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A priori estimates for elliptic equations with reaction terms involving the function and its gradient

Marie-Françoise Bidaut-Véron*,
Marta Garcia-Huidobro †
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Abstract We study local and global properties of positive solutions of $-\Delta u = u^p + M |\nabla u|^q$ in a domain Ω of \mathbb{R}^N , in the range $\min\{p, q\} > 1$ and $M \in \mathbb{R}$. We prove a priori estimates and existence or non-existence of ground states for the same equation.

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Key words. elliptic equations; Bernstein methods; ground states;

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1 Introduction

This article is concerned with local and global properties of positive solutions of the following type of equations

$$-\Delta u = M'|u|^{p-1}u + M|\nabla u|^q, \tag{1.1}$$

in $\Omega \setminus \{0\}$ where Ω is an open subset of \mathbb{R}^N containing 0, p and q are exponents larger than 1 and M, M' are real parameters. If $M' \leq 0$ the equation satisfies a comparison principle and a big part of the study can be carried via radial local supersolutions. This no longer the case when $M' > 0$ which will be assumed in all the article, and by homothety (1.1) becomes

$$-\Delta u = |u|^{p-1}u + M|\nabla u|^q. \tag{1.2}$$

If $M = 0$ (1.2) is called Lane-Emden equation

$$-\Delta u = |u|^{p-1}u. \tag{1.3}$$

It turns out that it plays an important role in modelling meteorological or astrophysical phenomena [15], [13], this is the reason for which the first study, in the radial case, goes back to the end of nineteenth century and the beginning of the twentieth. A fairly complete presentation can be found in [18]. If $N \geq 3$, This equations exhibits two main critical exponents $p = \frac{N}{N-2}$ and $p = \frac{N+2}{N-2}$ which play a key role in the description of the set of positive solutions which can be summarized by the following overview:

- 1- If $1 < p \leq \frac{N}{N-2}$, there exists no positive solution if Ω is the complement of a compact set. Even in that case solution can be replaced by supersolution. This is easy to prove by studying the inequality satisfied by the spherical average of a solution of the equation.
- 2- If $1 < p < \frac{N+2}{N-2}$, there exists no *ground state*, i.e. positive solution in \mathbb{R}^N . Furthermore any positive solution u in a ball $B_R = B_R(a)$ satisfies

$$u(x) \leq c(R - |x - a|)^{-\frac{2}{p-1}}, \tag{1.4}$$

where $c = c(N, p) > 0$, see [19].

3- If $p = \frac{N+2}{N-2}$ all the positive solutions in \mathbb{R}^N are radial with respect to some point a and endow the following form

$$u(x) := u_\lambda(x) = \frac{(N(N-2)\lambda)^{\frac{N-2}{4}}}{(\lambda + |x-a|^2)^{\frac{N-2}{2}}}. \quad (1.5)$$

All the positive solutions in $\mathbb{R}^N \setminus \{0\}$ are radial, see [12].

4- If $p > \frac{N+2}{N-2}$ there exist infinitely many positive ground states radial with respect to some points. They are obtained from one say v , radial for example with respect to 0 by the scaling transformation T_k where $k > 0$ with

$$T_k[v](x) = k^{\frac{2}{p-1}}v(kx). \quad (1.6)$$

Indeed, the first significant non-radial results deals with the case $1 < p \leq \frac{N}{N-2}$. They are based upon the Brezis-Lions lemma [11] which yields an estimate of solutions in the Lorentz space $L^{\frac{N}{N-2}, \infty}$, implying in turn the local integrability of u^q . Then a bootstrapping method as in [21] leads easily to some a priori estimate. Note that this subcritical case can be interpreted using the famous Serrin's results on quasilinear equations [24]. The first breakthrough in the study of Lane-Emden equation came in the treatment of the case $1 < p < \frac{N+2}{N-2}$; it is due to Gidas and Spruck [19]. Their analysis is based upon differentiating the equation and then obtaining sharp enough local integral estimates on the term u^{q-1} making possible the utilization of Harnack inequality as in [24]. The treatment of the critical case $p = \frac{N+2}{N-2}$, due to Caffarelli, Gidas and Spruck [12], was made possible thanks to a completely new approach based upon a combination of moving plane analysis and geometric measure theory. As for the supercritical case, not much is known and the existence of radial ground states is a consequence of Pohozaev's identity [22], using a shooting method.

The study of (1.2) when $M \neq 0$ presents some similarities with the one of Lane-Emden equation in the cases 1 and 2, except that the proof are much more involved. Actually the approach we develop in this article is much indebted to our recent paper [6] where we study local and global aspects of positive solutions of

$$-\Delta u = u^p |\nabla u|^q, \quad (1.7)$$

where $p \geq 0$, $0 \leq q < 2$, mostly in the superlinear case $p + q - 1 > 0$. Therein we prove the existence of a critical line of exponents

$$(\mathfrak{L}) := \{(p, q) \in \mathbb{R}_+ \times [0, 2) : (N-2)p + (N-1)q = N\}. \quad (1.8)$$

The subcritical range corresponds to the fact that (p, q) is below (\mathfrak{L}) . In this region Serrin's celebrated results [24] can be applied and we prove [6, Theorem A] that positive solutions of (1.7) in the punctured ball $B_2 \setminus \{0\}$ satisfy, for some constant $c > 0$ depending on the solution,

$$u(x) + |x| |\nabla u(x)| \leq c |x|^{2-N} \quad \text{for all } x \in B_1 \setminus \{0\}. \quad (1.9)$$

When (p, q) is above (\mathfrak{L}) , i.e. in the supercritical range, we introduced two methods for obtaining a priori estimate of solutions: The *pointwise Bernstein method* and the *integral Bernstein method*. The first one is based upon the change of unknown $u = v^{-\beta}$, and then to show

that $|\nabla v|$ satisfies an inequality of Keller-Osserman type. When (p, q) lies above (\mathfrak{L}) and verifies

- (i) either $1 \leq p < \frac{N+3}{N-1}$ and $p + q - 1 < \frac{4}{N-1}$,
- (ii) or $0 \leq p < 1$ and $p + q - 1 < \frac{(p+1)^2}{p(N-1)}$,

we prove that any positive solution of (1.7) in a domain $\Omega \subset \mathbb{R}^N$ satisfies

$$|\nabla u^a(x)| \leq c^* (\text{dist}(x, \partial\Omega))^{-1-a \frac{2-q}{p+q-1}} \quad \text{for all } x \in \Omega, \quad (1.10)$$

for some positive c^* and a depending on N, p and q [6, Theorem B]. As a consequence we prove that any positive solution of (1.7) in \mathbb{R}^N is constant. With the second method we combine the change of unknown $u = v^{-\beta}$ with integration and cut-off functions. We show the existence of a quadratic polynomial G in two variables such that for any $(p, q) \in \mathbb{R}_+ \times [0, 2)$ satisfying $G(p, q) < 0$ any positive solution of (1.7) in \mathbb{R}^N is constant [6, Theorem C]. The polynomial G is not simple but it is worth noting that if $0 \leq p < \frac{N+2}{N-2}$, there holds $G(p, 0) < 0$, which recovers Gidas and Spruck result [19].

For equation (1.2) we first observe that the equation is invariant under the scaling transformation (1.6) for any $k > 0$ if and only if q is *critical with respect to p* , i.e.

$$q = \frac{2p}{p+1}.$$

In general the transformation T_k exchanges (1.2) with

$$-\Delta v = v^p + Mk \frac{2p-q(p+1)}{p-1} |\nabla v|^q, \quad (1.11)$$

hence if $q < \frac{2p}{p+1}$, the limit equation when $k \rightarrow 0$ is (1.3). We say that the exponent p is dominant. We can also consider the transformation

$$S_k[v](x) = k^{\frac{2-q}{q-1}} v(kx), \quad (1.12)$$

when $q \neq 2$, which is the same as T_k if $q = \frac{2p}{p+1}$, and more generally transforms (1.2) into

$$-\Delta v = k^{\frac{q-p(2-q)}{q-1}} v^p + M |\nabla v|^q. \quad (1.13)$$

Hence if $q > \frac{2p}{p+1}$, the limit equation when $k \rightarrow 0$ is the Riccati equation

$$-\Delta v = M |\nabla v|^q. \quad (1.14)$$

It is also important to notice that the value of the coefficient M (and not only its sign) plays a fundamental role, only if $q = \frac{2p}{p+1}$. If $q \neq \frac{2p}{p+1}$ the transformation

$$u(x) = av(y) \quad \text{with } a = |M|^{-\frac{2}{(p+1)q-2p}} \text{ and } y = a^{\frac{p-1}{2}} x \quad (1.15)$$

allows to transform (1.2) into

$$-\Delta v = |v|^{p-1} v \pm |\nabla v|^q. \quad (1.16)$$

The equation (1.2) has been essentially studied in the *radial case* when $M < 0$ in connection with the parabolic equation

$$\partial_t u - \Delta u + M|\nabla u|^q = |u|^{p-1}u, \quad (1.17)$$

see [14], [16], [17], [25], [27], [30], [31]. The studies mainly deal with the case $q \neq \frac{2p}{p+1}$, although not complete when $q > \frac{2p}{p+1}$. When $q = \frac{2p}{p+1}$ the existence of a ground state is proved in dimension 1. Some partial results that we will improve, already exist in higher dimension. The case $M > 0$ attracted less attention.

In the *nonradial case*, any nonnegative nontrivial solution is positive since $p, q > 1$. We first observe, using a standard averaging method applied to positive supersolutions of (1.3), that if $M \geq 0$, $1 < p \leq \frac{N}{N-2}$ when $N \geq 3$, any $p > 1$ if $N = 1, 2$, then for any $q > 0$ there exists no positive solution in an exterior domain. When $0 < q < \frac{2p}{p+1}$ the equation endows some character of the pure Emden-Fowler equation (1.3) by the transformation T_k . In [23] it is proved that if $0 < q < \frac{2p}{p+1}$, $1 < p < \frac{N+2}{N-2}$ and $M \in \mathbb{R}$, any positive solution of (1.3) in an open domain satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c_{N,p,q,M} \left(1 + (\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} \right) \quad \text{for all } x \in \Omega. \quad (1.18)$$

Note that this does not imply the non-existence of ground state. In [1] Alarcón, García-Melián and Quass study the equation

$$-\Delta u = |\nabla u|^q + f(u), \quad (1.19)$$

in an exterior domain of \mathbb{R}^N emphasizing the fact that positive solutions are super harmonic functions. They prove that if $1 < q \leq \frac{N}{N-1}$ and if f is positive on $(0, \infty)$ and satisfies

$$\limsup_{s \rightarrow 0} s^{-p} f(s) > 0, \quad (1.20)$$

for some $p > \frac{N}{N-2}$, then (1.19) admits no positive supersolution. The same authors also study in [2] existence and non-existence of positive solutions of (1.19) in a bounded domain with Dirichlet condition.

The techniques we developed in this paper are based upon a delicate extension of the ones already introduced in [6]. Our first nonradial result dealing with the case $q > \frac{2p}{p+1}$ is the following:

Theorem A *Let $N \geq 1$, $p > 1$ and $q > \frac{2p}{p+1}$. Then for any $M > 0$, any solution of (1.2) in a domain $\Omega \subset \mathbb{R}^N$ satisfies*

$$|\nabla u(x)| \leq c_{N,p,q} \left(M^{-\frac{p+1}{(p+1)q-2p}} + (M \text{dist}(x, \partial\Omega))^{-\frac{1}{q-1}} \right) \quad \text{for all } x \in \Omega. \quad (1.21)$$

As a consequence, any ground state has at most a linear growth at infinity:

$$|\nabla u(x)| \leq c_{N,p,q} M^{-\frac{p+1}{(p+1)q-2p}} \quad \text{for all } x \in \mathbb{R}^N. \quad (1.22)$$

Our proof relies on a direct Bernstein method combined with Keller-Osserman's estimate applied to $|\nabla u|^2$. It is important to notice that the result holds for any $p > 1$, showing that, in some sense, the presence of the gradient term has a regularizing effect. In the case $q < \frac{2p}{p+1}$ we prove a non-existence result

Theorem A' *Let $N \geq 1$, $p > 1$, $1 < q < \frac{2p}{p+1}$ and $M > 0$. Then there exists a constant $c_{N,p,q} > 0$ such that there is no positive solution of (1.2) in \mathbb{R}^N satisfying*

$$u(x) \leq c_{N,p,q} M^{\frac{2}{2p-(p+1)q}} \quad \text{for all } x \in \mathbb{R}^N. \quad (1.23)$$

When q is critical with respect to p the situation is more delicate since the value of M plays a fundamental role. Our first statement is a particular case of a more general result in [1], but with a simpler proof which allows us to introduce techniques that we use later on.

Theorem B *Let $N \geq 2$, $p > 1$ if $N = 2$ or $1 < p \leq \frac{N}{N-2}$ if $N = 3$, $q = \frac{2p}{p+1}$ and $M > -\mu^*$ where*

$$\mu^* := \mu^*(N) = (p+1) \left(\frac{N - (N-2)p}{2p} \right)^{\frac{p}{p+1}}. \quad (1.24)$$

Then there exists no nontrivial nonnegative supersolution of (1.2) in an exterior domain.

In this range of values of p this result is optimal since for $M \leq -\mu^*$ there exists positive singular solutions. The constant μ^* will play an important role in the description developed in [7] of radial solutions of (1.2). Using a variant of the method used in the proof of Theorem B we obtain results of existence and nonexistence of large solutions.

Theorem B' *Let $N \geq 1$, $p > 1$ and $q = \frac{2p}{p+1}$.*

1- If Ω is a domain with a compact boundary satisfying the Wiener criterion and $M \geq -\mu^(2)$ there exists no positive supersolution of (1.2) in Ω satisfying*

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) = \infty. \quad (1.25)$$

2- If G is a bounded convex domain, $\Omega = \overline{G}^c$ and $M < -\mu^(1)$ there exists a positive solution of (1.2) in Ω satisfying (1.25).*

We show in [7] that the inequality $M < -\mu^*(1)$ is the necessary and sufficient condition for the existence of a radial large solution in the exterior of a ball.

Concerning ground states, we prove their nonexistence for any $p > 1$ provided $M > 0$ is large enough: indeed

Theorem C *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a domain, $p > 1$, $q = \frac{2p}{p+1}$. For any*

$$M > M_{\dagger} := \left(\frac{p-1}{p+1} \right)^{\frac{p-1}{p+1}} \left(\frac{N(p+1)^2}{4p} \right)^{\frac{p}{p+1}}, \quad (1.26)$$

and any $\nu > 0$ such that $(1 - \nu)M > M_{\dagger}$, there exists a positive constant $c_{N,p,\nu}$ such that any solution u in Ω satisfies

$$|\nabla u(x)| \leq c_{N,p,\nu} ((1 - \nu)M - M_{\dagger})^{-\frac{p+1}{p-1}} (\text{dist}(x, \partial\Omega))^{-\frac{p+1}{p-1}} \quad \text{for all } x \in \Omega. \quad (1.27)$$

Consequently there exists no nontrivial solution of (1.2) in \mathbb{R}^N .

