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# A priori estimates for elliptic equations with reaction terms involving the function and its gradient

Marie-Françoise Bidaut-Véron\*  
Marta Garcia-Huidobro †  
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**Abstract** We study local and global properties of solutions of  $-\Delta u = u^p + M |\nabla u|^q$  in a domain  $\Omega$  of  $\mathbb{R}^N$ , in the range  $\min\{p, q\} > 1$  and  $M \in \mathbb{R}$ . We prove a priori estimates and existence or non-existence of ground states.

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*Key words.* elliptic equations; Bernstein methods; ground states;

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# 1 Introduction

The aim of this article is to study local and global properties of positive solutions of the following type of equations

$$-\Delta u = |u|^{p-1}u + M |\nabla u|^q, \tag{1.1}$$

in  $\Omega \setminus \{0\}$  where  $\Omega$  is an open subset of  $\mathbb{R}^N$  containing 0,  $p$  and  $q$  are exponents larger than 1 and  $M$  is a real parameter. In the case  $M = 0$ , (1.1) reduces to the well-known Lane-Emden equation

$$-\Delta u = |u|^{p-1}u. \tag{1.2}$$

If  $N \geq 3$ , this equations exhibits two main critical exponents  $p = \frac{N}{N-2}$  and  $p = \frac{N+2}{N-2}$  which play a key role in the description of the set of positive solutions:

1- If  $1 < p \leq \frac{N}{N-2}$ , there exists no positive solution if  $\Omega$  is the complement of a compact set. Even in that case solution can be replaced by supersolution.

2- If  $1 < p < \frac{N+2}{N-2}$ , there exists no *ground state*, i.e. positive solution in  $\mathbb{R}^N$ . Furthermore any positive solution  $u$  in a ball  $B_R = B_R(a)$  satisfies

$$u(x) \leq c(R - |x - a|)^{-\frac{2}{p-1}}, \tag{1.3}$$

where  $c = c(N, p) > 0$ .

3- If  $p = \frac{N+2}{N-2}$  all the positive solutions in  $\mathbb{R}^N$  are radial with respect to some point  $a$  and endow the following form

$$u(x) := u_\lambda(x) = \frac{(N(N-2)\lambda)^{\frac{N-2}{4}}}{(\lambda + |x - a|^2)^{\frac{N-2}{4}}}. \tag{1.4}$$

All the positive solutions in  $\mathbb{R}^N \setminus \{0\}$  are radial.

4- If  $p > \frac{N+2}{N-2}$  there exist infinitely many positive ground states radial with respect to some points. They are obtained from one say  $v$ , radial for example with respect to 0 by the scaling transformation  $T_k$  where  $k > 0$  with

$$T_k[v](x) = k^{\frac{2}{p-1}}v(kx). \quad (1.5)$$

In a recent paper [5] the authors study local and global aspects of positive solutions of

$$-\Delta u = u^p |\nabla u|^q, \quad (1.6)$$

where  $p \geq 0$ ,  $0 \leq q < 2$ , mostly in the superlinear case  $p + q - 1 > 0$ . They prove the existence of a critical line of exponents

$$(\mathfrak{L}) := \{(p, q) \in \mathbb{R}_+ \times [0, 2) : (N - 2)p + (N - 1)q = N\}. \quad (1.7)$$

The subcritical range corresponds to the fact that  $(p, q)$  is below  $(\mathfrak{L})$ . In this region Serrin's celebrated results [17] can be applied and they prove that positive solutions of (1.6) in the punctured ball  $B_2 \setminus \{0\}$  satisfy

$$u(x) + |x| |\nabla u(x)| \leq c|x|^{2-N} \quad \text{for all } x \in B_1 \setminus \{0\}. \quad (1.8)$$

They introduce two methods for obtaining a priori estimate of solutions: The *pointwise Bernstein method* and the *integral Bernstein method*. The first one is based upon the change of unknown  $u = v^{-\beta}$  and then to show that  $|\nabla v|$  satisfies an inequality of Keller-Osserman type. They obtain that in the supercritical case, i.e. when  $(p, q)$  lies above  $(\mathfrak{L})$  and if

(i) either  $1 \leq p < \frac{N+3}{N-1}$  and  $p + q - 1 < \frac{4}{N-1}$ ,

(ii) or  $0 \leq p < 1$  and  $p + q - 1 < \frac{(p+1)^2}{p(N-1)}$ ,

any positive solution of (1.6) in a domain  $\Omega \subset \mathbb{R}^N$  satisfies

$$|\nabla u^a(x)| \leq c^* (\text{dist}(x, \partial\Omega))^{-1-a \frac{2-q}{p+q-1}} \quad \text{for all } x \in \Omega, \quad (1.9)$$

for some positive  $c^*$  and  $a$  depending on  $N$ ,  $p$  and  $q$ . As a consequence they prove that any positive solution of (1.6) in  $\mathbb{R}^N$  is constant. With the second method they combine the change of unknown  $u = v^{-\beta}$  with integration and cut-off functions. They show the existence of a polynomial  $G$  such that for any  $(p, q) \in \mathbb{R}_+ \times [0, 2)$  satisfying  $G(p, q) < 0$  any positive solution of (1.6) in  $\mathbb{R}^N$  is constant. The polynomial  $G$  is not simple but it is worth noting that if  $0 \leq p < \frac{N+2}{N-2}$ , there holds  $G(p, 0) < 0$ , which recovers Gidas and Spruck celebrated result [15].

For equation (1.1) we first observe that the equation is invariant under the scaling transformation (1.5) for any  $k > 0$  if and only if  $q$  is *critical with respect to  $p$* , i.e.

$$q = \frac{2p}{p+1}.$$

In general the transformation  $T_k$  exchanges (1.1) into

$$-\Delta v = v^p + Mk^{\frac{2p-q(p+1)}{p-1}} |\nabla v|^q, \quad (1.10)$$

hence if  $q < \frac{2p}{p+1}$ , the limit equation when  $k \rightarrow 0$  is (1.2). We say that the exponent  $p$  is dominant. We can also consider the transformation

$$S_k[v](x) = k^{\frac{2-q}{q-1}} v(kx), \quad (1.11)$$

when  $q \neq 2$ , which is the same as  $T_k$  if  $q = \frac{2p}{p+1}$  and more generally transforms (1.1) into

$$-\Delta v = k^{\frac{q-p(2-q)}{q-1}} v^p + M|\nabla v|^q. \quad (1.12)$$

Hence if  $q > \frac{2p}{p+1}$ , the limit equation when  $k \rightarrow 0$  is the Riccati equation

$$-\Delta v = M|\nabla v|^q. \quad (1.13)$$

It is also important to notice that the value of the coefficient  $M$  (and not only its sign) plays a fundamental role, only if  $q = \frac{2p}{p+1}$ . If  $q \neq \frac{2p}{p+1}$  the transformation

$$u(x) = av(y) \quad \text{with } a = |M|^{-\frac{2}{(p+1)q-2p}} \text{ and } y = a^{\frac{p-1}{2}} x \quad (1.14)$$

allows to transform (1.1) into

$$-\Delta v = |v|^{p-1} v \pm |\nabla v|^q. \quad (1.15)$$

The equation (1.1) has been essentially studied in the *radial case* when  $M < 0$  in connection with the parabolic equation

$$\partial_t u - \Delta u + M|\nabla u|^q = |u|^{p-1} u, \quad (1.16)$$

see [12], [13], [14], [18], [24], [25]. The studies mainly deal with the case  $q \neq \frac{2p}{p+1}$ , although not complete when  $q > \frac{2p}{p+1}$ . When  $q = \frac{2p}{p+1}$  the existence of a ground state is proved in dimension 1. Some partial results that we will improve already exist in higher dimension. Very few results are known in the case  $M > 0$ .

In the *nonradial case*, any nonnegative nontrivial solution is positive since  $p, q > 1$ . We first observe, using a classical result dealing with positive supersolutions of (1.2), that if  $M \geq 0$ ,  $1 < p \leq \frac{N}{N-2}$  when  $N \geq 3$ , any  $p > 1$  if  $N = 1, 2$ , then for any  $q > 0$  there exists no positive solution in an exterior domain. Another result of [16] deals with  $0 < q < \frac{2p}{p+1}$  where the equation endows some property of the pure Emden-Fowler equation (1.2). In this paper the authors prove that if  $0 < q < \frac{2p}{p+1}$ ,  $1 < p < \frac{N+2}{N-2}$  and  $M \in \mathbb{R}$ , any positive solution of (1.2) in an open domain satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c_{N,p,q,M} \left( 1 + (\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} \right) \quad \text{for all } x \in \Omega. \quad (1.17)$$

Note that this does not imply the non-existence of ground state.

Our first nonradial result dealing with the case  $q > \frac{2p}{p+1}$  is the following:

**Theorem A** *Let  $N \geq 1$ ,  $p > 1$  and  $q > \frac{2p}{p+1}$ . Then for any  $M > 0$ , any solution of (1.1) in a domain  $\Omega \subset \mathbb{R}^N$  satisfies*

$$|\nabla u(x)| \leq c_{N,p,q} \left( M^{-\frac{p+1}{(p+1)q-2p}} + (M \text{dist}(x, \partial\Omega))^{-\frac{1}{q-1}} \right) \quad \text{for all } x \in \Omega. \quad (1.18)$$

As a consequence, any ground state has at most a linear growth at infinity:

$$|\nabla u(x)| \leq c_{N,p,q} M^{-\frac{p+1}{(p+1)q-2p}} \quad \text{for all } x \in \mathbb{R}^N. \quad (1.19)$$

Our proof relies on a direct Bernstein method combined with Keller-Osserman's estimate applied to  $|\nabla u|^2$ . It is important to notice that the result holds for any  $p > 1$ . In some sense the presence of the gradient term has a regularizing effect. In the case  $q < \frac{2p}{p+1}$  we prove

**Theorem A'** *Let  $N \geq 1$ ,  $p > 1$ ,  $1 < q < \frac{2p}{p+1}$  and  $M > 0$ . Then there exists a constant  $c_{N,p,q} > 0$  such that no positive solution of (1.1) in  $\mathbb{R}^N$  satisfying*

$$u(x) \leq c_{N,p,q} M^{\frac{2}{2p-(p+1)q}} \quad \text{for all } x \in \mathbb{R}^N, \quad (1.20)$$

can exist.

When  $q$  is critical with respect to  $p$  the situation is more delicate. Our first statement is a particular case of a more general result in [1] with a simpler proof which allows to introduce techniques that we use later on.

**Theorem B** *Let  $N \geq 2$ ,  $p > 1$  if  $N = 2$  or  $1 < p \leq \frac{N}{N-2}$  if  $N = 3$ ,  $q = \frac{2p}{p+1}$  and  $-\mu^* < M$  where*

$$\mu^* := \mu^*(N) = (p+1) \left( \frac{N - (N-2)p}{2p} \right)^{\frac{p}{p+1}}. \quad (1.21)$$

*Then there exists no nontrivial nonnegative supersolution of (1.1) in an exterior domain.*

In this range of values of  $p$  this result is optimal since for  $M \leq -\mu^*$  there exists positive singular solutions. The constant  $\mu^*$  will play an important role in the description developed in [6] of radial solutions of (1.1). Using a variant of the method used in the proof we prove results of existence and nonexistence of large solutions.

**Theorem B'** *Let  $N \geq 1$ ,  $p > 1$  and  $q = \frac{2p}{p+1}$ .*

*1- If  $\Omega$  is a domain with a compact boundary satisfying the Wiener criterion and  $M \geq -\mu^*(2)$  there exists no positive supersolution of (1.1) in  $\Omega$  satisfying*

$$\lim_{\text{dist}(x, \partial\Omega) \rightarrow 0} u(x) = \infty. \quad (1.22)$$

*2- If  $G$  is a bounded convex domain,  $\Omega = \overline{G}^c$  and  $M < -\mu^*(1)$  there exists a positive solution of (1.1) in  $\Omega$  satisfying (1.22).*

We prove in [6] that for the necessary and sufficient condition for the existence of a radial large solution in the exterior of a ball is  $M < -\mu^*(1)$ .

Concerning ground states, we prove their nonexistence for any  $p > 1$  provided  $M > 0$  is large enough: indeed

**Theorem C** *Let  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 1$ , be a domain,  $p > 1$ ,  $q = \frac{2p}{p+1}$ . For any*

$$M > M_{\dagger} := \left( \frac{p-1}{p+1} \right)^{\frac{p-1}{p+1}} \left( \frac{N(p+1)^2}{4p} \right)^{\frac{p}{p+1}}, \quad (1.23)$$

and any  $\nu > 0$  such that  $(1 - \nu)M > M_{\dagger}$ , there exists a positive constant  $c_{N,p,\nu}$  such that any solution  $u$  in  $\Omega$  satisfies

$$|\nabla u(x)| \leq c_{N,p,\nu} ((1 - \nu)M - M_{\dagger})^{-\frac{p+1}{p-1}} (\text{dist}(x, \partial\Omega))^{-\frac{p+1}{p-1}} \quad \text{for all } x \in \Omega. \quad (1.24)$$

Consequently there exists no nontrivial solution of (1.1) in  $\mathbb{R}^N$ .

The next result, based upon an elaborate Bernstein method, complements Theorem C under a less restrictive assumption on  $M$  but a more restrictive assumption on  $p$ .

**Theorem D** *Let  $1 < p < \frac{N+3}{N-1}$ ,  $N \geq 2$ ,  $1 < q < \frac{N+2}{N}$  and  $\Omega \subset \mathbb{R}^N$  be a domain. Then there exist  $a > 0$  and  $c_{N,p,q} > 0$  such that for any  $M > 0$ , any positive solution  $u$  in  $\Omega$  satisfies*

$$|\nabla u^a(x)| \leq c_{N,p,q} (\text{dist}(x, \partial\Omega))^{-\frac{2a}{p-1}-1} \quad \text{for all } x \in \Omega. \quad (1.25)$$

Hence there exists no nontrivial nonnegative solution of (1.1) in  $\mathbb{R}^N$ .

It is remarkable that the constants  $a$  and  $c_{N,p,q}$  do not depend on  $M > 0$ , a fact which is clear when  $q \neq \frac{2p}{p+1}$  by using the transformation  $T_k$ , but much more delicate to prove when  $q = \frac{2p}{p+1}$  since (1.1) is invariant. When  $|M|$  is small, we use an integral method to obtain the following result which contains, as a particular case, the estimates in [6] and [8]. The key point of this method is to prove that the solutions in a punctured domain satisfy a local Harnack inequality.

