On the coupling of Spectral Element Method with Discontinuous Galerkin approximation for elasto-acoustic problems

Hélène Barucq, Henri Calandra, Aurélien Citrain, Julien Diaz, Christian Gout

On the coupling of Spectral Element Method with Discontinuous Galerkin approximation for elasto-acoustic problems

Hélène Barucq[1], Henri Calandra[2], Aurélien Citrain[3,1], Julien Diaz[1] and Christian Gout[3]

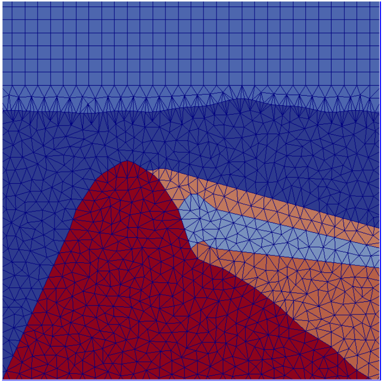
[1] Team project Magique.3D, INRIA.UPPA.CNRS, Pau, France.

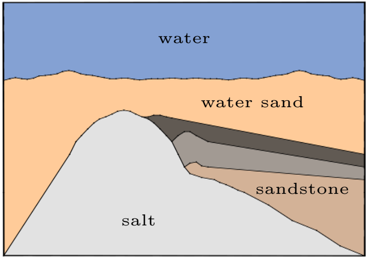
[2] TOTAL SA, CSTJF, Pau, France.

[3] INSA Rouen-Normandie Université, LMI EA 3226, 76000, Rouen.

Réunion d'équipe 13/09/2018

Why using hybrid meshes?





- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water)
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy

Elastodynamic system

$$\begin{cases} \rho(x)\dfrac{\partial v}{\partial t}(x,t) & = \nabla \cdot \underline{\underline{\sigma}}(x,t) \\ \dfrac{\partial \underline{\underline{\sigma}}}{\partial t}(x,t) & = \underline{\underline{C}}(x)\underline{\underline{\epsilon}}(v(x,t)) \end{cases}$$

With :

- $\rho(x)$ the density
- $C(x)$ the elasticity tensor
- $\epsilon(x,t)$ the deformation tensor
- $v(x,t)$, the wavespeed
- $\underline{\underline{\sigma}}(x,t)$ the strain tensor

Elasticus software

Written in **Fortran** for wave propagation simulation in the **time domain**

Features

- Using various types of meshes (**unstructured triangle and tetrahedra**)
- Modelling of various physics (**acoustic, elastic and elasto-acoustic**)

- **Discontinuous Galerkin Method** (DG) based on **unstructured triangle and unstrustured tetrahedra**
- with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**
- with **p-adaptivity**, **multi-order** computation...

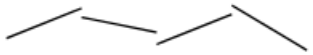
Table of contents

Discontinuous Galerkin Method

Use discontinuous functions :

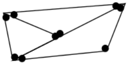
mesh

continuous

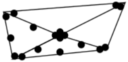


discontinuous

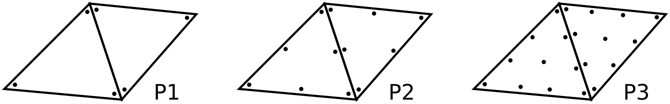
h adaptivity :



p adaptivity :



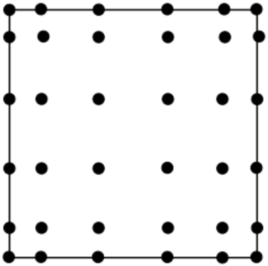
Degrees of freedom on each cell :



Spectral Element Method

General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature
- Gauss-Lobatto points as degrees of freedom (exponential convergence on L^2-norm)



- $\int f(x)dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j)$
- $\varphi_i(\xi_j) = \delta_{ij}$

Spectral Element Method

Main change with DG

- DG discontinuous, SEM continuous
- Need of defining local to global numbering
- Global matrices required by SEM
- Basis functions computed differently

Advantages of each method

DG

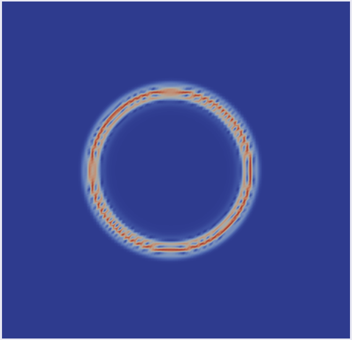
- Element per element computation (*hp*-adaptivity)
- Time discretization quasi explicit (block diagonal mass matrix)
- Simple to parallelize

SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method
- Reduces the computational cost when using structured meshes in comparison with DG

Description of the test cases

Physical parameters



P *wavespeed*	$1000\ m.s^{-1}$
Density	$1\ kg.m^{-3}$

Second order **Ricker Source** in *Pwave* ($f_{peak} = 10Hz$)