The next result, based upon an elaborate Bernstein method, complements Theorem C under a less restrictive assumption on M but a more restrictive assumption on p .

Theorem D *Let $1 < p < \frac{N+3}{N-1}$, $N \geq 2$, $1 < q < \frac{N+2}{N}$ and $\Omega \subset \mathbb{R}^N$ be a domain. Then there exist $a > 0$ and $c_{N,p,q} > 0$ such that for any $M > 0$, any positive solution u in Ω satisfies*

$$|\nabla u^a(x)| \leq c_{N,p,q} (\text{dist}(x, \partial\Omega))^{-\frac{2a}{p-1}-1} \quad \text{for all } x \in \Omega. \quad (1.28)$$

Hence there exists no nontrivial nonnegative solution of (1.2) in \mathbb{R}^N .

It is remarkable that the constants a and $c_{N,p,q}$ do not depend on $M > 0$, a fact which is clear when $q \neq \frac{2p}{p+1}$ by using the transformation T_k , but much more delicate to highlight when $q = \frac{2p}{p+1}$ since (1.2) is invariant. When $|M|$ is small, we use an integral method to obtain the following result which contains, as a particular case, the estimates in [19] and [7]. The key point of this method is to prove that the solutions in a punctured domain satisfy a local Harnack inequality.

Theorem E *Let $N \geq 3$, $1 < p < \frac{N+2}{N-2}$, $q = \frac{2p}{p+1}$. Then there exists $\epsilon_0 > 0$ depending on N and p such that for any M satisfying $|M| \leq \epsilon_0$, any positive solution u in $B_R \setminus \{0\}$ satisfies*

$$u(x) \leq c_{N,p} |x|^{-\frac{2}{p-1}} \quad \text{for all } x \in B_{\frac{R}{2}} \setminus \{0\}. \quad (1.29)$$

As a consequence there exists no positive solution of (1.2) in \mathbb{R}^N , and any positive solution u in a domain Ω satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c'_{N,p} (\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega. \quad (1.30)$$

Note that under the assumptions of Theorem E, there exist ground states for $|M|$ large enough when $1 < p < \frac{N}{N-2}$, or any $p > 1$ if $N = 1, 2$.

If u is a radial solutions of (1.2) in \mathbb{R}^N it satisfies

$$-u'' - \frac{N-1}{r}u' = |u|^{p-1}u + M|u|^q, \quad (1.31)$$

on $(0, \infty)$. Using several type of Lyapounov type functions introduced by Leighton [20] and Anderson and Leighton [3], we prove some results dealing with the case $M > 0$ which complement the ones of [25] relative to the case $M < 0$.

Theorem F *1- Let $p > 1$ and $q > \frac{2p}{p+1}$. Then there exists no radial ground state u satisfying $u(0) = 1$ when $M > 0$ is too large.*

2- Let $1 < p < \frac{N+2}{N-2}$. If $1 < q \leq p$ there exists no radial ground state for any $M > 0$. If $q > p$ there exists no radial ground state for $M > 0$ small enough.

3- Let $N \geq 3$, $p > \frac{N+2}{N-2}$ and $q \geq \frac{2p}{p+1}$. Then there exist radial ground states for $M > 0$ small enough.

We end the article in proving the existence of non-radial positive singular solutions of (1.2) in $\mathbb{R}^N \setminus \{0\}$ in the case $q = \frac{2p}{p+1}$ obtained by bifurcation from radial explicit positive singular solutions. Our result shows that the situation is very contrasted according $M > 0$ where a bifurcation from (M, X_M) occurs only if $p \geq \frac{N+1}{N-3}$ and $M \geq 0$ and $M < 0$ where there exists a countable set of bifurcations from (M_k, X_{M_k}) , $k \geq 1$, when $1 < p < \frac{N+1}{N-3}$.

In a subsequent article [7] we present a fairly complete description of the positive radial solutions of (1.2) in $\mathbb{R}^N \setminus \{0\}$ in the scaling invariant case $q = \frac{2p}{p+1}$.

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2 The direct Bernstein method

We begin with a simple property in the case $M \geq 0$ which is a consequence of the fact that the positive solutions of (1.2) are superharmonic.

Proposition 2.1 1- There exists no positive solution of (1.2) in $\mathbb{R}^N \setminus \overline{B}_R$, $R \geq 0$ if one of the two conditions is satisfied:

(i) $M \geq 0$, $q \geq 0$ and either $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$.

(ii) $M > 0$, $N \geq 3$, $p \geq 1$ and $1 < q \leq \frac{N}{N-1}$.

2- If $N \geq 3$, $q \geq 1$, $p > \frac{N}{N-2}$ and $u(x) = u(r, \sigma)$ is a positive solution of (1.2) in $\mathbb{R}^N \setminus \overline{B}_R$, $R \geq 0$. Then there exists $\rho \geq R$ such that

$$\frac{1}{N\omega_N} \int_{S^{N-1}} u(r, \sigma) dS := \bar{u}(r) \leq c_0 r^{-\frac{2}{p-1}} \quad \text{for all } r > \rho, \quad (2.1)$$

with $c_0 := \left(\frac{2N}{p-1}\right)^{\frac{1}{p-1}}$ and

$$\left| \frac{1}{N\omega_N} \int_{S^{N-1}} u_r(r, \sigma) dS \right| := |\bar{u}_r(r)| \leq (N-2)c_0 r^{-\frac{p+1}{p-1}} \quad \text{for all } r > \rho. \quad (2.2)$$

3- If $M > 0$, $p \geq 0$, and $q > \frac{N}{N-1}$ there holds for

$$|\bar{u}_r(r)| \leq \left(\frac{(q-1)(N-1)-1}{(q-1)M} \right)^{\frac{1}{q-1}} r^{-\frac{1}{q-1}} \quad \text{for all } r > \rho, \quad (2.3)$$

and

$$\bar{u}(r) \leq \left(\frac{q-1}{2-q} \right) \left(\frac{(q-1)(N-1)-1}{(q-1)M} \right)^{\frac{1}{q-1}} r^{\frac{q-2}{q-1}} \quad \text{for all } r > \rho, \quad (2.4)$$

Furthermore, if $R = 0$, inequalities (2.1), (2.2) and (2.3) hold with $\rho = 0$.

Proof. Assertion 1-(i) is not difficult to obtain by integrating the inequality satisfied by the spherical average of the solution and using Jensen's inequality. For the sake of completeness, we give a simple proof although the result is actually valid for much more general equations (see e.g. [8] and references therein). In this statement we denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in \mathbb{R}^N , by ω_N the volume of the unit N -ball and thus $N\omega_N$ is the $(N-1)$ -volume of the unit sphere S^{N-1} . Writing (1.2) in spherical coordinates and using Jensen formula, we get

$$-r^{1-N} (r^{N-1}\bar{u}_r)_r \geq \bar{u}^p + M |\bar{u}_r|^q. \quad (2.5)$$

It implies that $r \mapsto w(r) := -r^{N-1}\bar{u}_r$ is increasing on (R, ∞) , thus it admits a limit $\ell \in (-\infty, \infty]$. If $\ell \leq 0$, then $\bar{u}_r(r) > 0$ on (R, ∞) . Hence $\bar{u}(r) \geq \bar{u}(\rho) := c > 0$ for $r \geq \rho > R$. then

$$(r^{N-1}\bar{u}_r)_r \leq -c^p r^{N-1} \implies \bar{u}_r(r) \leq \frac{\rho^{N-1}}{r^{N-1}} \bar{u}_r(\rho) - \frac{c^p}{N} \left(r - \frac{\rho^N}{r^{N-1}} \right),$$

which implies $\bar{u}_r(r) \rightarrow -\infty$, thus $\bar{u}(r) \rightarrow -\infty$ as $r \rightarrow \infty$, contradiction. Therefore $\ell \in (0, \infty]$ and either $\bar{u}_r(r) < 0$ on (R, ∞) or there exists $r_\ell > R$ such that $\bar{u}_r(r_\ell) = 0$, \bar{u} is increasing on (R, r_ℓ) and decreasing on (r_ℓ, ∞) . If $\bar{u}_r(r) < 0$ on (R, ∞) , then we have for $r > 2R$

$$-r^{N-1}\bar{u}_r(r) \geq \int_{\frac{r}{2}}^r t^{N-1}\bar{u}^p(t)dt \geq \frac{r^N \bar{u}^p(r)}{2N} \implies (\bar{u}^{1-p})_r \geq \frac{(p-1)r}{2N} \implies \bar{u}(r) \leq \left(\frac{2N}{(p-1)r^2} \right)^{\frac{1}{p-1}},$$

which yields (2.1). If we are in the second case with $r_\ell > R$, we apply the same inequality with $r > 2r_\ell$ and again (2.1) for $r > 2r_\ell$. Since \bar{u} is superharmonic, the function $v(s) = \bar{u}(r)$ with $s = r^{2-N}$ is concave on $(0, R^{2-N})$ and it tends to 0 when $s \rightarrow 0$. Thus

$$v_s(s) \leq \frac{v}{s} \implies |\bar{u}_r(r)| \leq (N-2) \frac{\bar{u}(r)}{r} \leq (N-2)c_0 r^{-\frac{p+1}{p-1}}.$$

This implies (2.1) and (2.2). Note that the case $r_\ell > R$ cannot happen if $R = 0$, so in any case, if $R = 0$ then $\rho = 0$.

If $M > 0$, we have with $w(r) = -r^{N-1}\bar{u}_r$

$$w_r \geq M r^{(1-q)(N-1)} |w|^q.$$

We have seen that $w(r) > 0$ at infinity with limit $\ell \in (0, \infty]$, hence, on the maximal interval containing ∞ where $w > 0$, we have $(w^{1-q})_r \leq (1-q)M r^{(N-1)(1-q)}$. We have for $r > s > R$

$$w^{1-q}(r) - w^{1-q}(s) \leq M \ln \left(\frac{r}{s} \right),$$

if $q = \frac{N}{N-1}$ and

$$w^{1-q}(r) - w^{1-q}(s) \leq \frac{M(q-1)}{(q-1)(N-1)-1} \left(r^{1-(q-1)(N-1)} - s^{1-(q-1)(N-1)} \right)$$

if $q < \frac{N}{N-1}$, and both expressions which tend to $-\infty$ when $r \rightarrow \infty$, a contradiction. This proves 1-(ii). If $q > \frac{N}{N-1}$, the above expression yields, when $r \rightarrow \infty$,

$$\ell^{1-q} - w^{1-q}(s) \leq -\frac{(q-1)M}{(q-1)(N-1)-1} s^{1-(q-1)(N-1)}.$$

This implies

$$w(s) \leq \left(\frac{(q-1)(N-1)-1}{(q-1)M} \right)^{\frac{1}{q-1}} s^{N-1-\frac{1}{q-1}},$$

and (2.3). \square

Remark. The previous is a particular case of a much more general one dealing with quasilinear operators proved in [8, Theorem 3.1].

2.1 Proof of Theorems A, A' and C

The function u is at least $C^{3+\alpha}$ for some $\alpha \in (0, 1)$ since $p, q > 1$. Hence $z = |\nabla u|^2$ is $C^{2+\alpha}$. Since there holds by Bochner's identity and Schwarz's inequality

$$-\frac{1}{2}\Delta z + \frac{1}{N}(\Delta u)^2 + \langle \nabla \Delta u, \nabla u \rangle \leq 0, \quad (2.6)$$

we obtain from (1.2),

$$-\frac{1}{2}\Delta z + \frac{|u|^{2p}}{N} + \frac{2M}{N}|u|^{p-1}uz^{\frac{q}{2}} + \frac{M^2}{N}z^q - p|u|^{p-1}z - \frac{Mq}{2}z^{\frac{q}{2}-1}\langle \nabla z, \nabla u \rangle \leq 0.$$

Since for $\delta > 0$,

$$z^{\frac{q}{2}-1}|\langle \nabla z, \nabla u \rangle| \leq \left| z^{-\frac{1}{2}}\nabla z \right| z^{\frac{q-1}{2}}|\nabla u| = \left| z^{-\frac{1}{2}}\nabla z \right| z^{\frac{q}{2}} \leq \delta z^q + \frac{1}{4\delta} \frac{|\nabla z|^2}{z},$$

we obtain for any $\nu \in (0, 1)$, provided δ is small enough,

$$-\frac{1}{2}\Delta z + \frac{|u|^{2p}}{N} + \frac{2M}{N}|u|^{p-1}uz^{\frac{q}{2}} + \frac{M^2(1-\nu)^2}{N}z^q - p|u|^{p-1}z \leq c_1 \frac{|\nabla z|^2}{z}, \quad (2.7)$$

where $c_1 = c_1(M, N, \nu) > 0$.

2.1.1 Proof of Theorem A

We recall the following technical result proved in [6, Lemma 2.2] which will be used several times in the course of this article.

