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Existence of the anchored isoperimetric profile in supercritical bond percolation in dimension two and higher

Barbara Dembin *

Abstract: Let $d \geq 2$. We consider an i.i.d. supercritical bond percolation on $\mathbb{Z}^d$, every edge is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical point. We condition on the event that 0 belongs to the infinite cluster $\mathcal{C}_\infty$ and we consider connected subgraphs of $\mathcal{C}_\infty$ having at most n^d vertices and containing 0. Among these subgraphs, we are interested in the ones that minimize the open edge boundary size to volume ratio. These minimizers properly rescaled converge towards a translate of a deterministic shape and their open edge boundary size to volume ratio properly rescaled converges towards a deterministic constant.

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1 Introduction

Isoperimetric problems are among the oldest problems in mathematics. They consist in finding sets that maximize the volume given a constraint on the perimeter or equivalently that minimize the perimeter to volume ratio given a constraint on the volume. These problems can be formulated in the anisotropic case. Given a norm ν on $\mathbb{R}^d$ and S a continuous subset of $\mathbb{R}^d$, we define the tension exerted at a point x in the boundary ∂S of S to be $\nu(n_S(x))n_S(x)$, where $n_S(x)$ is the exterior unit normal vector of S at x. The quantity $\nu(n_S(x))$ corresponds to the intensity of the tension that is exerted at x. We define the surface energy of S as the integral of the intensity of the surface tension $\nu(n_S(x))$ over the boundary ∂S. An anisotropic isoperimetric problem consists in finding sets that minimize the surface energy to volume ratio given a constraint on the volume. To solve this problem, in [22], Wulff introduced through the Wulff construction a shape achieving the infimum. This shape is called the Wulff crystal, it corresponds to the unit ball for a norm built upon ν. Later, Taylor proved in [20] that this shape properly rescaled is the unique minimizer, up to translations and modifications on a null set, of the associated isoperimetric problem.

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The study of isoperimetric problems in the discrete setting is more recent. In the continuous setting, we study the perimeter to volume ratio, in the context of graphs, the analogous problem is the study of the size of edge boundary to volume ratio. This can be encoded by the Cheeger constant. For a finite graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, we define the edge boundary $\partial_{\mathcal{G}} A$ of a subset A of $V(\mathcal{G})$ as

$$\partial_{\mathcal{G}} A = \left\{ e = \langle x, y \rangle \in E(\mathcal{G}) : x \in A, y \notin A \right\}.$$

We denote by ∂A the edge boundary of A in $(\mathbb{Z}^d, \mathbb{E}^d)$ and by $|B|$ the cardinal of the finite set B. The isoperimetric constant, also called Cheeger constant, is defined as

$$\varphi_{\mathcal{G}} = \min \left\{ \frac{|\partial_{\mathcal{G}} A|}{|A|} \ : \ A \subset V(\mathcal{G}), 0 < |A| \leq \frac{|V(\mathcal{G})|}{2} \right\}.$$

This constant was introduced by Cheeger in his thesis [9] in order to obtain a lower bound for the smallest eigenvalue of the Laplacian. The isoperimetric constant of a graph gives information on its geometry.

Let $d \geq 2$. We consider an i.i.d. supercritical bond percolation on $\mathbb{Z}^d$, every edge is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster $\mathcal{C}_\infty$ [11]. In this paper, we want to study the geometry of $\mathcal{C}_\infty$ through its Cheeger constant. However, if we minimize the isoperimetric ratio over all possible subgraphs of $\mathcal{C}_\infty$ without any constraint on the size, one can show that $\varphi_{\mathcal{C}_\infty} = 0$ almost surely. For that reason, we shall minimize the isoperimetric ratio over all possible subgraphs of $\mathcal{C}_\infty$ given a constraint on the size. There are several ways to do it. We can for instance study the Cheeger constant of the graph $\mathcal{C}_n = \mathcal{C}_\infty \cap [-n, n]^d$ or of the largest connected component $\widetilde{\mathcal{C}}_n$ of $\mathcal{C}_n$ for $n \geq 1$. As we have $\varphi_{\mathcal{C}_\infty} = 0$ almost surely, the isoperimetric constants $\varphi_{\mathcal{C}_n}$ and $\varphi_{\widetilde{\mathcal{C}}_n}$ go to 0 when n goes to infinity. Benjamini and Mossel [1], Mathieu and Remy [14], Rau [17], Berger, Biskup, Hoffman and Kozma [2], Pete [15] proved that $\varphi_{\widetilde{\mathcal{C}}_n}$ is of order n^{-1}. Roughly speaking, by analogy with the full lattice, we expect that subgraphs of $\widetilde{\mathcal{C}}_n$ that minimize the isoperimetic ratio have an edge boundary size of order n^{d-1} and a size of order n^d, this is coherent with the fact that $\varphi_{\widetilde{\mathcal{C}}_n}$ is of order n^{-1}. This leads Benjamini to conjecture that for $p > p_c(d)$, the limit of $n\varphi_{\widetilde{\mathcal{C}}_n}$ when n goes to infinity exists and is a positive deterministic constant.

This conjecture was solved in dimension 2 by Biskup, Louidor, Procaccia and Rosenthal in [3] and by Gold in dimension 3 in [10]. They worked on a modified Cheeger constant. Instead of considering the open edge boundary of subgraphs within $\mathcal{C}_n$, they considered the open edge boundary within the whole infinite cluster $\mathcal{C}_\infty$, this is more natural because $\mathcal{C}_n$ has been artificially created by restricting $\mathcal{C}_\infty$ to the box $[-n, n]^d$. They also added a stronger constraint on the size of subgraphs of $\mathcal{C}_n$ to ensure that minimizers do not touch the boundary of the box $[-n, n]^d$. Moreover, the subgraphs achieving the minimum, properly rescaled, converge towards a deterministic shape that is the Wulff crystal. Namely, it is the shape solving the continuous anisotropic isoperimetric problem associated with a norm β_p corresponding to the surface tension in the percolation setting. The quantity $n\varphi_{\mathcal{C}_n}$ converges towards the solution of a continuous isoperimetric problem.

Dealing with the isoperimetric ratio within $\mathcal{C}_n$ needs to be done with caution. Indeed, we do not want minimizers to be close to the boundary of $\mathcal{C}_n$ because this boundary does not exist in $\mathcal{C}_\infty$. There is another way to define the Cheeger constant of $\mathcal{C}_\infty$, that is more natural in the sense that we do not restrict minimizers to remain in the box $[-n,n]^d$. This is called the anchored isoperimetric profile φ_n and it is defined by:

$$\varphi_n = \min\left\{ \frac{|\partial_{\mathcal{C}_\infty} H|}{|H|} : 0 \in H \subset \mathcal{C}_\infty,\ \text{H connected}, 0 < |H| \leq n^d \right\},$$

where we condition on the event $\{0 \in \mathcal{C}_\infty\}$. We say that H is a valid subgraph if $0 \in H \subset \mathcal{C}_\infty$, H is connected and $|H| \leq n^d$. We also define

$$\partial^o H = \left\{ e \in \partial H,\ e \text{ is open} \right\}.$$

Note that if $H \subset \mathcal{C}_\infty$, then $\partial_{\mathcal{C}_\infty} H = \partial^o H$. For each n, let $\mathcal{G}_n$ be the set of the valid subgraphs that achieve the infimum in φ_n. In this context, a minimizer $G_n \in \mathcal{G}_n$ can go potentially very far from 0. The minimizer G_n properly rescaled do not belong anymore to a compact set. This lack of compacity is the main issue to overcome to prove that the limit exists. It was done in dimension 2 in [3], with a specific norm that cannot be extended to higher dimensions. We need to introduce some definitions to be able to define properly a limit shape in dimension $d \geq 2$. In order to build a continuous limit shape, we shall define a continuous analogue of the open edge boundary. In fact, we will see that the open edge boundary may be interpreted in term of a surface tension $\mathcal{I}$, in the following sense. Given a norm τ on $\mathbb{R}^d$ and a subset E of $\mathbb{R}^d$ having a regular boundary, we define $\mathcal{I}_\tau(E)$ as

$$\mathcal{I}_\tau(E) = \int_{\partial E} \tau(n_E(x)) \mathcal{H}^{d-1}(dx),$$

where $\mathcal{H}^{d-1}$ denotes the Hausdorff measure in dimension $d-1$. The quantity $\mathcal{I}_\tau(E)$ represents the surface tension of E for the norm τ. At the point x, the tension has intensity $\tau(n_E(x))$ in the direction of the normal unit exterior vector $n_E(x)$. We denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure. We can associate with the norm τ the following isoperimetric problem:

$$\text{minimize } \frac{\mathcal{I}_\tau(E)}{\mathcal{L}^d(E)} \text{ subject to } \mathcal{L}^d(E) \leq 1.$$

We use the Wulff construction to build a minimizer for this anisotropic isoperimetric problem. We define the set $\widehat{W}_\tau$ as

$$\widehat{W}_\tau = \bigcap_{v \in \mathbb{S}^{d-1}} \left\{ x \in \mathbb{R}^d : x \cdot v \leq \tau(v) \right\},$$

where $\cdot$ denotes the standard scalar product and $\mathbb{S}^{d-1}$ is the unit sphere of $\mathbb{R}^d$. The set $\widehat{W}_\tau$ is a minimizer for the isoperimetric problem associated with τ. We will build in section 3 an appropriate norm β_p for our problem that will be directly related to the open edge boundary ratio. We define the Wulff crystal W_p as the dilate of $\widehat{W}_{\beta_p}$ such that $\mathcal{L}^d(W_p) = 1/\theta_p$, where $\theta_p = \mathbb{P}(0 \in \mathcal{C}_\infty)$.

In this paper, we adapt the proof of Gold to any dimension $d \geq 2$ to give a self-contained proof of the existence of the limit for the anchored isoperimetric profile. Note that this proof also holds in dimension 2, it gives an alternative proof of [3] with a simpler norm. The aim of this paper is the proof of the two following Theorems. The first theorem asserts the existence of the limit of $n\varphi_n$.

Theorem 1. *Let $d \geq 2$, $p > p_c(d)$ and let β_p be the norm that will be properly defined in section 3. Let W_p be the Wulff crystal for this norm, i.e., the dilate of $\widehat{W}_{\beta_p}$ such that $\mathcal{L}^d(W_p) = 1/\theta_p$. Then, conditionally on $\{0 \in \mathcal{C}_\infty\}$,*

$$\lim_{n\to\infty} n\varphi_n = \frac{\mathcal{I}_p(W_p)}{\theta_p \mathcal{L}^d(W_p)} = \mathcal{I}_p(W_p) \text{ a.s..}$$

The second theorem shows that the graphs G_n realizing the minimum converge in probability towards a translate of W_p.

Theorem 2. *Let $d \geq 2$ and $p > p_c(d)$. Let $\varepsilon > 0$. There exists positive constants C_1 and C_2 depending on d, p and ε such that, for all $n \geq 1$,*

$$\mathbb{P}\left[\max_{G_n \in \mathcal{G}_n} \inf_{x \in \mathbb{R}^d} \frac{1}{n^d} \left|G_n \Delta (n(x + W_p) \cap \mathcal{C}_\infty)\right| \geq \varepsilon \,\middle|\, 0 \in \mathcal{C}_\infty\right] \leq C_1 \, \mathrm{e}^{-C_2 n^{1-3/2d}},$$

where Δ denotes the symmetric difference.

To prove Theorem 1, we first prove a large deviations result from above for $n\varphi_n$ stated in the following Theorem.

Theorem 3. *Let $d \geq 2$. Let $p > p_c(d)$. For all $\varepsilon > 0$, there exist positive constants C_1 and C_2 depending on p, d, ε such that, for all $n \geq 1$,*

$$\mathbb{P}\left[n\varphi_n \geq (1+\varepsilon)\frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \leq C_1 \exp(-C_2 n)\,.$$

The proof of Theorem 3 is inspired from the proof of Theorem 5.4 in [10]. We shall build a valid subgraph that has an isoperimetric ratio close to φ_n. In order to do so, we approximate the Wulff shape W_p from the inside by a convex polytope P. We shall build a cutset Γ_n that cuts nP from infinity whose number of open edges is close to $n^{d-1}\mathcal{I}_p(P)$ with high probability. For each face F of P and v its associated exterior unit normal vector, we consider the cylinder $\mathrm{cyl}(F + \varepsilon v, \varepsilon)$ of basis $F + \varepsilon v$ and of height $\varepsilon > 0$. We build E by merging the cutsets from the top to the bottom of minimal capacity of the cylinders $\mathrm{cyl}(F + \varepsilon v, \varepsilon)$. The union of these cutsets is not yet a cutset itself because of the potential holes between these cutsets. We fix this issue by adding extra edges to fill the holes. We control next the number of extra edges we have added. We also need to control the capacity of the cutsets in a cylinder of polyhedral basis. We next build a valid subgraph $H_n \subset \mathbb{Z}^d$ from Γ_n by taking all the vertices of $\mathcal{C}_\infty \cap nP$ that are connected to 0 without using edges in Γ_n. We expect that $|H_n|$ is of order $\theta_p(d) n^d \mathcal{L}^d(P)$. We can bound from above $|\partial_{\mathcal{C}_\infty} H_n|$ thanks to the number of open edges in Γ_n and so we control its isoperimetric ratio. Finally, we control the upper large deviations for this number of open edges thanks to the upper large deviations for the flow in a cylinder of polyhedral basis. The next step is to obtain the large deviations result from below.

Theorem 4. *Let $d \geq 2$. Let $p > p_c(d)$. For all $\varepsilon > 0$, there exist positive constants C_1 and C_2 depending on p, d, ε such that, for all $n \geq 1$,*

$$\mathbb{P}\left[n\varphi_n \leq (1-\varepsilon)\frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)} \,\Big|\, 0 \in \mathcal{C}_\infty \right] \leq C_1 \exp(-C_2 n^{1-3/2d}) \,.$$

Remark 1.1. The deviation order in Theorem 4 is not optimal due to technical details of the proof. In this work we do not make any attempt to get the proper order of deviation. Our aim is mainly to obtain Theorems 1 and 2. The study of the large deviations order would be an interesting problem in itself.

Theorem 1 follows from Theorem 3 and Theorem 4 by a straightforward application of the Borel-Cantelli Lemma. Proving the large deviations result from below is the most difficult part of this work. To be able to compare discrete objects with continuous ones, we shall encode each optimizer $G_n \in \mathcal{G}_n$ as a measure μ_n defined as

$$\mu_n = \frac{1}{n^d} \sum_{x \in V(G_n)} \delta_{x/n} \,.$$

We first need to build from a minimizer G_n an appropriate continuous object P_n. To do so, we use the same method as in [10]. The main issue is that the boundary of G_n may be very tangled, we will have to build a smoother boundary of size of order n^{d-1}. This will enable us to build a continuous object P_n of finite perimeter such that, with high probability, its associated measure is close to μ_n in some sense to be specified later.

Let F be a subset of $\mathbb{R}^d$. We define its associated measure ν_F:

$$\forall E \in \mathcal{B}(\mathbb{R}^d), \qquad \nu_F(E) = \theta_p \mathcal{L}^d(F \cap E) \,.$$

We now define the set $\mathcal{W}$ of the measures associated with the translates of the Wulff shape as

$$\mathcal{W} = \left\{ \nu_{x+W_p} \,:\, x \in \mathbb{R}^d \right\} .$$

Note that μ_n belongs to $\mathcal{M}(\mathbb{R}^d)$, the set of finite measures on $\mathbb{R}^d$. We cannot use a metric as in [10] where μ_n was a measure on $[-1,1]^d$. In fact, we will not use a metric here. We first show that all the minimizers $G_n \in \mathcal{G}_n$ are with high probability in a local neighborhood of $\mathcal{W}$ for a weak topology. This is the key step before proving Theorem 4.

Theorem 5. *Let $d \geq 2$ and $p > p_c(d)$. Let $u :]0,+\infty[\to]0,+\infty[$ be a non-decreasing function such that $\lim_{t\to 0} u(t) = 0$. For all $\zeta > 0$, there exist positive constants C_1 and C_2 depending on d, p, u and ζ such that for all $n \geq 1$, for any finite set $\mathfrak{F}_n$ of uniformly continuous functions that satisfies:*

$$\forall f \in \mathfrak{F}_n \quad \|f\|_\infty \leq 1 \qquad \textit{and} \qquad \forall x, y \in \mathbb{R}^d \quad |f(x) - f(y)| \leq u(\|x-y\|_2) \,,$$

we have

$$\mathbb{P}\left[\exists G_n \in \mathcal{G}_n, \forall \nu \in \mathcal{W}, \sup_{f \in \mathfrak{F}_n} |\mu_n(f) - \nu(f)| > \zeta \,\Big|\, 0 \in \mathcal{C}_\infty \right] \leq C_1 \, \mathrm{e}^{-C_2 n^{1-3/2d}} \,.$$

The main difficulty of this paper lies in the proof of this theorem. In our context, an issue that was not present in [10] arises. Whereas the support of

the measure μ_n was included in a fixed compact set in [10], this is not the case here because we do not constrain $G_n \in \mathcal{G}_n$ to remain in the box $[-n,n]^d$. To fix this issue, we will use the method developed in [5]. We will first localize the set G_n in a finite number of balls of radius of order n up to a set of small fractional volume. We will study G_n only inside these balls, *i.e.*, the intersection of G_n with these balls. The intersection of G_n with the boundary of these balls will create an additional surface tension. However, this surface tension is not related to the open boundary edges of G_n but to the fact that we have cut G_n along these boundaries. Therefore, we should not take this surface tension into account for the isoperimetric constant. In fact, we will cut G_n in such a way to ensure that we do not create too much surface tension, *i.e.*, we will cut in regions where G_n is not concentrated. To conclude, we will link the probability that the measure μ_n corresponding to $G_n \in \mathcal{G}_n$ is far from a weak neighborhood of $\mathcal{W}$ with the probability that the surface tension of G_n is locally abnormally small.

Finally, to prove Theorem 2, we exhibit a set $\mathcal{F}_n$ of uniformly continuous functions such that we can bound from above the symmetric difference $|G_n \Delta (n(x+W_p) \cap \mathcal{C}_\infty)|$ by $\sup_{f \in \mathfrak{F}_n} |\mu_n(f) - \nu(f)|$ for some $\nu \in \mathcal{W}$ and then apply the result of Theorem 5.

The rest of the paper is organized as follows. In section 2, we give some definitions and useful results. We do the construction of the norm β_p in section 3. In section 4, we prove the upper large deviations in Theorem 3. We build a continuous object P_n from a minimizer $G_n \in \mathcal{G}_n$ and prove that its associated measure is close in some sense to the measure μ_n of G_n in section 5. Finally, in section 6, we prove Theorem 5 that is a preliminary work before proving the lower large deviations Theorem 4 and the convergence of G_n properly rescaled towards a limit shape in Theorem 2.

2 Some definitions and useful results

2.1 Geometric notations

For $x = (x_1, \dots, x_d)$, we define

$$\|x\|_2 = \sqrt{\sum_{i=1}^{d} x_i^2} \qquad \text{and} \qquad \|x\|_\infty = \max_{1 \le i \le d} |x_i| \,.$$

We say that $x, y \in \mathbb{Z}^d$ are $*$-connected if $\|x-y\|_\infty = 1$. We say that $\gamma = (x_0, \dots, x_n)$ is an $*$-path of $\mathbb{Z}^d$ if for any $0 \le i \le n-1$, the points x_i and x_{i+1} belong to $\mathbb{Z}^d$ and are $*$-connected. We say that Γ is $*$-connected or a lattice animal if any $x, y \in \Gamma$ are connected by an $*$-path in Γ. We denote by Animals_x the set of lattice animals containing the point $x \in \mathbb{Z}^d$.

Lemma 1. *[Kesten [12], p82 or Grimmett [11], p85] Let $x \in \mathbb{Z}^d$. For all positive integer l,*

$$|\{\Gamma \in \text{Animals}_x,\, |\Gamma| = l\}| \le (7^d)^l \,.$$

Let $S \subset \mathbb{R}^d$ and $r > 0$, we define $d_2(x,S) = \inf_{y \in S} \|x-y\|_2$ and $\mathcal{V}(S,r)$ the open r-neighborhood of S by

$$\mathcal{V}(S,r) = \Big\{ x \in \mathbb{R}^d \,:\, d_2(x,S) < r \Big\} \,.$$

Let $x \in \mathbb{R}^d$, $r > 0$ and a unit vector v. We denote by $B(x,r)$ the closed ball of radius r centered at x, by $\mathrm{disc}(x,r,v)$ the closed disc centered at x of radius r normal to v, and by $B^+(x,r,v)$ (respectively $B^-(x,r,v)$) the upper (resp. lower) half part of $B(x,r)$ along the direction of v, *i.e.*,

$$B^+(x,r,v) = \Big\{ y \in B(x,r) \,:\, (y-x)\cdot v \geq 0 \Big\},$$

and

$$B^-(x,r,v) = \Big\{ y \in B(x,r) \,:\, (y-x)\cdot v \leq 0 \Big\}.$$

We denote by α_d the $\mathcal{L}^d$ measure of a unit ball in $\mathbb{R}^d$. We denote by $\mathcal{H}^{d-1}$ the Hausdorff measure in dimension $d-1$. In particular, the $\mathcal{H}^{d-1}$ measure of a $d-1$ dimensional unit disc in $\mathbb{R}^d$ is equal to α_{d-1}. Let A be a non-degenerate hyperrectangle, *i.e.*, a rectangle of dimension $d-1$ in $\mathbb{R}^d$. Let $\overrightarrow{v}$ be one of the two unit vectors normal to A. Let $h > 0$, we denote by $\mathrm{cyl}(A,h)$ the cylinder of basis A and height h defined by

$$\mathrm{cyl}(A,h) = \Big\{ x + t\overrightarrow{v} \,:\, x \in A,\, t \in [-h,h] \Big\}.$$

The dependence on $\overrightarrow{v}$ is implicit in the notation $\mathrm{cyl}(A,h)$. Note that these definitions of cylinder may be extended in the case where A is of linear dimension $d-1$, *i.e.*, A is included in an hyperplane of $\mathbb{R}^d$, which is the affine span of A.

