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Bilal Al Taki

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NEW FUNCTIONAL INEQUALITY AND ITS APPLICATION

B. AL TAKI

ABSTRACT. In this short note, we prove by simple arguments a new kind of Logarithmic Sobolev inequalities generalizing two known inequalities founded in some papers related to fluid dynamics models (see for instance [6] and [3]). As a by product, we show how our inequality can help in obtaining some important a priori estimates for the solution of the Navier-Stokes-Korteweg system.

Keywords. Logarithmic Sobolev inequalities, Navier-Stokes-Korteweg equations, Quantum Navier-Stokes equations, A priori estimates.

AMS subject classification: 25A23, 35B45, 35Q30.

1. INTRODUCTION

Sobolev inequalities have played a fundamental role in the study of solution's existence for different kinds of partial differential equations. A distinguished type of these inequalities is the logarithmic type which can be reads as follows (see [5])

$$(1.1) \quad \int_{\Omega} |\nabla \varrho|^2 dx \geq C \int_{\Omega} (\varrho^2 \log \varrho - \varrho^2) dx$$

for ϱ sufficiently smooth function and Ω is a bounded smooth domain. Notice that, logarithmic Sobolev inequalities are amongst the most studied functional inequalities in Semigroups (see [5], [1]). They contain much more information than Poincaré inequalities and are at the same time sufficiently general to be available in numerous cases of interest, in particular in infinite dimensions (as limits of Sobolev inequalities on finite-dimensional spaces). Here, we prove a kind of Logarithmic Sobolev inequality that has an important feature on a mathematical model related to fluid dynamics. More precisely, the aim of this note is the following theorem¹.

Theorem 1.1. *Suppose that ρ is sufficiently smooth and n, m are two constants satisfy the following conditions*

$$(1.2) \quad n > 0 \quad m > 0 \quad \gamma_1 \gamma_2 (1 + n) + 2(\gamma_1 + \gamma_2)(c_n - 1 - n) > 0$$

where

$$\gamma_1 = \frac{4(m+1)}{2n+m+1} \quad \gamma_2 = \frac{4(2n-m+1)}{2n+m+1}$$

and

$$c_n = \frac{(1+n)(d+2)(d(1-n)+2n) - (d-1)^2(2n-1)^2}{(d+2)^2(1+n)},$$

then there exist two positive constants c_1 and c_2 such that

$$(1.3) \quad \begin{aligned} \mathcal{I} &= \int_{\Omega} \rho^{n+1} \nabla \nabla \rho^n : \nabla \nabla \rho^m dx + n \int_{\Omega} \rho^{n+1} \Delta \rho^n \Delta \rho^m dx \\ &\geq c_1 \int_{\Omega} |\nabla \nabla \rho^{\frac{2n+m+1}{2}}|^2 dx + c_2 \int_{\Omega} |\nabla \rho^{\frac{2n+m+1}{4}}|^4 dx. \end{aligned}$$

Remark 1.1. *Notice that this inequality can be viewed as a generalization of two known inequalities. The first one was proved by A. JÜNGEL and D. MATTEWS in [6]*

$$(1.4) \quad \int_{\Omega} \rho^2 \nabla \nabla \log \rho : \nabla \nabla \rho^m dx \gtrsim \int_{\Omega} |\Delta \rho^{\frac{m+2}{2}}|^2 dx \quad -2 < m < \frac{2d}{d+2}.$$

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¹To see the link between Inequality (1.1) and our Inequality (1.3), reader is invited to consult Remark 1.3

Indeed, using the fact that $\nabla \log \rho^\alpha = \alpha \nabla \log \rho$, then we can write ($r = \rho^2$)

$$(1.5) \quad \begin{aligned} \int_{\Omega} \rho^2 \nabla \nabla \log \rho : \nabla \nabla \rho^m dx &= \int_{\Omega} r \nabla \nabla \log r^{\frac{1}{2}} : \nabla \nabla r^{\frac{m}{2}} dx \\ &= \frac{1}{2} \int_{\Omega} r \nabla \nabla \log r : \nabla \nabla r^{\frac{m}{2}} dx. \end{aligned}$$

