A priori estimates for elliptic equations with reaction terms involving the function and its gradient

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Abstract We study local and global properties of positive solutions of $-\Delta u = u^p + M |\nabla u|^q$ in a domain Ω of $\mathbb{R}^N$, in the range $\min\{p, q\} > 1$ and $M \in \mathbb{R}$. We prove a priori estimates and existence or non-existence of ground states for the same equation.

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Contents

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1 Introduction

This article is concerned with local and global properties of positive solutions of the following type of equations

$$-\Delta u = M'|u|^{p-1}u + M\,|\nabla u|^q\,, \tag{1.1}$$

in $\Omega \setminus \{0\}$ where Ω is an open subset of $\mathbb{R}^N$ containing 0, p and q are exponents larger than 1 and M, M' are real parameters. If $M' \leq 0$ the equation satisfies a comparison principle and a big part of the study can be carried via radial local supersolutions. This no longer the case when $M' > 0$ which will be assumed in all the article, and by homothety (1.1) becomes

$$-\Delta u = |u|^{p-1}u + M\,|\nabla u|^q\,. \tag{1.2}$$

If $M = 0$ (1.2) is called Lane-Emden equation

$$-\Delta u = |u|^{p-1}u. \tag{1.3}$$

It turns out that it plays an important role in modelling meteorological or astrophysical phenomena [15], [13], this is the reason for which the first study, in the radial case, goes back to the end of nineteenth century and the beginning of the twentieth. A fairly complete presentation can be found in [18]. If $N \geq 3$, This equations exhibits two main critical exponents $p = \frac{N}{N-2}$ and $p = \frac{N+2}{N-2}$ which play a key role in the description of the set of positive solutions which can be summarized by the following overview:

1- If $1 < p \leq \frac{N}{N-2}$, there exists no positive solution if Ω is the complement of a compact set. Even in that case solution can be replaced by supersolution. This is easy to prove by studying the inequality satisfied by the spherical average of a solution of the equation.

2- If $1 < p < \frac{N+2}{N-2}$, there exists no *ground state*, i.e. positive solution in $\mathbb{R}^N$. Furthermore any positive solution u in a ball $B_R = B_R(a)$ satisfies

$$u(x) \leq c(R - |x-a|)^{-\frac{2}{p-1}}, \tag{1.4}$$

where $c = c(N,p) > 0$, see [19].
3- If $p = \frac{N+2}{N-2}$ all the positive solutions in $\mathbb{R}^N$ are radial with respect to some point a and endow the following form

$$u(x) := u_\lambda(x) = \frac{(N(N-2)\lambda)^{\frac{N-2}{4}}}{(\lambda + |x-a|^2)^{\frac{N-2}{2}}}. \tag{1.5}$$

All the positive solutions in $\mathbb{R}^N \setminus \{0\}$ are radial, see [12].
4- If $p > \frac{N+2}{N-2}$ there exist infinitely many positive ground states radial with respect to some points. They are obtained from one say v, radial for example with respect to 0 by the scaling transformation T_k where $k > 0$ with

$$T_k[v](x) = k^{\frac{2}{p-1}} v(kx). \tag{1.6}$$

Indeed, the first significant non-radial results deals with the case $1 < p \le \frac{N}{N-2}$. They are based upon the Brezis-Lions lemma [11] which yields an estimate of solutions in the Lorentz space $L^{\frac{N}{N-2},\infty}$, implying in turn the local integrability of u^q. Then a bootstrapping method as in [21] leads easily to some a priori estimate. Note that this subcritical case can be interpreted using the famous Serrin's results on quasilinear equations [24]. The first breakthrough in the study of Lane-Emden equation came in the treatment of the case $1 < p < \frac{N+2}{N-2}$; it is due to Gidas and Spruck [19]. Their analysis is based upon differentiating the equation and then obtaining sharp enough local integral estimates on the term u^{q-1} making possible the utilization of Harnack inequality as in [24]. The treatment of the critical case $p = \frac{N+2}{N-2}$, due to Caffarelli, Gidas and Spruck [12], was made possible thanks to a completely new approach based upon a combination of moving plane analysis and geometric measure theory. As for the supercritical case, not much is known and the existence of radial ground states is a consequence of Pohozaev's identity [22], using a shooting method.

The study of (1.2) when $M \neq 0$ presents some similarities with the one of Lane-Emden equation in the cases 1 and 2, except that the proof are much more involved. Actually the approach we develop in this article is much indebted to our recent paper [6] where we study local and global aspects of positive solutions of

$$-\Delta u = u^p \, |\nabla u|^q \,, \tag{1.7}$$

where $p \ge 0$, $0 \le q < 2$, mostly in the superlinear case $p + q - 1 > 0$. Therein we prove the existence of a critical line of exponents

$$(\mathfrak{L}) := \{(p,q) \in \mathbb{R}_+ \times [0,2) : (N-2)p + (N-1)q = N\}. \tag{1.8}$$

The subcritical range corresponds to the fact that (p,q) is below $(\mathfrak{L})$. In this region Serrin's celebrated results [24] can be applied and we prove [6, Theorem A] that positive solutions of (1.7) in the punctured ball $B_2 \setminus \{0\}$ satisfy, for some constant $c > 0$ depending on the solution,

$$u(x) + |x| \, |\nabla u(x)| \le c \, |x|^{2-N} \quad \text{for all } x \in B_1 \setminus \{0\}. \tag{1.9}$$

When (p,q) is above $(\mathfrak{L})$, i.e. in the supercritical range, we introduced two methods for obtaining a priori estimate of solutions: The *pointwise Bernstein method* and the *integral Bernstein method*. The first one is based upon the change of unknown $u = v^{-\beta}$, and then to show that $|\nabla v|$ satisfies an inequality of Keller-Osserman type. When (p,q) lies above $(\mathfrak{L})$ and verifies

(i) either $1 \leq p < \frac{N+3}{N-1}$ and $p+q-1 < \frac{4}{N-1}$,

(ii) or $0 \leq p < 1$ and $p+q-1 < \frac{(p+1)^2}{p(N-1)}$,

we prove that any positive solution of (1.7) in a domain $\Omega \subset \mathbb{R}^N$ satisfies

$$|\nabla u^a(x)| \leq c^* \left(\operatorname{dist}(x,\partial\Omega)\right)^{-1-a\frac{2-q}{p+q-1}} \quad \text{for all } x \in \Omega, \tag{1.10}$$

for some positive c^* and a depending on N, p and q [6, Theorem B]. As a consequence we prove that any positive solution of (1.7) in $\mathbb{R}^N$ is constant. With the second method we combine the change of unknown $u = v^{-\beta}$ with integration and cut-off functions. We show the existence of a quadratic polynomial G in two variables such that for any $(p,q) \in \mathbb{R}_+ \times [0,2)$ satisfying $G(p,q) < 0$ any positive solution of (1.7) in $\mathbb{R}^N$ is constant [6, Theorem C]. The polynomial G is not simple but it is worth noting that if $0 \leq p < \frac{N+2}{N-2}$, there holds $G(p,0) < 0$, which recovers Gidas and Spruck result [19].

For equation (1.2) we first observe that the equation is invariant under the scaling transformation (1.6) for any $k > 0$ if and only if q is *critical with respect to* p, i.e.

$$q = \frac{2p}{p+1}.$$

In general the transformation T_k exchanges (1.2) with

$$-\Delta v = v^p + M k^{\frac{2p-q(p+1)}{p-1}} |\nabla v|^q, \tag{1.11}$$

hence if $q < \frac{2p}{p+1}$, the limit equation when $k \to 0$ is (1.3). We say that the exponent p is dominant. We can also consider the transformation

$$S_k[v](x) = k^{\frac{2-q}{q-1}} v(kx), \tag{1.12}$$

when $q \neq 2$, which is the same as T_k if $q = \frac{2p}{p+1}$, and more generally transforms (1.2) into

$$-\Delta v = k^{\frac{q-p(2-q)}{q-1}} v^p + M|\nabla v|^q. \tag{1.13}$$

Hence if $q > \frac{2p}{p+1}$, the limit equation when $k \to 0$ is the Riccati equation

$$-\Delta v = M|\nabla v|^q. \tag{1.14}$$

It is also important to notice that the value of the coefficient M (and not only its sign) plays a fundamental role, only if $q = \frac{2p}{p+1}$. If $q \neq \frac{2p}{p+1}$ the transformation

$$u(x) = av(y) \quad \text{with } a = |M|^{-\frac{2}{(p+1)q-2p}} \text{ and } y = a^{\frac{p-1}{2}} x \tag{1.15}$$

allows to transform (1.2) into

$$-\Delta v = |v|^{p-1} v \pm |\nabla v|^q. \tag{1.16}$$

The equation (1.2) has been essentially studied in the *radial case* when $M < 0$ in connection with the parabolic equation

$$\partial_t u - \Delta u + M|\nabla u|^q = |u|^{p-1}u, \tag{1.17}$$

see [14], [16], [17], [25], [27], [30], [31]. The studies mainly deal with the case $q \neq \frac{2p}{p+1}$, although not complete when $q > \frac{2p}{p+1}$. When $q = \frac{2p}{p+1}$ the existence of a ground state is proved in dimension 1. Some partial results that we will improve, already exist in higher dimension. The case $M > 0$ attracted less attention.

In the *nonradial case*, any nonnegative nontrivial solution is positive since $p, q > 1$. We first observe, using a standard averaging method applied to positive supersolutions of (1.3), that if $M \geq 0$, $1 < p \leq \frac{N}{N-2}$ when $N \geq 3$, any $p > 1$ if $N = 1, 2$, then for any $q > 0$ there exists no positive solution in an exterior domain. When $0 < q < \frac{2p}{p+1}$ the equation endows some character of the pure Emden-Fowler equation (1.3) by the transformation T_k. In [23] it is proved that if $0 < q < \frac{2p}{p+1}$, $1 < p < \frac{N+2}{N-2}$ and $M \in \mathbb{R}$, any positive solution of (1.3) in an open domain satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c_{N,p,q,M}\left(1 + (\mathrm{dist}\,(x, \partial\Omega))^{-\frac{2}{p-1}}\right) \qquad \text{for all } x \in \Omega. \tag{1.18}$$

Note that this does not imply the non-existence of ground state. In [1] Alarcón, García-Melián and Quass study the equation

$$-\Delta u = |\nabla u|^q + f(u), \tag{1.19}$$

in an exterior domain of $\mathbb{R}^N$ emphasizing the fact that positive solutions are super harmonic functions. They prove that if $1 < q \leq \frac{N}{N-1}$ and if f is positive on $(0, \infty)$ and satisfies

$$\limsup_{s \to 0} s^{-p} f(s) > 0, \tag{1.20}$$

for some $p > \frac{N}{N-2}$, then (1.19) admits no positive supersolution. The same authors also study in [2] existence and non-existence of positive solutions of (1.19) in a bounded domain with Dirichlet condition.

The techniques we developed in this paper are based upon a delicate extension of the ones already introduced in [6]. Our first nonradial result dealing with the case $q > \frac{2p}{p+1}$ is the following:

Theorem A *Let $N \geq 1$, $p > 1$ and $q > \frac{2p}{p+1}$. Then for any $M > 0$, any solution of* (1.2) *in a domain $\Omega \subset \mathbb{R}^N$ satisfies*

$$|\nabla u(x)| \leq c_{N,p,q}\left(M^{-\frac{p+1}{(p+1)q-2p}} + (M\mathrm{dist}\,(x, \partial\Omega))^{-\frac{1}{q-1}}\right) \quad \textit{for all } x \in \Omega. \tag{1.21}$$

As a consequence, any ground state has at most a linear growth at infinity:

$$|\nabla u(x)| \leq c_{N,p,q} M^{-\frac{p+1}{(p+1)q-2p}} \qquad \textit{for all } x \in \mathbb{R}^N. \tag{1.22}$$

Our proof relies on a direct Bernstein method combined with Keller-Osserman's estimate applied to $|\nabla u|^2$. It is important to notice that the result holds for any $p>1$, showing that, in some sense, the presence of the gradient term has a regularizing effect. In the case $q<\frac{2p}{p+1}$ we prove a non-existence result

Theorem A' *Let $N\geq 1$, $p>1$, $1<q<\frac{2p}{p+1}$ and $M>0$. Then there exists a constant $c_{N,p,q}>0$ such that there is no positive solution of* (1.2) *in* $\mathbb{R}^N$ *satisfying*

$$u(x)\leq c_{N,p,q}M^{\frac{2}{2p-(p+1)q}} \qquad \text{for all } x\in\mathbb{R}^N. \tag{1.23}$$

When q is critical with respect to p the situation is more delicate since the value of M plays a fundamental role. Our first statement is a particular case of a more general result in [1], but with a simpler proof which allows us to introduce techniques that we use later on.

Theorem B *Let $N\geq 2$, $p>1$ if $N=2$ or $1<p\leq\frac{N}{N-2}$ if $N=3$, $q=\frac{2p}{p+1}$ and $M>-\mu^*$ where*

$$\mu^*:=\mu^*(N)=(p+1)\left(\frac{N-(N-2)p}{2p}\right)^{\frac{p}{p+1}}. \tag{1.24}$$

Then there exists no nontrivial nonnegative supersolution of (1.2) *in an exterior domain.*

In this range of values of p this result is optimal since for $M\leq-\mu^*$ there exists positive singular solutions. The constant μ^* will play an important role in the description developed in [7] of radial solutions of (1.2). Using a variant of the method used in the proof of Theorem B we obtain results of existence and nonexistence of large solutions.

Theorem B' *Let $N\geq 1$, $p>1$ and $q=\frac{2p}{p+1}$.*
1- If Ω is a domain with a compact boundary satisfying the Wiener criterion and $M\geq-\mu^*(2)$ there exists no positive supersolution of (1.2) *in Ω satisfying*

$$\lim_{\operatorname{dist}(x,\partial\Omega)\to 0} u(x)=\infty. \tag{1.25}$$

2- If G is a bounded convex domain, $\Omega=\overline{G}^c$ and $M<-\mu^*(1)$ there exists a positive solution of (1.2) *in Ω satisfying* (1.25).

We show in [7] that the inequality $M<-\mu^*(1)$ is the necessary and sufficient condition for the existence of a radial large solution in the exterior of a ball.

Concerning ground states, we prove their nonexistence for any $p>1$ provided $M>0$ is large enough: indeed

Theorem C *Let $\Omega\subset\mathbb{R}^N$, $N\geq 1$, be a domain, $p>1$, $q=\frac{2p}{p+1}$. For any*

$$M>M_\dagger:=\left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p+1}}\left(\frac{N(p+1)^2}{4p}\right)^{\frac{p}{p+1}}, \tag{1.26}$$

and any $\nu > 0$ such that $(1-\nu)M > M_\dagger$, there exists a positive constant $c_{N,p,\nu}$ such that any solution u in Ω satisfies

$$|\nabla u(x)| \leq c_{N,p,\nu} \left((1-\nu)M - M_\dagger\right)^{-\frac{p+1}{p-1}} \left(\operatorname{dist}(x,\partial\Omega)\right)^{-\frac{p+1}{p-1}} \quad \text{for all } x \in \Omega. \tag{1.27}$$

Consequently there exists no nontrivial solution of (1.2) *in* $\mathbb{R}^N$.

The next result, based upon an elaborate Bernstein method, complements Theorem C under a less restrictive assumption on M but a more restrictive assumption on p.

Theorem D *Let $1 < p < \frac{N+3}{N-1}$, $N \geq 2$, $1 < q < \frac{N+2}{N}$ and $\Omega \subset \mathbb{R}^N$ be a domain. Then there exist $a > 0$ and $c_{N,p,q} > 0$ such that for any $M > 0$, any positive solution u in Ω satisfies*

$$|\nabla u^a(x)| \leq c_{N,p,q} \left(\operatorname{dist}(x,\partial\Omega)\right)^{-\frac{2a}{p-1}-1} \quad \text{for all } x \in \Omega. \tag{1.28}$$

Hence there exists no nontrivial nonnegative solution of (1.2) *in* $\mathbb{R}^N$.

