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CLASSIFICATION OF DIFFERENTIAL SYMMETRY BREAKING OPERATORS FOR DIFFERENTIAL FORMS

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Abstract. We give a complete classification of conformally covariant differential operators between the spaces of differential i-forms on the sphere S^n and j-forms on the totally geodesic hypersphere S^{n-1} by analyzing the restriction of principal series representations of the Lie group $O(n+1,1)$. Further, we provide explicit formulæ for these matrix-valued operators in the flat coordinates and find factorization identities for them.

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1. Introduction

Suppose a Lie group G acts conformally on a Riemannian manifold (X, g). This means that there exists a positive-valued function $\Omega \in C^\infty(G\times X)$ (*conformal factor*) such that

$$L_h^* g_{h\cdot x} = \Omega(h,x)^2 g_x \quad \text{for all } h \in G \text{ and } x \in X,$$

where $L_h : X \to X, x \mapsto h\cdot x$ denotes the action of G on X. Since Ω satisfies a cocycle condition, we can form a family of representations $\varpi_u^{(i)}$ for $u \in \mathbb{C}$ and $0 \leq i \leq \dim X$ on the space $\mathcal{E}^i(X)$ of differential i-forms on X by

$$\varpi_u^{(i)}(h)\alpha := \Omega(h^{-1}, \cdot)^u L_{h^{-1}}^* \alpha \quad (h \in G). \tag{1.1}$$

The representation $\varpi_u^{(i)}$ of the conformal group G on $\mathcal{E}^i(X)$ will be simply denoted by $\mathcal{E}^i(X)_u$.

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If Y is a submanifold of X, then we can also define a family of representations $\varpi_v^{(j)}$ on $\mathcal{E}^j(Y)$ $(v \in \mathbb{C}, 0 \leq j \leq \dim Y)$ of the subgroup

$$G' := \{h \in G \,:\, h \cdot Y = Y\},$$

which acts conformally on the Riemannian submanifold $(Y, g\,|_Y)$.

We study differential operators $\mathcal{D} : \mathcal{E}^i(X) \longrightarrow \mathcal{E}^j(Y)$ that intertwine the two representations $\varpi_u^{(i)}|_{G'}$ and $\varpi_v^{(j)}$ of G'. Here $\varpi_u^{(i)}|_{G'}$ stands for the restriction of the G-representation $\varpi_u^{(i)}$ to the subgroup G'. We say that such $\mathcal{D}$ is a *differential symmetry breaking operator*, and denote by $\mathrm{Diff}_{G'}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$ the space of all differential symmetry breaking operators. We address the following problems:

Problem A. *Determine the dimension of the space* $\mathrm{Diff}_{G'}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$. *In particular, find a necessary and sufficient condition on a quadruple* (i, j, u, v) *such that there exist nontrivial differential symmetry breaking operators.*

Problem B. *Construct explicitly a basis of* $\mathrm{Diff}_{G'}(\mathcal{E}^i(X)_u, \mathcal{E}^j(Y)_v)$.

In the case where $X = Y$, $G = G'$, and $i = j = 0$, a classical prototype of such operators is a second order differential operator called the Yamabe operator

$$\Delta + \frac{n-2}{4(n-1)}\kappa \in \mathrm{Diff}_G(\mathcal{E}^0(X)_{\frac{n}{2}-1}, \mathcal{E}^0(X)_{\frac{n}{2}+1}),$$

where n is the dimension of X, Δ is the Laplace–Beltrami operator, and κ is the scalar curvature of X. Conformally covariant differential operators of higher order are also known: the Paneitz operator (fourth order) [11], which appears in four dimensional supergravity [2], or more generally, the so-called GJMS operators [3] are such examples. Analogous conformally covariant operators on forms ($i = j$ case) were studied by Branson [1]. On the other hand, the insight of representation theory of conformal groups is useful in studying Maxwell's equations, see [10], for instance.

Let us consider the more general case where $Y \neq X$ and $G' \neq G$. An obvious example of symmetry breaking operators is the restriction operator Rest_Y which belongs to $\mathrm{Diff}_{G'}(\mathcal{E}^i(X)_u, \mathcal{E}^i(Y)_u)$ for all $u \in \mathbb{C}$. Another elementary example is $\mathrm{Rest}_Y \circ \iota_{N_Y(X)} \in \mathrm{Diff}_{G'}(\mathcal{E}^i(X)_u, \mathcal{E}^{i-1}(Y)_v)$ if $v = u + 1$ where $\iota_{N_Y(X)}$ denotes the interior multiplication by the normal vector field to Y when Y is of codimension one in X.