**Theorem E** *Let  $N \geq 3$ ,  $1 < p < \frac{N+2}{N-2}$ ,  $q = \frac{2p}{p+1}$ . Then there exists  $\epsilon_0 > 0$  depending on  $N$  and  $p$  such that for any  $M$  satisfying  $|M| \leq \epsilon_0$ , any positive solution  $u$  in  $B_R \setminus \{0\}$  satisfies*

$$u(x) \leq c_{N,p} |x|^{-\frac{2}{p-1}} \quad \text{for all } x \in B_{\frac{R}{2}} \setminus \{0\}. \quad (1.26)$$

As a consequence there exists no positive solution of (1.1) in  $\mathbb{R}^N$  and any positive solution  $u$  in a domain  $\Omega$  satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c'_{N,p} (\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega. \quad (1.27)$$

Note that under the assumptions of Theorem E, there exist ground states for  $|M|$  large enough when  $1 < p < \frac{N}{N-2}$ , or any  $p > 1$  if  $N = 1, 2$ .

If  $u$  is a radial solutions of (1.1) in  $\mathbb{R}^N$  it satisfies

$$-u'' - \frac{N-1}{r}u' = |u|^{p-1}u + M|u'|^q, \quad (1.28)$$

on  $(0, \infty)$ . Using several type of Lyapounov functions we prove some results dealing with the case  $M > 0$  which complement the ones [18] relative to the case  $M < 0$ .

**Theorem F 1-** *Let  $p > 1$  and  $q > \frac{2p}{p+1}$ . Then there exists no radial ground state  $u$  satisfying  $u(0) = 1$  when  $M > 0$  is too large.*

2- Let  $1 < p < \frac{N+2}{N-2}$ . If  $1 < q \leq p$  there exists no radial ground state for any  $M > 0$ . If  $q > p$  there exists no radial ground state for  $M > 0$  small enough.

3- Let  $N \geq 3$ ,  $p > \frac{N+2}{N-2}$  and  $q \geq \frac{2p}{p+1}$ . Then there exist radial ground states for  $M > 0$  small enough.

We end the article in proving the existence of non-radial positive singular solutions of (1.1) in  $\mathbb{R}^N \setminus \{0\}$  in the case  $q = \frac{2p}{p+1}$  obtained by bifurcation from radial explicit positive singular solutions. Our result shows that the situation is very contrasted according  $M > 0$  where a bifurcation from  $(M, X_M)$  occurs only if  $p \geq \frac{N+1}{N-3}$  and  $M \geq 0$  and  $M < 0$  where there exists a countable set of bifurcations from  $(M_k, X_{M_k})$ ,  $k \geq 1$ , when  $1 < p < \frac{N+1}{N-3}$ .

In a subsequent article [6] we present a fairly complete description of the positive radial solutions of (1.1) in  $\mathbb{R}^N \setminus \{0\}$  in the scaling invariant case  $q = \frac{2p}{p+1}$ .

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## 2 The direct Bernstein method

We begin with a simple property in the case  $M \geq 0$  which is a consequence of the fact that the positive solutions of (1.1) are superharmonic.

**Proposition 2.1** 1- There exists no positive solution of (1.1) in  $\mathbb{R}^N \setminus \overline{B}_R$ ,  $R \geq 0$  if one of the two conditions is satisfied:

(i)  $M \geq 0$ ,  $q \geq 0$  and either  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p \leq \frac{N}{N-2}$ .

(ii)  $M > 0$ ,  $N \geq 3$ ,  $p \geq 1$  and  $1 < q \leq \frac{N}{N-1}$ .

2- If  $N \geq 3$ ,  $q \geq 1$ ,  $p > \frac{N}{N-2}$  and  $u(x) = u(r, \sigma)$  is a positive solution of (1.1) in  $\mathbb{R}^N \setminus \overline{B}_R$ ,  $R \geq 0$ . Then there exists  $\rho \geq R$  such that

$$\frac{1}{N\omega_N} \int_{S^{N-1}} u(r, \sigma) dS := \bar{u}(r) \leq c_0 r^{-\frac{2}{p-1}} \quad \text{for all } r > \rho, \quad (2.1)$$

with  $c_0 := \left(\frac{2N}{p-1}\right)^{\frac{1}{p-1}}$  and

$$\left| \frac{1}{N\omega_N} \int_{S^{N-1}} u_r(r, \sigma) dS \right| := |\bar{u}_r(r)| \leq (N-2)c_0 r^{-\frac{p+1}{p-1}} \quad \text{for all } r > \rho. \quad (2.2)$$

3- If  $M > 0$ ,  $p \geq 0$ , and  $q > \frac{N}{N-1}$  there holds for

$$|\bar{u}_r(r)| \leq \left( \frac{(q-1)(N-1)-1}{(q-1)M} \right)^{\frac{1}{q-1}} r^{-\frac{1}{q-1}} \quad \text{for all } r > \rho, \quad (2.3)$$

and

$$\bar{u}(r) \leq \frac{(q-1)(2-q)}{(q-1)M} \left( \frac{(q-1)(N-1)-1}{(q-1)M} \right)^{\frac{1}{q-1}} r^{\frac{q-2}{q-1}} \quad \text{for all } r > \rho, \quad (2.4)$$

Furthermore, if  $R = 0$ , inequalities (2.1), (2.2) and (2.3) hold with  $\rho = 0$ .

*Proof.* Assertion 1-(i) is wellknown and valid for much more general equations. In this statement we denote by  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  the spherical coordinates in  $\mathbb{R}^N$ , by  $\omega_N$  the volume of the unit  $N$ -ball and thus  $N\omega_N$  is the  $(N-1)$ -volume of the unit sphere  $S^{N-1}$ . Writing (1.1) in spherical coordinates and using Jensen formula, we get

$$-r^{1-N} (r^{N-1}\bar{u}_r)_r \geq \bar{u}^p + M |\bar{u}_r|^q. \quad (2.5)$$

It implies that  $r \mapsto w(r) := -r^{N-1}\bar{u}_r$  is increasing on  $(R, \infty)$ , thus it admits a limit  $\ell \in (-\infty, \infty]$ . If  $\ell \leq 0$ , then  $\bar{u}_r(r) > 0$  on  $(R, \infty)$ . Hence  $\bar{u}(r) \geq \bar{u}(\rho) := c > 0$  for  $r \geq \rho > R$ . then

$$(r^{N-1}\bar{u}_r)_r \leq -c^p r^{N-1} \implies \bar{u}_r(r) \leq \frac{\rho^{N-1}}{r^{N-1}} \bar{u}_r(\rho) - \frac{c^p}{N} \left( r - \frac{\rho^N}{r^{N-1}} \right),$$

which implies  $\bar{u}_r(r) \rightarrow -\infty$ , thus  $\bar{u}(r) \rightarrow -\infty$  as  $r \rightarrow \infty$ , contradiction. Therefore  $\ell \in (0, \infty]$  and either  $\bar{u}_r(r) < 0$  on  $(R, \infty)$  or there exists  $r_\ell > R$  such that  $\bar{u}_r(r_\ell) = 0$ ,  $\bar{u}$  is increasing on  $(R, r_\ell)$  and decreasing on  $(r_\ell, \infty)$ . If  $\bar{u}_r(r) < 0$  on  $(R, \infty)$ , then we have for  $r > 2R$

$$-r^{N-1}\bar{u}_r(r) \geq \int_{\frac{r}{2}}^r t^{N-1}\bar{u}^p(t) dt \geq \frac{r^N \bar{u}^p(r)}{2N} \implies (\bar{u}^{1-p})_r \geq \frac{(p-1)r}{2N} \implies \bar{u}(r) \leq \left( \frac{2N}{(p-1)r^2} \right)^{\frac{1}{p-1}},$$

which yields (2.1). If we are in the second case with  $r_\ell > R$ , we apply the same inequality with  $r > 2r_\ell$  and again (2.1) for  $r > 2r_\ell$ . Since  $\bar{u}$  is superharmonic, the function  $v(s) = \bar{u}(r)$  with  $s = r^{2-N}$  is concave on  $(0, R^{2-N})$  and it tends to 0 when  $s \rightarrow 0$ . Thus

$$v_s(s) \leq \frac{v}{s} \implies |\bar{u}_r(r)| \leq (N-2) \frac{\bar{u}(r)}{r} \leq (N-2)c_0 r^{-\frac{p+1}{p-1}}.$$

This implies (2.1) and (2.2). Note that the case  $r_\ell > R$  cannot happen if  $R = 0$ , so in any case, if  $R = 0$  then  $\rho = 0$ .

If  $M > 0$ , we have with  $w(r) = -r^{N-1}\bar{u}_r$

$$w_r \geq M r^{(1-q)(N-1)} |w|^q.$$

We have seen that  $w(r) > 0$  at infinity with limit  $\ell \in (0, \infty]$ , hence, on the maximal interval containing  $\infty$  where  $w > 0$ , we have  $(w^{1-q})_r \leq (1-q)M r^{(N-1)(1-q)}$ . We have for  $r > s > R$

$$w^{1-q}(r) - w^{1-q}(s) \leq M \ln \left( \frac{r}{s} \right),$$

if  $q = \frac{N}{N-1}$  and

$$w^{1-q}(r) - w^{1-q}(s) \leq \frac{M(q-1)}{(q-1)(N-1)-1} \left( r^{1-(q-1)(N-1)} - s^{1-(q-1)(N-1)} \right)$$

if  $q < \frac{N}{N-1}$ , and both expressions which tend to  $-\infty$  when  $r \rightarrow \infty$ , a contradiction. This proves 1-(ii). If  $q > \frac{N}{N-1}$ , the above expression yields, when  $r \rightarrow \infty$ ,

$$\ell^{1-q} - w^{1-q}(s) \leq -\frac{(q-1)M}{(q-1)(N-1)-1} s^{1-(q-1)(N-1)}.$$

This implies

$$w(s) \leq \left( \frac{(q-1)(N-1)-1}{(q-1)M} \right)^{\frac{1}{q-1}} s^{N-1-\frac{1}{q-1}},$$

and (2.3).  $\square$

*Remark.* The previous is a particular case of a much more general one dealing with quasilinear operators proved in [7, Th. 3.1].

## 2.1 Proof of Theorems A, A' and C

The function  $u$  is at least  $C^{3+\alpha}$  for some  $\alpha \in (0, 1)$  since  $p, q > 1$ . Hence  $z = |\nabla u|^2$  is  $C^{2+\alpha}$ . Since there holds by Bochner's identity and Schwarz's inequality

$$-\frac{1}{2}\Delta z + \frac{1}{N}(\Delta u)^2 + \langle \nabla \Delta u, \nabla u \rangle \leq 0, \quad (2.6)$$

we obtain from (1.1),

$$-\frac{1}{2}\Delta z + \frac{|u|^{2p}}{N} + \frac{2M}{N}|u|^{p-1}uz^{\frac{q}{2}} + \frac{M^2}{N}z^q - p|u|^{p-1}z - \frac{Mq}{2}z^{\frac{q}{2}-1}\langle \nabla z, \nabla u \rangle \leq 0.$$

Since for  $\delta > 0$ ,

$$z^{\frac{q}{2}-1}|\langle \nabla z, \nabla u \rangle| \leq \left| z^{-\frac{1}{2}}\nabla z \right| z^{\frac{q-1}{2}}|\nabla u| = \left| z^{-\frac{1}{2}}\nabla z \right| z^{\frac{q}{2}} \leq \delta z^q + \frac{1}{4\delta} \frac{|\nabla z|^2}{z},$$

we obtain for any  $\nu \in (0, 1)$ , provided  $\delta$  is small enough,

$$-\frac{1}{2}\Delta z + \frac{|u|^{2p}}{N} + \frac{2M}{N}|u|^{p-1}uz^{\frac{q}{2}} + \frac{M^2(1-\nu)^2}{N}z^q - p|u|^{p-1}z \leq c_1 \frac{|\nabla z|^2}{z}, \quad (2.7)$$

where  $c_1 = c_1(M, N, \nu) > 0$ .

### 2.1.1 Proof of Theorem A

Suppose  $\frac{2p}{p+1} < q$ . We set  $r = \frac{2p}{p-1}$ ,  $r' = \frac{r}{r-1}$ , then, for any  $\epsilon > 0$

$$p|u|^{p-1}z \leq \frac{\epsilon^r |u|^{(p-1)r}}{r} + \frac{z^{r'}}{\epsilon^{r'r'}} = (p-1) \frac{\epsilon^r |u|^{2p}}{2} + (p+1) \frac{z^{\frac{2p}{p+1}}}{2\epsilon^{r'}}.$$

We fix  $\eta \in (0, 1)$  and  $\epsilon$  so that  $\epsilon^r = \frac{2(1-\eta)}{N(p-1)}$  and get

$$p|u|^{p-1}z \leq (1-\eta) \frac{|u|^{2p}}{N} + c_2 z^{\frac{2p}{p+1}},$$

where  $c_2 = \frac{p+1}{2} \left( \frac{N(p-1)}{2(1-\eta)} \right)^{\frac{p+1}{p-1}}$ . We perform the change of scale (1.5) in order to reduce (1.1) to the case  $M = 1$  by setting  $u(x) = \alpha^{\frac{2}{p-1}} v(\alpha x)$  with  $\alpha = M^{-\frac{p-1}{(p+1)q-2p}}$ . Then the equation for  $z = |\nabla v|^2$  is considered in  $\Omega_\alpha = \alpha\Omega$ . Choosing now  $\eta = \frac{1}{2}$  we obtain

$$c_2 z^{\frac{2p}{p+1}} \leq \frac{1}{4N} z^q + c_3,$$

where  $c_3 = c_3(N, p, q) > 0$ , hence

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{2N} + \frac{1}{4N}z^q \leq c_3 + c_1 \frac{|\nabla z|^2}{z}.$$

Put  $\tilde{z} = \left(z - (4Nc_3)^{\frac{1}{q}}\right)_+$ , then

$$-\frac{1}{2}\Delta \tilde{z} + \frac{1}{4N}\tilde{z}^q \leq c_1 \frac{|\nabla \tilde{z}|^2}{\tilde{z}},$$

hence, from [5, Lemma 2.2], we derive

$$\tilde{z}(y) \leq c_4 (\text{dist}(y, \partial\Omega_\alpha))^{\frac{2}{q-1}}$$

where  $c_4 = c_4(N, q, c_1) > 0$  which implies

$$|\nabla v(y)| \leq c'_4 \left(1 + (\text{dist}(y, \partial\Omega_\alpha))^{-\frac{1}{q-1}}\right) \quad \forall y \in \Omega_\alpha. \quad (2.8)$$

Then (1.18) and (1.19) follow.