It is remarkable that the constants a and $c_{N,p,q}$ do not depend on $M > 0$, a fact which is clear when $q \neq \frac{2p}{p+1}$ by using the transformation T_k, but much more delicate to highlight when $q = \frac{2p}{p+1}$ since (1.2) is invariant. When $|M|$ is small, we use an integral method to obtain the following result which contains, as a particular case, the estimates in [19] and [7]. The key point of this method is to prove that the solutions in a punctured domain satisfy a local Harnack inequality.

Theorem E *Let $N \geq 3$, $1 < p < \frac{N+2}{N-2}$, $q = \frac{2p}{p+1}$. Then there exists $\epsilon_0 > 0$ depending on N and p such that for any M satisfying $|M| \leq \epsilon_0$, any positive solution u in $B_R \setminus \{0\}$ satisfies*

$$u(x) \leq c_{N,p} |x|^{-\frac{2}{p-1}} \quad \text{for all } x \in B_{\frac{R}{2}} \setminus \{0\}. \tag{1.29}$$

As a consequence there exists no positive solution of (1.2) *in* $\mathbb{R}^N$*, and any positive solution u in a domain Ω satisfies*

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c'_{N,p} \left(\operatorname{dist}(x,\partial\Omega)\right)^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega. \tag{1.30}$$

Note that under the assumptions of Theorem E, there exist ground states for $|M|$ large enough when $1 < p < \frac{N}{N-2}$, or any $p > 1$ if $N = 1, 2$.

If u is a radial solutions of (1.2) in $\mathbb{R}^N$ it satisfies

$$-u'' - \frac{N-1}{r} u' = |u|^{p-1} u + M \left|u'\right|^q, \tag{1.31}$$

on $(0,\infty)$. Using several type of Lyapounov type functions introduced by Leighton [20] and Anderson and Leighton [3], we prove some results dealing with the case $M > 0$ which complement the ones of [25] relative to the case $M < 0$.

Theorem F *1- Let $p > 1$ and $q > \frac{2p}{p+1}$. Then there exists no radial ground state u satisfying $u(0) = 1$ when $M > 0$ is too large.*

2- Let $1 < p < \frac{N+2}{N-2}$. If $1 < q \leq p$ there exists no radial ground state for any $M > 0$. If $q > p$ there exists no radial ground state for $M > 0$ small enough.
3- Let $N \geq 3$, $p > \frac{N+2}{N-2}$ and $q \geq \frac{2p}{p+1}$. Then there exist radial ground states for $M > 0$ small enough.

We end the article in proving the existence of non-radial positive singular solutions of (1.2) in $\mathbb{R}^N \setminus \{0\}$ in the case $q = \frac{2p}{p+1}$ obtained by bifurcation from radial explicit positive singular solutions. Our result shows that the situation is very contrasted according $M > 0$ where a bifurcation from (M, X_M) occurs only if $p \geq \frac{N+1}{N-3}$ and $M \geq 0$ and $M < 0$ where there exists a countable set of bifurcations from (M_k, X_{M_k}), $k \geq 1$, when $1 < p < \frac{N+1}{N-3}$.

In a subsequent article [7] we present a fairly complete description of the positive radial solutions of (1.2) in $\mathbb{R}^N \setminus \{0\}$ in the scaling invariant case $q = \frac{2p}{p+1}$.

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2 The direct Bernstein method

We begin with a simple property in the case $M \geq 0$ which is a consequence of the fact that the positive solutions of (1.2) are superharmonic.

Proposition 2.1 *1- There exists no positive solution of* (1.2) *in $\mathbb{R}^N \setminus \overline{B}_R$, $R \geq 0$ if one of the two conditions is satisfied:*
(i) $M \geq 0$, $q \geq 0$ and either $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$.
(ii) $M > 0$, $N \geq 3$, $p \geq 1$ and $1 < q \leq \frac{N}{N-1}$.
2- If $N \geq 3$, $q \geq 1$, $p > \frac{N}{N-2}$ and $u(x) = u(r, \sigma)$ is a positive solution of (1.2) *in $\mathbb{R}^N \setminus \overline{B}_R$, $R \geq 0$. Then there exists $\rho \geq R$ such that*

$$\frac{1}{N\omega_N}\int_{S^{N-1}} u(r,\sigma)dS := \overline{u}(r) \leq c_0 r^{-\frac{2}{p-1}} \qquad \text{for all } r > \rho, \tag{2.1}$$

with $c_0 := \left(\frac{2N}{p-1}\right)^{\frac{1}{p-1}}$ and

$$\left|\frac{1}{N\omega_N}\int_{S^{N-1}} u_r(r,\sigma)dS\right| := |\overline{u}_r(r)| \leq (N-2)c_0 r^{-\frac{p+1}{p-1}} \qquad \text{for all } r > \rho. \tag{2.2}$$

3- If $M > 0$, $p \geq 0$, and $q > \frac{N}{N-1}$ there holds for

$$|\overline{u}_r(r)| \leq \left(\frac{(q-1)(N-1)-1}{(q-1)M}\right)^{\frac{1}{q-1}} r^{-\frac{1}{q-1}} \qquad \text{for all } r > \rho, \tag{2.3}$$

and

$$\overline{u}(r) \leq \left(\frac{q-1}{2-q}\right)\left(\frac{(q-1)(N-1)-1}{(q-1)M}\right)^{\frac{1}{q-1}} r^{\frac{q-2}{q-1}} \qquad \text{for all } r > \rho, \tag{2.4}$$

Furthermore, if $R = 0$, inequalities (2.1), (2.2) *and* (2.3) *hold with $\rho = 0$.*

Proof. Assertion 1-(i) is not difficult to obtain by integrating the inequality satisfied by the spherical average of the solution and using Jensen's inequality. For the sake of completeness, we give a simple proof although the result is actually valid for much more general equations (see e.g. [8] and references therein). In this statement we denote by $(r,\sigma)\in\mathbb{R}_+\times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$, by ω_N the volume of the unit N-ball and thus $N\omega_N$ is the (N-1)-volume of the unit sphere S^{N-1}. Writing (1.2) in spherical coordinates and using Jensen formula, we get

$$-r^{1-N}\left(r^{N-1}\overline{u}_r\right)_r \geq \overline{u}^p + M\left|\overline{u}_r\right|^q. \tag{2.5}$$

It implies that $r\mapsto w(r):=-r^{N-1}\overline{u}_r$ is increasing on (R,∞), thus it admits a limit $\ell\in(-\infty,\infty]$. If $\ell\leq 0$, then $\overline{u}_r(r)>0$ on (R,∞). Hence $\overline{u}(r)\geq\overline{u}(\rho):=c>0$ for $r\geq\rho>R$. then

$$\left(r^{N-1}\overline{u}_r\right)_r \leq -c^p r^{N-1} \Longrightarrow \overline{u}_r(r)\leq \frac{\rho^{N-1}}{r^{N-1}}\overline{u}_r(\rho)-\frac{c^p}{N}\left(r-\frac{\rho^N}{r^{N-1}}\right),$$

which implies $\overline{u}_r(r)\to-\infty$, thus $\overline{u}(r)\to-\infty$ as $r\to-\infty$, contradiction. Therefore $\ell\in(0,\infty]$ and either $\overline{u}_r(r)<0$ on (R,∞) or there exists $r_\ell>R$ such that $\overline{u}_r(r_\ell)=0$, $\overline{u}$ is increasing on $(R,r_\ell,)$ and decreasing on (r_ℓ,∞). If $\overline{u}_r(r)<0$ on (R,∞), then we have for $r>2R$

$$-r^{N-1}\overline{u}_r(r)\geq\int_{\frac{r}{2}}^{r} t^{N-1}\overline{u}^p(t)dt \geq \frac{r^N\overline{u}^p(r)}{2N}\Longrightarrow\left(\overline{u}^{1-p}\right)_r\geq\frac{(p-1)r}{2N}\Longrightarrow\overline{u}(r)\leq\left(\frac{2N}{(p-1)r^2}\right)^{\frac{1}{p-1}},$$

which yields (2.1). If we are in the second case with $r_\ell>R$, we apply the same inequality with $r>2r_\ell$ and again (2.1) for $r>2r_\ell$. Since $\overline{u}$ is superharmonic, the function $v(s)=\overline{u}(r)$ with $s=r^{2-N}$ is concave on $(0,R^{2-N})$ and it tends to 0 when $s\to 0$. Thus

$$v_s(s)\leq\frac{v}{s}\Longrightarrow|\overline{u}_r(r)|\leq(N-2)\frac{\overline{u}(r)}{r}\leq(N-2)c_0 r^{-\frac{p+1}{p-1}}.$$

This implies (2.1) and (2.2). Note that the case $r_\ell>R$ cannot happen if $R=0$, so in any case, if $R=0$ then $\rho=0$.
If $M>0$, we have with $w(r)=-r^{N-1}\overline{u}_r$

$$w_r\geq Mr^{(1-q)(N-1)}\left|w\right|^q.$$

We have seen that $w(r)>0$ at infinity with limit $\ell\in(0,\infty]$, hence, on the maximal interval containing ∞ where $w>0$, we have $(w^{1-q})_r\leq(1-q)Mr^{(N-1)(1-q)}$. We have for $r>s>R$

$$w^{1-q}(r)-w^{1-q}(s)\leq M\ln\left(\frac{r}{s}\right),$$

if $q=\frac{N}{N-1}$ and

$$w^{1-q}(r)-w^{1-q}(s)\leq\frac{M(q-1)}{(q-1)(N-1)-1}\left(r^{1-(q-1)(N-1)}-s^{1-(q-1)(N-1)}\right)$$

if $q<\frac{N}{N-1}$, and both expressions which tend to $-\infty$ when $r\to\infty$, a contradiction. This proves 1-(ii). If $q>\frac{N}{N-1}$, the above expression yields, when $r\to\infty$,

$$\ell^{1-q}-w^{1-q}(s)\leq-\frac{(q-1)M}{(q-1)(N-1)-1}s^{1-(q-1)(N-1)}.$$

This implies

$$w(s) \leq \left(\frac{(q-1)(N-1)-1}{(q-1)M}\right)^{\frac{1}{q-1}} s^{N-1-\frac{1}{q-1}},$$

and (2.3). $\square$

Remark. The previous is a particular case of a much more general one dealing with quasilinear operators proved in [8, Theorem 3.1].

2.1 Proof of Theorems A, A' and C

The function u is at least $C^{3+\alpha}$ for some $\alpha \in (0,1)$ since $p, q > 1$. Hence $z = |\nabla u|^2$ is $C^{2+\alpha}$. Since there holds by Bochner's identity and Schwarz's inequality

$$-\frac{1}{2}\Delta z + \frac{1}{N}(\Delta u)^2 + \langle \nabla \Delta u, \nabla u\rangle \leq 0, \tag{2.6}$$

we obtain from (1.2),

$$-\frac{1}{2}\Delta z + \frac{|u|^{2p}}{N} + \frac{2M}{N}|u|^{p-1}uz^{\frac{q}{2}} + \frac{M^2}{N}z^q - p|u|^{p-1}z - \frac{Mq}{2}z^{\frac{q}{2}-1}\langle \nabla z, \nabla u\rangle \leq 0.$$

Since for $\delta > 0$,

$$z^{\frac{q}{2}-1}\left|\langle \nabla z, \nabla u\rangle\right| \leq \left|z^{-\frac{1}{2}}\nabla z\right| z^{\frac{q-1}{2}}\left|\nabla u\right| = \left|z^{-\frac{1}{2}}\nabla z\right| z^{\frac{q}{2}} \leq \delta z^q + \frac{1}{4\delta}\frac{|\nabla z|^2}{z},$$

we obtain for any $\nu \in (0,1)$, provided δ is small enough,

$$-\frac{1}{2}\Delta z + \frac{|u|^{2p}}{N} + \frac{2M}{N}|u|^{p-1}uz^{\frac{q}{2}} + \frac{M^2(1-\nu)^2}{N}z^q - p|u|^{p-1}z \leq c_1\frac{|\nabla z|^2}{z}, \tag{2.7}$$

where $c_1 = c_1(M, N, \nu) > 0$.

2.1.1 Proof of Theorem A

We recall the following technical result proved in [6, Lemma 2.2] which will be used several times in the course of this article.

Lemma 2.2 *Let $S > 1$, $R > 0$ and v be continuous and nonnegative in $\overline{B}_R$ and C^1 on the set $\mathcal{U}_+ = \{x \in B_R : v(x) > 0\}$. If v satisfies, for some real number a,*

$$-\Delta v + v^S \leq a\frac{|\nabla v|^2}{v} \tag{2.8}$$

on each connected component of $\mathcal{U}_+$, then

$$v(0) \leq c_{N,S,a}R^{-\frac{2}{S-1}}. \tag{2.9}$$

Abridged proof. Assuming $a > 0$, we set $W = v^\alpha$ for $0 < \alpha \leq \frac{1}{a+1}$, this transforms (2.8) into

$$-\Delta W + \frac{1}{\alpha} W^{\alpha(S-1)+1} \leq 0, \tag{2.10}$$

and then we apply Keller-Osserman inequality. $\square$

Proof of Theorem A. Suppose $\frac{2p}{p+1} < q$. We set $r = \frac{2p}{p-1}$, $r' = \frac{r}{r-1}$, then, for any $\epsilon > 0$

$$p|u|^{p-1} z \leq \frac{\epsilon^r |u|^{(p-1)r}}{r} + \frac{z^{r'}}{\epsilon^{r'} r'} = (p-1)\frac{\epsilon^r |u|^{2p}}{2} + (p+1)\frac{z^{\frac{2p}{p+1}}}{2\epsilon^{r'}}.$$

We fix $\eta \in (0,1)$ and ϵ so that $\epsilon^r = \frac{2(1-\eta)}{N(p-1)}$ and get

$$p|u|^{p-1} z \leq (1-\eta)\frac{|u|^{2p}}{N} + c_2 z^{\frac{2p}{p+1}},$$

where $c_2 = \frac{p+1}{2}\left(\frac{N(p-1)}{2(1-\eta)}\right)^{\frac{p+1}{p-1}}$. We perform the change of scale (1.6) in order to reduce (1.2) to the case $M = 1$ by setting $u(x) = \alpha^{\frac{2}{p-1}} v(\alpha x)$ with $\alpha = M^{-\frac{p-1}{(p+1)q-2p}}$. Then the equation for $z = |\nabla v|^2$ is considered in $\Omega_\alpha = \alpha\Omega$. Choosing now $\eta = \frac{1}{2}$ we obtain

$$c_2 z^{\frac{2p}{p+1}} \leq \frac{1}{4N} z^q + c_3,$$

where $c_3 = c_3(N,p,q) > 0$, hence

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{2N} + \frac{1}{4N} z^q \leq c_3 + c_1 \frac{|\nabla z|^2}{z}.$$

Put $\tilde{z} = \left(z - (4Nc_3)^{\frac{1}{q}}\right)_+$, then

$$-\frac{1}{2}\Delta \tilde{z} + \frac{1}{4N}\tilde{z}^q \leq c_1 \frac{|\nabla \tilde{z}|^2}{\tilde{z}},$$

hence, from Lemma 2.2, we derive

$$\tilde{z}(y) \leq c_4 \left(\operatorname{dist}(y, \partial\Omega_\alpha)\right)^{\frac{2}{q-1}}$$

where $c_4 = c_4(N, q, c_1) > 0$ which implies

$$|\nabla v(y)| \leq c_4' \left(1 + \left(\operatorname{dist}(y, \partial\Omega_\alpha)\right)^{-\frac{1}{q-1}}\right) \qquad \forall\, y \in \Omega_\alpha. \tag{2.11}$$

Then (1.21) and (1.22) follow.