Now, we can apply our Inequality (1.3) to deduce that²(we can view $\nabla \rho^n \simeq \rho^{n-1} \nabla \rho$ as $\nabla \log \rho$ for $n = 0$)

$$\int_{\Omega} r \nabla \nabla \log r : \nabla \nabla r^{\frac{m}{2}} dx \gtrsim \int_{\Omega} |\nabla \nabla r^{\frac{m+2}{4}}|^2 dx + \int_{\Omega} |\nabla r^{\frac{m+2}{8}}|^4 dx.$$

Therefore, using (1.5) and the fact that the domain is periodic, we can write

$$\begin{aligned} \int_{\Omega} \rho^2 \nabla \nabla \log \rho : \nabla \nabla \rho^m dx &\gtrsim \int_{\Omega} |\nabla \nabla \rho^{\frac{m+2}{2}}|^2 dx + \int_{\Omega} |\nabla \rho^{\frac{m+2}{4}}|^4 dx \\ &= \int_{\Omega} |\Delta \rho^{\frac{m+2}{2}}|^2 dx + \int_{\Omega} |\nabla \rho^{\frac{m+2}{4}}|^4 dx. \end{aligned}$$

Notice that the extra term in our inequality compared to inequality (1.4) comes from the fact that we use a method different to that introduced in [6].

The second one was proved by D. BRESCH, A. VASSEUR and C. YU in [3]

$$(1.6) \quad \int_{\Omega} \rho^{n+1} |\nabla \nabla \rho^n|^2 dx \gtrsim \int_{\Omega} |\nabla \nabla \rho^{\frac{3n+1}{2}}|^2 dx + \int_{\Omega} |\nabla \rho^{\frac{3n+1}{4}}|^4 dx \quad \frac{2}{d} - 1 < n < 1$$

which corresponds to take $n = m$ in our inequality (1.3).

Remark 1.2. Unfortunately, it seems complicated to interpret the condition (1.2) algebraically. For that, in Picture 1 below, we will show geometrically in which zone inequality (1.3) holds.

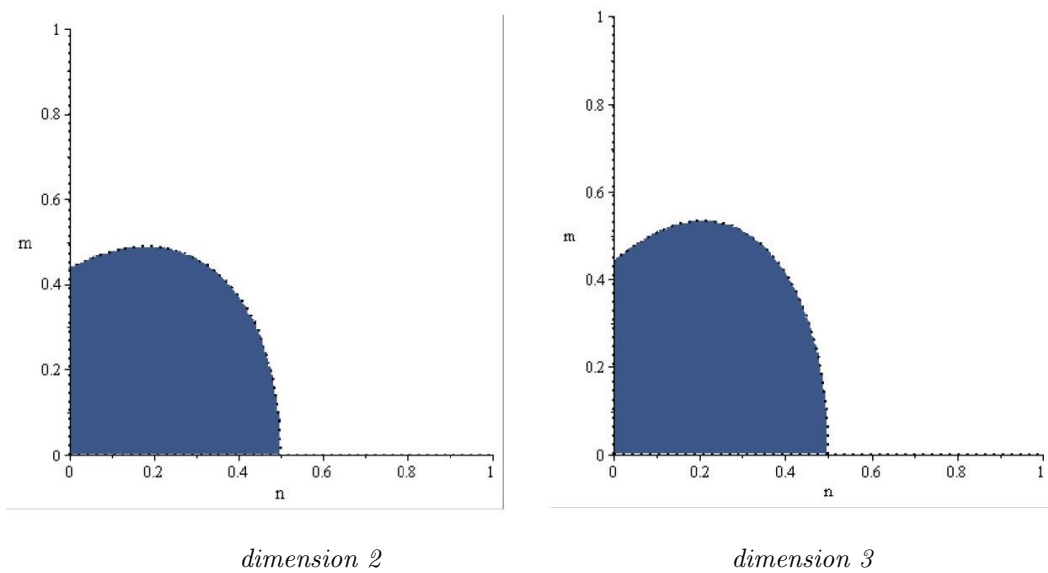


FIGURE 1

Remark 1.3. In order to make the link between the well known Logarithmic Sobolev inequality (1.1) and our Inequality (1.3), let us see that if we take for example $n = m = 0$ in (1.3), we get (taking in mind that $\nabla \rho^n \sim \nabla \log \rho$ when $n = 0$)

$$\int_{\Omega} \rho |\nabla \nabla \log \rho|^2 dx \gtrsim \int_{\Omega} |\nabla(\sqrt{\rho} \nabla \log \rho)|^2 dx + \int_{\Omega} \rho |\nabla \log \rho|^4 dx$$