In the model space where $(X, Y) = (S^n, S^{n-1})$, the pair (G, G') of conformal groups amounts to $(O(n+1, 1), O(n, 1))$ modulo center, and Problems A and B have been recently solved for $i = j = 0$ by Juhl [4], see also [5, 7] and [9] for different approaches by the F-method and the residue calculus, respectively.

Problems A and B for general i and j for the model space can be reduced to analogous problems for (nonspherical) principal series representations by the isomorphism

(2.2) below. In this note we shall give complete solutions to Problems A and B in those terms (see Theorems 3.1 and 4.1).

Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$.

2. Principal series representations of $G = O(n+1,1)$

We set up notations. Let $P = MAN$ be a Langlands decomposition of a minimal parabolic subgroup of $G = O(n+1,1)$. For $0 \leq i \leq n$, $\delta \in \mathbb{Z}/2\mathbb{Z}$, and $\lambda \in \mathbb{C}$, we extend the outer tensor product representation $\bigwedge^i(\mathbb{C}^n) \otimes (-1)^\delta \otimes \mathbb{C}_\lambda$ of $MA \simeq (O(n) \times O(1)) \times \mathbb{R}$ to P by letting N act trivially, and form a G-equivariant vector bundle $\mathcal{V}^i_{\lambda,\delta} := G \times_P \left(\bigwedge^i(\mathbb{C}^n) \otimes (-1)^\delta \otimes \mathbb{C}_\lambda\right)$ over the real flag variety $X = G/P \simeq S^n$. Then we define an unnormalized principal series representations

$$I(i,\lambda)_\delta := \mathrm{Ind}_P^G\left(\bigwedge^i(\mathbb{C}^n) \otimes (-1)^\delta \otimes \mathbb{C}_\lambda\right) \tag{2.1}$$

of G on the Fréchet space $C^\infty(X, \mathcal{V}^i_{\lambda,\delta})$ of smooth sections.

In our parametrization, $I(i, n-2i)_\delta$ and $I(i,i)_\delta$ have the same infinitesimal character with the trivial one-dimensional representation of G. Then, for all $u \in \mathbb{C}$, we have a natural G-isomorphism

$$\varpi_u^{(i)} \simeq I(i, u+i)_{i \bmod 2}. \tag{2.2}$$

Similarly, for $0 \leq j \leq n-1$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\nu \in \mathbb{C}$, we define an unnormalized principal series representation $J(j,\nu)_\varepsilon := \mathrm{Ind}_{P'}^{G'}\left(\bigwedge^j(\mathbb{C}^{n-1}) \otimes (-1)^\varepsilon \otimes \mathbb{C}_\nu\right)$ of the subgroup $G' = O(n,1)$ on $C^\infty(Y, \mathcal{W}^j_{\nu,\varepsilon})$, where $\mathcal{W}^j_{\nu,\varepsilon} := G' \times_{P'} \left(\bigwedge^j(\mathbb{C}^{n-1}) \otimes (-1)^\varepsilon \otimes \mathbb{C}_\nu\right)$ is a G'-equivariant vector bundle over $Y = G'/P' \simeq S^{n-1}$.

3. Existence condition for differential symmetry breaking operators

A continuous G'-intertwining operator $T : I(i,\lambda)_\delta \longrightarrow J(j,\nu)_\varepsilon$ is said to be a *symmetry breaking operator* (SBO). We say that T is a differential operator if T satisfies $\mathrm{Supp}(Tf) \subset \mathrm{Supp} f$ for all $f \in C^\infty(X, \mathcal{V}^i_{\lambda,\delta})$, and $\mathrm{Diff}_{G'}(I(i,\lambda)_\delta, J(j,\nu)_\varepsilon)$ denotes the space of differential SBOs. We give a complete solution to Problem A for $(X,Y) = (S^n, S^{n-1})$ in terms of principal series representations:

Theorem 3.1. *Let $n \geq 3$. Suppose $0 \leq i \leq n$, $0 \leq j \leq n-1$, $\lambda, \nu \in \mathbb{C}$, and $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on 6-tuple $(i,j,\lambda,\nu,\delta,\varepsilon)$ are equivalent:*

(i) $\mathrm{Diff}_{O(n,1)}(I(i,\lambda)_\delta, J(j,\nu)_\varepsilon) \neq \{0\}$.