2.2 Sets of finite perimeter and surface energy

The perimeter of a Borel set S of $\mathbb{R}^d$ in an open set O is defined as

$$\mathcal{P}(S,O) = \sup\left\{ \int_S \mathrm{div}\, f(x) d\mathcal{L}^d(x) \,:\, f \in C_c^\infty(O, B(0,1)) \right\},$$

where $C_c^\infty(O,B(0,1))$ is the set of the functions of class $\mathcal{C}^\infty$ from $\mathbb{R}^d$ to $B(0,1)$ having a compact support included in O, and div is the usual divergence operator. The perimeter $\mathcal{P}(S)$ of S is defined as $\mathcal{P}(S,\mathbb{R}^d)$. The topological boundary of S is denoted by ∂S. The reduced boundary $\partial^* S$ of S is a subset of ∂S such that, at each point x of $\partial^* S$, it is possible to define a normal vector $n_S(x)$ to S in a measure-theoretic sense, and moreover $\mathcal{P}(S) = \mathcal{H}^{d-1}(\partial^* S)$. Let ν be a norm on $\mathbb{R}^d$. We define its associated Wulff crystal $\mathcal{W}_\nu$ as

$$\mathcal{W}_\nu = \Big\{ x \in \mathbb{R}^d \,:\, \forall y,\ y\cdot x \leq \nu(y) \Big\}.$$

With the help of the Wulff crystal, we define the surface energy of a general set.

Definition 2.1. *The surface energy $\mathcal{I}(S,O)$ of a Borel set S of $\mathbb{R}^d$ in an open set O is defined as*

$$\mathcal{I}(S,O) = \sup\left\{ \int_S \mathrm{div}\, f(x) d\mathcal{L}^d(x) \,:\, f \in C_c^1(O, \mathcal{W}_\nu) \right\}.$$

We will note simply $\mathcal{I}(S) = \mathcal{I}(S,\mathbb{R}^d)$.

Proposition 2.1 (Proposition 14.3 in [5])**.** *The surface energy $\mathcal{I}(S,O)$ of a Borel set S of $\mathbb{R}^d$ of finite perimeter in an open set O is equal to*

$$\mathcal{I}(S,O) = \int_{\partial^* S \cap O} \nu(n_S(x)) d\mathcal{H}^{d-1}(x).$$

We recall the two following fundamental results.

Proposition 2.2 (Isoperimetric inequality)**.** *There exist two positive constants b_{iso}, c_{iso} which depend only on the dimension d, such that for any Cacciopoli set E, any ball $B(x,r) \subset \mathbb{R}^d$,*

$$\min\left(\mathcal{L}^d(E \cap B(x,r)), \mathcal{L}^d((\mathbb{R}^d \setminus E) \cap B(x,r))\right) \leq b_{iso}\mathcal{P}(E, \mathring{B}(x,r))^{d/d-1},$$

$$\min\left(\mathcal{L}^d(E), \mathcal{L}^d(\mathbb{R}^d \setminus E)\right) \leq c_{iso}\mathcal{P}(E)^{d/d-1}.$$

2.3 Approximation by convex polytopes

We recall here an important result, which allows to approximate adequately a set of finite perimeter by a convex polytope.

Definition 2.2 (Convex polytope)**.** *Let $P \subset \mathbb{R}^d$. We say that P is a convex polytope if there exist $v_1, \ldots, v_m$ unit vectors and $\varphi_1, \ldots, \varphi_m$ real numbers such that*

$$P = \bigcap_{1 \leq i \leq m} \left\{ x \in \mathbb{R}^d : x \cdot v_i \leq \varphi_i \right\}.$$

We denote by F_i the face of P associated with v_i, i.e.,

$$F_i = P \cap \left\{ x \in \mathbb{R}^d : x \cdot v_i = \varphi_i \right\}.$$

Any convex subset can be approximated from the outside and from the inside by a convex polytope with almost the same surface energy.

Lemma 2. *Let A be a bounded convex set. For each $\varepsilon > 0$, there exist convex polytopes P and Q such that $P \subset A \subset Q$ and $\mathcal{I}(Q) - \varepsilon \leq \mathcal{I}(A) \leq \mathcal{I}(P) + \varepsilon$.*

Proof. Let A be a bounded convex set. Let $\varepsilon > 0$. Let $(x_k)_{k \geq 1}$ be a dense family in ∂A. For $n \geq 1$, we define P_n as the convex hull of $x_1, \ldots, x_n$, *i.e.*, the smallest convex that contains the points $x_1, \ldots, x_n$. As A is convex, we have $P_n \subset A$ and P_n converges towards A when n goes to infinity for the $\mathcal{L}^1$ topology. The functional $\mathcal{I}$ is lower semi-continuous, thus

$$\mathcal{I}(A) \leq \liminf_{n \to \infty} \mathcal{I}(P_n),$$

so there exists n large enough such that

$$\mathcal{I}(A) \leq \mathcal{I}(P_n) + \varepsilon$$

and we take $P = P_n$. The existence of Q was shown in Lemma 5.1. in [7] for the Wulff shape. The proof may be easily adapted to a general convex bounded set A. □

3 Construction of the norm

Minimizing the open edge boundary is the analogue of minimizing a surface tension in the continuous setting. We shall build a norm β_p that represents the tension that is exerted on the surface, *i.e.*, any point x in a surface S having

$n_S(x)$ as a normal unit exterior vector has a tension $\beta_p(n_S(x))n_S(x)$ that exerts at the point x. To build this norm, let us consider $G_n \in \mathcal{G}_n$. We zoom on the boundary of G_n, we look at what happens in a small but macroscopic cube centered at a point x in the boundary ∂G_n (see figure 1). The cube is located in such a way that its bottom intersects G_n and its top intersects $\mathbb{Z}^d \setminus G_n$, and it is rotated so that its normal vector coincides with the normal exterior vector at the point x. As this cube is small, the portion of G_n in that cube does not affect much $|G_n|$, the total volume of G_n. Thus, if one would like to minimize the open edges to volume ratio, one needs to minimize the number of open edges of ∂G_n in that cube. This problem is equivalent to finding a set of edges that separates the top from the bottom of the cube with a minimal number of open edges.

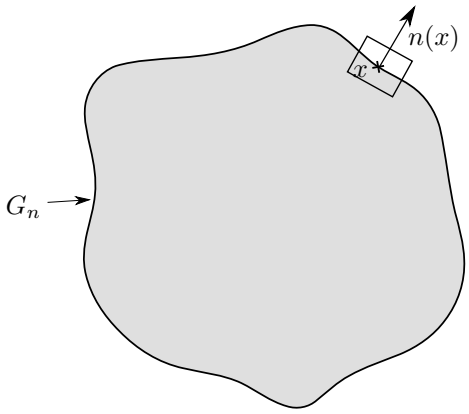

Figure 1 – A small box on the boundary ∂G_n of a minimizer $G_n \in \mathcal{G}_n$

Let us give now a more precise definition of the norm β_p. We consider a bond percolation on $\mathbb{Z}^d$ of parameter $p > p_c(d)$ with $d \geq 2$. We introduce many notations used for instance in [18] concerning flows through cylinders. Let A be a non-degenerate hyperrectangle, *i.e.*, a rectangle of dimension $d-1$ in $\mathbb{R}^d$. Let $\overrightarrow{v}$ be one of the two unit vectors normal to A. Let $h > 0$, we denote by $\mathrm{cyl}(A,h)$ the cylinder of basis A and height $2h$ defined by

$$\mathrm{cyl}(A,h) = \left\{ x + t\overrightarrow{v} \,:\, x \in A,\, t \in [-h,h] \right\}.$$

The set $\mathrm{cyl}(A,h) \setminus A$ has two connected components, denoted by $C_1(A,h)$ and $C_2(A,h)$. For $i = 1,2$, we denote by $C_i'(A,h)$ the discrete boundary of $C_i(A,h)$ defined by

$$C_i'(A,h) = \left\{ x \in \mathbb{Z}^d \cap C_i(A,h) \,:\, \exists y \notin \mathrm{cyl}(A,h),\, \langle x,y \rangle \in \mathbb{E}^d \right\}.$$

We say that a set of edges E cuts $C_1'(A,h)$ from $C_2'(A,h)$ in $\mathrm{cyl}(A,h)$ if any path γ from $C_1'(A,h)$ to $C_2'(A,h)$ in $\mathrm{cyl}(A,h)$ contains at least one edge of E. We call such a set a cutset. For any set of edges E, we denote by $|E|_o$ the number of open edges in E. We shall call it the capacity of E. We define

$$\tau_p(A,h) = \min \left\{ |E|_o : \;\; E \text{ cuts } C_1'(A,h) \text{ from } C_2'(A,h) \text{ in } \mathrm{cyl}(A,h) \right\}.$$

Note that this is a random quantity as $|E|_o$ is random, and that the cutsets in this definition are pinned near the boundary of A. Finding cutsets of minimal capacity is equivalent to the study of maximal flows, see [4]. To each edge e, we can associate the random variable $t(e) = \mathbb{1}_{e \text{ is open}}$. In the study of maximal flows, we interpret each $t(e)$ as the capacity of the edge e, *i.e.*, the maximal amount of water that can flow through e per unit of time. We are interested in the maximal amount of water that can flow through the cylinder given the constraint on the capacity. We refer to [19] for a rigorous definition of maximal flows. In the following, we will use the term flow to speak about the quantity τ_p. The following proposition is a corollary of Proposition 3.5 in [18], it enables us to give a rigorous definition of the norm β_p.

Proposition 3.1 (Definition of the norm β_p)**.** *Let $d \geq 2$, $p > p_c(d)$, A be a non-degenerate hyperrectangle and $\overrightarrow{v}$ one of the two unit vectors normal to A. Let h an height function such that $\lim_{n\to\infty} h(n) = \infty$. The limit*

$$\beta_p(\overrightarrow{v}) = \lim_{n\to\infty} \frac{\mathbb{E}[\tau_p(nA, h(n))]}{\mathcal{H}^{d-1}(nA)}$$

exists and is finite. Moreover, this limit is independent of A and h and β_p is a norm.

The norm β_p is called the flow constant. Roughly speaking, $\beta_p(\overrightarrow{v})$ corresponds to the expected maximal amount of water that can flow in the direction $\overrightarrow{v}$ on average. Actually, we can obtain a stronger convergence. A straightforward application of Theorem 3.8 in [18] gives the existence of the following almost sure limit:

$$\lim_{n\to\infty} \frac{\tau_p(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \beta_p(\overrightarrow{v})\,.$$

We define

$$\beta_{min} = \inf_{\overrightarrow{v}\in\mathbb{S}^{d-1}} \beta_p(\overrightarrow{v})\,, \qquad \beta_{max} = \sup_{\overrightarrow{v}\in\mathbb{S}^{d-1}} \beta_p(\overrightarrow{v})\,.$$

As β_p is a norm on $\mathbb{R}^d$, we have $\beta_{min} > 0$ and $\beta_{max} < \infty$. We will need the following upper large deviations result which is a straightforward application of Theorem 4 in [21].

Theorem 6. *Let $d \geq 2$ and $p > p_c(d)$. For every unit vector $\overrightarrow{v}$, for every non-degenerate hyperrectangle A normal to $\overrightarrow{v}$, for every $h > 0$ and for every $\lambda > \beta_p(\overrightarrow{v})$, there exist C_1 and C_2 depending only on λ and G, such that, for all $n \geq 0$,*

$$\mathbb{P}\left[\tau_p(nA, hn) \geq \lambda\mathcal{H}^{d-1}(A)n^{d-1}\right] \leq C_1 \exp(-C_2 hn^d)\,.$$

To ease the reading and lighten the notations, the value of the constants may change from appearance to appearance.

4 Upper large deviations

4.1 The case of a cylinder

The aim of this section is to prove Theorem 3. A convex polytope of dimension $d-1$ is a convex polytope F which is contained in an hyperplane of $\mathbb{R}^d$ and such that $\mathcal{H}^{d-1}(F) > 0$. We have the following Lemma.

Lemma 3. *Let $p > p_c(d)$. Let F be a convex polytope of dimension $d-1$. Let v be a unit vector normal to F. There exist positive real numbers C_1 and C_2 depending on F, p and d such that for all $n \geq 1$, for all $\lambda > \beta_p(v)\mathcal{H}^{d-1}(F)$, for all $h > 0$*

$$\mathbb{P}[\tau_p(nF, nh) \geq \lambda n^{d-1}] \leq C_1 \exp(-C_2 h n^d)\,.$$

Proof. Let $p > p_c(d)$. Let F be a convex polytope of dimension $d-1$ and v a unit vector normal to F. We shall cover F by a finite family of hypersquares and control the probability that the flow is abnormally big in $\mathrm{cyl}(nF, nh)$ by the probability that the flow is abnormally big in one of the cylinders of square basis. Let $\lambda > \beta_p(v)\mathcal{H}^{d-1}(F)$. Let $\kappa > 0$ be a real number that we will choose later. We denote by $S(\kappa)$ an hypersquare of dimension $d-1$ of side length κ and normal to v. We want to cover the following region of F by hypersquares isometric to $S(\kappa)$:

$$D(\kappa, F) = \Big\{ x \in F \,:\, d(x, \partial F) > 2\sqrt{d}\kappa \Big\}\,.$$

There exists a finite family $(S_i)_{i\in I}$ of closed hypersquares isometric to $S(\kappa)$ included in F having pairwise disjoint interiors, such that $D(\kappa, F) \subset \cup_{i\in I} S_i$ (see figure 2). Moreover, there exists a constant c_d depending only on the dimension d such that

$$\mathcal{H}^{d-1}\big(F \setminus D(\kappa, F)\big) \leq c_d \mathcal{H}^{d-2}(\partial F)\,\kappa\,. \tag{1}$$

We have then

$$|I| \leq \frac{\mathcal{H}^{d-1}(F)}{\mathcal{H}^{d-1}(S(\kappa))}\,. \tag{2}$$

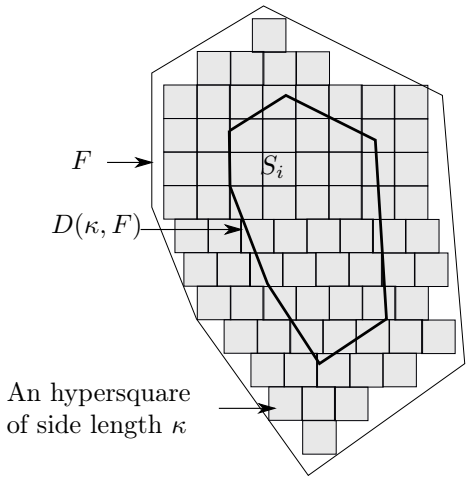

Figure 2 – Covering P with hypersquares

Let $h > 0$. We would like to build a cutset between $C_1'(nF, nh)$ and $C_2'(nF, nh)$ out of minimal cutsets for the flows $\tau_p(nS_i, nh)$, $i \in I$. Note that a cutset that

achieves the infimum defining $\tau_p(nS_i, nh)$ is pinned near the boundary ∂nS_i. However, if we pick up two hypersquares S_i and S_j that share a common side, due to the discretization, their corresponding minimal cutsets for the flow τ_p do not necessarily have the same trace on the common face of the associated cylinders $\mathrm{cyl}(nS_i, nh)$ and $\mathrm{cyl}(nS_j, nh)$. We shall fix this problem by adding extra edges around the boundaries of the hypersquares ∂S_i in order to glue properly the cutsets. We will need also to add extra edges around $n(F \setminus D(\kappa, F))$ in order to build a cutset between $C_1'(nF, nh)$ and $C_2'(nF, nh)$. For $i \in I$, let E_i be a minimal cutset for $\tau_p(nS_i, nh)$, *i.e.*, $E_i \subset \mathbb{E}^d$ cuts $C_1'(nS_i, nh)$ from $C_2'(nS_i, nh)$ in $\mathrm{cyl}(nS_i, nh)$ and $|E_i|_o = \tau_p(nS_i, nh)$. We fix $\zeta = 4d$. Let E_0 be the set of edges of $\mathbb{E}^d$ included in $\mathcal{E}_0$, where we define

$$\mathcal{E}_0 = \Big\{ x \in \mathbb{R}^d \,:\, d\Big(x, nF \setminus \bigcup_{i \in I} nS_i\Big) \le \zeta \Big\} \cup \bigcup_{i \in I} \Big\{ x \in \mathbb{R}^d \,:\, d(x, \partial nS_i) \le \zeta \Big\}.$$

The set of edges $E_0 \cup \bigcup_{i \in I} E_i$ separates $C_1'(nF, nh)$ from $C_2'(nF, nh)$ in the cylinder $\mathrm{cyl}(nF, nh)$, therefore,

$$\tau_p(nF, nh) \le |E_0|_o + \sum_{i \in I} |E_i|_o \le \mathrm{card}(E_0) + \sum_{i \in I} \tau_p(nS_i, nh)\,. \tag{3}$$

There exists a constant c_d' depending only on d such that, using inequalities (1) and (2),

$$\begin{aligned}
\mathrm{card}(E_0) &\le c_d' \left(\kappa n^{d-1} \mathcal{H}^{d-2}(\partial F) + |I| \mathcal{H}^{d-2}(\partial S(\kappa)) n^{d-2}\right) \\
&\le c_d' \left(\kappa n^{d-1} \mathcal{H}^{d-2}(\partial F) + \frac{\mathcal{H}^{d-1}(F)}{\mathcal{H}^{d-1}(S(\kappa))} \mathcal{H}^{d-2}(\partial S(\kappa)) n^{d-2}\right) \\
&\le c_d' \left(\kappa n^{d-1} \mathcal{H}^{d-2}(\partial F) + \frac{\mathcal{H}^{d-1}(F)}{\kappa} n^{d-2}\right).
\end{aligned}$$

Thus, for n large enough,

$$\mathrm{card}(E_0) \le 2c_d' \, \kappa \, \mathcal{H}^{d-2}(\partial F) n^{d-1}\,. \tag{4}$$

There exists $s > 0$ such that $\lambda > (1+s)\beta_p(v)\mathcal{H}^{d-1}(F)$. We choose κ small enough such that

$$2c_d' \kappa \mathcal{H}^{d-2}(\partial F) < \frac{s}{2} \beta_{min} \mathcal{H}^{d-1}(F)\,. \tag{5}$$

Inequalities (4) and (5) yield that

$$\mathrm{card}(E_0) \le \frac{s}{2} \beta_p(v) n^{d-1} \mathcal{H}^{d-1}(F)\,. \tag{6}$$

Thanks to inequality (6), we obtain

$$\begin{aligned}
&\mathbb{P}[\tau_p(nF, nh) \ge \lambda n^{d-1}] \\
&\qquad \le \mathbb{P}\left[\mathrm{card}(E_0) + \sum_{i \in I} \tau_p(nS_i, nh) \ge (1+s)\beta_p(v)\mathcal{H}^{d-1}(F) n^{d-1}\right] \\
&\qquad \le \sum_{i \in I} \mathbb{P}[\tau_p(nS_i, nh) \ge (1+s/2)\beta_p(v)\mathcal{H}^{d-1}(S_i) n^{d-1}]\,. \qquad (7)
\end{aligned}$$

Thanks to Theorem 6, there exist positive real numbers C_1, C_2 such that, for all $i \in I$,

$$\mathbb{P}[\tau_p(nS_i, nh) \geq (1+s/2)\beta_p(v)\mathcal{H}^{d-1}(S_i)n^{d-1}] \leq C_1 \exp(-C_2 hn^d)\,. \tag{8}$$

By combining inequalities (7) and (8), we obtain

$$\mathbb{P}[\tau_p(nF, nh) \geq \lambda n^{d-1}] \leq |I| C_1 \exp(-C_2 hn^d)\,,$$

and the result follows. □

We can now proceed to the proof of Theorem 3.

