Global null-controllability and nonnegative-controllability of slightly superlinear heat equations

Kévin Le Balc'h

GLOBAL NULL-CONTROLLABILITY AND NONNEGATIVE-CONTROLLABILITY OF SLIGHTLY SUPERLINEAR HEAT EQUATIONS

KÉVIN LE BALC'H

Abstract. We consider the semilinear heat equation posed on a smooth bounded domain Ω of $\mathbb{R}^N$ with Dirichlet or Neumann boundary conditions. The control input is a source term localized in some arbitrary nonempty open subset ω of Ω. The goal of this paper is to prove the uniform large time global null-controllability for semilinearities $f(s) = \pm|s|\log^\alpha(2+|s|)$ where $\alpha \in [3/2, 2)$ which is the case left open by Enrique Fernandez-Cara and Enrique Zuazua in 2000. It is worth mentioning that the free solution (without control) can blow-up. First, we establish the small-time global *nonnegative-controllability* (respectively *nonpositive-controllability*) of the system, i.e., one can steer any initial data to a nonnegative (respectively nonpositive) state in arbitrary time. In particular, one can act locally thanks to the control term in order to prevent the blow-up from happening. The proof relies on precise observability estimates for the linear heat equation with a bounded potential $a(t,x)$. More precisely, we show that observability holds with a sharp constant of the order $\exp\left(C\|a\|_\infty^{1/2}\right)$ for *nonnegative* initial data. This inequality comes from a new L^1 Carleman estimate. A Kakutani-Leray-Schauder's fixed point argument enables to go back to the semilinear heat equation. Secondly, the uniform large time null-controllability result comes from three ingredients: the global nonnegative-controllability, a comparison principle between the free solution and the solution to the underlying ordinary differential equation which provides the convergence of the free solution toward 0 in $L^\infty(\Omega)$-norm, and the local null-controllability of the semilinear heat equation.

Contents

1. Introduction

Let $T > 0$, $N \in \mathbb{N}^*$, Ω be a bounded, connected, open subset of $\mathbb{R}^N$ of class C^2 and n be the outer unit normal vector to $\partial\Omega$. We consider the semilinear heat equation with Neumann boundary conditions:

$$\begin{cases} \partial_t y - \Delta y + f(y) = h1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $f \in C^1(\mathbb{R};\mathbb{R})$.

Remark 1.1. All our results stay valid for Dirichlet boundary conditions (see Section 7).

In (1), $y = y(t,.) : \Omega \to \mathbb{R}$ is the *state* to be controlled and $h = h(t,.) : \Omega \to \mathbb{R}$ is the *control input* supported in ω, a nonempty open subset of Ω.

We assume that f satisfies

$$f(0) = 0. \tag{2}$$

In this case, $y = 0$ solves (1) with $y_0 = 0$ and $h = 0$.

In the following, we will also assume that f satisfies the restrictive growth condition

$$\exists \alpha > 0,\ \frac{f(s)}{|s|\log^\alpha(1+|s|)} \to 0 \text{ as } |s| \to +\infty. \tag{3}$$

Under the hypothesis (3), blow-up may occur if $h = 0$ in (1). Take for example $f(s) = -|s|\log^\alpha(1+|s|)$ with $\alpha > 1$. The mathematical theory of blow-up for

$$\begin{cases} \partial_t y - \Delta y = |y|\log^\alpha(1+|y|) & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{4}$$

was established in [24] and [25]. It was shown that blow-up

- occurs *globally* in the whole domain Ω if $\alpha < 2$,
- is of *pointwise nature* if $\alpha > 2$,
- is '*regional*', i.e., it occurs in an open subset of Ω if $\alpha = 2$.

See [26, Section 2 and Section 5] for a survey on this problem.

The goal of this paper is to analyze the *null-controllability* properties of (1).

Let us define $Q_T := (0,T)\times\Omega$. We recall two classical definitions of null-controllability.

Definition 1.2. Let $T > 0$. The system (1) is

- *globally null-controllable* in time T if for every $y_0 \in L^\infty(\Omega)$, there exists $h \in L^\infty(Q_T)$ such that the solution y of (1) satisfies $y(T,.) = 0$.
- *locally null-controllable* in time T if there exists $\delta_T > 0$ such that for every $y_0 \in L^\infty(\Omega)$ verifying $\|y_0\|_{L^\infty(\Omega)} \le \delta_T$, there exists $h \in L^\infty(Q_T)$ such that the solution y of (1) satisfies $y(T,.) = 0$.

We have the following well-known local null-controllability result.

Theorem 1.3. *For every $T > 0$, (1) is locally null-controllable in time T.*

The proof of Theorem 1.3 is a consequence of the (global) null-controllability of the linear heat equation with a bounded potential (due to Andrei Fursikov and Oleg Imanuvilov, see [23] or [21, Theorem 1.5]) and the *small L^∞ perturbations method* (see [3, Lemma 6] and [1], [5], [30], [33], [40] for other results in this direction).

The following global null-controllability (positive) result has been proved independently by Enrique Fernandez-Cara, Enrique Zuazua (see [22, Theorem 1.2]) and Viorel Barbu under a sign condition (see [4, Theorem 2] or [6, Theorem 3.6]) for Dirichlet boundary conditions. It has been extended to semilinearities which can depend on the gradient of the state and to Robin boundary conditions (then to Neumann boundary conditions) by Enrique Fernandez-Cara, Manuel Gonzalez-Burgos, Sergio Guerrero and Jean-Pierre Puel in [19] (see also [13] for the Dirichlet case).

Theorem 1.4. [19, Theorem 1]
We assume that (3) *holds for* $\alpha \leq 3/2$. *Then, for every* $T > 0$, (1) *is globally null-controllable in time* T.

Remark 1.5. Historically, the first global null-controllability (positive) result for (1) with f satisfying (3) was proved by Enrique Fernandez-Cara in [17] for $\alpha \leq 1$ and for Dirichlet boundary conditions.

The following global null-controllability (negative) result has been proved by Enrique Fernandez-Cara, Enrique Zuazua (see [22]).

Theorem 1.6. [22, Theorem 1.1]
We set $f(s) := \int_0^{|s|} \log^p(1+\sigma)d\sigma$ *with* $p > 2$ *and we assume that* $\Omega \setminus \overline{\omega} \neq \emptyset$. *Then, for every* $T > 0$, *there exists an initial datum* $y_0 \in L^\infty(\Omega)$ *such that for every* $h \in L^\infty(Q_T)$, *the maximal solution* y *of* (1) *blows-up in time* $T^* < T$.

Remark 1.7. Such a function f does satisfy (3) for any $\alpha > p$ because $|f(s)| \sim |s|\log^p(1+|s|)$ as $|s| \to +\infty$. Then, Theorem 1.6 shows that (1) can fail to be null-controllable for every $T > 0$ under the hypothesis (3) with $\alpha > 2$. Theorem 1.6 comes from a localized estimate in $\Omega \setminus \overline{\omega}$ that shows that the control cannot compensate the blow-up phenomena occurring in $\Omega \setminus \overline{\omega}$ (see [22, Section 2]).

When the nonlinear term f is dissipative, i.e., $sf(s) \geq 0$ for every $s \in \mathbb{R}$, then blow-up cannot occur. Furthermore, such a nonlinearity produces energy decay for the uncontrolled equation, therefore naively one may be led to believe that it can help in steering the solution to zero in arbitrary short time. The results of Sebastian Anita and Daniel Tataru show that this is false, more precisely that for 'strongly' superlinear f one needs a sufficiently large time in order to bring the solution to zero. An intuitive explanation for this is that the nonlinearity is also damping the effect of the control as it expands from the controlled region into the uncontrolled region (see [3]).

Theorem 1.8. [3, Theorem 3]
We set $f(s) := s\log^p(1+|s|)$ *with* $p > 2$ *and we assume that* $\Omega \setminus \overline{\omega} \neq \emptyset$. *Then, there exist* $x_0 \in \Omega \setminus \overline{\omega}$, $T_0 \in (0,1)$ *such that for every* $T \in (0,T_0)$, $h \in L^\infty(Q_T)$, *there exists* $y_0 \in L^\infty(\Omega)$ *such that the solution* y *to* (1) *satisfies* $y(T,x_0) < 0$.

Remark 1.9. In particular, for such a f as in Theorem 1.8, (1) is not globally null-controllable in small time T. Theorem 1.8 is due to pointwise upper bounds on the solution y of (1) which are independent of the control h (see [3, Section 3]).

2. Main results

2.1. **Small-time global nonnegative-controllability.** We introduce a new concept of controllability.

Definition 2.1. Let $T > 0$. The system (1) is *globally nonnegative-controllable* (respectively *globally nonpositive-controllable*) in time T if for every $y_0 \in L^\infty(\Omega)$, there exists $h \in L^\infty(Q_T)$ such that the solution y of (1) satisfies

$$y(T,.) \geq 0 \qquad \text{(respectively } y(T,.) \leq 0\text{)}. \tag{5}$$

The first main result of this paper is a small-time global nonnegative-controllability result for (1).

Theorem 2.2. *We assume that* (3) *holds for* $\alpha \leq 2$ *and* $f(s) \geq 0$ *for* $s \geq 0$ *(respectively* $f(s) \leq 0$ *for* $s \leq 0$*). Then, for every* $T > 0$, (1) *is globally nonnegative-controllable (respectively globally nonpositive-controllable) in time* T.

Remark 2.3. Theorem 2.2 is almost sharp because it does not hold for $\alpha > 2$ according to Theorem 1.8. The case where $|f(s)| \sim |s| \log^2(1 + |s|)$ as $|s| \to +\infty$ is open.

Remark 2.4. Theorem 2.2 does not treat the case $f(s) = -s \log^p(1+|s|)$ with $p < 2$ because of the sign condition.

2.2. **Large time global null-controllability.** The second main result of this paper is the following one.

Theorem 2.5. *We assume that* (3) *holds for* $\alpha \leq 2$, $f(s) > 0$ *for* $s > 0$ *or* $f(s) < 0$ *for* $s < 0$ *and* $1/f \in L^1([1, +\infty))$. *Then, there exists* T *sufficiently large such that* (1) *is globally null-controllable in time* T.

Remark 2.6. Theorem 2.5 proves that Theorem 1.6 is almost sharp. Indeed, let us take $f(s) = \int_0^{|s|} \log^p(1 + \sigma) d\sigma$ with $p < 2$, then by Theorem 2.5, there exists T sufficiently large such that (1) is globally null-controllable in time T. In particular, one can find a localized control which prevents the blow-up from happening. The case $f(s) = \int_0^{|s|} \log^2(1 + \sigma) d\sigma$ is open.

Remark 2.7. Theorem 2.5 does not treat the case $f(s) = -s \log^p(1+|s|)$ with $p < 2$ because of the sign condition.

Remark 2.8. The small-time global null-controllability of (1) remains open when (3) holds for $3/2 < \alpha \leq 2$.

2.3. **Proof strategy of the small-time global nonnegative-controllability.** We will only prove the global nonnegative-controllability result. The nonpositive-controllability result is an easy adaptation.

The proof strategy of Theorem 2.2 will follow Enrique Fernandez-Cara and Enrique Zuazua's proof of Theorem 1.4 (see [22]).

The starting point is to get some precise *observability estimates* for the linear heat equation with a bounded potential $a(t, x)$ for *nonnegative initial data*. More precisely, we show that observability holds with a sharp constant of the order $\exp\left(C \|a\|_\infty^{1/2}\right)$ for nonnegative initial data (see Theorem 4.4 below). This is done thanks to a *new Carleman estimate in* L^1 (see Theorem 4.9 below). This leads to a nonnegative-controllability result in L^∞ in the linear case with an estimate of the *control cost* of the order $\exp\left(C \|a\|_\infty^{1/2}\right)$ which is the key point of the proof (see Theorem 4.1 below).

We end the proof of Theorem 2.2 by a Kakutani-Leray-Schauder's fixed-point strategy. The idea of taking short control times to avoid blow-up phenomena is the same as in [22] and references therein. More precisely, the construction of the control follows two steps. The first step consists in steering the solution y of (1) to $y(T^*, .) \geq 0$ in time $T^* \leq T$ with an appropriate choice of the control. Then, the two conditions: $f(0) = 0$ and the dissipativity of f in $\mathbb{R}^+$ imply that the free solution y of (1) (with $h = 0$) defined in (T^*, T) stays nonnegative and bounded by using a comparison principle (see Section 5).

2.4. **Proof strategy of the large time global null-controllability.** We will only treat the case where $f(s) > 0$ for $s > 0$. The other case, i.e., $f(s) < 0$ for $s < 0$ is an easy adaptation.

The proof strategy of Theorem 2.5 is divided into three steps.

First, for every initial data $y_0 \in L^\infty(\Omega)$, one can steer the solution y of (1) in time $T_1 := 1$ (for instance) to a nonnegative state by using Theorem 2.2.

Secondly, we let evolve the system without control and we remark that

$$\forall (t,x) \in [T_1, +\infty) \times \Omega,\ 0 \leq y(t,x) \leq G(t),$$

with G independent of $\|y(T_1,.)\|_{L^\infty(\Omega)}$ and $G(t) \to 0$ when $t \to +\infty$. This kind of argument has already been used by Jean-Michel Coron in the context of the Burgers equation (see [10, Theorem 8]).