(ii) $\dim \mathrm{Diff}_{O(n,1)}(I(i,\lambda)_\delta, J(j,\nu)_\varepsilon) = 1$.

(iii) The 6-tuple belongs to one of the following six cases:

Case 1. $j = i$, $0 \leq i \leq n-1$, $\nu - \lambda \in \mathbb{N}$, $\varepsilon - \delta \equiv \nu - \lambda \mod 2$.
Case 2. $j = i-1$, $1 \leq i \leq n$, $\nu - \lambda \in \mathbb{N}$, $\varepsilon - \delta \equiv \nu - \lambda \mod 2$.
Case 3. $j = i+1$, $1 \leq i \leq n-2$, $(\lambda, \nu) = (i, i+1)$, $\varepsilon \equiv \delta + 1 \mod 2$.
Case 3′. $(i,j) = (0,1)$, $-\lambda \in \mathbb{N}$, $\nu = 1$, $\varepsilon \equiv \delta + \lambda + 1 \mod 2$.
Case 4. $j = i-2$, $2 \leq i \leq n-1$, $(\lambda, \nu) = (n-i, n-i+1)$, $\varepsilon \equiv \delta + 1 \mod 2$.
Case 4′. $(i,j) = (n, n-2)$, $-\lambda \in \mathbb{N}$, $\nu = 1$, $\varepsilon \equiv \delta + \lambda + 1 \mod 2$.

We set $\Xi := \{(i,j,\lambda,\nu)$: the 6-tuple $(i,j,\lambda,\nu,\delta,\varepsilon)$ satisfies one of the equivalent conditions of Theorem 3.1 for some $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}\}$.

4. Construction of differential symmetry breaking operators

In this section, we describe an explicit generator of the space of differential SBOs if one of the equivalent conditions in Theorem 3.1 is satisfied. For this we use the *flat picture* of the principal series representations $I(i,\lambda)_\delta$ of G which realizes the representation space $C^\infty(X, \mathcal{V}^i_{\lambda,\delta})$ as a subspace of $C^\infty(\mathbb{R}^n, \bigwedge^i(\mathbb{C}^n))$ by trivializing the bundle $\mathcal{V}^i_{\lambda,\delta} \longrightarrow X$ on the open Bruhat cell

$$\mathbb{R}^n \hookrightarrow X, \quad (x_1, \cdots, x_n) \mapsto \exp\left(\sum_{j=1}^n x_j N_j^-\right) P.$$

Here $\{N_1^-, \cdots, N_n^-\}$ is an orthonormal basis of the nilradical $\mathfrak{n}_-(\mathbb{R})$ of the opposite parabolic subalgebra with respect to an M-invariant inner product. Without loss of generality, we may and do assume that the open Bruhat cell $\mathbb{R}^{n-1} \hookrightarrow Y \simeq G'/P'$ is given by putting $x_n = 0$. Then the flat picture of the principal series representation $J(j,\nu)_\varepsilon$ of G' is defined by realizing $C^\infty(Y, \mathcal{W}^j_{\nu,\varepsilon})$ as a subspace of $C^\infty(\mathbb{R}^{n-1}, \bigwedge^j(\mathbb{C}^{n-1}))$. For the construction of explicit generators of matrix-valued SBOs, we begin with a scalar-valued differential operator. For $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$, we define a polynomial of two variables (s,t) by

$$\left(I_\ell \widetilde{C}^\alpha_\ell\right)(s,t) := s^{\frac{\ell}{2}} \widetilde{C}^\alpha_\ell\left(\frac{t}{\sqrt{s}}\right),$$

where $\widetilde{C}^\alpha_\ell(z)$ is the renormalized Gegenbauer polynomial given by

$$\widetilde{C}^\alpha_\ell(z) := \frac{1}{\Gamma\left(\alpha + \left[\frac{\ell+1}{2}\right]\right)} \sum_{k=0}^{\left[\frac{\ell}{2}\right]} (-1)^k \frac{\Gamma(\ell - k + \alpha)}{k!(\ell - 2k)!} (2z)^{\ell - 2k}.$$