²we pay no attention on the constant here

which is can be view as higher version of (1.1) (taking $\varrho := \sqrt{\rho}$ in (1.1))

$$\int_{\Omega} \rho |\nabla \log \rho|^2 dx \gtrsim \int_{\Omega} (\rho \log \sqrt{\rho} - \rho) dx.$$

Before proving our main theorem, we shall firstly prove the following Lemma.

Lemma 1.1. *Suppose that ρ is sufficiently smooth, $n > -1$ and the positive constant $c = c(n, d)$ is such that*

$$0 < c \leq \frac{(1+n)(d+2)(d(1-n)+2n) - (d-1)^2(2n-1)^2}{(d+2)^2(1+n)},$$

then we have

$$(1.7) \quad J = \int_{\Omega} \rho^2 |\nabla^2 \log \rho|^2 dx + n \int_{\Omega} \rho^2 |\Delta \log \rho|^2 dx \geq c \int_{\Omega} |2\nabla \sqrt{\rho}|^4 dx.$$

Proof. The proof is inspired by the extension of the entropy construction method introduced in [6]. To simplify the computations, we keep the same notation introduced in [6]

$$\theta = \frac{|\nabla \rho|}{\rho}, \quad \lambda = \frac{1}{d} \frac{\Delta \rho}{\rho}, \quad (\lambda + \xi)\theta^2 = \frac{1}{\rho^3} \nabla^2 \rho : (\nabla \rho)^2,$$

and $\eta \geq 0$ by

$$\|\nabla^2 \rho\|^2 = (d\lambda^2 + \frac{d}{d-1}\mu^2 + \eta^2)\rho^2.$$

We compute J using the above notation to obtain

$$J = \int_{\Omega} \rho^2 \left((1+nd)d\lambda^2 + \frac{d}{d-1}\xi^2 + \eta^2 - 2\lambda\theta^2(1+nd) - 2\xi\theta^2 + (1+n)\theta^4 \right) dx$$

We need to compare J to

$$K = 16 \int_{\Omega} |\nabla \sqrt{\rho}|^4 dx = \int_{\Omega} \rho^2 \theta^4 dx.$$

We shall rely on the following two dummy integral expressions:

$$F_1 = \int_{\Omega} \operatorname{div}((\nabla^2 \rho - \Delta \rho \mathbb{I}) \cdot \nabla \rho) dx,$$

$$F_2 = \int_{\Omega} \operatorname{div}(\rho^{-1} |\nabla \rho|^2 \nabla \rho) dx,$$

where I is the unit matrix in $\mathbb{R}^d \times \mathbb{R}^d$. Obviously, in view of the boundary conditions, $F_1 = F_2 = 0$. Our purpose now is to find constants c_0, c_1 and c_2 such that $J - c_0 K = J - c_0 K + c_1 F_1 + c_2 F_2 \geq 0$. The computation in [6] yields

$$F_1 = \int_{\Omega} \rho^2 \left(-d(d-1)\lambda^2 + \frac{d}{d-1}\xi^2 + \eta^2 \right) dx,$$

$$F_2 = \int_{\Omega} v^{2\gamma} \left((d+2)\lambda\theta^2 + 2\xi\theta^2 - \theta^4 \right) dx.$$

After simple calculation, we obtain that

$$\begin{aligned} J - c_0 K + c_1 F_1 + c_2 F_2 &= \int_{\Omega} \left[((1+nd) - c_1(d-1))d\lambda^2 + \frac{d}{d-1}(1+c_1)\xi^2 + \eta^2(1+c_1) \right. \\ &\quad \left. + \lambda\theta^2(-2(1+nd) + c_2(d+2)) + 2\xi\theta^2(c_2-1) + \theta^4(1+n-c_0-c_2) \right] dx. \end{aligned}$$

