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ON A CLASS OF SPACES OF SKEW-SYMMETRIC FORMS RELATED TO HAMILTONIAN SYSTEMS OF CONSERVATION LAWS

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Abstract. It was shown in [8] that the classification of n-component systems of conservation laws possessing a third-order Hamiltonian structure reduces to the following algebraic problem: classify n-planes H in $\wedge^2(V_{n+2})$ such that the induced map $Sym^2 H \longrightarrow \wedge^4 V_{n+2}$ has 1-dimensional kernel generated by a non-degenerate quadratic form on H^*.

This problem is trivial for $n = 2, 3$ and apparently wild for $n \geq 5$. In this paper we address the most interesting borderline case $n = 4$.

We prove that the variety $\mathcal{V}$ parametrizing those 4-planes H is an irreducible 38-dimensional $PGL(V_6)$-invariant subvariety of the Grassmannian $G(4, \wedge^2 V_6)$. With every $H \in \mathcal{V}$ we associate a *characteristic* cubic surface $S_H \subset \mathbb{P}H$, the locus of rank 4 two-forms in H. We demonstrate that the induced characteristic map $\sigma : \mathcal{V}/PGL(V_6) \dashrightarrow \mathcal{M}_c$, where $\mathcal{M}_c$ denotes the moduli space of cubic surfaces in $\mathbb{P}^3$, is dominant, hence generically finite. A complete classification of 4-planes $H \in \mathcal{V}$ with the reducible characteristic surface S_H is given.

Contents

1. Introduction

Our problem originates from the geometric theory of systems of conservation laws,

$$u^i_t = [V^i(\mathbf{u})]_x, \tag{1}$$

$i = 1, \dots, n$. PDEs of this type appear in a wide range of applications in continuum mechanics and mathematical physics, see e.g. [11]. A geometric counterpart of system (1) is a line congruence (n-parameter family of lines in projective space $\mathbb{P}^{n+1}$) specified by the equations

$$y^i = u^i y^{n+1} + V^i(\mathbf{u}) y^{n+2}, \tag{2}$$

here y^i are the homogeneous coordinates in $\mathbb{P}^{n+1}$. In the case $n = 2$ we obtain a two-parameter family, or a classical congruence of lines in $\mathbb{P}^3$. Since 19th century the theory of congruences has been one of the most popular chapters of projective differential geometry. It was observed in [1, 2] that all standard concepts of the theory of conservation laws such as rarefaction curves, shock curves, linear degeneracy, reciprocal transformations, etc, acquire a simple intuitive interpretation in the language of the projective theory of congruences. Algebro-geometric aspects of the correspondence (1) $\leftrightarrow$ (2) were thoroughly investigated in [4, 5].

Particularly interesting examples of systems (1) arise in the context of equations of associativity of 2D topological field theory (WDVV equations) [7, 10]. Such systems can be represented in Hamiltonian form

$$u^i_t = P^{ij} \frac{\delta H}{\delta u^j} \tag{3}$$

where P^{ij} is a third-order Hamiltonian operator of special type and H is a (necessarily nonlocal) Hamiltonian, see [8] for further details. It was shown in [8] that if system (1) possesses Hamiltonian representation (3) then the associated congruence (2) must necessarily be linear, that is, defined by n linear equations in the Plücker coordinates. Explicitly, congruence (2) must satisfy the relations

$$tr Y A^i = 0, \tag{4}$$

$i = 1, \dots, n$, where Y is the $(n+2) \times (n+2)$ skew-symmetric matrix formed by 2×2 minors (Plücker coordinates) of the $2 \times (n+2)$ matrix

$$\begin{pmatrix} u^i & \dots & u^n & 1 & 0 \\ V^i & \dots & V^n & 0 & 1 \end{pmatrix},$$

and A^i are constant $(n+2) \times (n+2)$ skew-symmetric matrices. Note that system (4) can be viewed as a linear system for the fluxes V^i of system (1). Furthermore, viewed as 2-forms, A^i must satisfy an additional relation of the form

$$\varphi_{ij} A^i \wedge A^j = 0 \tag{5}$$

where the matrix φ is symmetric and non-degenerate.

Introducing an n-plane $H = span\langle A^i \rangle$ in $\wedge^2(V_{n+2})$, interpreting relation (5) as a kernel of the natural map $Sym^2 H \longrightarrow \wedge^4 V_{n+2}$, and φ as a non-degenerate quadratic form on H^*, we arrive at the algebraic problem formulated above. In this paper we will concentrate on the particularly interesting case $n = 4$.

2. Problem and strategy

2.1. **The problem.** As outlined in the introduction, we would like to classify the four-planes H in $\wedge^2 V_6$ such that

(1) the induced map $Sym^2 H \longrightarrow \wedge^4 V_6 \simeq \wedge^2 V_6^*$ has rank exactly nine,
(2) its kernel is generated by a non-degenerate quadratic form on H^*.

The variety $\mathcal{V}$ parametrizing those four-planes is a $PGL(V_6)$-invariant subvariety of the Grassmannian $G(4, \wedge^2 V_6)$, whose dimension is 44. The condition that $Sym^2 H \longrightarrow \wedge^4 V_6$ be of rank at most nine describes a closed subvariety $\mathcal{V}^*$ whose expected codimension is $15 - 10 + 1 = 6$, so the expected dimension is $38 = \dim PGL(V_6) + 3$. Then $\mathcal{V}$ is defined by two open conditions in $\mathcal{V}^*$, so it must be open in $\mathcal{V}^*$ but not necessarily dense since it is not clear that $\mathcal{W}$ is irreducible. More precisely, $\mathcal{V}$ is a union of dense open subsets of irreducible components of $\mathcal{V}^*$. Moreover all these components have dimension at least 38.

2.2. **Ranks.** Elements of $\wedge^2 V_6$ are skew-symmetric forms on V_6^*, for which we have the usual notion of rank. For convenience we will also use the following terminology.

Definition 2.1. *Let H be a subspace of $\wedge^2 V_6$. Let θ be an element of $Sym^2 H$.*

(1) The q-rank of θ is the rank of the corresponding quadratic form on H^*;
(2) The rank of θ is the rank of its image in $\wedge^4 V_6 \simeq \wedge^2 V_6^*$, considered as a skew-symmetric form on V_6.

For a given θ, the q-rank depends on H, contrary to the rank.

Note that a four-plane H in $\mathcal{V}$ cannot contain any form ω of rank two, since such a form verifies $\omega \wedge \omega = 0$, hence produces a degenerate element (of q-rank one) in the kernel of the map $Sym^2 H \longrightarrow \wedge^4 V_6$. As a consequence, H contains elements of rank six: it was proved in [9] that the linear spaces whose non zero elements all have rank four have dimension at most three, and such three-planes were classified.