General context

- **Acoustic homogeneous** medium
- Four differents meshes : **10000 cells, 22500 cells**, 90000 cells, 250000 cells
- CFL computed using **power iteration** method
- **Leap-Frog** time scheme
- **Four threads** parallel execution with **OpenMP**

Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method

Quadrangle mesh 10000 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.99e-3	2.5e-2	19.30	500
SEM	4.9e-3	1.3e-1	0.36	204

Quadrangle mesh 22500 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	100.48	750
SEM	3.26e-3	7e-2	1.19	306

Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method

Quadrangle mesh 10000 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.99e-3	2.5e-2	19.30	500
SEM	4.9e-3	1.3e-1	0.36	204
SEM(DG CFL)	1.99e-3	4.8e-2	0.73	502

Quadrangle mesh 22500 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	100.48	750
SEM	3.26e-3	7e-2	1.19	306
SEM(DG CFL)	1.33e-3	1.2e-2	2.82	751

SEM fifty time faster than DG on a mesh with 22500 cells

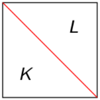
3 DG/SEM coupling

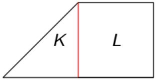
- Hybrid meshes structures
- Variational formulation
- Space discretization

Hybrid meshes structures

- Aim at coupling P_k and Q_k structures.
- Need to extend or split some of the structures (e.g. neighbour indexes)
- Define new face matrices

$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L$$

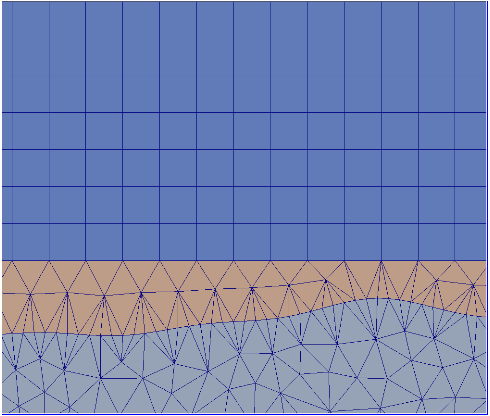




Variational formulation

Global context

- Domain in two parts : $\Omega_{h,1}$ (**structured quadrangle + SEM**), $\Omega_{h,2}$ (**unstructured triangle + DG**)



Variational formulation

SEM variational formulation :

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = -\int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C\xi_1 \mathbf{n}_1) \cdot v_1 \end{cases}$$

DG variational formulation :

$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot \mathbf{n}_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C\xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[[C\xi_2]] \cdot \mathbf{n}_2 \end{cases}$$

Variational formulation

Add the average of the solution of each part at the interface + put $\sigma_\star \mathbf{n}_\star = 0$

$$\begin{cases} \displaystyle\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \frac{1}{2}\int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2)\mathbf{n}_1 \cdot w_1 \\ \displaystyle\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = -\int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \frac{1}{2}\int_{\Gamma_{1/2}} (C\xi_1 \mathbf{n}_1) \cdot (v_1 + v_2) \end{cases}$$

$$\begin{cases} \displaystyle\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot \mathbf{n}_2 - \frac{1}{2}\int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2)\mathbf{n}_1 \cdot w_2 \\ \displaystyle\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[[C\xi_2]] \cdot \mathbf{n}_2 \\ \displaystyle -\frac{1}{2}\int_{\Gamma_{1/2}} (C\xi_2 \mathbf{n}_1) \cdot (v_1 + v_2) \end{cases}$$

Energy study

Goal : Show that our coupling preserves the energy

- We set $\xi_1 = \sigma_1, \xi_2 = \sigma_2, w_1 = v_1, w_2 = v_2$
- We add the equations of the two parts variational formulation

$$\frac{d}{dt}\mathbb{E} = 0$$

Space discretization : SEM part

- φ_i : SEM basis functions
- ψ_i : DG basis functions

$$\begin{cases} M_{\boldsymbol{v}_1}\partial_t \boldsymbol{v}_{h,1} + R_{\boldsymbol{\sigma}_1}\boldsymbol{\sigma}_{h,1} + R^{2,1}_{\boldsymbol{\sigma}_2}\boldsymbol{\sigma}_{h,2} = 0 \\ \\ M_{\boldsymbol{\sigma}_1}\partial_t \boldsymbol{\sigma}_{h,1} + R_{\boldsymbol{v}_1}\boldsymbol{v}_{h,1} + R^{2,1}_{\boldsymbol{v}_2}\boldsymbol{v}_{h,2} = 0 \end{cases}$$

- $M_{ij} = \int_\Omega \varphi_i \varphi_j \approx \sum_{e \in supp(\varphi_i) \cap supp(\varphi_j)} \sum_{k=1}^{(r+1)^d} \omega_k \varphi_i(\xi_k)\varphi_j(\xi_k) = \sum_{e \in supp(\varphi_i) \cap supp(\varphi_j)} \omega_i \delta_{i,j}$ the mass matrix
- $R_{p_{ij}} = \int_\Omega \varphi_i \frac{\partial \varphi_j}{\partial p}$ stiffness matrix