We tend to eliminate λ from the above integrand by defining c_1 and c_2 appropriately. The linear system

$$\begin{aligned} (1+nd) - c_1(d-1) &= 0, \\ -2(1+nd) + c_2(d+2) &= 0, \end{aligned}$$

has the solution

$$c_1 = \frac{(1+nd)}{d-1}, \quad c_2 = 2 \frac{(1+nd)}{d+2}.$$

Therefore we deduce that

$$J = \int_{\Omega} \rho^2 (b_1 \xi^2 + 2b_2 \xi \theta^2 + b_3 \theta^4 + b_4 \eta^2) dx$$

where we defined b_1, b_2, b_3 and b_4 as follows

$$b_1 = \frac{d^2}{(d-1)^2}(1+n) \quad b_2 = \frac{d}{(d+2)}(2n-1) \quad b_3 = \frac{d-nd+2n}{d+2} - c_0 \quad b_4 = \frac{d}{d-1}(1+n).$$

This integral is non negative if the integrand is pointwise non negative. This is the case if and only if

$$b_1 > 0 \quad b_4 > 0 \quad \text{and} \quad b_1 b_3 - b_2^2 \geq 0,$$

which is equivalent to

$$c_0 \leq \frac{(1+n)(d+2)(d(1-n)+2n) - (d-1)^2(2n-1)^2}{(d+2)^2(1+n)}. \quad \square$$

Now we are able to prove our main result, namely Theorem 1.1. Notice that, the procedure of proof introduced here is new and simple compared to that used by A. JÜNGEL and D. MATTHES in [6].

Proof of Theorem 1.1. First, we denote

$$J_1 := \int_{\Omega} \rho^{n+1} \nabla \nabla \rho^n : \nabla \nabla \rho^m dx \quad J_2 := \int_{\Omega} \Delta \rho^n \Delta \rho^m dx.$$

Below, we shall perform some computations on J_1 and J_2 . Indeed, we have

$$\begin{aligned} J_1 &= \int_{\Omega} \rho^{n+1} \nabla \nabla \rho^n : \nabla \nabla \rho^m dx \\ &= \int_{\Omega} \rho^{n+1} \nabla \left(\frac{n}{\theta} \rho^{n-\theta} \nabla \rho^\theta \right) : \nabla \left(\frac{m}{\theta} \rho^{m-\theta} \nabla \rho^\theta \right) dx \\ &= \frac{nm}{\theta} \left[\int_{\Omega} \rho^{2n+m-2\theta+1} |\nabla \nabla \rho^\theta|^2 dx + \int_{\Omega} \rho^{2n+1-\theta} \nabla \nabla \rho^\theta : \nabla \rho^{m-\theta} \otimes \nabla \rho^\theta dx \right. \\ &\quad \left. + \int_{\Omega} \rho^{n+1+m-\theta} \nabla \nabla \rho^\theta : \nabla \rho^{n-\theta} \otimes \nabla \rho^\theta dx + \int_{\Omega} \rho^{n+1} \nabla \rho^{n-\theta} \otimes \nabla \rho^\theta : \nabla \rho^{m-\theta} \otimes \nabla \rho^\theta dx \right] \\ &= \frac{nm}{\theta^2} \left[\int_{\Omega} \rho^{2n+m-2\theta+1} |\nabla \nabla \rho^\theta|^2 dx - \gamma_1 \int_{\Omega} \rho^{2n+m+1-\theta} \nabla \nabla \rho^\theta : \nabla \rho^{\theta/2} \otimes \nabla \rho^{\theta/2} dx \right. \\ &\quad \left. - \gamma_2 \int_{\Omega} \rho^{2n+1+m-\theta} \nabla \nabla \rho^\theta : \nabla \rho^{\theta/2} \otimes \nabla \rho^{\theta/2} dx + \gamma_1 \gamma_2 \int_{\Omega} \rho^{2n+m+1-\theta} (\nabla \rho^{\theta/2})^4 dx \right] \end{aligned}$$

where γ_1 and γ_2 are two constants defined by

$$\gamma_1 = \frac{4(\theta-m)}{\theta} \quad \gamma_2 = \frac{4(\theta-n)}{\theta}.$$