Finally, by using the second step, for T_2 sufficiently large, $y(T_2,.)$ belongs to a small ball of $L^\infty(\Omega)$ centered at 0, where the local null-controllability holds (see Theorem 1.3). Then, one can steer $y(T_2,.)$ to 0 with an appropriate choice of the control.

3. Parabolic equations: Well-posedness and regularity

The goal of this section is to state well-posedness results, dissipativity in time in L^p-norm, maximum principle and L^p-L^q estimates for linear parabolic equations. We also give the definition of a solution to the semilinear heat equation (1). The references of these results only treat the case of Dirichlet boundary conditions but the proofs can be easily adapted to Neumann boundary conditions.

3.1. **Well-posedness.** We introduce the functional space

$$W_T := L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{6}$$

which satisfies the following embedding (see [15, Section 5.9.2, Theorem 3])

$$W_T \hookrightarrow C([0,T];L^2(\Omega)). \tag{7}$$

3.1.1. *Linear parabolic equations.*

Definition 3.1. Let $a \in L^\infty(Q_T)$, $F \in L^2(Q_T)$ and $y_0 \in L^2(\Omega)$. A function $y \in W_T$ is a solution to

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = F & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{8}$$

if for every $w \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T (\partial_t y, w)_{((H^1(\Omega))',H^1(\Omega))} + \int_{Q_T} \nabla y.\nabla w + \int_{Q_T} ayw = \int_{Q_T} Fw, \tag{9}$$

and

$$y(0,.) = y_0 \text{ in } L^2(\Omega). \tag{10}$$

The following well-posedness result in L^2 holds for linear parabolic equations.

Proposition 3.2. *Let $a \in L^\infty(Q_T)$, $F \in L^2(Q_T)$ and $y_0 \in L^2(\Omega)$. The Cauchy problem (8) admits a unique weak solution $y \in W_T$. Moreover, there exists $C = C(\Omega) > 0$ such that*

$$\|y\|_{W_T} \leq C \exp\left(CT\|a\|_{L^\infty(Q_T)}\right)\left(\|y_0\|_{L^2(\Omega)} + \|F\|_{L^2(Q_T)}\right). \tag{11}$$

The proof of Proposition 3.2 is based on *Galerkin approximations, energy estimates and Gronwall's argument* (see [15, Section 7.1.2]).

We also have the following classical L^∞-estimate for (8).

Proposition 3.3. *Let $a \in L^\infty(Q_T)$, $F \in L^\infty(Q_T)$ and $y_0 \in L^\infty(\Omega)$. Then the solution y of (8) belongs to $L^\infty(Q_T)$ and there exists $C = C(\Omega) > 0$ such that*

$$\|y\|_{L^\infty(Q_T)} \leq C \exp\left(CT \|a\|_{L^\infty(Q_T)}\right) \left(\|y_0\|_{L^\infty(\Omega)} + \|F\|_{L^\infty(Q_T)}\right). \tag{12}$$

The proof of Proposition 3.3 is based on *Stampacchia's method* (see the proof of [28, Chapter 3, Paragraph 7, Theorem 7.1]).

Let us also mention the dissipativity in time of the L^p-norm of the heat equation with a bounded potential.

Proposition 3.4. *Let $a \in L^\infty(Q_T)$, $y_0 \in L^2(\Omega)$ and $t_1 < t_2 \in [0,T]$. Then, there exists $C = C(\Omega) > 0$ such that the solution $y \in W_T$ of (8) with $F = 0$, satisfies for every $p \in [1,2]$,*

$$\|y(t_2,.)\|_{L^p(\Omega)} \leq C \exp\left(CT \|a\|_{L^\infty(Q_T)}\right) \|y(t_1,.)\|_{L^p(\Omega)}. \tag{13}$$

The proof of Proposition 3.4 is based on the application of the variational formulation (9) with a cut-off of $w = |y|^{p-2}y$ and a Gronwall's argument.

3.1.2. *Nonlinear parabolic equations.* We give the definition of a solution of (1).

Definition 3.5. Let $y_0 \in L^\infty(\Omega)$, $h \in L^\infty(Q_T)$. A function $y \in W_T \cap L^\infty(Q_T)$ is the solution of (1) if for every $w \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T (\partial_t y, w)_{((H^1(\Omega))',H^1(\Omega))} + \int_{Q_T} \nabla y.\nabla w + \int_{Q_T} ayw = \int_{Q_T} (f(y) + h1_\omega)w, \tag{14}$$

and

$$y(0,.) = y_0 \text{ in } L^\infty(\Omega). \tag{15}$$

The uniqueness of a solution to (1) is an easy consequence of the fact that f is locally Lipschitz because $f \in C^1(\mathbb{R};\mathbb{R})$.

3.2. **Maximum principle.** We state the maximum principle for the heat equation.

Proposition 3.6. *Let $a \in L^\infty(Q_T)$, $F \leq G \in L^2(Q_T)$ and $y_0 \leq z_0 \in L^2(\Omega)$. Let y and z be the solutions to*

$$\left\{\begin{array}{l} \partial_t y - \Delta y + a(t,x)y = F, \\ \frac{\partial y}{\partial n} = 0, \\ y(0,.) = y_0, \end{array}\right. \quad \left\{\begin{array}{ll} \partial_t z - \Delta z + a(t,x)z = G & \text{in } (0,T)\times\Omega, \\ \frac{\partial z}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,.) = z_0 & \text{in } \Omega. \end{array}\right. \tag{16}$$

Then, we have the comparison principle

$$\forall t \in [0,T],\ \text{a.e. } x \in \Omega,\ y(t,x) \leq z(t,x). \tag{17}$$

The proof of Proposition 3.6 is based on the comparison principle for smooth solutions of (16) (see [41, Theorem 8.1.6]) and a regularization argument.

We state a comparison principle for the semilinear heat equation (1) without control h.

Proposition 3.7. *Let $y_0 \in L^\infty(\Omega)$, $h = 0$. We assume that there exist a subsolution $\underline{y}$ and a supersolution $\overline{y}$ in $L^\infty(Q_T)$ of (1), i.e., $\underline{y}$ (respectively $\overline{y}$) satisfies (14), (15) replacing the equality $=$ by the inequality $\leq$ (respectively by the inequality $\geq$). Moreover, we suppose that $\underline{y}$ and $\overline{y}$ are ordered in the following sense*

$$\forall t \in [0,T],\ \text{a.e. } x \in \Omega,\ \underline{y}(t,x) \leq \overline{y}(t,x).$$

Then, there exists a (unique) solution y of (1). Moreover, y satisfies the comparison principle

$$\forall t \in [0,T],\ \text{a.e. } x \in \Omega,\ \underline{y}(t,x) \leq y(t,x) \leq \overline{y}(t,x). \tag{18}$$

For the proof of Proposition 3.7, see [41, Corollary 12.1.1].

3.3. L^p-L^q **estimates.** We have the well-known regularizing effect of the heat semigroup.

Proposition 3.8. [8, Proposition 3.5.7]
Let $1 \leq q \leq p \leq +\infty$, $y_0 \in L^2(\Omega)$ and y be the solution to (8) with $(a, F) = (0, 0)$. Then, there exists $C = C(\Omega, p, q) > 0$ such that for every $t_1 < t_2 \in (0, T)$, we have

$$\|y(t_2, .)\|_{L^p(\Omega)} \leq C(t_2 - t_1)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|y(t_1, .)\|_{L^q(\Omega)} \tag{19}$$

4. Global nonnegative-controllability of the linear heat equation with a bounded potential

4.1. **Statement of the result.** Let $a \in L^\infty(Q_T)$. We consider the heat equation with a bounded potential

$$\begin{cases} \partial_t y - \Delta y + a(t,x)y = h1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{20}$$

and the following adjoint equation

$$\begin{cases} -\partial_t q - \Delta q + a(t,x)q = 0 & \text{in } (0,T)\times\Omega, \\ \frac{\partial q}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ q(T,.) = q_T & \text{in } \Omega. \end{cases} \tag{21}$$

The goal of this section is to prove the following theorem.

Theorem 4.1. *For every $T > 0$, (20) is globally nonnegative-controllable in time T. More precisely, for every $T > 0$, there exists $C = C(\Omega, \omega, T, a) > 0$, with*

$$C(\Omega,\omega,T,a) = \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + T\|a\|_{L^\infty(Q_T)} + \|a\|^{1/2}_{L^\infty(Q_T)}\right)\right) \tag{22}$$

such that for every $y_0 \in L^2(\Omega)$, there exists $h \in L^\infty(Q_T)$ such that

$$\|h\|_{L^\infty(Q_T)} \leq C(\Omega,\omega,T,a)\|y_0\|_{L^2(\Omega)}, \tag{23}$$

and

$$y(T,.) \geq 0. \tag{24}$$

Remark 4.2. Actually, by looking carefully at the proof of Theorem 4.1 (see Section 4.5 below), we can see that the control h in Theorem 4.1 can be chosen constant in the time and the space variables.

Remark 4.3. It is well-known that (20) is globally nonnegative-controllable in time T because it is globally null-controllable in time T (see [20, Theorem 2]) but the most interesting point is the *cost of nonnegative-controllability* given in Theorem 4.1. In particular, the exponent $1/2$ of the term $\|a\|^{1/2}_{L^\infty(Q_T)}$ will be the key point to prove Theorem 2.2 (see Section 5).

4.2. **A precise L^2-L^1 observability inequality for the linear heat equation with bounded potential and nonnegative initial data.** The proof of Theorem 4.1 is a consequence of this kind of observability inequality.

Theorem 4.4. *For every $T > 0$, there exists $C = C(\Omega, \omega, T, a) > 0$ of the form (22) such that for every $q_T \in L^2(\Omega; \mathbb{R}^+)$, the solution q to (21) satisfies*

$$\|q(0,.)\|^2_{L^2(\Omega)} \leq C\left(\int_0^T \int_\omega q dx dt\right)^2. \tag{25}$$

An immediate corollary of Theorem 4.4 is this observability inequality L^2-L^2 that we state to discuss it below, but that will not be used in the present article.

Corollary 4.5. *For every $T > 0$, there exists $C = C(\Omega, \omega, T, a) > 0$ of the form (22) such that for every $q_T \in L^2(\Omega; \mathbb{R}^+)$ the solution q to (21) satisfies*

$$\|q(0,.)\|^2_{L^2(\Omega)} \leq C \left(\int_0^T \int_\omega q^2 dx dt \right). \tag{26}$$

It is well-known that null-controllability in L^2 is equivalent to an observability inequality in L^2 for every $q_T \in L^2(\Omega; \mathbb{R})$ (see [9, Theorem 2.44]). The *main idea* behind Corollary 4.5 is the fact that nonnegative-controllability in L^2 is a consequence of an observability inequality in L^2 for every $q_T \in L^2(\Omega; \mathbb{R}^+)$ (see Section 4.5).

Remark 4.6. It is interesting to mention that (26) holds with C of the form

$$C(\Omega, \omega, T, a) = \exp\left(C(\Omega, \omega) \left(1 + \frac{1}{T} + T \|a\|_{L^\infty(Q_T)} + \|a\|^{2/3}_{L^\infty(Q_T)} \right) \right) \tag{27}$$

for every $q_T \in L^2(\Omega; \mathbb{R})$ (see [20, Theorem 2]). The exponent $2/3$ of the term $\|a\|^{2/3}_{L^\infty(Q_T)}$ is the key point to prove Theorem 1.4. Note that the optimality of the exponent $2/3$ has been proved by Thomas Duyckaerts, Xu Zhang and Enrique Zuazua in the context of parabolic systems in even space dimensions $N \geq 2$ and with Dirichlet boundary conditions (see [14, Theorem 1.1] and also [44, Theorem 5.2] for the main arguments of the proof). Corollary 4.5 shows that we can actually decrease the exponent $2/3$ to the exponent $1/2$ for nonnegative initial data. In some sense, we can make the connection between the recent preprint of Camille Laurent and Matthieu Léautaud who disprove the Miller's conjecture about the short-time observability constant of the heat equation in the general case and show that the conjecture holds true for nonnegative initial data by using Li-Yau estimates (see [29] and [32]).

Remark 4.7. In the context of the wave equation in one space dimension, the (optimal) constant of observability inequality for the linear wave equation with a bounded potential is actually $\exp\left(C\left(1 + \|a\|^{1/2}_{L^\infty(Q_T)}\right)\right)$ (see [42, Theorem 4]) which leads to the exact controllability of the semilinear wave equation in large time for semilinearities satisfying (3) with $\alpha < 2$ (see [42, Theorem 1] and also [7, Problem 5.5] for the presentation of the related open problem in the multidimensional case). Roughly speaking, as an ordinary differential argument would indicate, this constant of observability inequality is very natural because the wave operator is of order two in the time and the space variables. Then, by analogy and by taking into account that the heat operator is of order one in the time variable and of order two in the space variable, one could rather expect a constant of obervability inequality of the order $\exp\left(C \|a\|_{L^\infty(Q_T)}\right)$ or $\exp\left(C \|a\|^{1/2}_{L^\infty(Q_T)}\right)$ which seem to be more intuitive than the term $\exp\left(C \|a\|^{2/3}_{L^\infty(Q_T)}\right)$.

4.3. **A new L^1 Carleman estimate.** The goal of this section is to establish a L^1 Carleman estimate for nonnegative initial data (see Theorem 4.9 below). First, we introduce some classical weight functions for proving Carleman inequalities.