Proof of Theorem 3. Let $\varepsilon > 0$ and $\varepsilon' > 0$. By Lemma 2, there exists a convex polytope P such that $P \subset W_p$, $\mathcal{I}_p(P) \leq (1+\varepsilon')\mathcal{I}_p(W_p)$ and $\mathcal{L}^d(P) \geq (1-\varepsilon')\mathcal{L}^d(W_p)$. Up to multiplying P by a constant $\alpha < 1$ close to 1, we can assume without loss of generality that $\mathcal{L}^d(P) < \mathcal{L}^d(W_p)$. We have, for small enough ε' (depending on ε),

$$\begin{aligned}
\mathbb{P}\left[n\varphi_n \geq (1+\varepsilon)\frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)}\,\middle|\, 0 \in \mathcal{C}_\infty\right] & \\
&\leq \mathbb{P}\left[n\varphi_n \geq (1+\varepsilon/2)\left(\frac{1+\varepsilon'}{1-\varepsilon'}\right)\frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)}\,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\leq \mathbb{P}\left[n\varphi_n \geq (1+\varepsilon/2)\frac{\mathcal{I}_p(P)}{\theta_p(d)\mathcal{L}^d(P)}\,\middle|\, 0 \in \mathcal{C}_\infty\right]\,. \qquad (9)
\end{aligned}$$

Let us denote by $F_1, \ldots, F_m$ the faces of P and let $v_1, \ldots, v_m$ be the associated exterior unit vectors. Let $\delta > 0$. For $i \in \{1, \ldots, m\}$, we define

$$C_i = \mathrm{cyl}(F_i + \delta v_i, \delta)\,.$$

All the C_i are of disjoint interiors because P is convex. Indeed, assume there exists $z \in \mathring{C}_i \cap \mathring{C}_j$ for some $i \neq j$. Then there exist unique $x \in F_i$, $y \in F_j$ and $h, h' < 2\delta$ such that $z = x + hv_i = y + h'v_j$. The points x and y correspond to the orthogonal projection of z on P. As P is convex, the orthogonal projection on P is unique and so $x = y = z$. This contradicts the fact that z belongs to the interior of C_i. We now aim to build a cutset that cuts nP from infinity out of cutsets of minimal capacities for $\tau_p(n(F_i + \delta v_i), n\delta)$, $i \in \{1, \ldots, m\}$. The union of these cutsets is not enough to form a cutset from nP to infinity because there are holes between these cutsets. We shall add edges around the boundaries $\partial(n(F_i + \delta v_i))$ to close these holes (see figure 3). As the distance between two adjacent boundaries $\partial(n(F_i + \delta v_i))$ decreases with δ, by taking δ small enough, the size of the bridges and so their capacities are not too big. We recall that the capacity of a set, namely the number of open edges in the set, may be bounded from above by its size. Next, we control the maximal flow through the cylinders or equivalently the capacity of minimal cutsets in the cylinders with the help of Lemma 3.

For $i \in \{1, \ldots, m\}$, let E'_i be a minimal cutset for $\tau_p(n(F_i + \delta v_i), n\delta)$, *i.e.*, E'_i cuts $C'_1(n(F_i + \delta v_i), \delta)$ from $C'_2(n(F_i + \delta v_i), \delta)$ and $|E'_i|_o = \tau_p(n(F_i + \delta v_i), \delta n)$. We shall add edges to control the space between E'_i and the boundary $\partial(n(F_i + \delta v_i))$.

Let $\zeta = 4d$. Let $i, j \in \{1, \dots, m\}$ such that F_i and F_j share a common side. We define

$$\mathcal{M}(i,j) = (\mathcal{V}(nF_i \cap nF_j, n\delta + \zeta) \setminus \mathcal{V}(nF_i \cap nF_j, n\delta - \zeta)) \cap (nP)^c \,.$$

Let $M_{i,j}$ denote the set of the edges in $\mathbb{E}_n^d$ included in $\mathcal{M}(i,j)$ (see figure 3). There exists a constant c'_d depending only on the dimension d such that for all $i, j \in \{1, \dots, m\}$ such that F_i and F_j share a common side,

$$\mathrm{card}(M_{i,j}) \leq c_d \delta^{d-1} n^{d-1} \,. \tag{10}$$

We set

$$M = \bigcup_{i,j} M_{i,j} \,,$$

where the union is over $i, j \in \{1, \dots, m\}$ such that $i \neq j$ and F_i, F_j share a common side. The set $\Gamma_n = M \cup \bigcup_{i=1}^m E'_i$ cuts nP from infinity. We define H_n to be the set of the vertices connected to 0 by open paths which do not use an edge of Γ_n, *i.e.*,

$$H_n = \Big\{ x \in \mathbb{Z}^d,\, x \text{ is connected to 0 with open edges in } \mathbb{E}^d \setminus \Gamma_n \Big\} \,.$$

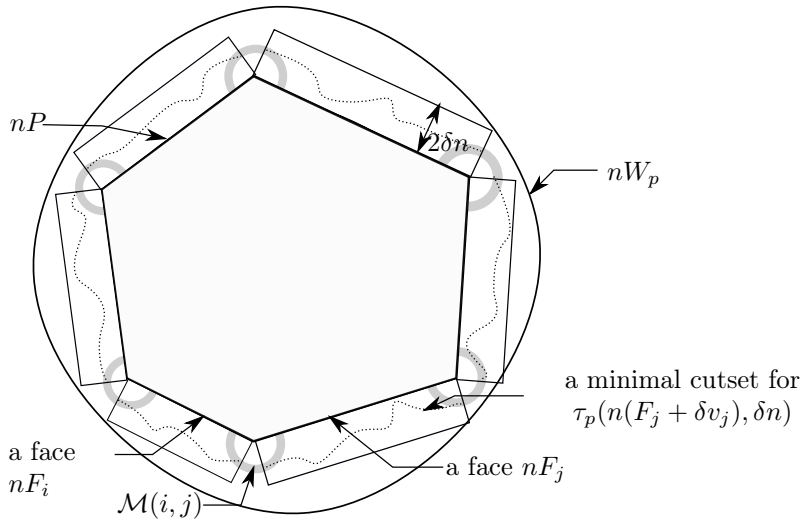

Figure 3 – Construction of a cutset Γ_n from nP to infinity

By definition, the set H_n is connected. As we condition on the event $\{0 \in \mathcal{C}_\infty\}$, the set H_n is a subgraph of $\mathcal{C}_\infty$. As P is a polytope,

$$\mathcal{I}_p(P) = \sum_{i=1}^m \beta_p(v_i) \mathcal{H}^{d-1}(F_i) \,.$$

Moreover, we have

$$|\partial_{\mathcal{C}_\infty} H_n| = |\partial^o H_n| \leq |\Gamma_n|_o \,,$$

where the last inequality comes from the fact that, by construction of H_n, if $e \in \partial H_n \setminus \Gamma_n$, then e is necessarily closed. Using (10), we have

$$\begin{aligned} |\Gamma_n|_o &\leq \mathrm{card}(M) + \sum_{i=1}^{m} |E_i'|_o \\ &\leq c_d m^2 \delta^{d-1} n^{d-1} + \sum_{i=1}^{m} \tau_p\big(n(F_i + \delta v_i), \delta n\big) \,. \end{aligned} \tag{11}$$

We choose δ small enough so that

$$m^2 c_d \delta^{d-1} < \delta \mathcal{I}_p(P)/2 \qquad \text{and} \qquad \mathcal{L}^d(\mathcal{V}(\partial P, 3\delta)) \leq \delta \mathcal{L}^d(P) \,. \tag{12}$$

Let us now estimate the probability that $|\Gamma_n|_o$ is abnormally big. Using inequalities (11) and (12), we get

$$\begin{aligned} &\mathbb{P}[|\Gamma_n|_o \geq (1+\delta)\mathcal{I}_p(P) n^{d-1} \,|\, 0 \in \mathcal{C}_\infty] \\ &\leq \frac{1}{\theta_p} \mathbb{P}\Big[\mathrm{card}(M) + \sum_{i=1}^{m} \tau_p(n(F_i + \delta v_i), \delta n) \geq (1+\delta) \sum_{i=1}^{m} \beta_p(v_i) \mathcal{H}^{d-1}(F_i) n^{d-1}\Big] \\ &\leq \frac{1}{\theta_p} \mathbb{P}\left[\sum_{i=1}^{m} \tau_p(n(F_i + \delta v_i), \delta n) \geq (1+\delta/2) \sum_{i=1}^{m} \beta_p(v_i) \mathcal{H}^{d-1}(F_i) n^{d-1}\right] \\ &\leq \frac{1}{\theta_p} \sum_{i=1}^{m} \mathbb{P}[\tau_p(n(F_i + \delta v_i), \delta n) \geq (1+\delta/2) \beta_p(v_i) \mathcal{H}^{d-1}(F_i) n^{d-1}] \,. \end{aligned} \tag{13}$$

By Lemma 3, there exist positive constants C_1, C_2 depending on d, p, P and δ such that, for all $1 \leq i \leq m$,

$$\mathbb{P}[\tau_p(n(F_i + \delta v_i), \delta n) \geq (1+\delta/2) \beta_p(v_i) \mathcal{H}^{d-1}(F_i) n^{d-1}] \leq C_1 \exp(-C_2 \delta n^d) \,. \tag{14}$$

Finally, combining inequalities (13) and (14), we obtain

$$\mathbb{P}[|\Gamma_n|_o \geq (1+\delta)\mathcal{I}_p(P) n^{d-1}] \leq \frac{m C_1}{\theta_p} \exp(-C_2 \delta n^d) \,. \tag{15}$$

We shall now estimate the number of vertices in H_n in order to check that H_n is a valid subgraph. For that purpose, we use a renormalization argument. Let $k > 0$. We partition $\mathbb{R}^d$ into disjoint cubes of side length $1/k$. We define B_j' as the union of B_j and all its $3^d - 1$ $*$-neighbors (the cubes B having at least one vertex at L^1 distance less than 1 from B_j). We consider $B_1, \ldots, B_{l_1}$ the cubes such that $B_1', \ldots, B_{l_1}'$ are contained in $P \setminus \mathcal{V}(\partial P, 2\delta)$ and $B_{l_1+1}, \ldots, B_{l_2}$ the cubes such that $B_{l_1+1}', \ldots, B_{l_2}'$ intersect $\mathcal{V}(\partial P, 2\delta)$. We can choose k large enough such that

$$\mathcal{L}^d\left(\bigcup_{i=l_1+1}^{l_2} B_i\right) \leq \mathcal{L}^d(\mathcal{V}(\partial P, 3\delta)) \leq \delta \mathcal{L}^d(P) \,. \tag{16}$$

We say that a cube B_j is good if the following event $\mathcal{E}_n^{(j)}$ occurs:

— There exists a unique open cluster of diameter larger than n/k in nB_j'.

— We have $\dfrac{|\mathcal{C}_\infty \cap nB_j|}{\mathcal{L}^d(nB_j)} \in (\theta_p - \delta, \theta_p + \delta)$.

There exist positive constants C_1 and C_2 depending on d, p, k and δ such that

$$\mathbb{P}[\mathcal{E}_n^{(j)c}] \leq C_1 \exp(-C_2 n)\,. \tag{17}$$

For a proof of the control of the probability of the first property see Theorem 7.68 in [11] or [16], for the second property see [16]. If the cube B_j is good, we denote by C_j its unique open cluster of diameter larger than n/k in nB'_j, for $1 \leq j \leq l_1$. On the event $\bigcap_{1\leq j\leq l_1} \mathcal{E}_n^{(j)} \cap \{\, 0 \in \mathcal{C}_\infty \,\}$, the set $\bigcup_{j=1}^{l_1} C_j$ is connected without using edges of Γ_n and contains 0, therefore, it is a subgraph of H_n. Furthermore, we claim that, on this event, we have $\mathcal{C}_\infty \cap (\bigcup_{1\leq j\leq l_1} nB_j) \subset H_n$. Indeed, let us assume that there exists $x \in \mathcal{C}_\infty \cap (\bigcup_{1\leq j\leq l_1} nB_j)$ that does not belong to H_n. Both 0 and x belong to $\mathcal{C}_\infty$, therefore, x is connected to 0 by a path $\gamma = (x_0, e_1, \ldots, e_l, x_l)$ with $x_0 = 0$ and $x_l = x$ that uses edges in Γ_n. We define

$$r = \sup\{\, i \geq 1,\, e_i \in \Gamma_n \,\}\,.$$

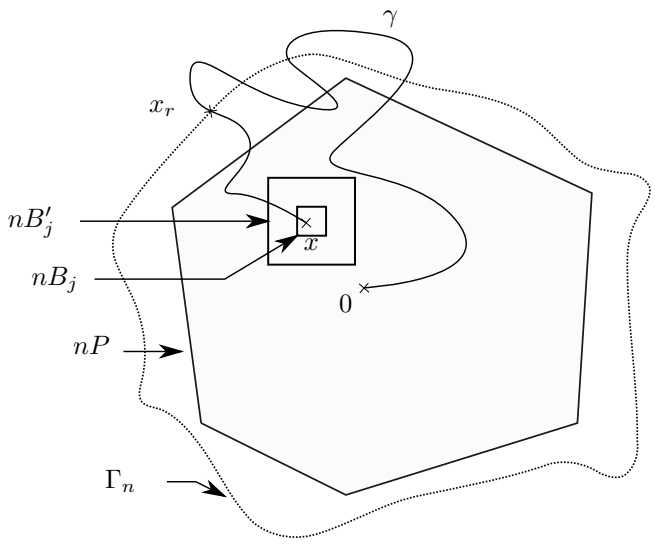

Figure 4 – Vertices in H_n

By construction, as $e_l \notin \Gamma_n$, we have $r < l$. Let us denote $\gamma' = (x_r, e_{r+1}, \ldots, x_l)$. The path γ' is not connected to H_n without using edges in Γ_n (see figure 4). Let j such that $x \in nB_j$, by construction x_r is outside nB'_j. Moreover, on the event $\mathcal{E}_n^{(j)}$, the cube nB'_j contains a unique cluster of diameter larger than n/k. As the path γ' starts outside nB'_j and ends inside nB_j, its intersection with nB'_j has a diameter larger than n/k. Besides, the path γ' is not connected to H_n in nB'_j by an open path, so the cube nB'_j contains two open clusters of diameter larger than n/k. This is a contradiction with the first property of a good cube.

Therefore, on the event $\bigcap_{1\le j\le l_1} \mathcal{E}_n^{(j)} \cap \{\, 0 \in \mathcal{C}_\infty \,\}$,

$$\begin{aligned} |H_n| &\ge |\mathcal{C}_\infty \cap (\cup_{1\le j\le l_1} nB_j)| \\ &\ge (\theta_p - \delta) \sum_{i=1}^{l_1} \mathcal{L}^d(nB_i)\,. \end{aligned} \tag{18}$$

Thanks to inequalities (16) and (18), we obtain

$$|H_n| \ge (\theta_p - \delta)(1-\delta)\mathcal{L}^d(nP)\,. \tag{19}$$

To ensure that H_n is a valid subgraph, it remains to check that $|H_n| \le n^d$, yet we have

$$\begin{aligned} |H_n| &\le (\theta_p + \delta) \sum_{i=1}^{l_1} \mathcal{L}^d(nB_i) + \sum_{i=l_1+1}^{l_2} \mathcal{L}^d(nB_i) \\ &\le (\theta_p + \delta) n^d \mathcal{L}^d(P) + n^d \delta \mathcal{L}^d(P) \\ &\le (\theta_p + 2\delta) n^d \mathcal{L}^d(P)\,. \end{aligned}$$

As $\mathcal{L}^d(P) < \mathcal{L}^d(W_p)$, we can choose δ small enough such that

$$|H_n| \le \theta_p \mathcal{L}^d(W_p) n^d \le n^d\,.$$

Finally, on the event

$$\bigcap_{1\le j\le l_1} \mathcal{E}_n^{(j)} \cap \{\, |\Gamma_n|_o \le (1+\delta)\mathcal{I}_p(P) n^{d-1} \,\} \cap \{\, 0 \in \mathcal{C}_\infty \,\}\,,$$

combining (11) and (19), we obtain, for small enough δ,

$$n\varphi_n \le \frac{|\Gamma_n|_o}{|H_n|} \le (1+\delta) \frac{\mathcal{I}_p(P)}{(\theta_p - 2\delta)(1-\delta)\mathcal{L}^d(P)} \le (1+\varepsilon/2) \frac{\mathcal{I}_p(P)}{\theta_p \mathcal{L}^d(P)}\,.$$

Combining the result of Lemma 3 and inequalities (9), (15) and (17), we obtain

$$\begin{aligned} &\mathbb{P}\left[n\varphi_n \ge (1+\varepsilon) \frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)} \middle| 0 \in \mathcal{C}_\infty \right] \\ &\qquad\qquad \le \frac{l_1 C_1}{\theta_p} \exp(-C_2 n) + \frac{m C_1}{\theta_p} \exp(-C_2 \delta n^d)\,. \end{aligned}$$

This yields the result. □

5 Construction of a continuous object

The aim of this section is to build a continuous object P_n from a minimizer $G_n \in \mathcal{G}_n$.

5.1 Some useful results on the minimizers

The following lemma ensures that the size of the minimizers $G_n \in \mathcal{G}_n$ are of order n^d.

Lemma 4. *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants D_1, D_2 and η_1 depending only on d and p such that, for all $n \geq 1$,*

$$\mathbb{P}\Big[\, \exists G_n \in \mathcal{G}_n,\, |G_n| \leq \eta_1 n^d \,\big|\, 0 \in \mathcal{C}_\infty \,\Big] \leq D_1 \exp(-D_2 n^{(d-1)/2d})\,.$$

To prove Lemma 4, we adapt the proof of Lemma A.8 in [10]. We need the following proposition that ensures that the open edge boundary of a large subgraph is not too small.

Proposition 5.1 (Berger-Biskup-Hoffman-Kozma, Proposition 5.2. in [2])**.** *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants c_1, c_2 and c_3 depending only on d and p such that, for all $t \geq 0$,*

$$\mathbb{P}\left[\begin{array}{c}\textit{There exists an open connected graph containing } 0 \\ \textit{such that } |G| \geq t^{d/(d-1)},\ |\partial^o G| \leq c_3 |G|^{(d-1)/d}\end{array}\right] \leq c_1 \exp(-c_2 t)\,.$$

Proof of Lemma 4. Thanks to Theorem 3, there exist positive constants c_1', c_2' and c_3' depending only on p and d such that for all $n \geq 1$,

$$\mathbb{P}\Big[\, \varphi_n \geq c_3' n^{-1} \,\big|\, 0 \in \mathcal{C}_\infty \,\Big] \leq c_1' \exp(-c_2' n)\,.$$

Let $G_n \in \mathcal{G}_n$. If $|G_n| \leq \sqrt{n}$, as $G_n \subset \mathcal{C}_\infty$ the set $\partial^o G_n$ is non empty on the event $\{0 \in \mathcal{C}_\infty\}$ and so $\varphi_n \geq n^{-1/2}$. This is impossible for large n. We now assume $|G_n| > \sqrt{n}$. Using Proposition 5.1 with $t = n^{(d-1)/2d}$, conditioning on $\{0 \in \mathcal{C}_\infty\}$, we obtain that $|\partial^o G_n| \geq c_3 |G_n|^{(d-1)/d}$ with probability at least $1 - c_1 \exp(-c_2 n^{(d-1)/2d})/\theta_p$. Moreover, on the event $\big\{\varphi_n \leq c_3' n^{-1}\big\} \cap \big\{0 \in \mathcal{C}_\infty\big\}$, we obtain

$$c_3 |G_n|^{-1/d} \leq \frac{|\partial^o G_n|}{|G_n|} = \varphi_n \leq c_3' n^{-1}.$$

So we set $\eta_1 = (c_3/c_3')^d$. Finally,

$$\begin{aligned}
&\mathbb{P}\Big[\exists G_n \in \mathcal{G}_n,\, |G_n| \leq \eta_1 n^d \,\big|\, 0 \in \mathcal{C}_\infty \,\Big] \\
&\qquad \leq \mathbb{P}\Big[\, \varphi_n \geq c_3' n^{-1} \,\big|\, 0 \in \mathcal{C}_\infty \,\Big] + \frac{c_1}{\theta_p} \exp(-c_2 n^{(d-1)/2d}) \\
&\qquad \leq c_1' \exp(-c_2' n) + \frac{c_1}{\theta_p} \exp(-c_2 n^{(d-1)/2d})\,.
\end{aligned}$$

This yields the result. □

5.2 Construction of a continuous set

To study the upper large deviations, we needed to go from a continuous object to a discrete object. In this section, we do the opposite. From now on, we will always condition on the event $\{0 \in \mathcal{C}_\infty\}$. We start with $G_n \in \mathcal{G}_n$ and we build a continuous object P_n. Our goal is to build a continuous object of finite perimeter which is close to $n^{-(d-1)} |\partial^o G_n|$. Although it seems natural

to take the continuous object $P_n = n^{-1}(G_n + [-1/2, 1/2]^d)$, this turns out to be a bad choice because the boundary ∂G_n may be very tangled and its size may be of higher order than n^{d-1}. We will build from G_n a graph F_n with a smoother boundary $\Gamma_n \subset \mathbb{E}^d$ in order to build the continuous object P_n. At this point, there is some work left. If we consider the subgraph F_n that contains all the vertices in $\mathcal{C}_\infty$ enclosed in Γ_n, the symmetric difference $F_n \Delta G_n$ may be big due to the presence of holes in G_n, more precisely portions of $\mathcal{C}_\infty$ enclosed in Γ_n but not contained in G_n (see Figure 5). Indeed, if these holes are too large, the symmetric difference $F_n \Delta G_n$ will be large too. However, we cannot keep all the holes in G_n to build F_n because when we will pass to a continuous object P_n, these holes will considerably increase the perimeter of P_n so that P_n may have a too large perimeter. The solution is to fill only the small holes to obtain F_n so that the perimeter of P_n remains of the correct order and the symmetric difference $F_n \Delta G_n$ remains small. In order to do so, we shall perform Zhang's construction in [23] to obtain a smooth boundary Γ_n for G_n but also to surgically remove these large holes from G_n by cutting along a smooth boundary. This work was done in [10]. We will only partially sketch Zhang's construction and we refer to [23] for a rigorous proof and more details about the construction. Although we did the same construction as Gold in [10], we do not use the same argument to conclude. Gold used a procedure called webbing to link all the different contours together in order to obtain a single connected object, this simplifies the combinatorial estimates. Here, we do not perform the webbing procedure, instead we use adequate combinatorial estimates. Avoiding the webbing procedure enables us to extend the result to dimension 2.