Now, let us choose θ such that

$$\theta = \frac{2n+m+1}{2}.$$

Thus the integral J_1 becomes

$$J_1 = \frac{nm}{\theta^2} \left[\int_{\Omega} |\nabla \nabla \rho^\theta|^2 dx - (\gamma_1 + \gamma_2) \int_{\Omega} \nabla \nabla \rho^\theta : \nabla \rho^{\theta/2} \otimes \nabla \rho^{\theta/2} dx + \gamma_1 \gamma_2 \int_{\Omega} (\nabla \rho^{\theta/2})^4 dx \right].$$

A similar computation on J_2 yields

$$\begin{aligned} J_2 &= \int_{\Omega} \rho^{n+1} \Delta \rho^n \Delta \rho^m dx \\ &= \frac{nm}{\theta^2} \left[\int_{\Omega} |\Delta \rho^\theta|^2 dx - (\gamma_1 + \gamma_2) \int_{\Omega} \Delta \rho^\theta (\nabla \rho^{\theta/2})^2 dx + \gamma_1 \gamma_2 \int_{\Omega} (\nabla \rho^{\theta/2})^4 dx \right]. \end{aligned}$$

Gathering J_1 and J_2 together and minding that for a periodic domain the following identity holds

$$\int_{\Omega} |\nabla \nabla \rho^\theta|^2 dx = \int_{\Omega} |\Delta \rho^\theta|^2 dx,$$

we infer that

$$\begin{aligned} (1.8) \quad \mathcal{I} &= J_1 + nJ_2 \\ &= \frac{nm}{\theta^2} \left[(1+n) \int_{\Omega} |\nabla \nabla \rho^\theta|^2 dx + \gamma_1 \gamma_2 (1+n) \int_{\Omega} (\nabla \rho^{\theta/2})^4 dx \right. \\ &\quad \left. - (\gamma_1 + \gamma_2) \left(\int_{\Omega} \nabla \nabla \rho^\theta : \nabla \rho^{\theta/2} \otimes \nabla \rho^{\theta/2} dx + n \int_{\Omega} \Delta \rho^\theta (\nabla \rho^{\theta/2})^2 dx \right) \right]. \end{aligned}$$

In the sequel, we want to establish an estimate of

$$-(\gamma_1 + \gamma_2) \left(\int_{\Omega} \nabla \nabla \rho^\theta : \nabla \rho^{\theta/2} \otimes \nabla \rho^{\theta/2} dx + n \int_{\Omega} \Delta \rho^\theta (\nabla \rho^{\theta/2})^2 dx \right).$$

To this purpose let us in the first place observe that the two following equalities hold

$$\rho \nabla \nabla \log \rho = \nabla \nabla \rho - 4 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}, \quad \rho \Delta \log \rho = \Delta \rho - 4 (\nabla \sqrt{\rho})^2.$$

This implies that

$$\begin{aligned} \int_{\Omega} \rho^2 |\nabla \nabla \log \rho|^2 dx &= \int_{\Omega} |\nabla \nabla \rho|^2 dx + \int_{\Omega} |2 \nabla \sqrt{\rho}|^4 dx - 8 \int_{\Omega} \nabla \nabla \rho : \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} dx \\ \int_{\Omega} \rho^2 |\Delta \log \rho|^2 dx &= \int_{\Omega} |\Delta \rho|^2 dx + \int_{\Omega} |2 \nabla \sqrt{\rho}|^4 dx - 8 \int_{\Omega} \Delta \rho (\nabla \sqrt{\rho})^2 dx. \end{aligned}$$

Therefore

$$\begin{aligned} &-8 \left(\int_{\Omega} \nabla \nabla \rho : \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} dx + n \int_{\Omega} \Delta \rho (\nabla \sqrt{\rho})^2 dx \right) \\ &= \int_{\Omega} \rho^2 |\nabla \nabla \log \rho|^2 dx + n \int_{\Omega} \rho^2 |\Delta \log \rho|^2 dx - (1+n) \int_{\Omega} |\nabla \nabla \rho|^2 dx - (1+n) \int_{\Omega} |2 \nabla \sqrt{\rho}|^4 dx. \end{aligned}$$

