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# Full Waveform Inversion Adjoint Studies MATHIAS 2018

Pierre Jacquet, Hélène Barucq, Julien Diaz, Henri Calandra

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# Full Waveform Inversion Adjoint Studies

MATHIAS 2018

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Pau, FRANCE



## The acoustic model

- The continuous model

- The discretized problem

- Assets of Bernstein polynomials

## Adjoint Studies

- FWI Introduction

- Adjoint then Discretized

- Discretize then Adjoint

## Some Results

- Consistency of the Adjoint Solution

- FWI Preliminary test

- Qualitative Cost Function Gradient Study

## The acoustic model

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Continuous Problem:

$$\begin{cases} \frac{1}{\rho_0 \mathbf{v}_p^2} \frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} = f_p & \text{in } \Omega \\ \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = f_v & \text{in } \Omega \end{cases}$$

$$\begin{cases} p(t=0) = 0 \\ v(t=0) = 0 \\ \frac{\partial p}{\partial t} + \mathbf{v}_p \frac{\partial p}{\partial x} \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

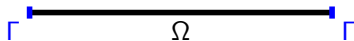


Figure: 1D Domain Model

Discretized Problem:

$$\begin{cases} \frac{\partial \bar{\mathbf{P}}}{\partial t} = A_{pv} \bar{\mathbf{V}} + A_{pp} \bar{\mathbf{P}} + \bar{\mathbf{F}}_p \\ \frac{\partial \bar{\mathbf{V}}}{\partial t} = A_{vp} \bar{\mathbf{P}} \end{cases}$$

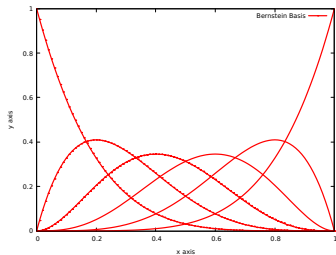


Figure: 1D Discretized Domain

- ▶ **Discontinuous Galerkin** space discretization
- ▶ Different **time-schemes** (RK4, AB3)
- ▶ Two polynomial basis (**Lagrange** and **Bernstein**)
- ▶ Constant velocity ( $\mathbf{v}_p$ ) per cells
- ▶ Constant density ( $\rho_0$ ) per cells

Bernstein formulation:

$$B_{ijkl}^N = C_{ijkl}^N \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^l \quad \text{with: } C_{ijkl}^N = \frac{N!}{i!j!k!l!}$$



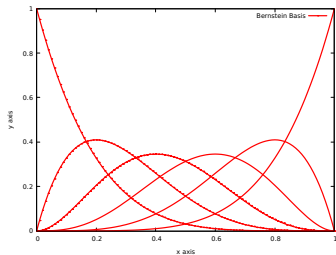
$P[X^5]$  Bernstein basis

Bernstein formulation:

$$B_{ijkl}^N = C_{ijkl}^N \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^l \quad \text{with: } C_{ijkl}^N = \frac{N!}{i!j!k!l!}$$

Easy Derivative expression:

$$\frac{\partial B_{\alpha}^N}{\partial \lambda_p} = N B_{\alpha - e_p}^{N-1} \quad \text{with: } \alpha = (i, j, k, l)$$



$P[X^5]$  Bernstein basis



Bernstein formulation:

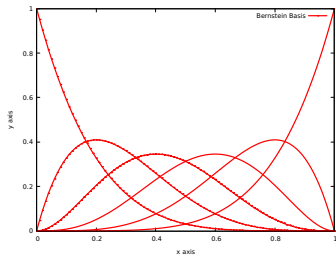
$$B_{ijkl}^N = C_{ijkl}^N \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^l \quad \text{with: } C_{ijkl}^N = \frac{N!}{i!j!k!l!}$$

Easy Derivative expression:

$$\frac{\partial B_{\alpha}^N}{\partial \lambda_p} = N B_{\alpha - e_p}^{N-1} \quad \text{with: } \alpha = (i, j, k, l)$$