Let us define a renormalization process. For a large integer k, that will be chosen later, we set $B_k = [-k, k[^d \cap \mathbb{Z}^d$ and define the following family of k-cubes, for $\mathbf{i} \in \mathbb{Z}^d$,

$$B_k(\mathbf{i}) = \tau_{\mathbf{i}(2k+1)}(B_k)\,,$$

where τ_b denotes the shift in $\mathbb{Z}^d$ with vector $b \in \mathbb{Z}^d$. The lattice $\mathbb{Z}^d$ is the disjoint union of this family: $\mathbb{Z}^d = \sqcup_{\mathbf{i}\in\mathbb{Z}^d} B_k(\mathbf{i})$. We introduce larger boxes B'_k, for $\mathbf{i} \in \mathbb{Z}^d$, we define

$$B'_k(\mathbf{i}) = \tau_{\mathbf{i}(2k+1)}(B_{3k}).$$

Underscore will be used to denote sets of cubes. For any set of k-cubes $\underline{A}$, the set $\underline{A}'$ denotes the set of the corresponding $3k$-cubes. Let $G_n \in \mathcal{G}_n$. We first use Zhang's construction to build a smooth cutset Γ_n that separates G_n from infinity. We denote by $\underline{A}$ the set of k-cubes that intersect $\partial_e G_n$, the exterior edge boundary of G_n. We then modify the current configuration ω into a configuration ω' by closing all the open edges in ∂G_n. This procedure is only formal as we will eventually reopen these edges. Zhang's construction enables us to extract a set of cubes $\underline{\Gamma} \subset \underline{A}$ such that $\underline{\Gamma}$ is $*$-connected and in the configuration ω', the union of the $3k$-cubes of $\underline{\Gamma}'$ contains a closed cutset Γ_n that isolates G_n from infinity and a rare event occurs in every cube of $\underline{\Gamma}$. These rare events are due to the existence of a closed cutset that creates a large interface of closed edges, this is a very unlikely event when $p > p_c(d)$. Of course, when we will eventually switch back to the configuration ω, these rare events will not occur anymore in some cubes.

Several connected components of $\mathcal{C}_\infty \setminus G_n$ in $\mathbb{Z}^d \setminus \Gamma_n$ are enclosed in Γ_n (see Figure 5). We say that a connected component C of $\mathcal{C}_\infty$ is surrounded by Γ_n if any path from C to infinity has to use an edge of Γ_n. We will say

that C is large if $|C| \geq n^{1-1/2(d-1)}$. We enumerate the large connected components $L_1, \dots, L_m$ and the small connected component $S_1, \dots, S_N$. We denote by $m(G_n)$ the number of large connected components of $\mathcal{C}_\infty \setminus G_n$ enclosed in Γ_n.

Remark 5.1. We insist here on the fact that these large components are not holes of the infinite cluster but holes of G_n (see Figure 5). Intuitively, we do not expect that a minimizer contains such holes because the graph obtained by filling all these holes have a smaller isoperimetric ratio. Indeed, by filling these holes, we reduce the open edge boundary and increase the volume. However, by filling these holes, the volume may exceed n^d and the graph we obtain by filling these holes may not be admissible. That is the reason why we cannot easily discard the presence of these large holes inside G_n. To obtain the proper order of large deviations, one would have to fix this issue.

We then build $F_n \subset \mathcal{C}_\infty$ by filling the small connected components $S_1, \dots, S_N$ of G_n, *i.e.*,

$$F_n = G_n \cup \bigcup_{i=1}^{N} S_i \,. \tag{20}$$

At this point, the boundary $\partial F_n \setminus \partial_e F_n$ of F_n may be still tangled around the large components. In the configuration ω', for each $1 \leq j \leq m$, there exists a closed cutset that separates L_j from infinity. We can apply Zhang's construction to each component L_j in order to build a smooth closed cutset $\widehat{\Gamma}_n^{(j)}$ and its corresponding set of k-cubes $\underline{\widehat{\Gamma}}_n^{(j)}$. Thanks to Zhang's construction, the set of cubes $\underline{\widehat{\Gamma}}_n^{(j)}$ is $*$-connected and in the configuration ω', a rare event occurs in each of its cubes. We denote the boundary of F_n by $\widetilde{\Gamma}_n$ and its associated set of k-cubes $\underline{\widetilde{\Gamma}}_n$ as

$$\widetilde{\Gamma}_n = \Gamma_n \cup \bigcup_{i=1}^{m} \widehat{\Gamma}_n^{(i)}, \qquad \underline{\widetilde{\Gamma}}_n = \underline{\Gamma}_n \cup \bigcup_{i=1}^{m} \underline{\widehat{\Gamma}}_n^{(i)} \,.$$

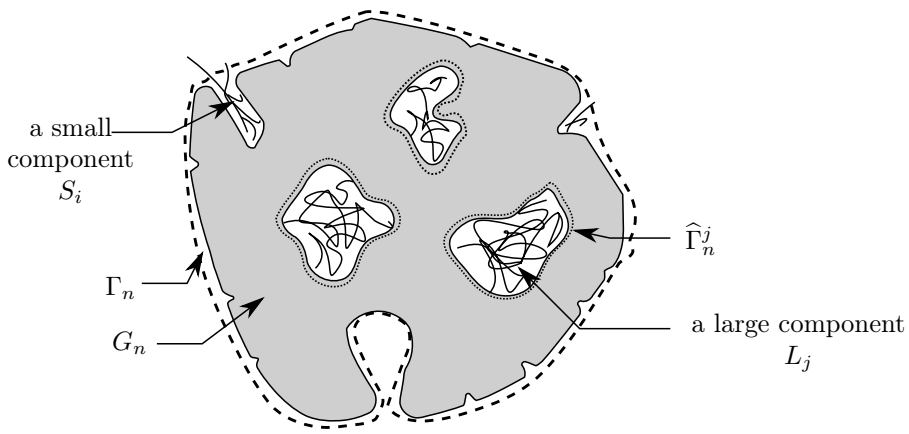

Figure 5 – Construction of $\widetilde{\Gamma}_n$ for a $G_n \in \mathcal{G}_n$

The set of k-cubes $\widetilde{\underline{\Gamma}}_n$ is not $*$-connected. It only contains cubes where a rare event occurs in the configuration ω'. Although for some cubes these events do not occur anymore in the configuration ω, we can bound from below the number of cubes that remain unchanged by $|\widetilde{\underline{\Gamma}}_n| - |\partial^o G_n|$. In these cubes, rare events still occur when we switch back to the original configuration ω. Using a Peierls argument, we can deduce that, with high probability, $|\widetilde{\underline{\Gamma}}_n|$ and $|\partial^o G_n|$ are of same order when k is taken large enough. To perform the combinatorial estimates we will need the two following propositions.

Proposition 5.2 (Lemmas 6, 7 in [23])**.** *Let $d \geq 2$ and let $p > p_c(d)$. There exist positive constants C_1 and C_2 depending only on p and d such that for each k-cube B_k,*

$$\mathbb{P}(\textit{a rare event occurs in } B_k) \leq C_1 \, \mathrm{e}^{-C_2 k} \ .$$

Moreover, this rare event depends only on the configuration of the $3k$-cube B'_k.

Remark 5.2. We do not define here what these rare events are, we refer to [23] for a precise definition of these rare events. For our purpose we only need to know that the decay is exponential in k. We say that a cube is abnormal if a rare event occurs in this cube.

Proposition 5.3. *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants c_1, c_2 and c_3 such that*

$$\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, m(G_n) > c_3 n^{d-2+3/2d} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \leq c_1 \exp(-c_2 n^{1-3/2d}) \,.$$

Proof. Thanks to Theorem 3, there exist positive constants C'_1, C'_2 and C'_3 depending only on p and d such that for all $n \geq 1$,

$$\mathbb{P}\Big[\, \varphi_n \geq C'_3 n^{-1} \,\Big|\, 0 \in \mathcal{C}_\infty \,\Big] \leq C'_1 \exp(-C'_2 n) \,.$$

Let $G_n \in \mathcal{G}_n$. We have with probability at least $1 - C'_1 \exp(-C'_2 n)$ that

$$|\partial^o G_n| \leq C'_3 n^{-1} |G_n| \leq C'_3 n^{d-1} \,.$$

Thanks to Proposition 5.1, there exist positive constants c'_1, c'_2 and c'_3 depending only on p and d such that, for all $t \geq 0$, we have

$$\mathbb{P}\left[\begin{array}{c}\text{There exists an open connected graph containing } 0 \\ \text{such that } |G| \geq t^{d/(d-1)},\ |\partial^o G| \leq c'_3 |G|^{(d-1)/d}\end{array}\right] \leq c'_1 \exp(-c'_2 t) \,. \tag{21}$$

In the following, we set $t = n^{(1-1/2(d-1))(d-1)/d} = n^{1-3/2d}$. First notice that by construction, each L_j is contained in $[-n^d, n^d] \cap \mathbb{Z}^d$. We have

$$\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, \exists i \in \{1,\dots,m(G_n)\},\, |\partial^o L_i| \leq c'_3 n^{(1-1/2(d-1))d/(d-1)} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\leq \mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, \exists i \in \{1,\dots,m(G_n)\},\, |\partial^o L_i| \leq c'_3 |L_i|^{d/(d-1)} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\leq \frac{1}{\theta_p} \mathbb{P}\left[\begin{array}{c}\text{There exists an open connected graph } G \text{ contained in} \\ [-n^d, n^d] \cap \mathbb{Z}^d \text{ such that } |G| \geq t^{d/(d-1)},\ |\partial^o G| \leq c'_3 |G|^{(d-1)/d}\end{array}\right] \\
&\leq \frac{1}{\theta_p} \sum_{x \in [-n^d, n^d] \cap \mathbb{Z}^d} \mathbb{P}\left[\begin{array}{c}\text{There exists an open connected graph } G \text{ containing} \\ x \text{ such that } |G| \geq t^{d/(d-1)},\ |\partial^o G| \leq c'_3 |G|^{(d-1)/d}\end{array}\right] .
\end{aligned}$$

Using the translation invariance together with inequality (21), we obtain

$$\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, \exists i \in \{1,\dots,m(G_n)\},\, |\partial^o L_i| \le c_3' n^{(1-\varepsilon)d/(d-1)} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\qquad \le \frac{(2n^d)^d}{\theta_p}\,\mathbb{P}\left[\begin{array}{c}\text{There exists an open connected graph } G \text{ containing } 0 \\ \text{such that } |G| \ge t^{d/(d-1)},\, |\partial^o G| \le c_3' |G|^{(d-1)/d}\end{array}\right] \\
&\qquad \le \frac{(2n^d)^d}{\theta_p} c_1' \exp(-c_2' n^{1-3/2d})\,.
\end{aligned}$$

By construction, for all $i \in \{1,\dots,m(G_n)\}$, we have $\partial^o L_i \subset \partial^o G_n$ and for all $j \in \{1,\dots,m(G_n)\}$ such that $i \neq j$, we have $\partial^o L_i \cap \partial^o L_j = \emptyset$. Thus, with high probability,

$$m(G_n) \le \frac{|\partial^o G_n|}{c_3' n^{(1-\varepsilon)d/(d-1)}} \le \frac{C_3' n^{d-1}}{c_3' n^{1-3/2d}} \le \frac{C_3'}{c_3'} n^{d-2+3/2d}\,.$$

Finally, by setting $c_3 = C_3'/c_3'$, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, m(G_n) > c_3 n^{d-2+3/2d} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\quad \le \mathbb{P}\Big[\varphi_n \ge c_3' n^{-1} \,\Big|\, 0 \in \mathcal{C}_\infty\Big] + \mathbb{P}\left[\begin{array}{c}\exists G_n \in \mathcal{G}_n,\, \exists i \in \{1,\dots,m(G_n)\}, \\ |\partial^o L_i| \le c_3' n^{(1-\varepsilon)d/(d-1)}\end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\quad \le C_1' \exp(-C_2' n) + \frac{(2n^d)^d}{\theta_p} c_1' \exp(-c_2' n^{1-3/2d})\,.
\end{aligned}$$

This yields the result. □

Using the control on the number of large components $m(G_n)$ of $\mathcal{C}_\infty$ enclosed in Γ_n and a Peierls argument, we obtain the following control of $|\widetilde{\Gamma}_n|$:

Proposition 5.4. *Let $d \ge 2$ and $p > p_c(d)$. There exist positive constants β_0, C_1, C_2 depending only on d and p such that, for all $n \ge 1$, for all $\beta \ge \beta_0$,*

$$\mathbb{P}\left[\max_{G_n \in \mathcal{G}_n} |\widetilde{\Gamma}_n| \ge \beta n^{d-1} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \le C_1 \exp(-C_2 n^{1-3/2d})\,.$$

Proof. Let k be a large integer that we will choose later. We consider a renormalization process of parameter k. Let $G_n \in \mathcal{G}_n$. First notice that as $\widetilde{\Gamma}_n \subset \bigcup_{B \in \underline{\widetilde{\Gamma}}_n} B'$, we have

$$|\widetilde{\Gamma}_n| \le (6k)^d |\underline{\widetilde{\Gamma}}_n|\,.$$

Thus, it is enough to control the quantity $|\underline{\widetilde{\Gamma}}_n|$ to prove Proposition 5.4. We can rewrite $\underline{\widetilde{\Gamma}}_n$ as

$$\underline{\widetilde{\Gamma}}_n = \bigcup_{i=1}^{m'} \underline{A}_i \text{ with } m' \le m(G_n)$$

where the $\underline{A}_i$ are pairwise disjoint $*$-connected sets of cubes. Thanks to Theorem 3, there exist positive constants C_1', C_2' and C_3' depending only on p and d such that for all $n \ge 1$,

$$\mathbb{P}\Big[\varphi_n \ge C_3' n^{-1} \,\Big|\, 0 \in \mathcal{C}_\infty\Big] \le C_1' \exp(-C_2' n)\,. \tag{22}$$

Let $G_n \in \mathcal{G}_n$. We have with probability at least $1 - C_1' \exp(-C_2' n)$ that

$$|\partial^o G_n| \leq C_3' n^{d-1} .$$

We choose β large enough such that

$$C_3' \leq \frac{\beta}{2 \cdot 4^d} ,$$

so that

$$|\partial^o G_n| \leq C_3' n^{d-1} \leq \frac{\beta}{2 \cdot 4^d} n^{d-1} .$$

We now want to sum over the possible realizations of $\underline{\widetilde{\Gamma}}_n$. Using Proposition 5.3 together with inequality (22), we get

$$\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, |\underline{\widetilde{\Gamma}}_n| \geq \beta n^{d-1} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\quad \leq \mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\ \begin{array}{c} \sum_{i=1}^{m'} |\underline{A}_i| \geq \beta n^{d-1},\, m' \leq c_3' n^{d-2+3/2d}, \\ |\partial^o G_n| \leq \frac{\beta}{2\cdot 4^d} n^{d-1} \end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\qquad + c_1 \exp(-c_2 n^{1-3/2d}) + C_1' \exp(-C_2' n) \\
&\quad \leq \sum_{j \geq \beta n^{d-1}} \sum_{m'=1}^{c_3' n^{d-2+3/2d}} \sum_{\substack{j_1+\cdots+j_{m'}=j \\ j_1>0,\,\ldots,\,j_{m'}>0}} \sum_{x_1,\ldots,x_{m'} \in [-n^d,n^d]^d} \sum_{\substack{A_1 \in \mathrm{Animals}_{x_1} \\ |A_1|=j_1}} \cdots \\
&\qquad \cdots \sum_{\substack{A_{m'} \in \mathrm{Animals}_{x_{m'}} \\ |A_{m'}|=j_{m'}}} \mathbb{P}\Big[\exists G_n \in \mathcal{G}_n,\, \underline{\widetilde{\Gamma}}_n = \bigcup_{i=1}^{m'} A_i,\, |\partial^o G_n| \leq \frac{\beta}{2 \cdot 4^d} n^{d-1} \,\Big|\, 0 \in \mathcal{C}_\infty\Big] \\
&\qquad + c_1 \exp(-c_2 n^{1-3/2d}) + C_1' \exp(-C_2' n) . \qquad (23)
\end{aligned}$$

Let us assume $\underline{\widetilde{\Gamma}}_n = \bigcup_{i=1}^{m'} A_i$. We can extract from $\underline{\widetilde{\Gamma}}_n$ a set of k-cubes $\underline{\widetilde{\Gamma}'}_n$ such that $|\underline{\widetilde{\Gamma}'}_n| \geq |\underline{\widetilde{\Gamma}}_n|/4^d$ and for any $\mathbf{i} \neq \mathbf{j}$ such that $B_k(\mathbf{i}), B_k(\mathbf{j}) \in \underline{\widetilde{\Gamma}'}_n$ we have $B_k'(\mathbf{i}) \cap B_k'(\mathbf{j}) = \emptyset$. As the rare event depends only on the configuration in the $3k$-cube $B_k'(\mathbf{j})$, the two following events $\big\{$a rare event occurs in $B_k(\mathbf{i})\big\}$ and $\big\{$a rare event occurs in $B_k(\mathbf{j})\big\}$ are independent. Using Proposition 5.2, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, \underline{\widetilde{\Gamma}}_n = \bigcup_{i=1}^{m'} A_i,\, |\partial^o G_n| \leq \frac{\beta}{2 \cdot 4^d} n^{d-1} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\quad \leq \mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, \underline{\widetilde{\Gamma}}_n = \bigcup_{i=1}^{m'} A_i,\, |\underline{\widetilde{\Gamma}'}_n| \geq j/4^d,\, |\partial^o G_n| \leq \frac{\beta}{2 \cdot 4^d} n^{d-1} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\quad \leq \mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\ \begin{array}{c} \underline{\widetilde{\Gamma}}_n = \bigcup_{i=1}^{m'} A_i,\, |\partial^o G_n| \leq \frac{\beta}{2\cdot 4^d} n^{d-1}, \\ |\{B \subset \underline{\widetilde{\Gamma}'}_n,\ B \text{ abnormal}\}| \geq j/4^d - |\partial^o G_n| \end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\quad \leq \mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, \underline{\widetilde{\Gamma}}_n = \bigcup_{i=1}^{m'} A_i,\, |\{B \subset \underline{\widetilde{\Gamma}'}_n,\ B \text{ abnormal}\}| \geq j/(2.4^d)\right] \cdot \frac{1}{\theta_p}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{4^d}{\theta_p} \sum_{l \geq j/(2.4^d)} \left(C_1 \, \mathrm{e}^{-C_2 k}\right)^l \\
&\leq \frac{2 \cdot 4^d}{\theta_p} \left(C_1 \, \mathrm{e}^{-C_2 k}\right)^{j/(2.4^d)}
\end{aligned}
$$

where k will be chosen large enough such that $C_1 \, \mathrm{e}^{-C_2 k} \leq 1/2$. So together with inequality (23) and using Lemma 1, we obtain

$$
\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n, \, |\widetilde{\underline{\Gamma}}_n| \geq \beta n^{d-1} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\leq \sum_{j \geq \beta n^{d-1}} \sum_{m'=1}^{c_3' n^{d-2+3/2d}} \sum_{\substack{j_1 + \cdots + j_{m'} = j \\ j_1 > 0, \ldots, j_{m'} > 0}} \sum_{x_1, \ldots, x_{m'} \in [-n^d, n^d]^d} \sum_{\substack{A_1 \in \mathrm{Animals}_{x_1} \\ |A_1| = j_1}} \cdots \\
&\cdots \sum_{\substack{A_{m'} \in \mathrm{Animals}_{x_{m'}} \\ |A_{m'}| = j_{m'}}} \frac{2 \cdot 4^d}{\theta_p} \left(C_1 \, \mathrm{e}^{-C_2 k}\right)^{j/(2.4^d)} + c_1 \, \mathrm{e}^{-c_2 n^{1-3/2d}} + C_1' \, \mathrm{e}^{-C_2' n} \\
&\leq \frac{2 \cdot 4^d}{\theta_p} \sum_{j \geq \beta n^{d-1}} \left(C_1 \, \mathrm{e}^{-C_2 k}\right)^{\frac{j}{2.4^d}} \sum_{m'=1}^{c_3' n^{d-2+3/2d}} \sum_{\substack{j_1 + \cdots + j_{m'} = j \\ j_1 > 0, \ldots, j_{m'} > 0}} (2n)^{d^2 m'} 7^{d j_1} \cdots 7^{d j_{m'}} \\
&\quad + c_1 \, \mathrm{e}^{-c_2 n^{1-3/2d}} + C_1' \, \mathrm{e}^{-C_2' n} \\
&\leq \frac{2 \cdot 4^d}{\theta_p} \sum_{j \geq \beta n^{d-1}} 7^{dj} \left(C_1 \, \mathrm{e}^{-C_2 k}\right)^{j/(2.4^d)} \sum_{m'=1}^{c_3' n^{d-2+3/2d}} (2n^d)^{dm'} \cdot \\
&\quad \times \left| \left\{ (j_1, \ldots, j_{m'}) : \begin{array}{c} j_1 + \cdots + j_{m'} = j, \\ j_1 > 0, \ldots, j_{m'} > 0 \end{array} \right\} \right| + c_1 \, \mathrm{e}^{-c_2 n^{1-3/2d}} + C_1' \, \mathrm{e}^{-C_2' n} \\
&\leq \frac{2 \cdot 4^d}{\theta_p} (2n^d)^{d(c_3' n^{d-2+3/2d} + 2)} \sum_{j \geq \beta n^{d-1}} (2 \cdot 7^d)^j \left(C_1 \, \mathrm{e}^{-C_2 k}\right)^{j/(2.4^d)} \\
&\quad + c_1 \, \mathrm{e}^{-c_2 n^{1-3/2d}} + C_1' \, \mathrm{e}^{-C_2' n}
\end{aligned}
$$

We now choose k large enough such that

$$
C_1 \, \mathrm{e}^{-C_2 k} \leq \frac{1}{2} \qquad \text{and} \qquad \left((2 \cdot 7^d)^{2 \cdot 4^d} C_1 \, \mathrm{e}^{-C_2 k}\right)^{1/(2.4^d)} \leq \mathrm{e}^{-1}
$$

Finally, we get

$$
\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n, \, |\widetilde{\underline{\Gamma}}_n| \geq \beta n^{d-1} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\leq \frac{2 (2n^d)^{d(c_3' n^{d-2+3/2d} + 2)} 4^d}{\theta_p} \sum_{j \geq \beta n^{d-1}} \left((2 \cdot 7^d)^{2 \cdot 4^d} C_1 \, \mathrm{e}^{-C_2 k}\right)^{j/(2.4^d)} \\
&\qquad + c_1 \, \mathrm{e}^{-c_2 n^{1-3/2d}} + C_1' \, \mathrm{e}^{-C_2' n} \\
&\leq \frac{4^{d+1} \exp\left(2 d^2 c_3' n^{d-2+3/2d} \log n - \beta n^{d-1}\right)}{\theta_p} + c_1 \, \mathrm{e}^{-c_2 n^{1-3/2d}} + C_1' \, \mathrm{e}^{-C_2' n} \, .
\end{aligned}
$$

This yields the result for $\beta \geq \beta_0$ where β_0 is such that for all $n \geq 0$, we have $\beta_0 > (4 d^2 c_3' \log n)/n^{1-3/2d}$. □

We can now build the relevant continuous object P_n. Given a finite set of edges S, we define

$$\mathrm{hull}(S) = \left\{ x \in \mathbb{Z}^d : \text{ any path from } x \text{ to infinity has to use an edge of } S \right\}$$

and

$$H_n = \mathrm{hull}(\Gamma_n) \setminus \left(\bigcup_{i=1}^{m} \mathrm{hull}(\widehat{\Gamma}_n^{(i)}) \right) .$$

We define P_n and its associated measure ν_n as

$$P_n = \frac{1}{n} \left(H_n + \left[-\frac{1}{2}, \frac{1}{2} \right]^d \right) ,$$

$$\forall E \in \mathcal{B}\left(\mathbb{R}^d\right), \, \nu_n(E) = \theta_p \mathcal{L}^d(P_n \cap E) .$$

We obtain a control on the size of the perimeter of P_n by a straightforward application of Proposition 5.4:

Corollary 5.1. *Let $d \geq 2$ and $p > p_c(d)$. There exist positive constants β_0, C_1, C_2 depending only on d and p such that for all $n \geq 1$, for all $\beta > \beta_0$,*

$$\mathbb{P}\left[\max_{G_n \in \mathcal{G}_n} \mathcal{P}(nP_n) \geq \beta n^{d-1} \,\middle|\, 0 \in \mathcal{C}_\infty \right] \leq C_1 \, \mathrm{e}^{-c_2 n^{1-3/2d}} .$$

The following Lemma will be useful to compare the measure ν_n with the measure associated to F_n.