Now using Lemma 1.1, we deduce that

$$\begin{aligned} (1.9) \quad &-8 \int_{\Omega} \left(\nabla \nabla \rho : \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} dx + n \int_{\Omega} \Delta \rho (\nabla \sqrt{\rho})^2 dx \right) \\ &\geq c_n \int_{\Omega} |2 \nabla \sqrt{\rho}|^4 dx - (1+n) \int_{\Omega} |\nabla \nabla \rho|^2 dx - (1+n) \int_{\Omega} |2 \nabla \sqrt{\rho}|^4 dx \end{aligned}$$

where c_n is given by

$$c_n = \frac{(1+n)(d+2)(d(1-n)+2n) - (d-1)^2(2n-1)^2}{(d+2)^2(1+n)}.$$

Hence assuming $\rho = \rho^\theta$ in inequality (1.9), we obtain

$$\begin{aligned} (1.10) \quad &- \left[\int_{\Omega} \nabla \nabla \rho^\theta : \nabla \rho^{\theta/2} \otimes \nabla \rho^{\theta/2} dx + n \int_{\Omega} \Delta \rho^\theta (\nabla \rho^{\theta/2})^2 dx \right] \\ &\geq 2(c_n - 1 - n) \int_{\Omega} |\nabla \rho^{\theta/2}|^4 dx - \frac{(1+n)}{8} \int_{\Omega} |\nabla \nabla \rho^\theta|^2 dx. \end{aligned}$$

Again assume that $\gamma_1 + \gamma_2 > 0$, substituting inequality (1.10) into (1.8), we obtain

$$\mathcal{I} \geq \frac{nm}{\theta^2} \left[(1+n) \left(1 - \frac{\gamma_1 + \gamma_2}{8} \right) \int_{\Omega} |\nabla \nabla \rho^\theta|^2 dx + (\gamma_1 \gamma_2 (1+n) + 2(\gamma_1 + \gamma_2)(c_n - 1 - n)) \int_{\Omega} |2 \nabla \rho^{\theta/2}|^4 dx \right]$$

Thus, contemplating the following constraints on the coefficients

$$0 < \gamma_1 + \gamma_2 < 8 \quad \gamma_1 \gamma_2 (1+n) + 2(\gamma_1 + \gamma_2)(c_n - 1 - n) > 0,$$

we finish the proof of inequality (1.3).

2. APPLICATION TO FLUID DYNAMICS SYSTEMS

In this section, we show how our functional inequality proved in the previous section can help us establish some important estimates on the solution of Navier-Stokes-Korteweg system. We point out here that we are not interested in the question of well posedness of such system since it needs more work (this will be the subject for a forthcoming paper). However, these estimates established here will be the main ingredient to treat this question. On the other hand, as we shall see, despite our fairly functional inequality, we are obliged to take a particular case of capillarity term and viscosity coefficient. We emphasize that we cover a more general case than that considered by many authors: see for instance the recent work of A. VASSEUR and I. L. VIOLET in [7] for more details. Indeed, the Navier-Stokes-Korteweg system is defined as:

$$\begin{aligned} (2.1) \quad &\partial_t \rho + \operatorname{div}(\rho u) = 0 \\ &\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) D(u) + \lambda(\rho) \operatorname{div} u \mathbb{I}) + \nabla p = \operatorname{div}(\mathbb{S}) \end{aligned}$$

where $\text{div}(\mathbb{S})$ is the capillary tensor which reads as follows

$$(2.2) \quad \mathbb{S} = (\rho \text{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho)|\nabla\rho|^2) \mathbb{I} - K(\rho)\nabla\rho \otimes \nabla\rho,$$

with $K : (0, \infty) \rightarrow (0, \infty)$ is a smooth function, u stands for the velocity of fluid, and $D(u) = \frac{1}{2}(\nabla u + \nabla^t u)$ is the strain tensor. The function $p(\rho)$ is a general increasing pressure that we assume in the sequel under the form $p(\rho) = a\rho^\gamma$ where $a > 0$, $\gamma > 0$. The viscosity coefficients μ and λ are the Lamé coefficients that should obey two mathematical restrictions³ (d is the dimension of spaces)

$$(2.3) \quad (i) \mu(\rho) > 0 \quad \mu(\rho) + d\lambda(\rho) > 0 \quad (ii) \lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)).$$