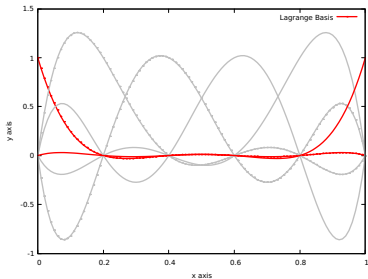
Sparse Degree Elevation operator:

$$B_{\alpha}^{N-1} = \sum_{p=0}^d \frac{\alpha_p + 1}{N} B_{\alpha + e_p}^N$$

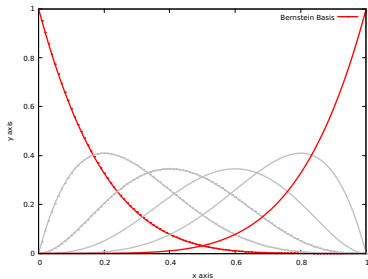


$P[X^5]$  Bernstein basis

Unique boundary condition values :



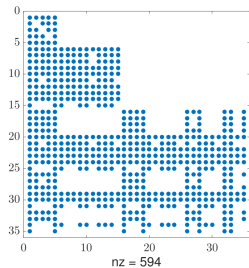
$P(X^5)$  Lagrange basis



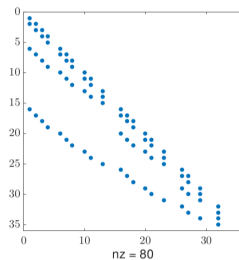
$P(X^5)$  Bernstein basis

⇒ Same Flux Management

# Derivative-Operator Analysis



3D Lagrange D matrix



3D Bernstein D matrix

- [1] Chan J. and Warburton T.  
GPU-Accelerated Bernstein Bézier Discontinuous Galerkin Methods for Wave Problems  
*SIAM Journal on Scientific Computing* 2017

# 1D Results

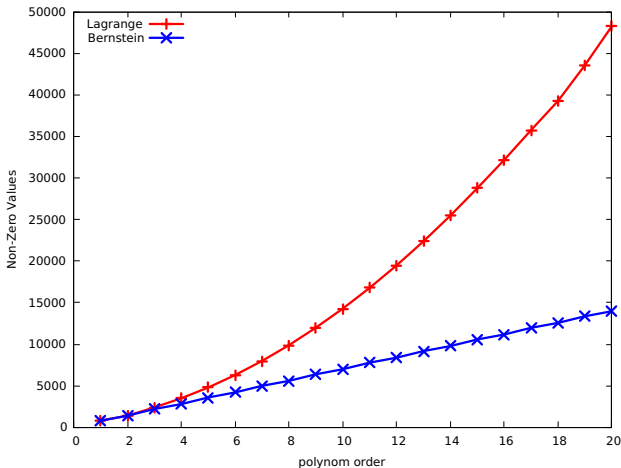


Figure: Operators NZVs as a function of the order

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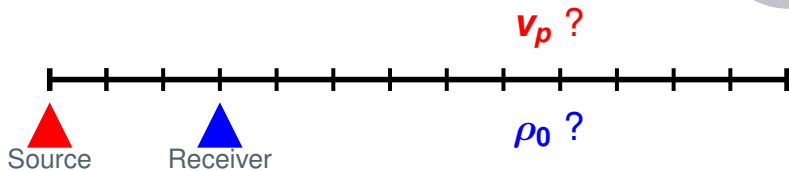
## Some Results

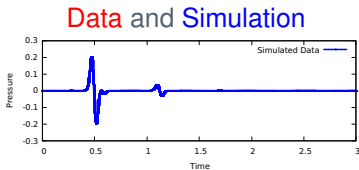
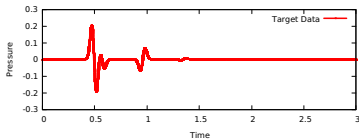
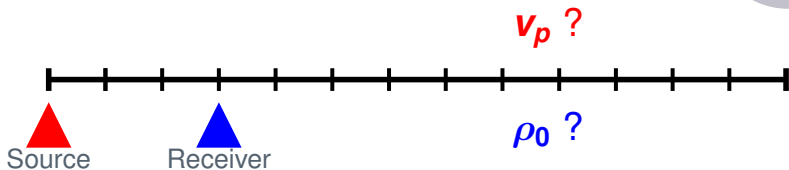
- Consistency of the Adjoint Solution