Assume now that there exists a ground state u. Fix $y \in \mathbb{R}^N$ and consider $\{y_n\} \subset \mathbb{R}^N$ such that $|y_n| = 2n > |y|$. We apply (2.11) with $\Omega_\alpha = B_n(y_n)$. Then

$$|\nabla v(y)| \leq c_4' \left(1 + |2n - |y||^{-\frac{1}{q-1}}\right),$$

and letting $n \to \infty$ we infer

$$|\nabla v(y)| \leq c_4' \qquad \forall\, y \in \mathbb{R}^N. \tag{2.12}$$

Hence, by the definition of v and y we see that

$$|\nabla u(x)| \leq c_4' M^{-\frac{p+1}{(p+1)q-2p}} \qquad \forall\, x \in \mathbb{R}^N$$

which is exactly (1.22). □

2.1.2 Proof of Theorem A'

Suppose $1 < q < \frac{2p}{p+1}$. By scaling we reduce to the case $M = 1$ and we replace u by v defined by (1.6) as in the proof of Theorem A with $\alpha = M^{\frac{p-1}{2p-(p+1)q}}$. From (2.7) with $\nu = \frac{1}{4}$ the function $z = |\nabla v|^2$ satisfies

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{N} + \frac{1}{2N} z^q - p v^{p-1} z \leq c_1 \frac{|\nabla z|^2}{z}. \tag{2.13}$$

By Hölder's inequality,

$$p v^{p-1} z \leq \frac{1}{4N} z^q + p(4Np)^{q'-1} v^{(p-1)q'}.$$

Since $(p-1)q' = 2p + \frac{2p-(p+1)q}{q-1}$ we derive

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{N}\left(1 - 4^{q'-1} p^{q'} N^{q'} v^{\frac{2p-(p+1)q}{q-1}}\right) + \frac{1}{4N} z^q \leq c_1 \frac{|\nabla z|^2}{z}.$$

If $\max v \leq c_{N,p,q} := (4^{q'-1} p^{q'} N^{q'})^{-\frac{q-1}{2p-(p+1)q}}$, we obtain

$$-\frac{1}{2}\Delta z + \frac{1}{4N} z^q \leq c_1 \frac{|\nabla z|^2}{z},$$

which implies that $z = 0$ by Lemma 2.2, hence v is constant and thus $v = 0$ from the equation. □

Remark. If u is a positive ground state of (1.2) radial with respect to 0, it satisfies $u_r(0) = 0$ and it is a decreasing function of r. The previous theorem asserts that it must satisfy

$$u(0) > c_{N,p,q} M^{\frac{2}{2p-(p+1)q}}. \tag{2.14}$$

2.1.3 Proof of Theorem C

Suppose $\frac{2p}{p+1} = q$. For $A > 0$ we consider the expression

$$\begin{aligned}(u^p + A\,|\nabla u|^q)^2 - N p u^{p-1} |\nabla u|^2 \\ = \left(u^p + A\,|\nabla u|^q - \sqrt{Np}\, u^{\frac{p-1}{2}} |\nabla u|\right)\left(u^p + A\,|\nabla u|^q + \sqrt{Np}\, u^{\frac{p-1}{2}} |\nabla u|\right).\end{aligned}$$

Now the function $Z \mapsto \Phi_A(Z) = u^p + AZ^q - \sqrt{Np}\, u^{\frac{p-1}{2}} Z$ achieves its minimum at $Z_0 = \left(\frac{\sqrt{Np}}{qA}\right)^{\frac{p+1}{p-1}} u^{\frac{p+1}{2}}$ and

$$\Phi_A(Z_0) = \left[1 - \frac{p-1}{p+1}\left(\frac{N(p+1)^2}{4p}\right)^{\frac{p}{p-1}} A^{-\frac{p+1}{p-1}}\right] u^p.$$

Thus setting

$$M_\dagger = \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p+1}} \left(\frac{N(p+1)^2}{4p}\right)^{\frac{p}{p+1}}, \tag{2.15}$$

we obtain that if $A \geq M_\dagger$, then $\Phi_A(Z) \geq 0$ for all Z. Put $M_\nu = (1-\nu)M$ for $\nu \in (0,1)$ such that $M_\dagger < M_\nu$, we derive from (2.7)

$$-\frac{1}{2}\Delta z + \frac{(u^p + M_\dagger z^{\frac{q}{2}})^2}{N} - pu^{p-1}z + \frac{M_\nu^2 - M_\dagger^2}{N} z^q \leq c_1 \frac{|\nabla z|^2}{z}, \tag{2.16}$$

which yields

$$-\frac{1}{2}\Delta z + \frac{M_\nu^2 - M_\dagger^2}{N} z^q \leq c_1 \frac{|\nabla z|^2}{z}.$$

Using again Lemma 2.2 we obtain

$$|\nabla u(x)| \leq c_1' \left((1-\nu)M - M_\dagger\right)^{-\frac{1}{q-1}} \left(\operatorname{dist}(x, \partial\Omega)\right)^{-\frac{1}{q-1}}, \tag{2.17}$$

which is equivalent to (1.27). □

2.2 Proof of Theorems B and B'

2.2.1 Proof of Theorem B

Since the result is known when $M \geq 0$ from Proposition 2.1, we can assume that $M = -m < 0$ and $N = 1, 2$ or $N \geq 3$ with $p < \frac{N}{N-2}$, u is a nonnegative supersolution of (1.2) in $\overline{B}_R^c$ and we set $u = v^b$ with $b > 1$. Then

$$-\Delta v \geq (b-1)\frac{|\nabla v|^2}{v} + \frac{1}{b} v^{1+b(p-1)} - mb^{q-1} v^{(b-1)(q-1)} |\nabla v|^q. \tag{2.18}$$

Here again $q = \frac{2p}{p+1}$, setting $z = |\nabla v|^2$ we obtain

$$-\Delta v \geq \frac{\Phi(z)}{bv}$$

where

$$\Phi(z) = b(b-1)z - mb^{\frac{2p}{p+1}} v^{\frac{2+b(p-1)}{p+1}} z^{\frac{p}{p+1}} + v^{2+b(p-1)}.$$

Thus Φ achieves it minimum for

$$z_0 = \left(\frac{mpb^{q-1}}{(b-1)(p+1)}\right)^{p+1} b^{p-1} v^{2+b(p-1)}$$

and

$$\Phi(z_0) = v^{2+b(p-1)}\left(1 - \frac{p^p}{(p+1)^{p+1}}\left(\frac{b}{b-1}\right)^p m^{p+1}\right). \tag{2.19}$$

In order to ensure the optimal choice, when $N \geq 3$ we take $1 + b(p-1) = \frac{N}{N-2}$, hence $b = \frac{2}{(N-2)(p-1)}$ which is larger than 1 because $p < \frac{N}{N-2}$. Finally

$$\Phi(z_0) = v^{\frac{N}{N-2}+1}\left(1 - \frac{1}{(p+1)^{p+1}}\left(\frac{2p}{N-p(N-2)}\right)^p m^{p+1}\right).$$

Hence, if

$$m < (p+1)\left(\frac{N-p(N-2)}{2p}\right)^{\frac{p}{p+1}} = \mu^*(N), \tag{2.20}$$

we have for some $\delta > 0$,

$$-\Delta v \geq \delta v^{\frac{N}{N-2}}, \tag{2.21}$$

and by Proposition 2.1 that is no positive solution in an exterior domain of $\mathbb{R}^N$.

If $N = 2$ for a given $b > 1$ we have from (2.19) that if

$$m < (p+1)\left(\frac{b-1}{bp}\right)^{\frac{p}{p+1}},$$

then, for some $\delta > 0$,

$$-\Delta v \geq \delta v^{1+b(p-1)}. \tag{2.22}$$

The result follows from Proposition 2.1 by choosing b large enough. $\square$

2.2.2 Proof of Theorem B'

1- We assume that such a supersolution u exists and we denote $u = e^v$, then

$$-\Delta v \geq F(|\nabla v|^2), \tag{2.23}$$

where

$$F(X) = X + e^{(p-1)v} + Me^{\frac{p-1}{p+1}v} X^{\frac{p}{p+1}}.$$

Clearly, if $M \geq 0$, then $F(X) \geq 0$ for any $X \geq 0$. Next we assume $M < 0$, then

$$F(X) \geq F(X_0) = e^{(p-1)v}\left(1 - p^p\left(\frac{|M|}{p+1}\right)^{p+1}\right) = e^{(p-1)v}\left(1 - \left(\frac{|M|}{\mu^*(2)}\right)^{p+1}\right).$$

Hence, if $|M| \leq \mu^*(2)$, v is a positive superharmonic function in Ω which tends to infinity on the boundary. Such a function is larger than the harmonic function with boundary value $k > 0$ for any k (and taking the value $\min_{|x|=R} v(x)$ for R large enough if Ω is an exterior domain). Letting $k \to \infty$ we derive a contradiction.

2- Let $R>0$ such that $\Omega^c\subset B_R$ and let w be the solution of

$$\begin{array}{ll}-\Delta w-ae^{(p-1)w}=0 & \text{in } B_R\cap\Omega\\ \displaystyle\lim_{\operatorname{dist}(x,\partial B_R)\to 0} w(x)=-\infty & \\ \displaystyle\lim_{\operatorname{dist}(x,\partial\Omega)\to 0} w(x)=\infty, & \end{array}\tag{2.24}$$

with $a=1-\left(\frac{|M|}{\mu^*(2)}\right)^{p+1}<0$, obtained by approximations. By the argument used in 1,

$$ae^{(p-1)w}\leq|\nabla w|^2+e^{(p-1)w}-|M|\,e^{\frac{p-1}{p+1}w}\,|\nabla w|^{\frac{2p}{p+1}},$$

hence

$$-\Delta w\leq|\nabla w|^2+e^{(p-1)w}-|M|\,e^{\frac{p-1}{p+1}w}\,|\nabla w|^{\frac{2p}{p+1}}.$$

Therefore $v=e^w$ is nonnegative and satisfies

$$\begin{array}{ll}-\Delta v-v^p+|M|\,|\nabla v|^{\frac{2p}{p+1}}\leq 0 & \text{in } B_R\cap\Omega\\ v=0 & \text{on } \partial B_R\\ \displaystyle\lim_{\operatorname{dist}(x,\partial\Omega)\to 0} v(x)=\infty. & \end{array}\tag{2.25}$$

Next we extend v by zero in B_R^c and denote by $\tilde v$ the new function. It is a nonnegative subsolution of (1.2) which tends to ∞ on $\partial\Omega$. For constructing a supersolution we recall that if $M\leq-\mu^*(1)$ there exist two types of explicit solutions of

$$-u''=u^p+M\,|u'|^{\frac{2p}{p+1}}\tag{2.26}$$

defined on $\mathbb{R}$ by $U_{j,M}(t)=\infty$ for $t\leq 0$ and $U_{j,M}(t)=X_{j,M}t^{-\frac{2}{p-1}}$, j=1,2, for $t>0$ where $X_{1,M}$ and $X_{2,M}$ are respectively the smaller and the larger positive root of

$$X^{p-1}-|M|\left(\frac{2}{p-1}\right)^{\frac{2}{p+1}}X^{\frac{p-1}{p+1}}+\frac{2(p+1)}{(p-1)^2}=0.\tag{2.27}$$

Since Ω^c is convex it is the intersection of all the closed half-spaces which contain it and we denote by $\mathcal{H}_\Omega$ the family of such hyperplanes which are touching $\partial\Omega$. If $H\in\mathcal{H}_\Omega$ let $\mathbf{n}_H$ be the normal direction to H, inward with respect to Ω, $\mathcal{H}_+=\{x\in\mathbb{R}^N:\langle\mathbf{n}_H,x-\mathbf{n}_H\rangle>0\}$ and we define U_H in the direction $\mathbf{n}_H$ by putting

$$U_H(x)=U_{2,M}(\langle\mathbf{n}_H,x-\mathbf{n}_H\rangle)=X_{2,M}\left(\langle\mathbf{n}_H,x-\mathbf{n}_H\rangle\right)^{-\frac{2}{p-1}}\quad\text{for all } x\in\mathcal{H}_+.$$

Hence and set, for $x\in\Omega:=\cap_{H\in\mathcal{H}_\Omega}\mathcal{H}_+$,

$$u_\Omega(x)=\inf_{H\in\mathcal{H}_\Omega}U_H(x).\tag{2.28}$$

Then u_Ω is a nonnegative supersolution of (1.2) in Ω and

$$u_\Omega(x)\geq X_{2,M}(\operatorname{dist}x,\Omega))^{-\frac{2}{p-1}}\qquad\forall x\in\Omega.$$

Next $v_\Omega = \ln u_\Omega$ blows up on $\partial\Omega$, is finite on ∂B_R and satisfies

$$-\Delta v_\Omega - ae^{(p-1)v_\Omega} \geq 0 \qquad \text{in } B_R \cap \Omega. \tag{2.29}$$

By comparison with w since $a < 0$, $v_\Omega \geq w$. Hence $u_\Omega \geq v$ in $B_R \setminus \Omega^c$. Extending v by zero as $\tilde{v}$ we obtain $u_\Omega \geq \tilde{v}$ in Ω^c. Hence u_Ω is a supersolution in Ω^c where it dominates the subsolution $\tilde{v}$. It follows by [29, Theorem 1-4-6] that there exists a solution u of (1.2) satisfying $\tilde{v} \leq u \leq u_\Omega$, which ends the proof. $\square$

3 The refined Bernstein method

The method is a combination of the one used in the previous proofs. It is based upon the replacement of the unknown by setting first $u = v^{-\beta}$ as in [19] and [10] and the study of the equation satisfied by $|\nabla v|$. However we do not use integral techniques. Since u is a positive solution of (1.2) in B_R, the function v is well defined and satisfies

$$-\Delta v + (1+\beta)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}v^{1-\beta(p-1)} + M\,|\beta|^{q-2}\,\beta v^{(\beta+1)(1-q)}\,|\nabla v|^q = 0 \tag{3.1}$$

in B_R. We set

$$z = |\nabla v|^2\ ,\ s = 1-q-\beta(q-1) = (1-q)(\beta+1)\,,\ \sigma = 1-\beta(p-1),$$

and derive

$$\Delta v = (1+\beta)\frac{z}{v} + \frac{1}{\beta}v^\sigma + M\,|\beta|^{q-2}\,\beta v^s z^{\frac{q}{2}}. \tag{3.2}$$

Combining Bochner's formula and Schwarz identity we have classically

$$\frac{1}{2}\Delta z \geq \frac{1}{N}(\Delta v)^2 + \langle \nabla\Delta v, \nabla v\rangle.$$

We explicit the different terms

$$\begin{aligned}(\Delta v)^2 = (1+\beta)^2\frac{z^2}{v^2} + M^2\beta^{2(q-1)}v^{2s}z^q + \frac{v^{2\sigma}}{\beta^2} + 2M(1+\beta)\,|\beta|^{q-2}\,\beta v^{s-1}z^{1+\frac{q}{2}}\\ + \frac{2(1+\beta)}{\beta}v^{\sigma-1}z + 2M\,|\beta|^{q-2}\,v^{s+\sigma}z^{\frac{q}{2}},\end{aligned}$$

$$\begin{aligned}\nabla\Delta v = (1+\beta)\frac{\nabla z}{v} - \frac{(1+\beta)z}{v^2}\nabla v + \frac{\sigma}{\beta}v^{\sigma-1}\nabla v + Ms\,|\beta|^{q-2}\,\beta v^{s-1}z^{\frac{q}{2}}\nabla v\\ + \frac{Mq}{2}\,|\beta|^{q-2}\,\beta v^s z^{\frac{q}{2}-1}\nabla z,\end{aligned}$$

$$\begin{aligned}\langle \nabla\Delta v, \nabla v\rangle = \left(\frac{1+\beta}{v} + \frac{Mq}{2}\,|\beta|^{q-2}\,\beta v^s z^{\frac{q}{2}-1}\right)\langle \nabla z, \nabla v\rangle - \frac{(1+\beta)z^2}{v^2} + \frac{\sigma}{\beta}v^{\sigma-1}z\\ + Ms\,|\beta|^{q-2}\,\beta v^{s-1}z^{\frac{q}{2}+1}.\end{aligned}$$