Then $\widetilde{C}^{\alpha}_{\ell}(z)$ is a nonzero polynomial for all $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$, and a (normalized) *Juhl's conformally covariant operator* $\widetilde{\mathbb{C}}_{\lambda,\nu} : C^{\infty}(\mathbb{R}^n) \longrightarrow C^{\infty}(\mathbb{R}^{n-1})$ is defined by

$$\widetilde{\mathbb{C}}_{\lambda,\nu} := \mathrm{Rest}_{x_n=0} \circ \left(I_{\ell}\widetilde{C}^{\lambda-\frac{n-1}{2}}_{\ell}\right)\left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}\right),$$

for $\lambda, \nu \in \mathbb{C}$ with $\ell := \nu - \lambda \in \mathbb{N}$. For instance,

$$\widetilde{\mathbb{C}}_{\lambda,\nu} = \mathrm{Rest}_{x_n=0} \circ \begin{cases} \mathrm{id} & \text{if } \nu = \lambda, \\ 2\frac{\partial}{\partial x_n} & \text{if } \nu = \lambda + 1, \\ \Delta_{\mathbb{R}^{n-1}} + (2\lambda - n + 3)\frac{\partial^2}{\partial x_n^2} & \text{if } \nu = \lambda + 2. \end{cases}$$

For $(i, j, \lambda, \nu) \in \Xi$, we introduce a new family of matrix-valued differential operators

$$\widetilde{\mathbb{C}}^{i,j}_{\lambda,\nu} : C^{\infty}(\mathbb{R}^n, \bigwedge^i(\mathbb{C}^n)) \longrightarrow C^{\infty}(\mathbb{R}^{n-1}, \bigwedge^j(\mathbb{C}^{n-1})),$$

by using the identifications $\mathcal{E}^i(\mathbb{R}^n) \simeq C^{\infty}(\mathbb{R}^n) \otimes \bigwedge^i(\mathbb{C}^n)$ and $\mathcal{E}^j(\mathbb{R}^{n-1}) \simeq C^{\infty}(\mathbb{R}^{n-1}) \otimes \bigwedge^j(\mathbb{C}^{n-1})$, as follows. Let $d^*_{\mathbb{R}^n}$ be the codifferential, which is the formal adjoint of the differential $d_{\mathbb{R}^n}$, and $\iota_{\frac{\partial}{\partial x_n}}$ the inner multiplication by the vector field $\frac{\partial}{\partial x_n}$. Both operators map $\mathcal{E}^i(\mathbb{R}^n)$ to $\mathcal{E}^{i-1}(\mathbb{R}^n)$. For $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$, let $\gamma(\alpha, \ell) := 1$ (ℓ is odd); $= \alpha + \frac{\ell}{2}$ (ℓ is even). Then we set

$$\mathbb{C}^{i,i}_{\lambda,\nu} := \widetilde{\mathbb{C}}_{\lambda+1,\nu-1} d_{\mathbb{R}^n} d^*_{\mathbb{R}^n} - \gamma(\lambda - \frac{n}{2}, \nu - \lambda)\widetilde{\mathbb{C}}_{\lambda,\nu-1} d_{\mathbb{R}^n}\iota_{\frac{\partial}{\partial x_n}} + \frac{1}{2}(\nu - i)\widetilde{\mathbb{C}}_{\lambda,\nu} \qquad \text{for } 0 \le i \le n-1.$$

$$\mathbb{C}^{i,i-1}_{\lambda,\nu} := -\widetilde{\mathbb{C}}_{\lambda+1,\nu-1} d_{\mathbb{R}^n} d^*_{\mathbb{R}^n}\iota_{\frac{\partial}{\partial x_n}} - \gamma(\lambda - \frac{n-1}{2}, \nu - \lambda)\widetilde{\mathbb{C}}_{\lambda+1,\nu} d^*_{\mathbb{R}^n} + \frac{1}{2}(\lambda + i - n)\widetilde{\mathbb{C}}_{\lambda,\nu}\iota_{\frac{\partial}{\partial x_n}} \text{ for } 1 \le i \le n.$$

We note that there exist isolated parameters (λ, ν) for which $\mathbb{C}^{i,i}_{\lambda,\nu} = 0$ or $\mathbb{C}^{i,i-1}_{\lambda,\nu} = 0$. For instance, $\mathbb{C}^{0,0}_{\lambda,\nu} = \frac{1}{2}\nu\widetilde{\mathbb{C}}_{\lambda,\nu}$, and thus $\mathbb{C}^{0,0}_{\lambda,\nu} = 0$ if $\nu = 0$. To be precise, we have the following:

$\mathbb{C}^{i,i}_{\lambda,\nu} = 0$ if and only if $\lambda = \nu = i$ or $\nu = i = 0$;
$\mathbb{C}^{i,i-1}_{\lambda,\nu} = 0$ if and only if $\lambda = \nu = n - i$ or $\nu = n - i = 0$.