Lemma 4.8. *Let $\omega_0 \subset\subset \omega$ be a nonempty open subset. Then there exists $\eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in Ω, $\eta^0 = 0$ in $\partial\Omega$, and $|\nabla \eta^0| > 0$ in $\overline{\Omega \setminus \omega_0}$.*

A proof of this lemma can be found in [9, Lemma 2.68].

Let ω_0 be a nonempty open set satisfying $\omega_0 \subset\subset \omega$ and let us set

$$\alpha(t,x) := \frac{e^{2\lambda \|\eta^0\|_\infty} - e^{\lambda \eta^0(x)}}{t(T-t)}, \tag{28}$$

$$\xi(t,x) := \frac{e^{\lambda \eta^0(x)}}{t(T-t)}, \tag{29}$$

for $(t,x) \in Q_T$, where η^0 is the function provided by Lemma 4.8 for this ω_0 and $\lambda \geq 1$ is a parameter.

We have the following new L^1 Carleman estimate.

Theorem 4.9. *There exist two constants $C := C(\Omega,\omega) > 0$ and $C_1 := C_1(\Omega,\omega) > 0$, such that,*

$$\forall \lambda \geq 1, \qquad \forall s \geq s_1(\lambda) := C(\Omega,\omega) e^{4\lambda \|\eta^0\|_\infty} \left(T + T^2 + T^2 \|a\|^{1/2}_{L^\infty(Q_T)} \right), \tag{30}$$

for every $q_T \in L^2(\Omega;\mathbb{R}^+)$, the nonnegative solution q of (21) satisfies

$$\int_{Q_T} e^{-s\alpha} \xi^2 q dx dt \leq C_1 \int_{(0,T)\times\omega} e^{-s\alpha} \xi^2 q dx dt. \tag{31}$$

Proof. Unless otherwise specified, we denote by C various positive constants varying from line to line which may depend on Ω, ω but independent of the parameters λ and s.

We introduce other weights which are similar to α and ξ

$$\widetilde{\alpha}(t,x) := \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{-\lambda\eta^0(x)}}{t(T-t)}, \tag{32}$$

$$\widetilde{\xi}(t,x) := \frac{e^{-\lambda\eta^0(x)}}{t(T-t)}. \tag{33}$$

The following estimates

$$\begin{array}{llll} |\partial_i \alpha| = |-\partial_i \xi| & \leq C\lambda\xi, & |\partial_i \widetilde{\alpha}| = |-\partial_i \widetilde{\xi}| & \leq C\lambda\widetilde{\xi}, \\ |\partial_t \alpha| & \leq 2T\xi^2 e^{2\lambda\|\eta^0\|_\infty}, & |\partial_t \widetilde{\alpha}| & \leq 2T\widetilde{\xi}^2 e^{4\lambda\|\eta^0\|_\infty}, \\ \xi(T/2)^2 & \geq 1, & \widetilde{\xi}(T/2)^2 & \geq e^{-\lambda\|\eta^0\|_\infty}, \end{array} \tag{34}$$

will be very useful for the proof.

Let $q_T \in C_0^\infty(\Omega;\mathbb{R}^+)$. The general case comes from an easy density argument by using the fact that $C_c^\infty(\Omega;\mathbb{R}^+)$ is dense in $L^2(\Omega;\mathbb{R}^+)$ for the $L^2(\Omega;\mathbb{R})$ topology.

The solution q of (21) is nonnegative by applying the maximum principle given in Proposition 3.6 with $y = 0$ and $z(t,x) = q(t-T,x)$.

We define

$$\psi := e^{-s\alpha} q \qquad \text{and} \qquad \widetilde{\psi} := e^{-s\widetilde{\alpha}} q.$$

The proof is divided into five steps:

- **Step 1:** We integrate over $(0,T)\times\Omega$ an identity satisfied by ψ.
- **Step 2:** We get an estimate which looks like to (31) up to some boundary terms.
- **Step 3:** We repeat the step 1 for $\widetilde{\psi}$.
- **Step 4:** We repeat the step 2 for $\widetilde{\psi}$.
- **Step 5:** We sum the estimates of the step 2 and the step 4 to get rid of the boundary terms.

Remark 4.10. The 'trick' of the proof to get rid of the boundary terms is inspired by the proof of the usual L^2 Carleman estimate for Neumann boundary conditions due to Andrei Fursikov and Oleg Imanuvilov (see [23, Chapter 1] and also [20, Appendix]).

Step 1: An identity satisfied by ψ. We readily obtain that

$$M\psi = 0, \tag{35}$$

where

$$\begin{aligned} M\psi = &-s\lambda^2 |\nabla\eta^0|^2 \xi\psi - 2s\lambda\xi\nabla\eta^0.\nabla\psi + \partial_t\psi \\ &+ s^2\lambda^2 |\nabla\eta^0|^2 \xi^2\psi + \Delta\psi + s\alpha_t\psi - a(t,x)\psi \\ &- s\lambda\Delta\eta^0\xi\psi. \end{aligned} \tag{36}$$

Remark 4.11. The starting point, i.e., the identity (35) is the same as in the classical proof developed by Andrei Fursikov and Oleg Imanuvilov in [23] (see also [21, Proof of Lemma 1.3] or [31, Section 7]). But, from now, the proof strategy of the L^1-Carleman estimate is very different from the usual one of the L^2-Carleman estimate. Indeed, we will focus on the fourth right hand side term of (36)

$$s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi.$$

It is nonnegative because ψ is nonnegative and it is of order two in the parameter s whereas the seventh right hand side term of (36)

$$a(t,x)\psi,$$

is of order 0 in the parameter s. This comparison suggests to integrate the identity (35) in order to obtain (31) for $\lambda \geq 1$ and $s \geq s_1(\lambda)$ as defined in (30).

We integrate (35) over $(0,T)\times\Omega$

$$\begin{aligned}
&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi - \int_{Q_T} 2s\lambda\xi\nabla\eta^0.\nabla\psi + \int_{Q_T}\partial_t\psi + \int_{Q_T}\Delta\psi \\
&= \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi - \int_{Q_T} s\alpha_t\psi + \int_{Q_T} a(t,x)\psi \\
&\quad + \int_{Q_T} s\lambda\Delta\eta^0\xi\psi.
\end{aligned} \tag{37}$$

Note that all the terms in (37) are well-defined. Indeed, by using $q_T \in C_c^\infty(\Omega)$ and the parabolic regularity in L^2 to (21) (see [12, Theorem 2.1]), we deduce that $q \in X_2 := L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ then $\psi \in X_2$.

Step 2: Estimates for ψ. As a consequence of the properties of η^0 (see Lemma 4.8), we have

$$m := \min\left\{|\nabla\eta^0(x)|^2 \ ; \ x \in \overline{\Omega\setminus\omega_0}\right\} > 0, \tag{38}$$

which yields

$$\begin{aligned}
&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi \\
&\geq \int_{(0,T)\times(\Omega\setminus\omega)} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi \geq m\int_{Q_T} s^2\lambda^2\xi^2\psi - m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi.
\end{aligned} \tag{39}$$

By combining (37) and (39), we have

$$\begin{aligned}
&m\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{Q_T} 2s\lambda\xi\nabla\eta^0.\nabla\psi + \int_{Q_T}\partial_t\psi + \int_{Q_T}\Delta\psi \\
&\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi + \int_{Q_T} s|\alpha_t|\psi + \int_{Q_T}|a(t,x)|\psi \\
&\quad + \int_{Q_T} s\lambda|\Delta\eta^0|\xi\psi + m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi.
\end{aligned} \tag{40}$$

We have the following integration by parts

$$-\int_{Q_T} 2s\lambda\xi\nabla\eta^0.\nabla\psi = \int_{Q_T} 2s\lambda\left(\nabla\xi.\nabla\eta^0\psi + \xi\Delta\eta^0\psi\right) - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi d\sigma dt, \tag{41}$$

$$\int_{Q_T}\partial_t\psi = \int_\Omega(\psi(T,.) - \psi(0,.)) = 0, \tag{42}$$

$$\int_{Q_T}\Delta\psi = \int_{\Sigma_T}\frac{\partial\psi}{\partial n}, \tag{43}$$

where $\Sigma_T := (0,T)\times\partial\Omega$.

From (40), (41), (42), (43), we have

$$\begin{aligned} & m\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n} \\ \leq & \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi + \int_{Q_T} s|\alpha_t|\psi + \int_{Q_T}|a(t,x)|\psi \\ & + \int_{Q_T} 3s\lambda|\Delta\eta^0|\xi\psi + \int_{Q_T} 2s\lambda|\nabla\xi||\nabla\eta^0|\psi + m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi. \end{aligned} \tag{44}$$

By using the first two lines of (34) and $\lambda\geq 1$, we have

$$\begin{aligned} & \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi + \int_{Q_T} s|\alpha_t|\psi + \int_{Q_T}|a(t,x)|\psi \\ & + \int_{Q_T} 3s\lambda|\Delta\eta^0|\xi\psi + \int_{Q_T} 2s\lambda|\nabla\xi||\nabla\eta^0|\psi \\ & \leq C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T}|a(t,x)|\psi + \int_{Q_T} s\lambda\xi\psi\right) \\ & \leq C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T}|a(t,x)|\psi\right). \end{aligned} \tag{45}$$

By combining (44) and (45), we get

$$\begin{aligned} & m\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n} \\ \leq & \ \leq C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T}|a(t,x)|\psi\right). \end{aligned} \tag{46}$$

Absorption. The goal of this intermediate step is to absorb the right hand side of (46) by the first left hand side term of (46) by taking s sufficiently large. In order to do this, it is useful to keep in mind the fact that $\lambda\geq 1$ and the third line of (34) for the next estimates.

By taking $s\geq (T/2)^2(4C/m)$, we have $Cs\xi\leq (m/4)(s\xi)^2$ and consequently

$$C\int_{Q_T} s\lambda^2\xi\psi \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{47}$$

By taking $s\geq Te^{2\lambda\|\eta^0\|_\infty}(4C/m)$, we have $Cse^{2\lambda\|\eta^0\|_\infty}T\xi^2\leq (m/4)(\lambda s\xi)^2$ and consequently

$$C\int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{48}$$

By taking $s\geq (T/2)^2\|a\|^{1/2}_{L^\infty(Q_T)}(4C/m)^{1/2}$, we have $C\|a\|_{L^\infty(Q_T)}\leq (m/4)(\lambda s\xi)^2$ and consequently

$$C\int_{Q_T}|a(t,x)|\psi \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{49}$$

Therefore, by taking $s\geq s_1(\lambda)$ as defined in (30), we have from (47), (48) and (49) that

$$C\left(\int_{Q_T} s\lambda^2\xi\psi + \int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\xi^2\psi + \int_{Q_T}|a(t,x)|\psi\right) \leq \frac{3m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi. \tag{50}$$

Then, from (46) and (50), for $s\geq s_1(\lambda)$, we get

$$\frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n} \leq m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi. \tag{51}$$

Step 3: An identity satisfied by $\widetilde{\psi}$. We readily obtain that

$$\widetilde{M}\widetilde{\psi} = 0, \tag{52}$$

where

$$\widetilde{M}\widetilde{\psi} = -s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} + \partial_t\widetilde{\psi} + s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi} + \Delta\widetilde{\psi} + s\widetilde{\alpha}_t\widetilde{\psi} - a(t,x)\widetilde{\psi} + s\lambda\Delta\eta^0\widetilde{\xi}\widetilde{\psi}. \tag{53}$$

We integrate (35) over $(0,T)\times\Omega$

$$\begin{aligned}&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} + \int_{Q_T}\partial_t\widetilde{\psi} + \int_{Q_T}\Delta\widetilde{\psi}\\ &= \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} - \int_{Q_T} s\widetilde{\alpha}_t\widetilde{\psi} + \int_{Q_T} a(t,x)\widetilde{\psi}\\ &\quad - \int_{Q_T} s\lambda\Delta\eta^0\widetilde{\xi}\widetilde{\psi}.\end{aligned} \tag{54}$$

Step 4: Estimates for $\widetilde{\psi}$. By using (38), we have

$$\begin{aligned}&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi}\\ &\geq \int_{(0,T)\times(\Omega\setminus\omega)} s^2\lambda^2|\nabla\eta^0|^2\widetilde{\xi}^2\widetilde{\psi} \geq m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} - m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}.\end{aligned} \tag{55}$$

By combining (54) and (55), we have

$$\begin{aligned}&m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} + \int_{Q_T}\partial_t\widetilde{\psi} + \int_{Q_T}\Delta\widetilde{\psi}\\ &\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} s|\widetilde{\alpha}_t|\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi}\\ &\quad + \int_{Q_T} s\lambda|\Delta\eta^0|\widetilde{\xi}\widetilde{\psi} + m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}.\end{aligned} \tag{56}$$

We have the following integration by parts

$$\int_{Q_T} 2s\lambda\widetilde{\xi}\nabla\eta^0.\nabla\widetilde{\psi} = -\int_{Q_T} 2s\lambda\left(\nabla\widetilde{\xi}.\nabla\eta^0\widetilde{\psi} + \widetilde{\xi}\Delta\eta^0\widetilde{\psi}\right) + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi}, \tag{57}$$

$$\int_{Q_T}\partial_t\widetilde{\psi} = \int_\Omega(\widetilde{\psi}(T,.) - \widetilde{\psi}(0,.)) = 0, \tag{58}$$

$$\int_{Q_T}\Delta\widetilde{\psi} = \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n}. \tag{59}$$

From (56), (57), (58), (59), we have

$$\begin{aligned}&m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n}\\ &\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} s|\widetilde{\alpha}_t|\widetilde{\psi} + \int_{Q_T}|a(t,x)|\widetilde{\psi}\\ &\quad + \int_{Q_T} 3s\lambda|\Delta\eta^0|\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} 2s\lambda|\nabla\widetilde{\xi}||\nabla\eta^0|\widetilde{\psi} + m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}.\end{aligned} \tag{60}$$

By using the first two lines of (34) and the fact that $\lambda \geq 1$, we have

$$\begin{aligned}
&\int_{Q_T} s\lambda^2|\nabla\eta^0|^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} s|\widetilde{\alpha}_t|\widetilde{\psi} + \int_{Q_T} |a(t,x)|\widetilde{\psi} \\
&+ \int_{Q_T} 3s\lambda|\Delta\eta^0|\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} 2s\lambda|\nabla\widetilde{\xi}||\nabla\eta^0|\widetilde{\psi} \\
&\leq C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} |a(t,x)|\widetilde{\psi} + \int_{Q_T} s\lambda\widetilde{\xi}\widetilde{\psi}\right) \\
&\leq C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} |a(t,x)|\widetilde{\psi}\right)
\end{aligned} \tag{61}$$

By combining (60) and (61), we get

$$\begin{aligned}
&m\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n} \\
&\leq C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} |a(t,x)|\widetilde{\psi}\right)
\end{aligned} \tag{62}$$

Absorption. Note that we will use the third line of (34) in the next four estimates.