Lemma 5. *Let $G_n \in \mathcal{G}_n$ and F_n as defined in* (20)*. We have $F_n = H_n \cap \mathcal{C}_\infty$.*

Proof. Let $G_n \in \mathcal{G}_n$. Let $x \in H_n \cap \mathcal{C}_\infty$, then x belongs to $\mathcal{C}_\infty \cap \mathrm{hull}\,\Gamma_n$ but is not in any of the large connected components $L_1, \ldots, L_m$. Therefore, x belongs to G_n or to one of the small components $S_1, \ldots, S_N$ and so $x \in F_n$.

Conversely, let $x \in F_n$. It is clear that $x \in \mathrm{hull}(\Gamma_n)$. Let us assume $x \in G_n$ and that there exists i such that $x \in \mathrm{hull}(\widehat{\Gamma}_n^{(i)})$. As G_n is connected there exists an open path γ in G_n that joins x with $G_n \setminus \widehat{\Gamma}_n^{(i)}$. As the edges of $\widehat{\Gamma}_n^{(i)} \setminus \partial^o L_i$ are closed, γ must use an edge of $\partial^o L_i$ and so go through a vertex of L_i. That is a contradiction as the path γ uses only vertices in G_n. Let us now assume that $x \in S_j$ and $x \in \mathrm{hull}(\widehat{\Gamma}_n^{(i)})$ for some i and j. As $x \in \mathcal{C}_\infty$, x is connected to infinity by an open path γ'. However, by the same arguments, to exit $\mathrm{hull}(\widehat{\Gamma}_n^{(i)})$, the path γ' has to go through a vertex of L_i. Thus, there exist an open path in $\mathcal{C}_\infty \setminus G_n$ that joins x to L_i. That is a contradiction as $x \notin L_i$.

Finally, $F_n \subset H_n \cap \mathcal{C}_\infty$. □

5.3 Closeness of measures

We shall show that for any ball of constant radius centered at a point $x \in \mathbb{Z}^d$, the measures ν_n and μ_n restricted to this ball are close to each other in some weak sense.

Proposition 5.5. *Let $p > p_c(d)$ and $r > 0$. Let $u :]0,+\infty[\to]0,+\infty[$ be a non-decreasing function such that $\lim_{t\to 0} u(t) = 0$. For all $\delta > 0$, there exist C_1 and C_2 depending on d, p, u and δ such that for all $n \geq 1$, for any finite set $\mathfrak{F}_n$ of uniformly continuous functions that satisfies:*

$$\forall f \in \mathfrak{F}_n \quad \|f\|_\infty \leq 1 \qquad \text{and} \qquad \forall x, y \in \mathbb{R}^d \quad |f(x) - f(y)| \leq u(\|x-y\|_2)\,,$$

we have

$$\mathbb{P}\left[\max_{G_n \in \mathcal{G}_n} \sup_{f\in\mathfrak{F}} |\mu_n(f\mathbb{1}_{B(x,r)}) - \nu_n(f\mathbb{1}_{B(x,r)})| > \delta \,\middle|\, 0 \in \mathcal{C}_\infty\right] \leq C_1\, \mathrm{e}^{-C_2 n^{1-3/2d}}\,.$$

Remark 5.3. We state here the result in a general form. In the following, we will apply this Proposition for the particular case of sets of functions that are translates of the same function. The function u is an upper bound on the modulus of continuity of the functions in $\mathfrak{F}_n$. If we think of $\mathfrak{F}_n$ as a set that grows with n, this condition may be interpreted as a sufficient condition to obtain compactness for the set $\mathfrak{F}_n$ in the limit.

To prove this result, we will need the following proposition that is a corollary of the results in [16]:

Proposition 5.6. *Let $d \geq 2$ and $p > p_c(d)$. Let $r > 0$, and let $Q \subset \mathbb{R}^d$ be a cube of side length $2r$. Let $\delta > 0$. There exist positive constants c_1 and c_2 depending on d, p and δ such that*

$$\mathbb{P}\left[\frac{|\mathcal{C}_\infty \cap Q|}{\mathcal{L}^d(Q)} \notin (\theta_p - \delta, \theta_p + \delta)\right] \leq c_1 \exp(-c_2 r^{d-1})\,.$$

Proof of Proposition 5.5 . Let $\delta > 0$ and $\varepsilon > 0$ that we will choose later. Let $u :]0,+\infty[\to]0,+\infty[$ be a non-decreasing function such that $\lim_{t\to 0} u(t) = 0$. Let $n \geq 1$. Let $\mathfrak{F}_n$ be a finite set of uniformly continuous function that satisfies:

$$\forall f \in \mathfrak{F}_n \quad \|f\|_\infty \leq 1 \qquad \text{and} \qquad \forall x, y \in \mathbb{R}^d \quad |f(x) - f(y)| \leq u(\|x-y\|_2)\,,$$

We define

$$\widetilde{\mu}_n = \frac{1}{n^d} \sum_{x \in V(F_n)} \delta_{x/n}\,.$$

Thanks to Theorem 3, there exists a constant η_3 depending only on the dimension such that

$$\mathbb{P}\left[n\varphi_n \geq \eta_3 \,\middle|\, 0 \in \mathcal{C}_\infty\right] \leq C_1 \exp(-C_2 n)\,.$$

Let $G_n \in \mathcal{G}_n$, with probability at least $1 - C_1 \exp(-C_2 n)$, we have

$$\frac{n|\partial^o G_n|}{|G_n|} \leq \eta_3,$$

and so $|\partial^o G_n| \leq \eta_3 n^{d-1}$. As each small component S_j is such that $\partial^o S_j \cap \partial^o G_n \neq \emptyset$, the number N of small components is at most $\eta_3 n^{d-1}$ and by definition of F_n,

$$|F_n \setminus G_n| \leq \sum_{j=1}^{N} |S_j| \leq \eta_3 n^{d-1/2(d-1)}\,.$$

Finally, with probability at least $1 - C_1 \exp(-C_2 n)$, for all $f \in \mathfrak{F}_n$,

$$|\mu_n(f) - \widetilde{\mu}_n(f)| \leq \frac{1}{n^d} \|f\|_\infty |F_n \setminus G_n| \leq \eta_3 n^{-1/2(d-1)} ,$$

and

$$\underline{\mathbb{P}} \left[\max_{G_n \in \mathcal{G}_n} \sup_{f \in \mathfrak{F}_n} |\mu_n(f \mathbb{1}_{B(x,r)}) - \nu_n(f \mathbb{1}_{B(x,r)})| > \eta_3 n^{-1/2(d-1)} \right] \leq C_1 \, \mathrm{e}^{-C_2 n} \tag{24}$$

where $\underline{\mathbb{P}}$ represents the probability measure conditioned on the event $\{0 \in \mathcal{C}_\infty\}$. Let $x \in \mathbb{R}^d$ and let $r > 0$. Let $f \in \mathfrak{F}_n$. We now would like to estimate the quantity

$$|\widetilde{\mu}_n(f \mathbb{1}_{B(x,r)}) - \nu_n(f \mathbb{1}_{B(x,r)})| .$$

We adapt the proof of 16.2 in [6]. We use again a renormalization argument but at a different scale $L = K \ln n$. We consider the lattice rescaled by this factor L. We say that a cluster C is crossing in a box B if for any two opposite faces of B, the cluster C contains an open path in B that joins these two faces. Let $\varepsilon > 0$. For $\underline{y} \in \mathbb{Z}^d$, we define $B_n(\underline{y}) = (2L\underline{y}/n) + [-L/n, L/n]^d$ and $B'_n(\underline{y}) = (2L\underline{y}/n) + [-3L/n, 3L/n]^d$. Let $X(\underline{y})$ be the indicator function of the event $\mathcal{E}_n(\underline{y})$. This event occurs if

• Inside $nB'_n(\underline{y})$, there is a unique crossing cluster C' that crosses the 3^d sub-boxes of $nB'_n(\underline{y})$. Moreover, C' is the only cluster in $nB'_n(\underline{y})$ of diameter larger than L.

• Inside $nB_n(\underline{y})$, there is a crossing cluster C^* such that

$$|C^*| \geq (\theta_p - \varepsilon) \mathcal{L}^d(nB_n(\underline{y})) .$$

• We have $\left|\{x \in nB_n(\underline{y}) : x \longleftrightarrow \partial nB_n(\underline{y}\}\right| \leq (\theta_p + \varepsilon) \mathcal{L}^d(nB_n(\underline{y}))$.

On the event $\mathcal{E}_n(\underline{y})$, any cluster $C \subset nB_n(\underline{y})$ that is connected by an open path to $\partial(nB'_n(\underline{y}))$ is the unique crossing cluster, *i.e.*, $C = C' = C^*$ and so it also satisfies

$$\frac{|C|}{\mathcal{L}^d(nB_n(\underline{y}))} \in [\theta_p - \varepsilon, \theta_p + \varepsilon] .$$

The family $(X(\underline{y}))_{\underline{y} \in \mathbb{Z}^d}$ is a site percolation process on the macroscopic lattice. The states of the sites are not independent from each other but there is only a short range dependency. Indeed, for any $\underline{y}$ and $\underline{z}$ such that $|\underline{y} - \underline{z}|_\infty \geq 3$, we have that $X(\underline{y})$ and $X(\underline{z})$ are independent. We define the connected component $\mathcal{C}(\underline{y})$ of $\underline{y}$ as

$$\mathcal{C}(\underline{y}) = \left\{ \underline{z} \in \mathbb{Z}^d \, : \, \underline{z} \text{ is connected to } \underline{y} \text{ by a macroscopic open path} \right\} .$$

Let

$$\underline{D} = \{\underline{y} \in \mathbb{Z}^d : \, B_n(\underline{y}) \subset B(x,r)\} .$$

We have

$$|\underline{D}| L^d \leq n^d \mathcal{L}^d(B(x,r)). \tag{25}$$

There exists an integer $n_0 = n_0(u(\varepsilon))$ such that, for $n \geq n_0(u(\varepsilon))$, we have $L/n \leq u(\varepsilon)$ so that

$$\mathcal{L}^d\left(B(x,r) \setminus \bigcup_{\underline{y}\in\underline{D}} B_n(\underline{y})\right) \leq \varepsilon \mathcal{L}^d(B(x,r)),$$

$$\forall w, z \in \mathbb{R}^d,\ \|w-z\|_2 \leq \frac{L}{n} \Rightarrow |f(x)-f(y)| \leq \varepsilon\,.$$

The last statement comes from the fact that f belongs to $\mathfrak{F}_n$. By decomposing $|\widetilde{\mu}_n(f\mathbb{1}_{B(x,r)}) - \nu_n(f\mathbb{1}_{B(x,r)})|$ on cubes of size L/n, we obtain:

$$\begin{aligned}
&|\widetilde{\mu}_n(f\mathbb{1}_{B(x,r)}) - \nu_n(f\mathbb{1}_{B(x,r)})| \\
&\quad \leq 2\mathcal{L}^d\left(B(x,r) \setminus \bigcup_{\underline{y}\in\underline{D}} B_n(\underline{y})\right) + \sum_{\underline{y}\in\underline{D}} \left|\int_{B_n(\underline{y})} f d\widetilde{\mu}_n - \int_{B_n(\underline{y})} f d\nu_n\right| \\
&\quad \leq 4\varepsilon\mathcal{L}^d(B(x,r)) + \sum_{\underline{y}\in\underline{D}} \left|\widetilde{\mu}_n(B_n(\underline{y})) - \nu_n(B_n(\underline{y}))\right|\,. \qquad (26)
\end{aligned}$$

Let $\underline{y} \in \underline{D}$. We need to distinguish several cases:
• If $B_n(\underline{y}) \cap P_n = \emptyset$, then $\nu_n(B_n(\underline{y})) = \widetilde{\mu}_n(B_n(\underline{y})) = 0$. From now on we will only consider cubes such that $B_n(\underline{y}) \cap P_n \neq \emptyset$.
• If $B_n(\underline{y}) \not\subset P_n$, then we bound

$$|\widetilde{\mu}_n(B_n(\underline{y})) - \nu_n(B_n(\underline{y}))| \leq \frac{1}{n^d}|B_n(\underline{y})|$$

and as $B_n(\underline{y}) \cap P_n \neq \emptyset$, the cube intersects the boundary of P_n. Thus,

$$B_n(\underline{y}) \subset \left\{z \in \mathbb{R}^d :\, d_\infty(z, \partial P_n \cap B(x,r)) \leq \frac{L}{n}\right\}.$$

Moreover,

$$\begin{aligned}
&\mathcal{L}^d\left(\left\{z \in \mathbb{R}^d :\, d_\infty(z, \partial P_n \cap B(x,r)) \leq \frac{L}{n}\right\}\right) \\
&\quad \leq \left|\left\{\, x \in H_n,\, \exists y \in \mathbb{Z}^d \setminus H_n,\, \|x-y\|_1 = 1 \,\right\} \cap B(nx, nr+d)\right| \left(\frac{2L+2}{n}\right)^d \\
&\quad \leq \mathcal{P}(nP_n, B(nx, nr+d)) \left(\frac{3L}{n}\right)^d \\
&\quad \leq \mathcal{P}(P_n, B(x, r+d)) \frac{(3L)^d}{n}\,.
\end{aligned}$$

• If $B_n(\underline{y}) \subset P_n$ and $|\mathcal{C}(\underline{y})| = \infty$, then the crossing cluster C^* of $B_n(\underline{y})$ is a portion of $\mathcal{C}_\infty$ and

$$\nu_n(B_n(\underline{y})) = \theta_p \frac{\mathcal{L}^d(nB_n(\underline{y}))}{n^d} \text{ and } \widetilde{\mu}_n(B_n(\underline{y})) = \frac{|(nB_n(\underline{y})) \cap C^*|}{n^d}\,.$$

Thus, we have

$$\widetilde{\mu}_n(B_n(\underline{y})) \in \left[(\theta_p - \varepsilon)\mathcal{L}^d(B_n(\underline{y})), (\theta_p + \varepsilon)\mathcal{L}^d(B_n(\underline{y}))\right]$$

and

$$|\widetilde{\mu}_n(B_n(\underline{y})) - \nu_n(B_n(\underline{y}))| \leq \varepsilon \mathcal{L}^d(B_n(\underline{y}))\,.$$

• If $B_n(\underline{y}) \subset P_n$ and $|\mathcal{C}(\underline{y})| < \infty$, then we bound

$$|\widetilde{\mu}_n(B_n(\underline{y})) - \nu_n(B_n(\underline{y}))| \leq \mathcal{L}^d(B_n(\underline{y}))\mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty}$$

By summing the previous inequalities over $\underline{y} \in \underline{D}$, thanks to inequality (25) and (26), we obtain

$$\begin{aligned}
&|\widetilde{\mu}_n(f\mathbb{1}_{B(x,r)}) - \nu_n(f\mathbb{1}_{B(x,r)})| \\
&\qquad \leq \mathcal{L}^d(B(x,r))\Big(5\varepsilon + \frac{1}{|\underline{D}|}\sum_{\underline{y}\in\underline{D}} \mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty}\Big) + \mathcal{P}(P_n, B(x,r+d))\frac{(3L)^d}{n}\,.
\end{aligned}$$

Let $c(r) = 6\mathcal{L}^d(B(0,r)) + 3^d$, we get

$$\begin{aligned}
&\underline{\mathbb{P}}\left[\max_{G_n\in\mathcal{G}_n}\sup_{f\in\mathfrak{F}_n} |\widetilde{\mu}_n(f\mathbb{1}_{B(x,r)}) - \nu_n(f\mathbb{1}_{B(x,r)})| > c(r)\varepsilon\right] \\
&\qquad \leq \frac{1}{\theta_p}\mathbb{P}\left[\frac{1}{|\underline{D}|}\sum_{\underline{y}\in\underline{D}} \mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty} \geq \varepsilon\right] + \underline{\mathbb{P}}\left[\max_{G_n\in\mathcal{G}_n} \mathcal{P}(P_n, B(x,r+d)) \geq \varepsilon\frac{n}{L^d}\right]\,.
\end{aligned} \tag{27}$$

Besides, using Corollary 5.1, for n large enough, we obtain

$$\underline{\mathbb{P}}\left[\max_{G_n\in\mathcal{G}_n} \mathcal{P}(P_n, B(x,r+d)) \geq \varepsilon\frac{n}{L^d}\right] \leq \underline{\mathbb{P}}\Big[\max_{G_n\in\mathcal{G}_n} \mathcal{P}(P_n) \geq \beta\Big] \leq c_1\,\mathrm{e}^{-c_2 n^{1-3/2d}}\,. \tag{28}$$

Let Λ be the cube centered at x of side length $2r$. We define

$$\underline{\Lambda} = \Big\{\,\underline{y} \in \mathbb{Z}^d :\, B_n(\underline{y}) \subset \Lambda\,\Big\}\,.$$

As $B(x,r) \subset \Lambda$, we have $\underline{D} \subset \underline{\Lambda}$ and

$$\frac{1}{|\underline{D}|}\sum_{\underline{y}\in\underline{D}} \mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty} \leq \frac{(2d)^d}{|\underline{\Lambda}|}\sum_{\underline{y}\in\underline{\Lambda}} \mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty}\,. \tag{29}$$

Let $q \in [0,1]$ be such that $\theta_q > 1 - \varepsilon/(2(2d)^d)$. As the family $(X(\underline{y}))_{\underline{y}\in\mathbb{Z}^d}$ is identically distributed, has a short range dependency and is such that $\mathbb{P}(X(\underline{0}) = 1)$ goes to 1 when n goes to infinity (see for instance Chapter 9 in [5]), then we can apply Liggett Schonmann and Stacey's result [13]: for n large enough, the family $(X(\underline{y}), \underline{y} \in \mathbb{Z}^d)$ stochastically dominates $(\widetilde{X}(\underline{y}), \underline{y} \in \mathbb{Z}^d)$ a family of independent Bernoulli variable of parameter q. We denote by $\widetilde{\underline{\mathcal{C}}}_\infty$ the unique infinite cluster of the Bernoulli field $(\widetilde{X}(\underline{y}))_{\underline{y}\in\mathbb{Z}^d}$. Using inequality (29) and the

stochastic domination, we get

$$\begin{aligned}\mathbb{P}\left[\frac{1}{|\underline{D}|}\sum_{\underline{y}\in\underline{D}}\mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty}\geq\varepsilon\right]&\leq\mathbb{P}\left[\frac{(2d)^d}{|\underline{\Lambda}|}\sum_{\underline{y}\in\underline{\Lambda}}\mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty}\geq\varepsilon\right]\\&\leq\mathbb{P}\left[\frac{1}{|\underline{\Lambda}|}\sum_{\underline{y}\in\underline{\Lambda}}\mathbb{1}_{\underline{y}\notin\widetilde{\underline{\mathcal{C}}}_\infty}\geq\frac{\varepsilon}{(2d)^d}\right]\\&\leq\mathbb{P}\left[\frac{\left|\underline{\Lambda}\cap\widetilde{\underline{\mathcal{C}}}_\infty\right|}{|\underline{\Lambda}|}\notin\left(\theta_q-\frac{\varepsilon}{2(2d)^d},\theta_q+\frac{\varepsilon}{2(2d)^d}\right)\right].\end{aligned}$$

Using Proposition 5.6, we obtain

$$\mathbb{P}\left[\frac{1}{|\underline{D}|}\sum_{\underline{y}\in\underline{D}}\mathbb{1}_{|\mathcal{C}(\underline{y})|<\infty}\geq\varepsilon\right]\leq c_1'\exp\left(-c_2'\left(\frac{rn}{L}\right)^{d-1}\right). \tag{30}$$

We set $\varepsilon=\delta/(2c(r))$. Finally, thanks to inequalities (24), (27), (28) and (30), we have for $n\geq n_0(u(\varepsilon))$

$$\begin{aligned}&\underline{\mathbb{P}}\left[\max_{G_n\in\mathcal{G}_n}\sup_{f\in\mathfrak{F}_n}|\widetilde{\mu}_n(f\mathbb{1}_{B(x,r)})-\nu_n(f\mathbb{1}_{B(x,r)})|>\delta\right]\\&\qquad\leq\underline{\mathbb{P}}\left[\max_{G_n\in\mathcal{G}_n}\sup_{f\in\mathfrak{F}_n}|\mu_n(f)-\widetilde{\mu}_n(f)|>\delta/2\right]\\&\qquad\quad+\underline{\mathbb{P}}\left[\max_{G_n\in\mathcal{G}_n}\sup_{f\in\mathfrak{F}_n}|\widetilde{\mu}_n(f\mathbb{1}_{B(x,r)})-\nu_n(f\mathbb{1}_{B(x,r)})|>c(r)\varepsilon\right]\\&\qquad\leq C_1\exp(-C_2n)+\frac{c_1'}{\theta_p}\exp\left(-c_2'\left(\frac{rn}{L}\right)^{d-1}\right)+c_1\,\mathrm{e}^{-c_2n^{1-3/2d}}.\end{aligned}$$

The result follows. □

6 Lower large deviations and shape Theorem

6.1 Closeness to the set of Wulff shapes

The aim of this section is to prove Theorem 5.