We renormalize these operators by

$$\widetilde{\mathbb{C}}^{i,i}_{\lambda,\nu} := \begin{cases} \mathrm{Rest}_{x_n=0} & \text{if } \lambda = \nu, \\ \widetilde{\mathbb{C}}_{\lambda,\nu} & \text{if } i = 0, \\ \mathbb{C}^{i,i}_{\lambda,\nu} & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{\mathbb{C}}^{i,i-1}_{\lambda,\nu} := \begin{cases} \mathrm{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } \lambda = \nu, \\ \widetilde{\mathbb{C}}_{\lambda,\nu} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } i = n, \\ \mathbb{C}^{i,i-1}_{\lambda,\nu} & \text{otherwise.} \end{cases}$$

Then $\widetilde{\mathbb{C}}^{i,i}_{\lambda,\nu}$ $(0 \le i \le n-1)$ and $\widetilde{\mathbb{C}}^{i,i-1}_{\lambda,\nu}$ $(1 \le i \le n)$ are nonzero differential operators of order $\nu - \lambda$ for any $\lambda, \nu \in \mathbb{C}$ with $\nu - \lambda \in \mathbb{N}$.

The differential operators $\widetilde{\mathbb{C}}^{i,i+1}_{\lambda,\nu}$ and $\widetilde{\mathbb{C}}^{i,i-2}_{\lambda,\nu}$ are defined only for special parameters (λ,ν) as follows.

$$\widetilde{\mathbb{C}}^{i,i+1}_{\lambda,i+1} := \begin{cases} \mathrm{Rest}_{x_n=0} \circ d_{\mathbb{R}^n} & \text{for } 1 \le i \le n-2, \lambda = i, \\ d_{\mathbb{R}^{n-1}} \circ \widetilde{\mathbb{C}}_{\lambda,0} & \text{for } i=0, \lambda \in -\mathbb{N}, \end{cases}$$

$$\widetilde{\mathbb{C}}^{i,i-2}_{\lambda,n-i+1} := \begin{cases} \mathrm{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} d^*_{\mathbb{R}^n} & \text{for } 2 \le i \le n, \lambda = n-i, \\ -d^*_{\mathbb{R}^{n-1}} \circ \mathbb{C}^{n,n-1}_{\lambda,0} & \text{for } i=n, \lambda \in -\mathbb{N}. \end{cases}$$

Then we give a complete solution to Problem B for the model space $(X,Y) = (S^n, S^{n-1})$ in terms of the flat picture of principal series representations as follows:

Theorem 4.1. *Suppose a 6-tuple $(i,j,\lambda,\nu,\delta,\varepsilon)$ satisfies one of the equivalent conditions in Theorem 3.1. Then the operators $\widetilde{\mathbb{C}}^{i,j}_{\lambda,\nu} : C^\infty(\mathbb{R}^n) \otimes \bigwedge^i(\mathbb{C}^n) \longrightarrow C^\infty(\mathbb{R}^{n-1}) \otimes \bigwedge^j(\mathbb{C}^{n-1})$ extend to differential SBOs $I(i,\lambda)_\delta \longrightarrow J(j,\nu)_\varepsilon$, to be denoted by the same letters. Conversely, any differential SBO from $I(i,\lambda)_\delta$ to $J(j,\nu)_\varepsilon$ is proportional to the following differential operators: $\widetilde{\mathbb{C}}^{i,i}_{\lambda,\nu}$ in Case 1, $\widetilde{\mathbb{C}}^{i,i-1}_{\lambda,\nu}$ in Case 2, $\widetilde{\mathbb{C}}^{i,i+1}_{i,i+1}$ in Case 3, $\widetilde{\mathbb{C}}^{0,1}_{\lambda,1}$ in Case 3′, $\widetilde{\mathbb{C}}^{i,i-2}_{n-i,n-i+1}$ in Case 4, and $\widetilde{\mathbb{C}}^{n,n-2}_{\lambda,1}$ in Case 4′.*