By taking $s \geq e^{\lambda\|\eta^0\|_\infty}(T/2)^2(4C/m)$, we have $Cs\widetilde{\xi} \leq (m/4)(s\widetilde{\xi})^2$ and consequently

$$C\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{63}$$

By taking $s \geq Te^{4\lambda\|\eta^0\|_\infty}(4C/m)$, we have $Cse^{2\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2 \leq (m/4)(\lambda s\widetilde{\xi})^2$ and consequently

$$C\int_{Q_T} se^{2\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{64}$$

By taking $s \geq e^{\lambda\|\eta^0\|_\infty}(T/2)^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}(4C/m)^{1/2}$, we have $C\left\|a\right\|_{L^\infty(Q_T)} \leq (m/4)(\lambda s\widetilde{\xi})^2$ and consequently

$$C\int_{Q_T} |a(t,x)|\widetilde{\psi} \leq \frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{65}$$

Therefore, by taking $s \geq s_1(\lambda)$ as defined in (30), we have from (47), (48) and (65) that

$$C\left(\int_{Q_T} s\lambda^2\widetilde{\xi}\widetilde{\psi} + \int_{Q_T} se^{4\lambda\|\eta^0\|_\infty}T\widetilde{\xi}^2\widetilde{\psi} + \int_{Q_T} |a(t,x)|\widetilde{\psi}\right) \leq \frac{3m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{66}$$

Then, from (62) and (66), for $s \geq s_1(\lambda)$, we get

$$\frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n} \leq m\int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}. \tag{67}$$

Step 5: Elimination of the boundary terms. From now, we take $s \geq s_1(\lambda)$. By summing (51) and (67), we get

$$\begin{aligned}
&\frac{m}{4}\int_{Q_T} s^2\lambda^2\xi^2\psi - \int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi + \int_{\Sigma_T}\frac{\partial\psi}{\partial n} \\
&+\frac{m}{4}\int_{Q_T} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi} + \int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi} + \int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n} \\
&\leq m\left(\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi + \int_{(0,T)\times\omega} s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}\right).
\end{aligned} \tag{68}$$

Since $\eta^0 = 0$ on $\partial\Omega$, we have

$$\xi = \widetilde{\xi},\ \alpha = \widetilde{\alpha} \text{ and } \psi = \widetilde{\psi} \qquad \text{on } \Sigma_T,$$

which leads to

$$-\int_{\Sigma_T} 2s\lambda\xi\frac{\partial\eta^0}{\partial n}\psi+\int_{\Sigma_T} 2s\lambda\widetilde{\xi}\frac{\partial\eta^0}{\partial n}\widetilde{\psi}=0. \tag{69}$$

Moreover, we have

$$\partial_i\psi=e^{-s\alpha}(\partial_i q+s\lambda\partial_i\eta^0\xi q),\ \partial_i\widetilde{\psi}=e^{-s\widetilde{\alpha}}(\partial_i q-s\lambda\partial_i\eta^0\widetilde{\xi}q),$$

whence by using $\frac{\partial q}{\partial n}=0$ on Σ_T, we get

$$\frac{\partial\psi}{\partial n}=s\lambda\frac{\partial\eta^0}{\partial n}\xi e^{-s\alpha}q,\ \frac{\partial\widetilde{\psi}}{\partial n}=-s\lambda\frac{\partial\eta^0}{\partial n}\widetilde{\xi}e^{-s\widetilde{\alpha}}q\qquad\text{on }\Sigma_T.$$

This leads to

$$\int_{\Sigma_T}\frac{\partial\psi}{\partial n}+\int_{\Sigma_T}\frac{\partial\widetilde{\psi}}{\partial n}=0. \tag{70}$$

We get from (68), (69) and (70)

$$\begin{aligned}&\frac{m}{4}\left(\int_{Q_T}s^2\lambda^2\xi^2\psi+\int_{Q_T}s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}\right)\\&\qquad\le C\left(\int_{(0,T)\times\omega}s^2\lambda^2\xi^2\psi+\int_{(0,T)\times\omega}s^2\lambda^2\widetilde{\xi}^2\widetilde{\psi}\right).\end{aligned} \tag{71}$$

By using the fact that $\widetilde{\xi}\le\xi$, $e^{-s\widetilde{\alpha}}\le e^{-s\alpha}$ in Q_T, we get from (71) the Carleman estimate (31). This concludes the proof of Theorem 4.9. □

4.4. **Proof of the L^2-L^1 observability inequality: Theorem 4.4.** The goal of this subsection is to prove Theorem 4.4, which is a consequence of Theorem 4.9, L^p-L^q estimates and the dissipativity in time of the L^p norm of (21).

Proof. **Step 1: L^1-L^1 observability inequality.** We fix $\lambda=1$ and $s=s_1$ in Theorem 4.9 to get

$$\int_{Q_T}t^{-2}(T-t)^{-2}e^{-s\alpha}qdxdt\le C_1(\Omega,\omega)\int_{(0,T)\times\omega}t^{-2}(T-t)^{-2}e^{-s\alpha}qdxdt. \tag{72}$$

First, we observe that in $(T/4,3T/4)\times\Omega$,

$$\begin{aligned}t^{-2}(T-t)^{-2}e^{-s\alpha}&\ge\frac{C}{T^4}\exp\left(-\frac{C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)}{T^2}\right)\\&\ge\frac{C}{T^4}e^{-C(\Omega,\omega)\left(1+\frac{1}{T}+\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)}.\end{aligned} \tag{73}$$

Secondly, from the fact that $x^2e^{-Mx}\le C/M^2$ for every $x,M\ge0$ used with $x=t^{-1}(T-t)^{-1}$ and $M=C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)$, we remark that in $(0,T)\times\omega$,

$$\begin{aligned}&t^{-2}(T-t)^{-2}e^{-s\alpha}\\&\le t^{-2}(T-t)^{-2}\exp\left(-C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)t^{-1}(T-t)^{-1}\right)\\&\le\frac{C}{\left(C(\Omega,\omega)\left(T+T^2+T^2\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)\right)^2}\\&\le\frac{C(\Omega,\omega)}{T^4}.\end{aligned} \tag{74}$$

Then, we get from (72), (73) and (74)

$$\int_{(T/4,3T/4)\times\Omega}qdxdt\le e^{C(\Omega,\omega)\left(1+\frac{1}{T}+\left\|a\right\|_{L^\infty(Q_T)}^{1/2}\right)}\int_{(0,T)\times\omega}qdxdt. \tag{75}$$

On the other hand, we obtain by the dissipativity in time of the L^1-norm (see Proposition 3.4 with $p=1$)

$$\|q(T/4,.)\|_{L^1(\Omega)} \leq \frac{2C\exp\left(CT\|a\|_{L^\infty(Q_T)}\right)}{T}\int_{T/4}^{3T/4}\|q(t,.)\|_{L^1(\Omega)}\,dt. \tag{76}$$

By using (75) and (76), we get

$$\|q(T/4,.)\|_{L^1(\Omega)} \leq C(\Omega,\omega,T,a)\int_{(0,T)\times\omega} q dxdt, \tag{77}$$

where $C(\Omega,\omega,T,a)$ is defined in (22).

From now, we denote by $C(\Omega,\omega,T,a)$ various positive constants varying from line to line which are of the form (22).

Step 2: Global L^2-L^1 estimate. The goal of this step is to prove that

$$\|q(0,.)\|_{L^2(\Omega)} \leq C(\Omega,\omega,T,a)\,\|q(T/4,.)\|_{L^1(\Omega)}\,. \tag{78}$$

To simplify the notations, we set $\widehat{q}(t) := q(T-t)$ for $t\in[0,T]$. Then, (78) rewrites as follows

$$\left\|\widehat{q}(\widehat{T_2},.)\right\|_{L^2(\Omega)} \leq C(\Omega,\omega,T,a)\left\|\widehat{q}(\widehat{T_1},.)\right\|_{L^1(\Omega)}\,. \tag{79}$$

with $\widehat{T_2} := T > \widehat{T_1} := 3T/4$.

We introduce the following sequence

$$r_0 := 1, \qquad \forall k\geq 0,\ r_{k+1} := \begin{cases} \frac{Nr_k}{N-r_k} & \text{if } r_k < N,\\ 2r_k & \text{if } r_k \geq N.\end{cases} \tag{80}$$

We readily have from the definition (80) that

$$\forall k\geq 0,\ \beta_k := \frac{N}{2}\left(\frac{1}{r_k}-\frac{1}{r_{k+1}}\right) \leq \frac{1}{2} < 1, \tag{81}$$

and

$$\exists l\geq 1,\ r_l \geq 2. \tag{82}$$

We also introduce a sequence of times

$$\forall k\in\{0,\dots,l\},\ \tau_k := \widehat{T_1} + \frac{k}{l}(\widehat{T_2}-\widehat{T_1}). \tag{83}$$

Let us remark that

$$\forall k\in\{0,\dots,l\},\ \tau_{k+1}-\tau_k = \frac{\widehat{T_2}-\widehat{T_1}}{l} = \frac{T}{2l}. \tag{84}$$

By induction, we will show that

$$\forall k\in\{0,\dots,l\},\ \|\widehat{q}(\tau_k,.)\|_{L^{r_k}(\Omega)} \leq C(\Omega,\omega,T,a)\,\|\widehat{q}(\tau_0,.)\|_{L^1(\Omega)}\,. \tag{85}$$

The case $k=0$ is obvious (take $C_0=1$). Then, by denoting by $S(t)=e^{t\Delta}$ the heat-semigroup with Neumann boundary conditions, we have for every $k\geq 0$,

$$\widehat{q}(\tau_{k+1}) = S(\tau_{k+1}-\tau_k)\widehat{q}(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} S(\tau_{k+1}-s)(-a(s,.)\widehat{q}(s))ds, \tag{86}$$

from the equation satisfied by $\widehat{q}$ (see (21)).

We assume that (85) holds for $k\in\{0,\dots,l\}$. From (86), (81) and the regularizing effect L^{r_k}-$L^{r_{k+1}}$ of the heat-semigroup (see Proposition 3.8), we have

$$\begin{aligned}\|\widehat{q}(\tau_{k+1})\|_{L^{r_{k+1}}(\Omega)} &\leq (\tau_{k+1}-\tau_k)^{-\beta_k}\,\|\widehat{q}(\tau_k)\|_{L^{r_k}(\Omega)}\\ &\quad + \int_{\tau_k}^{\tau_{k+1}}(\tau_{k+1}-s)^{-\beta_k}\,\|a\|_{L^\infty(Q_T)}\,\|\widehat{q}(s)\|_{L^{r_k}(\Omega)}\,ds\\ &\leq A_{1,k}+A_{2,k},\end{aligned} \tag{87}$$

where

$$A_{1,k} := (\tau_{k+1} - \tau_k)^{-\beta_k} \left\| \widehat{q}(\tau_k) \right\|_{L^{r_k}(\Omega)}, \tag{88}$$

and

$$A_{2,k} := \int_{\tau_k}^{\tau_{k+1}} (\tau_{k+1} - s)^{-\beta_k} \left\| a \right\|_{L^\infty(Q_T)} \left\| \widehat{q}(s) \right\|_{L^{r_k}(\Omega)} ds. \tag{89}$$

From (88), (84), (81) and (85), we have

$$A_{1,k} \leq CT^{-\beta_k} C(\Omega, \omega, T, a) \left\| \widehat{q}(\tau_0, .) \right\|_{L^1(\Omega)} \leq C(\Omega, \omega, T, a) \left\| \widehat{q}(\tau_0, .) \right\|_{L^1(\Omega)}. \tag{90}$$

From (89), the dissipativity in time of the L^{r_k}-norm (see Proposition 3.4), the induction assumption (85), (81) and (84), we have

$$\begin{aligned} A_{2,k} &\leq \left\| a \right\|_\infty \int_{\tau_k}^{\tau_{k+1}} (\tau_{k+1} - s)^{-\beta_k} C e^{CT\|a\|_\infty} \left\| \widehat{q}(\tau_k) \right\|_{L^{r_k}(\Omega)} ds \\ &\leq C \left\| a \right\|_\infty e^{CT\|a\|_\infty} C(\Omega, \omega, T, a) \left\| \widehat{q}(\tau_0, .) \right\|_{L^1(\Omega)} (\tau_{k+1} - \tau_k)^{-\beta_k + 1} \\ &\leq C(\Omega, \omega, T, a) \left\| a \right\|_\infty T^{-\beta_k + 1} \left\| \widehat{q}(\tau_0, .) \right\|_{L^1(\Omega)} \\ &\leq C(\Omega, \omega, T, a) \left\| a \right\|_\infty (T+1) \left\| \widehat{q}(\tau_0, .) \right\|_{L^1(\Omega)} \\ &\leq C(\Omega, \omega, T, a) \left(e^{T\|a\|_\infty} + 2e^{\|a\|_\infty^{1/2}} \right) \left\| \widehat{q}(\tau_0, .) \right\|_{L^1(\Omega)} \\ &\leq C(\Omega, \omega, T, a) \left\| \widehat{q}(\tau_0, .) \right\|_{L^1(\Omega)}. \end{aligned} \tag{91}$$

The estimates (87), (90) and (91) prove (85) for $(k+1)$ and concludes the induction. Thus, (85) holds for $k = l$, which combined with (82) and (83), yields (79).