Assume now that there exists a ground state  $u$ . Fix  $y \in \mathbb{R}^N$  and consider  $\{y_n\} \subset \mathbb{R}^N$  such that  $|y_n| = 2n > |y|$ . We apply (2.8) with  $\Omega_\alpha = B_n(y_n)$ . Then

$$|\nabla v(y)| \leq c'_4 \left(1 + |2n - y|^{-\frac{1}{q-1}}\right),$$

and letting  $n \rightarrow \infty$  we infer

$$|\nabla v(y)| \leq c'_4 \quad \forall y \in \mathbb{R}^N, \quad (2.9)$$

### 2.1.2 Proof of Theorem A'

Suppose  $1 < q < \frac{2p}{p+1}$ . By scaling we reduce to the case  $M = 1$  and we replace  $u$  by  $v$  defined by (1.5) as in the proof of Theorem A with  $\alpha = M^{\frac{p-1}{2p-(p+1)q}}$ . From (2.7) with  $\nu = \frac{1}{4}$  the function  $z = |\nabla v|^2$  satisfies

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{N} + \frac{1}{2N}z^q - v^{p-1}z \leq c_1 \frac{|\nabla z|^2}{z}, \quad (2.10)$$

By Hölder's inequality,

$$pv^{p-1}z \leq \frac{1}{4N}z^q + p(4Np)^{q'-1}v^{(p-1)q'}.$$

Since  $(p-1)q' = 2p + \frac{2p-(p+1)q}{q-1}$  we derive

$$-\frac{1}{2}\Delta z + \frac{v^{2p}}{N} \left(1 - 4^{q'-1}p^{q'}N^{q'}v^{\frac{2p-(p+1)q}{q-1}}\right) + \frac{1}{4N}z^q \leq c_1 \frac{|\nabla z|^2}{z}.$$

If  $\max v \leq c_{N,p,q} := (4^{q'-1} p^{q'} N^{q'})^{-\frac{q-1}{2p-(p+1)q}}$ , we obtain

$$-\frac{1}{2}\Delta z + \frac{1}{4N}z^q \leq c_1 \frac{|\nabla z|^2}{z},$$

which implies classically that  $z = 0$ , hence  $v$  is constant and thus  $v = 0$  from the equation.  $\square$

*Remark.* If  $u$  is a positive ground state of (1.1) radial with respect to 0, it satisfies  $u_r(0) = 0$  and it is a decreasing function of  $r$ . The previous theorem asserts that it must satisfy

$$u(0) > c_{N,p,q} M^{\frac{2}{2p-(p+1)q}}. \quad (2.11)$$

### 2.1.3 Proof of Theorem C

Suppose  $\frac{2p}{p+1} = q$ . For  $A > 0$  we consider the expression

$$\begin{aligned} & (u^p + A|\nabla u|^q)^2 - Npu^{p-1}|\nabla u|^2 \\ &= \left(u^p + A|\nabla u|^q - \sqrt{Np}u^{\frac{p-1}{2}}|\nabla u|\right) \left(u^p + A|\nabla u|^q + \sqrt{Np}u^{\frac{p-1}{2}}|\nabla u|\right) \end{aligned}$$

Now the function  $Z \mapsto \Phi_A(Z) = u^p + AZ^q - \sqrt{Np}u^{\frac{p-1}{2}}Z$  achieves its minimum at  $Z_0 = \left(\frac{\sqrt{Np}}{qA}\right)^{\frac{p+1}{p-1}}u^{\frac{p+1}{2}}$  and

$$\Phi_A(Z_0) = \left[1 - \frac{p-1}{p+1} \left(\frac{N(p+1)^2}{4p}\right)^{\frac{p}{p-1}} A^{-\frac{p+1}{p-1}}\right] u^p;$$

thus setting

$$M_{\dagger} = \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p+1}} \left(\frac{N(p+1)^2}{4p}\right)^{\frac{p}{p+1}}, \quad (2.12)$$

we obtain that if  $A \geq M_{\dagger}$ , then  $\Phi_A(Z) \geq 0$  for all  $Z$ . Put  $M_{\nu} = (1-\nu)M$  for  $\nu \in (0,1)$  such that  $M_{\dagger} < M_{\nu}$ , we derive from (2.7)

$$-\frac{1}{2}\Delta z + \frac{(u^p + M_{\dagger}z^{\frac{q}{2}})^2}{N} - pu^{p-1}z + \frac{M_{\nu}^2 - M_{\dagger}^2}{N}z^q \leq c_1 \frac{|\nabla z|^2}{z}, \quad (2.13)$$

which yields

$$-\frac{1}{2}\Delta z + \frac{M_{\nu}^2 - M_{\dagger}^2}{N}z^q \leq c_1 \frac{|\nabla z|^2}{z}.$$

Using again [5, Lemma 2.2] we obtain

$$|\nabla u(x)| \leq c'_1 ((1-\nu)M - M_{\dagger})^{-\frac{1}{q-1}} (\text{dist}(x, \partial\Omega))^{-\frac{1}{q-1}}, \quad (2.14)$$

which is equivalent to (1.24).  $\square$

## 2.2 Proof of Theorems B and B'

### 2.2.1 Proof of Theorem B

Since the result is known when  $M \geq 0$  from Proposition 2.1, we can assume that  $M = -m < 0$  and  $N = 1, 2$  or  $N \geq 3$  with  $p < \frac{N}{N-2}$ ,  $u$  is a nonnegative supersolution of (1.1) in  $\overline{B}_R^c$  and we set  $u = v^b$  with  $b > 1$ . Then

$$-\Delta v \geq (b-1) \frac{|\nabla v|^2}{v} + \frac{1}{b} v^{1+b(p-1)} - mb^{q-1} v^{(b-1)(q-1)} |\nabla v|^q. \quad (2.15)$$

Here again  $q = \frac{2p}{p+1}$ , setting  $z = |\nabla v|^2$  we obtain

$$-\Delta v \geq \frac{\Phi(z)}{bv}$$

where

$$\Phi(z) = b(b-1)z - mb^{\frac{2p}{p+1}} v^{\frac{2+b(p-1)}{p+1}} z^{\frac{p}{p+1}} + v^{2+b(p-1)}.$$

Thus  $\Phi$  achieves its minimum for

$$z_0 = \left( \frac{mpb^{q-1}}{(b-1)(p+1)} \right)^{p+1} b^{p-1} v^{2+b(p-1)}$$

and

$$\Phi(z_0) = v^{2+b(p-1)} \left( 1 - \frac{p^p}{(p+1)^{p+1}} \left( \frac{b}{b-1} \right)^p m^{p+1} \right). \quad (2.16)$$

In order to ensure the optimal choice, when  $N \geq 3$  we take  $1 + b(p-1) = \frac{N}{N-2}$ , hence  $b = \frac{2}{(N-2)(p-1)}$  which is larger than 1 because  $p < \frac{N}{N-2}$ . Finally

$$\Phi(z_0) = v^{\frac{N}{N-2}+1} \left( 1 - \frac{1}{(p+1)^{p+1}} \left( \frac{2p}{N-p(N-2)} \right)^p m^{p+1} \right)$$

Hence, if

$$m < (p+1) \left( \frac{N-p(N-2)}{2p} \right)^{\frac{p}{p+1}} = \mu^*(N), \quad (2.17)$$

we have for some  $\delta > 0$ ,

$$-\Delta v \geq \delta v^{\frac{N}{N-2}}, \quad (2.18)$$

and by Proposition 2.1 that is no positive solution in an exterior domain of  $\mathbb{R}^N$ .

If  $N = 2$  for a given  $b > 1$  we have from (2.16) that if

$$m < (p+1) \left( \frac{b-1}{bp} \right)^{\frac{p}{p+1}},$$

then, for some  $\delta > 0$ ,

$$-\Delta v \geq \delta v^{1+b(p-1)}. \quad (2.19)$$

The result follows from Proposition 2.1 by choosing  $b$  large enough.  $\square$

### 2.2.2 Proof of Theorem B'

1- We assume that such a supersolution  $u$  exists and we denote  $u = e^v$ , then

$$-\Delta v \geq F(|\nabla v|^2), \quad (2.20)$$

where

$$F(X) = X + e^{(p-1)v} + M e^{\frac{p-1}{p+1}v} X^{\frac{p}{p+1}}.$$

Clearly, if  $M \geq 0$ , then  $F(X) \geq 0$  for any  $X \geq 0$ . Next we assume  $M < 0$ , then

$$F(X) \geq F(X_0) = e^{(p-1)v} \left( 1 - p^p \left( \frac{|M|}{p+1} \right)^{p+1} \right) = e^{(p-1)v} \left( 1 - \left( \frac{|M|}{\mu^*(2)} \right)^{p+1} \right).$$

Hence, if  $|M| \leq \mu^*(2)$ ,  $v$  is a positive superharmonic function in  $\Omega$  which tends to infinity on the boundary. Such a function is larger than the harmonic function with boundary value  $k > 0$  for any  $k$  (and taking the value  $\min_{|x|=R} v(x)$  for  $R$  large enough if  $\Omega$  is an exterior domain). Letting  $k \rightarrow \infty$  we derive a contradiction.

2- Let  $R > 0$  such that  $\Omega^c \subset B_R$  and let  $w$  be the solution of

$$\begin{aligned} -\Delta w - a e^{(p-1)w} &= 0 && \text{in } B_R \cap \Omega \\ \lim_{\text{dist}(x, \partial B_R) \rightarrow 0} w(x) &= -\infty \\ \lim_{\text{dist}(x, \partial \Omega) \rightarrow 0} w(x) &= \infty. \end{aligned} \quad (2.21)$$

with  $a = 1 - \left( \frac{|M|}{\mu^*(2)} \right)^{p+1} < 0$ , obtained by approximations. By the argument used in 1,

$$a e^{(p-1)w} \leq |\nabla w|^2 + e^{(p-1)w} - |M| e^{\frac{p-1}{p+1}w} |\nabla w|^{\frac{2p}{p+1}},$$

hence

$$-\Delta w \leq |\nabla w|^2 + e^{(p-1)w} - |M| e^{\frac{p-1}{p+1}w} |\nabla w|^{\frac{2p}{p+1}}.$$

Therefore  $v = e^w$  is nonnegative and satisfies

$$\begin{aligned} -\Delta v - v^p + |M| |\nabla v|^{\frac{2p}{p+1}} &\leq 0 && \text{in } B_R \cap \Omega \\ v &= 0 && \text{on } \partial B_R \\ \lim_{\text{dist}(x, \partial \Omega) \rightarrow 0} v(x) &= \infty. \end{aligned} \quad (2.22)$$

Next we extend  $v$  by zero in  $B_R^c$  and denote by  $\tilde{v}$  the new function. It is a nonnegative subsolution of (1.1) which tends to  $\infty$  on  $\partial \Omega$ . For constructing a supersolution we recall that if  $M \leq -\mu^*(1)$  there exist two types of explicit solutions of

$$-u'' = u^p + M |u'|^{\frac{2p}{p+1}} \quad (2.23)$$

defined on  $\mathbb{R}$  by  $U_{j,M}(t) = \infty$  for  $t \leq 0$  and  $U_{j,M}(t) = X_{j,M} t^{-\frac{2}{p-1}}$ ,  $j=1,2$ , for  $t > 0$  where  $X_{1,M}$  and  $X_{2,M}$  are respectively the smaller and the larger positive root of

$$X^{p-1} - |M| \left( \frac{2}{p-1} \right)^{\frac{2}{p+1}} X^{\frac{p-1}{p+1}} + \frac{2(p+1)}{(p-1)^2} = 0. \quad (2.24)$$

Since  $\Omega^c$  is convex it is the intersection of all the closed half-spaces which contain it and we denote by  $\mathcal{H}_\Omega$  the family of such hyperplanes which are touching  $\partial\Omega$ . If  $H \in \mathcal{H}_\Omega$  let  $\mathbf{n}_H$  be the normal direction to  $H$ , inward with respect to  $\Omega$ ,  $\mathcal{H}_+ = \{x \in \mathbb{R}^N : \langle \mathbf{n}_H, x - \mathbf{n}_H \rangle > 0\}$  and we define  $U_H$  in the direction  $\mathbf{n}_H$  by putting

$$U_H(x) = U_{2,M}(\langle \mathbf{n}_H, x - \mathbf{n}_H \rangle) = X_{2,M}(\langle \mathbf{n}_H, x - \mathbf{n}_H \rangle)^{-\frac{2}{p-1}} \quad \text{for all } x \in \mathcal{H}_+.$$

Hence and set, for  $x \in \Omega := \cap_{H \in \mathcal{H}_\Omega} \mathcal{H}_+$ ,

$$u_\Omega(x) = \inf_{H \in \mathcal{H}_\Omega} U_H(x). \quad (2.25)$$

Then  $u_\Omega$  is a nonnegative supersolution of (1.1) in  $\Omega$  and

$$u_\Omega(x) \geq X_{2,M}(\text{dist } x, \Omega)^{-\frac{2}{p-1}} \quad \forall x \in \Omega.$$

Next  $v_\Omega = \ln u_\Omega$  blows up on  $\partial\Omega$ , is finite on  $\partial B_R$  and satisfies

$$-\Delta v_\Omega - a e^{(p-1)v_\Omega} \geq 0 \quad \text{in } B_R \cap \Omega. \quad (2.26)$$

By comparison with  $w$  since  $a < 0$ ,  $v_\Omega \geq w$ . Hence  $u_\Omega \geq v$  in  $B_R \setminus \Omega^c$ . Extending  $v$  by zero as  $\tilde{v}$  we obtain  $u_\Omega \geq \tilde{v}$  in  $\Omega^c$ . Hence  $u_\Omega$  is a supersolution in  $\Omega^c$  where it dominates the subsolution  $\tilde{v}$ . It follows by [23, Th 1-4-6] that there exists a solution  $u$  of (1.1) satisfying  $\tilde{v} \leq u \leq u_\Omega$ , which ends the proof.  $\square$

### 3 The refined Bernstein method

The method is a combination of the one used in the previous proofs. It is based upon the replacement of the unknown by setting first  $u = v^{-\beta}$  as in [15] and [9] and the study of the equation satisfied by  $|\nabla v|$ . However we do not use integral techniques. Since  $u$  is a positive solution of (1.1) in  $B_R$ , the function  $v$  is well defined and satisfies

$$-\Delta v + (1 + \beta) \frac{|\nabla v|^2}{v} + \frac{1}{\beta} v^{1-\beta(p-1)} + M |\beta|^{q-2} \beta v^{(\beta+1)(1-q)} |\nabla v|^q, \quad (3.1)$$

in  $B_R$ . We set

$$z = |\nabla v|^2, \quad s = 1 - q - \beta(q-1) = (1-q)(\beta+1), \quad \sigma = 1 - \beta(p-1),$$

and derive

$$\Delta v = (1 + \beta) \frac{z}{v} + \frac{1}{\beta} v^\sigma + M |\beta|^{q-2} \beta v^s z^{\frac{q}{2}}. \quad (3.2)$$