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# FWI Introduction





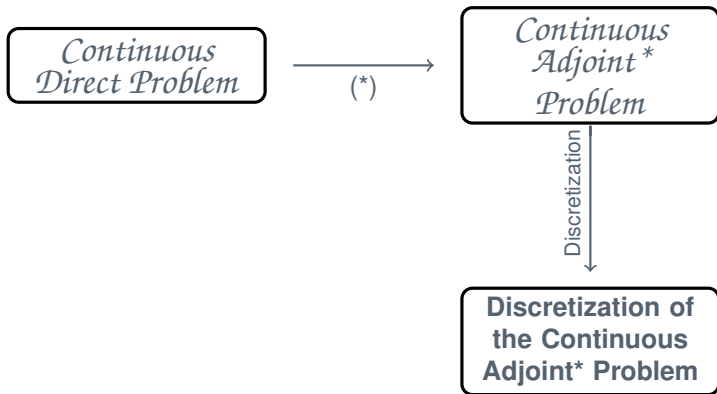
$$J(p) = \frac{1}{2} \|Rp - \text{data}\|^2$$

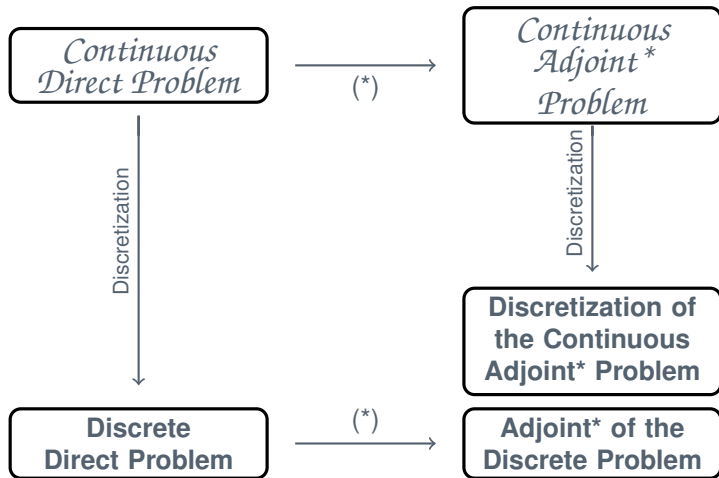
- Find  $v_p$  and  $\rho_0$  to minimize  $J$

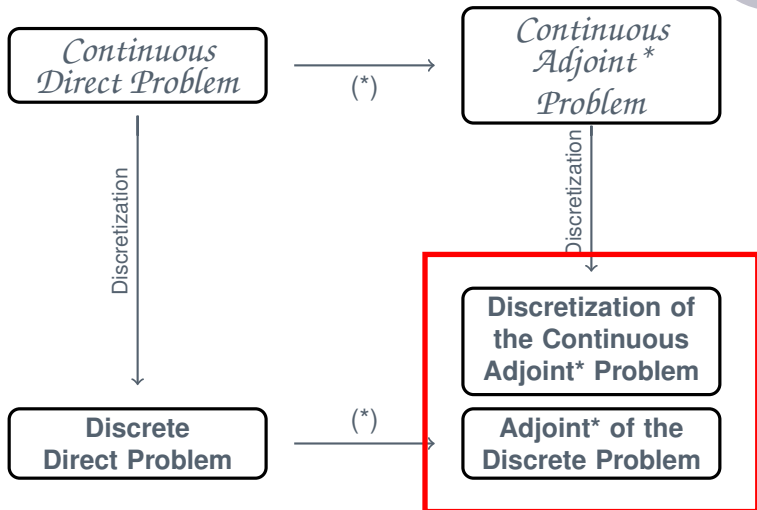
*Continuous  
Direct Problem*











$$J(p) = \frac{1}{2} \|Rp - data\|^2$$

$$\left\{ \begin{array}{l} \frac{1}{\rho_0 \mathbf{v}_p^2} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f_p \\ \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 \\ p(t=0) = 0 \\ \mathbf{v}(t=0) = 0 \\ \frac{\partial p}{\partial t} + \mathbf{v}_p \nabla p \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \end{array} \right.$$