Hence

$$\begin{aligned}-\frac{1}{2}\Delta z + \frac{1}{N}(\Delta v)^2 + \left(\frac{1+\beta}{v} + \frac{Mq}{2}\,|\beta|^{q-2}\,\beta v^s z^{\frac{q}{2}-1}\right)\langle \nabla z, \nabla v\rangle\\ - \frac{(1+\beta)z^2}{v^2} + \frac{\sigma}{\beta}v^{\sigma-1}z + Ms\,|\beta|^{q-2}\,\beta v^{s-1}z^{\frac{q}{2}+1} \leq 0.\end{aligned} \tag{3.3}$$

3.1 Proof of Theorem D

We develop the term $(\Delta v)^2$ in (3.3) and get

$$-\frac{1}{2}\Delta z+\left(\frac{(1+\beta)^2}{N}-(1+\beta)\right)\frac{z^2}{v^2}+\frac{M^2\beta^{2(q-1)}}{N}v^{2s}z^q+M\left(s+\frac{2(1+\beta)}{N}\right)|\beta|^{q-2}\beta v^{s-1}z^{1+\frac{q}{2}}$$
$$+\frac{v^{2\sigma}}{N\beta^2}+\left(\frac{1+\beta}{v}+\frac{Mq}{2}|\beta|^{q-2}\beta v^s z^{\frac{q}{2}-1}\right)\langle\nabla z,\nabla v\rangle+\frac{N\sigma+2(1+\beta)}{N\beta}v^{\sigma-1}z+\frac{2M|\beta|^{q-2}}{N}v^{s+\sigma}z^{\frac{q}{2}}$$
$$\leq 0. \tag{3.4}$$

Next we set $z=v^{-k}Y$ where k is a real parameter. Then $\nabla z=-kv^{-k-1}Y\nabla v+v^{-k}\nabla Y$,

$$\langle\nabla z,\nabla v\rangle=-kv^{-k-1}Yz+v^{-k}\langle\nabla Y,\nabla v\rangle=-kv^{-2k-1}Y^2+v^{-k}\langle\nabla Y,\nabla v\rangle,$$
$$\frac{\langle\nabla z,\nabla v\rangle}{v}=-kv^{-2k-2}Y^2+v^{-k-1}\langle\nabla Y,\nabla v\rangle,$$
$$Mv^s z^{\frac{q}{2}-1}\langle\nabla z,\nabla v\rangle=-kMv^{s-\frac{qk}{2}-k-1}Y^{\frac{q}{2}+1}+Mv^{s-\frac{qk}{2}}Y^{\frac{q}{2}-1}\langle\nabla Y,\nabla v\rangle,$$
$$\begin{aligned}-\Delta z&=\operatorname{div}\left(kv^{-k-1}Y\nabla v-v^{-k}\nabla Y\right)\\&=kv^{-k-1}Y\Delta v-k(k+1)v^{-k-2}Yz+2kv^{-k-1}\langle\nabla Y,\nabla v\rangle-v^{-k}\Delta Y\\&=kv^{-k-1}Y\Delta v-k(k+1)v^{-2k-2}Y^2+2kv^{-k-1}\langle\nabla Y,\nabla v\rangle-v^{-k}\Delta Y.\end{aligned}$$

From (3.2)

$$\Delta v=(1+\beta)v^{-k-1}Y+\frac{1}{\beta}v^\sigma+M|\beta|^{q-2}\beta v^{s-k\frac{q}{2}}Y^{\frac{q}{2}},$$

therefore

$$\begin{aligned}-\Delta z=k(\beta-k)v^{-2k-2}Y^2+\frac{k}{\beta}v^{\sigma-k-1}Y+kM|\beta|^{q-2}\beta v^{s-k\frac{q}{2}-k-1}Y^{\frac{q}{2}+1}\\+2kv^{-k-1}\langle\nabla Y,\nabla v\rangle-v^{-k}\Delta Y.\end{aligned}$$

Replacing $\langle\nabla z,\nabla v\rangle$ and Δz given by the above expressions in (3.4) and z by $v^{-k}Y$, leads to

$$-\Delta Y+\left(\frac{k(\beta-k)}{2}+\frac{(1+\beta)^2}{N}-(k+1)(\beta+1)\right)v^{-k-2}Y^2+\frac{v^{2\sigma+k}}{N\beta^2}+\frac{M^2\beta^{2(q-1)}}{N}v^{2s+k-kq}Y^q$$
$$+\left(\frac{k+\beta+1}{v}+\frac{Mq|\beta|^{q-2}\beta}{2}v^{s+k-k\frac{q}{2}}Y^{\frac{q}{2}-1}\right)\langle\nabla Y,\nabla v\rangle+\frac{2M|\beta|^{q-2}}{N}v^{s+\sigma+k-k\frac{q}{2}}Y^{\frac{q}{2}}$$
$$+\left(s+\frac{2(1+\beta)}{N}-\frac{k(q-1)}{2}\right)M|\beta|^{q-2}\beta v^{s-k\frac{q}{2}-1}Y^{1+\frac{q}{2}}+\frac{1}{\beta}\left(\frac{k}{2}+\sigma+\frac{2(1+\beta)}{N}\right)v^{\sigma-1}Y\leq 0.$$

For $\epsilon_1,\epsilon_2>0$,

$$\frac{1}{v}|\langle\nabla Y,\nabla v\rangle|\leq\epsilon_1 v^{-k-2}Y^2+\frac{1}{4\epsilon_1}\frac{|\nabla Y|^2}{Y},$$
$$v^{s+k-k\frac{q}{2}}Y^{\frac{q}{2}-1}|\langle\nabla Y,\nabla v\rangle|\leq\epsilon_2 v^{2s-kq+k}Y^q+\frac{1}{4\epsilon_2}\frac{|\nabla Y|^2}{Y}.$$

Hence

$$\begin{aligned}
&-\Delta Y+\frac{v^{2\sigma+k}}{N\beta^2}+\frac{2M\left|\beta\right|^{q-2}}{N}v^{s+\sigma+k-k\frac{q}{2}}Y^{\frac{q}{2}}+\left(\frac{M^2\beta^{2(q-1)}}{N}-\frac{Mq\epsilon_2\left|\beta\right|^{q-1}}{2}\right)v^{2s+k-kq}Y^q\\
&\quad+\left(\frac{k(\beta-k)}{2}+\frac{(1+\beta)^2}{N}-(k+1)(\beta+1)-\left|k+\beta+1\right|\epsilon_1\right)v^{-k-2}Y^2\\
&\quad+\frac{1}{\beta}\left(\frac{k}{2}+\sigma+\frac{2(1+\beta)}{N}\right)v^{\sigma-1}Y+\left(s+\frac{2(1+\beta)}{N}-\frac{k(q-1)}{2}\right)M\left|\beta\right|^{q-2}\beta v^{s-k\frac{q}{2}-1}Y^{1+\frac{q}{2}}\\
&\quad\leq\left(\frac{\left|k+\beta+1\right|}{\epsilon_1}+\frac{Mq\left|\beta\right|^{q-1}}{2\epsilon_2}\right)\frac{\left|\nabla Y\right|^2}{4Y}.
\end{aligned}\tag{3.5}$$

We first choose $\epsilon_2=\frac{M|\beta|^{q-1}}{qN}$, then

$$\begin{aligned}
&-\Delta Y+\frac{v^{2\sigma+k}}{N\beta^2}+\left(\frac{k(\beta-k)}{2}+\frac{(1+\beta)^2}{N}-(k+1)(\beta+1)-\left|k+\beta+1\right|\epsilon_1\right)v^{-k-2}Y^2\\
&\quad+\frac{1}{\beta}\left(\frac{k}{2}+\sigma+\frac{2(1+\beta)}{N}\right)v^{\sigma-1}Y+\frac{M^2\beta^{2(q-1)}}{2N}v^{2s+k-kq}Y^q+\frac{2M\left|\beta\right|^{q-2}}{N}v^{s+\sigma+k-k\frac{q}{2}}Y^{\frac{q}{2}}\\
&\quad+\left(s+\frac{2(1+\beta)}{N}-\frac{k(q-1)}{2}\right)M\left|\beta\right|^{q-2}\beta v^{s-k\frac{q}{2}-1}Y^{1+\frac{q}{2}}\\
&\quad\leq\left(\frac{\left|k+\beta+1\right|}{\epsilon_1}+\frac{Nq^2}{2}\right)\frac{\left|\nabla Y\right|^2}{4Y}.
\end{aligned}\tag{3.6}$$

In order to show the sign of the terms on the left in (3.5), we separate the terms containing the coefficient M from the ones which do not contain it. Indeed these last terms are associated to the mere Lane-Emden equation (1.3) which is treated, as a particular case, in [6, Theorem B] where the exponents therein are $q=0$, and $p\in\left(1,\frac{N+3}{N-1}\right)$. We set

$$\begin{aligned}
H_{\epsilon_1,1}&=\frac{v^{2\sigma+k}}{N\beta^2}+\left(\frac{k(\beta-k)}{2}+\frac{(1+\beta)^2}{N}-(k+1)(\beta+1)-\left|k+\beta+1\right|\epsilon_1\right)v^{-k-2}Y^2\\
&\qquad+\frac{1}{\beta}\left(\frac{k}{2}+\sigma+\frac{2(1+\beta)}{N}\right)v^{\sigma-1}Y\\
&=v^{2\sigma+k}\tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma}Y),
\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}
\tilde{H}_{\epsilon_1,1}(t)&=\left(\frac{k(\beta-k)}{2}+\frac{(1+\beta)^2}{N}-(k+1)(\beta+1)-\left|k+\beta+1\right|\epsilon_1\right)t^2\\
&\qquad+\frac{1}{\beta}\left(\frac{k}{2}+\sigma+\frac{2(1+\beta)}{N}\right)t+\frac{1}{N\beta^2},
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
H_{M,2}&=\frac{M^2\beta^{2(q-1)}}{2N}v^{2s+k-kq}Y^q+\frac{2M\left|\beta\right|^{q-2}}{N}v^{s+\sigma+k-k\frac{q}{2}}Y^{\frac{q}{2}}\\
&\qquad+\left(s+\frac{2(1+\beta)}{N}-\frac{k(q-1)}{2}\right)M\left|\beta\right|^{q-2}\beta v^{s-k\frac{q}{2}-1}Y^{1+\frac{q}{2}}.
\end{aligned}\tag{3.9}$$

Then

$$-\Delta Y + v^{2\sigma+k}\tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma}Y) + H_{M,2} \leq \left(\frac{|k+\beta+1|}{\epsilon_1} + \frac{Nq^2}{2}\right)\frac{|\nabla Y|^2}{4Y}.$$

The sign of $\tilde{H}_{\epsilon_1,1}$ depends on its discriminant $\mathcal{D}_{\epsilon_1}$ which is a polynomial in its coefficients. Then if for $\epsilon_1 = 0$ this discriminant is negative $\mathcal{D}_0$ is negative, the discriminant $\mathcal{D}_{\epsilon_1}$ of $\tilde{H}_{\epsilon_1,1}$ shares this property for $\epsilon_1 > 0$ small enough and therefore $H_{\epsilon_1,1}$ is positive. The proof is similar as the one of [6, Theorem B] in case (i) but for the sake of completeness we recall the main steps. Firstly

$$\mathcal{D}'_0 := N^2\beta^2\mathcal{D}_0 = \left(\frac{Nk}{2} + \sigma N + 2(1+\beta)\right)^2 - 4\left(\frac{Nk(\beta-k)}{2} + (1+\beta)^2 - N(k+1)(\beta+1)\right).$$

Then

$$\mathcal{D}'_0 = \left(\frac{N(p-1)}{4} - 1\right)(2\sigma+k)^2 + 2(p-1)(2\sigma+k) + \tilde{L}$$

where $\tilde{L} = (p-1)k^2 + p(\lambda+2)^2 > 0$. Put

$$S = \frac{2\sigma+k}{k+2} = 1 - \frac{2\beta(p-1)}{k+2} \quad \text{and} \quad \mathcal{T}(S) = \left(\frac{(N-1)(p-1)}{4} - 1\right)S^2 + (p-1)S + p.$$

After some computations we get, if $k \neq -2$,

$$\mathcal{D}'_1 := \frac{(p-1)\mathcal{D}'_0}{(k+2)^2} = (p-1)\left(\frac{k}{k+2} - \frac{S}{2}\right)^2 + \mathcal{T}(S). \tag{3.10}$$

We choose $S > 2$ such that $\frac{k}{k+2} - \frac{S}{2} = 0$, hence $\beta = \frac{2-k}{2(p-1)}$. If $p < \frac{N+3}{N-1}$ the coefficient of S^2 in $\mathcal{T}(S)$ is negative. Hence $\mathcal{T}(S) < 0$ provided S is large enough which is satisfied if $k < -2$ with $|k+2|$ small enough. We infer from this that $\beta > 0$, $\mathcal{D}_0 < 0$ and $\tilde{H}_{\epsilon_1,1} > 0$ if ϵ_1 is small enough. In particular $\tilde{H}_{\epsilon_1,1}(t) \geq c_6(t^2+1)$ for some $c_6 = c_6(N,p,q) > 0$, which means

$$v^{2\sigma+k}\tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma}Y) \geq c_6\left(v^{-k-2}Y^2 + v^{2\sigma+k}\right). \tag{3.11}$$

Secondly the positivity of $H_{M,2}$ is ensured, as β and M are positive, by the positivity of

$$\mathcal{A} := s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2}.$$

Replacing s by its value, we obtain, since $1 < q < \frac{N+2}{N}$ and $\beta + \frac{2+k}{2} > 0$, which can be assume by taking $|k+2|$ small enough,

$$\mathcal{A} = 2\frac{1+\beta}{N} - (q-1)\left(\beta + 1 + \frac{k}{2}\right) > -\frac{k}{N}$$

Then we deduce that

$$-\Delta Y + c_6\left(v^{-k-2}Y^2 + v^{2\sigma+k}\right) \leq c_7\frac{|\nabla Y|^2}{Y}, \tag{3.12}$$

and $c_7 = c_7(N,p,q) > 0$ is independent of M. Since $S = 1 - \frac{2\beta(p-1)}{k+2} = 1 - \frac{2-k}{k+2} = \frac{2k}{k+2} > 0$, we have

$$2Y^{\frac{2S}{S+1}} = 2\left(\frac{Y^2}{v^{k+2}}\right)^{\frac{S}{S+1}} v^{\frac{(k+2)S}{S+1}} \leq \frac{Y^2}{v^{k+2}} + v^{(k+2)S} = \frac{Y^2}{v^{k+2}} + v^{2\sigma+k}. \tag{3.13}$$

From this we infer the inequality

$$-\Delta Y + 2c_6 Y^{\frac{2S}{S+1}} \leq c_7 \frac{|\nabla Y|^2}{Y}. \tag{3.14}$$

Then we derive from Lemma 2.2 that in the ball B_R there holds

$$Y(0) \leq c_8 R^{-\frac{2(S+1)}{S-1}} = c_8 R^{-2+\frac{2(k+2)}{\beta(p-1)}}. \tag{3.15}$$

From this it follows

$$\left|\nabla u^{-\frac{2+k}{2\beta}}(0)\right| \leq \frac{|k+2|}{2}\sqrt{c_8}\, R^{-1+\frac{k+2}{\beta(p-1)}}. \tag{3.16}$$

Setting $a = -\frac{k+2}{2\beta} > 0$ we get that for any domain $\Omega \subset \mathbb{R}^N$ any positive solution in Ω satisfies

$$|\nabla u^a(x)| \leq \frac{|k+2|}{2}\sqrt{c_8}\,(\mathrm{dist}\,(x,\partial\Omega))^{-1-\frac{2a}{p-1}} \qquad \text{for all } x \in \Omega. \tag{3.17}$$

The non existence of any positive of (1.2) solution in $\mathbb{R}^N$ follows classically. □

Corollary 3.1 *Let Ω be a smooth domain in $\mathbb{R}^N$, $N \geq 2$ with a bounded boundary, $1 < p < \frac{N+3}{N-1}$, $1 < q < \frac{N+2}{N}$ and $M > 0$. If u is a positive solution of (1.2) in Ω there exists d_0 depending on Ω and $c_9 = c_9(N,p,q) > 0$ such that*

$$u(x) \leq c_9 \left((\mathrm{dist}\,(x,\partial\Omega))^{-\frac{2}{p-1}} + \max_{\mathrm{dist}\,(z,\partial\Omega)=d_0} u(z)\right) \qquad \text{for all } x \in \Omega. \tag{3.18}$$

Proof. It is similar to the one of [6, Corollary B-2]. □

4 The integral method

4.1 Preliminary inequalities

We recall the next inequality [9, Lemma 3.1].