Matrix of DG/SEM coupling :

$$R^{2,1}_{\sigma_2,ij} = \frac{1}{2}\int_{\partial\Omega_1 \cap \partial\Omega_2} \psi_i \varphi_j$$

Space discretization : DG part

$$\begin{cases} \rho M_{\boldsymbol{v}_2}\partial_t \boldsymbol{v}_{h,2} + R_{\boldsymbol{\sigma}_2}\boldsymbol{\sigma}_{h,2} - R^{1,2}_{\boldsymbol{\sigma}_1}\boldsymbol{\sigma}_{h,1} = 0 \\ M_{\boldsymbol{\sigma}_2}\partial_t \boldsymbol{\sigma}_{h,2} + R_{\boldsymbol{v}_2}\boldsymbol{v}_{h,2} - R^{1,2}_{\boldsymbol{v}_1}\boldsymbol{v}_{h,1} = 0 \end{cases}$$

- $M^K_{ij} = \int_K \psi^K_i \psi^K_j$ mass matrix,
- $R^K_{p_{ij}} = \int_K \psi^K_i \dfrac{\partial \psi^K_j}{\partial p}$ stiffness matrix,
- $R^{K,L}_{p_{ij}} = \int_{\partial K \cap \partial L} \psi^K_i \psi^L_j n_K \cdot e_p$ the mass-face matrix

Two new matrices which come from the DG/SEM coupling $R^{1,2}_{\star}$. Block composed :

$$R^{1,2}_{v_1} = R^{1,2}_{\sigma_1} = -\frac{1}{2}\int_{\partial\Omega_2 \cap \partial K_1} \psi^{K_2}_j \varphi_i \tag{1}$$

4 Comparison between DG/SEM an DG on hybrid meshes
- Experimentation context
- Comparative tables

Context

- Acoustic homogeneous medium
- 54000 triangles
- 21000 quadrangles
- Using Leap-Frog time scheme
- Parallel computation using OpenMP
- Done with different orders of discretization

Comparative tables

$P_1 - Q_3$ computation :

	CFL	L_2-error	CPU-time
DG	1e-5	0.03	7343.92
DG/SEM	1e-5	0.03	823.22

$P_2 - Q_3$ computation :

	CFL	L_2-error	CPU-time
DG	1e-5	0.002	9452.73
DG/SEM	1e-5	0.003	1393.80

$P_3 - Q_1$ computation :

	CFL	L_2-error	CPU-time
DG	3e-5	0.009	3078.15
DG/SEM	3e-5	0.01	2951

$P_3 - Q_2$ computation :

	CFL	L_2-error	CPU-time
DG	1e-5	5.4e-4	9951.60
DG/SEM	1e-5	0.007	3122

General settings

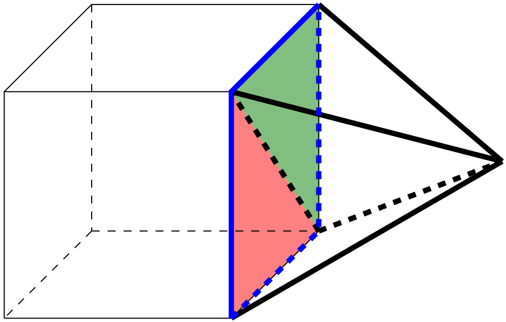
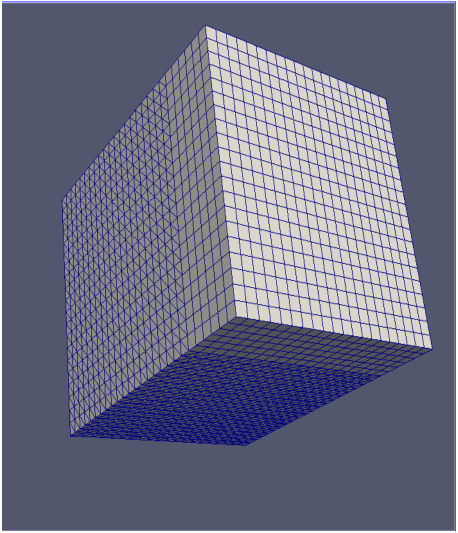


Figure: Hexa/Tet boundary configuration

- Only deal with a simple case of 3D hybrid meshes : one hexahedra has only two tetrahedra as neighbour
- Extend SEM in 3D (basis functions...)
- Require introducing a new matrix which handles the rotation cases between two elements



Conclusion and perspectives

Conclusion

1. Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
2. As expected, SEM is more efficient on structured quadrangle mesh than DG
3. Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Add a local time-stepping scheme
- Develop PML in the hexahedral part

Thank you for your attention !

Questions?