Proof of Theorem 5. Let $\varepsilon>0$. Let $\xi>0$ that we will choose later depending on ε. We define λ such that

$$1-\lambda=\frac{1}{1+\xi}.$$

We denote by W_ξ:

$$W_\xi=\left\{\nu_{W+x}:\ \begin{array}{c}x\in\mathbb{R}^d,\ W\text{ is a dilate of }W_p\text{ such that}\\ \mathcal{L}^d((1-\lambda)W_p)\leq\mathcal{L}^d(W)\leq\mathcal{L}^d((1+2\xi)W_p)\end{array}\right\}.$$

Let $u :]0,+\infty[\to]0,+\infty[$ be a non-decreasing function such that $\lim_{t\to 0} u(t) = 0$. Let $n \geq 1$. Let $\mathfrak{F}_n$ be a finite set of uniformly continuous function that satisfies for all $f \in \mathfrak{F}_n$,

$$\|f\|_\infty \leq 1 \qquad \text{and} \qquad \forall x, y \in \mathbb{R}^d,\ |f(x) - f(y)| \leq u(\|x-y\|_2)\,.$$

We define the weak neighborhood $\mathcal{V}(\mathcal{W}_\xi, \mathfrak{F}_n, \varepsilon)$ of $\mathcal{W}_\xi$ given $\mathfrak{F}_n$ and ε as

$$\mathcal{V}(\mathcal{W}_\xi, \mathfrak{F}_n, \varepsilon) = \Big\{ \nu \in \mathcal{M}(\mathbb{R}^d)\ : \exists \mu \in \mathcal{W}_\xi,\ \sup_{f\in\mathfrak{F}_n} |\nu(f) - \mu(f)| \leq \varepsilon \Big\}\,.$$

Our goal is to show that μ_n is in the set $\mathcal{V}(\mathcal{W}_\xi, \mathfrak{F}_n, \varepsilon)$ with high probability.
Step *(i)*: Let $G_n \in \mathcal{G}_n$. Thanks to Proposition 5.5, the measures μ_n and ν_n associated with P_n and G_n are locally close to each other. In the following, it will be more convenient to work with the continuous object P_n instead of G_n. We can localize almost all the volume of P_n in a random region that is a union of balls of constant radius. We follow the method in Chapter 17 in [5]. We can cover P_n in $\mathbb{R}^d$, up to a small fractional volume, by a finite number of random disjoint balls of constant size. Thanks to the isoperimetric inequalities, we can then control the volume of P_n outside of these balls. Let $\delta > 0$ be a real number that we will choose later. We denote by X:

$$X = \big\{ x \in \mathbb{Z}^d : \mathcal{L}^d(B(x,1) \cap P_n) \geq \delta \big\}\,.$$

On the event $\big\{ |\widetilde{\Gamma}_n| \leq \beta n^{d-1} \big\}$, the set X is included in $B(0, \beta n^{d-2})$ and is therefore finite. As each point in $\mathbb{R}^d$ belongs to at most 2^d balls among the $B(x,1)$, $x \in \mathbb{Z}^d$, then using Proposition 2.2

$$\delta|X| \leq \sum_{x\in X} \mathcal{L}^d(B(x,1)\cap P_n) \leq 2^d \mathcal{L}^d(P_n) \leq 2^d c_{iso} \mathcal{P}(P_n)^{\frac{d}{d-1}} \leq 2^d c_{iso} \beta^{\frac{d}{d-1}}$$

and finally $|X| \leq M$ where $M = 2^d c_{iso} \beta^{\frac{d}{d-1}}/\delta$. We now would like to control the volume of P_n outside the balls $B(x,1)$ in X, *i.e.*, to bound the measure of $P_n \setminus \bigcup_{x\in X} B(x,1)$. For $x \in \mathbb{Z}^d \setminus X$, by the isoperimetric inequality in Proposition 2.2, we obtain as in section 17 in [5]

$$\begin{aligned}
\mathcal{L}^d\left(P_n \setminus \bigcup_{x\in X} B(x,1)\right) &\leq \sum_{x\in\mathbb{Z}^d\setminus X} \mathcal{L}^d(P_n \cap B(x,1)) \\
&\leq \delta^{1/d} b_{iso}^{\frac{d}{d-1}} \sum_{x\in\mathbb{Z}^d\setminus X} \mathcal{P}(P_n, \mathring{B}(x,1)) \\
&= \delta^{1/d} b_{iso}^{\frac{d}{d-1}} \sum_{x\in\mathbb{Z}^d\setminus X} \mathcal{H}^{d-1}(\partial^* P_n \cap \mathring{B}(x,1)) \\
&\leq 2^d \delta^{1/d} b_{iso}^{\frac{d}{d-1}} \mathcal{H}^{d-1}(\partial^*(P_n)) = 2^d \delta^{1/d} b_{iso}^{\frac{d}{d-1}} \mathcal{P}(P_n) \\
&\leq 2^d \delta^{1/d} b_{iso}^{\frac{d}{d-1}} \beta. \qquad (31)
\end{aligned}$$

We note $\eta = 2^d \delta^{1/d} b_{iso}^{\frac{d}{d-1}} \beta$. Therefore, if $\mathcal{P}(P_n) \leq \beta$, then $X \subset B(0, \beta n^{d-2})$, $|X| \leq M$ and $\mathcal{L}^d(P_n \setminus \cup_{x\in X} B(x,1)) \leq \eta$.

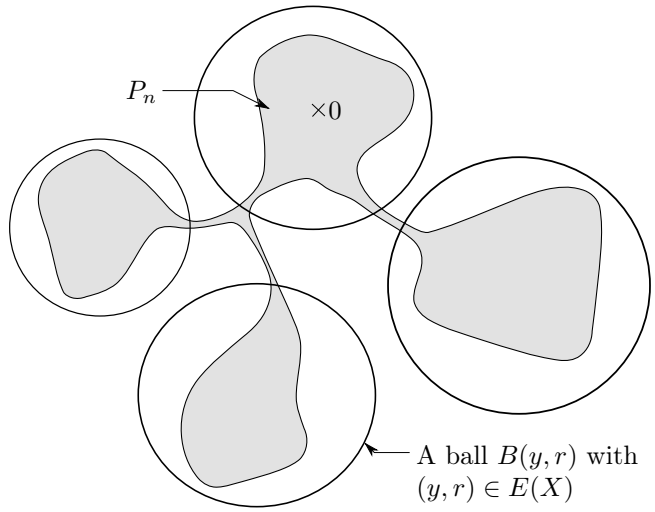

Figure 6 – Covering almost all the volume of P_n by balls of constant radius

We next would like to perform a kind of surgery between the balls. To do so, we first shall build from the balls $(B(x,1))_{x\in X}$ a family of balls that covers $\cup_{x\in X} B(x,1)$ and such that the balls are far apart (see Figure 6). This is the purpose of Lemma 17.1. in [5]. We obtain a subset

$$E(X) = \{(y_1,r_1),\dots,(y_m,r_m)\} \subset X \times \left\{1,\dots,3^{|X|}\right\}$$

such that $|E(X)| \le |X|$ and

- $\forall (a,r) \in E(X)$, $B(a,r) \cap X \neq \emptyset$
- $\cup_{x\in X} B(x,1) \subset \cup_{(a,r)\in E(X)} B(a,r)$
- $\forall (a,r)$, $(b,s) \in E(X)$, $(a,r) \neq (b,s) \Rightarrow B(a,r+1) \cap B(b,s+1) = \emptyset$

We set

$$\varphi_{W_p} = \frac{\mathcal{I}_p(W_p)}{\theta_p \mathcal{L}^d(W_p)}\,.$$

Let $\delta' > 0$ be a real number that we will choose later. By applying Corollary 5.1 and Theorem 3, we obtain by conditioning on $E(X)$,

$$\begin{aligned}
&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\ \mu_n \notin \mathcal{V}(\mathcal{W}_\xi, \mathfrak{F}_n, \varepsilon) \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\
&\quad \le \underline{\mathbb{P}}[\max_{G_n\in\mathcal{G}_n} \mathcal{P}(nP_n) \ge \beta n^{d-1}] + \underline{\mathbb{P}}[n\varphi_n > (1+\delta')\varphi_{W_p}] \\
&\qquad + \underline{\mathbb{P}}[\exists G_n \in \mathcal{G}_n,\ \mu_n \notin \mathcal{V}(\mathcal{W}_\xi, \mathfrak{F}_n, \varepsilon),\ \mathcal{P}(P_n) \le \beta, n\varphi_n \le (1+\delta')\varphi_{W_p}] \\
&\quad \le b_1 \exp(-b_2 n^{1-3/2d}) + b_1' \exp(-b_2' n) \\
&\qquad + \sum_{1\le m\le M} \sum_{y_1,\dots,y_m} \sum_{r_1,\dots,r_m} \underline{\mathbb{P}} \left[\begin{array}{c} \exists G_n \in \mathcal{G}_n,\ \mu_n \notin \mathcal{V}(\mathcal{W}_\xi, \mathfrak{F}_n, \varepsilon), \\ E(X) = \{(y_1,r_1),\dots,(y_m,r_m)\} \\ \mathcal{P}(P_n) \le \beta,\ n\varphi_n \le (1+\delta')\varphi_{W_p} \end{array} \right], \qquad (32)
\end{aligned}$$

where the second summation is over $y_1,\dots,y_m$ in $\mathbb{Z}^d \cap B(0,\beta n^{d-2})$ and the third summation is over $r_1,\dots,r_m$ in $\{1,\dots,3^M\}$. The number of ways to choose m

and $r_1, \dots, r_m$ is bounded from above by a constant depending only on M, while the number of ways of choosing the centers $y_1, \dots, y_m$ is polynomial in n. We next control the probability inside the sums. We will only focus on what happens inside the balls.

Step *(ii)*: Let $\{(y_1, r_1), \dots, (y_m, r_m)\}$ be a value for the random set $E(X)$ which occurs with positive probability. We define $\Omega = \Omega(E(X))$ as

$$\Omega = \bigcup_{i=1}^{m} \mathring{B}(y_i, r_i + 1)\,,$$

and the restriction $\overline{P}_n$ of P_n to the balls determined by $E(X)$:

$$\overline{P}_n = P_n \cap \left(\bigcup_{i=1}^{m} \mathring{B}(y_i, r_i + 1) \right)\,.$$

Thus, using inequality (31), we have

$$\mathcal{L}^d(P_n \setminus \overline{P}_n) \le \eta\,. \tag{33}$$

We show now that $\nu_{\overline{P}_n}(f)$ is close to $\mu_n(f)$ with high probability on the event

$$\left\{\, E(X) = \{(y_1, r_1), \dots, (y_m, r_m)\} \,\right\}.$$

It is easy to check that $\mathfrak{F}_n \cup \{1\}$ associated with the function u satisfies the conditions required in Proposition 5.5. So that applying Proposition 5.5 for every $r \in \{1, \dots, 3^M\}$, there exist positive constants c_1, c_2 depending on M, u, and δ such that for all $x \in \mathbb{Z}^d$

$$\max_{r \in \{1,\dots,3^M\}} \underline{\mathbb{P}} \left[\max_{G_n \in \mathcal{G}_n} \sup_{f \in \mathfrak{F}_n \cup \{1\}} |\nu_n(f \mathbb{1}_{B(x,r)}) - \mu_n(f \mathbb{1}_{B(x,r)})| > \frac{\eta}{M} \right] \le c_1\, \mathrm{e}^{-c_2 n^{1-3/2d}}\,.$$

Thus, using inequality (33), we obtain

$$\begin{aligned}
&\underline{\mathbb{P}} \left[\max_{G_n \in \mathcal{G}_n} \sup_{f \in \mathfrak{F}_n \cup \{1\}} |\nu_{\overline{P}_n}(f) - \mu_n(f)| > 2\eta,\ E(X) = \{(y_1, r_1), \dots, (y_m, r_m)\} \right] \\
&\quad \le \sum_{i=1}^{m} \underline{\mathbb{P}} \left[\max_{G_n \in \mathcal{G}_n} \sup_{f \in \mathfrak{F}_n \cup \{1\}} |\nu_{\overline{P}_n}(f \mathbb{1}_{B(y_i,r_i)}) - \mu_n(f \mathbb{1}_{B(y_i,r_i)})| > \eta/M \right] \\
&\quad \le M \max_{r \in \{1,\dots,3^M\}} \underline{\mathbb{P}} \left[\max_{G_n \in \mathcal{G}_n} \sup_{f \in \mathfrak{F}_n \cup \{1\}} |\nu_n(f \mathbb{1}_{B(y_1,r)}) - \mu_n(f \mathbb{1}_{B(y_1,r)})| > \eta/M \right] \\
&\quad \le M c_1\, \mathrm{e}^{-c_2 n^{1-3/2d}}\,.
\end{aligned} \tag{34}$$

In particular, on the event $\{E(X) = \{(y_1, r_1), \dots, (y_m, r_m)\}\}$, with probability at least $1 - M c_1 \exp(-c_2 n^{1-3/2d})$, we have

$$\left| \theta_p \mathcal{L}^d(\overline{P}_n) - \frac{|G_n|}{n^d} \right| \le 2\eta\,. \tag{35}$$

Moreover, by Lemma 4, there exist positive constants η_1, D_1 and D_2 such that

$$\underline{\mathbb{P}}\left[\min_{G_n\in\mathcal{G}_n}|G_n|\le\eta_1 n^d\right]\le D_1\exp(-D_2 n^{(d-1)/2d})\,.$$

We recall that η is a function of δ. We will choose δ small enough such that

$$\eta\le\min\left(\frac{\eta_1}{4},\frac{\xi^d}{2},\frac{\varepsilon}{8},\frac{\eta_1}{3\theta_p}\right)\,. \tag{36}$$

Other conditions will be imposed later on δ.

On the event $\left\{\min_{G_n\in\mathcal{G}_n}|G_n|>\eta_1 n^d\right\}$, using inequalities (35) and (36), we obtain

$$\mathcal{L}^d(\overline{P}_n)\ge\frac{1}{\theta_p}\left(\frac{|G_n|}{n^d}-2\eta\right)\ge\frac{1}{\theta_p}\left(\eta_1-2\eta\right)\ge\frac{\eta_1}{2\theta_p} \tag{37}$$

and as $\mathcal{L}^d(W_p)=1/\theta_p$, using inequality (35), we have

$$\mathcal{L}^d(\overline{P}_n)\le\frac{1}{\theta_p}\left(\frac{|G_n|}{n^d}+2\eta\right)\le\frac{1}{\theta_p}(1+\xi^d)=\mathcal{L}^d(W_p)(1+\xi^d)\le\mathcal{L}^d((1+\xi)W_p)\,. \tag{38}$$

For $\nu\in\mathcal{W}_\xi$, we have

$$\sup_{f\in\mathfrak{F}_n}|\nu_{\overline{P}_n}(f)-\nu(f)|\ge\sup_{f\in\mathfrak{F}_n}|\mu_n(f)-\nu(f)|-\sup_{f\in\mathfrak{F}_n}|\mu_n(f)-\nu_{\overline{P}_n}(f)|\,,$$

so that, together with inequalities (34) and (36), with high probability,

$$\mu_n\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,\varepsilon)\implies\nu_{\overline{P}_n}\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,3\varepsilon/4)\,.$$

Thus, combining with inequalities (37) and (38), we have

$$\begin{aligned}
&\underline{\mathbb{P}}\left[\begin{array}{c}\exists G_n\in\mathcal{G}_n,\ \mu_n\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,\varepsilon),\ \mathcal{P}(P_n)\le\beta\,,\\ E(X)=\{(y_1,r_1),\dots,(y_m,r_m)\},\ n\varphi_n\le(1+\delta')\varphi_{W_p}\end{array}\right]\\
&\quad\le\underline{\mathbb{P}}\left[\begin{array}{c}\exists G_n\in\mathcal{G}_n,\ \nu_{\overline{P}_n}\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,3\varepsilon/4),\ n\varphi_n\le(1+\delta')\varphi_{W_p},\\ \frac{\eta_1}{2\theta_p}\le\mathcal{L}^d(\overline{P}_n)\le\mathcal{L}^d((1+\xi)W_p),\ E(X)=\{(y_1,r_1),\dots,(y_m,r_m)\}\end{array}\right]\\
&\quad+Mc_1\,\mathrm{e}^{-c_2 n^{1-3/2d}}+D_1\exp(-D_2 n^{(d-1)/2d})\,.
\end{aligned} \tag{39}$$

We do not cover $\overline{P}_n$ directly but we cover separately each $\overline{P}_n\cap B(y_k,r_k+1)$ for $k\in\{1,\dots,m\}$. For any $r\in\{1,\dots,3^M\}$, we define the space

$$\mathcal{C}_\beta^{(r)}=\left\{F\subset\mathring{B}(0,r+1),\ \mathcal{P}(F,\mathring{B}(0,r+1))\le\beta\right\}$$

endowed with the topology L^1 associated to the distance $d(F,F')=\mathcal{L}^d(F\Delta F')$, where Δ is the symmetric difference between sets. For this topology, the space $\mathcal{C}_\beta^{(r)}$ is compact. Suppose that we associate to each $F\in\mathcal{C}_\beta^{(r)}$ a positive number $\varepsilon_F\le\min(\eta,\mathcal{L}^d(\xi W_p))/M$. The collection of open sets

$$\left\{H\text{ Borel subset of }\mathring{B}(0,r+1)\ :\ \mathcal{L}^d(H\Delta F)<\varepsilon_F\right\},\ F\in\mathcal{C}_\beta^{(r)},$$

is then an open covering of $\mathcal{C}_\beta^{(r)}$. By compactness, we can extract a finite covering $(F_i^{(r)}, \varepsilon_{F_i^{(r)}})_{1\le i\le N^{(r)}}$ of $\mathcal{C}_\beta^{(r)}$. By union bound, we obtain

$$\underline{\mathbb{P}}\left[\begin{array}{c}\exists G_n\in\mathcal{G}_n,\ \nu_{\overline{P}_n}\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,3\varepsilon/4),\ n\varphi_n\le(1+\delta')\varphi_{W_p},\\ \frac{\eta_1}{2\theta_p}\le\mathcal{L}^d(\overline{P}_n)\le\mathcal{L}^d((1+\xi)W_p),\ E(X)=\{(y_1,r_1),\dots,(y_m,r_m)\}\end{array}\right]$$
$$\le\sum_{i_1=1}^{N^{(r_1)}}\cdots\sum_{i_m=1}^{N^{(r_m)}}\underline{\mathbb{P}}[\mathcal{F}_{i_1,\dots,i_m}]\qquad(40)$$

where

$$\mathcal{F}_{i_1,\dots,i_m}=\left\{\begin{array}{c}\exists G_n\in\mathcal{G}_n\ :\ \forall\,1\le k\le m,\\ \mathcal{L}^d((F_{i_k}^{(r_k)}+y_k)\Delta(\overline{P}_n\cap B(y_k,r_k+1)))\le\varepsilon_{F_{i_k}^{(r_k)}},\\ \nu_{\overline{P}_n}\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,3\varepsilon/4),\ n\varphi_n\le(1+\delta')\varphi_{W_p},\\ \frac{\eta_1}{2\theta_p}\le\mathcal{L}^d(\overline{P}_n)\le\mathcal{L}^d((1+\xi)W_p),\\ E(X)=\{(y_1,r_1),\dots,(y_m,r_m)\}\end{array}\right\}.$$