Combining Bochner's formula and Schwarz identity we have classically

$$\frac{1}{2}\Delta z \geq \frac{1}{N}(\Delta v)^2 + \langle \nabla \Delta v, \nabla v \rangle.$$

We explicit the different terms

$$\begin{aligned} (\Delta v)^2 &= (1 + \beta)^2 \frac{z^2}{v^2} + M^2 \beta^{2(q-1)} v^{2s} z^q + \frac{v^{2\sigma}}{\beta^2} + 2M(1 + \beta) |\beta|^{q-2} \beta v^{s-1} z^{1+\frac{q}{2}} \\ &\quad + \frac{2(1 + \beta)}{\beta} v^{\sigma-1} z + 2M |\beta|^{q-2} v^{s+\sigma} z^{\frac{q}{2}}, \\ \nabla \Delta v &= (1 + \beta) \frac{\nabla z}{v} - \frac{(1 + \beta)z}{v^2} \nabla v + \frac{\sigma}{\beta} v^{\sigma-1} \nabla v + M_s |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}} \nabla v \\ &\quad + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \nabla z, \\ \langle \nabla \Delta v, \nabla v \rangle &= \left( \frac{1 + \beta}{v} + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \right) \langle \nabla z, \nabla v \rangle - \frac{(1 + \beta)z^2}{v^2} + \frac{\sigma}{\beta} v^{\sigma-1} z \\ &\quad + M_s |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1}. \end{aligned}$$

Hence

$$\begin{aligned} -\frac{1}{2}\Delta z + \frac{1}{N}(\Delta v)^2 + \left( \frac{1 + \beta}{v} + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \right) \langle \nabla z, \nabla v \rangle \\ - \frac{(1 + \beta)z^2}{v^2} + \frac{\sigma}{\beta} v^{\sigma-1} z + M_s |\beta|^{q-2} \beta v^{s-1} z^{\frac{q}{2}+1} \leq 0. \end{aligned} \quad (3.3)$$

The following extension of the Keller-Osserman inequality is proved in [5, Lemma 2.2].

**Lemma 3.1** *Let  $S > 1$  and  $R > 0$ . Assume  $v$  is continuous in  $\overline{B}_R$  and  $C^1$  on the set  $\mathcal{U}_+ := \{x \in B_R : v(x) > 0\}$ . If  $v$  satisfies for some real number  $a$*

$$-\Delta v + v^S \leq a \frac{|\nabla v|^2}{v}, \quad (3.4)$$

on each connected component of  $\mathcal{U}_+$ , there holds

$$v(0) \leq C_{N,S,a} R^{-\frac{2}{S-1}}. \quad (3.5)$$

for some positive constant  $C_{N,S,a}$ .

### 3.1 Proof of Theorem D

We develop the term  $(\Delta v)^2$  in (3.2) and get

$$\begin{aligned} -\frac{1}{2}\Delta z + \left( \frac{(1 + \beta)^2}{N} - (1 + \beta) \right) \frac{z^2}{v^2} + \frac{M^2 \beta^{2(q-1)}}{N} v^{2s} z^q + M \left( s + \frac{2(1 + \beta)}{N} \right) |\beta|^{q-2} \beta v^{s-1} z^{1+\frac{q}{2}} \\ + \frac{v^{2\sigma}}{N\beta^2} + \left( \frac{1 + \beta}{v} + \frac{Mq}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{2}-1} \right) \langle \nabla z, \nabla v \rangle + \frac{N\sigma + 2(1 + \beta)}{N\beta} v^{\sigma-1} z + \frac{2M |\beta|^{q-2}}{N} v^{s+\sigma} z^{\frac{q}{2}} \\ \leq 0. \end{aligned} \quad (3.6)$$

Next we set  $z = v^{-k}Y$  where  $k$  is a real parameter. Then  $\nabla z = -kv^{-k-1}Y\nabla v + v^{-k}\nabla Y$ ,

$$\langle \nabla z, \nabla v \rangle = -kv^{-k-1}Yz + v^{-k}\langle \nabla Y, \nabla v \rangle = -kv^{-2k-1}Y^2 + v^{-k}\langle \nabla Y, \nabla v \rangle,$$

$$\frac{\langle \nabla z, \nabla v \rangle}{v} = -kv^{-2k-2}Y^2 + v^{-k-1}\langle \nabla Y, \nabla v \rangle,$$

$$\begin{aligned} Mv^s z^{\frac{q}{2}-1} \langle \nabla z, \nabla v \rangle &= -kMv^{s-\frac{qk}{2}-k-1}Y^{\frac{q}{2}+1} + Mv^{s-\frac{qk}{2}}Y^{\frac{q}{2}-1} \langle \nabla Y, \nabla v \rangle, \\ -\Delta z &= \operatorname{div} (kv^{-k-1}Y\nabla v - v^{-k}\nabla Y) \\ &= kv^{-k-1}Y\Delta v - k(k+1)v^{-k-2}Yz + 2kv^{-k-1}\langle \nabla Y, \nabla v \rangle - v^{-k}\Delta Y \\ &= kv^{-k-1}Y\Delta v - k(k+1)v^{-2k-2}Y^2 + 2kv^{-k-1}\langle \nabla Y, \nabla v \rangle - v^{-k}\Delta Y. \end{aligned}$$

From (3.2)

$$\Delta v = (1 + \beta)v^{-k-1}Y + \frac{1}{\beta}v^\sigma + M|\beta|^{q-2}\beta v^{s-k\frac{q}{2}}Y^{\frac{q}{2}},$$

therefore

$$\begin{aligned} -\Delta z &= k(\beta - k)v^{-2k-2}Y^2 + \frac{k}{\beta}v^{\sigma-k-1}Y + kM|\beta|^{q-2}\beta v^{s-k\frac{q}{2}-k-1}Y^{\frac{q}{2}+1} \\ &\quad + 2kv^{-k-1}\langle \nabla Y, \nabla v \rangle - v^{-k}\Delta Y. \end{aligned}$$

Replacing  $\langle \nabla z, \nabla v \rangle$  and  $\Delta z$  given by the above expressions in (3.6) and  $z$  by  $v^{-k}Y$ , leads to

$$\begin{aligned} -\Delta Y &+ \left( \frac{k(\beta - k)}{2} + \frac{(1 + \beta)^2}{N} - (k + 1)(\beta + 1) \right) v^{-k-2}Y^2 + \frac{v^{2\sigma+k}}{N\beta^2} + \frac{M^2\beta^{2(q-1)}}{N}v^{2s+k-kq}Y^q \\ &+ \left( \frac{k + \beta + 1}{v} + \frac{Mq|\beta|^{q-2}\beta}{2}v^{s+k-k\frac{q}{2}}Y^{\frac{q}{2}-1} \right) \langle \nabla Y, \nabla v \rangle + \frac{2M|\beta|^{q-2}}{N}v^{s+\sigma+k-k\frac{q}{2}}Y^{\frac{q}{2}} \\ &+ \left( s + \frac{2(1 + \beta)}{N} - \frac{k(q-1)}{2} \right) M|\beta|^{q-2}\beta v^{s-k\frac{q}{2}-1}Y^{1+\frac{q}{2}} + \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1 + \beta)}{N} \right) v^{\sigma-1}Y \leq 0. \end{aligned}$$

For  $\epsilon_1, \epsilon_2 > 0$ ,

$$\begin{aligned} \frac{1}{v} |\langle \nabla Y, \nabla v \rangle| &\leq \epsilon_1 v^{-k-2}Y^2 + \frac{1}{4\epsilon_1} \frac{|\nabla Y|^2}{Y}, \\ v^{s+k-k\frac{q}{2}}Y^{\frac{q}{2}-1} |\langle \nabla Y, \nabla v \rangle| &\leq \epsilon_2 v^{2s-kq+k}Y^q + \frac{1}{4\epsilon_2} \frac{|\nabla Y|^2}{Y}. \end{aligned}$$

Hence

$$\begin{aligned} -\Delta Y &+ \frac{v^{2\sigma+k}}{N\beta^2} + \frac{2M|\beta|^{q-2}}{N}v^{s+\sigma+k-k\frac{q}{2}}Y^{\frac{q}{2}} + \left( \frac{M^2\beta^{2(q-1)}}{N} - \frac{Mq\epsilon_2|\beta|^{q-1}}{2} \right) v^{2s+k-kq}Y^q \\ &+ \left( \frac{k(\beta - k)}{2} + \frac{(1 + \beta)^2}{N} - (k + 1)(\beta + 1) - |k + \beta + 1|\epsilon_1 \right) v^{-k-2}Y^2 \\ &+ \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1 + \beta)}{N} \right) v^{\sigma-1}Y + \left( s + \frac{2(1 + \beta)}{N} - \frac{k(q-1)}{2} \right) M|\beta|^{q-2}\beta v^{s-k\frac{q}{2}-1}Y^{1+\frac{q}{2}} \\ &\leq \left( \frac{|k + \beta + 1|}{\epsilon_1} + \frac{Mq|\beta|^{q-1}}{2\epsilon_2} \right) \frac{|\nabla Y|^2}{4Y}, \end{aligned} \tag{3.7}$$

We first choose  $\epsilon_2 = \frac{M|\beta|^{q-1}}{qN}$ , then

$$\begin{aligned}
 -\Delta Y + \frac{v^{2\sigma+k}}{N\beta^2} + \left( \frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1|\epsilon_1 \right) v^{-k-2} Y^2 \\
 + \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y + \frac{M^2 \beta^{2(q-1)}}{2N} v^{2s+k-kq} Y^q + \frac{2M|\beta|^{q-2}}{N} v^{s+\sigma+k-k\frac{q}{2}} Y^{\frac{q}{2}} \\
 + \left( s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right) M |\beta|^{q-2} \beta v^{s-k\frac{q}{2}-1} Y^{1+\frac{q}{2}} \\
 \leq \left( \frac{|k+\beta+1|}{\epsilon_1} + \frac{Nq^2}{2} \right) \frac{|\nabla Y|^2}{4Y},
 \end{aligned} \tag{3.8}$$

In order to show the sign of the terms on the left in (3.7), we separate the terms containing the coefficient  $M$  from the ones which do not contain it. Indeed these last terms are associated to the mere Lane-Emden equation (1.2) which is treated, as a particular case, in [5, Theorem B] where the exponents therein are  $q = 0$ , and  $p \in \left(1, \frac{N+3}{N-1}\right)$ . We set

$$\begin{aligned}
 H_{\epsilon_1,1} &= \frac{v^{2\sigma+k}}{N\beta^2} + \left( \frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1|\epsilon_1 \right) v^{-k-2} Y^2 \\
 &\quad + \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y \\
 &= v^{2\sigma+k} \tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma} Y),
 \end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
 \tilde{H}_{\epsilon_1,1}(t) &= \left( \frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1|\epsilon_1 \right) t^2 \\
 &\quad + \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) t + \frac{1}{N\beta^2},
 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
 H_{M,2} &= \frac{M^2 \beta^{2(q-1)}}{2N} v^{2s+k-kq} Y^q + \frac{2M|\beta|^{q-2}}{N} v^{s+\sigma+k-k\frac{q}{2}} Y^{\frac{q}{2}} \\
 &\quad + \left( s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right) M |\beta|^{q-2} \beta v^{s-k\frac{q}{2}-1} Y^{1+\frac{q}{2}}.
 \end{aligned} \tag{3.11}$$

Then

$$-\Delta Y + v^{2\sigma+k} \tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma} Y) + H_{M,2} \leq \left( \frac{|k+\beta+1|}{\epsilon_1} + \frac{Nq^2}{2} \right) \frac{|\nabla Y|^2}{4Y}.$$

The sign of  $\tilde{H}_{\epsilon_1,1}$  depends on its discriminant  $\mathcal{D}_{\epsilon_1}$  which is a polynomial in its coefficients. Then if for  $\epsilon_1 = 0$  this discriminant is negative  $\mathcal{D}_0$  is negative, the discriminant  $\mathcal{D}_{\epsilon_1}$  of  $\tilde{H}_{\epsilon_1,1}$  shares this property for  $\epsilon_1 > 0$  small enough and therefore  $H_{\epsilon_1,1}$  is positive. The proof is similar as the one of [5, Th B] in case (i) but for the sake of completeness we recall the main steps. Firstly

$$\mathcal{D}'_0 := N^2 \beta^2 \mathcal{D}_0 = \left( \frac{Nk}{2} + \sigma N + 2(1+\beta) \right)^2 - 4 \left( \frac{Nk(\beta-k)}{2} + (1+\beta)^2 - N(k+1)(\beta+1) \right).$$

Then

$$\mathcal{D}'_0 = \left( \frac{N(p-1)}{4} - 1 \right) (2\sigma + k)^2 + 2(p-1)(2\sigma + k) + \tilde{L}$$

where  $\tilde{L} = (p-1)k^2 + p(\lambda+2)^2 > 0$ . Put

$$S = \frac{2\sigma + k}{k+2} = 1 - \frac{2\beta(p-1)}{k+2} \quad \text{and} \quad \mathcal{T}(S) = \left( \frac{(N-1)(p-1)}{4} - 1 \right) S^2 + (p-1)S + p.$$

After some computations we get, if  $k \neq -2$ ,

$$\mathcal{D}'_1 := \frac{(p-1)\mathcal{D}'_0}{(k+2)^2} = (p-1) \left( \frac{k}{k+2} - \frac{S}{2} \right)^2 + \mathcal{T}(S). \quad (3.12)$$

We choose  $S > 2$  such that  $\frac{k}{k+2} - \frac{S}{2} = 0$ , hence  $\beta = \frac{2-k}{p-1}$ . If  $p < \frac{N+3}{N-1}$  the coefficient of  $S^2$  in  $\mathcal{T}(S)$  is negative. Hence  $\mathcal{T}(S) < 0$  provided  $S$  is large enough which is satisfied if  $k < -2$  with  $|k+2|$  small enough. We infer from this that  $\beta > 0$ ,  $\mathcal{D}_0 < 0$  and  $\tilde{H}_{\epsilon_1,1} > 0$  if  $\epsilon_1$  is small enough. In particular  $\tilde{H}_{\epsilon_1,1}(t) \geq c_6(t^2 + 1)$  for some  $c_6 = c_6(N, p, q) > 0$ , which means

$$v^{2\sigma+k} \tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma}Y) \geq c_6 \left( v^{-k-2}Y^2 + v^{2\sigma+k} \right). \quad (3.13)$$