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a domain. Then for any positive $u \in C^2(\Omega)$, any nonnegative $\eta \in C_0^\infty(\Omega)$ and any real numbers m and d such that $d \neq m+2$, the following inequality holds*

$$A\int_\Omega \eta u^{m-2}|\nabla u|^4 dx - \frac{N-1}{N}\int_\Omega \eta u^m(\Delta u)^2 dx - B\int_\Omega \eta u^{m-1}|\nabla u|^2\Delta u dx \leq R, \tag{4.1}$$

where

$$A = \frac{1}{4N}\left(2(N-m)d - (N-1)(m^2+d^2)\right),\ B = \frac{1}{2N}\left(2(N-1)m + (N+2)d\right),$$

and

$$R = \frac{m+d}{2}\int_\Omega u^{m-1}|\nabla u|^2\langle\nabla u,\nabla\eta\rangle dx + \int_\Omega u^m\Delta u\langle\nabla u,\nabla\eta\rangle dx + \frac{1}{2}\int_\Omega u^m|\nabla u|^2\Delta\eta dx.$$

It is noticeable that d is a free parameter which plays a role only in the coefficients of the integral terms. The following technical result is useful to deal with the multi-parameter constraints problems which occur in our construction. It was first used in [10] under a simpler form and extended in [9, Lemma 3.4].

Lemma 4.2 *For any $N \in \mathbb{N}$, $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$ there exist real numbers m and d verifying*

$$\begin{array}{ll} (i) & d \neq m+2, \\[2mm] (ii) & \dfrac{2(N-1)p}{N+2} < d, \\[2mm] (iii) & \max\left\{-2, 1-p, \dfrac{(N-4)p-N}{2}\right\} < m \leq 0, \\[2mm] (iv) & 2(N-m)d - (N-1)(m^2+d^2) > 0. \end{array} \tag{4.2}$$

4.2 Proof of Theorem E

Step 1: The integral estimates. Let $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$. We apply Lemma 4.1 to a positive solution $u \in C^2(\Omega)$ of (1.2), firstly with $q > 1$ and then with $q = \frac{2p}{p+1}$.

$$\begin{aligned} A\int_\Omega \eta u^{m-2} |\nabla u|^4 \, dx - \frac{N-1}{N} \int_\Omega \eta \left(u^{m+2p} + 2Mu^{m+p} |\nabla u|^q + M^2 u^m |\nabla u|^{2q} \right) dx \\ - B\int_\Omega \eta u^{m-1} |\nabla u|^2 \Delta u dx \leq R. \end{aligned} \tag{4.3}$$

We multiply (1.2) by ηu^{m+p} and integrate over Ω. Then

$$\begin{aligned} \int_\Omega \eta \left(u^{m+2p} + Mu^{m+p} |\nabla u|^q \right) dx &= -\int_\Omega \eta u^{m+p} \Delta u dx \\ &= \int_\Omega u^{m+p} \langle \nabla u, \nabla \eta \rangle dx + (m+p) \int_\Omega \eta u^{m+p-1} |\nabla u|^2 \, dx. \end{aligned}$$

We set

$$\begin{aligned} &F = \int_\Omega \eta u^{m-2} |\nabla u|^4 \, dx \, , \; P = \int_\Omega \eta u^{m-1} |\nabla u|^{q+2} \, dx \, , \; V = \int_\Omega \eta u^{m+2p} dx, \\ &T = \int_\Omega \eta u^{m+p-1} |\nabla u|^2 \, dx \, , \; W = \int_\Omega \eta u^{m+p} |\nabla u|^q \, dx \, , \; U = \int_\Omega \eta u^m |\nabla u|^{2q} \, dx, \\ &S = \int_\Omega u^{m+p} \langle \nabla u, \nabla \eta \rangle dx, \end{aligned}$$

so that there holds

$$AF - \frac{N-1}{N} \left(V + 2MW + M^2 U \right) + BT + BMP \leq R, \tag{4.4}$$

and

$$V + MW = (m+p)T + S. \tag{4.5}$$

Eliminating V between (4.4) and (4.5), we get

$$AF + B_0T + M\left(BP - \frac{N-1}{N}W - \frac{N-1}{N}MU\right) \le R - \frac{N-1}{N}S, \tag{4.6}$$

where

$$B_0 = B - \frac{N-1}{N}(m+p) = \frac{N+2}{2N}d - \frac{N-1}{N}p.$$

Also

$$2P = 2\int_\Omega \eta u^m \frac{|\nabla u|^2}{u}|\nabla u|^q\,dx \le \int_\Omega \eta u^m\left(\frac{|\nabla u|^4}{u^2} + |\nabla u|^{2q}\right)dx = F + U.$$

We fix now $q = \frac{2p}{p+1}$, then

$$\begin{aligned} U = \int_\Omega \eta u^m |\nabla u|^{2q}\,dx &= \int_\Omega \eta u^m\left(\frac{|\nabla u|}{\sqrt{u}}\right)^{4(q-1)} u^{2(q-1)}|\nabla u|^{4-2q}\,dx \\ &\le \frac{p-1}{p+1}\int_\Omega \eta u^{m-2}|\nabla u|^4\,dx + \frac{2}{p+1}\int_\Omega \eta u^{m+p-1}|\nabla u|^2\,dx \\ &\le \frac{p-1}{p+1}F + \frac{2}{p+1}T, \end{aligned} \tag{4.7}$$

hence

$$P \le \frac{1}{2}F + \frac{1}{2}U \le \frac{p}{p+1}F + \frac{1}{p+1}T \tag{4.8}$$

and

$$\begin{aligned} 2W = 2\int_\Omega \eta u^{m+p}|\nabla u|^q\,dx &\le \int_\Omega \eta u^{m+2p}dx + \int_\Omega \eta u^m|\nabla u|^{2q}\,dx = V + U \\ &\le U + (m+p)T + S - MW. \end{aligned} \tag{4.9}$$

Next we assume that $|M| \le 1$. From (4.7), (4.9), it follows that

$$W \le U + (m+p)T + S \le F + (m+p+1)T + S. \tag{4.10}$$

From now we fix m and d according Lemma 4.2. Therefore $A > 0$ by (4.2)-(iv) and $B > 0$ by combining (4.2)-(ii) and (4.2)-(iii). Furthermore $B_0 > 0$ by (4.2)-(ii). Hence, from (4.7), (4.8) and (4.10) we derive, since $\frac{N-1}{N} < 1$ and $m \le 0$ from (4.2)-(ii)

$$\begin{aligned} \left|BP - \frac{N-1}{N}W - \frac{N-1}{N}MU\right| &\le B(F+T) + F + (p+1)T + S + F + T, \\ &\le (B+2)F + (B+p+2)T + S. \end{aligned}$$

Plugging these estimates into (4.6) we infer

$$AF + B_0T - |M|\Big((B+2)F + (B+p+2)T + S\Big) \le R - \frac{N-1}{N}S. \tag{4.11}$$

Since A and B_0 are positive, there exists $\mu_1 \in (0,1)$ such that for any $|M| < \mu_1$,

$$A_1 := A - |M|\,(B+2) > \frac{A}{2} \quad \text{and} \quad B_1 := B_0 - |M|\,(B+p+2) > \frac{B_0}{2}.$$

Set $A_2 = \min\{A_1, B_1\}$, then, and whatever is the sign of S,

$$A_2(F+T) \leq |R| + |S|\,.$$

Using (4.7) and (4.8) we have

$$A_2(U+P) \leq 2A_2(F+T) \leq 2(|R|+|S|). \tag{4.12}$$

In the sequel we denote by c_j some positive constants depending on N and p. Then

$$U+P+F+T+W \leq c_1(|R|+|S|). \tag{4.13}$$

On the other hand, we have

$$|R| \leq c_2 \int_\Omega \left(u^{m-1}\,|\nabla u|^3\,|\nabla\eta| + u^{m+p}\,|\nabla u|\,|\nabla\eta| + u^m\,|\nabla u|^{q+1}\,|\nabla\eta| + u^m\,|\nabla u|^2\,|\Delta\eta| \right) dx.$$

Since

$$|\nabla u|^q = \left(\frac{|\nabla u|}{\sqrt{u}}\right)^q u^{\frac{q}{2}} \leq \frac{|\nabla u|^2}{u} + u^{\frac{q}{2-q}} = \frac{|\nabla u|^2}{u} + u^p,$$

we deduce

$$\int_\Omega u^m |\nabla u|^{q+1} |\nabla\eta| dx \leq \int_\Omega u^{m-1} |\nabla u|^3 |\nabla\eta| dx + \int_\Omega u^{m+p} |\nabla u||\nabla\eta| dx.$$

Thus we derive from (4.13)

$$\begin{aligned} U+P+F+T+W \leq 2c_3 \Bigg(\int_\Omega u^{m-1}|\nabla u|^3|\nabla\eta|dx &+ \int_\Omega u^{m+p}|\nabla u||\nabla\eta|dx \\ &+ \int_\Omega u^m\,|\nabla u|^2\,|\Delta\eta|\,dx \Bigg). \end{aligned} \tag{4.14}$$

From this point we can use the method developed in [10, p 599] for proving the Harnack inequality satisfied by positive solutions of (1.3) in Ω. We set $\eta = \xi^\lambda$ with $\xi \in C_0^\infty(\Omega)$ with value in $[0,1]$ and $\lambda > 4$. For $\epsilon \in (0,1)$ we have by the Hölder-Young inequality

$$\int_\Omega u^{m-1}|\nabla u|^3|\nabla\xi^\lambda|dx \leq \frac{\epsilon}{4c_3}\int_\Omega u^{m-2}|\nabla u|^4\xi^\lambda dx + C(\epsilon, c_3)\int_\Omega u^{m+2}|\nabla\xi|^4\xi^{\lambda-4}dx, \tag{4.15}$$

$$\int_\Omega u^{m+p}|\nabla u||\nabla\xi^p|dx \leq \frac{\epsilon}{4c_3}\int_\Omega u^{m+p-1}|\nabla u|^2\xi^p dx + C(\epsilon, c_3)\int_\Omega u^{m+p+1}|\nabla\xi|^2\xi^{\lambda-2}dx, \tag{4.16}$$

and

$$\int_\Omega u^m|\nabla u|^2|\Delta\xi^p|dx \leq \frac{\epsilon}{4c_3}\int_\Omega u^{m-2}|\nabla u|^4\xi^p dx + C(\epsilon, c_3)\int_\Omega u^{m+2}\left(|\nabla\xi|^4 + |\Delta\xi|^2\right)\xi^{\lambda-4}dx. \tag{4.17}$$

Hence

$$U + P + F + T + W \leq c_4 \left(\int_\Omega u^{m+2} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} dx + \int_\Omega u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx \right). \tag{4.18}$$

Let us denote by $c_4 X$ the right-hand side of (4.18). Combining (4.5), (4.16) and (4.18) we also get

$$S := \int_\Omega u^{m+p} |\nabla u| |\nabla \xi^p| dx \leq c_5 X \Longrightarrow V := \int_\Omega u^{m+2p} \xi^p dx \leq c_6 X, \tag{4.19}$$

and we finally obtain

$$U + V + P + F + S + T + W \leq c_7 X. \tag{4.20}$$

Finally we estimate the different terms in X, using that $m + p > 0$ from (4.2)-(iii). For $\epsilon > 0$

$$\begin{aligned} \int_\Omega u^{m+2} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} dx \leq \epsilon & \int_\Omega u^{m+2p} \xi^\lambda dx \\ & + C(\epsilon, c_7) \int_\Omega \xi^{\lambda - 2\frac{m+2p}{p-1}} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx, \end{aligned} \tag{4.21}$$

and

$$\int_\Omega u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx \leq \epsilon \int_\Omega u^{m+2p} \xi^\lambda dx + C(\epsilon, c_7) \int_\Omega \xi^{\lambda - 2\frac{m+2p}{p-1}} |\nabla \xi|^{\frac{2(m+2p)}{p-1}} dx. \tag{4.22}$$

At end we obtain

$$U + V + P + F + S + T + W \leq c_8 \int_\Omega \xi^{\lambda - 2\frac{m+2p}{p-1}} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx. \tag{4.23}$$

Step 2: The Harnack inequality. We suppose that $\Omega = B_R \setminus \{0\} := B_R^*$, fix $y \in B^*_{\frac{R}{2}}$, set $r = |y|$, then $B_r(y) \subset B_R^*$. Let $\xi \in C_0^\infty(B_r(y))$ such that $0 \leq \xi \leq 1$, $\xi = 1$ in $B_{\frac{r}{2}}(y)$, $|\nabla \xi| \leq cr^{-1}$ and $|\Delta \xi| \leq cr^{-2}$. We choose $\lambda > \max\left\{4, \frac{m+2p}{p+1}\right\}$, then

$$\int_{B_r(y)} \xi^{\lambda - 2\frac{m+2p}{p-1}} \left(|\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx \leq c_9 r^{N - \frac{2(m+2p)}{p-1}},$$

and hence

$$\int_{B_{\frac{r}{2}}(y)} u^{m+2p} dx \leq V \leq c_{10} r^{N - \frac{2(m+2p)}{p-1}}. \tag{4.24}$$

We write (1.2) under the form

$$\Delta u + D(x) u + M \langle G(x). \nabla u \rangle = 0, \tag{4.25}$$

with

$$D(x) = u^{p-1} \quad \text{and} \quad G(x) = |\nabla u|^{-\frac{2}{p+1}} \nabla u.$$

Set $\sigma = \frac{m+2p}{p-1}$, then $\sigma > \frac{N}{2}$ by (4.2)-(iii) and

$$\int_{B_{\frac{r}{2}}(y)} D^\sigma(x)dx \le V \le c_{10} r^{N-\frac{2(m+2p)}{p-1}} = c_{10} r^{N-2\sigma}. \tag{4.26}$$

Next we estimate G. For $\tau, \omega, \gamma > 0$ and $\theta > 1$, we have with $\theta' = \frac{\theta}{\theta-1}$,

$$|\nabla u|^{(q-1)\tau} = u^\omega |\nabla u|^\gamma u^{-\omega} |\nabla u|^{(q-1)\tau-\gamma} \le u^{\omega\theta'} |\nabla u|^{\gamma\theta} + u^{-\omega\theta} |\nabla u|^{((q-1)\tau-\gamma)\theta'}.$$