So we need to study the quantity $\underline{\mathbb{P}}[\mathcal{F}]$ for a generic m-uplet $(F_1,\dots,F_m)\in\mathcal{C}_\beta^{(r_1)}\times\cdots\times\mathcal{C}_\beta^{(r_m)}$ and their associated $\varepsilon_{F_1},\dots,\varepsilon_{F_m}$. By definition of the Cheeger constant φ_n, we obtain

$$\underline{\mathbb{P}}[\mathcal{F}]=\underline{\mathbb{P}}\left[\exists G_n\in\mathcal{G}_n\ :\ \begin{array}{c}\forall 1\le i\le m,\\ \mathcal{L}^d((F_i+y_i)\Delta(\overline{P}_n\cap B(y_i,r_i+1)))\le\varepsilon_{F_i},\\ \nu_{\overline{P}_n}\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,3\varepsilon/4),\\ |\partial^o G_n|\le(1+\delta')n^{-1}|G_n|\varphi_{W_p},\\ \frac{\eta_1}{2\theta_p}\le\mathcal{L}^d(\overline{P}_n)\le\mathcal{L}^d((1+\xi)W_p),\\ E(X)=\{(y_1,r_1),\dots,(y_m,r_m)\}\end{array}\right].$$

To lighten the notations, we set

$$F=\bigcup_{i=1}^m(F_i+y_i)\,.$$

We have

$$\begin{aligned}\mathcal{L}^d(F\Delta\overline{P}_n)&=\sum_{i=1}^m\mathcal{L}^d((\overline{P}_n\cap B(y_i,r_i+1))\Delta(F_i+y_i))\\ &\le\sum_{i=1}^m\varepsilon_{F_i}\le\min(\eta,\mathcal{L}^d(\xi W_p))\,.\end{aligned}\qquad(41)$$

Whereas the surface tension of F in the interior of these balls corresponds to the surface tension of our minimizer G_n, the surface tension of F along the boundary of the balls $B(y_j,r_j+1)$ does not correspond to the surface tension of G_n because we have artificially created it. Roughly speaking, F is the continuous object corresponding to the graph G_n intersected with the $nB(y_j,r_j+1)$. This new graph has extra surface tension compared to G_n due to the fact that we have built it by cutting G_n along the boundary of these balls. However, our hope is to cut along the boundary of these balls in such a way that the surface

tension we create is negligible. We do not work on G_n but on the continuous object F, but we have to keep in mind that these two objects are close. The idea is to cut F in the regions $B(y_i, r_i + 1) \setminus B(y_i, r_i)$, $i \in \{1, \dots, m\}$. These regions contain a negligible volume of G_n and so of F, we want to cut F in these regions along a surface of negligible perimeter and so of negligible surface tension. By Lemma 14.4 in [5], for $i \in \{1, \dots, m\}$, for $\mathcal{H}^1$ almost all t in $]0, 1[$,

$$\mathcal{I}(F \cap B(y_i, r_i + t)) \leq \mathcal{I}(F \cap \mathring{B}(y_i, r_i + t)) + \beta_{max} \mathcal{H}^{d-1}(F \cap \partial B(y_i, r_i + t)) \,. \tag{42}$$

Let T be a subset of $]0, 1[$ where all the above inequalities hold simultaneously. We recall that for any $i \in \{1, \dots, m\}$, $\varepsilon_{F_i} \leq \eta/M$. We have $\mathcal{H}^1(T) = 1$ and when we integrate in polar coordinates, using inequality (41),

$$\begin{aligned}
\int_T \sum_{i=1}^m \mathcal{H}^{d-1}(F \cap \partial B(y_i, r_i + t)) dt &= \sum_{i=1}^m \mathcal{L}^d(F \cap B(y_i, r_i + 1) \setminus B(y_i, r_i)) \\
&\leq \sum_{i=1}^m \mathcal{L}^d((F_i + y_i) \setminus B(y_i, r_i)) \\
&\leq \mathcal{L}^d\Big(\overline{P}_n \setminus \bigcup_{i=1}^m B(y_i, r_i)\Big) + \mathcal{L}^d(\overline{P}_n \Delta F) \\
&\leq 2\eta \,.
\end{aligned}$$

Thus, there exists $t \in T$ such that

$$\sum_{i=1}^m \mathcal{H}^{d-1}(F \cap \partial B(y_i, r_i + t)) \leq 3\eta \,. \tag{43}$$

We next set

$$\overline{F} = F \cap \left(\bigcup_{i=1}^m B(y_i, r_i + t) \right) \,.$$

Using inequality (43), we get

$$\begin{aligned}
\mathcal{P}(\overline{F}) &\leq \mathcal{P}\left(\overline{F}, \bigcup_{i=1}^m \mathring{B}(y_i, r_i + t) \right) + \sum_{i=1}^m \mathcal{H}^{d-1}(F \cap \partial B(y_i, r_i + t)) \\
&\leq \mathcal{P}\left(\overline{F}, \bigcup_{i=1}^m \mathring{B}(y_i, r_i + t) \right) + 3\eta \,,
\end{aligned} \tag{44}$$

and using Proposition 2.1,

$$\mathcal{I}_p(\overline{F}) \leq \mathcal{I}_p\left(\overline{F}, \bigcup_{i=1}^m \mathring{B}(y_i, r_i + t) \right) + 3\beta_{max}\eta \,. \tag{45}$$

On the event $\mathcal{F}$, using inequality (41), we obtain

$$\begin{aligned}
\mathcal{L}^d(F) &\leq \mathcal{L}^d(\overline{F}) + \mathcal{L}^d\left(F \setminus \bigcup_{i=1}^m B(y_i, r_i) \right) \\
&\leq \mathcal{L}^d(\overline{F}) + \mathcal{L}^d(F \Delta \overline{P}_n) + \mathcal{L}^d\left(\overline{P}_n \setminus \bigcup_{i=1}^m B(y_i, r_i) \right) \leq \mathcal{L}^d(\overline{F}) + 2\eta \,.
\end{aligned} \tag{46}$$

Finally, using inequalities (44) and (46), we obtain

$$\mathcal{I}_p\left(\overline{F}, \bigcup_{i=1}^m \mathring{B}(y_i, r_i + t)\right) \geq \beta_{min}\, \mathcal{P}\left(\overline{F}, \bigcup_{i=1}^m \mathring{B}(y_i, r_i + t)\right)$$
$$\geq \beta_{min}(\mathcal{P}(\overline{F}) - 3\eta)\,. \tag{47}$$

and using again inequality (41),

$$\mathcal{L}^d(F) \geq \mathcal{L}^d(\overline{P}_n) - \mathcal{L}^d(\overline{P}_n \Delta F) \geq \frac{\eta_1}{2\theta_p} - \eta\,. \tag{48}$$

Using the isoperimetric inequalities of Proposition 2.2 and inequalities (46) and (48), we get

$$\mathcal{P}(\overline{F}) \geq \left(\frac{\mathcal{L}^d(\overline{F})}{c_{iso}}\right)^{1-1/d} \geq \left(\frac{\mathcal{L}^d(F) - 2\eta}{c_{iso}}\right)^{1-1/d} \geq \left(\frac{\eta_1 - 6\eta\theta_p}{2\theta_p c_{iso}}\right)^{1-1/d}. \tag{49}$$

Next, we choose δ small enough to obtain a η that satisfies the following inequalities:

$$3\beta_{max}\eta \leq \frac{\lambda\beta_{min}}{2}\left(\left(\frac{\eta_1 - 6\eta\theta_p}{2\theta_p c_{iso}}\right)^{1-1/d} - 3\eta\right), \tag{50}$$

and also

$$\eta_1 \geq 6\eta\theta_p\,.$$

With this choice of δ, we obtain with high probability, using inequalities (45), (47) and (49),

$$\mathcal{I}_p(\overline{F}) \leq (1 + \lambda/2)\mathcal{I}_p\left(\overline{F}, \bigcup_{i=1}^m \mathring{B}(y_i, r_i + t)\right) \leq (1 + \lambda/2)\mathcal{I}_p(F, \Omega)\,. \tag{51}$$

Let $G_n \in \mathcal{G}_n$, on the event $\mathcal{F}$, we have

$$\left|\theta_p \mathcal{L}^d(\overline{P}_n) - \frac{|G_n|}{n^d}\right| \leq 2\eta\,.$$

So that, together with inequality (46),

$$\begin{aligned}|G_n| &\leq n^d(\theta_p\mathcal{L}^d(F) + \theta_p\mathcal{L}^d(\overline{P}_n\Delta F) + 2\eta)\\ &\leq n^d(\theta_p\mathcal{L}^d(\overline{F}) + \varepsilon_F + 4\eta)\\ &\leq n^d\theta_p\mathcal{L}^d(\overline{F})\left(1 + \frac{5\eta}{\theta_p\mathcal{L}^d(\overline{F})}\right).\end{aligned}$$

Let us now choose δ small enough so that

$$\frac{5\eta}{\eta_1/2 - 3\eta\theta_p} \leq \delta'\,. \tag{52}$$

Using inequalities (46) and (48), we obtain

$$\begin{aligned}|G_n| &\leq n^d\theta_p\mathcal{L}^d(\overline{F})\left(1 + \frac{5\eta}{\eta_1/2 - 3\eta\theta_p}\right)\\ &\leq n^d\theta_p\mathcal{L}^d(\overline{F})(1 + \delta').\end{aligned}$$

Finally, let r be such that $\mathcal{L}^d(\overline{F}) = \mathcal{L}^d(rW_p)$, we get

$$\begin{aligned}(1+\delta')n^{-1}|G_n|\varphi_{W_p} &\leq (1+\delta')^2 n^{d-1}\frac{\varphi_{W_p}}{\varphi_{\overline{F}}}\mathcal{I}_p(\overline{F})\\ &\leq (1+\delta')^2\frac{\mathcal{I}_p(rW_p)}{\mathcal{I}_p(\overline{F})}rn^{d-1}\mathcal{I}_p(\overline{F})\,.\end{aligned}$$

We now choose δ' small enough such that

$$(1+\delta')^2(1-\lambda) \leq 1-\frac{\lambda}{2}\,. \tag{53}$$

Using inequality (41), we obtain

$$\mathcal{L}^d(\overline{F}) \leq \mathcal{L}^d(\overline{P}_n) + \mathcal{L}^d(\overline{P_n}\Delta F) \leq \mathcal{L}^d((1+\xi)W_p) + \mathcal{L}^d(\xi W_p) \leq \mathcal{L}^d((1+2\xi)W_p)$$

and so $r \leq 1+2\xi$. We distinguish now two cases:
• If $r \leq 1-\lambda$, using inequality (53)

$$(1+\delta')^2\frac{\mathcal{I}_p(rW_p)}{\mathcal{I}_p(\overline{F})}rn^{d-1}\mathcal{I}_p(\overline{F}) \leq (1-\lambda/2)n^{d-1}\mathcal{I}_p(\overline{F})$$

where we used the fact that the Wulff crystal is a minimizer for $\mathcal{I}_p$, *i.e.*, that $\mathcal{I}_p(rW_p) \leq \mathcal{I}_p(\overline{F})$.
• Let us assume that $r \in (1-\lambda, 1+2\xi]$. We recall that on the event $\mathcal{F}$, for all $\nu \in \mathcal{W}_\xi$,

$$\sup_{f\in\mathfrak{F}_n}|\nu_{\overline{P}_n}(f) - \nu(f)| \geq 3\varepsilon/4\,.$$

Thus, for all $x \in \mathbb{R}^d$, for $f \in \mathfrak{F}_n$ we have

$$\begin{aligned}|\nu_{\overline{P}_n}(f) - \nu_{rW_p+x}(f)| &\leq \left|\int_{\overline{P}_n\setminus(rW_p+x)} f(x)d\mathcal{L}^d(x) - \int_{(rW_p+x)\setminus\overline{P}_n} f(x)d\mathcal{L}^d(x)\right|\\ &\leq \int_{\overline{P}_n\setminus(rW_p+x)}|f(x)|d\mathcal{L}^d(x) + \int_{(rW_p+x)\setminus\overline{P}_n}|f(x)|d\mathcal{L}^d(x)\\ &\leq \int_{\overline{P}_n\setminus(rW_p+x)}1d\mathcal{L}^d(x) + \int_{(rW_p+x)\setminus\overline{P}_n}1d\mathcal{L}^d(x)\\ &\leq \mathcal{L}^d\big(\overline{P}_n\Delta(rW_p+x)\big)\,,\end{aligned}$$

and so,

$$\mathcal{L}^d\big(\overline{P}_n\Delta(rW_p+x)\big) \geq \sup_{f\in\mathfrak{F}_n}|\nu_{\overline{P}_n}(f) - \nu_{rW_p+x}(f)| \geq 3\varepsilon/4$$

and as η satisfies inequality (36), we obtain

$$\mathcal{L}^d\big(\overline{F}\Delta(rW_p+x)\big) \geq \mathcal{L}^d\big(\overline{P}_n\Delta(rW_p+x)\big) - \mathcal{L}^d\big(\overline{F}\Delta\overline{P}_n\big) \geq 3\varepsilon/4 - \eta \geq \varepsilon/2.$$

Moreover, as rW_p is a minimizer for the isoperimetric problem, there exists a constant $c(\varepsilon) > 0$, that is a non-decreasing function of ε depending also on p and r, that goes to 0 when ε goes to 0, such that

$$\begin{aligned}\inf\Big\{\mathcal{I}(E)\,:\,\forall x\in\mathbb{R}^d,\,\mathcal{L}^d(E\Delta(x+rW_p)) \geq \varepsilon/2,\,\mathcal{L}^d(E) = \mathcal{L}^d(rW_p)\Big\}\\ \geq \mathcal{I}_p(rW_p)(1+c(\varepsilon))\,.\end{aligned}$$

Finally,

$$\frac{\mathcal{I}_p(rW_p)}{\mathcal{I}_p(\overline{F})} \leq \frac{1}{1+c(\varepsilon)}$$

and so,

$$(1+\delta')^2 \frac{\mathcal{I}_p(rW_p)}{\mathcal{I}_p(\overline{F})} rn^{d-1}\mathcal{I}_p(\overline{F}) \leq \frac{(1+\delta')^2}{1+c(\varepsilon)}(1+2\xi)n^{d-1}\mathcal{I}_p(\overline{F})\,.$$

We choose ξ small enough depending on ε such that

$$\frac{1+2\xi}{1+c(\varepsilon)} \leq 1-\lambda = \frac{1}{1+\xi}\,.$$

This is equivalent to choose ξ such that

$$3\xi + 2\xi^2 \leq c(\varepsilon)\,. \tag{54}$$

We obtain using inequality (53)

$$(1+\delta')^2 \frac{\mathcal{I}_p(rW_p)}{\mathcal{I}_p(\overline{F})} rn^{d-1}\mathcal{I}_p(\overline{F}) \leq (1-\lambda/2)n^{d-1}\mathcal{I}_p(\overline{F})\,.$$

Finally, combining the two cases, with ε and δ' properly chosen and inequality (51), we obtain

$$\underline{\mathbb{P}}[\mathcal{F}] \leq \underline{\mathbb{P}}\left[\exists G_n \in \mathcal{G}_n : \begin{array}{c} \forall 1 \leq i \leq m, \\ \mathcal{L}^d((\overline{P}_n \cap B(y_i, r_i+1))\Delta(F_i+y_i)) \leq \varepsilon_{F_i}, \\ |\partial^o G_n| \leq \left(1-\frac{\lambda^2}{4}\right) n^{d-1}\mathcal{I}_p(F,\Omega), \\ E(X) = \{(y_1,r_1),\dots,(y_m,r_m)\} \end{array}\right]. \tag{55}$$

Step *(iii)*: The remaining of the proof follows the same ideas as in [8]. We link the probability defined in the right hand side of (55) with the probability that the flow is abnormally small in some local region of $\partial F \cap \Omega$. We now want to cover ∂F by balls of small radius such that ∂F is "almost flat" in each ball, this is the purpose of the following Lemma:

Lemma 6. *[Lemma 1 in [8]] Let $R > 0$. Let F be a subset of $\mathring{B}(0,R)$ of finite perimeter. For every positive constants δ' and η', there exists a finite family of closed disjoint balls $(B(x_i,\rho_i))_{i\in I\cup K}$ and vectors $(v_i)_{i\in I\cup K}$, such that, letting $B_i = B(x_i,\rho_i)$ and $B_i^- = B^-(x_i,\rho_i,v_i)$, we have for all $i \in I$*

$$x_i \in \partial^* F \cap \mathring{B}(0,R),\ \rho_i \in]0,1[,\ B_i \subset \mathring{B}(0,R),\ \mathcal{L}^d((F\cap B_i)\Delta B_i^-) \leq \delta'\alpha_d\rho_i^d,$$

and

$$\left|\mathcal{I}_p(F,\mathring{B}(0,R)) - \sum_{i\in I}\alpha_{d-1}\rho_i^{d-1}(\nu(n_F(x_i))\right| \leq \eta'.$$

We apply Lemma 6 to each $F_k \subset \mathring{B}(0,r_k+1)$, with $\delta_2 > 0$ that will be chosen later and $\eta' = \lambda^4\mathcal{I}_p(F,\Omega)/16M$. We obtain for each k, a family $\left(B_i^{(k)}\left(x_i^{(k)},\rho_i^{(k)},v_i^{(k)}\right)\right)_{i\in I^{(k)}}$ that does not depend on $y_1,\dots,y_m$, so that

$$\left|\mathcal{I}_p(F_k,\mathring{B}(0,r_k+1)) - \sum_{i\in I}\alpha_{d-1}(\rho_i^{(k)})^{d-1}(\nu(n_{F_k}(x_i^{(k)}))\right| \leq \eta'. \tag{56}$$

We now choose

$$\varepsilon_{F_k} \leq \min\left(\min_{i\in I^{(k)}} \alpha_d(\rho_i^{(k)})^d \delta_2,\ \frac{\eta}{M},\ \frac{\mathcal{L}^d(\xi W_p)}{M} \right), \tag{57}$$

for a fixed δ_2 that we will choose later. Besides, as the balls $B(y_k, r_k+1)$ are disjoint, for $k \in \{1,\dots,m\}$, we have

$$\mathcal{I}_p(F,\Omega) = \sum_{k=1}^m \mathcal{I}_p(F\cap B(y_k,r_k+1),\Omega) = \sum_{k=1}^m \mathcal{I}_p(F_k, \mathring{B}(0,r_k+1))\,.$$

Using inequality (56), we obtain

$$\left| \mathcal{I}_p(F,\Omega) - \sum_{k=1}^m \sum_{i\in I^{(k)}} \alpha_{d-1}(\rho_i^{(k)})^{d-1} \nu(n_{F_k}(x_i^{(k)})) \right| \leq m\eta' \leq \lambda^4 \mathcal{I}_p(F,\Omega)/16\,.$$

So, we get

$$\mathcal{I}_p(F,\Omega) \leq \frac{1}{1-\lambda^4/16} \left(\sum_{k=1}^m \sum_{i\in I^{(k)}} \alpha_{d-1}(\rho_i^{(k)})^{d-1} \nu(n_{F_k}(x_i^{(k)})) \right)$$

and

$$\left(1-\frac{\lambda^2}{4}\right) \mathcal{I}_p(F,\Omega) \leq \frac{1-\lambda^2/4}{1-\lambda^4/16} \left(\sum_{k=1}^m \sum_{i\in I^{(k)}} \alpha_{d-1}(\rho_i^{(k)})^{d-1} \nu(n_{F_k}(x_i^{(k)})) \right).$$

Whence setting $w = \lambda^2/(4+\lambda^2) < 1$,

$$\left(1-\frac{\lambda^2}{4}\right) \mathcal{I}_p(F,\Omega) \leq (1-w) \left(\sum_{k=1}^m \sum_{i\in I^{(k)}} \alpha_{d-1}(\rho_i^{(k)})^{d-1} \nu(n_{F_k}(x_i^{(k)})) \right). \tag{58}$$

Since the balls $(B_i^{(k)} + y_k)_{1\leq k\leq m,\, i\in I^{(k)}}$ are pairwise disjoint, we have

$$|\partial^o G_n| \geq \sum_{k=1}^m \sum_{i\in I^{(k)}} |(\partial^o G_n) \cap (n(B_i^{(k)} + y_k))|\,. \tag{59}$$

Using inequalities (58) and (59), we get

$$\begin{aligned} &\underline{\mathbb{P}}\left[\exists G_n \in \mathcal{G}_n,\ \begin{array}{c} \mathcal{L}^d((\overline{P}_n \cap B(y_i,r_i+1))\Delta(F_i+y_i)) \leq \varepsilon_{F_i},\ 1\leq i\leq m, \\ |\partial^o G_n| \leq (1-\lambda^2/4) n^{d-1} \mathcal{I}_p(F,\Omega), \\ E(X) = \{(y_1,r_1),\dots,(y_m,r_m)\} \end{array} \right] \\ &\leq \underline{\mathbb{P}}\left[\begin{array}{l} \exists G_n \in \mathcal{G}_n,\ \mathcal{L}^d((\overline{P}_n \cap B(y_i,r_i+1))\Delta(F_i+y_i)) \leq \varepsilon_{F_i},\ 1\leq i\leq m, \\ \sum_{k=1}^m \sum_{i\in I^{(k)}} |(\partial^o G_n) \cap (n(B_i^{(k)}+y_k))| \\ \qquad \leq (1-w) n^{d-1} \left(\sum_{k=1}^m \sum_{i\in I^{(k)}} \alpha_{d-1}(\rho_i^{(k)})^{d-1} \nu(n_{F_k}(x_i^{(k)})) \right) \end{array} \right]. \end{aligned} \tag{60}$$