Lemma 2.2 *Let $S > 1$, $R > 0$ and v be continuous and nonnegative in \overline{B}_R and C^1 on the set $\mathcal{U}_+ = \{x \in B_R : v(x) > 0\}$. If v satisfies, for some real number a ,*

$$-\Delta v + v^S \leq a \frac{|\nabla v|^2}{v} \quad (2.8)$$

on each connected component of \mathcal{U}_+ , then

$$v(0) \leq c_{N,S,a} R^{-\frac{2}{S-1}}. \quad (2.9)$$

Abridged proof. Assuming $a > 0$, we set $W = v^\alpha$ for $0 < \alpha \leq \frac{1}{a+1}$, this transforms (2.8) into

$$-\Delta W + \frac{1}{\alpha} W^{\alpha(S-1)+1} \leq 0, \quad (2.10)$$

and then we apply Keller-Osserman inequality. \square

Proof of Theorem A. Suppose $\frac{2p}{p+1} < q$. We set $r = \frac{2p}{p-1}$, $r' = \frac{r}{r-1}$, then, for any $\epsilon > 0$

$$p|u|^{p-1}z \leq \frac{\epsilon^r |u|^{(p-1)r}}{r} + \frac{z^{r'}}{\epsilon^{r'} r'} = (p-1) \frac{\epsilon^r |u|^{2p}}{2} + (p+1) \frac{z^{\frac{2p}{p+1}}}{2\epsilon^{r'}}.$$

We fix $\eta \in (0, 1)$ and ϵ so that $\epsilon^r = \frac{2(1-\eta)}{N(p-1)}$ and get

$$p|u|^{p-1}z \leq (1-\eta) \frac{|u|^{2p}}{N} + c_2 z^{\frac{2p}{p+1}},$$

where $c_2 = \frac{p+1}{2} \left(\frac{N(p-1)}{2(1-\eta)} \right)^{\frac{p+1}{p-1}}$. We perform the change of scale (1.6) in order to reduce (1.2) to the case $M = 1$ by setting $u(x) = \alpha^{\frac{2}{p-1}} v(\alpha x)$ with $\alpha = M^{-\frac{p-1}{(p+1)q-2p}}$. Then the equation for $z = |\nabla v|^2$ is considered in $\Omega_\alpha = \alpha\Omega$. Choosing now $\eta = \frac{1}{2}$ we obtain

$$c_2 z^{\frac{2p}{p+1}} \leq \frac{1}{4N} z^q + c_3,$$

where $c_3 = c_3(N, p, q) > 0$, hence

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{2N} + \frac{1}{4N} z^q \leq c_3 + c_1 \frac{|\nabla z|^2}{z}.$$

Put $\tilde{z} = \left(z - (4Nc_3)^{\frac{1}{q}} \right)_+$, then

$$-\frac{1}{2}\Delta \tilde{z} + \frac{1}{4N} \tilde{z}^q \leq c_1 \frac{|\nabla \tilde{z}|^2}{\tilde{z}},$$

hence, from Lemma 2.2, we derive

$$\tilde{z}(y) \leq c_4 (\text{dist}(y, \partial\Omega_\alpha))^{\frac{2}{q-1}}$$

where $c_4 = c_4(N, q, c_1) > 0$ which implies

$$|\nabla v(y)| \leq c'_4 \left(1 + (\text{dist}(y, \partial\Omega_\alpha))^{-\frac{1}{q-1}} \right) \quad \forall y \in \Omega_\alpha. \quad (2.11)$$

Then (1.21) and (1.22) follow.

Assume now that there exists a ground state u . Fix $y \in \mathbb{R}^N$ and consider $\{y_n\} \subset \mathbb{R}^N$ such that $|y_n| = 2n > |y|$. We apply (2.11) with $\Omega_\alpha = B_n(y_n)$. Then

$$|\nabla v(y)| \leq c'_4 \left(1 + |2n - |y||^{-\frac{1}{q-1}} \right),$$

and letting $n \rightarrow \infty$ we infer

$$|\nabla v(y)| \leq c'_4 \quad \forall y \in \mathbb{R}^N. \quad (2.12)$$

Hence, by the definition of v and y we see that

$$|\nabla u(x)| \leq c'_4 M^{-\frac{p+1}{(p+1)q-2p}} \quad \forall x \in \mathbb{R}^N$$

which is exactly (1.22). \square

2.1.2 Proof of Theorem A'

Suppose $1 < q < \frac{2p}{p+1}$. By scaling we reduce to the case $M = 1$ and we replace u by v defined by (1.6) as in the proof of Theorem A with $\alpha = M^{\frac{p-1}{2p-(p+1)q}}$. From (2.7) with $\nu = \frac{1}{4}$ the function $z = |\nabla v|^2$ satisfies

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{N} + \frac{1}{2N}z^q - pv^{p-1}z \leq c_1 \frac{|\nabla z|^2}{z}. \quad (2.13)$$

By Hölder's inequality,

$$pv^{p-1}z \leq \frac{1}{4N}z^q + p(4Np)^{q'-1}v^{(p-1)q'}.$$

Since $(p-1)q' = 2p + \frac{2p-(p+1)q}{q-1}$ we derive

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{N} \left(1 - 4^{q'-1}p^{q'}N^{q'}v^{\frac{2p-(p+1)q}{q-1}}\right) + \frac{1}{4N}z^q \leq c_1 \frac{|\nabla z|^2}{z}.$$

If $\max v \leq c_{N,p,q} := (4^{q'-1}p^{q'}N^{q'})^{-\frac{q-1}{2p-(p+1)q}}$, we obtain

$$-\frac{1}{2}\Delta z + \frac{1}{4N}z^q \leq c_1 \frac{|\nabla z|^2}{z},$$

which implies that $z = 0$ by Lemma 2.2, hence v is constant and thus $v = 0$ from the equation. \square

Remark. If u is a positive ground state of (1.2) radial with respect to 0, it satisfies $u_r(0) = 0$ and it is a decreasing function of r . The previous theorem asserts that it must satisfy

$$u(0) > c_{N,p,q} M^{\frac{2}{2p-(p+1)q}}. \quad (2.14)$$

2.1.3 Proof of Theorem C

Suppose $\frac{2p}{p+1} = q$. For $A > 0$ we consider the expression

$$\begin{aligned} & (u^p + A|\nabla u|^q)^2 - Npu^{p-1}|\nabla u|^2 \\ &= \left(u^p + A|\nabla u|^q - \sqrt{Np}u^{\frac{p-1}{2}}|\nabla u|\right) \left(u^p + A|\nabla u|^q + \sqrt{Np}u^{\frac{p-1}{2}}|\nabla u|\right). \end{aligned}$$

Now the function $Z \mapsto \Phi_A(Z) = u^p + AZ^q - \sqrt{Np} u^{\frac{p-1}{2}} Z$ achieves its minimum at $Z_0 = \left(\frac{\sqrt{Np}}{qA}\right)^{\frac{p+1}{p-1}} u^{\frac{p+1}{2}}$ and

$$\Phi_A(Z_0) = \left[1 - \frac{p-1}{p+1} \left(\frac{N(p+1)^2}{4p} \right)^{\frac{p}{p-1}} A^{-\frac{p+1}{p-1}} \right] u^p.$$

Thus setting

$$M_{\dagger} = \left(\frac{p-1}{p+1} \right)^{\frac{p-1}{p+1}} \left(\frac{N(p+1)^2}{4p} \right)^{\frac{p}{p+1}}, \quad (2.15)$$

we obtain that if $A \geq M_{\dagger}$, then $\Phi_A(Z) \geq 0$ for all Z . Put $M_{\nu} = (1-\nu)M$ for $\nu \in (0, 1)$ such that $M_{\dagger} < M_{\nu}$, we derive from (2.7)

$$-\frac{1}{2}\Delta z + \frac{(u^p + M_{\dagger} z^{\frac{q}{2}})^2}{N} - pu^{p-1}z + \frac{M_{\nu}^2 - M_{\dagger}^2}{N} z^q \leq c_1 \frac{|\nabla z|^2}{z}, \quad (2.16)$$

which yields

$$-\frac{1}{2}\Delta z + \frac{M_{\nu}^2 - M_{\dagger}^2}{N} z^q \leq c_1 \frac{|\nabla z|^2}{z}.$$

Using again Lemma 2.2 we obtain

$$|\nabla u(x)| \leq c'_1 ((1-\nu)M - M_{\dagger})^{-\frac{1}{q-1}} (\text{dist}(x, \partial\Omega))^{-\frac{1}{q-1}}, \quad (2.17)$$

which is equivalent to (1.27). \square

2.2 Proof of Theorems B and B'

2.2.1 Proof of Theorem B

Since the result is known when $M \geq 0$ from Proposition 2.1, we can assume that $M = -m < 0$ and $N = 1, 2$ or $N \geq 3$ with $p < \frac{N}{N-2}$, u is a nonnegative supersolution of (1.2) in \overline{B}_R^c and we set $u = v^b$ with $b > 1$. Then

$$-\Delta v \geq (b-1) \frac{|\nabla v|^2}{v} + \frac{1}{b} v^{1+b(p-1)} - mb^{q-1} v^{(b-1)(q-1)} |\nabla v|^q. \quad (2.18)$$

Here again $q = \frac{2p}{p+1}$, setting $z = |\nabla v|^2$ we obtain

$$-\Delta v \geq \frac{\Phi(z)}{bv}$$

where

$$\Phi(z) = b(b-1)z - mb^{\frac{2p}{p+1}} v^{\frac{2+b(p-1)}{p+1}} z^{\frac{p}{p+1}} + v^{2+b(p-1)}.$$

Thus Φ achieves its minimum for

$$z_0 = \left(\frac{mpb^{q-1}}{(b-1)(p+1)} \right)^{p+1} b^{p-1} v^{2+b(p-1)}$$

and

$$\Phi(z_0) = v^{2+b(p-1)} \left(1 - \frac{p^p}{(p+1)^{p+1}} \left(\frac{b}{b-1} \right)^p m^{p+1} \right). \quad (2.19)$$

In order to ensure the optimal choice, when $N \geq 3$ we take $1 + b(p-1) = \frac{N}{N-2}$, hence $b = \frac{2}{(N-2)(p-1)}$ which is larger than 1 because $p < \frac{N}{N-2}$. Finally

$$\Phi(z_0) = v^{\frac{N}{N-2}+1} \left(1 - \frac{1}{(p+1)^{p+1}} \left(\frac{2p}{N-p(N-2)} \right)^p m^{p+1} \right).$$

Hence, if

$$m < (p+1) \left(\frac{N-p(N-2)}{2p} \right)^{\frac{p}{p+1}} = \mu^*(N), \quad (2.20)$$

we have for some $\delta > 0$,

$$-\Delta v \geq \delta v^{\frac{N}{N-2}}, \quad (2.21)$$

and by Proposition 2.1 that is no positive solution in an exterior domain of \mathbb{R}^N .

If $N = 2$ for a given $b > 1$ we have from (2.19) that if

$$m < (p+1) \left(\frac{b-1}{bp} \right)^{\frac{p}{p+1}},$$

then, for some $\delta > 0$,

$$-\Delta v \geq \delta v^{1+b(p-1)}. \quad (2.22)$$

The result follows from Proposition 2.1 by choosing b large enough. \square

2.2.2 Proof of Theorem B'

1- We assume that such a supersolution u exists and we denote $u = e^v$, then

$$-\Delta v \geq F(|\nabla v|^2), \quad (2.23)$$

where

$$F(X) = X + e^{(p-1)v} + M e^{\frac{p-1}{p+1}v} X^{\frac{p}{p+1}}.$$

Clearly, if $M \geq 0$, then $F(X) \geq 0$ for any $X \geq 0$. Next we assume $M < 0$, then

$$F(X) \geq F(X_0) = e^{(p-1)v} \left(1 - p^p \left(\frac{|M|}{p+1} \right)^{p+1} \right) = e^{(p-1)v} \left(1 - \left(\frac{|M|}{\mu^*(2)} \right)^{p+1} \right).$$

Hence, if $|M| \leq \mu^*(2)$, v is a positive superharmonic function in Ω which tends to infinity on the boundary. Such a function is larger than the harmonic function with boundary value $k > 0$ for any k (and taking the value $\min_{|x|=R} v(x)$ for R large enough if Ω is an exterior domain). Letting $k \rightarrow \infty$ we derive a contradiction.

2- Let $R > 0$ such that $\Omega^c \subset B_R$ and let w be the solution of

$$\begin{aligned} -\Delta w - ae^{(p-1)w} &= 0 && \text{in } B_R \cap \Omega \\ \lim_{\text{dist}(x, \partial B_R) \rightarrow 0} w(x) &= -\infty \\ \lim_{\text{dist}(x, \partial \Omega) \rightarrow 0} w(x) &= \infty, \end{aligned} \quad (2.24)$$

with $a = 1 - \left(\frac{|M|}{\mu^*(2)}\right)^{p+1} < 0$, obtained by approximations. By the argument used in 1,

$$ae^{(p-1)w} \leq |\nabla w|^2 + e^{(p-1)w} - |M| e^{\frac{p-1}{p+1}w} |\nabla w|^{\frac{2p}{p+1}},$$

hence

$$-\Delta w \leq |\nabla w|^2 + e^{(p-1)w} - |M| e^{\frac{p-1}{p+1}w} |\nabla w|^{\frac{2p}{p+1}}.$$

Therefore $v = e^w$ is nonnegative and satisfies

$$\begin{aligned} -\Delta v - v^p + |M| |\nabla v|^{\frac{2p}{p+1}} &\leq 0 && \text{in } B_R \cap \Omega \\ v &= 0 && \text{on } \partial B_R \\ \lim_{\text{dist}(x, \partial \Omega) \rightarrow 0} v(x) &= \infty. \end{aligned} \quad (2.25)$$

Next we extend v by zero in B_R^c and denote by \tilde{v} the new function. It is a nonnegative subsolution of (1.2) which tends to ∞ on $\partial \Omega$. For constructing a supersolution we recall that if $M \leq -\mu^*(1)$ there exist two types of explicit solutions of

$$-u'' = u^p + M |u'|^{\frac{2p}{p+1}} \quad (2.26)$$

defined on \mathbb{R} by $U_{j,M}(t) = \infty$ for $t \leq 0$ and $U_{j,M}(t) = X_{j,M} t^{-\frac{2}{p-1}}$, $j=1,2$, for $t > 0$ where $X_{1,M}$ and $X_{2,M}$ are respectively the smaller and the larger positive root of

$$X^{p-1} - |M| \left(\frac{2}{p-1}\right)^{\frac{2}{p+1}} X^{\frac{p-1}{p+1}} + \frac{2(p+1)}{(p-1)^2} = 0. \quad (2.27)$$

Since Ω^c is convex it is the intersection of all the closed half-spaces which contain it and we denote by \mathcal{H}_Ω the family of such hyperplanes which are touching $\partial \Omega$. If $H \in \mathcal{H}_\Omega$ let \mathbf{n}_H be the normal direction to H , inward with respect to Ω , $\mathcal{H}_+ = \{x \in \mathbb{R}^N : \langle \mathbf{n}_H, x - \mathbf{n}_H \rangle > 0\}$ and we define U_H in the direction \mathbf{n}_H by putting

$$U_H(x) = U_{2,M}(\langle \mathbf{n}_H, x - \mathbf{n}_H \rangle) = X_{2,M} (\langle \mathbf{n}_H, x - \mathbf{n}_H \rangle)^{-\frac{2}{p-1}} \quad \text{for all } x \in \mathcal{H}_+.$$

Hence and set, for $x \in \Omega := \cap_{H \in \mathcal{H}_\Omega} \mathcal{H}_+$,

$$u_\Omega(x) = \inf_{H \in \mathcal{H}_\Omega} U_H(x). \quad (2.28)$$

Then u_Ω is a nonnegative supersolution of (1.2) in Ω and

$$u_\Omega(x) \geq X_{2,M} (\text{dist } x, \Omega)^{-\frac{2}{p-1}} \quad \forall x \in \Omega.$$

Next $v_\Omega = \ln u_\Omega$ blows up on $\partial\Omega$, is finite on ∂B_R and satisfies

$$-\Delta v_\Omega - ae^{(p-1)v_\Omega} \geq 0 \quad \text{in } B_R \cap \Omega. \quad (2.29)$$

