can be used for *time-independent* modifications, indeed note that the functional  $v$  in [3] always constant.

Let us see why this is the case. The approach of [3] is based on a cointeraction property studied by [5, 6, 18] between the Butcher-Connes-Kreimer coproduct and another *extraction-contraction* coproduct  $\delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ . The formula is the following

$$(\text{id} \otimes \Delta)\delta = \mathcal{M}_{1,3}(\delta \otimes \delta)\Delta.$$

Let us consider now a character  $v \in \mathcal{H}^*$ . If we multiply both sides by  $(v \otimes \text{id} \otimes \text{id})$  and set  $M_v^* = (v \otimes \text{id})\delta : \mathcal{H} \rightarrow \mathcal{H}$  as in [3, Proposition 17], then we obtain

$$\Delta M_v^* = (M_v^* \otimes M_v^*)\Delta,$$

namely  $M_v^*$  is a coalgebra morphism on  $\mathcal{H}$ . Then one can define a modified rough path as  $vX := M_v X = X \circ M_v^*$ . The crucial Chen property is still satisfied since

$$\begin{aligned} (vX)_{st} &= (v \otimes X_{st})\delta = (v \otimes X_{su} \otimes X_{ut})(\text{id} \otimes \Delta)\delta \\ &= (v \otimes X_{su} \otimes X_{ut})\mathcal{M}_{1,3}(\delta \otimes \delta)\Delta \\ &= ((v \otimes X_{su}) \otimes (v \otimes X_{ut}))(\delta \otimes \delta)\Delta \\ &= ((vX)_{su} \otimes (vX)_{ut})\Delta \end{aligned}$$

However this does not work if  $v : [0, 1]^2 \rightarrow \mathcal{H}^*$  is a time-dependent character. Indeed in this case we set  $(vX)_{st} := (v_{st} \otimes \text{id})\delta$  and we obtain

$$\begin{aligned} (vX)_{st} &= (v_{st} \otimes X_{st})\delta = (v_{st} \otimes X_{su} \otimes X_{ut})(\text{id} \otimes \Delta)\delta \\ &= (v_{st} \otimes X_{su} \otimes X_{ut})\mathcal{M}_{1,3}(\delta \otimes \delta)\Delta \\ &= ((v_{st} \otimes X_{su}) \otimes (v_{st} \otimes X_{ut}))(\delta \otimes \delta)\Delta \end{aligned}$$

but we can not conclude that this is equal to  $((vX)_{su} \otimes (vX)_{ut})\Delta$ . Our construction, as explained after formula (1.8), is not purely algebraic but is based on a (non-canonical) choice of generalised Young integrals with respect to the rough path  $X$ . Moreover our transformation group, infinite-dimensional, is much larger than that finite-dimensional group studied in [3].

## 8. PERSPECTIVES

In this paper we have shown that the space of branched  $\gamma$ -rough paths has is a principal homogeneous space with respect to the linear group  $\mathcal{C}^\gamma$ . This is related to the analytical properties of the operator  $\delta$  defined in (1.2), which is invertible under the conditions of Gubinelli's Sewing Lemma, but not in general, and in particular not in the context of the Chen relation on trees with low degree.

It would be now interesting to see how this action can be translated on the level of controlled paths [21]. The space of paths controlled by a rough path  $X \in \mathbf{BRP}^\gamma$  should be interpreted as the tangent space to  $\mathbf{BRP}^\gamma$  at  $X$ , and the action on rough paths should induce an action on controlled paths. In particular it should be possible to write an action on solutions to rough differential equations.

The proof of Proposition 7.2, and in particular (7.1), gives a recursive way of computing the unique  $g \in \mathcal{C}^\gamma$  translating a given branched  $\gamma$ -rough path into another. An interesting feature of the BCFP scheme is that is given in terms of a coaction so explicit calculations are somewhat easier in this more restricted case as one can compute  $g^\tau$  for each tree  $\tau \in \mathcal{T}_N$  directly by extracting and contracting subforests of  $\tau$  without doing any recursions (see (7.5).) However, we do not have a computational rule for an important case: suppose that  $\mathbb{X}$  is branched rough path lift of a stochastic process with a.s.  $C^{\gamma-}$  trajectories; it would be nice to have a way of finding  $g \in \mathcal{C}^\gamma$  such that  $g\mathbb{X}$  is centred with respect to the underlying distribution of the process, provided this is possible. Even this last problem, namely giving precise conditions under which this centring is possible is interesting in itself. This should be related to the notion of Wick polynomials and deformations of products as considered in [16].

More generally, in the physics literature there are various renormalisation procedures which allow to obtain convergent iterated integrals from divergent ones by subtracting suitable ‘‘counterterms’’. In the context of rough paths, implementing one of the most accepted such procedures due to Bogoliubov–Parasiuk–Hepp–Zimmerman (BPHZ) has been carried out by J. Unterberger in [34, 35] by means of the Fourier normal ordering algorithm and using a technique relating the trees in the Butcher–Connes–Kreimer Hopf algebra to certain Feynman diagrams. In our context, this could provide a canonical

choice for  $g \in \mathcal{C}^\gamma$  implementing the BPHZ renormalization procedure in a way analogous to what is done in [5] for Regularity Structures.

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