Notice that, as mentioned in [4], the Korteweg tensor can be written in the form

$$(2.4) \quad \text{div}(\mathbb{S}) = \rho\nabla(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)} ds)).$$

The global existence of the solution of system (2.1) for a general Korteweg tensor is as far as we know an open problem. The main difficulties of such models is twofold: the first one goes back to the difficulty of establishing the necessary a priori estimates. The second one lies in the strongly non linear third-order differential operator and the dispersive structure of the momentum equation. For these reasons, the only known result around this system is limited to the so called quantum Navier-Stokes system which corresponds to the case when (see for instance [7])

$$(2.5) \quad \mu(\rho) = \rho \quad \lambda(\rho) = 0 \quad K(\rho) = \rho^{-1}.$$

By virtue of our Inequality (1.3), we are able to prove that under more general case of capillarity and viscosity coefficients

$$\mu(\rho) = \rho^{n+1} \quad \lambda(\rho) = 2n\rho^{n+1} \quad K(\rho) = \rho^{2m-1}$$

where n and m should satisfy the constraint (1.2), the Navier-Stokes-Korteweg system has the following two inequalities. The first one is the classical energy estimate while the second one is the so-called B-D entropy.

Lemma 2.1. *For ρ and u sufficiently smooth, we have*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u|^2 dx + 2 \int_{\Omega} \rho^{n+1} |D(u)|^2 dx + 2n \int_{\Omega} \rho^{n+1} |\text{div} u|^2 dx \\ & \quad + \frac{d}{dt} \int_{\Omega} \frac{a}{\gamma-1} \rho^\gamma dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \nabla \left(\int_0^\rho \sqrt{s^{2m-1}} ds \right) \right|^2 dx \leq 0 \\ & \frac{d}{dt} \int_{\Omega} \rho |u + \frac{2\nabla\rho^n}{\rho}| dx + 2 \int_{\Omega} \rho^{n+1} |A(u)|^2 dx + \frac{d}{dt} \int_{\Omega} \frac{a}{\gamma-1} \rho^\gamma dx + 2a\gamma(n+1) \int_{\Omega} \rho^{n+\gamma-2} |\nabla\rho|^2 dx \\ & \quad + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \nabla \left(\int_0^\rho \sqrt{s^{2m-1}} ds \right) \right|^2 dx + \alpha \int_{\Omega} |\nabla\nabla\rho^{\frac{2n+m+1}{2}}|^2 + \beta \int_{\Omega} |\nabla\rho^{\frac{2n+m+1}{4}}|^4 dx \leq 0. \end{aligned}$$

where α and β are two positive constants and $A(u) = \frac{1}{2}(\nabla u - \nabla^t u)$.

Before starting the proof of Lemma 2.1, let us prove the following identity which will be used later.

Lemma 2.2. *For any smooth function $\rho(x)$, we have*

$$(2.6) \quad \rho\nabla(\sqrt{\rho^{2m-1}}\Delta(\int_0^\rho \sqrt{s^{2m-1}} ds)) = \frac{1}{m(m+1)} [\text{div}(\rho^{m+1}\nabla\nabla\rho^m) + m\nabla(\rho^{m+1}\Delta\rho^m)].$$

³(i) is a cosequence of physical restriction however (ii) is called the Bresch-Desjardins relation introduced in [2] for compressible Navier-Stokes equations