By comparison with w since $a < 0$, $v_\Omega \geq w$. Hence $u_\Omega \geq v$ in $B_R \setminus \Omega^c$. Extending v by zero as \tilde{v} we obtain $u_\Omega \geq \tilde{v}$ in Ω^c . Hence u_Ω is a supersolution in Ω^c where it dominates the subsolution \tilde{v} . It follows by [29, Theorem 1-4-6] that there exists a solution u of (1.2) satisfying $\tilde{v} \leq u \leq u_\Omega$, which ends the proof. \square

3 The refined Bernstein method

The method is a combination of the one used in the previous proofs. It is based upon the replacement of the unknown by setting first $u = v^{-\beta}$ as in [19] and [10] and the study of the equation satisfied by $|\nabla v|$. However we do not use integral techniques. Since u is a positive solution of (1.2) in B_R , the function v is well defined and satisfies

$$-\Delta v + (1 + \beta) \frac{|\nabla v|^2}{v} + \frac{1}{\beta} v^{1-\beta(p-1)} + M |\beta|^{q-2} \beta v^{(\beta+1)(1-q)} |\nabla v|^q = 0 \quad (3.1)$$

in B_R . We set

$$z = |\nabla v|^2, \quad s = 1 - q - \beta(q-1) = (1-q)(\beta+1), \quad \sigma = 1 - \beta(p-1),$$

and derive

$$\Delta v = (1 + \beta) \frac{z}{v} + \frac{1}{\beta} v^\sigma + M |\beta|^{q-2} \beta v^s z^{\frac{q}{2}}. \quad (3.2)$$

Combining Bochner's formula and Schwarz identity we have classically

$$\frac{1}{2} \Delta z \geq \frac{1}{N} (\Delta v)^2 + \langle \nabla \Delta v, \nabla v \rangle.$$

We explicit the different terms

$$\begin{aligned} (\Delta v)^2 &= (1 + \beta)^2 \frac{z^2}{v^2} + M^2 \beta^{2(q-1)} v^{2s} z^q + \frac{v^{2\sigma}}{\beta^2} + 2M(1 + \beta) |\beta|^{q-2} \beta v^{s-1} z^{1+\frac{q}{2}} \\ &\quad + \frac{2(1 + \beta)}{\beta} v^{\sigma-1} z + 2M |\beta|^{q-2} v^{s+\sigma} z^{\frac{q}{2}}, \\ \nabla \Delta v &= (1 + \beta) \frac{\nabla z}{v} - \frac{(1 + \beta)z}{v^2} \nabla v + \frac{\sigma}{\beta} v^{\sigma-1} \nabla v + Ms |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}} \nabla v \\ &\quad + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \nabla z, \\ \langle \nabla \Delta v, \nabla v \rangle &= \left(\frac{1 + \beta}{v} + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \right) \langle \nabla z, \nabla v \rangle - \frac{(1 + \beta)z^2}{v^2} + \frac{\sigma}{\beta} v^{\sigma-1} z \\ &\quad + Ms |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1}. \end{aligned}$$

Hence

$$\begin{aligned} -\frac{1}{2} \Delta z + \frac{1}{N} (\Delta v)^2 + \left(\frac{1 + \beta}{v} + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \right) \langle \nabla z, \nabla v \rangle \\ - \frac{(1 + \beta)z^2}{v^2} + \frac{\sigma}{\beta} v^{\sigma-1} z + Ms |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} \leq 0. \end{aligned} \quad (3.3)$$

3.1 Proof of Theorem D

We develop the term $(\Delta v)^2$ in (3.3) and get

$$\begin{aligned} & -\frac{1}{2}\Delta z + \left(\frac{(1+\beta)^2}{N} - (1+\beta) \right) \frac{z^2}{v^2} + \frac{M^2\beta^{2(q-1)}}{N} v^{2s} z^q + M \left(s + \frac{2(1+\beta)}{N} \right) |\beta|^{q-2} \beta v^{s-1} z^{1+\frac{q}{2}} \\ & + \frac{v^{2\sigma}}{N\beta^2} + \left(\frac{1+\beta}{v} + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \right) \langle \nabla z, \nabla v \rangle + \frac{N\sigma + 2(1+\beta)}{N\beta} v^{\sigma-1} z + \frac{2M|\beta|^{q-2}}{N} v^{s+\sigma} z^{\frac{q}{2}} \\ & \leq 0. \end{aligned} \tag{3.4}$$

Next we set $z = v^{-k}Y$ where k is a real parameter. Then $\nabla z = -kv^{-k-1}Y\nabla v + v^{-k}\nabla Y$,

$$\langle \nabla z, \nabla v \rangle = -kv^{-k-1}Yz + v^{-k}\langle \nabla Y, \nabla v \rangle = -kv^{-2k-1}Y^2 + v^{-k}\langle \nabla Y, \nabla v \rangle,$$

$$\frac{\langle \nabla z, \nabla v \rangle}{v} = -kv^{-2k-2}Y^2 + v^{-k-1}\langle \nabla Y, \nabla v \rangle,$$

$$Mv^s z^{\frac{q}{2}-1} \langle \nabla z, \nabla v \rangle = -kMv^{s-\frac{qk}{2}-k-1}Y^{\frac{q}{2}+1} + Mv^{s-\frac{qk}{2}}Y^{\frac{q}{2}-1} \langle \nabla Y, \nabla v \rangle,$$

$$\begin{aligned} -\Delta z &= \operatorname{div} (kv^{-k-1}Y\nabla v - v^{-k}\nabla Y) \\ &= kv^{-k-1}Y\Delta v - k(k+1)v^{-k-2}Yz + 2kv^{-k-1}\langle \nabla Y, \nabla v \rangle - v^{-k}\Delta Y \\ &= kv^{-k-1}Y\Delta v - k(k+1)v^{-2k-2}Y^2 + 2kv^{-k-1}\langle \nabla Y, \nabla v \rangle - v^{-k}\Delta Y. \end{aligned}$$

From (3.2)

$$\Delta v = (1+\beta)v^{-k-1}Y + \frac{1}{\beta}v^\sigma + M|\beta|^{q-2}\beta v^{s-k\frac{q}{2}}Y^{\frac{q}{2}},$$

therefore

$$\begin{aligned} -\Delta z &= k(\beta-k)v^{-2k-2}Y^2 + \frac{k}{\beta}v^{\sigma-k-1}Y + kM|\beta|^{q-2}\beta v^{s-k\frac{q}{2}-k-1}Y^{\frac{q}{2}+1} \\ &\quad + 2kv^{-k-1}\langle \nabla Y, \nabla v \rangle - v^{-k}\Delta Y. \end{aligned}$$

Replacing $\langle \nabla z, \nabla v \rangle$ and Δz given by the above expressions in (3.4) and z by $v^{-k}Y$, leads to

$$\begin{aligned} & -\Delta Y + \left(\frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) \right) v^{-k-2}Y^2 + \frac{v^{2\sigma+k}}{N\beta^2} + \frac{M^2\beta^{2(q-1)}}{N} v^{2s+k-kq}Y^q \\ & + \left(\frac{k+\beta+1}{v} + \frac{Mq|\beta|^{q-2}\beta}{2} v^{s+k-k\frac{q}{2}}Y^{\frac{q}{2}-1} \right) \langle \nabla Y, \nabla v \rangle + \frac{2M|\beta|^{q-2}}{N} v^{s+\sigma+k-k\frac{q}{2}}Y^{\frac{q}{2}} \\ & + \left(s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right) M|\beta|^{q-2}\beta v^{s-k\frac{q}{2}-1}Y^{1+\frac{q}{2}} + \frac{1}{\beta} \left(\frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1}Y \leq 0. \end{aligned}$$

For $\epsilon_1, \epsilon_2 > 0$,

$$\frac{1}{v} |\langle \nabla Y, \nabla v \rangle| \leq \epsilon_1 v^{-k-2}Y^2 + \frac{1}{4\epsilon_1} \frac{|\nabla Y|^2}{Y},$$

$$v^{s+k-k\frac{q}{2}}Y^{\frac{q}{2}-1} |\langle \nabla Y, \nabla v \rangle| \leq \epsilon_2 v^{2s-kq+k}Y^q + \frac{1}{4\epsilon_2} \frac{|\nabla Y|^2}{Y}.$$

Hence

$$\begin{aligned}
 -\Delta Y &+ \frac{v^{2\sigma+k}}{N\beta^2} + \frac{2M|\beta|^{q-2}}{N} v^{s+\sigma+k-k\frac{q}{2}} Y^{\frac{q}{2}} + \left(\frac{M^2\beta^{2(q-1)}}{N} - \frac{Mq\epsilon_2|\beta|^{q-1}}{2} \right) v^{2s+k-kq} Y^q \\
 &+ \left(\frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1|\epsilon_1 \right) v^{-k-2} Y^2 \\
 &+ \frac{1}{\beta} \left(\frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y + \left(s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right) M|\beta|^{q-2} \beta v^{s-k\frac{q}{2}-1} Y^{1+\frac{q}{2}} \\
 &\leq \left(\frac{|k+\beta+1|}{\epsilon_1} + \frac{Mq|\beta|^{q-1}}{2\epsilon_2} \right) \frac{|\nabla Y|^2}{4Y}.
 \end{aligned} \tag{3.5}$$

We first choose $\epsilon_2 = \frac{M|\beta|^{q-1}}{qN}$, then

$$\begin{aligned}
 -\Delta Y &+ \frac{v^{2\sigma+k}}{N\beta^2} + \left(\frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1|\epsilon_1 \right) v^{-k-2} Y^2 \\
 &+ \frac{1}{\beta} \left(\frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y + \frac{M^2\beta^{2(q-1)}}{2N} v^{2s+k-kq} Y^q + \frac{2M|\beta|^{q-2}}{N} v^{s+\sigma+k-k\frac{q}{2}} Y^{\frac{q}{2}} \\
 &+ \left(s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right) M|\beta|^{q-2} \beta v^{s-k\frac{q}{2}-1} Y^{1+\frac{q}{2}} \\
 &\leq \left(\frac{|k+\beta+1|}{\epsilon_1} + \frac{Nq^2}{2} \right) \frac{|\nabla Y|^2}{4Y}.
 \end{aligned} \tag{3.6}$$

In order to show the sign of the terms on the left in (3.5), we separate the terms containing the coefficient M from the ones which do not contain it. Indeed these last terms are associated to the mere Lane-Emden equation (1.3) which is treated, as a particular case, in [6, Theorem B] where the exponents therein are $q = 0$, and $p \in \left(1, \frac{N+3}{N-1}\right)$. We set

$$\begin{aligned}
 H_{\epsilon_1,1} &= \frac{v^{2\sigma+k}}{N\beta^2} + \left(\frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1|\epsilon_1 \right) v^{-k-2} Y^2 \\
 &\quad + \frac{1}{\beta} \left(\frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y \\
 &= v^{2\sigma+k} \tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma} Y),
 \end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
 \tilde{H}_{\epsilon_1,1}(t) &= \left(\frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1|\epsilon_1 \right) t^2 \\
 &\quad + \frac{1}{\beta} \left(\frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) t + \frac{1}{N\beta^2},
 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
 H_{M,2} &= \frac{M^2\beta^{2(q-1)}}{2N} v^{2s+k-kq} Y^q + \frac{2M|\beta|^{q-2}}{N} v^{s+\sigma+k-k\frac{q}{2}} Y^{\frac{q}{2}} \\
 &\quad + \left(s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right) M|\beta|^{q-2} \beta v^{s-k\frac{q}{2}-1} Y^{1+\frac{q}{2}}.
 \end{aligned} \tag{3.9}$$

Then

$$-\Delta Y + v^{2\sigma+k} \tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma} Y) + H_{M,2} \leq \left(\frac{|k + \beta + 1|}{\epsilon_1} + \frac{Nq^2}{2} \right) \frac{|\nabla Y|^2}{4Y}.$$

The sign of $\tilde{H}_{\epsilon_1,1}$ depends on its discriminant \mathcal{D}_{ϵ_1} which is a polynomial in its coefficients. Then if for $\epsilon_1 = 0$ this discriminant is negative \mathcal{D}_0 is negative, the discriminant \mathcal{D}_{ϵ_1} of $\tilde{H}_{\epsilon_1,1}$ shares this property for $\epsilon_1 > 0$ small enough and therefore $H_{\epsilon_1,1}$ is positive. The proof is similar as the one of [6, Theorem B] in case (i) but for the sake of completeness we recall the main steps. Firstly

$$\mathcal{D}'_0 := N^2 \beta^2 \mathcal{D}_0 = \left(\frac{Nk}{2} + \sigma N + 2(1 + \beta) \right)^2 - 4 \left(\frac{Nk(\beta - k)}{2} + (1 + \beta)^2 - N(k + 1)(\beta + 1) \right).$$

Then

$$\mathcal{D}'_0 = \left(\frac{N(p-1)}{4} - 1 \right) (2\sigma + k)^2 + 2(p-1)(2\sigma + k) + \tilde{L}$$

where $\tilde{L} = (p-1)k^2 + p(\lambda + 2)^2 > 0$. Put

$$S = \frac{2\sigma + k}{k + 2} = 1 - \frac{2\beta(p-1)}{k + 2} \quad \text{and} \quad \mathcal{T}(S) = \left(\frac{(N-1)(p-1)}{4} - 1 \right) S^2 + (p-1)S + p.$$

After some computations we get, if $k \neq -2$,

$$\mathcal{D}'_1 := \frac{(p-1)\mathcal{D}'_0}{(k+2)^2} = (p-1) \left(\frac{k}{k+2} - \frac{S}{2} \right)^2 + \mathcal{T}(S). \quad (3.10)$$

We choose $S > 2$ such that $\frac{k}{k+2} - \frac{S}{2} = 0$, hence $\beta = \frac{2-k}{2(p-1)}$. If $p < \frac{N+3}{N-1}$ the coefficient of S^2 in $\mathcal{T}(S)$ is negative. Hence $\mathcal{T}(S) < 0$ provided S is large enough which is satisfied if $k < -2$ with $|k + 2|$ small enough. We infer from this that $\beta > 0$, $\mathcal{D}_0 < 0$ and $\tilde{H}_{\epsilon_1,1} > 0$ if ϵ_1 is small enough. In particular $\tilde{H}_{\epsilon_1,1}(t) \geq c_6(t^2 + 1)$ for some $c_6 = c_6(N, p, q) > 0$, which means

$$v^{2\sigma+k} \tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma} Y) \geq c_6 \left(v^{-k-2} Y^2 + v^{2\sigma+k} \right). \quad (3.11)$$