We fix

$$\tau = 2\frac{2p+m}{p-1} = 2\sigma\,,\ \omega = \frac{(2-m)(p+m-1)}{p+1} \ \text{ and } \ \theta = \frac{p+1}{2-m}.$$

Then $\omega > 0$ and $\theta > 1$ from (4.2)-(iii), $\omega > 0$. Then $u^{\omega\theta'} |\nabla u|^{\gamma\theta} = u^{p+m-1} |\nabla u|^2$ and $u^{-\omega\theta} |\nabla u|^{((q-1)\tau-\gamma)\theta'} = u^{m-2} |\nabla u|^4$, thus

$$\int_{B_{\frac{r}{2}}(y)} |\nabla u|^{(q-1)\tau} dx \le F + T \le c_{11} \int_\Omega \xi^{\lambda-2\frac{m+2p}{p-1}} \left(|\nabla\xi|^4 + |\Delta\xi|^2 \xi^2\right)^{\frac{m+2p}{2(p-1)}} dx.$$

This implies

$$\int_{B_{\frac{r}{2}}(y)} G^\tau(x)dx \le c_{12} r^{N-\tau}, \tag{4.27}$$

with $\tau > N$. Using the results of [28, Section 5], we infer that a Harnack inequality, uniform with respect to r, is satisfied. Hence there exists $c_{13} > 0$ depending on N, p such that for any $r \in (0, \frac{R}{2}]$ and y such that $|y| = r$ there holds

$$\max_{z \in B_{\frac{r}{2}}(y)} u(z) \le c_{13} \min_{z \in B_{\frac{r}{2}}(y)} u(z) \quad \forall 0 < r \le \tfrac{R}{2} \ \ \forall y \ \text{ s.t. } \ |y| = r, \tag{4.28}$$

which implies

$$u(x) \le c_{14} u(x') \quad \forall x, x' \in \mathbb{R}^N \quad \text{s.t. } \ |x| = |x'| \le \frac{R}{2}, \tag{4.29}$$

and actually $c_{14} = c_{13}^7$ by a simple geometric construction. By (4.24)

$$r^N \omega_N r^N \left(\min_{z \in B_{\frac{r}{2}}(y)} u(z)\right)^{m+2p} \le 4^N c_{10} r^{N-\frac{2(m+2p)}{p-1}},$$

where ω_N is the volume of the unit N-ball. This implies

$$u(x) \le c_{14} |x|^{-\frac{2}{p-1}} \qquad \forall x \in B^*_{\frac{R}{2}}. \tag{4.30}$$

The proof follows. $\square$

Remark. Using standard rescaling techniques (see e.g. [29, Lemma 3.3.2]) the gradient estimate holds

$$|\nabla u(x)| \le c_{15} |x|^{-\frac{p+1}{p-1}} \qquad \forall x \in B^*_{\frac{R}{3}}. \tag{4.31}$$

And the next estimate for a solution u in a domain Ω satisfying the interior sphere condition with radius R is valid

$$u(x) \le c_{14} (\operatorname{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} \quad \forall x \in \Omega \ \text{ s.t. } \operatorname{dist}(x, \partial\Omega) \le \frac{R}{2}. \tag{4.32}$$

5 Radial ground states

We recall that if $q \neq \frac{2p}{p+1}$ and $M \neq 0$, (1.2) can be reduced to the case $M = \pm 1$ by using the transformation (1.15). Since any ground state u of (1.2) radial with respect to 0 is decreasing (this is classical and straightforward), it achieves its maximum at 0 and the following equivalence holds if v is defined by (1.15)

$$\begin{array}{lll} -u'' - \dfrac{N-1}{r}u' = |u|^{p-1}u + M\,|u_r|^q & \text{s.t.} & \max u = u(0) = 1 \\ \Longleftrightarrow & & \\ -v'' - \dfrac{N-1}{r}v' = |v|^{p-1}v \pm |v_r|^q & \text{s.t.} & \max v = v(0) = |M|^{\frac{2}{(p+1)q-2p}}. \end{array} \tag{5.1}$$

Hence large or small values of M for u are exchanged into large or small values of $v(0)$ for v and in the sequel we will essentially express our results using the function u.

5.1 Energy functions

We consider first the energy function

$$r \mapsto H(r) = \frac{u^{p+1}}{p+1} + \frac{u'^2}{2}. \tag{5.2}$$

Then

$$H'(r) = M\left|u'\right|^{q+1} - \frac{N-1}{r}u'^2.$$

Hence, if $M \leq 0$, H is decreasing, a property often used in [25]. This implies in particular that a radial ground state satisfies

$$\left|u'(r)\right| \leq \sqrt{\frac{2}{p+1}}\,(u(0))^{\frac{p+1}{2}}. \tag{5.3}$$

A similar estimate holds in all the cases.

Proposition 5.1 *Let $M > 0$, $p, q > 1$. If u is a radial ground state solution of* (1.2)*, then the function H defined in* (5.2) *is decreasing and in particular* (5.3) *holds.*

Proof. Let u be such a radial ground state. By Proposition 2.1 we must have $q > \frac{N}{N-1}$ and

$$\frac{r}{u'^2}H' = Mr\left|u'\right|^{q-1} + 1 - N \leq \frac{(N-1)q - N}{q-1} + 1 - N = -\frac{1}{q-1},$$

this implies the claim. □

5.1.1 Exponential perturbations

As we have seen it in the introduction, if $q < \frac{2p}{p+1}$ equation (1.2) can be seen as a perturbation of the Lane-Emden equation (1.3) while if $q > \frac{2p}{p+1}$ it can be seen as a perturbation of the Ricatti equation (1.14). Two types of transformations can emphasize these aspects.

1) For $p > 1$ set

$$u(r) = r^{-\frac{2}{p-1}} x(t), \quad u'(r) = -r^{-\frac{p+1}{p-1}} y(t), \quad t = \ln r, \tag{5.4}$$

then

$$\begin{aligned} x_t &= \frac{2}{p-1} x - y \\ y_t &= -Ky + x^p + Me^{-\omega t} y^q \end{aligned} \tag{5.5}$$

with

$$K = \frac{(N-2)p - N}{p-1}, \tag{5.6}$$

and

$$\omega = \frac{(p+1)q - 2p}{p+1}. \tag{5.7}$$

If $q > \frac{2p}{p+1}$ (resp. $q < \frac{2p}{p+1}$), then $\omega > 0$ (resp. $\omega < 0$) system (5.7) is a perturbation of the Lane-Emden system

$$\begin{aligned} x_t &= \frac{2}{p-1} x - y \\ y_t &= -Ky + x^p, \end{aligned} \tag{5.8}$$

at ∞ (resp. $-\infty$). The following energy type function introduced in [20] is natural with (5.8)

$$\mathcal{N}(t) = \mathcal{L}(x(t), y(t)) = \frac{K}{p-1} x^2 - \frac{x^{p+1}}{p+1} - \left(\frac{2}{p-1}\right)^q Me^{-\omega t} \frac{x^{q+1}}{q+1} - \frac{1}{2}\left(\frac{2x}{p-1} - y\right)^2, \tag{5.9}$$

and it satisfies

$$\begin{aligned} \mathcal{N}'(t) = \left(\frac{2x}{p-1} - y\right) &\left[L\left(\frac{2x}{p-1} - y\right) - Me^{-\omega t}\left(\left(\frac{2x}{p-1}\right)^q - y^q\right)\right] \\ &+ \omega\left(\frac{2}{p-1}\right)^q Me^{-\omega t} \frac{x^{q+1}}{q+1}, \end{aligned} \tag{5.10}$$

where $L = N - 2 - \dfrac{4}{p-1} = K - \dfrac{2}{p-1}$. Relation (5.10) will be used later on.

2) For $p, q > 1$ set

$$u(r) = r^{-\frac{2-q}{q-1}} \xi(t), \quad u'(r) = -r^{-\frac{1}{q-1}} \eta(t), \quad t = \ln r, \tag{5.11}$$

then

$$\begin{aligned} \xi_t &= \frac{2-q}{q-1} \xi - \eta \\ \eta_t &= -\frac{(N-1)q - N}{q-1} \eta + e^{\overline{\omega} t} \xi^p + M\eta^q \end{aligned} \tag{5.12}$$

where

$$\overline{\omega} = \frac{p-1}{q-1} \omega. \tag{5.13}$$

Note that if $q < \frac{2p}{p+1}$ this system at ∞ endows the form

$$\begin{aligned} \xi_t &= \frac{2-q}{q-1}\xi - \eta \\ \eta_t &= -\frac{(N-1)q-N}{q-1}\eta + M\eta^q. \end{aligned} \tag{5.14}$$

It is therefore autonomous and much easier to study.

5.1.2 Pohozaev-Pucci-Serrin type functions

Let $\alpha, \gamma, \theta, \kappa$ be real parameters with $\alpha, \kappa > 0$. Set

$$\mathcal{Z}(r) = r^\kappa \left(\frac{u'^2}{2} + \frac{u^{p+1}}{p+1} + \alpha \frac{uu'}{r} - \gamma u' \left|u'\right|^q \right). \tag{5.15}$$

This type of function has been introduced in [25] in their study of equation (1.2) with $M = 1$ with specific parameters. We use it here to embrace all the values of M. We define $\mathcal{U}$ by the identity

$$\mathcal{Z}' + \theta \left|u'\right|^{q-1} \mathcal{Z} = r^{\kappa-1}\mathcal{U}. \tag{5.16}$$

Then

$$\begin{aligned} \mathcal{U} = &\left(\frac{\kappa}{2} + \alpha + 1 - N\right) u'^2 + \left(\frac{\kappa}{p+1} - \alpha\right) u^{p+1} + \alpha(\kappa - N)\frac{uu'}{r} + \left(\frac{\theta}{p+1} - \gamma q\right) r u^{p+1} \left|u'\right|^{q-1} \\ &+ \left(M + \gamma + \frac{\theta}{2}\right) r \left|u'\right|^{q+1} + \Big(\left((N-1)q - \kappa\right)\gamma - \alpha(\theta + M)\Big) u \left|u'\right|^q - \gamma(\theta + qM) r u \left|u'\right|^{2q-1}. \end{aligned} \tag{5.17}$$

5.2 Some known results in the case $M < 0$

We recall the results of [14], [25] and [23] relative to the case $M < 0$.

Theorem 5.2 *1- Let $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$.*
1-(i) If $q > \frac{2p}{p+1}$, there is no ground state for any $M < 0$ ([25, Theorem C]).
1-(ii) If $1 < q < \frac{2p}{p+1}$ there exists a ground state when $|M|$ is large [14, Proposition 5.7] and there exists no ground state when $|M|$ is small ([23]).
2- Assume $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and let $\overline{q}$ be the unique root in $(\frac{2p}{p+1}, p)$ of the quadratic equation

$$(N-1)(X-p)^2 - (N+2-(N-2)p)((p+1)X - 2p)X = 0.$$

2-(i) If $\overline{q} \leq q < p$ there exists no ground state for any $M < 0$ ([25, Theorem C]).
2-(ii) If $\frac{2p}{p+1} < q < \overline{q}$, there exists no ground state for $|M|$. It is an open question whether there could exist a finite number of M for which there exists a ground state ([25, Theorem 4]).
2-(iii) If $1 < q < \frac{2p}{p+1}$, there exists a ground state for large $|M|$ ([14, Proposition 5.7]) and no ground state when $|M|$ is small ([23]).

3- Assume $p > \frac{N+2}{N-2}$ and $q > 1$ and let $Q_{N,p} = \frac{2(N-1)p}{2N+p+1} \in (\frac{2p}{p+1}, p)$.
3-(i) If $Q_{N,p} < q < p$ there exists a ground state for $|M|$ small.
3-(ii) If $1 < q \le Q_{N,p}$ there exists a ground state for any $M < 0$ ([25, Theorem A]).

4- Assume $p = \frac{N+2}{N-2}$. There exists at least one $M < 0$ such that there exists a ground state if and only if $1 < q < p$. More precisely:
4-(i) If $\frac{2p}{p+1} < q < p$ there exists ground state if $|M|$ is small ([25, Theorem B]).
4-(ii) If $q \ge \frac{2p}{p+1}$ there exists a ground state for any $M < 0$ ([25, Theorem A]).

Remark. It is interesting to quote that when $M < 0$ and $q \ge \frac{2p}{p+1}$, there holds [25, Theorem 3],

$$u(r) = O(r^{-\frac{2}{p-1}}) \quad \text{and} \quad u'(r) = O(r^{-\frac{p+1}{p-1}}) \quad \text{when } r \to \infty.$$

5.3 The case $M > 0$

The next result is a consequence of Theorem A.

Theorem 5.3 *Let $M > 0$, $p > 1$ and $q > \frac{2p}{p+1}$ then there exists no radial ground state satisfying $u(0) = 1$ when M is large.*

Proof. Suppose that such a solution u exists. From Theorem A and Proposition 2.1 there holds

$$\sup_{r>0} \left|u'(r)\right| \le c_{N,p,q}|M|^{-\frac{p+1}{(p+1)q-2p}} \quad \text{and} \quad \sup_{r>0} r^{\frac{p+1}{p-1}} \left|u'(r)\right| \le c_{N,p}. \tag{5.18}$$

As a consequence, if $r > R > 0$,

$$\begin{aligned} 1 - u(r) = u(0) - u(r) = u(0) - u(R) + u(R) - u(r) &\le c_{N,p,q}|M|^{-\frac{p+1}{(p+1)q-2p}} R + \int_R^\infty |u'(s)|\, ds \\ &\le c_{N,p,q}|M|^{-\frac{p+1}{(p+1)q-2p}} R + c'_{N,p} R^{-\frac{2}{p-1}}, \end{aligned}$$

with $c'_{N,p} = \frac{p-1}{2} c_{N,p}$. Since $u(r) \to 0$ when $r \to \infty$, we take $R = |M|^{\frac{p-1}{(p+1)q-2p}}$ and derive

$$1 \le \left(c_{N,p,q} + c'_{N,p}\right) |M|^{-\frac{2}{(p+1)q-2p}}, \tag{5.19}$$

and the conclusion follows. $\square$

Remark. If we use Proposition 5.1 we can make estimate (5.19) more precise.

5.3.1 The case $M > 0$, $1 < p \le \frac{N+2}{N-2}$

It is a consequence of our general results that there is no radial ground state for large M or for small M when $1 < q \le \frac{2p}{p+1}$ and $1 < p < \frac{N+2}{N-2}$. Indeed, if $1 < q < \frac{2p}{p+1}$ is a consequence of the equivalence statement between a priori estimate and non-existence of ground state proved in [23], and if $q = \frac{2p}{p+1}$ it follows from Theorems C and E. Actually in the radial case, the result is more general.