2.3. **The birational involution.** Recall that the Pfaffian quadratic map

$$\begin{array}{rccc} pf : \wedge^2 V_6 & \longrightarrow & \wedge^4 V_6 \\ \omega & \mapsto & \omega \wedge \omega \end{array}$$

is birational, and essentially an involution. Indeed, if we identify $\wedge^4 V_6$ to $\wedge^2 V_6^*$, hence $\wedge^4 V_6^*$ to $\wedge^2 V_6$, we get a map

$$\begin{array}{rccc} pf^* : \wedge^4 V_6 & \longrightarrow & \wedge^2 V_6 \\ \omega^* & \mapsto & \omega^* \wedge \omega^*, \end{array}$$

and $pf^* \circ pf(\omega) = Pf(\omega)\omega$, where $Pf(\omega)$ is essentially $\omega \wedge \omega \wedge \omega$, or the Pfaffian of ω (which is defined only up to a non zero scalar). So if ω has rank six, we can recover it from $\omega \wedge \omega$ which has also rank six. Note that if ω has rank four (resp. two), then $\omega \wedge \omega$ has rank two (resp. zero). In particular, $\omega \wedge \omega$ never has rank four.

2.4. **Irreducibility.**

Theorem 2.2. *$\mathcal{V}$ is irreducible of dimension* 38.

Proof. Consider a four-plane H that belongs to $\mathcal{V}$, and denote by q a generator of the kernel of the map $Sym^2 H \to \wedge^4 V_6$. As a quadratic form on H^*, the tensor q is non degenerate, hence identifies H^* with H, yielding a quadratic form q^* on H.

As we observed, a generic ω in H has rank six and q-rank four. Let $L \subset H$ be its orthogonal with respect to q^*. Then we can write, up to scalar,

$$q = \omega^2 - \theta, \qquad \text{with} \quad \Omega \in Sym^2 L.$$

In other words, $\Omega = \omega \wedge \omega$ belongs to $Sym^2 L \simeq L \wedge L$ (the map $Sym^2 L \to L \wedge L$ is injective, hence an isomorphism, because its kernel is contained in the kernel of the map $Sym^2 H \longrightarrow \wedge^4 V_6$, which by hypothesis does not contain any non zero element of q-rank three or less). Moreover, since ω has rank six, Ω also has rank six, and therefore $\Omega \wedge \Omega$ is a non zero multiple of ω. In particular we can recover H from L and $\Omega \in Sym^2 L$.

This suggests to consider the diagram

$$\begin{array}{ccccc} \mathbb{P}(Sym^2 U) & \hookleftarrow & \mathcal{W} & \xrightarrow{\ \pi\ } & \mathcal{V} \subset G(4, \wedge^2 V_6), \\ \downarrow & & \downarrow{\scriptstyle p_2} & & \\ G(3, \wedge^2 V_6) & \hookleftarrow & \mathcal{T} & & \end{array}$$

our notations being the following. First, $\mathcal{T}$ is the open subset of $G(3, \wedge^2 V_6)$ parametrizing the three-planes L such that

(1) the map $Sym^2 L \longrightarrow \wedge^4 V_6$ is injective,
(2) the general element in $L \wedge L$ has rank six.

Second, U denotes the tautological rank three vector bundle on $G(3, \wedge^2 V_6)$ and $\mathcal{W}$ is the open subset of $\mathbb{P}(Sym^2 U)$ parametrizing pairs $(L, [\Omega])$ such that

(1) L belongs to $\mathcal{T}$,
(2) $\Omega \in Sym^2 L \simeq L \wedge L$ has maximal rank and q-rank,
(3) $(\omega \wedge L) \cap (L \wedge L) = 0$, where $\omega := \Omega \wedge \Omega$.

The map π from $\mathcal{W}$ to $\mathcal{V}$ sends the pair $(L, [\Omega])$ to $H = \langle L, \omega \rangle$. This is well-defined because

- ω cannot belong to L: otherwise, we would have $\omega \wedge \omega = Pf(\Omega)\Omega$, yielding the relation $\omega^2 = Pf(\Omega)\Omega$ in $Sym^2 L$, and then Ω would have q-rank one, a contradiction;
- the kernel of the map $Sym^2 H \to \wedge^4 V_6$ is generated by $q = \omega^2 - Pf(\Omega)\Omega$, which is non degenerate since Ω has q-rank three: indeed, if there was another element in this kernel, we could write it, after substracting a suitable multiple of q if necessary, as $q' = \omega\lambda - \Phi$ for some λ in L and some Φ in $Sym^2 L \simeq L \wedge L$; this would yield a relation $\omega \wedge \lambda = \Phi$ in $\wedge^4 V_6$, contradicting the hypothesis that $(\omega \wedge L) \cap (L \wedge L) = 0$.

By the remarks at the beginning of this proof, π is surjective, so the irreducibility of $\mathcal{V}$ will be a consequence of the irreducibility of $\mathcal{W}$. But the latter is obviously an open subset of the projective bundle $\mathbb{P}(Sym^2 U)$, hence certainly irreducible, of dimension $3 \times 12 + 5 = 41$. Finally, the fibers of π are open subsets in three-dimensional projective spaces, and therefore the dimension of $\mathcal{V}$ is $41 - 3 = 38$. □

2.5. **Characteristic surfaces.** For $H \in \mathcal{V} \subset G(4, \wedge^2 V_6)$, the locus of rank four two-forms defines a cubic surface $S_H \subset \mathbb{P}H$, that we call the *characteristic surface*. This surface is the intersection of $\mathbb{P}H$ with the Pfaffian hypersurface $Pf \subset \mathbb{P} \wedge^2 V_6$. Recall that the singular locus of this hypersurface is the Grassmannian $G(2, V_6)$, the locus of two-forms of rank two. Although $\mathbb{P}H$ does not meet this Grassmannian

when H belongs to $\mathcal{V}$, will we see later on that S_H can be badly singular, and even reducible. Nevertheless, we have a natural map

$$\sigma : \mathcal{V}/PGL(V_6) \dashrightarrow \mathcal{M}_c,$$

where $\mathcal{M}_c$ denotes the moduli space of cubic surfaces in $\mathbb{P}^3$. Both spaces are four-dimensional, and we will see later on that σ is dominant, hence generically finite.

Note that by construction, $H \in \mathcal{V}$ defines not only the cubic surface $S_H \subset \mathbb{P}H$, but also a Pfaffian representation of this surface. This is a classical topic. For example, any cubic surface (possibly singular) admits a Pfaffian representation [3, Proposition 7.6]. Moreover a generic such representation can be determined by five points in general position on the surface[3, Example 7.4]. A simple algorithm for that is presented in [12].