Secondly the positivity of  $H_{M,2}$  is ensured, as  $\beta$  and  $M$  are positive, by the positivity of

$$\mathcal{A} := s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2}.$$

Replacing  $s$  by its value, we obtain, since  $1 < q < \frac{N+2}{N}$  and  $\beta + \frac{2+k}{2} > 0$ , which can be assume by taking  $|k+2|$  small enough,

$$\mathcal{A} = 2\frac{1+\beta}{N} - (q-1) \left( \beta + 1 + \frac{k}{2} \right) > -\frac{k}{N}$$

Then we deduce that

$$-\Delta Y + c_6 \left( v^{-k-2}Y^2 + v^{2\sigma+k} \right) \leq c_7 \frac{|\nabla Y|^2}{Y}, \quad (3.14)$$

and  $c_7 = c_7(N, p, q) > 0$  is independent of  $M$ . Since  $S = 1 - \frac{2\beta(p-1)}{k+2} = 1 - 2\frac{2-k}{k+2} > 0$ , we have

$$2Y^{\frac{2S}{S+1}} = 2 \left( \frac{Y^2}{v^{k+2}} \right)^{\frac{S}{S+1}} v^{\frac{(k+2)S}{S+1}} \leq \frac{Y^2}{v^{k+2}} + v^{(k+2)S} = \frac{Y^2}{v^{k+2}} + v^{2\sigma+k}. \quad (3.15)$$

From this we infer the inequality

$$-\Delta Y + 2c_6 Y^{\frac{2S}{S+1}} \leq c_7 \frac{|\nabla Y|^2}{Y}. \quad (3.16)$$

Then we derive from Lemma 3.1 that in the ball  $B_R$  there holds

$$Y(0) \leq c_8 R^{-\frac{2(S+1)}{S-1}} = c_8 R^{-2 + \frac{2(k+2)}{\beta(p-1)}}. \quad (3.17)$$

From this it follows

$$\left| \nabla u^{-\frac{2+k}{2\beta}}(0) \right| \leq \frac{|k+2|}{2} \sqrt{c_8} R^{-1+\frac{k+2}{\beta(p-1)}}. \quad (3.18)$$

Setting  $a = -\frac{k+2}{2\beta} > 0$  we get that for any domain  $\Omega \subset \mathbb{R}^N$  any positive solution in  $\Omega$  satisfies

$$|\nabla u^a(x)| \leq \frac{|k+2|}{2} \sqrt{c_8} (\text{dist}(x, \partial\Omega))^{-1-\frac{2a}{p-1}} \quad \text{for all } x \in \Omega. \quad (3.19)$$

The non existence of any positive of (1.1) solution in  $\mathbb{R}^N$  follows classically.  $\square$

**Corollary 3.2** *Let  $\Omega$  be a smooth domain in  $\mathbb{R}^N$ ,  $N \geq 2$  with a bounded boundary,  $1 < p < \frac{N+3}{N-1}$ ,  $1 < q < \frac{N+2}{N}$  and  $M > 0$ . If  $u$  is a positive solution of (1.1) in  $\Omega$  there exists  $d_0$  depending on  $\Omega$  and  $c_9 = c_9(N, p, q) > 0$  such that*

$$u(x) \leq c_9 \left( (\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} + \max_{\text{dist}(z, \partial\Omega)=d_0} u(z) \right) \quad \text{for all } x \in \Omega. \quad (3.20)$$

*Proof.* It is similar as the one of [5, Cor. B-2].  $\square$

## 4 The integral method

### 4.1 Preliminary inequalities

We recall the next inequality [8, Lemma 3.1].

**Lemma 4.1** *Let  $\Omega \subset \mathbb{R}^N$  be a domain. Then for any positive  $u \in C^2(\Omega)$ , any nonnegative  $\eta \in C_0^\infty(\Omega)$  and any real numbers  $m$  and  $d$  such that  $d \neq m+2$ , the following inequality holds*

$$A \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx - \frac{N-1}{N} \int_{\Omega} \eta u^m (\Delta u)^2 dx - B \int_{\Omega} \eta u^{m-1} |\nabla u|^2 \Delta u dx \leq R, \quad (4.1)$$

where

$$A = \frac{1}{4N} (2(N-m)d - (N-1)(m^2 + d^2)), \quad B = \frac{1}{2N} (2(N-1)m + (N+2)d),$$

and

$$R = \frac{m+d}{2} \int_{\Omega} u^{m-1} |\nabla u|^2 \langle \nabla u, \nabla \eta \rangle dx + \int_{\Omega} u^m \Delta u \langle \nabla u, \nabla \eta \rangle dx + \frac{1}{2} \int_{\Omega} u^m |\nabla u|^2 \Delta \eta dx.$$

It is noticeable that  $d$  is a free parameter which plays a role only in the the coefficients of the integral terms. The following technical result is useful to deal with the multi-parameter constraints problems which occur in our construction. It was first used in [9] under a simpler form and extended in [8, Lemma 3.4].

**Lemma 4.2** For any  $N \in \mathbb{N}$ ,  $N \geq 3$  and  $1 < p < \frac{N+2}{N-2}$  there exist real numbers  $m$  and  $d$  verifying

$$\begin{aligned}
(i) \quad & d \neq m + 2, \\
(ii) \quad & \frac{2(N-1)p}{N+2} < d, \\
(iii) \quad & \max \left\{ -2, 1-p, \frac{(N-4)p-N}{2} \right\} < m \leq 0, \\
(iv) \quad & 2(N-m)d - (N-1)(m^2 + d^2) > 0.
\end{aligned} \tag{4.2}$$

## 4.2 Proof of Theorem E

*Proof of Theorem E.*

*Step 1: The integral estimates.* Let  $\eta \in C_0^\infty(\Omega)$ ,  $\eta \geq 0$ . We apply Lemma 4.1 to a positive solution  $u \in C^2(\Omega)$  of (1.1), firstly with  $q > 1$  and then with  $q = \frac{2p}{p+1}$ .

$$\begin{aligned}
A \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx - \frac{N-1}{N} \int_{\Omega} \eta \left( u^{m+2p} + 2Mu^{m+p} |\nabla u|^q + M^2 u^m |\nabla u|^{2q} \right) dx \\
- B \int_{\Omega} \eta u^{m-1} |\nabla u|^2 \Delta u dx \leq R.
\end{aligned} \tag{4.3}$$

We multiply (1.1) by  $\eta u^{m+p}$  and integrate over  $\Omega$ . Then

$$\begin{aligned}
\int_{\Omega} \eta \left( u^{m+2p} + Mu^{m+p} |\nabla u|^q \right) dx &= - \int_{\Omega} \eta u^{m+p} \Delta u dx \\
&= \int_{\Omega} u^{m+p} \langle \nabla u, \nabla \eta \rangle dx + (m+p) \int_{\Omega} \eta u^{m+p-1} |\nabla u|^2 dx.
\end{aligned}$$

We set

$$\begin{aligned}
F &= \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx, \quad P = \int_{\Omega} \eta u^{m-1} |\nabla u|^{q+2} dx, \quad V = \int_{\Omega} \eta u^{m+2p} dx, \\
T &= \int_{\Omega} \eta u^{m+p-1} |\nabla u|^2 dx, \quad W = \int_{\Omega} \eta u^{m+p} |\nabla u|^q dx, \quad U = \int_{\Omega} \eta u^m |\nabla u|^{2q} dx, \\
S &= \int_{\Omega} u^{m+p} \langle \nabla u, \nabla \eta \rangle dx,
\end{aligned}$$

so that there holds

$$AF - \frac{N-1}{N} (V + 2MW + M^2U) + BT + BMP \leq R, \tag{4.4}$$

and

$$V + MW = (m+p)T + S. \tag{4.5}$$

Eliminating  $V$  between (4.4) and (4.5), we get

$$AF + B_0T + M \left( BP - \frac{N-1}{N}W - \frac{N-1}{N}MU \right) \leq R - \frac{N-1}{N}S, \quad (4.6)$$

where

$$B_0 = B - \frac{N-1}{N}(m+p) = \frac{N+2}{2N}d - \frac{N-1}{N}p.$$

Also

$$2P = 2 \int_{\Omega} \eta u^m \frac{|\nabla u|^2}{u} |\nabla u|^q dx \leq \int_{\Omega} \eta u^m \left( \frac{|\nabla u|^4}{u^2} + |\nabla u|^{2q} \right) dx = F + U.$$

We fix now  $q = \frac{2p}{p+1}$ , then

$$\begin{aligned} U &= \int_{\Omega} \eta u^m |\nabla u|^{2q} dx = \int_{\Omega} \eta u^m \left( \frac{|\nabla u|}{\sqrt{u}} \right)^{4(q-1)} u^{2(q-1)} |\nabla u|^{4-2q} dx, \\ &\leq \frac{p-1}{p+1} \int_{\Omega} \eta u^{m-2} |\nabla u|^4 dx + \frac{2}{p+1} \int_{\Omega} \eta u^{m+p-1} |\nabla u|^2 dx, \\ &\leq \frac{p-1}{p+1} F + \frac{2}{p+1} T, \end{aligned} \quad (4.7)$$

hence

$$P \leq \frac{1}{2}F + \frac{1}{2}U \leq \frac{p}{p+1}F + \frac{1}{p+1}T \quad (4.8)$$

and

$$\begin{aligned} 2W &= 2 \int_{\Omega} \eta u^{m+p} |\nabla u|^q dx \leq \int_{\Omega} \eta u^{m+2p} dx + \int_{\Omega} \eta u^m |\nabla u|^{2q} dx = V + U \\ &\leq U + (m+p)T + S - MW. \end{aligned} \quad (4.9)$$

Next we assume that  $|M| \leq 1$ . From (4.7), (4.8), it follows that

$$W \leq U + (m+p)T + S \leq F + (m+p+1)T + 2S. \quad (4.10)$$

From now we fix  $m$  and  $d$  according Lemma 4.2. Therefore  $A > 0$  by (4.2)-(iv) and  $B > 0$  by combining (4.2)-(ii) and (4.2)-(iii). Furthermore  $B_0 > 0$  by (4.2)-(ii). Hence, from (4.7), (4.10) and (4.10) we derive, since  $\frac{N-1}{N} < 1$  and  $m \leq 0$  from (4.2)-(ii)

$$\begin{aligned} \left| BP - \frac{N-1}{N}W - \frac{N-1}{N}MU \right| &\leq B(F+T) + F + (p+1)T + S + F + T, \\ &\leq (B+2)F + (B+p+1)T + S. \end{aligned}$$

Plugging these estimates into (4.6) we infer

$$AF + B_0T - |M| \left( (B+2)F + (B+p+1)T + S \right) \leq R - \frac{N-1}{N}S. \quad (4.11)$$

Since  $A$  and  $B_0$  are positive, there exists  $\mu_1 \in (0, 1)$  such that for any  $|M| < \mu_1$ ,

$$A_1 := A - |M|(B + 2) > \frac{A}{2} \quad \text{and} \quad B_1 := B_0 - |M|(B + p + 1) > \frac{B_0}{2}.$$

Set  $A_2 = \min\{A_1, B_1\}$ , then, and whatever is the sign of  $S$ ,

$$A_2(F + T) \leq |R| + |S|.$$

Using (4.7) and (4.8) we have

$$A_2(U + P) \leq 2A_2(F + T) \leq 2(|R| + |S|). \quad (4.12)$$

In the sequel we denote by  $c_j$  some positive constants depending on  $N$  and  $p$ . Then

$$U + P + F + T + W \leq c_1(|R| + |S|). \quad (4.13)$$

On the other hand, we have

$$|R| \leq c_2 \int_{\Omega} \left( u^{m-1} |\nabla u|^3 |\nabla \eta| + u^{m+p} |\nabla u| |\nabla \eta| + u^m |\nabla u|^{q+1} |\nabla \eta| + u^m |\nabla u|^2 |\Delta \eta| \right) dx.$$

Since

$$|\nabla u|^q = \left( \frac{|\nabla u|}{\sqrt{u}} \right)^q u^{\frac{q}{2}} \leq \frac{|\nabla u|^2}{u} + u^{\frac{q}{2}-\frac{q}{2}} = \frac{|\nabla u|^2}{u} + u^p,$$

we deduce

$$\int_{\Omega} u^m |\nabla u|^{q+1} |\nabla \eta| dx \leq \int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \eta| dx + \int_{\Omega} u^{m+p} |\nabla u| |\nabla \eta| dx.$$

Thus we derive from (4.13)

$$U + P + F + T + W \leq 2c_3 \left( \int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \eta| dx + \int_{\Omega} u^{m+p} |\nabla u| |\nabla \eta| dx + \int_{\Omega} u^m |\nabla u|^2 |\Delta \eta| dx \right). \quad (4.14)$$

From this point we can use the method developed in [9, p 599] for proving the Harnack inequality satisfied by positive solutions of (1.2) in  $\Omega$ . We set  $\eta = \xi^\lambda$  with  $\xi \in C_0^\infty(\Omega)$  with value in  $[0, 1]$  and  $\lambda > 4$ . For  $\epsilon \in (0, 1)$  we have by the Hölder-Young inequality

$$\int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \xi^\lambda| dx \leq \frac{\epsilon}{4c_3} \int_{\Omega} u^{m-2} |\nabla u|^4 \xi^\lambda dx + C(\epsilon, c_3) \int_{\Omega} u^{m+2} |\nabla \xi|^4 \xi^{\lambda-4} dx, \quad (4.15)$$

$$\int_{\Omega} u^{m+p} |\nabla u| |\nabla \xi^p| dx \leq \frac{\epsilon}{4c_3} \int_{\Omega} u^{m+p-1} |\nabla u|^2 \xi^p dx + C(\epsilon, c_3) \int_{\Omega} u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx, \quad (4.16)$$

and

$$\int_{\Omega} u^m |\nabla u|^2 |\Delta \xi^p| dx \leq \frac{\epsilon}{4c_3} \int_{\Omega} u^{m-2} |\nabla u|^4 \xi^p dx + C(\epsilon, c_3) \int_{\Omega} u^{m+2} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \right) \xi^{\lambda-4} dx. \quad (4.17)$$

Hence

$$U + P + F + T + W \leq c_4 \left( \int_{\Omega} u^{m+2} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} dx + \int_{\Omega} u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx \right). \quad (4.18)$$