$$t \in [0, T]$$

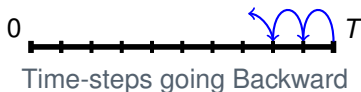
$$\left\{ \begin{array}{l} \frac{1}{\rho_0 \mathbf{v}_p^2} \frac{\partial \lambda_1}{\partial t} + \nabla \cdot \lambda_2 = \frac{\partial J}{\partial p} \\ \rho_0 \frac{\partial \lambda_2}{\partial t} + \nabla \lambda_1 = 0 \\ \lambda_1(t=T) = 0 \\ \lambda_2(t=T) = 0 \\ \frac{\partial \lambda_1}{\partial t} + \mathbf{v}_p \nabla \lambda_1 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \end{array} \right.$$

$$t \in [T, 0]$$

$$J(\bar{\mathbf{P}}) = \frac{1}{2} \|R\bar{\mathbf{P}} - data\|^2$$

$$\begin{cases} \frac{\partial \bar{\mathbf{P}}^n}{\partial t} = A_{pv} \bar{\mathbf{V}}^n + A_{pp} \bar{\mathbf{P}}^n + \bar{\mathbf{F}}_p^n \\ \frac{\partial \bar{\mathbf{V}}^n}{\partial t} = A_{vp} \bar{\mathbf{P}}^n \end{cases}$$

$$\begin{cases} \frac{\partial \bar{\Lambda}_1^n}{\partial t} = +A_{pv} \bar{\Lambda}_2^n + A_{pp} \bar{\Lambda}_1^n + \bar{\mathbf{D}}_p^n \\ \frac{\partial \bar{\Lambda}_2^n}{\partial t} = A_{vp} \bar{\Lambda}_1^n \end{cases}$$



$$\frac{\partial \bar{\mathbf{U}}^n}{\partial t} = A \bar{\mathbf{U}}^n + \bar{\mathbf{F}}^n \quad \text{With : } \bar{\mathbf{U}} = \begin{pmatrix} \bar{\mathbf{P}} \\ \bar{\mathbf{V}} \end{pmatrix}, A = \begin{pmatrix} A_{pp} & A_p \\ A_v & 0 \end{pmatrix}, \bar{\mathbf{F}} = \begin{pmatrix} \bar{\mathbf{F}}_p \\ 0 \end{pmatrix}$$

All time scheme can be summed-up such as :

$$L \bar{\mathbf{U}} = E \bar{\mathbf{F}}$$

We are looking for a Discrete Adjoint state satisfying :

$$L^* \bar{\boldsymbol{\lambda}} = -R^*(R \bar{\mathbf{U}} - data)$$

# DtA : Discretize then Adjoint Strategy

Example with RK4



RK4 time-scheme leads to :

$$\bar{U}^{n+1} = B\bar{U}^n + C_0\bar{F}^n + C_{\frac{1}{2}}\bar{F}^{n+\frac{1}{2}} + C_1\bar{F}^{n+1}$$

$$L\bar{U} = E\bar{F} = \bar{G}$$
$$\begin{pmatrix} I & & & & \\ -B & I & & & \\ & -B & I & & \\ & & \ddots & \ddots & \\ & & & -B & I \end{pmatrix} \begin{pmatrix} \bar{U}^0 \\ \bar{U}^1 \\ \bar{U}^2 \\ \vdots \\ \bar{U}^n \end{pmatrix} = \begin{pmatrix} \bar{G}^0 \\ \bar{G}^1 \\ \bar{G}^2 \\ \vdots \\ \bar{G}^n \end{pmatrix}$$

So :

$$L^* = \begin{pmatrix} I & -B^* & & & \\ & I & -B^* & & \\ & & \ddots & \ddots & \\ & & & I & -B^* \\ & & & & I \end{pmatrix}$$