Theorem 5.4 *Let $M > 0$ and $1 < p < \frac{N+2}{N-2}$. If $1 < q \leq p$, there exists no radial ground state for any M. If $q > p$ there exists no radial ground state for M small enough.*

Proof. By Proposition 2.1, we may assume $N \geq 3$ and

$$\frac{N}{N-2} < p \leq \frac{N+2}{N-2} \quad \text{and} \quad q > \frac{N}{N-1}. \tag{5.20}$$

(i) Assume first $q < \frac{2p}{p+1}$. We use the system (5.5). Then ω, defined by (5.7) is negative. Hence the Leighton function $\mathcal{N}$ defined by (5.9) is nonincreasing since $L \leq 0$ when $p \leq \frac{N+2}{N-2}$. Furthermore since $(x(t), y(t)) \to (0,0)$ when $t \to -\infty$ and $e^{-\omega t} \to 0$, we get $\mathcal{N}(-\infty) = 0$ it follows that $\mathcal{N}(t) < 0$ for $t \in \mathbb{R}$. Moreover, by Proposition 2.1,

$$u(r) = O(r^{-\frac{2-q}{q-1}}) \quad \text{as } r \to \infty \Longleftrightarrow x(t) = O(e^{\frac{q(p+1)-2p}{(p-1)(q-1)}t}) = o(1) \quad \text{as } t \to \infty$$

This implies $e^{-\omega t} x^{q+1}(t) = O(e^{2\frac{q(p+1)-2p}{(p-1)(q-1)}t}) = o(1)$ as $t \to \infty$ and $\mathcal{N}(\infty) = 0$, contradiction.
(ii) Assume next $\frac{2p}{p+1} \leq q \leq p$. We consider the function (5.15) with the parameters

$$\kappa = \frac{2(p+1)(N-1)}{p+3} = (p+1)\alpha \quad \text{and} \quad \gamma = -\frac{2M}{q(p+1)+2} = \frac{\theta}{q(p+1)},$$

already used by [25] when $M = -1$, and we get with $\mathcal{U}$ defined by (5.16),

$$\mathcal{U} = \frac{2}{(p+3)^2} \frac{u\,|u'|}{r} \left(A + BM\chi + CM\chi^2\right) \quad \text{with } \chi = \frac{p+3}{2+q(p+1)} r\,|u'|^{q-1},$$

where

$$A = (N-1)(N+2-(N-2)p)\,,\ B = 2(N-1)(p-q)\,,\ C = q(q(p+1)-2p). \tag{5.21}$$

By our assumptions $A \geq 0$, $B \geq 0$ and $C > 0$. Hence $\mathcal{U} > 0$. This implies

$$\mathcal{Z}(r) = e^{-\int_0^r \theta |u'|^{q-1} ds} \mathcal{Z}(0) + \int_0^r e^{-\theta \int_s^r |u'|^{q-1} d\sigma} s^{\kappa-1} \mathcal{U}(s) ds = \int_0^r e^{-\theta \int_s^r |u'|^{q-1} d\sigma} s^{\kappa-1} \mathcal{U}(s) ds,$$

since $\mathcal{Z}(0) = 0$. If u is a ground state, then $u'(r) \to 0$ as $r \to \infty$, thus $u\,|u'|^q \leq u\,|u'|^{\frac{2p}{p+1}}$. Hence, from Proposition 2.1, $u'^2(r) = O(r^{-2\frac{p+1}{p-1}})$ as $r \to \infty$. The other terms $u^{p+1}(r)$, $r^{-1}u(r)u'(r)$ and $u\,|u'|^{\frac{2p}{p+1}}$ satisfy the same bound, hence

$$\mathcal{Z}(r) = O(r^{\kappa - \frac{2(p+1)}{p-1}}) = O(r^{\frac{2(p+3)(N-1)}{p+3} - \frac{2(p+1)}{p-1}}) = O(r^{\frac{2(p+1)((N-2)p-(N+2))}{(p+3)(p-1)}}).$$

Then $\mathcal{Z}(r) \to 0$ when $r \to \infty$, contradiction.
(iii) Suppose $q > p$ and u is a ground state. By Proposition 5.1 and (5.18), there holds

$$r\,|u'|^{q-1} = r\,|u'|^{\frac{p-1}{p+1}}\,|u'|^{q-\frac{2p}{p+1}} \leq c_{N,p}.$$

Then $\chi = \frac{p+3}{2+q(p+1)} r\,|u'|^{q-1} \leq c_{N,p}$. Hence, if $M \leq M_{N,p}$ for some $M_{N,p} > 0$, $\mathcal{U}$ is positive as A is. We conclude as above. $\square$

5.3.2 The case $M > 0$, $p > \frac{N+2}{N-2}$

We recall that in Theorem C if $q = \frac{2p}{p+1}$ and $p > 1$ there is no ground state whenever $M > M_{N,p}$, see (1.26). In Theorem A' if $1 < q < \frac{2p}{p+1}$ and $p > 1$ there is no ground state u such that $u(0) = 1$ if M is too large. In the next result we complement Theorem 5.3 for small value of M in assuming $q > \frac{2p}{p+1}$.

Theorem 5.5 *If $p > \frac{N+2}{N-2}$ and $q \geq \frac{2p}{p+1}$ then there exist radial ground states for $M > 0$ small enough.*

Proof. First we consider the function $\mathcal{Z}$ with $k = N$ and obtain

$$\mathcal{Z}(r) = r^N \left(\frac{u'^2}{2} + \frac{u^{p+1}}{p+1} + \alpha \frac{uu'}{r} - \gamma u \left| u' \right|^q \right).$$

The function vanishes at the origin. We compute $\mathcal{U}$ from the identity $\mathcal{Z}' + \theta \left| u' \right|^{q-1} \mathcal{Z} = r^{N-1} \mathcal{U}$ and get

$$\begin{aligned} \mathcal{U} = & \left(\alpha - \frac{N-2}{2} \right) u'^2 + \left(\frac{N}{p+1} - \alpha \right) u^{p+1} + \left(\frac{\theta}{p+1} - \gamma q \right) r u^{p+1} \left| u' \right|^{q-1} \\ & + \left(M + \gamma + \frac{\theta}{2} \right) r \left| u' \right|^{q+1} + \Big[\left((N-1)q - N \right) \gamma - \alpha(\theta + M) \Big] u \left| u' \right|^q - \gamma(\theta + qM) r u \left| u' \right|^{2q-1}. \end{aligned}$$

If $\gamma = 0$ and $\theta = -2M$, then

$$\mathcal{U} = \left(\alpha - \frac{N-2}{2} \right) u'^2 + \left(\frac{N}{p+1} - \alpha \right) u^{p+1} - \frac{2M}{p+1} r u^{p+1} \left| u' \right|^{q-1} + \alpha M u \left| u' \right|^q.$$

If u is a regular solution which vanishes at some $r_0 > 0$, then $\mathcal{Z}(r_0) = 2^{-1} r_0^2 u'^N(r_0) > 0$. As $p > \frac{N+2}{N-2}$, by choosing $\alpha = \frac{1}{2} \left(\frac{N}{p+1} + \frac{N-2}{2} \right)$ we have $\frac{N}{p+1} < \alpha < \frac{N-2}{2}$. We define $\ell > 0$ by $(N-2)p - (N+2) = 4(p+1)\ell$, then $\frac{N-2}{2} - \alpha = \alpha - \frac{N}{p+1} = \ell$ and then

$$\mathcal{U} \leq -\ell(u'^2 + u^{p+1}) + M\alpha u \left| u' \right|^q.$$

Assume first $q < 2$, we have from Hölder's inequality and $0 < r \leq r_0$ where u is positive

$$u \left| u' \right|^q \leq \frac{q}{2} u'^2 + \frac{2-q}{2} |u|^{\frac{2}{2-q}} \leq u'^2 + |u|^{\frac{2}{2-q}},$$

and

$$\mathcal{U} + (\ell - M) u'^2 \leq M\alpha u^{\frac{2}{2-q}} - \ell u^{p+1} = \ell u^{p+1} \left(\frac{M\alpha}{\ell} u^{\frac{q(p+1)-2p}{2-q}} - 1 \right) \leq \ell u^{p+1} \left(\frac{M\alpha}{\ell} - 1 \right)$$

since $q \geq \frac{2p}{p+1}$ and $u \leq u(0) = 1$. Taking $M \leq \frac{\ell}{\alpha} = \frac{(N-2)p-N-2}{(N-2)p+3N-2}$, $\mathcal{U}$ is negative implying that $r \mapsto e^{-2M \int_0^r |u'|^{q-1} ds} \mathcal{Z}(r)$ is nonincreasing. Since $\mathcal{Z}(0) = 0$, $\mathcal{Z}(r) \leq 0$, a contradiction.

If $q=2$, then $\mathcal{U}\leq -\ell(u'^2+u^{p+1})+M\alpha u'^2$ since $u\leq 1$ on $[0,r_0]$. We still infer that $\mathcal{U}\leq 0$ if we choose $M\leq \frac{\ell}{\alpha}$.

Finally, if $q>2$, we have from Theorem A, $u'\leq C_{N,p,q}M^{-\frac{p+1}{(p+1)q-2p}}$. Therefore, using again the decay of u from $u(0)=1$,

$$M\alpha u\left|u'\right|^q\leq M\alpha u\left|u'\right|^{q-2}u'^2\leq M\alpha C^{q-2}_{N,p,q}M^{-\frac{(p+1)(q-2)}{(p+1)q-2p}}u'^2=\alpha C^{q-2}_{N,p,q}M^{\frac{2}{(p+1)q-2p}}u'^2.$$

Hence $\mathcal{U}\leq -\left(\ell-\alpha C^{q-2}_{N,p,q}M^{\frac{2}{(p+1)q-2p}}\right)u'^2$. Taking

$$M^{\frac{2}{(p+1)q-2p}}\leq C^{2-q}_{N,p,q}\frac{(N-2)p-N-2}{(N-2)p+3N-2}$$

we conclude that $\mathcal{U}<0$ which ends the proof as in the previous cases. $\square$

Theorem F is the combination of Theorem 5.3, Theorem 5.4 and Theorem 5.5.

6 Separable solutions

We denote by $(r,\sigma)\in\mathbb{R}_+\times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$. Then equation (1.2) takes the form

$$-u_{rr}-\frac{N-1}{r}u_r-\frac{1}{r^2}\Delta' u=|u|^{p-1}+M\left(u_r^2+\frac{1}{r^2}\left|\nabla' u\right|^2\right)^{\frac{q}{2}},\tag{6.1}$$

where Δ' is the Laplace-Beltrami operator on S^{N-1} and ∇' the tangential gradient. If we look for separable nonnegative solutions of (1.2) i.e. solutions under the form $u(r,\sigma)=\psi(r)\omega(\sigma)$, then $q=\frac{2p}{p+1}$, $\psi(r)=r^{-\frac{2}{p-1}}$, and ω is a solution of

$$-\Delta'\omega+\frac{2K}{p-1}\omega=\omega^p+M\left(\left(\frac{2}{p-1}\right)^2\omega^2+\left|\nabla'\omega\right|^2\right)^{\frac{p}{p+1}},\tag{6.2}$$

where K is defined in (5.6). Throughout this section we assume

$$p>1\quad\text{and}\quad q=\frac{2p}{p+1}.\tag{6.3}$$

6.1 Constant solutions

The constant function $\omega=X$ is a solution of (6.2) if

$$X^{p-1}+M\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}}-\frac{2K}{p-1}=0.\tag{6.4}$$

For $N=1,2$ and $p>1$ or $N\geq 3$ and $1<p<\frac{N}{N-2}$, we recall that $\mu^*=\mu^*(N)$ has been defined in (1.24). The following result is easy to prove

Proposition 6.1 *1- Let $M \geq 0$ then there exists a unique positive root X_M to (6.4) if and only if $p > \frac{N}{N-2}$. Moreover the mapping $M \mapsto X_M$ is continuous and decreasing from $[0, \infty)$ onto $(0, \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}]$.*

2- Let $M < 0$, $N \geq 3$ and $p \geq \frac{N}{N-2}$ then there exists a unique positive root X_M to (6.4) and the mapping $M \mapsto X_M$ is continuous and decreasing from $(-\infty, 0]$ onto $[\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \infty)$.

3- Let $M < 0$, $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$ then there exists no positive root to (6.4) if $-\mu^* < M \leq 0$. If $M = M^* := -\mu^*$ there exists a unique positive root $X_{M^*} = \left(\frac{2|K|}{p(p-1)}\right)^{\frac{1}{p-1}}$. If $M < -\mu^*$ there exist two positive roots $X_{1,M} < X_{2,M}$. The mapping $M \mapsto X_{1,M}$ is continuous and increasing from $(-\infty, \mu^*]$ onto $(0, X_{M^*}]$. The mapping $M \mapsto X_{2,M}$ is continuous and decreasing from $(-\infty, \mu^*]$ onto $[X_{M^*}, \infty)$.

Abridged proof. Set

$$f_M(X) = X^{p-1} + M\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} - \frac{2K}{p-1}, \tag{6.5}$$

then $f'_M(X) = (p-1)X^{p-2} + M\frac{p-1}{p+1}\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X^{-\frac{2}{p+1}}$.

1- If M is nonnegative, f_M is increasing from $-\frac{2K}{p-1} = -\frac{2[(N-2)p-N]}{(p-1)^2}$ to ∞; hence, if $p > \frac{N}{N-2}$ there exists a unique $X_M > 0$ such that $f_M(X_M) = 0$, while if $1 < p < \frac{N}{N-2}$, f_M admits no zero on $[0, \infty)$. Since $f_M > f_{M'}$ for $M > M' > 0$, there holds $X_M > X_{M'}$, By the implicit function theorem the mapping $M \mapsto X_M$ is C^1 and decreasing from $[0, \infty)$ onto $(0, \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}]$. Actually it can be proved that (see [7, Proposition 2.2])

$$X_M = \frac{p-1}{2}\left(\frac{K}{M}\right)^{\frac{p+1}{p-1}}(1 + o(1)) \quad \text{as} \quad M \to \infty. \tag{6.6}$$

2- If M is negative, f_M achieves it minimum on $[0, \infty)$ at $X_0 = \left(\frac{-M}{p+1}\right)^{\frac{p+1}{p(p-1)}}\left(\frac{2}{p-1}\right)^{\frac{2}{p-1}}$, and

$$\begin{aligned} f_M(X_0) &= -\frac{p}{(p+1)^{\frac{p+1}{p}}}\left(\frac{2}{p-1}\right)^2 (-M)^{\frac{p+1}{p}} - \frac{2K}{p-1} \\ &= -\left(\frac{2}{p-1}\right)^2\left(\frac{p}{(p+1)^{\frac{p+1}{p}}}(-M)^{\frac{p+1}{p}} + \frac{(N-2)p-N}{2}\right). \end{aligned}$$

Since $K > 0$, there exists a unique $X_M > 0$ such that $f_M(X_M) = 0$ and $X_M > X_0$. The mapping

$M \mapsto X_M$ is C^1 and decreasing from $(-\infty, 0]$ onto $[\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \infty)$. The following estimate holds

$$\begin{aligned}\max\left\{\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}}|M|^{\frac{p+1}{p(p-1)}}\right\} &\leq X_M \\ &\leq 2^{\frac{2}{p-1}}\left(\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}} + \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}}|M|^{\frac{p+1}{p(p-1)}}\right).\end{aligned} \tag{6.7}$$

3- If $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, then $f_M(0) > 0$. Hence, if $f_M(X_0) > 0$ there exists no positive root to $f_M(X) = 0$. Equivalently, if $-\mu^* < M < 0$. If $f_M(X_0) = 0$, X_0 is a double root and this is possible only if $M = -\mu^*$, hence $X_{-\mu^*} = \left(\frac{2|K|}{p(p-1)}\right)^{\frac{1}{p-1}}$. If $f_M(X) < 0$, or equivalently, if $M < -\mu^*$, the equation $f_M(X) = 0$ admits two positive roots $X_{1,M} < X_0 < X_{2,M}$. The monotonicity of the $X_{j,M}$, j=1,2, and their range follows easily from the monotonicity of $M \mapsto f_M(X)$ for $M < 0$. Actually the following asymptotics hold when $M \to -\infty$,

$$X_{1,M} = \frac{p-1}{2}\left(\frac{K}{M}\right)^{\frac{p+1}{p-1}}(1 + o(1)) \text{ and } \; X_{2,M} = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}}(-M)^{\frac{p+1}{p(p-1)}}(1 + o(1)). \tag{6.8}$$

□

6.2 Bifurcations

We set

$$A(\omega) = -\Delta'\omega + \frac{2K}{p-1}\omega - \omega^p - M\left(\left(\frac{2}{p-1}\right)^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}}, \tag{6.9}$$

If $\eta \in C^\infty(S^{N-1})$ and if there exists a constant positive solution X to $A(X) = 0$ we have

$$\frac{d}{d\tau}A(X + \tau\eta)\lfloor_{\tau=0} = -\Delta'\eta + \left(\frac{2K}{p-1} - pX^{p-1} - M\frac{2p}{p+1}\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}}\right)\eta.$$

Hence the problem is singular if

$$-\frac{2K}{p-1} + pX^{p-1} + M\frac{2p}{p+1}\left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}}X^{\frac{p-1}{p+1}} = \lambda_k, \tag{6.10}$$

where $\lambda_k = k(k + N - 2)$ is the k-th nonzero eigenvalue of $-\Delta'$ in $H^1(S^{N-1})$. The following result follows classically from the standard bifurcation theorem from a simple eigenvalue (which can always be assumed if we consider functions depending only on the azimuthal angle on S^{N-1} reducing the eigenvalue problem to a simple Legendre type ordinary differential equation) see e.g. [26, Chapter 13] and identity (6.4).