Secondly the positivity of $H_{M,2}$ is ensured, as β and M are positive, by the positivity of

$$\mathcal{A} := s + \frac{2(1 + \beta)}{N} - \frac{k(q-1)}{2}.$$

Replacing s by its value, we obtain, since $1 < q < \frac{N+2}{N}$ and $\beta + \frac{2+k}{2} > 0$, which can be assume by taking $|k + 2|$ small enough,

$$\mathcal{A} = 2 \frac{1 + \beta}{N} - (q-1) \left(\beta + 1 + \frac{k}{2} \right) > -\frac{k}{N}$$

Then we deduce that

$$-\Delta Y + c_6 \left(v^{-k-2} Y^2 + v^{2\sigma+k} \right) \leq c_7 \frac{|\nabla Y|^2}{Y}, \quad (3.12)$$

and $c_7 = c_7(N, p, q) > 0$ is independent of M . Since $S = 1 - \frac{2\beta(p-1)}{k+2} = 1 - \frac{2-k}{k+2} = \frac{2k}{k+2} > 0$, we have

$$2Y^{\frac{2S}{S+1}} = 2 \left(\frac{Y^2}{v^{k+2}} \right)^{\frac{S}{S+1}} v^{\frac{(k+2)S}{S+1}} \leq \frac{Y^2}{v^{k+2}} + v^{(k+2)S} = \frac{Y^2}{v^{k+2}} + v^{2\sigma+k}. \quad (3.13)$$

From this we infer the inequality

$$-\Delta Y + 2c_6 Y^{\frac{2S}{S+1}} \leq c_7 \frac{|\nabla Y|^2}{Y}. \quad (3.14)$$

Then we derive from Lemma 2.2 that in the ball B_R there holds

$$Y(0) \leq c_8 R^{-\frac{2(S+1)}{S-1}} = c_8 R^{-2+\frac{2(k+2)}{\beta(p-1)}}. \quad (3.15)$$

From this it follows

$$\left| \nabla u^{-\frac{2+k}{2\beta}}(0) \right| \leq \frac{|k+2|}{2} \sqrt{c_8} R^{-1+\frac{k+2}{\beta(p-1)}}. \quad (3.16)$$

Setting $a = -\frac{k+2}{2\beta} > 0$ we get that for any domain $\Omega \subset \mathbb{R}^N$ any positive solution in Ω satisfies

$$|\nabla u^a(x)| \leq \frac{|k+2|}{2} \sqrt{c_8} (\text{dist}(x, \partial\Omega))^{-1-\frac{2a}{p-1}} \quad \text{for all } x \in \Omega. \quad (3.17)$$

The non existence of any positive of (1.2) solution in \mathbb{R}^N follows classically. \square

Corollary 3.1 *Let Ω be a smooth domain in \mathbb{R}^N , $N \geq 2$ with a bounded boundary, $1 < p < \frac{N+3}{N-1}$, $1 < q < \frac{N+2}{N}$ and $M > 0$. If u is a positive solution of (1.2) in Ω there exists d_0 depending on Ω and $c_9 = c_9(N, p, q) > 0$ such that*

$$u(x) \leq c_9 \left((\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} + \max_{\text{dist}(z, \partial\Omega)=d_0} u(z) \right) \quad \text{for all } x \in \Omega. \quad (3.18)$$

Proof. It is similar to the one of [6, Corollary B-2]. \square

4 The integral method

4.1 Preliminary inequalities

We recall the next inequality [9, Lemma 3.1].

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a domain. Then for any positive $u \in C^2(\Omega)$, any nonnegative $\eta \in C_0^\infty(\Omega)$ and any real numbers m and d such that $d \neq m+2$, the following inequality holds*

$$A \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx - \frac{N-1}{N} \int_{\Omega} \eta u^m (\Delta u)^2 dx - B \int_{\Omega} \eta u^{m-1} |\nabla u|^2 \Delta u dx \leq R, \quad (4.1)$$

where

$$A = \frac{1}{4N} (2(N-m)d - (N-1)(m^2 + d^2)), \quad B = \frac{1}{2N} (2(N-1)m + (N+2)d),$$

and

$$R = \frac{m+d}{2} \int_{\Omega} u^{m-1} |\nabla u|^2 \langle \nabla u, \nabla \eta \rangle dx + \int_{\Omega} u^m \Delta u \langle \nabla u, \nabla \eta \rangle dx + \frac{1}{2} \int_{\Omega} u^m |\nabla u|^2 \Delta \eta dx.$$

It is noticeable that d is a free parameter which plays a role only in the coefficients of the integral terms. The following technical result is useful to deal with the multi-parameter constraints problems which occur in our construction. It was first used in [10] under a simpler form and extended in [9, Lemma 3.4].

Lemma 4.2 *For any $N \in \mathbb{N}$, $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$ there exist real numbers m and d verifying*

$$\begin{aligned}
 (i) \quad & d \neq m + 2, \\
 (ii) \quad & \frac{2(N-1)p}{N+2} < d, \\
 (iii) \quad & \max \left\{ -2, 1-p, \frac{(N-4)p-N}{2} \right\} < m \leq 0, \\
 (iv) \quad & 2(N-m)d - (N-1)(m^2 + d^2) > 0.
 \end{aligned} \tag{4.2}$$

4.2 Proof of Theorem E

Step 1: The integral estimates. Let $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$. We apply Lemma 4.1 to a positive solution $u \in C^2(\Omega)$ of (1.2), firstly with $q > 1$ and then with $q = \frac{2p}{p+1}$.

$$\begin{aligned}
 A \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx - \frac{N-1}{N} \int_{\Omega} \eta \left(u^{m+2p} + 2Mu^{m+p} |\nabla u|^q + M^2 u^m |\nabla u|^{2q} \right) dx \\
 - B \int_{\Omega} \eta u^{m-1} |\nabla u|^2 \Delta u dx \leq R.
 \end{aligned} \tag{4.3}$$

We multiply (1.2) by ηu^{m+p} and integrate over Ω . Then

$$\begin{aligned}
 \int_{\Omega} \eta \left(u^{m+2p} + Mu^{m+p} |\nabla u|^q \right) dx &= - \int_{\Omega} \eta u^{m+p} \Delta u dx \\
 &= \int_{\Omega} u^{m+p} \langle \nabla u, \nabla \eta \rangle dx + (m+p) \int_{\Omega} \eta u^{m+p-1} |\nabla u|^2 dx.
 \end{aligned}$$

We set

$$\begin{aligned}
 F &= \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx, \quad P = \int_{\Omega} \eta u^{m-1} |\nabla u|^{q+2} dx, \quad V = \int_{\Omega} \eta u^{m+2p} dx, \\
 T &= \int_{\Omega} \eta u^{m+p-1} |\nabla u|^2 dx, \quad W = \int_{\Omega} \eta u^{m+p} |\nabla u|^q dx, \quad U = \int_{\Omega} \eta u^m |\nabla u|^{2q} dx, \\
 S &= \int_{\Omega} u^{m+p} \langle \nabla u, \nabla \eta \rangle dx,
 \end{aligned}$$

so that there holds

$$AF - \frac{N-1}{N} (V + 2MW + M^2U) + BT + BMP \leq R, \tag{4.4}$$

and

$$V + MW = (m + p)T + S. \quad (4.5)$$

Eliminating V between (4.4) and (4.5), we get

$$AF + B_0T + M \left(BP - \frac{N-1}{N}W - \frac{N-1}{N}MU \right) \leq R - \frac{N-1}{N}S, \quad (4.6)$$

where

$$B_0 = B - \frac{N-1}{N}(m+p) = \frac{N+2}{2N}d - \frac{N-1}{N}p.$$

Also

$$2P = 2 \int_{\Omega} \eta u^m \frac{|\nabla u|^2}{u} |\nabla u|^q dx \leq \int_{\Omega} \eta u^m \left(\frac{|\nabla u|^4}{u^2} + |\nabla u|^{2q} \right) dx = F + U.$$

We fix now $q = \frac{2p}{p+1}$, then

$$\begin{aligned} U &= \int_{\Omega} \eta u^m |\nabla u|^{2q} dx = \int_{\Omega} \eta u^m \left(\frac{|\nabla u|}{\sqrt{u}} \right)^{4(q-1)} u^{2(q-1)} |\nabla u|^{4-2q} dx \\ &\leq \frac{p-1}{p+1} \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx + \frac{2}{p+1} \int_{\Omega} \eta u^{m+p-1} |\nabla u|^2 dx \\ &\leq \frac{p-1}{p+1} F + \frac{2}{p+1} T, \end{aligned} \quad (4.7)$$

hence

$$P \leq \frac{1}{2}F + \frac{1}{2}U \leq \frac{p}{p+1}F + \frac{1}{p+1}T \quad (4.8)$$

and

$$\begin{aligned} 2W &= 2 \int_{\Omega} \eta u^{m+p} |\nabla u|^q dx \leq \int_{\Omega} \eta u^{m+2p} dx + \int_{\Omega} \eta u^m |\nabla u|^{2q} dx = V + U \\ &\leq U + (m+p)T + S - MW. \end{aligned} \quad (4.9)$$

Next we assume that $|M| \leq 1$. From (4.7), (4.9), it follows that

$$W \leq U + (m+p)T + S \leq F + (m+p+1)T + S. \quad (4.10)$$

From now we fix m and d according Lemma 4.2. Therefore $A > 0$ by (4.2)-(iv) and $B > 0$ by combining (4.2)-(ii) and (4.2)-(iii). Furthermore $B_0 > 0$ by (4.2)-(ii). Hence, from (4.7), (4.8) and (4.10) we derive, since $\frac{N-1}{N} < 1$ and $m \leq 0$ from (4.2)-(ii)

$$\begin{aligned} \left| BP - \frac{N-1}{N}W - \frac{N-1}{N}MU \right| &\leq B(F+T) + F + (p+1)T + S + F + T, \\ &\leq (B+2)F + (B+p+2)T + S. \end{aligned}$$

Plugging these estimates into (4.6) we infer

$$AF + B_0T - |M| \left((B+2)F + (B+p+2)T + S \right) \leq R - \frac{N-1}{N}S. \quad (4.11)$$

Since A and B_0 are positive, there exists $\mu_1 \in (0, 1)$ such that for any $|M| < \mu_1$,

$$A_1 := A - |M|(B + 2) > \frac{A}{2} \quad \text{and} \quad B_1 := B_0 - |M|(B + p + 2) > \frac{B_0}{2}.$$

Set $A_2 = \min\{A_1, B_1\}$, then, and whatever is the sign of S ,

$$A_2(F + T) \leq |R| + |S|.$$

Using (4.7) and (4.8) we have

$$A_2(U + P) \leq 2A_2(F + T) \leq 2(|R| + |S|). \quad (4.12)$$

In the sequel we denote by c_j some positive constants depending on N and p . Then

$$U + P + F + T + W \leq c_1(|R| + |S|). \quad (4.13)$$

On the other hand, we have

$$|R| \leq c_2 \int_{\Omega} \left(u^{m-1} |\nabla u|^3 |\nabla \eta| + u^{m+p} |\nabla u| |\nabla \eta| + u^m |\nabla u|^{q+1} |\nabla \eta| + u^m |\nabla u|^2 |\Delta \eta| \right) dx.$$

Since

$$|\nabla u|^q = \left(\frac{|\nabla u|}{\sqrt{u}} \right)^q u^{\frac{q}{2}} \leq \frac{|\nabla u|^2}{u} + u^{\frac{q}{2}-\frac{q}{2}} = \frac{|\nabla u|^2}{u} + u^p,$$

we deduce

$$\int_{\Omega} u^m |\nabla u|^{q+1} |\nabla \eta| dx \leq \int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \eta| dx + \int_{\Omega} u^{m+p} |\nabla u| |\nabla \eta| dx.$$

Thus we derive from (4.13)

$$U + P + F + T + W \leq 2c_3 \left(\int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \eta| dx + \int_{\Omega} u^{m+p} |\nabla u| |\nabla \eta| dx + \int_{\Omega} u^m |\nabla u|^2 |\Delta \eta| dx \right). \quad (4.14)$$

From this point we can use the method developed in [10, p 599] for proving the Harnack inequality satisfied by positive solutions of (1.3) in Ω . We set $\eta = \xi^\lambda$ with $\xi \in C_0^\infty(\Omega)$ with value in $[0, 1]$ and $\lambda > 4$. For $\epsilon \in (0, 1)$ we have by the Hölder-Young inequality

$$\int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \xi^\lambda| dx \leq \frac{\epsilon}{4c_3} \int_{\Omega} u^{m-2} |\nabla u|^4 \xi^\lambda dx + C(\epsilon, c_3) \int_{\Omega} u^{m+2} |\nabla \xi|^4 \xi^{\lambda-4} dx, \quad (4.15)$$

$$\int_{\Omega} u^{m+p} |\nabla u| |\nabla \xi^p| dx \leq \frac{\epsilon}{4c_3} \int_{\Omega} u^{m+p-1} |\nabla u|^2 \xi^p dx + C(\epsilon, c_3) \int_{\Omega} u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx, \quad (4.16)$$

and

$$\int_{\Omega} u^m |\nabla u|^2 |\Delta \xi^p| dx \leq \frac{\epsilon}{4c_3} \int_{\Omega} u^{m-2} |\nabla u|^4 \xi^p dx + C(\epsilon, c_3) \int_{\Omega} u^{m+2} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \right) \xi^{\lambda-4} dx. \quad (4.17)$$

Hence

$$U + P + F + T + W \leq c_4 \left(\int_{\Omega} u^{m+2} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} dx + \int_{\Omega} u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx \right). \quad (4.18)$$

Let us denote by $c_4 X$ the right-hand side of (4.18). Combining (4.5), (4.16) and (4.18) we also get

$$S := \int_{\Omega} u^{m+p} |\nabla u| |\nabla \xi|^p dx \leq c_5 X \implies V := \int_{\Omega} u^{m+2p} \xi^p dx \leq c_6 X, \quad (4.19)$$

and we finally obtain

$$U + V + P + F + S + T + W \leq c_7 X. \quad (4.20)$$

Finally we estimate the different terms in X , using that $m + p > 0$ from (4.2)-(iii). For $\epsilon > 0$

$$\begin{aligned} \int_{\Omega} u^{m+2} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} dx &\leq \epsilon \int_{\Omega} u^{m+2p} \xi^{\lambda} dx \\ &+ C(\epsilon, c_7) \int_{\Omega} \xi^{\lambda-2 \frac{m+2p}{p-1}} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx, \end{aligned} \quad (4.21)$$

and

$$\int_{\Omega} u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx \leq \epsilon \int_{\Omega} u^{m+2p} \xi^{\lambda} dx + C(\epsilon, c_7) \int_{\Omega} \xi^{\lambda-2 \frac{m+2p}{p-1}} |\nabla \xi|^{\frac{2(m+2p)}{p-1}} dx. \quad (4.22)$$