# Adjoint test

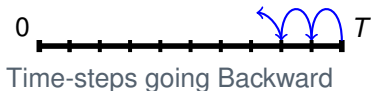
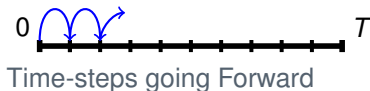


$$\langle L\bar{U}, \bar{\Lambda} \rangle = \langle \bar{U}, L^*\bar{\Lambda} \rangle$$

$$\langle L\bar{U}, \bar{\Lambda} \rangle = \langle \bar{U}, L^*\bar{\Lambda} \rangle$$

$$\begin{cases} L\bar{U} = E\bar{F} = \bar{G} \\ \bar{U}(t=0) = 0 \end{cases}$$

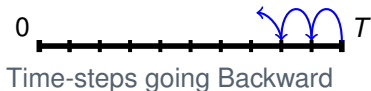
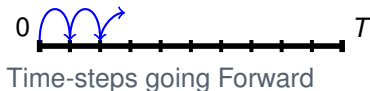
$$\begin{cases} L^*\bar{\Lambda} = -R^*(R\bar{U} - data) = \bar{D} \\ \bar{\Lambda}(t=T) = 0 \end{cases}$$



$$\langle L\bar{U}, \bar{\Lambda} \rangle = \langle \bar{U}, L^*\bar{\Lambda} \rangle$$

$$\begin{cases} L\bar{U} = E\bar{F} = \bar{G} \\ \bar{U}(t=0) = 0 \end{cases}$$

$$\begin{cases} L^*\bar{\Lambda} = -R^*(R\bar{U} - data) = \bar{D} \\ \bar{\Lambda}(t=T) = 0 \end{cases}$$



$$\langle E\bar{F}, \bar{\Lambda} \rangle = \langle \bar{U}, -R^*(R\bar{U} - data) \rangle$$

$$\langle L\bar{U}, \bar{\Lambda} \rangle = \langle \bar{U}, L^*\bar{\Lambda} \rangle$$

$$\begin{cases} L\bar{U} = E\bar{F} = \bar{G} \\ \bar{U}(t=0) = 0 \end{cases}$$

$$\begin{cases} L^*\bar{\Lambda} = -R^*(R\bar{U} - data) = \bar{D} \\ \bar{\Lambda}(t=T) = 0 \end{cases}$$



$$\langle E\bar{F}, \bar{\Lambda} \rangle = \langle \bar{U}, -R^*(R\bar{U} - data) \rangle$$

$$\langle \bar{G}, \bar{\Lambda} \rangle = \langle \bar{U}, \bar{D} \rangle$$

**Adjoint test succeeds  $\iff$  operator  $L^*$  well established**

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Adjoint test passed for :

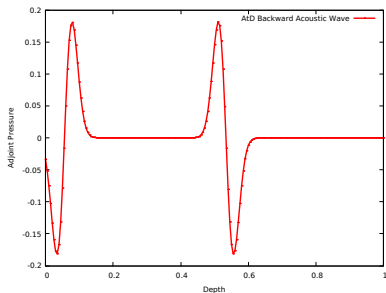
- ▶ Lagrange Operators
- ▶ Bernstein Operators
- ▶ Runge Kutta 4 time-scheme
- ▶ Adams Bashforth 3 time-scheme
- ▶ With a canonical space inner-product  
 $\langle u, v \rangle_X = \sum_i u_i v_i$
- ▶ With a M-space inner product  
 $\langle u, v \rangle_X^M = \langle Mu, v \rangle_X$

Adjoint test passed for :