The family of Pfaffian representations of a smooth cubic surface is five dimensional. In comparison, there are only finitely many determinantal representations. They are in bijection with the 72 linear systems of twisted cubics on the surface, or the 72 sixes of non incident lines (this was already known in the 19th century, see [3, Corollary 6.4]).

3. Semisimple three planes

3.1. **A grading of $\mathfrak{e}_7$.** Although the classification of orbits in $G(4, \wedge^2 V_6)$ is not known (a priori a wild problem), the classification of orbits in $G(3, \wedge^2 V_6)$ is known. This is because $V_3 \otimes \wedge^2 V_6$ is part of a $\mathbb{Z}_3$-grading of $\mathfrak{e}_7$:

$$\mathfrak{e}_7 \simeq \mathfrak{sl}(V_3) \times \mathfrak{sl}(V_6) \oplus V_3 \otimes \wedge^2 V_6 \oplus V_3^* \otimes \wedge^2 V_6^*.$$

Then it makes sense to talk about semisimple or nilpotent elements in $V_3 \otimes \wedge^2 V_6$, by considering them as elements of $\mathfrak{e}_7$. The set of semisimple elements is the union of the Cartan subspaces, which are all equivalent under the action of $GL(V_3) \times GL(V_6)$. In order to exhibit one of those, we fix a basis v_1, v_2, v_3 of V_3, and a basis $e_1, \dots, e_6$ of V_6. We let $e_{ij} = e_i \wedge e_j \in \wedge^2 V_6$.

Proposition 3.1. *The subspace C generated by the following three vectors is a Cartan subspace of $V_3 \otimes \wedge^2 V_6$:*

$$c_1 = v_1 \otimes e_{12} + v_2 \otimes e_{34} + v_3 \otimes e_{56},$$

$$c_2 = v_1 \otimes e_{36} + v_2 \otimes e_{52} + v_3 \otimes e_{14},$$

$$c_3 = v_1 \otimes e_{54} + v_2 \otimes e_{16} + v_3 \otimes e_{32}.$$

We say that a three plane L in $\wedge^2 V_6$ is semisimple if it admits a basis $\omega_1, \omega_2, \omega_3$ such that $v_1 \otimes \omega_1 + v_2 \otimes \omega_2 + v_3 \otimes \omega_3$ is semisimple in $V_3 \otimes \wedge^2 V_6$. By the previous proposition, this is the case if and only if there exists a basis $e_1, \dots, e_6$ of V_6, and coefficients a, b, c, not all zero, such that

$$\omega_1 = ae_{12} + be_{36} + ce_{54},$$

$$\omega_2 = ae_{34} + be_{52} + ce_{16},$$

$$\omega_3 = ae_{56} + be_{14} + ce_{32}.$$

3.2. **Four planes containing a semisimple three plane.** . We would like to understand four planes $H \in \mathcal{V}$ that contain a semisimple three plane L. Necessarily, the map $Sym^2 L \longrightarrow \wedge^4 V_6$ must be injective. Note that

$$\frac{1}{2}\omega_1^2 = bce_{3654} + ace_{1254} + abe_{1236}, \qquad -\omega_2\omega_3 = a^2e_{3654} + b^2e_{1254} + c^2e_{1236},$$

$$\frac{1}{2}\omega_2^2 = bce_{1652} + ace_{1634} + abe_{3452}, \qquad -\omega_1\omega_3 = a^2e_{1652} + b^2e_{1634} + c^2e_{3452},$$

$$\frac{1}{2}\omega_3^2 = bce_{1432} + ace_{3256} + abe_{1456}, \qquad -\omega_2\omega_3 = a^2e_{1432} + b^2e_{3256} + c^2e_{1456},$$

so this injectivity condition is equivalent to the fact that the vectors (a^2, b^2, c^2) and (bc, ac, ab) are not colinear.

Now we would like to complete L with a vector $\omega_0 \notin L$ such that the four plane H they span belongs to $\mathcal{V}$. We can choose ω_0 such that it generates the orthogonal to L in H with respect to its non degenerate quadratic form (indeed, generically we may suppose that the restriction of this quadratic form to L is non degenerate). This implies that $\omega_0 \wedge \omega_0$ belongs to $L \wedge L$.

Now observe that we can decompose $V_6 = V_+ \oplus V_-$, where $V_+ = \langle e_1, e_3, e_5\rangle$ and $V_- = \langle e_2, e_4, e_6\rangle$. With respect to this decomposition, we have

$$\wedge^2 V_6 = \wedge^2 V_+ \oplus (V_+ \otimes V_-) \oplus \wedge^2 V_-,$$

and $L \subset V_+ \otimes V_-$. Let us decompose accordingly $\omega_0 = \omega_+ + \Omega + \omega_-$. We get

$$\frac{1}{2}\omega_0^2 = \omega_+\Omega + \Big(\omega_+\omega_- + \frac{1}{2}\Omega^2\Big) + \Omega\omega_-,$$

with respect to the decomposition

$$\wedge^4 V_6 = (\wedge^3 V_+ \otimes V_-) \oplus (\wedge^2 V_+ \otimes \wedge^2 V_-) \oplus (V_- \otimes \wedge^3 V_+).$$

Of course $L \wedge L$ is contained in $\wedge^2 V_+ \otimes \wedge^2 V_-$ so we need in particular that

$$\omega_+\Omega = \Omega\omega_- = 0.$$

Generically, Ω has rank three, and then the previous equations imply $\omega_+ = \omega_- = 0$. This means that $H \subset V_+ \otimes V_-$. Then H belongs to $\mathcal{W}$, since the image of $Sym^2 H$ is contained in $\wedge^2 V_+ \otimes \wedge^2 V_-$, which has dimension nine. One can check that a generic such H does belong to $\mathcal{V}$. In fact it suffices to produce an explicit example; here is one:

$$\omega_1 = e_{34} - e_{56},$$
$$\omega_2 = e_{52} - e_{14},$$
$$\omega_3 = e_{16} - e_{32},$$
$$\omega_0 = e_{12} + e_{14} + e_{32} + e_{36} + e_{54} - e_{56}.$$

The relation is

$$\omega_0^2 + \omega_1^2 - \omega_2^2 + \omega_3^2 + \omega_1\omega_2 + \omega_1\omega_3 - \omega_2\omega_3 = 0,$$

which is non degenerate.