Proof. By straightforward computation, we can write

$$\begin{aligned}
& \rho \partial_j (\rho^{m-1/2} \partial_i^2 (\int_0^\rho s^{m-1/2})) \\
&= \frac{1}{m} \rho \partial_j (\rho^{m-1/2} \partial_i (\rho^{1/2} \partial_i \rho^m)) \\
&= \frac{1}{m} \rho \partial_j (\rho^m \partial_i^2 \rho^m + \frac{1}{2} \rho^{m-1} \partial_i \rho \partial_i \rho^m) \\
&= \frac{1}{m} \partial_j (\rho^{m+1} \partial_i^2 \rho^m) - \frac{1}{m} \rho^m \partial_j \rho \partial_i^2 \rho^m + \frac{1}{2m^2} \rho \partial_j ((\partial_i \rho^m)^2) \\
&= \frac{1}{m} \partial_j (\rho^{m+1} \partial_i^2 \rho^m) - \frac{1}{m(m+1)} \partial_j \rho^{m+1} \partial_i^2 \rho^m + \frac{1}{m(m+1)} \partial_i \rho^{m+1} \partial_j \partial_i \rho^m \\
&= \frac{1}{m(m+1)} [(m+1) \partial_j (\rho^{m+1} \partial_i^2 \rho^m) - \partial_j \rho^{m+1} \partial_i^2 \rho^m + \partial_i (\rho^{m+1} \partial_i \partial_j \rho^m) - \rho^{m+1} \partial_j \partial_i^2 \rho^m] \\
&= \frac{1}{m(m+1)} [\partial_i (\rho^{m+1} \partial_i \partial_j \rho^m) + m \partial_j (\rho^{m+1} \partial_i^2 \rho^m)].
\end{aligned}$$

Proof of Lemma 2.1 The proof of the first inequality is too classical. It sufficient to multiply the equation of conservation of momentum (2.1)₂ by u , integrate by parts and use the mass conservation equation. For the second one, we start by multiplying the mass conservation by $(n+1)\rho^n$ to obtain

$$\partial_t \rho^{n+1} + \operatorname{div}(\rho^{n+1} u) + n \rho^{n+1} \operatorname{div} u = 0.$$

Now, differentiating with respect to x , we get

$$\partial_t \nabla \rho^{n+1} + \operatorname{div}(u \otimes \nabla \rho^{n+1}) + \operatorname{div}(\rho^{n+1} \nabla^t u) + n \nabla(\rho^{n+1} \operatorname{div} u) = 0.$$

Multiplying the above equation by 2 and adding it to (2.1)₂, we obtain

$$(2.7) \quad \partial_t \left(\rho \left(u + \frac{2\nabla \rho^n}{\rho} \right) \right) + \operatorname{div} \left(\rho u \otimes \left(u + \frac{2\nabla \rho^n}{\rho} \right) \right) - 2 \operatorname{div}(\rho^{n+1} A(u)) + a \nabla \rho^\gamma = \rho \nabla \left(\sqrt{\rho^{2m-1}} \Delta \left(\int_0^\rho \sqrt{s^{2m-1}} ds \right) \right)$$

Now, multiplying Equation (2.7) by $(u + \frac{2\nabla \rho^n}{\rho})$ and integrating by parts, we deduce

$$\frac{d}{dt} \int_\Omega \rho \left| u + \frac{2\nabla \rho^n}{\rho} \right|^2 dx + 2 \int_\Omega \rho^{n+1} |A(u)|^2 dx + \frac{d}{dt} \int_\Omega \frac{a \rho^\gamma}{\gamma-1} dx + 2a\gamma(n+1) \int_\Omega \rho^{n+\gamma-2} |\nabla \rho|^2 dx = I,$$

where we denote

$$I := \int_\Omega \rho \nabla \left(\sqrt{\rho^{2m-1}} \Delta \left(\int_0^\rho \sqrt{s^{2m-1}} ds \right) \right) \cdot \left(u + \frac{2\nabla \rho^n}{\rho} \right) dx$$

Now, using the mass conservation equation and identity (2.6), we have

$$I = \frac{1}{2} \frac{d}{dt} \int_\Omega \left| \nabla \left(\int_0^\rho \sqrt{s^{2m-1}} ds \right) \right|^2 dx + \frac{2(n+1)}{n(m+1)} \left[\int_\Omega \rho^{m+1} \nabla \nabla \rho^m : \nabla \nabla \rho^n dx + m \int_\Omega \rho^m \Delta \rho^m \cdot \Delta \rho^n dx \right].$$

By virtue of our Inequality (1.3), the proof of Lemma 2.1 is finished.

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INRIA, RESEARCH CENTER OF PARIS, FRANCE.

E-mail address: `bilal.al-taki@inria.fr`