At end we obtain

$$U + V + P + F + S + T + W \leq c_8 \int_{\Omega} \xi^{\lambda-2 \frac{m+2p}{p-1}} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx. \quad (4.23)$$

Step 2: The Harnack inequality. We suppose that $\Omega = B_R \setminus \{0\} := B_R^*$, fix $y \in B_{\frac{R}{2}}^*$, set $r = |y|$, then $B_r(y) \subset B_R^*$. Let $\xi \in C_0^\infty(B_r(y))$ such that $0 \leq \xi \leq 1$, $\xi = 1$ in $B_{\frac{r}{2}}(y)$, $|\nabla \xi| \leq cr^{-1}$ and $|\Delta \xi| \leq cr^{-2}$. We choose $\lambda > \max \left\{ 4, \frac{m+2p}{p+1} \right\}$, then

$$\int_{B_r(y)} \xi^{\lambda-2 \frac{m+2p}{p-1}} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx \leq c_9 r^{N - \frac{2(m+2p)}{p-1}},$$

and hence

$$\int_{B_{\frac{r}{2}}(y)} u^{m+2p} dx \leq V \leq c_{10} r^{N - \frac{2(m+2p)}{p-1}}. \quad (4.24)$$

We write (1.2) under the form

$$\Delta u + D(x)u + M \langle G(x), \nabla u \rangle = 0, \quad (4.25)$$

with

$$D(x) = u^{p-1} \quad \text{and} \quad G(x) = |\nabla u|^{-\frac{2}{p+1}} \nabla u.$$

Set $\sigma = \frac{m+2p}{p-1}$, then $\sigma > \frac{N}{2}$ by (4.2)-(iii) and

$$\int_{B_{\frac{r}{2}}(y)} D^\sigma(x) dx \leq V \leq c_{10} r^{N - \frac{2(m+2p)}{p-1}} = c_{10} r^{N-2\sigma}. \quad (4.26)$$

Next we estimate G . For $\tau, \omega, \gamma > 0$ and $\theta > 1$, we have with $\theta' = \frac{\theta}{\theta-1}$,

$$|\nabla u|^{(q-1)\tau} = u^\omega |\nabla u|^\gamma u^{-\omega} |\nabla u|^{(q-1)\tau-\gamma} \leq u^{\omega\theta'} |\nabla u|^{\gamma\theta} + u^{-\omega\theta} |\nabla u|^{((q-1)\tau-\gamma)\theta'}.$$

We fix

$$\tau = 2 \frac{2p+m}{p-1} = 2\sigma, \quad \omega = \frac{(2-m)(p+m-1)}{p+1} \quad \text{and} \quad \theta = \frac{p+1}{2-m}.$$

Then $\omega > 0$ and $\theta > 1$ from (4.2)-(iii), $\omega > 0$. Then $u^{\omega\theta'} |\nabla u|^{\gamma\theta} = u^{p+m-1} |\nabla u|^2$ and $u^{-\omega\theta} |\nabla u|^{((q-1)\tau-\gamma)\theta'} = u^{m-2} |\nabla u|^4$, thus

$$\int_{B_{\frac{r}{2}}(y)} |\nabla u|^{(q-1)\tau} dx \leq F + T \leq c_{11} \int_{\Omega} \xi^{\lambda-2\frac{m+2p}{p-1}} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right)^{\frac{m+2p}{2(p-1)}} dx.$$

This implies

$$\int_{B_{\frac{r}{2}}(y)} G^\tau(x) dx \leq c_{12} r^{N-\tau}, \quad (4.27)$$

with $\tau > N$. Using the results of [28, Section 5], we infer that a Harnack inequality, uniform with respect to r , is satisfied. Hence there exists $c_{13} > 0$ depending on N, p such that for any $r \in (0, \frac{R}{2}]$ and y such that $|y| = r$ there holds

$$\max_{z \in B_{\frac{r}{2}}(y)} u(z) \leq c_{13} \min_{z \in B_{\frac{r}{2}}(y)} u(z) \quad \forall 0 < r \leq \frac{R}{2} \quad \forall y \text{ s.t. } |y| = r, \quad (4.28)$$

which implies

$$u(x) \leq c_{14} u(x') \quad \forall x, x' \in \mathbb{R}^N \quad \text{s.t. } |x| = |x'| \leq \frac{R}{2}, \quad (4.29)$$

and actually $c_{14} = c_{13}^7$ by a simple geometric construction. By (4.24)

$$r^N \omega_N r^N \left(\min_{z \in B_{\frac{r}{2}}(y)} u(z) \right)^{m+2p} \leq 4^N c_{10} r^{N - \frac{2(m+2p)}{p-1}},$$

where ω_N is the volume of the unit N -ball. This implies

$$u(x) \leq c_{14} |x|^{-\frac{2}{p-1}} \quad \forall x \in B_{\frac{R}{2}}^*. \quad (4.30)$$

The proof follows. \square

Remark. Using standard rescaling techniques (see e.g. [29, Lemma 3.3.2]) the gradient estimate holds

$$|\nabla u(x)| \leq c_{15} |x|^{-\frac{p+1}{p-1}} \quad \forall x \in B_{\frac{R}{3}}^*. \quad (4.31)$$

And the next estimate for a solution u in a domain Ω satisfying the interior sphere condition with radius R is valid

$$u(x) \leq c_{14} (\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} \quad \forall x \in \Omega \text{ s.t. } \text{dist}(x, \partial\Omega) \leq \frac{R}{2}. \quad (4.32)$$

5 Radial ground states

We recall that if $q \neq \frac{2p}{p+1}$ and $M \neq 0$, (1.2) can be reduced to the case $M = \pm 1$ by using the transformation (1.15). Since any ground state u of (1.2) radial with respect to 0 is decreasing (this is classical and straightforward), it achieves its maximum at 0 and the following equivalence holds if v is defined by (1.15)

$$\begin{aligned} -u'' - \frac{N-1}{r}u' &= |u|^{p-1}u + M|u_r|^q & \text{s.t.} \quad \max u &= u(0) = 1 \\ \iff -v'' - \frac{N-1}{r}v' &= |v|^{p-1}v \pm |v_r|^q & \text{s.t.} \quad \max v &= v(0) = |M|^{\frac{2}{(p+1)q-2p}}. \end{aligned} \quad (5.1)$$

Hence large or small values of M for u are exchanged into large or small values of $v(0)$ for v and in the sequel we will essentially express our results using the function u .

5.1 Energy functions

We consider first the energy function

$$r \mapsto H(r) = \frac{u^{p+1}}{p+1} + \frac{u'^2}{2}. \quad (5.2)$$

Then

$$H'(r) = M|u'|^{q+1} - \frac{N-1}{r}u'^2.$$

Hence, if $M \leq 0$, H is decreasing, a property often used in [25]. This implies in particular that a radial ground state satisfies

$$|u'(r)| \leq \sqrt{\frac{2}{p+1}} (u(0))^{\frac{p+1}{2}}. \quad (5.3)$$

A similar estimate holds in all the cases.

Proposition 5.1 *Let $M > 0$, $p, q > 1$. If u is a radial ground state solution of (1.2), then the function H defined in (5.2) is decreasing and in particular (5.3) holds.*

Proof. Let u be such a radial ground state. By Proposition 2.1 we must have $q > \frac{N}{N-1}$ and

$$\frac{r}{u'^2}H' = Mr|u'|^{q-1} + 1 - N \leq \frac{(N-1)q - N}{q-1} + 1 - N = -\frac{1}{q-1},$$

this implies the claim. \square

5.1.1 Exponential perturbations

As we have seen it in the introduction, if $q < \frac{2p}{p+1}$ equation (1.2) can be seen as a perturbation of the Lane-Emden equation (1.3) while if $q > \frac{2p}{p+1}$ it can be seen as a perturbation of the Riccati

equation (1.14). Two types of transformations can emphasize these aspects.

1) For $p > 1$ set

$$u(r) = r^{-\frac{2}{p-1}}x(t), \quad u'(r) = -r^{-\frac{p+1}{p-1}}y(t), \quad t = \ln r, \quad (5.4)$$

then

$$\begin{aligned} x_t &= \frac{2}{p-1}x - y \\ y_t &= -Ky + x^p + Me^{-\omega t}y^q \end{aligned} \quad (5.5)$$

with

$$K = \frac{(N-2)p - N}{p-1}, \quad (5.6)$$

and

$$\omega = \frac{(p+1)q - 2p}{p+1}. \quad (5.7)$$

If $q > \frac{2p}{p+1}$ (resp. $q < \frac{2p}{p+1}$), then $\omega > 0$ (resp. $\omega < 0$) system (5.7) is a perturbation of the Lane-Emden system

$$\begin{aligned} x_t &= \frac{2}{p-1}x - y \\ y_t &= -Ky + x^p, \end{aligned} \quad (5.8)$$

at ∞ (resp. $-\infty$). The following energy type function introduced in [20] is natural with (5.8)

$$\mathcal{N}(t) = \mathcal{L}(x(t), y(t)) = \frac{K}{p-1}x^2 - \frac{x^{p+1}}{p+1} - \left(\frac{2}{p-1}\right)^q Me^{-\omega t} \frac{x^{q+1}}{q+1} - \frac{1}{2} \left(\frac{2x}{p-1} - y\right)^2, \quad (5.9)$$

and it satisfies

$$\begin{aligned} \mathcal{N}'(t) &= \left(\frac{2x}{p-1} - y\right) \left[L \left(\frac{2x}{p-1} - y\right) - Me^{-\omega t} \left(\left(\frac{2x}{p-1}\right)^q - y^q \right) \right] \\ &\quad + \omega \left(\frac{2}{p-1}\right)^q Me^{-\omega t} \frac{x^{q+1}}{q+1}, \end{aligned} \quad (5.10)$$

where $L = N - 2 - \frac{4}{p-1} = K - \frac{2}{p-1}$. Relation (5.10) will be used later on.

2) For $p, q > 1$ set

$$u(r) = r^{-\frac{2-q}{q-1}}\xi(t), \quad u'(r) = -r^{-\frac{1}{q-1}}\eta(t), \quad t = \ln r, \quad (5.11)$$

then

$$\begin{aligned} \xi_t &= \frac{2-q}{q-1}\xi - \eta \\ \eta_t &= -\frac{(N-1)q - N}{q-1}\eta + e^{\bar{\omega}t}\xi^p + M\eta^q \end{aligned} \quad (5.12)$$

where

$$\bar{\omega} = \frac{p-1}{q-1}\omega. \quad (5.13)$$

Note that if $q < \frac{2p}{p+1}$ this system at ∞ endows the form

$$\begin{aligned}\xi_t &= \frac{2-q}{q-1}\xi - \eta \\ \eta_t &= -\frac{(N-1)q-N}{q-1}\eta + M\eta^q.\end{aligned}\tag{5.14}$$

It is therefore autonomous and much easier to study.

5.1.2 Pohozaev-Pucci-Serrin type functions

Let $\alpha, \gamma, \theta, \kappa$ be real parameters with $\alpha, \kappa > 0$. Set

$$\mathcal{Z}(r) = r^\kappa \left(\frac{u'^2}{2} + \frac{u^{p+1}}{p+1} + \alpha \frac{uu'}{r} - \gamma u' |u'|^q \right).\tag{5.15}$$

This type of function has been introduced in [25] in their study of equation (1.2) with $M = 1$ with specific parameters. We use it here to embrace all the values of M . We define \mathcal{U} by the identity

$$\mathcal{Z}' + \theta |u'|^{q-1} \mathcal{Z} = r^{\kappa-1} \mathcal{U}.\tag{5.16}$$

Then

$$\begin{aligned}\mathcal{U} &= \left(\frac{\kappa}{2} + \alpha + 1 - N \right) u'^2 + \left(\frac{\kappa}{p+1} - \alpha \right) u^{p+1} + \alpha(\kappa - N) \frac{uu'}{r} + \left(\frac{\theta}{p+1} - \gamma q \right) r u^{p+1} |u'|^{q-1} \\ &\quad + \left(M + \gamma + \frac{\theta}{2} \right) r |u'|^{q+1} + \left(((N-1)q - \kappa) \gamma - \alpha(\theta + M) \right) u |u'|^q - \gamma(\theta + qM) r u |u'|^{2q-1}.\end{aligned}\tag{5.17}$$

5.2 Some known results in the case $M < 0$

We recall the results of [14], [25] and [23] relative to the case $M < 0$.

Theorem 5.2 1- Let $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$.

1-(i) If $q > \frac{2p}{p+1}$, there is no ground state for any $M < 0$ ([25, Theorem C]).

1-(ii) If $1 < q < \frac{2p}{p+1}$ there exists a ground state when $|M|$ is large [14, Proposition 5.7] and there exists no ground state when $|M|$ is small ([23]).

2- Assume $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and let \bar{q} be the unique root in $(\frac{2p}{p+1}, p)$ of the quadratic equation

$$(N-1)(X-p)^2 - (N+2 - (N-2)p)((p+1)X - 2p)X = 0.$$

2-(i) If $\bar{q} \leq q < p$ there exists no ground state for any $M < 0$ ([25, Theorem C]).

2-(ii) If $\frac{2p}{p+1} < q < \bar{q}$, there exists no ground state for $|M|$. It is an open question whether there could exist a finite number of M for which there exists a ground state ([25, Theorem 4]).

2-(iii) If $1 < q < \frac{2p}{p+1}$, there exists a ground state for large $|M|$ ([14, Proposition 5.7]) and no ground state when $|M|$ is small ([23]).

3- Assume $p > \frac{N+2}{N-2}$ and $q > 1$ and let $Q_{N,p} = \frac{2(N-1)p}{2N+p+1} \in (\frac{2p}{p+1}, p)$.

3-(i) If $Q_{N,p} < q < p$ there exists a ground state for $|M|$ small.

3-(ii) If $1 < q \leq Q_{N,p}$ there exists a ground state for any $M < 0$ ([25, Theorem A]).

4- Assume $p = \frac{N+2}{N-2}$. There exists at least one $M < 0$ such that there exists a ground state if and only if $1 < q < p$. More precisely:

4-(i) If $\frac{2p}{p+1} < q < p$ there exists ground state if $|M|$ is small ([25, Theorem B]).

4-(ii) If $q \geq \frac{2p}{p+1}$ there exists a ground state for any $M < 0$ ([25, Theorem A]).

Remark. It is interesting to quote that when $M < 0$ and $q \geq \frac{2p}{p+1}$, there holds [25, Theorem 3],

$$u(r) = O(r^{-\frac{2}{p-1}}) \quad \text{and} \quad u'(r) = O(r^{-\frac{p+1}{p-1}}) \quad \text{when } r \rightarrow \infty.$$

5.3 The case $M > 0$

The next result is a consequence of Theorem A.