Step 3: By using (77) and (78), we prove (25) and consequently Theorem 4.4. □

4.5. **Proof of the linear global nonnegative-controllability: Theorem 4.1.** The goal of this section is to prove Theorem 4.1. The following proof is inspired by the so-called *Hilbert Uniqueness method* due to Jacques-Louis Lions (see [34] and more precisely [43, Section 2.1]).

Proof. The proof is divided into two steps. First, we build a sequence of controls $h_\varepsilon \in L^\infty((0,T) \times \omega)$ with $\varepsilon > 0$ which provide the approximate nonnegative-controllability of (20). Secondly, we pass to the limit when ε tends to 0.

Step 1. Let us fix $T > 0$, $a \in L^\infty(Q_T)$ and $y_0 \in L^2(\Omega)$. For any $\varepsilon \in (0,1)$, we consider the following functional: for every $q_T \in L^2(\Omega; \mathbb{R}^+)$,

$$J_\varepsilon(q_T) = \frac{1}{2} \left(\int_{(0,T)\times\omega} q dx dt \right)^2 + \varepsilon \left\| q_T \right\|_{L^2(\Omega)} + \int_\Omega q(0,x) y_0(x) dx, \tag{92}$$

where q is the solution to (21).

The functional J_ε is continuous, convex and coercive on the unbounded closed convex set $L^2(\Omega; \mathbb{R}^+)$. More precisely, we will show that

$$\liminf_{\|q_T\|_{L^2(\Omega)} \to +\infty} \frac{J_\varepsilon(q_T)}{\|q_T\|_{L^2(\Omega)}} \geq \varepsilon. \tag{93}$$

Indeed, given a sequence $(q_{T,k})_{k\geq 0} \in L^2(\Omega)$ with $\|q_{T,k}\|_{L^2(\Omega)} \to +\infty$, we normalize it:

$$\widetilde{q}_{T,k} := \frac{q_{T,k}}{\|q_{T,k}\|_{L^2(\Omega)}},$$

and we denote by $\widetilde{q}_k$ the solution to (21) associated to the initial data $\widetilde{q}_{T,k}$. We have

$$\frac{J_\varepsilon(q_{T,k})}{\|q_{T,k}\|_{L^2(\Omega)}} = \frac{\|q_{T,k}\|_{L^2(\Omega)}}{2} \left(\int_{(0,T)\times\omega} \widetilde{q}_k dx dt \right)^2 + \varepsilon + \int_\Omega \widetilde{q}_k(0,x) y_0(x) dx. \tag{94}$$

We distinguish the following two cases.

Case 1:

$$\liminf_{k\to+\infty} \int_{(0,T)\times\omega} \widetilde{q}_k dxdt > 0. \tag{95}$$

When (95) holds, we clearly have

$$\liminf_{k\to+\infty} \frac{J_\varepsilon(q_{T,k})}{\|q_{T,k}\|_{L^2(\Omega)}} = +\infty \geq \varepsilon$$

Case 2:

$$\liminf_{k\to+\infty} \int_{(0,T)\times\omega} \widetilde{q}_k dxdt = 0. \tag{96}$$

In this case, by using the estimate (11) of Proposition 3.2, the embedding (7) and (96), extracting subsequences (that we denote by the index k to simplify the notation), we deduce that there exists $\widetilde{q} \in W_T$ such that

$$\widetilde{q}_k \rightharpoonup \widetilde{q} \text{ in } W_T, \tag{97}$$

$$\widetilde{q}_k(0,.) \rightharpoonup \widetilde{q}(0,.) \text{ in } L^2(\Omega), \tag{98}$$

$$\int_{(0,T)\times\omega} \widetilde{q}_k dxdt \to 0. \tag{99}$$

By using Aubin Lions' lemma (see [38, Section 8, Corollary 4]) and (97), $(\widetilde{q}_k)_{k\in\mathbb{N}}$ is relatively compact in $L^2(Q_T)$, then up to a subsequence we have

$$\widetilde{q}_k \to \widetilde{q} \text{ in } L^2(Q_T;\mathbb{R}^+). \tag{100}$$

In view of (99) and (100), we have

$$\widetilde{q} = 0 \text{ in } (0,T)\times\omega. \tag{101}$$

Then, by using (101) and the observability inequality (25), we have

$$\widetilde{q}(0,.) = 0. \tag{102}$$

Consequently, by combining (98) and (102), we have

$$\int_\Omega \widetilde{q}_k(0,x) y_0(x) dx \to 0,$$

which yields (93) thanks to (94).

We deduce that J_ε admits a minimum $q_{\varepsilon,T} \in L^2(\Omega;\mathbb{R}^+)$. We take

$$h_\varepsilon := \left(\int_{(0,T)\times\omega} q_\varepsilon \right) 1_\omega, \tag{103}$$

and we denote by $y_\varepsilon \in W_T \cap L^\infty(Q_T)$ the solution to

$$\begin{cases} \partial_t y_\varepsilon - \Delta y_\varepsilon + a(t,x) y_\varepsilon = h_\varepsilon 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y_\varepsilon}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_\varepsilon(0,.) = y_0 & \text{in } \Omega. \end{cases} \tag{104}$$

We use the fact that $J_\varepsilon(q_{T,\varepsilon}) \leq J_\varepsilon(0) = 0$ to get

$$\frac{1}{2}\left(\int_{(0,T)\times\omega} q_\varepsilon \right)^2 + \varepsilon \|q_{\varepsilon,T}\|_{L^2(\Omega)} \leq -\int_\Omega q_\varepsilon(0,x) y_0(x) dx. \tag{105}$$

By using the observability inequality (25), (103), (105) and Young's inequality, we obtain the following bound on the sequence of controls

$$\|h_\varepsilon\|^2_{L^\infty(Q_T)} \leq C(\Omega,\omega,T,a) \|y_0\|^2_{L^2(\Omega)}, \tag{106}$$

where $C(\Omega, \omega, T, a)$ is of the form (22).

For $\lambda > 0$ and $p_T \in L^2(\Omega; \mathbb{R}^+)$, we have

$$J_\varepsilon(q_{\varepsilon,T}) \leq J_\varepsilon(q_{\varepsilon,T} + \lambda p_T). \tag{107}$$

Dividing the inequality (107) by λ and letting $\lambda \to 0^+$, we easily obtain from (103),

$$\begin{aligned} -(y_0, p(0,.))_{L^2(\Omega)} &\leq \int_{(0,T)\times\omega} h_\varepsilon p + \varepsilon \liminf_{\lambda\to 0^+} \frac{\|q_{\varepsilon,T} + \lambda p_T\|_{L^2(\Omega)} - \|q_{\varepsilon,T}\|_{L^2(\Omega)}}{\lambda} \\ &\leq \int_{(0,T)\times\omega} h_\varepsilon p + \varepsilon \|p_T\|_{L^2(\Omega)}, \end{aligned} \tag{108}$$

where p is the solution to (21) with initial data p_T. Since systems (20) and (21) are in duality, we have

$$\int_{(0,T)\times\omega} h_\varepsilon p = (y_\varepsilon(T,.), p_T)_{L^2(\Omega)} - (y_0, p(0,.))_{L^2(\Omega)}, \tag{109}$$

which, combined with (108), yields

$$(y_\varepsilon(T,.), p_T)_{L^2(\Omega)} \geq -\varepsilon \|p_T\|_{L^2(\Omega)}, \ \forall p_T \in L^2(\Omega; \mathbb{R}^+). \tag{110}$$

Step 2. By using (106), (104), Proposition 3.2, Proposition 3.3 and the embedding (7), up to a subsequence, we get that there exist $h \in L^\infty(Q_T)$ and $y \in W_T \cap L^\infty(Q_T)$ such that

$$h_\varepsilon \rightharpoonup^* h \text{ in } L^\infty(Q_T) \text{ as } \varepsilon \to 0, \tag{111}$$

$$y_\varepsilon \rightharpoonup y \text{ in } W_T \ \Rightarrow y_\varepsilon(0,.) \rightharpoonup y(0,.), \ y_\varepsilon(T,.) \rightharpoonup y(T,.) \text{ in } L^2(\Omega) \qquad \text{as } \varepsilon \to 0. \tag{112}$$

Then, by using (104), (111) and (112), we obtain that y is the solution of (20) associated to the control h satisfying (23) (by letting ε goes to 0 in (106)) and

$$(y(T,.), p_T)_{L^2(\Omega)} \geq 0, \ \forall p_T \in L^2(\Omega; \mathbb{R}^+). \tag{113}$$

Then, we deduce from (113) that y satisfies (24), which concludes the proof of Theorem 4.1. □

5. A fixed-point argument to prove the small-time nonlinear global nonnegative controllability

The goal of this section is to prove Theorem 2.2. We assume that (3) holds for $\alpha \leq 2$ and $f(s) \geq 0$ for $s \geq 0$.

5.1. **A comparison principle.**

First, we begin with this lemma, which is a consequence of the comparison principle for subsolutions and supersolutions of (1) with control $h = 0$ stated in Proposition 3.7.

Lemma 5.1. *Let $T > 0$, $y_0 \in L^\infty(\Omega)$. Assume that there exists $T^* \in (0, T]$ and a control $h^* \in L^\infty(Q_{T^*})$ such that the solution $y \in L^\infty(Q_{T^*})$ to (1) satisfies (5) (replacing $T \leftarrow T^*$). Then, if we set*

$$h(t,.) := \begin{cases} h^*(t,.) & \text{for } t \in (0, T^*), \\ 0 & \text{for } t \in (T^*, T), \end{cases}$$

the solution y of (1) belongs to $L^\infty(Q_T)$ and satisfies (5). Moreover, there exists $C := C(\Omega) > 0$ such that

$$\|y\|_{L^\infty(Q_T)} \leq C \|y\|_{L^\infty(Q_{T^*})}. \tag{114}$$

Proof. By using the fact that $f(0) = 0$, $f(s) \geq 0$ for $s \geq 0$ and the comparison principle (see Proposition 3.7), we have

$$\forall t \in [T^*, T],\ \text{a.e. } x \in \Omega,\ 0 \leq y(t,x) \leq \widetilde{y}(t,x), \tag{115}$$

where $\widetilde{y}$ is the nonnegative solution to

$$\begin{cases} \partial_t \widetilde{y} - \Delta \widetilde{y} = 0 & \text{in } (T^*, T) \times \Omega, \\ \frac{\partial \widetilde{y}}{\partial n} = 0 & \text{on } (T^*, T) \times \partial\Omega, \\ \widetilde{y}(T^*, .) = y(T^*, .) & \text{in } \Omega. \end{cases} \tag{116}$$

Therefore, by using Proposition 3.3 for (116), we get that there exists $C := C(\Omega) > 0$ such that

$$\|\widetilde{y}\|_{L^\infty((T^*,T)\times\Omega)} \leq C \|y(T^*, .)\|_{L^\infty(\Omega)} \leq C \|y\|_{L^\infty(Q_{T^*})}. \tag{117}$$

By using (115) and (117), we obtain that $y \in L^\infty(Q_T)$, (5) and (114) hold. □

5.2. **The fixed-point: definition of the application.** We begin with some notations. Let us set

$$g(s) = \begin{cases} \dfrac{f(s)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases} \tag{118}$$

The function g is continuous and by using the fact that f satisfies (3) with $\alpha \leq 2$, we deduce that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\forall s \in \mathbb{R},\ |g(s)|^{1/2} \leq \varepsilon \log(2 + |s|) + C_\varepsilon. \tag{119}$$

The end of the section is devoted to the proof of Theorem 2.2.

Proof. Let $T > 0$, $y_0 \in L^\infty(\Omega)$.

Unless otherwise specified, we denote by C various positive constants varying from line to line which may depend on Ω, ω, T.