Let us denote by  $c_4 X$  the right-hand side of (4.18). Combining (4.5), (4.16) and (4.18) we also get

$$S := \int_{\Omega} u^{m+p} |\nabla u| |\nabla \xi|^p dx \leq c_5 X \implies V := \int_{\Omega} u^{m+2p} \xi^p dx \leq c_6 X. \quad (4.19)$$

and we finally obtain

$$U + V + P + F + S + T + W \leq c_7 X. \quad (4.20)$$

Finally we estimate the different terms in  $X$ , using that  $m + p > 0$  from (4.2)-(iii). For  $\epsilon > 0$

$$\begin{aligned} \int_{\Omega} u^{m+2} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} dx &\leq \epsilon \int_{\Omega} u^{m+2p} \xi^{\lambda} dx \\ &+ C(\epsilon, c_7) \int_{\Omega} \xi^{\lambda-2 \frac{m+2p}{p-1}} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx. \end{aligned} \quad (4.21)$$

and

$$\int_{\Omega} u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} dx \leq \epsilon \int_{\Omega} u^{m+2p} \xi^{\lambda} dx + C(\epsilon, c_7) \int_{\Omega} \xi^{\lambda-2 \frac{m+2p}{p-1}} |\nabla \xi|^{\frac{2(m+2p)}{p-1}} dx. \quad (4.22)$$

At end we obtain

$$U + V + P + F + S + T + W \leq c_8 \int_{\Omega} \xi^{\lambda-2 \frac{m+2p}{p-1}} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx. \quad (4.23)$$

*Step 2: The Harnack inequality.* We suppose that  $\Omega = B_R \setminus \{0\} := B_R^*$ , fix  $y \in B_{\frac{R}{2}}^*$ , set  $r = |y|$ , then  $B_r(y) \subset B_R^*$ . Let  $\xi \in C_0^\infty(B_r(y))$  such that  $0 \leq \xi \leq 1$ ,  $\xi = 1$  in  $B_{\frac{r}{2}}(y)$ ,  $|\nabla \xi| \leq cr^{-1}$  and  $|\Delta \xi| \leq cr^{-2}$ . We choose  $\lambda > \max \left\{ 4, \frac{m+2p}{p+1} \right\}$ , then

$$\int_{B_r(y)} \xi^{\lambda-2 \frac{m+2p}{p-1}} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} dx \leq c_{22} r^{N - \frac{2(m+2p)}{p-1}},$$

and hence

$$\int_{B_{\frac{r}{2}}(y)} u^{m+2p} dx \leq V \leq c_9 r^{N - \frac{2(m+2p)}{p-1}}. \quad (4.24)$$

We write (1.1) under the form

$$\Delta u + D(x)u + M \langle G(x), \nabla u \rangle = 0, \quad (4.25)$$

with

$$D(x) = u^{p-1} \quad \text{and} \quad G(x) = |\nabla u|^{-\frac{2}{p+1}} \nabla u.$$

Set  $\sigma = \frac{m+2p}{p-1}$ , then  $\sigma > \frac{N}{2}$  by (4.2)-(iii) and

$$\int_{B_{\frac{r}{2}}(y)} D^\sigma(x) dx \leq V \leq c_9 r^{N - \frac{2(m+2p)}{p-1}} = c_{22} r^{N-2\sigma}. \quad (4.26)$$

Next we estimate  $G$ . For  $\tau, \omega, \gamma > 0$  and  $\theta > 1$ , we have with  $\theta' = \frac{\theta}{\theta-1}$ ,

$$|\nabla u|^{(q-1)\tau} = u^\omega |\nabla u|^\gamma u^{-\omega} |\nabla u|^{(q-1)\tau-\gamma} \leq u^{\omega\theta'} |\nabla u|^{\gamma\theta} + u^{-\omega\theta} |\nabla u|^{((q-1)\tau-\gamma)\theta'}.$$

We fix

$$\tau = 2 \frac{2p+m}{p-1} = 2\sigma, \quad \omega = \frac{(2-m)(p+m-1)}{p+1} \quad \text{and} \quad \theta = \frac{p+1}{2-m},$$

Then  $\omega > 0$  and  $\theta > 1$  from (4.2)-(iii),  $\omega > 0$ . Then  $u^{\omega\theta'} |\nabla u|^{\gamma\theta} = u^{p+m-1} |\nabla u|^2$  and  $u^{-\omega\theta} |\nabla u|^{((q-1)\tau-\gamma)\theta'} = u^{m-2} |\nabla u|^4$ , thus

$$\int_{B_{\frac{r}{2}}(y)} |\nabla u|^{(q-1)\tau} dx \leq F + T \leq c_{10} \int_{\Omega} \xi^{\lambda-2\frac{m+2p}{p-1}} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right)^{\frac{m+2p}{2(p-1)}} dx.$$

This implies

$$\int_{B_{\frac{r}{2}}(y)} G^\tau(x) dx \leq c_{11} r^{N-\tau}, \quad (4.27)$$

with  $\tau > N$ . Using the results of [21, Sec. 5], a Harnack inequality, uniform with respect to  $r$ , is satisfied. Hence there exists  $c_{12} > 0$  depending on  $N, p$  such that for any  $r \in (0, \frac{R}{2}]$  and  $y$  such that  $|y| = r$  there holds

$$\max_{z \in B_{\frac{r}{2}}(y)} u(z) \leq c_{12} \min_{z \in B_{\frac{r}{2}}(y)} u(z) \quad \forall 0 < r \leq \frac{R}{2} \quad \forall y \text{ s.t. } |y| = r, \quad (4.28)$$

which implies, with  $c_{13} = c_{12}^3$

$$u(x) \leq c_{13} u(x') \quad \forall x, x' \in \mathbb{R}^N \quad \text{s.t. } |x| = |x'| \leq \frac{R}{2}. \quad (4.29)$$

By (4.24)

$$r^N \omega_N r^N \left( \min_{z \in B_{\frac{r}{2}}(y)} u(z) \right)^{m+2p} \leq 4^N c_{10} r^{N - \frac{2(m+2p)}{p-1}},$$

where  $\omega_N$  is the volume of the unit  $N$ -ball. This implies

$$u(x) \leq c_{14} |x|^{-\frac{2}{p-1}} \quad \forall x \in B_{\frac{R}{2}}^*. \quad (4.30)$$

The proof follows.  $\square$

*Remark.* Using standard rescaling techniques (see e.g. [23, Lemma 3.3.2]) the gradient estimate holds

$$|\nabla u(x)| \leq c_{15} |x|^{-\frac{p+1}{p-1}} \quad \forall x \in B_{\frac{R}{3}}^*. \quad (4.31)$$

And the next estimate for a solution  $u$  in a domain  $\Omega$  satisfying the interior sphere condition with radius  $R$  is valid

$$u(x) \leq c_{14} (\text{dist}(x, \partial\Omega))^{-\frac{2}{p-1}} \quad \forall x \in \Omega \quad \text{s.t. } \text{dist}(x, \partial\Omega) \leq \frac{R}{2}. \quad (4.32)$$

## 5 Radial ground states

We recall that if  $q \neq \frac{2p}{p+1}$  and  $M \neq 0$ , (1.1) can be reduced to the case  $M = \pm 1$  by using the transformation (1.14). Since any ground state  $u$  of (1.1) radial with respect to 0 is decreasing (this is classical and straightforward), it achieves its maximum at 0 and the following equivalence holds if  $v$  is defined by (1.14)

$$\begin{aligned} -u'' - \frac{N-1}{r}u' &= |u|^{p-1}u + M|u_r|^q & \text{s.t.} \quad \max u &= u(0) = 1 \\ \iff -v'' - \frac{N-1}{r}v' &= |v|^{p-1}v \pm |v_r|^q & \text{s.t.} \quad \max v &= v(0) = |M|^{\frac{2}{(p+1)q-2p}}. \end{aligned} \quad (5.1)$$

Hence large or small values of  $M$  for  $u$  are exchanged into large or small values of  $v(0)$  for  $v$  and in the sequel we will essentially express our results using the function  $u$ .

### 5.1 Energy functions

We consider first the energy function

$$r \mapsto H(r) = \frac{u^{p+1}}{p+1} + \frac{u'^2}{2}. \quad (5.2)$$

Then

$$H'(r) = M|u'|^{q+1} - \frac{N-1}{r}u'^2.$$

Hence, if  $M \leq 0$ ,  $H$  is decreasing, a property often used in [18]. This implies in particular that a radial ground state satisfies

$$|u'(r)| \leq \sqrt{\frac{2}{p+1}} (u(0))^{\frac{p+1}{2}}. \quad (5.3)$$

A similar estimate holds in all the cases.

**Proposition 5.1** *Let  $M > 0$ ,  $p, q > 1$ . If  $u$  is a radial ground state solution of (1.1), then the function  $H$  defined in (5.2) is decreasing and in particular (5.3) holds.*

*Proof.* Let  $u$  be such a radial ground state. By Proposition 2.1 we must have  $q > \frac{N}{N-1}$  and

$$\frac{r}{u'^2}H' = Mr|u'|^{q-1} + 1 - N \leq \frac{(N-1)q - N}{q-1} + 1 - N = -\frac{1}{q-1},$$

this implies the claim.  $\square$

#### 5.1.1 Exponential perturbations

As we have seen it in the introduction, if  $q < \frac{2p}{p+1}$  equation (1.1) can be seen as a perturbation of the Lane-Emden equation (1.2) while if  $q > \frac{2p}{p+1}$  it can be seen as a perturbation of the Riccati

equation (1.13). Two types of transformations can emphasize these aspects.

1) For  $p > 1$  set

$$u(r) = r^{-\frac{2}{p-1}}x(t), \quad u'(r) = -r^{-\frac{p+1}{p-1}}y(t), \quad t = \ln r, \quad (5.4)$$

then

$$\begin{aligned} x_t &= \frac{2}{p-1}x - y \\ y_t &= -Ky + x^p + Me^{-\omega t}y^q \end{aligned} \quad (5.5)$$

with

$$K = \frac{(N-2)p - N}{p-1}, \quad (5.6)$$

and

$$\omega = \frac{(p+1)q - 2p}{p+1}. \quad (5.7)$$

If  $q > \frac{2p}{p+1}$  (resp.  $q < \frac{2p}{p+1}$ ), then  $\omega > 0$  (resp.  $\omega < 0$ ) system (5.7) is a perturbation of the Lane-Emden system

$$\begin{aligned} x_t &= \frac{2}{p-1}x - y \\ y_t &= -Ky + x^p, \end{aligned} \quad (5.8)$$

at  $\infty$  (resp.  $-\infty$ ). The following Leighton function is natural with (5.8)

$$\mathcal{N}(t) = \mathcal{L}(x(t), y(t)) = \frac{K}{p-1}x^2 - \frac{x^{p+1}}{p+1} - \left(\frac{2}{p-1}\right)^q Me^{-\omega t} \frac{x^{q+1}}{q+1} - \frac{1}{2} \left(\frac{2x}{p-1} - y\right)^2, \quad (5.9)$$

and it satisfies

$$\begin{aligned} \mathcal{N}'(t) &= \left(\frac{2x}{p-1} - y\right) \left[ L \left(\frac{2x}{p-1} - y\right) - Me^{-\omega t} \left( \left(\frac{2x}{p-1}\right)^q - y^q \right) \right] \\ &\quad + \omega \left(\frac{2}{p-1}\right)^q Me^{-\omega t} \frac{x^{q+1}}{q+1}, \end{aligned} \quad (5.10)$$

where  $L = N - 2 - \frac{4}{p-1} = K - \frac{2}{p-1}$ . Relation (5.10) will be used later on.

2) For  $p, q > 1$  set

$$u(r) = r^{-\frac{2-q}{q-1}}\xi(t), \quad u'(r) = -r^{-\frac{1}{q-1}}\eta(t), \quad t = \ln r, \quad (5.11)$$

then

$$\begin{aligned} \xi_t &= \frac{2-q}{q-1}\xi - \eta \\ \eta_t &= -\frac{(N-1)q - N}{q-1}\eta + e^{\bar{\omega}t}\xi^p + M\eta^q \end{aligned} \quad (5.12)$$

where

$$\bar{\omega} = \frac{p-1}{q-1}\omega. \quad (5.13)$$

Note that if  $q < \frac{2p}{p+1}$  this system at  $\infty$  endows the form

$$\begin{aligned}\xi_t &= \frac{2-q}{q-1}\xi - \eta \\ \eta_t &= -\frac{(N-1)q-N}{q-1}\eta + M\eta^q,\end{aligned}\tag{5.14}$$

and is therefore completely integrable.

### 5.1.2 Pohozaev-Pucci-Serrin type functions

Let  $\alpha, \gamma, \theta, \kappa$  be real parameters with  $\alpha, \kappa > 0$ . Set

$$\mathcal{Z}(r) = r^\kappa \left( \frac{u'^2}{2} + \frac{u^{p+1}}{p+1} + \alpha \frac{uu'}{r} - \gamma u' |u'|^q \right).\tag{5.15}$$

This type of function has been introduced in [18] in their study of equation (1.1) with  $M = 1$  with specific parameters. We use it here to embrace all the values of  $M$ . We define  $\mathcal{U}$  by the identity

$$\mathcal{Z}' + \theta |u'|^{q-1} \mathcal{Z} = r^{\kappa-1} \mathcal{U}.\tag{5.16}$$

Then

$$\begin{aligned}\mathcal{U} &= \left( \frac{\kappa}{2} + \alpha + 1 - N \right) u'^2 + \left( \frac{\kappa}{p+1} - \alpha \right) u^{p+1} + \alpha(\kappa - N) \frac{uu'}{r} + \left( \frac{\theta}{p+1} - \gamma q \right) r u^{p+1} |u'|^{q-1} \\ &\quad + \left( M + \gamma + \frac{\theta}{2} \right) r |u'|^{q+1} + \left( ((N-1)q - \kappa) \gamma - \alpha(\theta + M) \right) u |u'|^q - \gamma(\theta + qM) r u |u'|^{2q-1}.\end{aligned}\tag{5.17}$$

## 5.2 Some known results in the case $M < 0$

We recall the results of [12], [18] and [16] relative to the case  $M < 0$ .

**Theorem 5.2** 1) Let  $N \geq 3$  and  $1 < p \leq \frac{N}{N-2}$ .

If  $q > \frac{2p}{p+1}$ , there is no ground state [18, Th. C] for any  $M < 0$

If  $1 < q < \frac{2p}{p+1}$  there exists a ground state when  $|M|$  is large [12, Prop. 5.7] and there exists no ground state when  $|M|$  is small [16].