When $H \subset V_+ \otimes V_-$, we get a 3×3 matrix of linear forms in four variables, and the characteristic surface S_H is given by the determinant of this matrix. Moreover the quadratic relation in $Sym^2 H$ is orthogonal to the image of the map

$$\wedge^2 V_+^* \otimes \wedge^2 V_-^* \longrightarrow Sym^2 H^*$$

given by the 2×2 minors of our matrix. In the classical terminology, the quadratic relation in $Sym^2 H$ is thus nothing else than the *Schur quadric* associated to the given determinantal representation of the cubic surface S_H. When this surface is smooth, it is known that the Schur quadric is unique and smooth (see [6, 9.1.3]). In particular this implies:

Theorem 3.2. *The image of the characteristic map $\sigma : \mathcal{V}/PGL(V_6) \longrightarrow \mathcal{M}_c$ contains the open subset parametrizing smooth cubic surfaces. In particular σ is dominant, hence generically finite.*

Remark. The stabilizer in $PGL(V_6)$ of a general point in $\mathcal{V}$ is the copy of $\mathbb{C}^*$ given by the automorphisms of the form $tId_{V_+} + Id_{V_-}$. This explains that the quotient $\mathcal{V}/PGL(V_6)$ has dimension four, rather than $3 = 38 - 35$.

We can also conclude from the previous discussion that in general, our special Pfaffian representations are in fact determinantal. Since σ is generically finite, this implies that $\mathcal{V}/PGL(V_6)$ is birational to the moduli space of determinantal representations of cubic surfaces. As a consequence, the degree of σ must be 72.

3.3. **More examples.** Consider the three-plane L generated by

$$\begin{aligned} \omega_0 &= e_0 \wedge e_2 + e_1 \wedge e_3, \\ \omega_1 &= e_0 \wedge e_4 + e_1 \wedge e_5, \\ \omega_2 &= e_0 \wedge e_1 + e_2 \wedge e_5 + e_3 \wedge e_4. \end{aligned}$$

This is triple tritangent plane, the triple line being generic. Let us complete it into a four-plane $H \in \mathcal{V}$. For that we consider a general element of $L \wedge L$,

$$\Omega = xf_{23} + yf_{45} + z(f_{25} - f_{34}) + a(f_{04} - f_{15}) + b(f_{02} - f_{13}) + c(f_{01} + f_{25} + f_{34}).$$

Then we need to compute $\frac{1}{2}\Omega^2$. After simplyfying a little bit the result by adding a suitable element of L, we get

$$\begin{aligned} \omega &= (xy - z^2)e_{01} + (yc + a^2)e_{23} + (xc + b^2)e_{45} + \\ &\quad + (yb + za)(e_{02} - e_{13}) + (zb + xa)(e_{04} - e_{15}) + (zc + ab)(e_{34} - e_{25}). \end{aligned}$$

Then we need to check that $H = \langle \omega, L \rangle$ belongs to $\mathcal{V}$ when the parameters are general. We know that ω^2 is a general element of $L \wedge L$, so the only thing we need to check is that the intersection of $L \wedge L$ with $\omega \wedge L$ is zero. Let $\Omega_1 = f_{04} + f_{15}$, $\Omega_2 = f_{02} + f_{13}$, $\Omega_3 = f_{34} + f_{25}$. Note the space $\langle \Omega_1, \Omega_2, \Omega_3 \rangle$ is transverse to $L \wedge L$. Moreover, modulo the latter we have

$$\begin{aligned} \omega\omega_0 &= -(zc + ab)\Omega_1 - (xc + b^2)\Omega_2 + (zb + xa)\Omega_3, \\ \omega\omega_1 &= -(yc + a^2)\Omega_1 - (zc + ab)\Omega_2 - (yb + za)\Omega_3, \\ \omega\omega_2 &= -(yb + za)\Omega_1 - (zb + xa)\Omega_2 - (zc + ab)\Omega_3. \end{aligned}$$

In general those three combinations of $\Omega_1, \Omega_2, \Omega_3$ are independent, and we are done.

Note that the characteristic surface S_H cannot be smooth. Indeed, by construction L is a triple tritangent plane, and therefore S_H must admit two singular points of type A_2, or even worse singularities (see e.g. [6, 9.2.2] Dolgachev).

Example 1. Let z, c be non zero and the other coefficients be zero, so (after factoring out a z) $\omega = -ze_{01} - c(e_{25} - e_{34})$. Here the (unique) quadratic relation is

$$\frac{z}{2}\omega^2 + c^2\omega\omega_2 = c(c^2 - z^2)\omega_0\omega_1 - \frac{zc^2}{2}\omega_2^2$$

and is non degenerate in general. The equation of the surface S_H is

$$Pf(t\omega + x_0\omega_0 + x_1\omega_1 + x_2\omega_2) = zc^2t^3 - c^2t^2x_2 - ztx_2^2 - 2ctx_0x_1 + x_2^3 = 0.$$

This surface has two singular points $[0,0,1,0]$ and $[0,1,0,0]$, each yielding an A_2 singularity.

Example 2. Let $b = y = 1$ and the other coefficients be zero, so $\omega = e_{02} - e_{13} + e_{45}$. Here $\Omega = f_{02} - f_{13} + f_{45}$, so the (unique) quadratic relation is

$$\frac{1}{2}\omega^2 = \omega_1\omega_2 + \frac{1}{2}\omega_0^2$$

and is non degenerate. The equation of the characteristic surface S_H is

$$Pf(t\omega + x_0\omega_0 + x_1\omega_1 + x_2\omega_2) = t^3 - tx_0^2 - 2tx_1x_2 + x_2^3 = 0.$$

This surface has a unique singularity at $[0,0,1,0]$ and this is an A_4 singularity.

3.4. **Special Pfaffian representations of Cayley's ruled surface.** It seems difficult to determine precisely what is the image of the characteristic map. We have just found some singular cubic surfaces with A_2 and even A_4 singularities, and we will consider reducible surfaces in the next section. Among irreducible surfaces, there are two types of non normal ones [6, Theorem 9.2.1], and the most degenerate one is *Cayley's ruled surface*, of equation

$$x_0^2x_2 + x_1^2x_3 = 0.$$

In this section we discuss the special Pfaffian representations of this surface.

So we consider $H = \langle\omega_0, \omega_1, \omega_2, \omega_3\rangle$ in $\mathcal{V}$ and suppose that its characteristic surface S_H has equation

$$Pf(x_0\omega_0 + x_1\omega_1 + x_2\omega_2 + x_3\omega_3) = x_0^2x_2 + x_1^2x_3.$$

In particular, among the wedge products $\omega_i\omega_j\omega_k$, only $\omega_0^2\omega_2$ and $\omega_1^2\omega_3$ are non zero. Since H belongs to $\mathcal{V}$, there exists a non-degenerate relation $q = \sum_{ij} q_{ij}\omega_i\omega_j = 0$. Taking the products with ω_k for $k = 0, \dots, 3$ we get $q_{00} = q_{02} = q_{11} = q_{13} = 0$. So the relation q must be a combination of $\omega_0\omega_1, \omega_0\omega_3, \omega_1\omega_2, \omega_2^2, \omega_2\omega_3, \omega_3$.