We will perform a Kakutani-Leray-Schauder's fixed-point argument in $L^\infty(Q_T)$.

For each $z \in L^\infty(Q_T)$, we consider the linear system

$$\begin{cases} \partial_t y - \Delta y + g(z) y = h 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0, .) = y_0 & \text{in } \Omega. \end{cases} \tag{120}$$

We set

$$T_z^* := \min\left(T, \|g(z)\|_{L^\infty(Q_T)}^{-1/2}\right). \tag{121}$$

According to Theorem 4.1, there exists a control $h_z \in L^\infty(Q_{T_z^*})$ satisfying

$$\begin{aligned} &\|h_z\|_{L^\infty(Q_{T_z^*})} \\ &\leq \exp\left(C\left(1 + \frac{1}{T_z^*} + T_z^* \|g(z)\|_{L^\infty(Q_T)} + \|g(z)\|_{L^\infty(Q_T)}^{1/2}\right)\right) \|y_0\|_{L^2(\Omega)} \\ &\leq \exp\left(C\left(1 + \|g(z)\|_{L^\infty(Q_T)}^{1/2}\right)\right) \|y_0\|_{L^2(\Omega)}, \end{aligned} \tag{122}$$

such that the solution y of (120) in $(0, T_z^*) \times \Omega$ with $h = h_z$ satisfies

$$y(T_z^*, .) \geq 0. \tag{123}$$

By extending by 0 the control h_z in (T_z^*, T), we get from (122)

$$\|h_z\|_{L^\infty(Q_T)} \leq \exp\left(C\left(1 + \|g(z)\|_{L^\infty(Q_T)}^{1/2}\right)\right) \|y_0\|_{L^2(\Omega)}. \tag{124}$$

For each $z \in L^\infty(Q_T)$, we introduce the set of controls

$$H(z) := \{h_z \in L^\infty(Q_T)\ ;\ h_z \text{ fulfills (124) and } h_z \equiv 0 \text{ in } (T_z^*, T) \times \Omega\}. \tag{125}$$

We have the following facts.

Fact 5.2. For every $z \in L^\infty(Q_T)$, $H(z)$ is compact for the weak-star topology of $L^\infty(Q_T)$.

Fact 5.3. Assume that $z_k \to z$ in $L^\infty(Q_T)$ and $h_k \in H(z_k) \rightharpoonup^* h$ in $L^\infty(Q_T)$ as $k \to +\infty$. Then, we have $h \in H(z)$.

We define the set-valued mapping $\Phi : L^\infty(Q_T) \to \mathcal{P}(L^\infty(Q_T))$ in the following way. For every $z \in L^\infty(Q_T)$, $\Phi(z)$ is the set of $y \in L^\infty(Q_T)$ such that for some $h_z \in H(z)$, y is the solution of (120) and this solution satisfies (123).

We recall the Kakutani-Leray-Schauder's fixed point theorem (see [27, Theorem 2.2, Theorem 2.4]).

Theorem 5.4 (Kakutani-Leray-Schauder's fixed point theorem)**.** *If*

(1) Φ is a Kakutani map, that is to say for every $z \in L^\infty(Q_T)$, $\Phi(z)$ is a nonempty convex and closed subset of $L^\infty(Q_T)$,

(2) Φ is compact, that is to say for every bounded set $B \subset L^\infty(Q_T)$, there exists a compact set $K \subset L^\infty(Q_T)$ such that for every $z \in B$, $\Phi(z) \subset K$,

(3) Φ is upper semicontinuous in $L^\infty(Q_T)$, that is to say for all closed subset $\mathcal{A} \subset L^\infty(Q_T)$, $\Phi^{-1}(\mathcal{A}) = \{z \in L^\infty(Q_T)\ ;\ \Phi(z) \cap \mathcal{A} \neq \emptyset\}$ is closed,

(4) $\mathcal{F} := \{y \in L^\infty(Q_T)\ ;\ \exists \lambda \in (0,1),\ y \in \lambda\Phi(y)\}$ is bounded in $L^\infty(Q_T)$,

hold.

Then Φ has a fixed point, i.e, there exists $y \in L^\infty(Q_T)$ such that $y \in \Phi(y)$.

5.3. **Hypotheses of Kakutani-Leray-Schauder's fixed point theorem.** We will check that the four hypotheses of Theorem 5.4 hold.

The point (1) **holds.** Indeed, for every $z \in L^\infty(Q_T)$, we have seen that $\Phi(z)$ is nonempty. The convexity of $\Phi(z)$ comes from the fact that the inequality (123) is stable by convex combinations. Let us show that $\Phi(z)$ is closed. Let $(y_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q_T)$, such that for every $k \in \mathbb{N}$, $y_k \in \Phi(z)$ and $y_k \to y$ in $L^\infty(Q_T)$. Then, for every $k \in \mathbb{N}$, there exists a control $h_k \in H(z)$ such that y_k is the solution to

$$\left\{\begin{array}{ll} \partial_t y_k - \Delta y_k + g(z)y_k = h_k 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y_k}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_k(0,.) = y_0 & \text{in } \Omega, \end{array}\right. \tag{126}$$

and this solution satisfies

$$y_k(T_z^*,.) \geq 0. \tag{127}$$

By using Fact 5.2, Proposition 3.2 and the embedding (7), we get that there exist a strictly increasing sequence $(k_l)_{l\in\mathbb{N}}$ of integers and $h \in H(z)$ such that

$$h_{k_l} \rightharpoonup^* h \text{ in } L^\infty(Q_T) \text{ as } l \to +\infty, \tag{128}$$

$$y_{k_l} \rightharpoonup y \text{ in } W_T \ \Rightarrow y_{k_l}(0,.) \rightharpoonup y_0,\ y_{k_l}(T_z^*,.) \rightharpoonup y(T_z^*,.) \text{ in } L^2(\Omega) \quad \text{as } l \to +\infty. \tag{129}$$

By passing to the limit as $l \to +\infty$ in (126), (127) and by using (128) and (129), we get that $y \in \Phi(z)$. This concludes the proof of the point (1).

The point (2) **holds.** Let B be a bounded set of $L^\infty(Q_T)$. By using (124) and Proposition 3.3 applied to (120), we deduce that there exists $R > 0$ such that for every $z \in B$, for every $y \in \Phi(z)$ associated to a control $h_z \in H(z)$, we have

$$z, y, h_z \in B_R := \{\zeta \in L^\infty(Q_T)\ ;\ \|\zeta\|_{L^\infty(Q_T)} \leq R\}. \tag{130}$$

Let $Y \in L^\infty(Q_T)$ be the solution to the Cauchy problem

$$\left\{\begin{array}{ll} \partial_t Y - \Delta Y = 0 & \text{in } (0,T)\times\Omega, \\ \frac{\partial Y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ Y(0,.) = y_0 & \text{in } \Omega. \end{array}\right. \tag{131}$$

Let $y^* = y - Y$, where $y \in \Phi(z)$, with $z \in B$, associated to a control $h_z \in H(z)$. Then, y^* is the solution to

$$\begin{cases} \partial_t y^* - \Delta y^* + g(z)y = h_z 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y^*}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y^*(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{132}$$

From (130), we have

$$\|-g(z)y + h_z 1_\omega\|_{L^\infty(Q_T)} \leq C_R. \tag{133}$$

From (133), a maximal parabolic regularity theorem in L^p (see [12, Theorem 2.1]), with $p = N+2$, applied to y^*, solution of (132), we deduce that

$$y^* \in X_p := W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)) \text{ and } \|y^*\|_{X_p} \leq C_R. \tag{134}$$

By the Sobolev embedding theorem $X_p \hookrightarrow C^{\beta/2,\beta}(\overline{Q_T})$ with $\beta > 0$ (see [41, Theorem 1.4.1]), we deduce that $y^* \in C^0(\overline{Q_T})$ and

$$\forall (t,x) \in \overline{Q_T},\ \forall (t',x') \in \overline{Q_T},\ |y^*(t,x) - y^*(t',x')| \leq C_R(|t-t'|^{\beta/2} + |x-x'|^\beta). \tag{135}$$

Let K^* be the set of y^* such that (135) holds. Then, we have $K := (Y + K^*) \cap B_R$ is a compact convex subset of $L^\infty(Q_T)$ by Ascoli's theorem and

$$\forall z \in B,\ \Phi(z) \subset K.$$

This concludes the proof of the point (2).

The point (3) holds. Let $\mathcal{A}$ be a closed subset of $L^\infty(Q_T)$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q_T)$, $(y_k)_{k\in\mathbb{N}}$ be a sequence of elements in $L^\infty(Q_T)$, and $z \in L^\infty(Q_T)$ be such that

$$z_k \to z \text{ in } L^\infty(Q_T) \text{ as } k \to +\infty, \tag{136}$$

$$\forall k \in \mathbb{N},\ y_k \in \mathcal{A}, \tag{137}$$

$$\forall k \in \mathbb{N},\ y_k \in \Phi(z_k). \tag{138}$$

By (138) and (124), for every $k \in \mathbb{N}$, there exists a control $h_k \in H(z_k)$ such that y_k is the solution to

$$\begin{cases} \partial_t y_k - \Delta y_k + g(z_k)y_k = h_k 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y_k}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_k(0,.) = y_0 & \text{in } \Omega, \end{cases} \tag{139}$$

and this solution satisfies

$$y_k(T^*_{z_k},.) \geq 0. \tag{140}$$

By (136), Fact 5.3 and the point (2) of Theorem 5.4, we get that there exist a strictly increasing sequence $(k_l)_{l\in\mathbb{N}}$ of integers, $h \in H(z)$ and $y \in L^\infty(Q_T)$ such that

$$h_{k_l} \rightharpoonup^* h \text{ in } L^\infty(Q_T) \text{ as } l \to +\infty, \tag{141}$$

$$y_{k_l} \to y \text{ in } L^\infty(Q_T) \text{ as } l \to +\infty. \tag{142}$$

Since $\mathcal{A}$ is closed, (137) and (142) imply that $y \in \mathcal{A}$. Hence, it suffices to check that

$$y \in \Phi(z). \tag{143}$$

Letting $l \to +\infty$ in (139) and (140) and using (136), (141) and (142), we get that y satisfies (120) and (123). Hence, (143) holds. This concludes the proof of the point (3).

The point (4) holds. Let $y \in \mathcal{F}$. Then, for some $\lambda \in (0,1)$ and $h_y \in H(y)$, we have

$$\begin{cases} \partial_t y - \Delta y + f(y) = \lambda h_y 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,.) = \lambda y_0 & \text{in } \Omega. \end{cases}$$

and

$$y(T_y^*, .) \geq 0.$$

Therefore, by using Lemma 5.1 and Proposition 3.3, we have

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq C \|y\|_{L^\infty(Q_{T_y^*})} \\ &\leq C \exp\left(C T_y^* \|g(y)\|_{L^\infty(Q_T)}\right) \left(\|y_0\|_{L^\infty(\Omega)} + \|h_y\|_{L^\infty(Q_T)}\right). \end{aligned} \tag{144}$$

Consequently, by taking into account the definition of T_y^*, i.e., (121) and using (124), (144), (119), we deduce that

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq \exp\left(C\left(1 + \|g(y)\|_{L^\infty(Q_T)}^{1/2}\right)\right) \|y_0\|_{L^\infty(\Omega)} \\ &\leq \exp\left(C\left(1 + \varepsilon \log\left(2 + \|y\|_{L^\infty(Q_T)}\right) + C_\varepsilon\right)\right) \|y_0\|_{L^\infty(\Omega)} \\ &\leq \exp\left(C_\varepsilon\right) \left(2 + \|y\|_{L^\infty(Q_T)}\right)^{\varepsilon C} \|y_0\|_{L^\infty(\Omega)}. \end{aligned} \tag{145}$$

Therefore, by taking ε sufficiently small such that $\varepsilon C = 1/2$, we deduce from (145) that $\mathcal{F}$ is bounded in $L^\infty(Q_T)$. This concludes the proof of the point (4).

By Theorem 5.4, Φ has a fixed point y. We denote by h_y the associated control. Then, by using Lemma 5.1, y is the solution to (1) with control h_y such that (5) holds. This concludes the proof of Theorem 2.2. □

6. Application of the global nonnegative-controllability to the large time global null-controllability

In this section, we prove Theorem 2.5. We assume that (3) holds for $\alpha \in [3/2, 2]$, $f(s) > 0$ for $s > 0$ and $1/f \in L^1([1, +\infty))$.

Proof. Let $y_0 \in L^\infty(\Omega)$. The proof is divided into three steps.

Step 1: Steer the solution to a nonnegative state in time $T_1 := 1$. By using Theorem 2.2, there exists $h_1 \in L^\infty(Q_{T_1})$ such that the solution y to (1) replacing $T \leftarrow T_1$ satisfies

$$y_{T_1} := y(T_1, .) \geq 0.$$

Step 2: Dissipation of f on $\mathbb{R}^+$ and comparison to an ordinary differential equation. We set

$$h_2(t, .) := 0, \text{ for } t \in [T_1, T_2],$$

with T_2 which will be determined later.