2) Assume  $\frac{N}{N-2} < p < \frac{N+2}{N-2}$  and let  $\bar{q}$  be the unique root in  $(\frac{2p}{p+1}, p)$  of the quadratic equation

$$(N-1)(X-p)^2 - (N+2 - (N-2)p)((p+1)X - 2p)X = 0.$$

If  $\bar{q} \leq q < p$  there exists no ground state [18, Th. C] for any  $M < 0$

If  $\frac{2p}{p+1} < q < \bar{q}$ , there exists no ground state for  $|M|$ . It is an open question whether there could exist a finite number of  $M$  for which there exists a ground state [18, Th. 4].

If  $1 < q < \frac{2p}{p+1}$ , there exists a ground state for large  $|M|$  [12, Prop. 5.7] and no ground state when  $|M|$  is small [16].

3) Assume  $p > \frac{N+2}{N-2}$  and  $q > 1$  and let  $Q_{N,p} = \frac{2(N-1)p}{2N+p+1} \in (\frac{2p}{p+1}, p)$

If  $Q_{N,p} < q < p$  there exists a ground state for  $|M|$  small.

If  $1 < q \leq Q_{N,p}$  there exists a ground state for any  $M < 0$  [18, Th. A].

4) Assume  $p = \frac{N+2}{N-2}$ . There exists at least one  $M < 0$  such that there exists a ground state if and only if  $1 < q < p$ . More precisely:

If  $\frac{2p}{p+1} < q < p$  there exists ground state if  $|M|$  is small [18, Th. B].

If  $q \geq \frac{2p}{p+1}$  there exists a ground state for any  $M < 0$  [18, Th. A].

*Remark.* It is interesting to quote that when  $M < 0$  and  $q \geq \frac{2p}{p+1}$ , there holds [18, Th. 3],

$$u(r) = O(r^{-\frac{2}{p-1}}) \quad \text{and} \quad u'(r) = O(r^{-\frac{p+1}{p-1}}) \quad \text{when } r \rightarrow \infty.$$

### 5.3 The case $M > 0$

The next result is a consequence of Theorem A.

**Theorem 5.3** *Let  $M > 0$ ,  $p > 1$  and  $q > \frac{2p}{p+1}$  then there exists no ground state satisfying  $u(0) = 1$  when  $M$  is large.*

*Proof.* Suppose that such a solution  $u$  exists. From Theorem A and Proposition 2.1 there holds

$$\sup_{r>0} |u'(r)| \leq c_{N,p,q} |M|^{-\frac{p+1}{(p+1)q-2p}} \quad \text{and} \quad \sup_{r>0} r^{\frac{p+1}{p-1}} |u'(r)| \leq c_{N,p}. \quad (5.18)$$

As a consequence, if  $r > R > 0$ ,

$$\begin{aligned} 1 - u(r) &= u(0) - u(r) = u(0) - u(R) + u(R) - u(r) \leq c_{N,p,q} |M|^{-\frac{p+1}{(p+1)q-2p}} R + \int_R^\infty |u'(s)| ds \\ &\leq c_{N,p,q} |M|^{-\frac{p+1}{(p+1)q-2p}} R + c'_{N,p} R^{-\frac{2}{p-1}}, \end{aligned}$$

with  $c'_{N,p} = \frac{p-1}{2} c_{N,p}$ . Since  $u(r) \rightarrow 0$  when  $r \rightarrow \infty$ , we take  $R = |M|^{\frac{p-1}{(p+1)q-2p}}$  and derive

$$1 \leq (c_{N,p,q} + c'_{N,p}) |M|^{-\frac{2}{(p+1)q-2p}}, \quad (5.19)$$

and the conclusion follows.  $\square$

*Remark.* If we use Proposition 5.1 we can make estimate (5.19) more precise.

#### 5.3.1 The case $M > 0$ , $1 < p \leq \frac{N+2}{N-2}$

It is a consequence of our general results that there is no radial ground state for large  $M$  or for small  $M$  when  $1 < q \leq \frac{2p}{p+1}$  and  $1 < p < \frac{N+2}{N-2}$ . Indeed, if  $1 < q < \frac{2p}{p+1}$  is a consequence of the equivalence statement between a priori estimate and non-existence of ground state proved in [16], and if  $q = \frac{2p}{p+1}$  it follows from Theorems C and E. Actually in the radial case, the result is more general.

**Theorem 5.4** *Let  $M > 0$  and  $1 < p < \frac{N+2}{N-2}$ . If  $q \leq p$  there exists no radial ground state for any  $M$ . If  $q > p$  there exists no radial ground state for  $M$  small enough.*

*Proof.* By Proposition 2.1, we may assume  $N \geq 3$  and

$$\frac{N}{N-2} < p \leq \frac{N+2}{N-2} \quad \text{and} \quad q > \frac{N}{N-1}. \quad (5.20)$$

(i) Assume first  $q < \frac{2p}{p+1}$ . We use the system (5.5). Then  $\omega$ , defined by (5.7) is negative. Hence the Leighton function  $\mathcal{N}$  defined by (5.9) is nonincreasing since  $L \leq 0$  when  $p \leq \frac{N+2}{N-2}$ . Furthermore since  $(x(t), y(t)) \rightarrow (0, 0)$  when  $t \rightarrow -\infty$  and  $e^{-\omega t} \rightarrow 0$ , we get  $\mathcal{N}(-\infty) = 0$  it follows that  $\mathcal{N}(t) < 0$  for  $t \in \mathbb{R}$ . Moreover, by Proposition 2.1,

$$u(r) = O(r^{-\frac{2-q}{q-1}}) \quad \text{as } r \rightarrow \infty \iff x(t) = O(e^{\frac{q(p+1)-2p}{(p-1)(q-1)}t}) = o(1) \quad \text{as } t \rightarrow \infty$$

This implies  $e^{-\omega t} x^{q+1}(t) = O(e^{\frac{2q(p+1)-2p}{(p-1)(q-1)}t}) = o(1)$  as  $t \rightarrow \infty$  and  $\mathcal{N}(\infty) = 0$ , contradiction.

(ii) Assume next  $\frac{2p}{p+1} \leq q \leq p$ . We consider the function (5.15) with the parameters

$$\kappa = \frac{2(p+1)(N-1)}{p+3} = (p+1)\alpha \quad \text{and} \quad \gamma = -\frac{2M}{q(p+1)+2} = \frac{\theta}{q(p+1)},$$

already used by [18] when  $M = -1$ , and we get with  $\mathcal{U}$  defined by (5.16),

$$\mathcal{U} = \frac{2}{(p+3)^2} \frac{u|u'|}{r} (A + BM\chi + CM\chi^2) \quad \text{with } \chi = \frac{p+3}{2+q(p+1)} r |u'|^{q-1},$$

where

$$A = (N-1)(N+2-(N-2)p), \quad B = 2(N-1)(p-q), \quad C = q(q(p+1)-2p). \quad (5.21)$$

By our assumptions  $A \geq 0$ ,  $B \geq 0$  and  $C > 0$ . Hence  $\mathcal{U} > 0$ . This implies

$$\mathcal{Z}(r) = e^{-\int_0^r \theta |u'|^{q-1} ds} \mathcal{Z}(0) + \int_0^r e^{-\theta \int_s^r |u'|^{q-1} d\sigma} s^{\kappa-1} \mathcal{U}(s) ds = \int_0^r e^{-\theta \int_s^r |u'|^{q-1} d\sigma} s^{\kappa-1} \mathcal{U}(s) ds,$$

since  $\mathcal{Z}(0) = 0$ . If  $u$  is a ground state, then  $u'(r) \rightarrow 0$  as  $r \rightarrow \infty$ , thus  $u|u'|^q \leq u|u'|^{\frac{2p}{p+1}}$ . Hence, from Proposition 2.1,  $u'^2(r) = O(r^{-\frac{2p+1}{p-1}})$  as  $r \rightarrow \infty$ . The other terms  $u^{p+1}(r)$ ,  $r^{-1}u(r)u'(r)$  and  $u|u'|^{\frac{2p}{p+1}}$  satisfy the same bound, hence

$$\mathcal{Z}(r) = O(r^{\kappa - \frac{2(p+1)}{p-1}}) = O(r^{\frac{2(p+3)(N-1)}{p+3} - \frac{2(p+1)}{p-1}}) = O(r^{\frac{2(p+1)((N-2)p-(N+2))}{(p+3)(p-1)}}).$$

Then  $\mathcal{Z}(r) \rightarrow 0$  when  $r \rightarrow \infty$ , contradiction.

(iii) Suppose  $q > p$  and  $u$  is a ground state. By Proposition 5.1 and (5.18), there holds

$$r |u'|^{q-1} = r |u'|^{\frac{p-1}{p+1}} |u'|^{q - \frac{2p}{p+1}} \leq c_{N,p}.$$

Then  $\chi = \frac{p+3}{2+q(p+1)} r |u'|^{q-1} \leq c_{N,p}$ . Hence, if  $M \leq M_{N,p}$  for some  $M_{N,p} > 0$ ,  $\mathcal{U}$  is positive as  $A$  is. We conclude as above.  $\square$

### 5.3.2 The case $M > 0$ , $p > \frac{N+2}{N-2}$

We recall that in Theorem C if  $q = \frac{2p}{p+1}$  and  $p > 1$  there is no ground state whenever  $M > M_N(p)$  see (1.23). In Theorem A' if  $1 < q < \frac{2p}{p+1}$  and  $p > 1$  there is no ground state  $u$  such that  $u(0) = 1$  if  $M$  is too large. In the next result we complement Theorem 5.3 for small value of  $M$  in assuming  $q > \frac{2p}{p+1}$ .

**Theorem 5.5** *If  $p > \frac{N+2}{N-2}$  and  $q \geq \frac{2p}{p+1}$  then there exist ground states for  $M > 0$  small enough.*

*Proof.* First we consider the function  $\mathcal{Z}$  with  $k = N$  and obtain

$$\mathcal{Z}(r) = r^N \left( \frac{u'^2}{2} + \frac{u^{p+1}}{p+1} + \alpha \frac{uu'}{r} - \gamma u |u'|^q \right).$$

The function vanishes at the origin. We compute  $\mathcal{U}$  from the identity  $\mathcal{Z}' + \theta |u'|^{q-1} \mathcal{Z} = r^{N-1} \mathcal{U}$  and get

$$\begin{aligned} \mathcal{U} &= \left( \alpha - \frac{N-2}{2} \right) u'^2 + \left( \frac{N}{p+1} - \alpha \right) u^{p+1} + \left( \frac{\theta}{p+1} - \gamma q \right) r u^{p+1} |u'|^{q-1} \\ &\quad + \left( M + \gamma + \frac{\theta}{2} \right) r |u'|^{q+1} + \left[ ((N-1)q - N) \gamma - \alpha(\theta + M) \right] u |u'|^q - \gamma(\theta + qM) r u |u'|^{2q-1}. \end{aligned}$$

If  $\gamma = 0$  and  $\theta = -2M$ , then

$$\mathcal{U} = \left( \alpha - \frac{N-2}{2} \right) u'^2 + \left( \frac{N}{p+1} - \alpha \right) u^{p+1} - \frac{2M}{p+1} r u^{p+1} |u'|^{q-1} + \alpha M u |u'|^q.$$

If  $u$  is a regular solution which vanishes at some  $r_0 > 0$ , then  $\mathcal{Z}(r_0) = 2^{-1} r_0^2 u'^N(r_0) > 0$ . As  $p > \frac{N+2}{N-2}$ , by choosing  $\alpha = \frac{1}{2} \left( \frac{N}{p+1} + \frac{N-2}{2} \right)$  we have  $\frac{N}{p+1} < \alpha < \frac{N-2}{2}$ . We define  $\ell > 0$  by  $(N-2)p - (N+2) = 4(p+1)\ell$ , then  $\frac{N-2}{2} - \alpha = \alpha - \frac{N}{p+1} = \ell$  and then

$$\mathcal{U} \leq -\ell(u'^2 + u^{p+1}) + M\alpha u |u'|^q.$$

Assume first  $q < 2$ , we have from Hölder's inequality and  $0 < r \leq r_0$  where  $u$  is positive

$$u |u'|^q \leq \frac{q}{2} u'^2 + \frac{2-q}{2} |u|^{\frac{2}{2-q}} \leq u'^2 + |u|^{\frac{2}{2-q}},$$

and

$$\mathcal{U} + (\ell - M)u'^2 \leq M\alpha u^{\frac{2}{2-q}} - \ell u^{p+1} = \ell u^{p+1} \left( \frac{M\alpha}{\ell} u^{\frac{q(p+1)-2p}{2-q}} - 1 \right) \leq \ell u^{p+1} \left( \frac{M\alpha}{\ell} - 1 \right)$$

since  $q \geq \frac{2p}{p+1}$  and  $u \leq u(0) = 1$ . Taking  $M \leq \frac{\ell}{\alpha} = \frac{(N-2)p - N - 2}{(N-2)p + 3N - 2}$ ,  $\mathcal{U}$  is negative implying that  $r \mapsto e^{-2M \int_0^r |u'|^{q-1} ds} \mathcal{Z}(r)$  is nonincreasing. Since  $\mathcal{Z}(0) = 0$ ,  $\mathcal{Z}(r) \leq 0$ , a contradiction.

If  $q = 2$ , then  $\mathcal{U} \leq -\ell(u'^2 + u^{p+1}) + M\alpha u'^2$  since  $u \leq 1$  on  $[0, r_0]$ . We still infer that  $\mathcal{U} \leq 0$  if we choose  $M \leq \frac{\ell}{\alpha}$ .

Finally, if  $q > 2$ , we have from Theorem A,  $u' \leq C_{N,p,q} M^{-\frac{p+1}{(p+1)q-2p}}$ . Therefore, using again the decay of  $u$  from  $u(0) = 1$ ,

$$M\alpha u |u'|^q \leq M\alpha u |u'|^{q-2} u'^2 \leq M\alpha C_{N,p,q}^{q-2} M^{-\frac{(p+1)(q-2)}{(p+1)q-2p}} u'^2 = \alpha C_{N,p,q}^{q-2} M^{\frac{2}{(p+1)q-2p}} u'^2.$$

Hence  $\mathcal{U} \leq -\left(\ell - \alpha C_{N,p,q}^{q-2} M^{\frac{2}{(p+1)q-2p}}\right) u'^2$ . Taking

$$M^{\frac{2}{(p+1)q-2p}} \leq C_{N,p,q}^{2-q} \frac{(N-2)p - N - 2}{(N-2)p + 3N - 2}$$

we conclude that  $\mathcal{U} < 0$  which ends the proof as in the previous cases.  $\square$

Theorem F is the combination of Theorem 5.3, Theorem 5.4 and Theorem 5.5.