The pencil $\langle\omega_2, \omega_3\rangle$ is made of forms of constant rank four, and is therefore of one of the two possible types found in [9]:

$$\omega_2 = e_{02} + e_{13}, \qquad \omega_3 = e_{04} + e_{15}$$

for the generic type, will the special type is

$$\omega_2 = e_{02} + e_{13}, \qquad \omega_3 = e_{03} + e_{14}.$$

Note that the pencil $\langle\omega_2, \omega_3\rangle$ spans the singular locus of the characteristic surface. In particular it is uniquely defined by H and we call it *the singular pencil*.

Let us decompose the two other two-forms as

$$\omega_0 = ae_{01} + e_0 \wedge f_0 + e_1 \wedge f_1 + \mu, \qquad \omega_1 = be_{01} + e_0 \wedge g_0 + e_1 \wedge g_1 + \nu, \tag{6}$$

where $f_0, f_1, g_0, g_1, \mu, \nu$ do not involve e_0, e_1.

Lemma 3.3. *The singular pencil $\langle\omega_2, \omega_3\rangle$ must be special.*

Proof. We proceed by contradiction. Suppose the singular pencil is of generic type, and choose an adapted basis as above.

If $q_{01} \neq 0$, we can modify ω_0 and ω_1 by suitable linear combinations of ω_2 and ω_3, in such a way that the relation takes the form $q = q_{01}\omega_0\omega_1 + Q(\omega_2, \omega_3)$. (The equation of the characteristic surface itself will not change.) In particular, we get the relations

$$f_0 \wedge \nu + g_0 \wedge \mu = f_1 \wedge \nu + g_1 \wedge \mu = 0, \qquad \mu \wedge \nu = 0. \tag{7}$$

Suppose that the vectors f_0, f_1, g_0, g_1 are linearly dependant. Then there is a three-plane P such that f_0, f_1, g_0, g_1 belong to P and μ, ν belong to $\wedge^2 P$. But then $\omega_0 \wedge \omega_1 = e_{01} \wedge \theta$ for some θ in $\wedge^2 P$, which has therefore rank at most two. Now the linear relation q yields

$$q_{01}\theta = 2q_{22}e_{23} + 2q_{23}(e_{25} - e_{34}) + 2q_{33}e_{45}.$$

The right hand side is a form of rank at most two when $q_{23}^2 = q_{22}q_{33}$. But then q itself has q-rank at most three, a contradiction.

Suppose now that the vectors f_0, f_1, g_0, g_1 are linearly independant. Then the first two equations in (7) are verified if and only if we can write μ, ν in the form

$$\mu = xf_0 \wedge f_1 + y(f_0 \wedge g_1 + g_0 \wedge f_1) + zg_0 \wedge g_1,$$

$$\nu = yf_0 \wedge f_1 + z(f_0 \wedge g_1 + g_0 \wedge f_1) + tg_0 \wedge g_1.$$

Then the relation $\mu \wedge \nu = 0$ amounts to $xt = yz$. Moreover the conditions that $\omega_0^3 = \omega_1^3 = 0$ amount to $z = a(xz - y^2)$ and $y = b(yt - z^2)$. If $yz \neq 0$, then $ab \neq 0$ and we can write

$$\omega_0 = \frac{1}{a}(ae_0 - f_1) \wedge (ae_1 + f_0) + \frac{1}{z}(yf_0 + zg_0) \wedge (yf_1 + zg_1),$$

$$\omega_1 = \frac{1}{b}(be_0 - g_1) \wedge (be_1 + g_0) + \frac{1}{y}(yf_0 + zg_0) \wedge (yf_1 + zg_1).$$

Then the conditions $\omega_0^2\omega_1 = 0$ and $\omega_0\omega_1^2 = 0$ reduce to $ay + bz = 0$. In this case, both triples $\langle ae_0 - f_1, be_0 - g_1, yf_1 + zg_1\rangle$ and $\langle ae_1 + f_0, be_1 + g_0, yf_0 + zg_0\langle$ are linearly dependent. This implies that this six vectors generate a space $Q \subset V_6$ of dimension at most four. But then since ω_0 and ω_1 belong to $\wedge^2 Q$, the pencil they generate will necessarily contain a two-form of rank two, which is a contradiction.

We also need to consider the case where for example $y = 0$, hence also $bz = 0$. If $y = 0$ and $b = 0$, then the condition $\omega_0\omega_1^2 = 0$ amounts to $x = 0$, and then $z = a(xz - y^2) = 0$. If $y = z = 0$, the conditions that $\omega_0^2\omega_1 = 0$ and $\omega_0\omega_1^2 = 0$ imply that $x = t = 0$. But then all the two-forms in H are of the form $e_0 \wedge h_0 + e_1 \wedge h_1$, hence they have rank at most four and the rank will necessary drop somewhere, again a contradiction.

There remains to consider the case where $q_{01} = 0$. In that case we need $q_{03}q_{12}$ to be non zero for the q to remain non degenerate, and we can suppose that $q_{03} = q_{12} = 1$. Then the relation q implies that

$$\mu \wedge e_4 + \nu \wedge e_2 = \mu \wedge e_5 + \nu \wedge e_3 = 0. \tag{8}$$

On the other hand, the fact that the products of ω_0, ω_1 with ω_2^2, $\omega_2\omega_3$, ω_3^2 vanish imply that we can write μ and ν in the form

$$\mu = xe_{24} + y(e_{25} + e_{34}) + ze_{35}, \qquad \nu = x'e_{24} + y'(e_{25} + e_{34}) + z'e_{35}.$$

But then (8) readily imply that $\mu = \nu = 0$. We are then again in a situation where all the two-forms in H are of the form $e_0 \wedge h_0 + e_1 \wedge h_1$, which is not possible. $\square$

We can therefore suppose that the singular pencil $\langle \omega_2, \omega_3 \rangle$ is of special type, as given above. Then ω_2^2, $\omega_2\omega_3$, ω_3^2 give $e_{0123}, e_{0124}, e_{0134}$. Let us decompose ω_0 and ω_1 as in (6). The fact that their products with ω_2^2, $\omega_2\omega_3$, ω_3^2 vanish means that μ and ν do not involve e_5.