Theorem 5.3 *Let $M > 0$, $p > 1$ and $q > \frac{2p}{p+1}$ then there exists no radial ground state satisfying $u(0) = 1$ when M is large.*

Proof. Suppose that such a solution u exists. From Theorem A and Proposition 2.1 there holds

$$\sup_{r>0} |u'(r)| \leq c_{N,p,q} |M|^{-\frac{p+1}{(p+1)q-2p}} \quad \text{and} \quad \sup_{r>0} r^{\frac{p+1}{p-1}} |u'(r)| \leq c_{N,p}. \quad (5.18)$$

As a consequence, if $r > R > 0$,

$$\begin{aligned} 1 - u(r) &= u(0) - u(r) = u(0) - u(R) + u(R) - u(r) \leq c_{N,p,q} |M|^{-\frac{p+1}{(p+1)q-2p}} R + \int_R^\infty |u'(s)| ds \\ &\leq c_{N,p,q} |M|^{-\frac{p+1}{(p+1)q-2p}} R + c'_{N,p} R^{-\frac{2}{p-1}}, \end{aligned}$$

with $c'_{N,p} = \frac{p-1}{2} c_{N,p}$. Since $u(r) \rightarrow 0$ when $r \rightarrow \infty$, we take $R = |M|^{\frac{p-1}{(p+1)q-2p}}$ and derive

$$1 \leq (c_{N,p,q} + c'_{N,p}) |M|^{-\frac{2}{(p+1)q-2p}}, \quad (5.19)$$

and the conclusion follows. \square

Remark. If we use Proposition 5.1 we can make estimate (5.19) more precise.

5.3.1 The case $M > 0$, $1 < p \leq \frac{N+2}{N-2}$

It is a consequence of our general results that there is no radial ground state for large M or for small M when $1 < q \leq \frac{2p}{p+1}$ and $1 < p < \frac{N+2}{N-2}$. Indeed, if $1 < q < \frac{2p}{p+1}$ is a consequence of the equivalence statement between a priori estimate and non-existence of ground state proved in [23], and if $q = \frac{2p}{p+1}$ it follows from Theorems C and E. Actually in the radial case, the result is more general.

Theorem 5.4 *Let $M > 0$ and $1 < p < \frac{N+2}{N-2}$. If $1 < q \leq p$, there exists no radial ground state for any M . If $q > p$ there exists no radial ground state for M small enough.*

Proof. By Proposition 2.1, we may assume $N \geq 3$ and

$$\frac{N}{N-2} < p \leq \frac{N+2}{N-2} \quad \text{and} \quad q > \frac{N}{N-1}. \quad (5.20)$$

(i) Assume first $q < \frac{2p}{p+1}$. We use the system (5.5). Then ω , defined by (5.7) is negative. Hence the Leighton function \mathcal{N} defined by (5.9) is nonincreasing since $L \leq 0$ when $p \leq \frac{N+2}{N-2}$. Furthermore since $(x(t), y(t)) \rightarrow (0, 0)$ when $t \rightarrow -\infty$ and $e^{-\omega t} \rightarrow 0$, we get $\mathcal{N}(-\infty) = 0$ it follows that $\mathcal{N}(t) < 0$ for $t \in \mathbb{R}$. Moreover, by Proposition 2.1,

$$u(r) = O(r^{-\frac{2-q}{q-1}}) \quad \text{as } r \rightarrow \infty \iff x(t) = O(e^{\frac{q(p+1)-2p}{(p-1)(q-1)}t}) = o(1) \quad \text{as } t \rightarrow \infty$$

This implies $e^{-\omega t} x^{q+1}(t) = O(e^{\frac{2q(p+1)-2p}{(p-1)(q-1)}t}) = o(1)$ as $t \rightarrow \infty$ and $\mathcal{N}(\infty) = 0$, contradiction.

(ii) Assume next $\frac{2p}{p+1} \leq q \leq p$. We consider the function (5.15) with the parameters

$$\kappa = \frac{2(p+1)(N-1)}{p+3} = (p+1)\alpha \quad \text{and} \quad \gamma = -\frac{2M}{q(p+1)+2} = \frac{\theta}{q(p+1)},$$

already used by [25] when $M = -1$, and we get with \mathcal{U} defined by (5.16),

$$\mathcal{U} = \frac{2}{(p+3)^2} \frac{u|u'|}{r} (A + BM\chi + CM\chi^2) \quad \text{with } \chi = \frac{p+3}{2+q(p+1)} r |u'|^{q-1},$$

where

$$A = (N-1)(N+2-(N-2)p), \quad B = 2(N-1)(p-q), \quad C = q(q(p+1)-2p). \quad (5.21)$$

By our assumptions $A \geq 0$, $B \geq 0$ and $C > 0$. Hence $\mathcal{U} > 0$. This implies

$$\mathcal{Z}(r) = e^{-\int_0^r \theta |u'|^{q-1} ds} \mathcal{Z}(0) + \int_0^r e^{-\theta \int_s^r |u'|^{q-1} d\sigma} s^{\kappa-1} \mathcal{U}(s) ds = \int_0^r e^{-\theta \int_s^r |u'|^{q-1} d\sigma} s^{\kappa-1} \mathcal{U}(s) ds,$$

since $\mathcal{Z}(0) = 0$. If u is a ground state, then $u'(r) \rightarrow 0$ as $r \rightarrow \infty$, thus $u|u'|^q \leq u|u'|^{\frac{2p}{p+1}}$. Hence, from Proposition 2.1, $u'^2(r) = O(r^{-2\frac{p+1}{p-1}})$ as $r \rightarrow \infty$. The other terms $u^{p+1}(r)$, $r^{-1}u(r)u'(r)$ and $u|u'|^{\frac{2p}{p+1}}$ satisfy the same bound, hence

$$\mathcal{Z}(r) = O(r^{\kappa - \frac{2(p+1)}{p-1}}) = O(r^{\frac{2(p+3)(N-1)}{p+3} - \frac{2(p+1)}{p-1}}) = O(r^{\frac{2(p+1)((N-2)p-(N+2))}{(p+3)(p-1)}}).$$

Then $\mathcal{Z}(r) \rightarrow 0$ when $r \rightarrow \infty$, contradiction.

(iii) Suppose $q > p$ and u is a ground state. By Proposition 5.1 and (5.18), there holds

$$r |u'|^{q-1} = r |u'|^{\frac{p-1}{p+1}} |u'|^{q - \frac{2p}{p+1}} \leq c_{N,p}.$$

Then $\chi = \frac{p+3}{2+q(p+1)} r |u'|^{q-1} \leq c_{N,p}$. Hence, if $M \leq M_{N,p}$ for some $M_{N,p} > 0$, \mathcal{U} is positive as A is. We conclude as above. \square

5.3.2 The case $M > 0$, $p > \frac{N+2}{N-2}$

We recall that in Theorem C if $q = \frac{2p}{p+1}$ and $p > 1$ there is no ground state whenever $M > M_{N,p}$, see (1.26). In Theorem A' if $1 < q < \frac{2p}{p+1}$ and $p > 1$ there is no ground state u such that $u(0) = 1$ if M is too large. In the next result we complement Theorem 5.3 for small value of M in assuming $q > \frac{2p}{p+1}$.

Theorem 5.5 *If $p > \frac{N+2}{N-2}$ and $q \geq \frac{2p}{p+1}$ then there exist radial ground states for $M > 0$ small enough.*

Proof. First we consider the function \mathcal{Z} with $k = N$ and obtain

$$\mathcal{Z}(r) = r^N \left(\frac{u'^2}{2} + \frac{u^{p+1}}{p+1} + \alpha \frac{uu'}{r} - \gamma u |u'|^q \right).$$

The function vanishes at the origin. We compute \mathcal{U} from the identity $\mathcal{Z}' + \theta |u'|^{q-1} \mathcal{Z} = r^{N-1} \mathcal{U}$ and get

$$\begin{aligned} \mathcal{U} = & \left(\alpha - \frac{N-2}{2} \right) u'^2 + \left(\frac{N}{p+1} - \alpha \right) u^{p+1} + \left(\frac{\theta}{p+1} - \gamma q \right) r u^{p+1} |u'|^{q-1} \\ & + \left(M + \gamma + \frac{\theta}{2} \right) r |u'|^{q+1} + \left[((N-1)q - N) \gamma - \alpha(\theta + M) \right] u |u'|^q - \gamma(\theta + qM) r u |u'|^{2q-1}. \end{aligned}$$

If $\gamma = 0$ and $\theta = -2M$, then

$$\mathcal{U} = \left(\alpha - \frac{N-2}{2} \right) u'^2 + \left(\frac{N}{p+1} - \alpha \right) u^{p+1} - \frac{2M}{p+1} r u^{p+1} |u'|^{q-1} + \alpha M u |u'|^q.$$

If u is a regular solution which vanishes at some $r_0 > 0$, then $\mathcal{Z}(r_0) = 2^{-1} r_0^2 u'^N(r_0) > 0$. As $p > \frac{N+2}{N-2}$, by choosing $\alpha = \frac{1}{2} \left(\frac{N}{p+1} + \frac{N-2}{2} \right)$ we have $\frac{N}{p+1} < \alpha < \frac{N-2}{2}$. We define $\ell > 0$ by $(N-2)p - (N+2) = 4(p+1)\ell$, then $\frac{N-2}{2} - \alpha = \alpha - \frac{N}{p+1} = \ell$ and then

$$\mathcal{U} \leq -\ell(u'^2 + u^{p+1}) + M\alpha u |u'|^q.$$

Assume first $q < 2$, we have from Hölder's inequality and $0 < r \leq r_0$ where u is positive

$$u |u'|^q \leq \frac{q}{2} u'^2 + \frac{2-q}{2} |u|^{\frac{2}{2-q}} \leq u'^2 + |u|^{\frac{2}{2-q}},$$

and

$$\mathcal{U} + (\ell - M)u'^2 \leq M\alpha u^{\frac{2}{2-q}} - \ell u^{p+1} = \ell u^{p+1} \left(\frac{M\alpha}{\ell} u^{\frac{q(p+1)-2p}{2-q}} - 1 \right) \leq \ell u^{p+1} \left(\frac{M\alpha}{\ell} - 1 \right)$$

since $q \geq \frac{2p}{p+1}$ and $u \leq u(0) = 1$. Taking $M \leq \frac{\ell}{\alpha} = \frac{(N-2)p - N - 2}{(N-2)p + 3N - 2}$, \mathcal{U} is negative implying that $r \mapsto e^{-2M \int_0^r |u'|^{q-1} ds} \mathcal{Z}(r)$ is nonincreasing. Since $\mathcal{Z}(0) = 0$, $\mathcal{Z}(r) \leq 0$, a contradiction.

If $q = 2$, then $\mathcal{U} \leq -\ell(u'^2 + u^{p+1}) + M\alpha u'^2$ since $u \leq 1$ on $[0, r_0]$. We still infer that $\mathcal{U} \leq 0$ if we choose $M \leq \frac{\ell}{\alpha}$.

Finally, if $q > 2$, we have from Theorem A, $u' \leq C_{N,p,q} M^{-\frac{p+1}{(p+1)q-2p}}$. Therefore, using again the decay of u from $u(0) = 1$,

$$M\alpha u |u'|^q \leq M\alpha u |u'|^{q-2} u'^2 \leq M\alpha C_{N,p,q}^{q-2} M^{-\frac{(p+1)(q-2)}{(p+1)q-2p}} u'^2 = \alpha C_{N,p,q}^{q-2} M^{\frac{2}{(p+1)q-2p}} u'^2.$$

Hence $\mathcal{U} \leq -\left(\ell - \alpha C_{N,p,q}^{q-2} M^{\frac{2}{(p+1)q-2p}}\right) u'^2$. Taking

$$M^{\frac{2}{(p+1)q-2p}} \leq C_{N,p,q}^{2-q} \frac{(N-2)p - N - 2}{(N-2)p + 3N - 2}$$

we conclude that $\mathcal{U} < 0$ which ends the proof as in the previous cases. \square

Theorem F is the combination of Theorem 5.3, Theorem 5.4 and Theorem 5.5.

6 Separable solutions

We denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in \mathbb{R}^N . Then equation (1.2) takes the form

$$-u_{rr} - \frac{N-1}{r} u_r - \frac{1}{r^2} \Delta' u = |u|^{p-1} + M \left(u_r^2 + \frac{1}{r^2} |\nabla' u|^2 \right)^{\frac{q}{2}}, \quad (6.1)$$

where Δ' is the Laplace-Beltrami operator on S^{N-1} and ∇' the tangential gradient. If we look for separable nonnegative solutions of (1.2) i.e. solutions under the form $u(r, \sigma) = \psi(r)\omega(\sigma)$, then $q = \frac{2p}{p+1}$, $\psi(r) = r^{-\frac{2}{p-1}}$, and ω is a solution of

$$-\Delta' \omega + \frac{2K}{p-1} \omega = \omega^p + M \left(\left(\frac{2}{p-1} \right)^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{p+1}}, \quad (6.2)$$

where K is defined in (5.6). Throughout this section we assume

$$p > 1 \quad \text{and} \quad q = \frac{2p}{p+1}. \quad (6.3)$$

6.1 Constant solutions

The constant function $\omega = X$ is a solution of (6.2) if

$$X^{p-1} + M \left(\frac{2}{p-1} \right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} - \frac{2K}{p-1} = 0. \quad (6.4)$$

For $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, we recall that $\mu^* = \mu^*(N)$ has been defined in (1.24). The following result is easy to prove

Proposition 6.1 1- Let $M \geq 0$ then there exists a unique positive root X_M to (6.4) if and only if $p > \frac{N}{N-2}$. Moreover the mapping $M \mapsto X_M$ is continuous and decreasing from $[0, \infty)$ onto $(0, \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}]$.

2- Let $M < 0$, $N \geq 3$ and $p \geq \frac{N}{N-2}$ then there exists a unique positive root X_M to (6.4) and the mapping $M \mapsto X_M$ is continuous and decreasing from $(-\infty, 0]$ onto $[\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \infty)$.

3- Let $M < 0$, $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$ then there exists no positive root to (6.4) if $-\mu^* < M \leq 0$. If $M = M^* := -\mu^*$ there exists a unique positive root $X_{M^*} = \left(\frac{2|K|}{p(p-1)}\right)^{\frac{1}{p-1}}$. If $M < -\mu^*$ there exist two positive roots $X_{1,M} < X_{2,M}$. The mapping $M \mapsto X_{1,M}$ is continuous and increasing from $(-\infty, \mu^*]$ onto $(0, X_{M^*}]$. The mapping $M \mapsto X_{2,M}$ is continuous and decreasing from $(-\infty, \mu^*]$ onto $[X_{M^*}, \infty)$.