Then, by using the comparison principle given in Proposition 3.7, we deduce that the solution y to

$$\begin{cases} \partial_t y - \Delta y = -f(y) & \text{in } (T_1, T_2) \times \Omega, \\ \frac{\partial y}{\partial n} = 0 & \text{on } (T_1, T_2) \times \partial\Omega, \\ y(T_1, .) = y_{T_1} & \text{in } \Omega, \end{cases}$$

satisfies

$$\forall t \in [T_1, T_2], \text{ a.e. } x \in \Omega, \ 0 \leq y(t, x) \leq v(t), \tag{146}$$

where v is the (global) nonnegative solution to the ordinary differential equation

$$\begin{cases} \dot{v}(t) = -f(v(t)) & \text{in } (T_1, +\infty), \\ v(T_1) = \|y_{T_1}\|_{L^\infty(\Omega)} + 1 & . \end{cases} \tag{147}$$

A straightforward calculation leads to

$$\forall t \in [T_1, +\infty), \ v(t) > 0 \text{ and } F(v(t)) - F(v(T_1)) = t - T_1, \tag{148}$$

where F is defined as follows

$$\forall s > 0, \ F(s) = \int_{+\infty}^{s} \frac{-1}{f(\sigma)} d\sigma = \int_{s}^{+\infty} \frac{1}{f(\sigma)} d\sigma. \tag{149}$$

Note that F is well-defined because $f(\sigma) > 0$ for every $\sigma > 0$ and $1/f \in L^1([1, +\infty))$ by hypothesis. We check that F is a C^1 strictly decreasing function. Moreover, we have $1/f \notin L^1((0, 1])$ because $f \in C^1(\mathbb{R}; \mathbb{R})$ and $f(0) = 0$. Hence, we have by (149)

$$\lim_{s\to 0^+} F(s) = +\infty \text{ and } \lim_{s\to+\infty} F(s) = 0. \tag{150}$$

Therefore, we deduce that $F : (0, +\infty) \to (0, +\infty)$ is a C^1-diffeomorphism. We denote by $F^{-1} : (0, +\infty) \to (0, +\infty)$ its inverse, which is strictly decreasing. Then, by (148), we have

$$\forall t \in [T_1, +\infty),\ v(t) = F^{-1}(t - T_1 + F(v(T_1)) \leq F^{-1}(t - T_1). \tag{151}$$

The estimate (151) is the key point because it states that we can upperbound v by a function independent of the size of $v(T_1)$ and we also have

$$F^{-1}(t - T_1) \to 0 \text{ as } t \to +\infty, \tag{152}$$

by using (150).

Let $\delta > 0$ be such that the null-controllability of (1) holds in $B_{L^\infty(\Omega)}(0, \delta)$ in time $T = 1$. The existence of δ is given by Theorem 1.3.

By (152), we deduce that there exists T_2 sufficiently large such that

$$F^{-1}(T_2 - T_1) \leq \delta. \tag{153}$$

Consequently, by using (146), (151), (153), we have

$$\text{a.e. } x \in \Omega,\ 0 \leq y(T_2, x) \leq \delta. \tag{154}$$

Step 3: Local null-controllability. By using Theorem 1.3 with $T = 1$, we deduce from (154) that there exists a control $h_3 \in L^\infty((T_2, T_3) \times \Omega)$ with $T_3 := T_2 + 1$ such that the solution y of (1) replacing $(0, T) \leftarrow (T_2, T_3)$ satisfies $y(T_3, .) = 0$.

To sum up, the control

$$h(t, .) := \begin{cases} h_1(t, .) & \text{for } t \in (0, T_1), \\ h_2(t, .) & \text{for } t \in (T_1, T_2), \\ h_3(t, .) & \text{for } t \in (T_2, T_3), \end{cases}$$

steers the initial data $y_0 \in L^\infty(\Omega)$ to 0. It is worth mentioning that the final time of control T_3 does not depend on y_0. This concludes the proof of Theorem 2.5. □

7. Dirichlet boundary conditions

Theorem 2.2 and Theorem 2.5 remain valid for Dirichlet boundary conditions, as to say for

$$\begin{cases} \partial_t y - \Delta y + f(y) = h 1_\omega & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, .) = y_0 & \text{in } \Omega. \end{cases} \tag{155}$$

The main point is to establish a L^1-Carleman estimate similar to Theorem 4.9 for

$$\begin{cases} -\partial_t q - \Delta q + a(t, x) q = 0 & \text{in } (0, T) \times \Omega, \\ q = 0 & \text{on } (0, T) \times \partial\Omega, \\ q(T, .) = q_T & \text{in } \Omega. \end{cases} \tag{156}$$

We keep the notations of Section 4.3.

Theorem 7.1. *There exists two constants $C = C(\Omega, \omega) > 0$ and $C_1 := C_1(\Omega, \omega) > 0$, such that,*

$$\forall \lambda \geq 1, \qquad \forall s \geq s_1(\lambda) := C(\Omega, \omega) \left(e^{2\lambda \|\eta^0\|_\infty} T + T^2 + T^2 \|a\|^{1/2}_{L^\infty(Q_T)} \right), \tag{157}$$

for every $q_T \in L^2(\Omega; \mathbb{R}^+)$, the nonnegative solution q of (156) satisfies

$$\lambda \int_{Q_T} e^{-s\alpha} s\xi^2 \eta^0 q + \int_{Q_T} e^{-s\alpha} \xi q \leq C_1 \lambda \int_{(0,T)\times\omega} e^{-s\alpha} s\xi^2 q dx dt. \tag{158}$$

Proof. The proof follows the one of Theorem 4.9. This is why we omit some details. We multiply the identity (35) by η^0 and we integrate over $(0,T)\times\Omega$

$$\begin{aligned}&\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi\eta^0 - \int_{Q_T} 2s\lambda\xi(\nabla\eta^0.\nabla\psi)\eta^0 + \int_{Q_T}(\partial_t\psi)\eta^0 + \int_{Q_T}(\Delta\psi)\eta^0\\ &= \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi\eta^0 - \int_{Q_T} s\alpha_t\psi\eta^0 + \int_{Q_T} a(t,x)\psi\eta^0\\ &\quad + \int_{Q_T} s\lambda\Delta\eta^0\xi\psi\eta^0.\end{aligned} \tag{159}$$

By the properties of η^0, we have

$$\int_{Q_T} s^2\lambda^2|\nabla\eta^0|^2\xi^2\psi\eta^0 \geq m\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 - m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi\eta^0, \tag{160}$$

where m is defined in (38).

By combining (159) and (160), we have

$$\begin{aligned}&m\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 - \int_{Q_T} 2s\lambda\xi(\nabla\eta^0.\nabla\psi)\eta^0 + \int_{Q_T}(\partial_t\psi)\eta^0 + \int_{Q_T}(\Delta\psi)\eta^0\\ &\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi\eta^0 + \int_{Q_T} s|\alpha_t|\psi\eta^0 + \int_{Q_T}|a(t,x)|\psi\eta^0\\ &\quad + \int_{Q_T} s\lambda|\Delta\eta^0|\xi\psi\eta^0 + m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi\eta^0.\end{aligned} \tag{161}$$

We have the following integration by parts

$$-\int_{Q_T} 2s\lambda\xi(\nabla\eta^0.\nabla\psi)\eta^0 = \int_{Q_T} 2s\lambda\left((\nabla\xi.\nabla\eta^0)\eta^0\psi + \xi(\Delta\eta^0)\eta^0\psi + \underbrace{\xi|\nabla\eta^0|^2\psi}_{\geq 0}\right). \tag{162}$$

$$\int_{Q_T}(\partial_t\psi)\eta^0 = \int_\Omega \eta^0(.)(\psi(T,.)-\psi(0,.)) = 0, \tag{163}$$

$$\int_{Q_T}(\Delta\psi)\eta^0 = \int_{Q_T}\psi\Delta\eta^0. \tag{164}$$

From (161), (162), (163), (164) and the properties of η^0, we have

$$\begin{aligned}&m\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 + 2m\int_{Q_T} s\lambda\xi\psi\\ &\leq \int_{Q_T} s\lambda^2|\nabla\eta^0|^2\xi\psi\eta^0 + \int_{Q_T} s|\alpha_t|\psi\eta^0 + \int_{Q_T}|a(t,x)|\psi\eta^0\\ &\quad +3\int_{Q_T} s\lambda|\Delta\eta^0|\xi\psi\eta^0 + 2\int_{Q_T} s\lambda|\nabla\xi||\nabla\eta^0|\psi\eta^0 + \int_{Q_T}\psi|\Delta\eta^0|\\ &\quad +m\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi\eta^0 + 2m\int_{(0,T)\times\omega} s\lambda\xi\psi.\end{aligned} \tag{165}$$

The first five right hand side terms of (165) can be absorbed by the first left hand side term provided $s\geq s_1(\lambda)$ as defined in (157) (see 'Step 2, Absorption' of the proof of Theorem 4.9 for details: it is exactly the same mechanism as in the proof for the Neumann case). The sixth right hand side term of (165) can be absorbed by the second left hand side term provided $s\geq C(\Omega,\omega)T^2$. The two last right hand side terms of (165) are smaller than $\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi$ provided $s\geq C(\Omega,\omega)T^2$. This leads to

$$\int_{Q_T} s^2\lambda^2\xi^2\psi\eta^0 + \int_{Q_T} s\lambda\xi\psi \leq C\int_{(0,T)\times\omega} s^2\lambda^2\xi^2\psi,$$

which yields (158) by dividing by $s\lambda$. □

From Theorem 7.1, we deduce a precise L^2-L^1 observability inequality as in Theorem 4.4 by using the second left hand side term of (158). It is an easy adaptation of Section 4.4.

The proof of the linear global nonnegative-controllability result as Theorem 4.1

and the fixed-point argument (see Section 5) remain unchanged. This leads to the small-time global nonnegative controllability for (155).

The proof of the large time global null-controllability result for (155) follows the same lines as Section 6. In particular, the comparison principle between the free solution and the solution to the ordinary differential equation, i.e., (146) stays valid because $v(t) > 0$ on $(T_1, T_2) \times \partial\Omega$.

8. Comments

8.1. **Nonlinearities depending on the gradient of the state.** We do not treat semilinearties $F(y, \nabla y)$ as considered in [19] (see also [13]) because the left hand side of the L^1-Carleman estimate (31) established in Theorem 4.9 does not provide estimates on the gradient of the state.

8.2. **Nonlinear reaction-diffusion systems.** We may wonder to what extent our main results, i.e., Theorem 2.2 and Theorem 2.5 for (1), can be adapted to the $m \times m$ semilinear reaction-diffusion system

$$\forall 1 \leq i \leq m, \ \begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u_1, \dots, u_m) + h_i 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{166}$$

with $(d_1, \dots, d_m) \in (0, +\infty)^m$ and $(f_1, \dots, f_m) \in C^1(\mathbb{R}^m; \mathbb{R})^m$ satisfying

$$\forall i \in \{1, \dots, m\}, \ f_i(0, \dots, 0) = 0. \tag{167}$$

We assume that the nonlinearity is *strongly quasi-positive*, i.e.,

$$\forall u \in \mathbb{R}^m, \ \forall i \neq j \in \{1, \dots, m\}, \ \frac{\partial f_i}{\partial u_j}(u_1, \dots, u_m) \geq 0. \tag{168}$$

and satisfies a '*mass-control structure*'

$$\forall u \in [0, +\infty)^m, \ \sum_{i=1}^m f_i(u) \leq C \left(1 + \sum_{i=1}^m u_i \right). \tag{169}$$

Lots of systems come naturally with the two properties (168) and (169) in applications (see [36, Section 2]).

We have the following global-nonnegative controllability result in small time.

Theorem 8.1. *For each f_i, we assume that* (3) *holds for $\alpha \leq 2$. For every $T > 0$, the system* (166) *is globally nonnegative-controllable in time T.*

Application 8.2. Let $\alpha \in (0, 2)$. The system

$$\begin{cases} \partial_t u - \Delta u = -u \log^\alpha(2 + |u|) + h_1 1_\omega & \text{in } (0,T) \times \Omega, \\ \partial_t v - \Delta v = u \log^\alpha(2 + |u|) + h_2 1_\omega & \text{in } (0,T) \times \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ (u,v)(0,.) = (u_0, v_0) & \text{in } \Omega, \end{cases} \tag{170}$$

is globally nonnegative-controllable for every time $T > 0$.

Proof. As the proof is very similar to that of Theorem 2.2, we limit ourselves to pointing out only the differences.

Difference 1: A L^1-Carleman estimate for a linear parabolic system. Let $A \in L^\infty(Q_T; \mathbb{R}^{m \times m})$ be such that

$$\forall i \neq j \in \{1, \dots, m\}, \text{ a.e. } (t,x) \in Q_T, \ A_{i,j}(t,x) \geq 0. \tag{171}$$

Remark 8.3. The condition (171) is satisfied by the linearized system of (166) around $(0,0)$ thanks to (168).

We consider the adjoint system

$$\left\{\begin{array}{ll} -\partial_t\zeta-\Delta\zeta=A(t,x)\zeta & \text{in } (0,T)\times\Omega,\\ \frac{\partial\zeta}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega,\\ \zeta(T,.)=\zeta_T & \text{in } \Omega. \end{array}\right. \tag{172}$$

Our goal is to establish this L^1-Carleman inequality: for every $\zeta_T\in L^2(\Omega;\mathbb{R}^+)^m$, the nonnegative solution ζ of (172) satisfies

$$\sum_{i=1}^m\int_{Q_T}e^{-s\alpha}\xi^2\zeta_i dxdt\leq C(\Omega,\omega)\left(\sum_{i=1}^m\int_{(0,T)\times\omega}e^{-s\alpha}\xi^2\zeta_i dxdt\right), \tag{173}$$

for any $\lambda\geq 1$, $s\geq s_1(\lambda):=C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}\left(T+T^2+T^2\|A\|^{1/2}_{L^\infty(Q_T;\mathbb{R}^{m\times m})}\right)$.