Suppose that $q_{01} \neq 0$, so that the relation q can be normalized as before to the form $q = \omega_0\omega_1 + Q(\omega_2, \omega_3)$. We deduce once again that $f_0 \wedge \nu + g_0 \wedge \mu = f_1 \wedge \nu + g_1 \wedge \mu = 0$, and moreover that $f_1 \wedge g_0 - f_0 \wedge g_1$ does not involve e_5. In particular we get the relations

$$f_{05}\nu + g_{05}\mu = f_{15}\nu + g_{15}\mu = 0, \qquad f_{15}g_0 - g_{05}f_1 - f_{05}g_1 + g_{15}f_0 = 0. \tag{9}$$

Note that μ and ν cannot both be zero, since otherwise all the two-forms in H are of the form $e_0 \wedge h_0 + e_1 \wedge h_1$, which is not possible. Hence the first two relations imply that $f_{05}g_{15} = f_{15}g_{05}$, and the third one, multiplied by f_{05}, yields

$$f_{15}(g_{05}f_0 + f_{05}g_0) = f_{05}(g_{05}f_1 + f_{05}g_1).$$

But then the two-form $g_{05}\omega_0 + f_{05}\omega_1$ can be expressed only in terms of e_0, e_1 and $g_{05}f_0 + f_{05}g_0$ (is $f_{05} \neq 0$, or $g_{05}f_1 + f_{05}g_1$ if $f_{15} \neq 0$), hence has rank at most two: a contradiction!

If $f_{05} = f_{15} = 0$, then g_{05} and g_{15} cannot both be zero, for otherwise the two-forms in H would not involve e_5 at all, and thus the rank would be at most four on H and would have to drop somewhere. Then the equations (9) imply that $\mu = 0$ and that f_0 and f_1 are proportional. But then ω_0 can be written in terms of three vectors only, and thus has rank two, a contradiction again.

Finally, suppose that $q_{01} = 0$, and then normalize as before the relation q to $q = \omega_0\omega_3 + \omega_1\omega_2 + Q(\omega_2, \omega_3)$, which is of maximal q-rank for any Q. The existence of such a relation is equivalent to the equations

$$\mu \wedge e_3 + \nu \wedge e_2 = \mu \wedge e_4 + \nu \wedge e_3 = 0, \quad f_{05} = g_{15} = f_{15} - g_{05} = 0. \tag{10}$$

The fact that the pencil $\langle \omega_0, \omega_1 \rangle$ contains only forms of rank four is equivalent to the following conditions:

$$\mu \wedge f_0 = \nu \wedge g_1 = 0, \quad \nu \wedge f_0 = \mu \wedge (f_1 - g_0), \quad \mu \wedge g_1 = \nu \wedge (g_0 - f_1). \tag{11}$$

Moreover we need $\omega_0^2\omega_3 = \omega_1^2\omega_2 = 0$, which amount to $\mu \wedge_3 = \nu \wedge e_3 = 0$, and $\omega_0\omega_1\omega_2 = \omega_0\omega_1\omega_3 = 0$, which amount to $\mu \wedge e_4 = \nu \wedge e_2 = 0$. Therefore there exists two scalars m, n such that $\mu = me_{34}$ and $\nu = ne_{23}$, and the first two equations of (10) are then verified. Moreover $\omega_0^2\omega_2 = \omega_1^2\omega_3 \neq 0$ if and only if $m = -n \neq 0$. The conditions (11) then reduce to

$$g_{14} = f_{02} = 0, \qquad f_{04} = g_{02} - f_{12}, \qquad g_{12} = f_{14} - g_{04}. \tag{12}$$

We also need that $f_{15} = g_{05} \neq 0$ (for otherwise e_5 is not involved at all and the rank has to drop somewhere). If those conditions are satisfied, then the characteristic surface has the correct equation and the relation q is verified for some Q. The last condition to check, in order to get a four plane H that belongs to $\mathcal{V}$, is that this is the only relation in $Sym^2 H$.

Lemma 3.4. *There is no other relation than q.*

Proof. We grade $\wedge^4 V_6$ by the degrees on e_0, e_1, respectively e_2, e_3, e_4, and e_5. For example ω_2^2, $\omega_2\omega_3$, ω_3^2 span the space of forms of multidegree $(2,2,0)$. We note that among the binomials in the ω_i, the only ones having terms of multidegree $(1,2,1)$ are ω_0^2 (which gives e_{1345}), ω_1^2 (which gives e_{0235}), and $\omega_0\omega_1$ (which gives $e_{0345} + e_{1235}$); those three terms are independent. Moreover, among the remaining monomials, only $\omega_0\omega_3$, $\omega_1\omega_2$, $\omega_0\omega_2$, $\omega_1\omega_3$ have terms of multidegree $(2,1,1)$ (e_{0135} for the first two, e_{0125} for the third one, e_{0145} for the last one). This readily implies that the dimension of $H \wedge H$ is nine. □

Multiplying e_0, e_1 by m and rescaling, we may suppose that $m = 1$. We get the following result:

Theorem 3.5. *Suppose that the characteristic surface S_H of a four plane H in $\mathcal{V}$ is Cayley's ruled cubic surface. Then there exists a basis of V_6, and coefficients A, B, C, D, E, such that*

$$\omega_0 = e_{15} + e_{34} + Ae_{03} + 2Be_{04} - Be_{12} + Ce_{13} + De_{14},$$
$$\omega_1 = e_{05} - e_{23} + Be_{02} + Ee_{03} - De_{04} + 2De_{12} + Ee_{13},$$
$$\omega_2 = e_{02} + e_{13}, \qquad \omega_3 = e_{03} + e_{14}.$$

Conversely, any such four plane belongs to $\mathcal{V}$ and its characteristic surface is Cayley's ruled surface.

In particular, letting $A = B = C = D = E = 0$ we get the following four plane of skew-symmetric matrices:

$$\begin{pmatrix} 0 & 0 & c & d & 0 & b \\ 0 & 0 & 0 & c & d & a \\ -c & 0 & 0 & -b & 0 & 0 \\ -d & -c & b & 0 & a & 0 \\ 0 & -d & 0 & -a & 0 & 0 \\ -b & -a & 0 & 0 & 0 & 0 \end{pmatrix}$$

The dependance relation is simply $q = \omega_0\omega_3 + \omega_1\omega_2 = 0$.

3.5. **Cones.** Our final result in this section shows that the characteristic map is not surjective:

Proposition 3.6. *Let $H \in \mathcal{V}$ be such that the characteristic surface S_H is a cone. Then S_H is reducible.*

Proof. Suppose that S_H is a cone over ω_0. In particular $\omega_0^3 = 0$, so ω_0 has rank four and there is a unique $V_4 \subset V_6$ such that ω_0 belongs to $\wedge^2 V_4$. Moreover ω_0^2 is a generator of $\wedge^4 V_4$, and the fact that $\omega_0^2\omega = 0$ for all $\omega \in H$ is equivalent to the fact that $H \subset V_4 \wedge V_6$. More concretely, we can complete V_4 with two independent vectors e_5, e_6 such that every $\omega \in H$ can be written in the form

$$\omega = e_5 \wedge \alpha + e_6 \wedge \beta + \theta,$$

where α, β belong to V_4 and θ to $\wedge^2 V_4$.