Abridged proof. Set

$$f_M(X) = X^{p-1} + M \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} - \frac{2K}{p-1}, \quad (6.5)$$

then $f'_M(X) = (p-1)X^{p-2} + M \frac{p-1}{p+1} \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X^{-\frac{2}{p+1}}$.

1- If M is nonnegative, f_M is increasing from $-\frac{2K}{p-1} = -\frac{2[(N-2)p-N]}{(p-1)^2}$ to ∞ ; hence, if $p > \frac{N}{N-2}$ there exists a unique $X_M > 0$ such that $f_M(X_M) = 0$, while if $1 < p < \frac{N}{N-2}$, f_M admits no zero on $[0, \infty)$. Since $f_M > f_{M'}$ for $M > M' > 0$, there holds $X_M > X_{M'}$. By the implicit function theorem the mapping $M \mapsto X_M$ is C^1 and decreasing from $[0, \infty)$ onto $(0, \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}]$. Actually it can be proved that (see [7, Proposition 2.2])

$$X_M = \frac{p-1}{2} \left(\frac{K}{M}\right)^{\frac{p+1}{p-1}} (1 + o(1)) \quad \text{as } M \rightarrow \infty. \quad (6.6)$$

2- If M is negative, f_M achieves its minimum on $[0, \infty)$ at $X_0 = \left(\frac{-M}{p+1}\right)^{\frac{p+1}{p-1}} \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}}$, and

$$\begin{aligned} f_M(X_0) &= -\frac{p}{(p+1)^{\frac{p+1}{p}}} \left(\frac{2}{p-1}\right)^2 (-M)^{\frac{p+1}{p}} - \frac{2K}{p-1} \\ &= -\left(\frac{2}{p-1}\right)^2 \left(\frac{p}{(p+1)^{\frac{p+1}{p}}} (-M)^{\frac{p+1}{p}} + \frac{(N-2)p-N}{2}\right). \end{aligned}$$

Since $K > 0$, there exists a unique $X_M > 0$ such that $f_M(X_M) = 0$ and $X_M > X_0$. The mapping

$M \mapsto X_M$ is C^1 and decreasing from $(-\infty, 0]$ onto $\left[\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \infty\right)$. The following estimate holds

$$\begin{aligned} \max \left\{ \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}} |M|^{\frac{p+1}{p(p-1)}} \right\} &\leq X_M \\ &\leq 2^{\frac{2}{p-1}} \left(\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}} + \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}} |M|^{\frac{p+1}{p(p-1)}} \right). \end{aligned} \quad (6.7)$$

3- If $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, then $f_M(0) > 0$. Hence, if $f_M(X_0) > 0$ there exists no positive root to $f_M(X) = 0$. Equivalently, if $-\mu^* < M < 0$. If $f_M(X_0) = 0$, X_0 is a double root and this is possible only if $M = -\mu^*$, hence $X_{-\mu^*} = \left(\frac{2|K|}{p(p-1)}\right)^{\frac{1}{p-1}}$. If $f_M(X) < 0$, or equivalently, if $M < -\mu^*$, the equation $f_M(X) = 0$ admits two positive roots $X_{1,M} < X_0 < X_{2,M}$. The monotonicity of the $X_{j,M}$, $j=1,2$, and their range follows easily from the monotonicity of $M \mapsto f_M(X)$ for $M < 0$. Actually the following asymptotics hold when $M \rightarrow -\infty$,

$$X_{1,M} = \frac{p-1}{2} \left(\frac{K}{M}\right)^{\frac{p+1}{p-1}} (1 + o(1)) \text{ and } X_{2,M} = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}} (-M)^{\frac{p+1}{p(p-1)}} (1 + o(1)). \quad (6.8)$$

□

6.2 Bifurcations

We set

$$A(\omega) = -\Delta' \omega + \frac{2K}{p-1} \omega - \omega^p - M \left(\left(\frac{2}{p-1}\right)^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{p+1}}, \quad (6.9)$$

If $\eta \in C^\infty(S^{N-1})$ and if there exists a constant positive solution X to $A(X) = 0$ we have

$$\frac{d}{d\tau} A(X + \tau\eta)|_{\tau=0} = -\Delta' \eta + \left(\frac{2K}{p-1} - pX^{p-1} - M \frac{2p}{p+1} \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} \right) \eta.$$

Hence the problem is singular if

$$-\frac{2K}{p-1} + pX^{p-1} + M \frac{2p}{p+1} \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} = \lambda_k, \quad (6.10)$$

where $\lambda_k = k(k + N - 2)$ is the k -th nonzero eigenvalue of $-\Delta'$ in $H^1(S^{N-1})$. The following result follows classically from the standard bifurcation theorem from a simple eigenvalue (which can always be assumed if we consider functions depending only on the azimuthal angle on S^{N-1} reducing the eigenvalue problem to a simple Legendre type ordinary differential equation) see e.g. [26, Chapter 13] and identity (6.4).

Theorem 6.2 *Let $M_0 \in \mathbb{R}$ and X_{M_0} be a constant solution of (6.2). If X_{M_0} satisfies for some $k \in \mathbb{N}^*$,*

$$M_0 \left(\frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_{M_0}^{\frac{p-1}{p+1}} = \frac{p+1}{p(p-1)} (2K - \lambda_k), \quad (6.11)$$

there exists a continuous branch of nonconstant positive solutions (M, ω_M) of (6.2) bifurcating from the (M_0, X_{M_0}) .

Since $M \left(\frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_M^{\frac{p-1}{p+1}} = \frac{2K}{p-1} - X_M^{p-1}$ by (6.4) the following statements follow immediately from Proposition 6.1.

Lemma 6.3 *Set $\Phi(M) = M \left(\frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_M^{\frac{p-1}{p+1}}$ when X_M is uniquely determined, and $\Phi_j(M) = M \left(\frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_{j,M}^{\frac{p-1}{p+1}}$, $j=1,2$, if there exist two equilibria. Then*

1- *If $N \geq 3$ and $p > \frac{N}{N-2}$, the mapping $M \mapsto \Phi(M)$ is continuous and increasing from $[0, \infty)$ onto $[0, \frac{2K}{p-1})$.*

2- *If $N \geq 3$ and $p \geq \frac{N}{N-2}$, the mapping $M \mapsto \Phi(M)$ is continuous and increasing from $(-\infty, 0]$ onto $(-\infty, 0]$.*

3- *If $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, the mapping $M \mapsto \Phi_1(M)$ (resp $M \mapsto \Phi_2(M)$) is continuous and decreasing (resp. increasing) from $(-\infty, -\mu^*]$ onto $[\frac{2K}{p-1}, 0)$ (resp. $(-\infty, \frac{2K}{p-1}]$).*

If we analyse the range $R[\Phi]$ of Φ or $R[\Phi_j]$ of Φ_j , we prove the following result.

Theorem 6.4 1- *Let $N \geq 3$ and $p \geq \frac{N}{N-2}$.*

1-(i) *There exists a continuous curve of bifurcation (M, ω_M) issued from (M_0, X_{M_0}) for some $M_0 = M_0(p) \geq 0$ if and only if $p \geq \frac{N+1}{N-3}$ and $k = 1$. Furthermore $M_0(\frac{N+1}{N-3}) = 0$.*

1-(ii) *The bifurcation curve $s \mapsto (M(s), \omega_{M(s)})$, is defined on $(-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$ and verifies $(M(0), \omega_{M(0)}) = (M_0, X_{M_0})$.*

2- *Let $N \geq 3$ and $p \geq \frac{N}{N-2}$.*

2-(i) *For any $k \geq 1$ there exist $M_k < 0$ and a continuous branch of bifurcation (M, ω_M) issued from (M_k, X_{M_k}) , with the restriction that $p < \frac{N+1}{N-3}$ if $k = 1$.*

2-(ii) *The bifurcation curve $s \mapsto (M(s), \omega_{M(s)})$, is defined on $(-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$ and verifies $(M(0), \omega_{M(0)}) = (M_0, X_{M_0})$. Finally $M_k \rightarrow -\infty$ when $k \rightarrow \infty$.*

3- *let $N = 1, 2$ and $p > 1$, or $N \geq 3$ and $1 < p < \frac{N}{N-2}$.*

3-(i) *There exists no $M < 0$ such that $\frac{2K}{p-1} < \Phi_1(M) < 0$, and a countable set of $M_k < 0$, $k \geq 1$, such that $\Phi_2(M_k) = \frac{p+1}{p(p-1)} (2K - \lambda_k)$.*

3-(ii) *There exist a countable branches of bifurcation of solutions $(M_k(s), \omega_{M_k(s)})$ issued from (M_k, X_{2,M_k}) .*

Proof. Assertion 1. Since from Lemma 6.3, $R[\Phi] = [0, \frac{2K}{p-1})$ for $M \geq 0$, we have to see under what condition on $p \geq \frac{N}{N-2}$ one can find $k \geq 1$ such that

$$0 \leq \frac{p+1}{p(p-1)} (2K - \lambda_k) < \frac{2K}{p-1} \iff \frac{2K}{p+1} < \lambda_k \leq 2K.$$

Since $K < N$ and $\lambda_k \geq 2N$ for $k \geq 2$, the only possibility for this last inequality to hold is $k = 1$. The inequality $\frac{2K}{p+1} < N - 1$ always holds since $p > 1$, while the inequality $N - 1 = \lambda_1 \leq 2K$ is equivalent to $p \geq \frac{N+1}{N-3}$. Therefore $M_0 = 0$ and $X_{M_0} = \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}$. If we consider only functions on the sphere S^{N-1} which depend uniquely on the azimuthal angle $\theta = \tan^{-1}(x_N \lfloor_{S^{N-1}})$, the function $\psi_1(\sigma) = x_N \lfloor_{S^{N-1}}$ is a eigenfunction of $-\Delta'$ in $H^1(S^{N-1})$ with multiplicity one. Hence the bifurcation branch is locally a regular curve $s \mapsto (M(s), \omega_{M(s)})$ with $0 \leq s < \epsilon'_0$ and we construct the bifurcating solution on S^{N-1} using the classical tangency condition [26, Theorem 13.5],

$$\omega_{M(s)} = X_{M_0} + s(\psi_1 + \zeta_s) \quad (6.12)$$

where $\zeta_s \in H^1(S^{N-1})$, is orthogonal to ψ_1 in $H^1(S^{N-1})$ and satisfies $\|\zeta_s\|_{C^1} = o(1)$ when $s \rightarrow 0$. This implies the claim.

Assertion 2. Since $R[\Phi] = (-\infty, 0)$ for $M < 0$, we have to find $k \geq 1$ such that

$$\frac{p+1}{p(p-1)} (2K - \lambda_k) < 0 \iff 2K < \lambda_k.$$

As in Case 1, $K < 2N$, then inequality $2K \leq \lambda_k$ holds for all $k \geq 2$, and if $k = 1$ this is possible only if $p < \frac{N+1}{N-3}$. The construction of the bifurcating curve is the same as in Case 1.

Assertion 3. We have $R[\Phi_1] = [\frac{2K}{p-1}, 0)$ for $M \leq -\mu^*$. If we look for the existence of some $k \geq 1$ such that

$$\frac{2K}{p-1} \leq \frac{p+1}{p(p-1)} (2K - \lambda_k) < 0 \iff 2K \leq \lambda_k < \frac{2K}{p+1};$$

we get an impossibility since $K < 0$. Hence there exists no $M_0 < 0$ such that (M_0, X_{1, M_0}) is a bifurcation point. We have also $R[\Phi_2] = (-\infty, \frac{2K}{p-1}]$ for $M \leq -\mu^*$. Now the condition for the existence of a bifurcation branch issued from (M_0, X_{2, M_0}) for some $M_0 \leq -\mu^*$ is

$$\frac{p+1}{p(p-1)} (2K - \lambda_k) \leq \frac{2K}{p-1} \iff \lambda_k \geq \frac{2K}{p+1},$$

which is always true for any $k \geq 1$ and $1 < p < \frac{N}{N-2}$. \square

Remark. The exponent $p = \frac{N+1}{N-3}$ is the Sobolev critical exponent on S^{N-1} . If one consider the Lane-Emden equation with a Leray potential

$$-\Delta u + \lambda|x|^{-2}u = u^{\frac{N+1}{N-3}}, \quad (6.13)$$

with $\lambda \in \mathbb{R}$, then the separable solutions $u(r, \sigma) = r^{-\frac{N-3}{2}}\omega(\sigma)$ verify

$$-\Delta'\omega + \left(\frac{(N-1)(N-3)}{4} - \lambda\right)\omega - \omega^{\frac{N+1}{N-3}} = 0 \quad \text{on } S^{N-1}. \quad (6.14)$$

It was observed in [10] that there exists a branch of bifurcation $(\lambda, \omega_\lambda)$ with $\lambda > 0$ issued from $(0, \omega_0)$, where ω_0 is the constant explicit solution of (6.14).

Remark. In Theorem 6.4-1- and the above remark, we conjectured that on the bifurcating curve there holds locally $M(s) < M_0$, and that for any $p \geq \frac{N+1}{N-3}$ there exists $M_0 := M_0(p)$ such that for $M > M_0$ all the positive solutions to (6.2) are constant, furthermore M_0 is defined by (6.11). When $p = \frac{N+1}{N-3}$, then $M = 0$ and there exists infinitely many positive solutions to (6.2) [10, Proposition 5.1]. When $\frac{N}{N-2} < p < \frac{N+1}{N-3}$, it is unclear if the branches of bifurcation $(M(s), \omega_{M(s)})$ issued from (M_0, ω_{M_0}) with $M_0 < 0$ are such that $M(s)$ keeps a constant sign. If it is the case one could expect that if $M \geq 0$ and $\frac{N}{N-2} < p < \frac{N+1}{N-3}$, all the positive solutions to (6.2) are constant.

The following statement is an immediate consequence of Theorem 6.4.

Corollary 6.5 *1-If $p > 1$ and $q = \frac{2p}{p+1}$ there always exist nonradial positive singular solutions of (1.2) in $\mathbb{R}^N \setminus \{0\}$ under the form $u(r, \sigma) = r^{-\frac{2}{p-1}}\omega(\sigma)$.*

2- If $N \geq 4$ and $p > \frac{N+1}{N-3}$, the functions are obtained by bifurcation from X_M with $M > 0$.

3- If $N \geq 3$ and $\frac{N}{N-2} \leq p < \frac{N+1}{N-3}$, the functions are obtained by bifurcation from X_M with $M < 0$.

4-If $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, the functions are obtained by bifurcation from (M_k, X_{2, M_k}) with $M_k < -\mu^$ and $k \geq 1$.*

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