5. Matrix-valued factorization identities

Suppose that $T_X : I(i,\lambda')_\delta \to I(i,\lambda)_\delta$ or $T_Y : J(j,\nu)_\varepsilon \to J(j,\nu')_\varepsilon$ are G- or G'-intertwining operators, respectively. Then the composition $T_Y \circ D_{X\to Y}$ or $D_{X\to Y} \circ T_X$ of a symmetry breaking operator $D_{X\to Y} : I(i,\lambda)_\delta \to J(j,\nu)_\varepsilon$ gives another symmetry breaking operator:

$$\begin{tikzcd} I(i,\lambda)_\delta \arrow[r, "D_{X\to Y}"] \arrow[dr, dashed] & J(j,\nu)_\varepsilon \arrow[d, "T_Y"] \\ I(i,\lambda')_\delta \arrow[u, "T_X"] \arrow[ur, dashed] & J(j,\nu')_\varepsilon \end{tikzcd}$$

The multiplicity-free property (see Theorem 3.1 (ii)) assures the existence of matrix-valued factorization identities for differential SBOs, namely, $D_{X\to Y} \circ T_X$ must be a scalar multiple of $\widetilde{\mathbb{C}}^{i,j}_{\lambda',\nu}$, and $T_Y \circ D_{X\to Y}$ must be a scalar multiple of $\widetilde{\mathbb{C}}^{i,j}_{\lambda,\nu'}$. We shall determine these constants explicitly when T_X or T_Y are Branson's conformally covariant operators [1] defined below. Let $0 \le i \le n$. For $\ell \in \mathbb{N}_+$, we set

$$\mathcal{T}^{(i)}_{2\ell} := ((\frac{n}{2}-i-\ell)d_{\mathbb{R}^n} d^*_{\mathbb{R}^n} + (\frac{n}{2}-i+\ell) d^*_{\mathbb{R}^n} d_{\mathbb{R}^n}) \Delta^{\ell-1}_{\mathbb{R}^n} = (-2\ell\, d_{\mathbb{R}^n} d^*_{\mathbb{R}^n} - (\frac{n}{2}-i+\ell)\Delta_{\mathbb{R}^n}) \Delta^{\ell-1}_{\mathbb{R}^n}.$$

Then the differential operator $\mathcal{T}^{(i)}_{2\ell} : \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^i(\mathbb{R}^n)$ induces a nonzero $O(n+1,1)$-intertwining operator, to be denoted by the same letter $\mathcal{T}^{(i)}_{2\ell}$, from $I\left(i, \frac{n}{2}-\ell\right)_\delta$ to $I\left(i, \frac{n}{2}+\ell\right)_\delta$, for $\delta \in \mathbb{Z}/2\mathbb{Z}$. Similarly, we define a G'-intertwining operator $\mathcal{T}'^{(j)}_{2\ell}$:

$J\left(j, \frac{n-1}{2}-\ell\right)_\varepsilon \longrightarrow J\left(j, \frac{n-1}{2}+\ell\right)_\varepsilon$ for $0 \leq j \leq n-1$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ as the lift of the differential operator ${\mathcal{T}'}^{(j)}_{2\ell} : \mathcal{E}^j(\mathbb{R}^{n-1}) \longrightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$ which is given by

$$ {\mathcal{T}'}^{(j)}_{2\ell} = ((\frac{n-1}{2} - j - \ell) d_{\mathbb{R}^{n-1}} d^*_{\mathbb{R}^{n-1}} + (\frac{n-1}{2} - j + \ell) d^*_{\mathbb{R}^{n-1}} d_{\mathbb{R}^{n-1}}) \Delta^{\ell-1}_{\mathbb{R}^{n-1}}. $$