Theorem 6.2 *Let $M_0 \in \mathbb{R}$ and X_{M_0} be a constant solution of* (6.2)*. If X_{M_0} satisfies for some $k \in \mathbb{N}^*$,*

$$M_0 \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X_{M_0}^{\frac{p-1}{p+1}} = \frac{p+1}{p(p-1)} (2K - \lambda_k), \tag{6.11}$$

there exists a continuous branch of nonconstant positive solutions (M, ω_M) of (6.2) *bifurcating from the (M_0, X_{M_0}).*

Since $M \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X_M^{\frac{p-1}{p+1}} = \frac{2K}{p-1} - X_M^{p-1}$ by (6.4) the following statements follow immediately from Proposition 6.1.

Lemma 6.3 *Set $\Phi(M) = M \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X_M^{\frac{p-1}{p+1}}$ when X_M is uniquely determined, and $\Phi_j(M) = M \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X_{j,M}^{\frac{p-1}{p+1}}$, j=1,2, if there exist two equilibria. Then*

1- If $N \geq 3$ and $p > \frac{N}{N-2}$, the mapping $M \mapsto \Phi(M)$ is continuous and increasing from $[0, \infty)$ onto $[0, \frac{2K}{p-1})$.

2- If $N \geq 3$ and $p \geq \frac{N}{N-2}$, the mapping $M \mapsto \Phi(M)$ is continuous and increasing from $(-\infty, 0]$ onto $(-\infty, 0]$.

3- If $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, the mapping $M \mapsto \Phi_1(M)$ (resp $M \mapsto \Phi_2(M)$) is continuous and decreasing (resp. increasing) from $(-\infty, -\mu^*]$ onto $[\frac{2K}{p-1}, 0)$ (resp. $(-\infty, \frac{2K}{p-1}]$).

If we analyse the range $R[\Phi]$ of Φ or $R[\Phi_j]$ of Φ_j, we prove the following result.

Theorem 6.4 *1- Let $N \geq 3$ and $p \geq \frac{N}{N-2}$.*

1-(i) There exists a continuous curve of bifurcation (M, ω_M) issued from (M_0, X_{M_0}) for some $M_0 = M_0(p) \geq 0$ if and only if $p \geq \frac{N+1}{N-3}$ and $k = 1$. Furthermore $M_0(\frac{N+1}{N-3}) = 0$.

1-(ii) The bifurcation curve $s \mapsto (M(s), \omega_{M(s)})$, is defined on $(-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$ and verifies $(M(0), \omega_{M(0)}) = (M_0, X_{M_0})$.

2- Let $N \geq 3$ and $p \geq \frac{N}{N-2}$.

2-(i) For any $k \geq 1$ there exist $M_k < 0$ and a continuous branch of bifurcation (M, ω_M) issued from (M_k, X_{M_k}), with the restriction that $p < \frac{N+1}{N-3}$ if $k = 1$.

2-(ii) The bifurcation curve $s \mapsto (M(s), \omega_{M(s)})$, is defined on $(-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$ and verifies $(M(0), \omega_{M(0)}) = (M_0, X_{M_0})$. Finally $M_k \to -\infty$ when $k \to \infty$.

3- let $N = 1, 2$ and $p > 1$, or $N \geq 3$ and $1 < p < \frac{N}{N-2}$.

3-(i) There exists no $M < 0$ such that $\frac{2K}{p-1} < \Phi_1(M) < 0$, and a countable set of $M_k < 0$, $k \geq 1$, such that $\Phi_2(M_k) = \frac{p+1}{p(p-1)} (2K - \lambda_k)$.

3-(ii) There exist a countable branches of bifurcation of solutions $(M_k(s), \omega_{M_k(s)})$ issued from (M_k, X_{2,M_k}).

Proof. Assertion 1. Since from Lemma 6.3, $R[\Phi] = [0, \frac{2K}{p-1})$ for $M \geq 0$, we have to see under what condition on $p \geq \frac{N}{N-2}$ one can find $k \geq 1$ such that

$$0 \leq \frac{p+1}{p(p-1)}\,(2K - \lambda_k) < \frac{2K}{p-1} \Longleftrightarrow \frac{2K}{p+1} < \lambda_k \leq 2K.$$

Since $K < N$ and $\lambda_k \geq 2N$ for $k \geq 2$, the only possibility for this last inequality to hold is $k = 1$. The inequality $\frac{2K}{p+1} < N-1$ always holds since $p > 1$, while the inequality $N - 1 = \lambda_1 \leq 2K$ is equivalent to $p \geq \frac{N+1}{N-3}$. Therefore $M_0 = 0$ and $X_{M_0} = \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}$. If we consider only functions on the sphere S^{N-1} which depend uniquely on the azimuthal angle $\theta = \tan^{-1}(x_N\lfloor_{S^{N-1}})$, the function $\psi_1(\sigma) = x_N\lfloor_{S^{N-1}}$ is a eigenfunction of $-\Delta'$ in $H^1(S^{N-1})$ with multiplicity one. Hence the bifurcation branch is locally a regular curve $s \mapsto (M(s), \omega_{M(s)})$ with $0 \leq s < \epsilon_0'$ and we construct the bifurcating solution on S^{N-1} using the classical tangency condition [26, Theorem 13.5],

$$\omega_{M(s)} = X_{M_0} + s(\psi_1 + \zeta_s) \tag{6.12}$$

where $\zeta_s \in H^1(S^{N-1})$, is orthogonal to ψ_1 in $H^1(S^{N-1})$ and satisfies $\|\zeta_s\|_{C^1} = o(1)$ when $s \to 0$. This implies the claim.

Assertion 2. Since $R[\Phi] = (-\infty, 0)$ for $M < 0$, we have to find $k \geq 1$ such that

$$\frac{p+1}{p(p-1)}\,(2K - \lambda_k) < 0 \Longleftrightarrow 2K < \lambda_k.$$

As in Case 1, $K < 2N$, then inequality $2K \leq \lambda_k$ holds for all $k \geq 2$, and if $k = 1$ this is possible only if $p < \frac{N+1}{N-3}$. The construction of the bifurcating curve is the same as in Case 1.

Assertion 3. We have $R[\Phi_1] = [\frac{2K}{p-1}, 0)$ for $M \leq -\mu^*$. If we look for the existence of some $k \geq 1$ such that

$$\frac{2K}{p-1} \leq \frac{p+1}{p(p-1)}\,(2K - \lambda_k) < 0 \Longleftrightarrow 2K \leq \lambda_k < \frac{2K}{p+1};$$

we get an impossibility since $K < 0$. Hence there exists no $M_0 < 0$ such that (M_0, X_{1,M_0}) is a bifurcation point. We have also $R[\Phi_2] = (-\infty, \frac{2K}{p-1}]$ for $M \leq -\mu^*$. Now the condition for the existence of a bifurcation branch issued from (M_0, X_{2,M_0}) for some $M_0 \leq -\mu^*$ is

$$\frac{p+1}{p(p-1)}\,(2K - \lambda_k) \leq \frac{2K}{p-1} \Longleftrightarrow \lambda_k \geq \frac{2K}{p+1},$$

which is always true for any $k \geq 1$ and $1 < p < \frac{N}{N-2}$. □

Remark. The exponent $p = \frac{N+1}{N-3}$ is the Sobolev critical exponent on S^{N-1}. If one consider the Lane-Emden equation with a Leray potential

$$-\Delta u + \lambda |x|^{-2} u = u^{\frac{N+1}{N-3}}, \tag{6.13}$$

with $\lambda \in \mathbb{R}$, then the separable solutions $u(r, \sigma) = r^{-\frac{N-3}{2}}\omega(\sigma)$ verify

$$-\Delta'\omega + \left(\frac{(N-1)(N-3)}{4} - \lambda\right)\omega - \omega^{\frac{N+1}{N-3}} = 0 \quad \text{on } S^{N-1}. \tag{6.14}$$

It was observed in [10] that there exists a branch of bifurcation (λ,ω_λ) with $\lambda>0$ issued from $(0,\omega_0)$, where ω_0 is the constant explicit solution of (6.14).

Remark. In Theorem 6.4-1- and the above remark, we conjectured that on the bifurcating curve there holds locally $M(s)<M_0$, and that for any $p\geq\frac{N+1}{N-3}$ there exists $M_0:=M_0(p)$ such that for $M>M_0$ all the positive solutions to (6.2) are constant, furthermore M_0 is defined by (6.11). When $p=\frac{N+1}{N-3}$, then $M=0$ and there exists infinitely many positive solutions to (6.2) [10, Proposition 5.1]. When $\frac{N}{N-2}<p<\frac{N+1}{N-3}$, it is unclear if the branches of bifurcation $(M(s),\omega_{M(s)})$ issued from (M_0,ω_{M_0}) with $M_0<0$ are such that $M(s)$ keeps a constant sign. If it is the case one could expect that if $M\geq 0$ and $\frac{N}{N-2}<p<\frac{N+1}{N-3}$, all the positive solutions to (6.2) are constant.

The following statement is an immediate consequence of Theorem 6.4.

Corollary 6.5 *1-If $p>1$ and $q=\frac{2p}{p+1}$ there always exist nonradial positive singular solutions of* (1.2) *in $\mathbb{R}^N\setminus\{0\}$ under the form $u(r,\sigma)=r^{-\frac{2}{p-1}}\omega(\sigma)$.*
2- If $N\geq 4$ and $p>\frac{N+1}{N-3}$, the functions are obtained by bifurcation from X_M with $M>0$.
3- If $N\geq 3$ and $\frac{N}{N-2}\leq p<\frac{N+1}{N-3}$, the functions are obtained by bifurcation from X_M with $M<0$.
4-If $N=1,2$ and $p>1$ or $N\geq 3$ and $1<p<\frac{N}{N-2}$, the functions are obtained by bifurcation from (M_k,X_{2,M_k}) with $M_k<-\mu^*$ and $k\geq 1$.

References

[1] Alarcón S., García-Melián J., Quass A., *Nonexistence of positive supersolutions to some nonlinear elliptic problems.* J. Math. Pures Appl. **99** (2013), 618-634.

[2] Alarcón S., García-Melián J., Quass A., *Existence and non-existence of solutions to elliptic equations with a general convection term.* Proc. Royal Soc. Edinburgh A **144** (2014), 225-239.

[3] Anderson L. R., Leighton W., *Lyapounov functions for autonomous systems of second order.* J. Math. Anal. Appl. **23** (1968), 645-664.

[4] Bidaut-Véron M.F., *Local and global behaviour of solutions of quasilinear elliptic equations of Emden-Fowler type.* Arch. Rat. Mech. Anal. **107** (1989), 3293-324.

[5] Bidaut-Véron M.F., Garcia-Huidobro M., Véron L., *Local and global properties of solutions of quasilinear Hamilton-Jacobi equations.* J. Funct. Anal. **267** (2014), no. 9, 3294–3331.

[6] Bidaut-Véron M.F., Garcia-Huidobro M., Véron L., *Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient.* Duke Math. J. **168** (2019), 1487-1537.

[7] Bidaut-Véron M.F., Garcia-Huidobro M., Véron L, *Radial solutions of scaling invariant nonlinear elliptic equations with mixed reaction terms.* Submitted, **arXiv:1812.08418**.

[8] Bidaut-Véron M. F., Pohozaev S., *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*. Journal d'Analyse Mathématique **84** (2001), 1-49.

[9] Bidaut-Véron M. F., Raoux Th., *Asymptotic of solutions of some nonlinear elliptic systems.* Comm. Part. Diff. Equ. **21** (1996), 1035-1086.

[10] Bidaut-Véron M.F., Véron L., *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*. Invent. Math. **106** (1991), 489-539.

[11] Brezis H., Lions P. L., *A note on isolated singularities for linear elliptc equations*. Math. Anal. Appl. Adv., Suppl. Stud. **7A** (1981), 263-266.

[12] Caffarelli L., Gidas B., Spruck J., *A*symptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. **42** (1989), 271-297.

[13] Chandrasekar S., *An introduction to the Study of Steller Structure*. Dover Publ. Inc. (1967).

[14] Chipot M., Weissler, F., *Some blow-up results for a nonlinear parabolic equation with a gradient term*. S.I.A.M. J. of Num. Anal. **20** (4) (1989), 886-907.

[15] Emden V. R., *Gaskugeln*. G. B. Teubner, Leipzig (1897).

[16] Fila M., *Remarks on blow-up for a nonlinear parabolic equation with a gradient term.* Proc. A.M.S. **111** (1991), 785-801.

[17] Fila M., Quittner, *Radial positive solutions for a semilinear elliptic equation with a gradient term*. Adv. Math. Sci. Appl. **2** (1993), 3945.

[18] Fowler R. H., *Further studies and similar differential equations*. Quart. J. Math. **2** (1931), 259-288.

[19] Gidas B., Spruck J., *Global and local behaviour of positive solutions of nonlinear elliptic equations*. Comm. Pure Appl. Math. **34** (1981). 525-598.

[20] Leighton W., *On the construction of Liapunov functions for certain autonomous nonlinear differential equations*. In Contributions to Differential Equations **2** (1963), 367-383.

[21] Lions P. L., *Isolated singularities in semilinear problems*. J. Diff. Eq. **38** (1980), 441-450.

[22] Pohozaev S. I. *On the eigenfunctions of the equation* $\Delta u = \lambda f(u)$. Dokl. Akad. Nauk. SSSR **165** (1965), 33-39.

[23] Polacik P., Quittner P., Souplet P.. *Singularity and decay estimates in superlinear problems via Liouville-type theorems, Part.I: Elliptic equations and systems.* Duke Math. J. **139** (2007), 555-579.

[24] Serrin J., *Local behavior of solutions of quasi-linear equations*. Acta Math. **111** (1964), 247-302.

[25] Serrin J., Zou H., *Existence and non-existence results for ground states of quasilinear elliptic equations*. Arch. Rat. Mech. Anal. **121** (1992), 101-130.

[26] Smoller J., *Schock Waves and Reaction-Diffusion Equations*. Grundlehrender mathematischen Wissenschaften **228**, 2nd Edit., Springer-Verlag (1994).

[27] Souplet Ph., *Recent results and open problems on parabolic equations with gradient nonlinearities*. E.J.D.E. **20** (2001), 1-19.

[28] Trudinger N., *Local Estimates for Subsolutions and Supersolutions*. Annali Sc. Norm. Sup. Pisa **27** (1973), 265-308.

[29] Véron L., *Local and global aspects of quasilinear degenerate elliptic equations. Quasilinear elliptic singular problems*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2017). xv+ pp. 1-457.

[30] Voirol F. X., Thèse de Doctorat, Université de Metz, 1994.

[31] Voirol F. X., *Coexistence of singular and regular solutions for the equation of Chipot and Weissler*. Acta Math. Univ-Comenianae **65** (1996), 53-64.