## 6 Separable solutions

We denote by  $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$  the spherical coordinates in  $\mathbb{R}^N$ . Then equation (1.1) takes the form

$$-u_{rr} - \frac{N-1}{r} u_r - \frac{1}{r^2} \Delta' u = |u|^{p-1} + M \left( u_r^2 + \frac{1}{r^2} |\nabla' u|^2 \right)^{\frac{q}{2}}, \quad (6.1)$$

where  $\Delta'$  is the Laplace-Beltrami operator on  $S^{N-1}$  and  $\nabla'$  the tangential gradient. If we look for separable nonnegative solutions of (1.1) i.e. solutions under the form  $u(r, \sigma) = \psi(r)\omega(\sigma)$ , then  $q = \frac{2p}{p+1}$ ,  $\psi(r) = r^{-\frac{2}{p-1}}$ , and  $\omega$  is a solution of

$$-\Delta' \omega + \frac{2K}{p-1} \omega = \omega^p + M \left( \left( \frac{2}{p-1} \right)^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{p+1}}, \quad (6.2)$$

where  $K$  is defined in (5.6). Throughout this section we assume

$$p > 1 \quad \text{and} \quad q = \frac{2p}{p+1}. \quad (6.3)$$

### 6.1 Constant solutions

The constant function  $\omega = X$  is a solution if

$$X^{p-1} + M \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} - \frac{2K}{p-1} = 0. \quad (6.4)$$

For  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , we recall that  $\mu^* = \mu^*(N)$  has been defined in (1.21). The following result is easy to prove

**Proposition 6.1** *1- Let  $M \geq 0$  then there exists a unique positive root  $X_M$  to (6.4) if and only if  $p > \frac{N}{N-2}$ . Moreover the mapping  $M \mapsto X_M$  is continuous and decreasing from  $[0, \infty)$  onto  $(0, \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}]$ .*

2- Let  $M < 0$ ,  $N \geq 3$  and  $p \geq \frac{N}{N-2}$  then there exists a unique positive root  $X_M$  to (6.4) and the mapping  $M \mapsto X_M$  is continuous and decreasing from  $(-\infty, 0]$  onto  $[\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \infty)$ .

3- Let  $M < 0$ ,  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$  then there exists no positive root to (6.4) if  $-\mu^* < M \leq 0$ . If  $M = M^* := -\mu^*$  there exists a unique positive root  $X_{M^*} = \left(\frac{2|K|}{p(p-1)}\right)^{\frac{1}{p-1}}$ . If  $M < -\mu^*$  there exist two positive roots  $X_{1,M} < X_{2,M}$ . The mapping  $M \mapsto X_{1,M}$  is continuous and increasing from  $(-\infty, \mu^*]$  onto  $(0, X_{M^*}]$ . The mapping  $M \mapsto X_{2,M}$  is continuous and decreasing from  $(-\infty, \mu^*]$  onto  $[X_{M^*}, \infty)$ .

## 6.2 Bifurcations

We set

$$A(\omega) = -\Delta'\omega + \frac{2K}{p-1}\omega - \omega^p - M \left( \left( \frac{2}{p-1} \right)^2 \omega^2 + |\nabla'\omega|^2 \right)^{\frac{p}{p+1}}, \quad (6.5)$$

If  $\eta \in C^\infty(S^{N-1})$  and if there exists a constant positive solution  $X$  to  $A(X) = 0$  we have

$$\frac{d}{d\tau} A(X + \tau\eta)|_{\tau=0} = -\Delta'\eta + \left( \frac{2K}{p-1} - pX^{p-1} - M \frac{2p}{p+1} \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} \right) \eta.$$

Hence the problem is singular if

$$-\frac{2K}{p-1} + pX^{p-1} + M \frac{2p}{p+1} \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} = \lambda_k, \quad (6.6)$$

where  $\lambda_k = k(k + N - 2)$  is the  $k$ -th nonzero eigenvalue of  $-\Delta'$  in  $H^1(S^{N-1})$ . The following result follows classically from the standard bifurcation theorem from a simple eigenvalue (which can always be assumed using symmetries) and identity (6.4).

**Theorem 6.2** *Let  $M_0 \in \mathbb{R}$  and  $X_{M_0}$  be a constant solution of (6.2). If  $X_{M_0}$  satisfies for some  $k \in \mathbb{N}^*$ ,*

$$M_0 \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_{M_0}^{\frac{p-1}{p+1}} = \frac{p+1}{p(p-1)} (2K - \lambda_k), \quad (6.7)$$

*there exists a continuous branch of nonconstant positive solutions  $(M, \omega_M)$  of (6.2) bifurcating from the  $(M_0, X_{M_0})$ .*

Since  $M \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_M^{\frac{p-1}{p+1}} = \frac{2K}{p-1} - X_M^{p-1}$  by (6.4) the following statements follow immediately from Proposition 6.1.

**Lemma 6.3** *Set  $\Phi(M) = M \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_M^{\frac{p-1}{p+1}}$  when  $X_M$  is uniquely determined, and  $\Phi_j(M) = M \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X_{j,M}^{\frac{p-1}{p+1}}$ ,  $j=1, 2$ , if there exist two equilibria. Then*

- 1- If  $N \geq 3$  and  $p > \frac{N}{N-2}$ , the mapping  $M \mapsto \Phi(M)$  is continuous and increasing from  $[0, \infty)$  onto  $[0, \frac{2K}{p-1})$ .
- 2- If  $N \geq 3$  and  $p \geq \frac{N}{N-2}$ , the mapping  $M \mapsto \Phi(M)$  is continuous and increasing from  $(-\infty, 0]$  onto  $(-\infty, 0]$ .
- 3- If  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , the mapping  $M \mapsto \Phi_1(M)$  (resp.  $M \mapsto \Phi_2(M)$ ) is continuous and decreasing (resp. increasing) from  $(-\infty, -\mu^*]$  onto  $[\frac{2K}{p-1}, 0)$  (resp.  $(-\infty, \frac{2K}{p-1}]$ ).

If we analyse the range  $R[\Phi]$  of  $\Phi$  or  $R[\Phi_j]$  of  $\Phi_j$ , we prove the following result.

**Theorem 6.4** 1- If  $N \geq 3$  and  $p \geq \frac{N}{N-2}$  there exists a continuous curve of bifurcation  $(M, \omega_M)$  issued from  $(M_0, X_{M_0})$  for some  $M_0 = M_0(p) \geq 0$  if and only if  $p \geq \frac{N+1}{N-3}$  and  $k = 1$ . Furthermore  $M_0(\frac{N+1}{N-3}) = 0$ . The bifurcation curve  $s \mapsto (M(s), \omega_{M(s)})$ , is defined on  $(-\epsilon_0, \epsilon_0)$  for some  $\epsilon_0 > 0$  and verifies  $(M(0), \omega_{M(0)}) = (M_0, X_{M_0})$ .

2- If  $N \geq 3$  and  $p \geq \frac{N}{N-2}$ , then for any  $k \geq 1$  there exist  $M_k < 0$  and a continuous branch of bifurcation  $(M, \omega_M)$  issued from  $(M_k, X_{M_k})$ , with the restriction that  $p < \frac{N+1}{N-3}$  if  $k = 1$ . In that case, the bifurcation curve  $s \mapsto (M(s), \omega_{M(s)})$ , is defined on  $(-\epsilon_0, \epsilon_0)$  for some  $\epsilon_0 > 0$  and verifies  $(M(0), \omega_{M(0)}) = (M_0, X_{M_0})$ . Finally  $M_k \rightarrow -\infty$  when  $k \rightarrow \infty$ .

3- If  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$  there exists no  $M < 0$  such that  $\frac{2K}{p-1} < \Phi_1(M) < 0$  and a countable set of  $M_k < 0$ ,  $k \geq 1$ , such that  $\Phi_2(M_k) = \frac{p+1}{p(p-1)}(2K - \lambda_k)$ . Therefore there exist a countable branches of bifurcation of solutions  $(M_k(s), \omega_{M_k(s)})$  issued from  $(M_k, X_{2, M_k})$ .

*Proof.* Assertion 1. Since from Lemma 6.3,  $R[\Phi] = [0, \frac{2K}{p-1})$  for  $M \geq 0$ , we have to see under what condition on  $p \geq \frac{N}{N-2}$  one can find  $k \geq 1$  such that

$$0 \leq \frac{p+1}{p(p-1)}(2K - \lambda_k) < \frac{2K}{p-1} \iff \frac{2K}{p+1} < \lambda_k \leq 2K.$$

Since  $K < N$  and  $\lambda_k \geq 2N$  for  $k \geq 2$ , the only possibility for this last inequality to hold is  $k = 1$ . The inequality  $\frac{2K}{p+1} < N - 1$  always holds since  $p > 1$ , while the inequality  $N - 1 = \lambda_1 \leq 2K$  is equivalent to  $p \geq \frac{N+1}{N-3}$ . Therefore  $M_0 = 0$  and  $X_{M_0} = \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}$ . If we consider only functions on the sphere  $S^{N-1}$  which depend uniquely on the azimuthal angle  $\theta = \tan^{-1}(x_N|_{S^{N-1}})$ , the function  $\psi_1(\sigma) = x_N|_{S^{N-1}}$  is an eigenfunction of  $-\Delta'$  in  $H^1(S^{N-1})$  with multiplicity one. Hence the bifurcation branch is locally a regular curve  $s \mapsto (M(s), \omega_{M(s)})$  with  $0 \leq s < \epsilon'_0$  and we construct the bifurcating solution on  $S^{N-1}$  using the classical tangency condition [19, Th. 13.5],

$$\omega_{M(s)} = X_{M_0} + s(\psi_1 + \zeta_s) \tag{6.8}$$

where  $\zeta_s \in H^1(S^{N-1})$ , is orthogonal to  $\psi_1$  in  $H^1(S^{N-1})$  and satisfies  $\|\zeta_s\|_{C^1} = o(1)$  when  $s \rightarrow 0$ . this implies the claim.

*Assertion 2.* Since  $R[\Phi] = (-\infty, 0)$  for  $M < 0$ , we have to find  $k \geq 1$  such that

$$\frac{p+1}{p(p-1)}(2K - \lambda_k) < 0 \iff 2K < \lambda_k.$$

As in Case 1,  $K < 2N$ , then inequality  $2K \leq \lambda_k$  holds for all  $k \geq 2$ , and if  $k = 1$  this is possible only if  $p < \frac{N+1}{N-3}$ . The construction of the bifurcating curve is the same as in Case 1.

*Assertion 3.* We have  $R[\Phi_1] = [\frac{2K}{p-1}, 0)$  for  $M \leq -\mu^*$ . If we look for the existence of some  $k \geq 1$  such that

$$\frac{2K}{p-1} \leq \frac{p+1}{p(p-1)}(2K - \lambda_k) < 0 \iff 2K \leq \lambda_k < \frac{2K}{p+1};$$

we get an impossibility since  $K < 0$ . Hence there exists no  $M_0 < 0$  such that  $(M_0, X_{1,M_0})$  is a bifurcation point. We have also  $R[\Phi_2] = (-\infty, \frac{2K}{p-1}]$  for  $M \leq -\mu^*$ . Now the condition for the existence of a bifurcation branch issued from  $(M_0, X_{2,M_0})$  for some  $M_0 \leq -\mu^*$  is

$$\frac{p+1}{p(p-1)}(2K - \lambda_k) \leq \frac{2K}{p-1} \iff \lambda_k \geq \frac{2K}{p+1},$$

which is always true for any  $k \geq 1$  and  $1 < p < \frac{N}{N-2}$ .  $\square$

*Remark.* The exponent  $p = \frac{N+1}{N-3}$  is the Sobolev critical exponent on  $S^{N-1}$ . If one consider the Lane-Emden equation with a Leray potential

$$-\Delta u + \lambda|x|^{-2}u = u^{\frac{N+1}{N-3}}, \quad (6.9)$$

with  $\lambda \in \mathbb{R}$ , then the separable solutions  $u(r, \sigma) = r^{-\frac{N-3}{2}}\omega(\sigma)$  verify

$$-\Delta' \omega + \left( \frac{(N-1)(N-3)}{4} - \lambda \right) \omega - \omega^{\frac{N+1}{N-3}} = 0 \quad \text{on } S^{N-1}. \quad (6.10)$$

It was observed in [9] that there exists a branch of bifurcation  $(\lambda, \omega_\lambda)$  with  $\lambda > 0$  issued from  $(0, \omega_0)$ , where  $\omega_0$  is the constant explicit solution of (6.10).

*Remark.* In Theorem 6.4-(i) and Remark above, we conjectured that on the bifurcating curve there holds locally  $M(s) < M_0$ , and that for any  $p \geq \frac{N+1}{N-3}$  there exists  $M_0 := M_0(p)$  such that for  $M > M_0$  all the positive solutions to (6.2) are constant, furthermore  $M_0$  is defined by (6.7). When  $p = \frac{N+1}{N-3}$ , then  $M = 0$  and there exists infinitely many positive solutions to (6.2) [9, Prop. 5.1]. When  $\frac{N}{N-2} < p < \frac{N+1}{N-3}$ , it is unclear if the branches of bifurcation  $(M(s), \omega_{M(s)})$  issued from  $(M_0, \omega_{M_0})$  with  $M_0 < 0$  are such that  $M(s)$  keeps a constant sign. If it is the case one could expect that if  $M \geq 0$  and  $\frac{N}{N-2} < p < \frac{N+1}{N-3}$ , all the positive solutions to (6.2) are constant.

The following statement is an immediate consequence of Theorem 6.4.

**Corollary 6.5** *If  $p > 1$  and  $q = \frac{2p}{p+1}$  there always exist nonradial positive singular solutions of (1.1) in  $\mathbb{R}^N \setminus \{0\}$  under the form  $u(r, \sigma) = r^{-\frac{2}{p-1}}\omega(\sigma)$ . If  $N \geq 4$  and  $p > \frac{N+1}{N-3}$  the functions are obtained by bifurcation from  $X_M$  with  $M > 0$ . If  $N \geq 3$  and  $\frac{N}{N-2} \leq p < \frac{N+1}{N-3}$  the functions are obtained by bifurcation from  $X_M$  with  $M < 0$ . If  $N = 1, 2$  and  $p > 1$  or  $N \geq 3$  and  $1 < p < \frac{N}{N-2}$ , the functions are obtained by bifurcation from  $(M_k, X_{2,M_k})$  with  $M_k < -\mu^*$  and  $k \geq 1$ .*

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