In order to prove (173), we first remark that the nonnegativity of ζ comes from (171) (see [37, Chapter 3, Theorem 13]). Then, by applying the same proof strategy to each line of (172) as performed in Theorem 4.9 and by forgetting for the moment the terms involving $A_{i,j}(t,x)\zeta_j$, we get

$$\begin{aligned}\sum_{i=1}^m\int_{Q_T}e^{-s\alpha}\lambda^2(s\xi)^2\zeta_i dxdt\leq C(\Omega,\omega)\Bigg(&\|A\|_{L^\infty(Q_T)}\int_{Q_T}e^{-s\alpha}|\zeta|dxdt\\ &+\sum_{i=1}^m\int_{(0,T)\times\omega}e^{-s\alpha}\lambda^2(s\xi)^2\zeta_i dxdt\Bigg),\end{aligned} \tag{174}$$

for $\lambda\geq 1$, $s\geq C(\Omega,\omega)e^{4\lambda\|\eta^0\|_\infty}\left(T+T^2\right)$. We conclude the proof of (173) by absorbing the first right hand side term of (174) provided $s\geq C(\Omega,\omega)T^2\|A\|^{1/2}_{L^\infty(Q_T)}$.

Difference 2: Without control, the free solution associated to a nonnegative initial data of (166) **stays nonnegative and remains bounded.** An adaptation of Lemma 5.1 to the system (166) holds true. But, the reason is different. It comes from [16, Theorem 1.1] which ensures global existence of classical solutions associated to nonnegative initial data for nonlinear reaction-diffusion systems with semilinearities satisfying (168), (169) and a (super)-quadratic growth (see also [39] under an additional structure assumption, the so-called dissipation of entropy).

Remark 8.4. It is worth mentioning that if the nonlinearities of (166) are bounded in $L^1(Q_T)$ for all $T>0$ (which is the case of (170) for instance), then the solutions exist globally because the growth of the semilinearity $(f_i)_{1\leq i\leq m}$ is less than $|u|^{\frac{N+2}{N}}$ (see [36, Section 1]).

This concludes the proof of Theorem 8.1. □

In the following result, we give a sufficient condition to ensure the global null-controllability of (166).

Theorem 8.5. *Let $\alpha\in(1,2)$. For each f_i, we assume that* (3) *holds with α and*

$$\exists C>0,\ \forall r\in[0,+\infty)^m,\ \sum_{i=1}^m f_i(r)\leq -C\left(\sum_{i=1}^m r_i\right)\log^\alpha\left(2+\left(\sum_{i=1}^m r_i\right)\right). \tag{175}$$

Then, there exists T sufficiently large such that (166) *is globally null-controllable in time T.*

Application 8.6. Let $\alpha\in(1,2)$. There exists $T>0$ such that the system

$$\left\{\begin{array}{ll} \partial_t u-\Delta u=-u\log^\alpha(2+|u|+|v|)+h_1 1_\omega & \text{in } (0,T)\times\Omega,\\ \partial_t v-\Delta v=-v\log^\alpha(2+|u|+|v|)+h_2 1_\omega & \text{in } (0,T)\times\Omega,\\ \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0 & \text{on } (0,T)\times\partial\Omega,\\ (u,v)(0,.)=(u_0,v_0) & \text{in } \Omega, \end{array}\right.$$

is globally null-controllable in time $T>0$.

Proof. As the proof is very similar to that of Theorem 2.2, we omit the details.

The first step consists in steering the initial data to a nonnegative state in time $T_1 := 1$. This is possible thanks to Theorem 8.1. After that, we use the following comparison principle between u, the solution to

$$\forall 1 \leq i \leq m, \ \begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u_1, \dots, u_m) & \text{in } (T_1, T_2) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (T_1, T_2) \times \partial\Omega, \\ u_i(T_1, .) = u_{i,T_1} & \text{in } \Omega, \end{cases}$$

and v, the nonnegative (global) solution to the ordinary differential system

$$\forall 1 \leq i \leq m, \ \begin{cases} \dot{v_i}(t) = -f_i(v(t)) & \text{in } (T_1, +\infty), \\ v_i(T_1) = \|u_{i,T_1}\|_{L^\infty(\Omega)} + 1 & , \end{cases} \tag{176}$$

that is to say

$$\forall i \in \{1, \dots, m\}, \ \forall t \in [T_1, T_2], \ \text{a.e. } x \in \Omega, \ 0 \leq u_i(t, x) \leq v_i(t). \tag{177}$$

This comes from the quasi-monotone nondecreasing of $(f_i)_{1 \leq i \leq m}$ which is a consequence of (168) (see [41, Theorem 12.2.1] or also [35, Chapter 8, Theorem 3.1]).

Then, by using (175), (176), (177) and the arguments of the step 2 of the proof of Theorem 2.5, we readily get

$$\forall i \in \{1, \dots, m\}, \ \text{a.e. } x \in \Omega, \ 0 \leq u_i(T_2, x) \leq \delta,$$

where T_2 is chosen sufficiently large and $\delta > 0$ is the radius of the ball of $L^\infty(\Omega)^m$ centered at 0 where the local null-controllability of (166) holds in time $T = 1$ (see for instance [18, Theorem 1.1] and the small L^∞ perturbations method).

Then, one can steer $u(T_2, .)$ to 0 with an appropriate choice of the control. □

Another interesting problem could be to determine if Theorem 8.1 and Theorem 8.5 can be generalized with fewer controls than equations in (166). The usual strategy of Luz de Teresa to 'eliminate controls' in a linear parabolic system (see [11] or [2, Theorem 4.1]) seems to be difficult to implement because the Carleman inequality in L^1 (see Theorem 4.9) only provide estimates on the function (and not on its partial derivatives in time and space).

Acknowledgments. I would like to very much thank Karine Beauchard and Michel Pierre (Ecole Normale Supérieure de Rennes) for many fruitful, stimulating discussions, helpful advice. I am also grateful to Frédéric Marbach (Ecole Normale Supérieure de Rennes) for suggesting me to look at [10, Lemma 9] and to Sylvain Ervedoza (Institut de Mathématiques de Toulouse) for reading a preliminary draft of this paper and for pointing me the two references [29] and [32].

References

[1] Farid Ammar Khodja, Assia Benabdallah, and Cédric Dupaix. Null-controllability of some reaction-diffusion systems with one control force. *J. Math. Anal. Appl.*, 320(2):928–943, 2006.

[2] Farid Ammar-Khodja, Assia Benabdallah, Manuel González-Burgos, and Luz de Teresa. Recent results on the controllability of linear coupled parabolic problems: a survey. *Math. Control Relat. Fields*, 1(3):267–306, 2011.

[3] Sebastian Anita and Daniel Tataru. Null controllability for the dissipative semilinear heat equation. *Appl. Math. Optim.*, 46(2-3):97–105, 2002. Special issue dedicated to the memory of Jacques-Louis Lions.

[4] Viorel Barbu. Exact controllability of the superlinear heat equation. *Appl. Math. Optim.*, 42(1):73–89, 2000.

[5] Viorel Barbu. Local controllability of the phase field system. *Nonlinear Anal.*, 50(3, Ser. A: Theory Methods):363–372, 2002.

[6] Viorel Barbu. *Controllability and stabilization of parabolic equations*, volume 90 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2018. Subseries in Control.

[7] Vincent D. Blondel and Alexandre Megretski, editors. *Unsolved problems in mathematical systems and control theory*. Princeton University Press, Princeton, NJ, 2004.

[8] Thierry Cazenave and Alain Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[9] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[10] Jean-Michel Coron. Some open problems on the control of nonlinear partial differential equations. In *Perspectives in nonlinear partial differential equations*, volume 446 of *Contemp. Math.*, pages 215–243. Amer. Math. Soc., Providence, RI, 2007.

[11] Luz de Teresa. Insensitizing controls for a semilinear heat equation. *Comm. Partial Differential Equations*, 25(1-2):39–72, 2000.

[12] Robert Denk, Matthias Hieber, and Jan Prüss. Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data. *Math. Z.*, 257(1):193–224, 2007.

[13] Anna Doubova, Enrique Fernández-Cara, Manuel González-Burgos, and Enrique Zuazua. On the controllability of parabolic systems with a nonlinear term involving the state and the gradient. *SIAM J. Control Optim.*, 41(3):798–819, 2002.

[14] Thomas Duyckaerts, Xu Zhang, and Enrique Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(1):1–41, 2008.

[15] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[16] Klemens Fellner and Bao Quoc Tang. Global classical solutions to quadratic systems with mass conservation in arbitrary dimensions. *ArXiv e-prints: 1808.01315*, August 2018.

[17] Enrique Fernández-Cara. Null controllability of the semilinear heat equation. *ESAIM Control Optim. Calc. Var.*, 2:87–103, 1997.

[18] Enrique Fernández-Cara, Manuel González-Burgos, and Luz de Teresa. Controllability of linear and semilinear non-diagonalizable parabolic systems. *ESAIM Control Optim. Calc. Var.*, 21(4):1178–1204, 2015.

[19] Enrique Fernández-Cara, Manuel González-Burgos, Sergio Guerrero, and Jean-Pierre Puel. Exact controllability to the trajectories of the heat equation with Fourier boundary conditions: the semilinear case. *ESAIM Control Optim. Calc. Var.*, 12(3):466–483, 2006.

[20] Enrique Fernández-Cara, Manuel González-Burgos, Sergio Guerrero, and Jean-Pierre Puel. Null controllability of the heat equation with boundary Fourier conditions: the linear case. *ESAIM Control Optim. Calc. Var.*, 12(3):442–465, 2006.

[21] Enrique Fernández-Cara and Sergio Guerrero. Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.*, 45(4):1399–1446 (electronic), 2006.

[22] Enrique Fernández-Cara and Enrique Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(5):583–616, 2000.

[23] Andrei V. Fursikov and Oleg Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[24] Victor A. Galaktionov and Juan L. Vázquez. Regional blow up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation. *SIAM J. Math. Anal.*, 24(5):1254–1276, 1993.

[25] Victor A. Galaktionov and Juan L. Vazquez. Blow-up for quasilinear heat equations described by means of nonlinear Hamilton-Jacobi equations. *J. Differential Equations*, 127(1):1–40, 1996.

[26] Victor A. Galaktionov and Juan L. Vázquez. The problem of blow-up in nonlinear parabolic equations. *Discrete Contin. Dyn. Syst.*, 8(2):399–433, 2002. Current developments in partial differential equations (Temuco, 1999).

[27] Andrzej Granas. On the Leray-Schauder alternative. *Topol. Methods Nonlinear Anal.*, 2(2):225–231, 1993.

[28] Olga Aleksandrovna. Ladyzenskaja, Vsevolod Alekseevich Solonnikov, and Nina Nikolaevna Uralceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[29] Camille Laurent and Matthieu Léautaud. Observability of the heat equation, geometric constants in control theory, and a conjecture of Luc Miller. *ArXiv e-prints: 1806.00969*, June 2018.

[30] Kévin Le Balc'h. Controllability of a 4 x 4 quadratic reaction–diffusion system. *Journal of Differential Equations*, 2018, In press, doi:10.1016/j.jde.2018.08.046.

[31] Jérôme Le Rousseau and Gilles Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 18(3):712–747, 2012.

[32] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.

[33] Juan Límaco, Marcondes Clark, Alexandro Marinho, Silvado B. de Menezes, and Aldo T. Louredo. Null controllability of some reaction-diffusion systems with only one control force in moving domains. *Chin. Ann. Math. Ser. B*, 37(1):29–52, 2016.

[34] Jacques-Louis Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1):1–68, 1988.

[35] C. V. Pao. *Nonlinear parabolic and elliptic equations*. Plenum Press, New York, 1992.

[36] Michel Pierre. Global existence in reaction-diffusion systems with control of mass: a survey. *Milan J. Math.*, 78(2):417–455, 2010.

[37] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

[38] Jacques Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[39] Philippe Souplet. Global existence for reaction-diffusion systems with dissipation of mass and quadratic growth. *Journal of Evolution Equations, In press, ArXiv e-prints:1804.05193*, April 2018.

[40] Gensheng Wang and Liang Zhang. Exact local controllability of a one-control reaction-diffusion system. *J. Optim. Theory Appl.*, 131(3):453–467, 2006.

[41] Zhuoqun Wu, Jingxue Yin, and Chunpeng Wang. *Elliptic & parabolic equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[42] Enrique Zuazua. Exact controllability for semilinear wave equations in one space dimension. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(1):109–129, 1993.

[43] Enrique Zuazua. Finite-dimensional null controllability for the semilinear heat equation. *J. Math. Pures Appl. (9)*, 76(3):237–264, 1997.

[44] Enrique Zuazua. Controllability and observability of partial differential equations: some results and open problems. In *Handbook of differential equations: evolutionary equations. Vol. III*, Handb. Differ. Equ., pages 527–621. Elsevier/North-Holland, Amsterdam, 2007.

Kévin Le Balc'h, Univ Rennes, ENS Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

E-mail address: kevin.lebalch@ens-rennes.fr