Lemma 3.7. *The characteristic surface S_H is a cone if and only if ω_0 is H-symmetric, in the sense that $\omega_0 \wedge \alpha \wedge \beta = 0$ for all $\omega = e_5 \wedge \alpha + e_6 \wedge \beta + \theta \in H$.*

A convenient way to use this lemma is to denote $\tilde{\omega}_0(\alpha, \beta) = \omega_0 \wedge \alpha \wedge \beta$, where $\tilde{\omega}_0$ is now a non degenerate skew-symmetric form on V_4 (although defined only up to scalar).

4. The reducible case

In this section we discuss the case where the characteristic surface S_H is a reducible cubic. Equivalently, S_H contains a plane P. Since H belongs to $\mathcal{V}$, this plane must be made of two-forms of constant rank four. Such planes have been classified in [9]; up to the action of PGL_6 there are exactly four different types, which we consider case by case.

Type 1. The first type consists in three-planes contained in $\wedge^2\mathbb{C}^5$; it is represented by the plane L generated by

$$\begin{aligned}\omega_1 &= e_1\wedge e_4 + e_2\wedge e_3,\\ \omega_2 &= e_1\wedge e_5 + e_2\wedge e_4,\\ \omega_3 &= e_2\wedge e_5 + e_3\wedge e_4.\end{aligned}$$

Note that $Sym^2L\subset\wedge^4\mathbb{C}^5\subset\wedge^4\mathbb{C}^6$, and therefore any tensor in $L\wedge L$ has rank two. Suppose there exists $\lambda\in L$ and $\omega\in\wedge^2\mathbb{C}^6$ such that $\omega\wedge\lambda$ belongs to $L\wedge L$. Then necessarily ω belongs to $\wedge^2\mathbb{C}^5$. But then H must be a four plane in$\wedge^2\mathbb{C}^5$, in which the variety of tensors of rank at most two has codimension 3. So H necessarily contains some tensors of rank two, a contradiction.

Now suppose that $\omega\wedge\omega$ belongs to $L\wedge L$. Let us decompose $\omega=\phi+\alpha\wedge e_6$. Then $\omega\wedge\omega=\phi\wedge\phi+2\phi\wedge\alpha\wedge e_6$. In particular we need $\phi\wedge\alpha=0$. If $\alpha=0$, then ω belongs to $\wedge^2\mathbb{C}^5$ and we get a contradiction as before. Otherwise, α must divide ϕ and then ω has rank two, a contradiction again. We conclude that no three plane of type 1 can be contained in a four plane of $\mathcal{V}$.

Type 2. The second type is provided by the plane L generated by

$$\begin{aligned}\omega_1 &= e_0\wedge e_4 - e_1\wedge e_3,\\ \omega_2 &= e_0\wedge e_5 - e_2\wedge e_3,\\ \omega_3 &= e_1\wedge e_5 - e_2\wedge e_4,\end{aligned}$$

An easy computation shows that the L is stabilized by a subgroup $Stab_L$ of GL_6 isomorphic to $\mathbb{C}^*\times GL_3$. Moreover the action of $Stab_L$ on L is equivalent to the action of $\mathbb{C}^*\times GL_3$ on $A_3(\mathbb{C})$, the space of 3×3 skew-symmetric matrices, given by $(z,A):X\mapsto zA^tXA$. In particular the action of $Stab_L$ on L has exactly two orbits, the origin and its complement; and the action of $Stab_L$ on Sym^2L has exactly four orbits, given by the q-rank.

For H to belong to $\mathcal{V}$, we need $\omega\wedge\omega$ to be an element of $L\wedge L$ of q-rank three. We may suppose this element, up to the action of $Stab_L$, to be

$$-\frac{1}{2}(\omega_1^2+\omega_2^2+\omega_3^2) = e_{0134}+e_{0235}+e_{1245}.$$

Then we recover ω by applying pf^*, which yields (up to scalar)

$$\omega = e_0\wedge e_3 + e_1\wedge e_4 + e_2\wedge e_5.$$

We also need to consider the case where there exists $\lambda\in L$ such that $\omega\wedge\lambda$ belongs to $L\wedge L$. Up to the action of $Stab_L$ we may suppose that $\lambda=\omega_1$. Then a straightforward computation shows that (modulo L) ω must be a combination of $e_0\wedge e_1,\ e_0\wedge e_3,\ e_1\wedge e_4,\ e_3\wedge e_4$. In particular ω belongs to $\wedge^2M$, where $M=\langle e_0,e_1,e_3,e_4\rangle$. Therefore the line joining ω to ω_1, which also belongs to $\wedge^2M$, will necessary contain an element of rank two, and we get a contradiction.

The conclusion of this discussion is that up to equivalence, a three plane L of type two can be uniquely extended to a four plane H in $\mathcal{V}$, which in a suitable basis is the four plane of matrices of the form:

$$\begin{pmatrix} 0 & 0 & 0 & d & a & b \\ 0 & 0 & 0 & -a & d & c \\ 0 & 0 & 0 & -b & -c & d \\ d & a & b & 0 & 0 & 0 \\ -a & d & c & 0 & 0 & 0 \\ -b & -c & d & 0 & 0 & 0 \end{pmatrix}$$

The associated cubic surface has equation $d(a^2+b^2+c^2+d^2)=0$. It is the union of the plane L and of a smooth quadric.

Type 3. The third type is provided by the plane L generated by

$$\begin{aligned} \omega_1 &= e_0\wedge e_2 + e_1\wedge e_3, \\ \omega_2 &= e_0\wedge e_3 + e_1\wedge e_4, \\ \omega_3 &= e_0\wedge e_4 + e_1\wedge e_5, \end{aligned}$$

In this case, $L\wedge L \simeq Sym^2 L$ is the space of four-forms divisible by $e_0\wedge e_1$. In particular $L\wedge L$ does not contain any rank six element, so our general strategy does not apply.