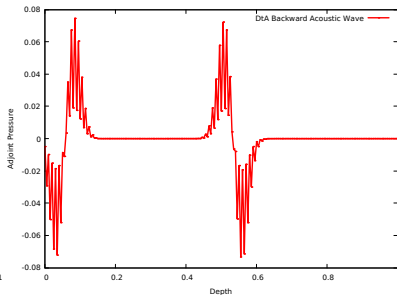
- ▶ Lagrange Operators
- ▶ Bernstein Operators
- ▶ Runge Kutta 4 time-scheme
- ▶ Adams Bashforth 3 time-scheme
- ▶ With a canonical space inner-product  
( $\langle u, v \rangle_X = \sum_i u_i v_i$ )
- ▶ With a M-space inner product  
( $\langle u, v \rangle_X^M = \langle Mu, v \rangle_X$ )

```
./run
--- Adjoint test ----
inner product UP/DUDP 553123.57586755091
inner product GPGU/QPQU 553123.57586756046
./run
--- Adjoint test ----
inner product UP/DUDP -75077.332007383695
inner product GPGU/QPQU -75077.332007386358
./run
--- Adjoint test ----
inner product UP/DUDP 125669.89223600870
inner product GPGU/QPQU 125669.89223600952
./run
--- Adjoint test ----
inner product UP/DUDP -132852.64215701097
inner product GPGU/QPQU -132852.64215701059
```

# Non consistency of the Adjoint solution



With the AtD strategy



With the DtA strategy using the canonical inner-product (Lagrange+RK4)

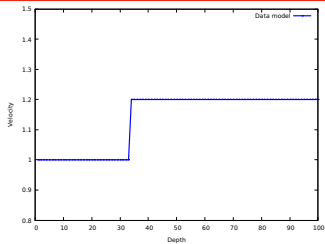
## Adjoint test succeeds !

[1] Sei Alain and Symes William

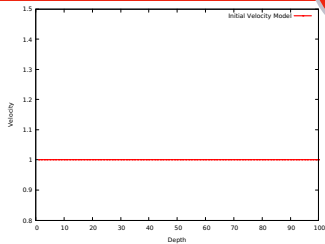
A Note on Consistency and Adjointness for Numerical Schemes  
1997



# FWI Preliminary test (for all strategies)

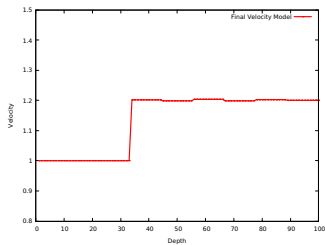
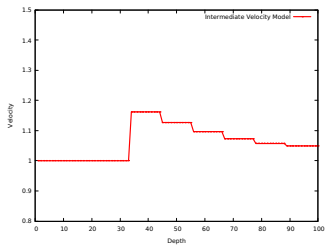
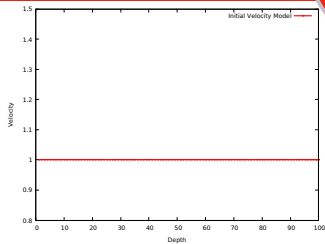
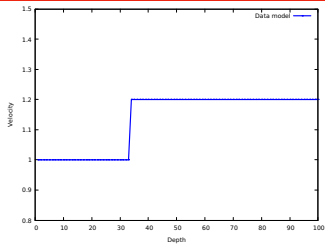


Target model

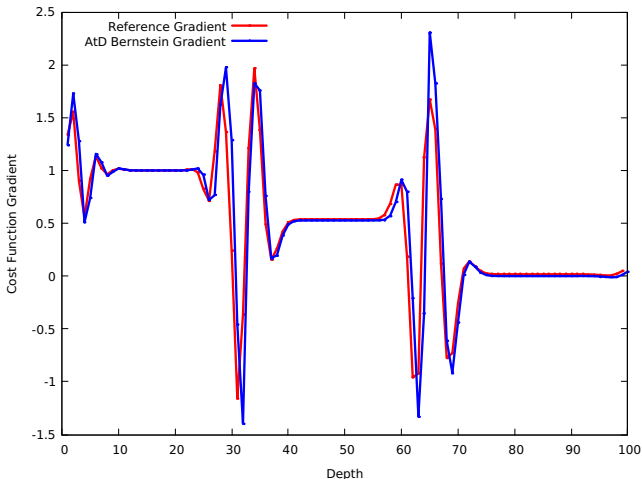


Initial model (iter=0)