Consider the following diagrams for $j = i$ and $j = i-1$:

$$
\begin{tikzcd}
I\left(i, \frac{n}{2}-\ell\right)_\delta \arrow[d, "\mathcal{T}^{(i)}_{2\ell}"'] \arrow[dr, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n}{2}-\ell, \frac{n}{2}+a+\ell}"] & \\
I\left(i, \frac{n}{2}+\ell\right)_\delta \arrow[r, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n}{2}+\ell, \frac{n}{2}+a+\ell}"'] & J\left(j, \frac{n}{2}+a+\ell\right)_\varepsilon ,
\end{tikzcd}
\qquad
\begin{tikzcd}
I\left(i, \frac{n-1}{2}-a-\ell\right)_\delta \arrow[r, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n-1}{2}-a-\ell, \frac{n-1}{2}-\ell}"] \arrow[dr, "\widetilde{\mathbb{C}}^{i,j}_{\frac{n-1}{2}-a-\ell, \frac{n-1}{2}+\ell}"'] & J\left(j, \frac{n-1}{2}-\ell\right)_\varepsilon \arrow[d, "{\mathcal{T}'}^{(j)}_{2\ell}"] \\
& J\left(j, \frac{n-1}{2}+\ell\right)_\varepsilon ,
\end{tikzcd}
$$

where parameters δ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ are chosen according to Theorem 3.1 (iii). In what follows, we put

$$ p_\pm = \begin{cases} i \pm \ell - \frac{n}{2} & \text{if } a \neq 0 \\ \pm 2 & \text{if } a = 0 \end{cases}, \quad q = \begin{cases} i + \ell - \frac{n-1}{2} & \text{if } i \neq 0, a \neq 0 \\ -2 & \text{if } i \neq 0, a = 0 \\ -\left(\ell + \frac{n-1}{2}\right) & \text{if } i = 0 \end{cases}, \quad r = \begin{cases} i - \ell - \frac{n+1}{2} & \text{if } i \neq n, a \neq 0 \\ 2 & \text{if } i \neq n, a = 0 \\ -\left(\ell + \frac{n+1}{2}\right) & \text{if } i = n \end{cases}, $$

$$ K_{\ell,a} := \prod_{k=1}^{\ell} \left(\left[\frac{a}{2}\right] + k \right). $$

Then the factorization identities for differential SBOs $\widetilde{\mathbb{C}}^{i,j}_{\lambda,\nu}$ for $j \in \{i-1, i\}$ and Branson's conformally covariant operators $\mathcal{T}^{(i)}_{2\ell}$ or ${\mathcal{T}'}^{(j)}_{2\ell}$ are given as follows.

Theorem 5.1. *Suppose* $0 \leq i \leq n-1, a \in \mathbb{N}$ *and* $\ell \in \mathbb{N}_+$. *Then*

(1) $\widetilde{\mathbb{C}}^{i,i}_{\frac{n}{2}+\ell, a+\ell+\frac{n}{2}} \circ \mathcal{T}^{(i)}_{2\ell} = p_- K_{\ell,a} \widetilde{\mathbb{C}}^{i,i}_{\frac{n}{2}-\ell, a+\ell+\frac{n}{2}}$.

(2) ${\mathcal{T}'}^{(i)}_{2\ell} \circ \widetilde{\mathbb{C}}^{i,i}_{\frac{n-1}{2}-a-\ell, \frac{n-1}{2}-\ell} = q K_{\ell,a} \widetilde{\mathbb{C}}^{i,i}_{\frac{n-1}{2}-a-\ell, \frac{n-1}{2}+\ell}$.

Theorem 5.2. *Suppose* $1 \leq i \leq n$, $a \in \mathbb{N}$ *and* $\ell \in \mathbb{N}_+$. *Then*

(1) $\widetilde{\mathbb{C}}^{i,i-1}_{\frac{n}{2}+\ell, a+\ell+\frac{n}{2}} \circ \mathcal{T}^{(i)}_{2\ell} = p_+ K_{\ell,a} \widetilde{\mathbb{C}}^{i,i-1}_{\frac{n}{2}-\ell, a+\ell+\frac{n}{2}}$.

(2) ${\mathcal{T}'}^{(i-1)}_{2\ell} \circ \widetilde{\mathbb{C}}^{i,i-1}_{\frac{n-1}{2}-a-\ell, \frac{n-1}{2}-\ell} = r K_{\ell,a} \widetilde{\mathbb{C}}^{i,i-1}_{\frac{n-1}{2}-a-\ell, \frac{n-1}{2}+\ell}$.

In the case where $i = 0$, $\widetilde{\mathbb{C}}^{i,i}_{\lambda,\nu}$ is a scalar-valued operator, and the corresponding factorization identities in Theorem 5.1 were studied in [4, 8, 9].

The main results are proved by using the F-method [5, 6, 9]. Details will appear elsewhere.

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