Instead, let us try directly to complete L into a four-plane $H\in\mathcal{V}$, with some two-form ω such that $\omega\wedge\omega$ belongs to $L\wedge L$. We can decompose $\omega = e_0\wedge u_0 + e_1\wedge u_1 + \psi$, where ψ does not involve e_0, e_1. Then $\omega\wedge\omega$ is divisible by $e_0\wedge e_1$ if and only if

$$u_0\wedge\psi = u_1\wedge\psi = 0 \quad \text{and} \quad \psi\wedge\psi = 0.$$

This means that ψ has rank two (ψ cannot be zero, for otherwise $H\wedge H = L\wedge L$) and is divisible by u_0 and u_1. Note that u_0, u_1 cannot be dependent, since otherwise ω would have rank two. So ψ must be a multiple of $u_0\wedge u_1$ and after a suitable normalization we can suppose that

$$\omega = e_0\wedge u_0 + e_1\wedge u_1 + u_0\wedge u_1.$$

Note that an element in L is of the form $\lambda = e_0\wedge\ell_0 + e_1\wedge\ell_1$, and the planes $\langle \ell_0, \ell_1\rangle$ span a Veronese surface in $G(2,5)$. It is easy to see that $(\omega\wedge L)\cap(L\wedge L) = 0$ if and only if the plane $\langle u_0, u_1\rangle$ does not belong to this surface. This ensures that the kernel of the map $Sym^2 H \to H\wedge H$ is one-dimensional. For H to belong to $\mathcal{V}$, we finally need that a generator of this kernel be of maximal q-rank, which is easily checked to be true in general: take for example $u_0 = e_5$ and $u_1 = e_2$, then the relation is

$$\omega^2 + \omega_1\omega_3 + \omega_2^2 = 0,$$

which is non degenerate. The four-plane of skew-symmetric matrices is:

$$\begin{pmatrix} 0 & 0 & b & c & d & a \\ 0 & 0 & a & b & c & d \\ -b & -a & 0 & 0 & 0 & -a \\ -c & -b & 0 & 0 & 0 & 0 \\ -d & -c & 0 & 0 & 0 & 0 \\ -a & -d & a & 0 & 0 & 0 \end{pmatrix}$$

A general point in H is of the form $\phi = e_0\wedge(su_0+\ell_0) + e_1\wedge(su_1+\ell_1) + su_0\wedge u_1$, and has rank four or less when $s\ell_0\wedge\ell_1\wedge u_0\wedge u_1 = 0$. This implies that the

characteristic surface is the union of L and a cone over a conic in L. This conic may be singular: consider for example the case where $u_0 = e_5$ and $u_1 = e_3$; we get the following four plane of skew-symmetric matrices:

$$\begin{pmatrix} 0 & 0 & b & c & d & a \\ 0 & 0 & 0 & a+b & c & d \\ -b & 0 & 0 & 0 & 0 & 0 \\ -c & -a-b & 0 & 0 & 0 & -a \\ -d & -c & 0 & 0 & 0 & 0 \\ -a & -d & 0 & a & 0 & 0 \end{pmatrix}$$

Type 4. The fourth type is represented by the plane L generated by

$$\begin{aligned} \omega_1 &= e_0 \wedge e_3 + e_1 \wedge e_2, \\ \omega_2 &= e_0 \wedge e_4 + e_2 \wedge e_3, \\ \omega_3 &= e_0 \wedge e_5 + e_1 \wedge e_3. \end{aligned}$$

Then $L \wedge L \simeq Sym^2 L$ is generated by $f_{14} + f_{25}, f_{24}, f_{34}, f_{15}, f_{35}, f_{45}$. In particular an element of $L \wedge L$ can be written as $f_4 \wedge g_4 + f_5 \wedge g_5$, hence never has rank six.

So let us directly try to complete L into a four-plane $H \in \mathcal{V}$ with a two-form ω such that $\omega \wedge \omega$ belongs to $L \wedge L$. Let us decompose

$$\omega = \psi + u_4 \wedge e_4 + u_5 \wedge e_5 + z e_4 \wedge e_5,$$

where ψ, u_4, u_5 do not involve e_4, e_5. Then we have

$$\frac{1}{2}\omega \wedge \omega = \frac{1}{2}\psi \wedge \psi + \psi \wedge u_4 \wedge e_4 + \psi \wedge u_5 \wedge e_5 + (z\psi - u_4 \wedge u_5) \wedge e_4 \wedge e_5.$$

For this to belong to $L \wedge L$, we need that $z\psi = u_4 \wedge u_5$. If $z \neq 0$, we simply get that $\omega \wedge \omega = 0$, which is excluded. If $z = 0$, then u_4 and u_5 must be colinear, and after a change of basis we may suppose that $u_5 = 0$. Then we need $\psi \wedge u_4$ to be a combination of e_{012} and e_{023}, in which case $\psi \wedge u_4 \wedge e_4$ is a linear combination of $\omega_1 \wedge \omega_2$ and $\omega_2 \wedge \omega_2$. But then the kernel of $Sym^2 H \longrightarrow H \wedge H$ is generated by a quadratic relation of the form $\omega^2 = Q(\omega_1, \omega_2)$, which contradicts the non degeneracy condition.

There remains the possibility that $(\omega \wedge L) \cap (L \wedge L) \neq 0$. Another computation leads to the same conclusion.

We summarize this discussion as follows:

Theorem 4.1. *Let $H \in \mathcal{V}$ be such that the characteristic surface S_H contains a plane L. Then either:*

(1) L is of type 2 and the residual component of S_H is a smooth quadric;
(2) L is of type 3 and the residual component of S_H is a quadratic cone, whose vertex is outside P.

References

[1] S.I. Agafonov and E.V. Ferapontov, *Systems of conservation laws from the point of view of the projective theory of congruences*, Izv. RAN, ser. mat. **60** (1996) N.6, 3–30.

[2] S. I. Agafonov and E. V. Ferapontov, *Systems of conservation laws of Temple class, equations of associativity and linear congruences in P^4*, Manuscripta Math. **106**, p. 4 (2001) 461–488.

[3] Beauville A., *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64.

[4] P. De Poi, E. Mezzetti, *Linear congruences and hyperbolic systems of conservation laws. Projective varieties with unexpected properties*, 209–230, Walter de Gruyter GmbH & Co. KG, Berlin, 2005.

[5] P. De Poi, E. Mezzetti, *Congruences of lines in $\mathbb{P}^5$, quadratic normality, and completely exceptional Monge-Ampère equations*, Geom. Dedicata **131** (2008) 213–230.
[6] Dolgachev I., Classical Algebraic Geometry, a modern view. Cambridge University Press, 2012.
[7] *E.V. Ferapontov, C.A.P. Galvao, O. Mokhov, Y. Nutku*, Bi-Hamiltonian structure of equations of associativity in 2-d topological field theory, Comm. Math. Phys. **186** (1997) 649-669.
[8] E.V. Ferapontov, M.V. Pavlov and R.F. Vitolo, *Systems of conservation laws with third-order Hamiltonian structures*, Lett. Math. Phys. **108**, no. 6 (2018) 1525–1550.
[9] Manivel L., Mezzetti E., *On linear spaces of skew-symmetric matrices of constant rank*, Manuscripta Math. **117** (2005) 319–331.
[10] *M.V. Pavlov, R.F. Vitolo*, On the bi-Hamiltonian geometry of WDVV equations, Lett. Math. Phys. **105**, no. 8 (2015) 1135–1163.
[11] *D. Serre*, Systems of conservation laws. 1, 2. Cambridge University Press, Cambridge, 1999-2000.
[12] Tanturri F., *Pfaffian representations of cubic surfaces*, Geom. Dedicata **168** (2014) 69–86.

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