# FWI Preliminary test (for all strategies)

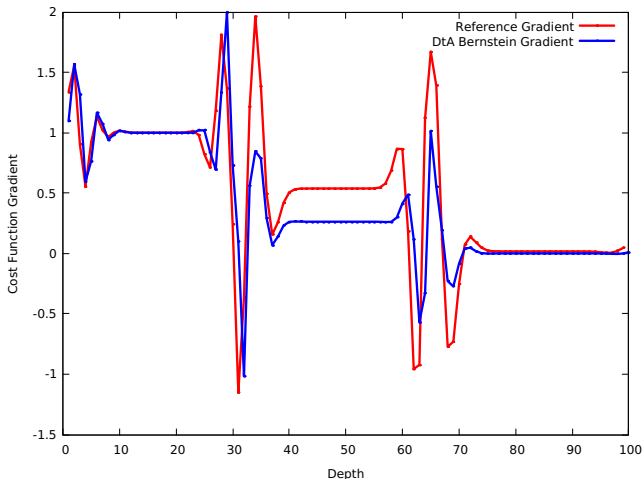


# Qualitative Cost Function Gradient Study



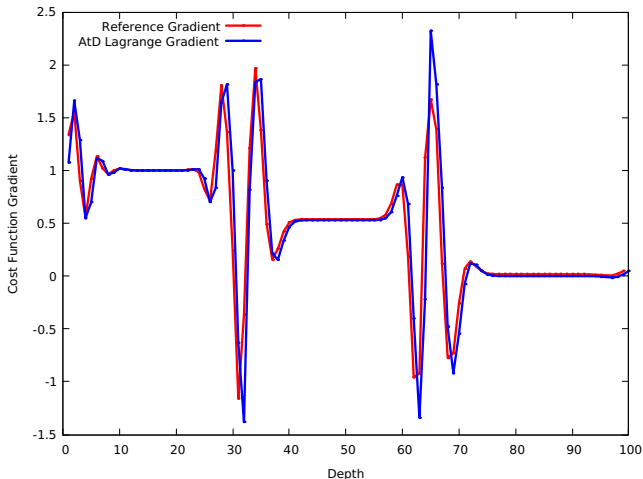
**Figure:** Comparison between a Reference Gradient and the FWI Gradient with AtD strategy (Bernstein elements and RK4 time scheme)

# Qualitative Cost Function Gradient Study



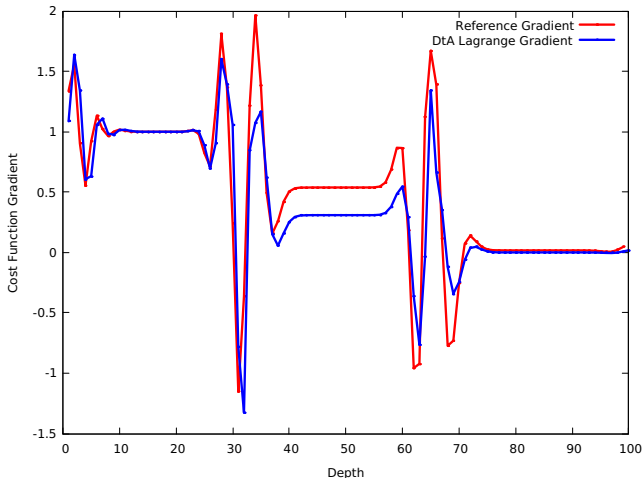
**Figure:** Comparison between a Reference Gradient and the FWI Gradient with DtA strategy (Bernstein elements and RK4 time scheme)

# Qualitative Cost Function Gradient Study



**Figure:** Comparison between a Reference Gradient and the FWI Gradient with AtD strategy (Lagrange elements and RK4 time scheme)

# Qualitative Cost Function Gradient Study



**Figure:** Comparison between a Reference Gradient and the FWI Gradient with DtA strategy (Lagrange elements and RK4 time scheme)

## Conclusion :

- ▶ Adjoint then Discretized strategy works
- ▶ Discretized then Adjoint strategy has unexpected results (Gradient formulation ? Bug ?)
- ▶ The adjoint state is not consistent by using the Discretized and Adjoint strategy (but Adjoint test succeeds)

## Conclusion :

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## Perspectives :

- ▶ Complementary 1D tests
- ▶ 2D FWI + tests
- ▶ 3D FWI + tests
- ▶ Coupling SEM/DG elements (Aurélien Citrain's thesis)