Let $k \in \{1,\dots,m\}$. We aim to control $\mathrm{card}((G_n \cap n(B_i^{(k)}+y_k))\Delta(n(B_i^{(k)}+y_k)^- \cap \mathbb{Z}^d))$. To do so, it is more convenient to work with the graph F_n. In

the following, we drop the superscript (k) for clarity. With high probability, we have

$$\begin{aligned}
&\mathrm{card}((G_n \cap n(B_i^{(k)} + y_k))\Delta(n(B_i^{(k)} + y_k)^- \cap \mathbb{Z}^d)) \\
&\qquad \le \mathrm{card}((F_n \cap n(B_i + y_k))\Delta(n(B_i + y_k)^- \cap \mathbb{Z}^d)) + \mathrm{card}(F_n \setminus G_n) \\
&\qquad \le \mathrm{card}((F_n \cap n(B_i + y_k))\Delta(n(B_i + y_k)^- \cap \mathbb{Z}^d)) + \eta_3 n^{d-1/2(d-1)} .
\end{aligned}$$

As $B_i + y_k \subset B(y_k, r_k + 1)$, we have

$$\begin{aligned}
\mathcal{L}^d((nP_n \cap n(B_i + y_k))\Delta(n(B_i + y_k)^-)) &\le \mathcal{L}^d((nF_k \cap nB_i)\Delta(nB_i^-)) \\
&\quad + n^d \mathcal{L}^d(P_n \Delta(F_k + y_k)) \\
&\le n^d \alpha_d \rho_i^d \delta_2 + \varepsilon_{F_k} \le 2n^d \alpha_d \rho_i^d \delta_2 .
\end{aligned}$$

By the same arguments as in section 5.2 in [8],

$$\begin{aligned}
&\mathrm{card}((F_n \cap n(B_i + y_k))\Delta n(B_i + y_k)^-)) \\
&\qquad \le \mathcal{L}^d(((nP_n \cap n(B_i + y_k))\Delta n(B_i + y_k)^-) \cap \mathbb{Z}^d + [-1/2, 1/2]^d) \\
&\qquad \le 2n^d \alpha_d \rho_i^d \delta_2 + n^{d-1} 4d(\mathcal{H}^{d-1}(\partial B_i) + \mathcal{H}^{d-1}(\partial B_i^-)) .
\end{aligned}$$

Finally, for n large enough,

$$\mathrm{card}((G_n \cap n(B_i + y_k))\Delta(n(B_i + y_k)^- \cap \mathbb{Z}^d)) \le 4n^d \alpha_d \rho_i^d \delta_2 .$$

Thus, using inequality (60), for large enough n,

$$\begin{aligned}
&\underline{\mathbb{P}}\left[\begin{array}{c} \exists G_n \in \mathcal{G}_n, \mathcal{L}^d((\overline{P}_n \cap B(y_i, r_i + 1))\Delta(F_i + y_i)) \le \varepsilon_{F_i},\ 1 \le i \le m, \\ |\partial^o G_n| \le (1 - \lambda^2/4) n^{d-1} \mathcal{I}_p(F, \Omega),\ E(X) = \{(y_1, r_1), \dots, (y_m, r_m)\} \end{array} \right] \\
&\quad \le \sum_{k=1}^m \sum_{i \in I^{(k)}} \underline{\mathbb{P}} \left[\begin{array}{c} \exists G_n \in \mathcal{G}_n, \\ \left|(G_n \cap n(B_i + y_k))\Delta(n(B_i^- + y_k) \cap \mathbb{Z}^d)\right| \le 4\delta_2 \alpha_d \rho_i^d n^d, \\ |(\partial^o G_n) \cap n(B_i + y_k)| \\ \qquad\qquad \le (1 - w) n^{d-1} \left(\alpha_{d-1} \rho_i^{d-1} \nu(n_{F_k}(x_i^{(k)})) \right) \end{array} \right] \\
&\quad \le \frac{1}{\theta_p} \sum_{k=1}^m \sum_{i \in I^{(k)}} \mathbb{P}[G(x_i^{(k)} + y_k, \rho_i^{(k)}, n_{F_k}(x_i^{(k)}), w, \delta_2)] \qquad (61)
\end{aligned}$$

where $G(x, r, v, w, \delta_2)$ is the event that there exists a set $U \subset B \cap \mathbb{Z}^d$ such that:

$$\mathrm{card}(U \Delta (nB^-(x, r, v) \cap \mathbb{Z}^d)) \le 4\delta_2 \alpha_d r^d n^d$$

and

$$|(\partial^o G_n) \cap nB| \le (1 - w) \alpha_{d-1} r^{d-1} (\nu(v) n^{d-1} .$$

This event depends only on the edges inside $B(x, r, v)$ and is invariant under integer translation. So that,

$$\begin{aligned}
&\underline{\mathbb{P}}\left[\begin{array}{c} \exists G_n \in \mathcal{G}_n, \mathcal{L}^d((\overline{P}_n \cap B(y_i, r_i + 1))\Delta(F_i + y_i)) \le \varepsilon_{F_i},\ 1 \le i \le m, \\ |\partial^o G_n| \le (1 - \lambda^2/4) n^{d-1} \mathcal{I}_p(F, \Omega),\ E(X) = \{(y_1, r_1), \dots, (y_m, r_m)\} \end{array} \right] \\
&\quad \le \frac{1}{\theta_p} \sum_{k=1}^m \sum_{i \in I^{(k)}} \mathbb{P}[G(x_i^{(k)}, \rho_i^{(k)}, n_{F_k}(x_i^{(k)}), w, \delta_2)] . \qquad (62)
\end{aligned}$$

This event is a rare event. Indeed, if this event occurs, we can show that the capacity of the minimal cutset that separates the upper half part of $B(x,r,v)$ (upper half part according to the direction v) from the lower half part is abnormally small. To do so, we build from the set U an almost flat cutset in the ball. The fact that $\mathrm{card}(U\Delta B^-(x,r,v))$ is small implies that $\partial_e U$ is almost flat and is close to $\mathrm{disc}(x,r,v)$. However, this does not prevent the existence of long thin strands that might escape the ball and prevent U from being a cutset in the ball. The idea is to cut these strands by adding edges at a fixed height. We have to choose the appropriate height to ensure that the extra edges we needed to add to cut these strands are not too many, so that we can control their capacity. The new set of edges we create by adding to U these edges will be in a sense a cutset. The last thing to do is then to cover the $\mathrm{disc}(x,r,v)$ by hyperrectangles in order to use the estimate that the flow is abnormally small in a cylinder. This work was done in section 6 in [8]. It is possible to choose δ_2 depending on $F_1,\dots,F_m$, G and w such that for all $k\in\{1,\dots,m\}$, there exist positive constants $C_{1,i}^{F_k}$ and $C_{2,i}^{F_k}$ depending on G, d, F_k, i and w so that for all $i\in I^{(k)}$,

$$\mathbb{P}[G(x_i,\rho_i,n_{F_k}(x_i),w,\delta_2)]\le C_{1,i}^{F_k}\exp(-C_{2,i}^{F_k}n^{d-1})\,.$$

Note that this upper bound is uniform on $y_1,\dots,y_m$ but still depends on $r_1,\dots,r_m$. Together with inequalities (55) and (62), we obtain

$$\begin{aligned}\underline{\mathbb{P}}[\mathcal{F}]&\le\underline{\mathbb{P}}\left[\begin{array}{c}\exists G_n\in\mathcal{G}_n,\ \mathcal{L}^d((\overline{P}_n\cap B(y_i,r_i+1))\Delta(F_i+y_i))\le\varepsilon_{F_i},\ 1\le i\le m,\\ |\partial^o G_n|\le(1-\lambda^2/4)n^{d-1}\mathcal{I}_p(F,\Omega),\\ E(X)=\{(y_1,r_1),\dots,(y_m,r_m)\}\end{array}\right]\\ &\le\frac{1}{\theta_p}\sum_{k=1}^m\sum_{i\in I^{(k)}}C_{1,i}^{F_k}\exp(-C_{2,i}^{F_k}n^{d-1})\,.\end{aligned}$$

So there exist positive constants $C_1^{F_1},\dots,C_1^{F_m}$ and $C_2^{F_1},\dots,C_2^{F_m}$ such that

$$\underline{\mathbb{P}}[\mathcal{F}]\le\sum_{k=1}^m C_1^{F_k}\exp(-C_2^{F_k}n^{d-1})\,. \tag{63}$$

Combining inequalities (32), (39), (40) and (63), we obtain for small enough δ_2,

$$\begin{aligned}&\mathbb{P}\left[\exists G_n\in\mathcal{G}_n,\ \mu_n\notin\mathcal{V}(\mathcal{W}_\xi,\mathfrak{F}_n,\varepsilon)\,|\,0\in\mathcal{C}_\infty\right]\\ &\quad\le b_1\,\mathrm{e}^{-b_2n^{1-3/2d}}+b_1'\,\mathrm{e}^{-b_2'n}+\sum_{m=1}^M\sum_{y_1,\dots,y_m}\sum_{r_1,\dots,r_m}\sum_{i_1=1}^{N^{(r_1)}}\cdots\sum_{i_m=1}^{N^{(r_m)}}\underline{\mathbb{P}}[\mathcal{F}_{i_1,\dots,i_m}]\\ &\quad+M3^{M^2}C_dn^{M(d-2)}\Big(Mc_1\,\mathrm{e}^{-c_2n^{1-3/2d}}+D_1\,\mathrm{e}^{-D_2n^{(d-1)/2d}}\Big)\\ &\quad\le b_1\,\mathrm{e}^{-b_2n^{1-3/2d}}+b_1'\,\mathrm{e}^{-b_2'n}\\ &\quad+\sum_{m=1}^M\sum_{y_1,\dots,y_m}\sum_{r_1,\dots,r_m}\sum_{i_1=1}^{N^{(r_1)}}\cdots\sum_{i_m=1}^{N^{(r_m)}}\sum_{k=1}^m\frac{C_1^{F_{i_k}^{(r_k)}}}{\theta_p}\,\mathrm{e}^{-C_2^{F_{i_k}^{(r_k)}}n^{d-1}}\\ &\qquad+M3^{M^2}C_dn^{M(d-2)}\Big(Mc_1\,\mathrm{e}^{-c_2n^{1-3/2d}}+D_1\,\mathrm{e}^{-D_2n^{(d-1)/2d}}\Big)\\ &\quad\le b_1\,\mathrm{e}^{-b_2n^{1-3/2d}}+b_1'\,\mathrm{e}^{-b_2'n}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{M}\sum_{y_1,\dots,y_m} 3^{M^2}\max_{r_1,\dots,r_m}\left\{\sum_{i_1=1}^{N^{(r_1)}}\cdots\sum_{i_m=1}^{N^{(r_m)}}\sum_{k=1}^{m}\frac{C_1^{F_{i_k}^{(r_k)}}}{\theta_p}\,\mathrm{e}^{-C_2^{F_{i_k}^{(r_k)}}}n^{d-1}\right\}\\
&\qquad + M3^{M^2}C_d n^{M(d-2)}\Big(Mc_1\,\mathrm{e}^{-c_2 n^{1-3/2d}}+D_1\,\mathrm{e}^{-D_2 n^{(d-1)/2d}}\Big)\\
&\le b_1\,\mathrm{e}^{-b_2 n^{1-3/2d}}+b_1'\,\mathrm{e}^{-b_2' n}\\
&+C_d n^{M(d-2)}\sum_{m=1}^{M}3^{M^2}\max_{r_1,\dots,r_m}\left\{\sum_{i_1=1}^{N^{(r_1)}}\cdots\sum_{i_m=1}^{N^{(r_m)}}\sum_{k=1}^{m}\frac{C_1^{F_{i_k}^{(r_k)}}}{\theta_p}\,\mathrm{e}^{-C_2^{F_{i_k}^{(r_k)}}}n^{d-1}\right\}\\
&+M3^{M^2}C_d n^{M(d-2)}\Big(Mc_1\,\mathrm{e}^{-c_2 n^{1-3/2d}}+D_1\,\mathrm{e}^{-D_2 n^{(d-1)/2d}}\Big) \qquad (64)
\end{aligned}
$$

where C_d is a constant depending only on the dimension and the maximum is over $r_1,\dots,r_m\in\{1,\dots,3^M\}$. We recall that M, N and the number of ways of choosing $r_1,\dots,r_m$ are finite and independent of n.

Remark 6.1. To obtain inequality (64), it is crucial to use a covering of $\mathcal{C}_\beta$ that is uniform in $y_1,\dots,y_m$.

Let us assume $\mu_n\notin\mathcal{V}(\mathcal{W},\mathfrak{F}_n,2\varepsilon)$. Let $\nu\in\mathcal{W}_\xi$, we can write $\nu=\nu_{x+rW_p}$ with $x\in\mathbb{R}^d$ and $r\in[1-\lambda,1+2\xi]$. We have for all $f\in\mathfrak{F}_n$

$$
\begin{aligned}
|\nu_{x+W_p}(f)-\nu_{x+rW_p}(f)| &\le \max\big(\mathcal{L}^d(W_p\setminus(1-\lambda)W_p),\,\mathcal{L}^d((1+2\xi)W_p\setminus W_p)\big)\\
&\le c(p,d,\xi) \qquad (65)
\end{aligned}
$$

where $c(p,d,\xi)$ is a constant that goes to 0 when ξ goes to 0. So that

$$\sup_{f\in\mathfrak{F}_n}|\nu_{x+W_p}(f)-\nu_{x+rW_p}(f)|\le c(p,d,\xi)\,.$$

As $\mu_n\notin\mathcal{V}(\mathcal{W},\mathfrak{F}_n,2\varepsilon)$, we have

$$\sup_{f\in\mathfrak{F}_n}|\mu_n(f)-\nu_{x+W_p}(f)|>2\varepsilon\,.$$

So that up to choosing a smaller ξ, we have

$$c(p,d,\xi)\le\varepsilon \qquad (66)$$

and so

$$
\begin{aligned}
&\underline{\mathbb{P}}[\exists G_n\in\mathcal{G}_n,\,\forall\nu\in\mathcal{W},\,\sup_{f\in\mathfrak{F}_n}|\mu_n(f)-\nu_{x+W_p}(f)|>2\varepsilon]\\
&\qquad\le\underline{\mathbb{P}}[\exists G_n\in\mathcal{G}_n,\,\forall\nu\in\mathcal{W}_\xi,\,\sup_{f\in\mathfrak{F}_n}|\mu_n(f)-\nu_{x+W_p}(f)|>\varepsilon]\,.
\end{aligned}
$$

Finally, using (64), there exist positive constants C_1 and C_2 depending on ε, u, p and d such that for all $n\ge1$,

$$\mathbb{P}\left[\exists G_n\in\mathcal{G}_n,\,\mu_n\notin\mathcal{V}(\mathcal{W},\mathfrak{F}_n,2\varepsilon)\,\middle|\,0\in\mathcal{C}_\infty\right]\le C_1\,\mathrm{e}^{-c_2 n^{1-3/2d}}$$

and the result follows.

To conclude, let us sum up the order in which the constants are chosen. We first choose $\varepsilon>0$. Next, we choose ξ small enough such that it satisfies both inequalities (54) and (66), and δ' such that it satisfies inequality (53). Next,

we choose δ such that $\eta(\delta)$ satisfies inequalities (36), (50) and (52). We choose δ_2 depending on w (and so on ε) and G. The parameter δ_2 has to satisfy some inequalities that we do not detail here, we refer to section 7 in [8]. Finally, to each r in $\{1, \dots, 3^M\}$, to each $F \in \mathcal{C}_\beta^{(r)}$, we choose ε_F in such a way it satifies inequality (57). $\square$

6.2 Proof of Theorem 4

In this section we prove Theorem 4. Thanks to Theorem 5, we know that with high probability μ_n is close to the set $\mathcal{W}$ and so it is close to the measure of a translate of the Wulff shape. In fact, as μ_n has its support included in $B(0, n^{d-1})$, the measure μ_n is close to $\mathcal{W}_n$, the set of measures defined as:

$$\mathcal{W}_n = \left\{ \nu_{x+W_p},\, x \in B(0, n^{d-1}) \right\}.$$

The continuous set $\mathcal{W}_n$ can be approximated by a finite set $\widetilde{\mathcal{W}}$ containing a polynomial number of measures such that μ_n is close to $\widetilde{\mathcal{W}}$ and so is close to at least one measure in $\widetilde{\mathcal{W}}$. Let $\varepsilon > 0$ and let $w > 0$ be a real number depending on ε that we will choose later. We first use Lemma 6, to cover W_p by a finite number of balls of small radius such that W_p is almost flat in each ball. Let δ_2 that will be chosen later and let $(B(x_k, \rho_k, v_k))_{k \in J}$ be a family associated to $W_p, \delta_2, \varepsilon$ that satisfies the conditions stated in Lemma 6. We will use this covering for all the translates of the Wulff shape. We set $\varepsilon_W = \min_{k \in J} \alpha_d \rho_k^d \delta_2$. We now cover $\mathcal{W}_n$ by a polynomial in n number of balls of radius less than ε_W. Let $\xi > 0$ small enough such that

$$\forall x, y \in \mathbb{R}^d,\ \|x - y\|_2 \le \xi \implies \mathcal{L}^d\left((x + W_p)\Delta(y + W_p)\right) \le \frac{\varepsilon_W}{4}.$$

By construction, μ_n has its support included in $B(0, n^{d-1})$. We can cover $B(0, n^{d-1})$ by a polynomial in n number of balls of radius ξ. More precisely, there exist $z_1, \dots, z_{M'} \in B(0, n^{d-1})$, such that M' is polynomial in n and

$$B(0, n^{d-1}) \subset \bigcup_{i=1}^{M'} B(z_i, \xi).$$

We set

$$\widetilde{\mathcal{W}} = \left\{ \nu_{z_i + W_p}, i = 1, \dots, M' \right\}.$$

Let $\delta > 0$ we will choose later. We define W_p^δ and $W_p^{-\delta}$ as

$$W_p^\delta = \{x \in \mathbb{R}^d \,:\, d_2(x, W_p) \le \delta\} \text{ and } W_p^{-\delta} = \{x \in W_p \,:\, d_2(x, \partial W_p) \ge \delta\}.$$

Let us define g as

$$g(x) = \begin{cases} \min(d_2(x, W_p)/\delta, 1) & \text{if } x \in \mathbb{R}^d \setminus W_p \\ -\min(d_2(x, \partial W_p)/\delta, 1) & \text{if } x \in W_p \end{cases}.$$

The function g is uniformly continuous and satisfies $\|g\|_\infty \le 1$. For each $i \in \{1, \dots, M'\}$, we define g_i by $g_i(x) = g(x - z_i)$ for $x \in \mathbb{R}^d$, and $\mathfrak{F} = \{g_i,\, 1 \le i \le M'\} \cup \{1\}$. The set $\mathfrak{F}$ is a set made of translates of g and the constant function

equal to 1. If the measure μ_n is in the local weak neighborhood $\mathcal{V}(\mathcal{W}, \mathfrak{F}, \frac{\varepsilon_W}{4})$, then there exists ν_{x+W_p} in $\mathcal{V}(\mathcal{W}_n, \mathfrak{F}, \frac{\varepsilon_W}{4})$ such that

$$\sup_{f\in\mathfrak{F}} |\nu_{x+W_p}(f) - \mu_n(f)| \leq \frac{\varepsilon_W}{4}\,.$$

Moreover there exists an $i \in \{1, \dots, M'\}$ such that $x \in B(z_i, \xi)$ and so

$$\sup_{f\in\mathfrak{F}} |\nu_{x+W_p}(f) - \nu_{z_i+W_p}(f)| \leq \mathcal{L}^d\left((x+W_p)\Delta(z_i+W_p)\right) \leq \frac{\varepsilon_W}{4}$$

and also

$$\mu_n \in \mathcal{V}\big(\widetilde{\mathcal{W}}, \mathfrak{F}, \varepsilon_W/2\big)\,.$$

Let us choose $r > 0$ large enough so that the ball $B(0, r-2d)$ contains W_p. For $x \in \mathbb{R}^d$, we define $\lfloor x \rfloor$ to be the closest point to x in $\mathbb{Z}^d$ for the Euclidean distance. For any $i \in \{1, \dots, M'\}$, we have

$$W + z_i \subset B(\lfloor z_i \rfloor, r)\,.$$

Let us define the function u such that for all $\iota > 0$,

$$u(\iota) = \min\big(\sup\big\{\,\delta > 0,\ \forall x, y \in \mathbb{R}^d,\ \|x-y\|_2 \leq \delta \implies |g(x) - g(y)| \leq \iota\,\big\}, 1\big)\,.$$

As the function g is uniformly continuous, the function u is positive. Moreover, as $\mathfrak{F}$ is made of translated of g and the constant function equal to 1, it is clear that this set satisfies the condition stated in Proposition 5.5 associated with the function u. Using Proposition 5.5 with the function u, there exist positive constants C_1, C_2 depending only on r, u, p and ε_W such that for all $i \in \{1, \dots, M'\}$

$$\underline{\mathbb{P}}\left[\max_{G_n\in\mathcal{G}_n} \sup_{f\in\mathfrak{F}} |\mu_n(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)}) - \nu_n(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)})| > \varepsilon_W/4\right] \leq C_1\,\mathrm{e}^{-c_2 n^{1-3/2d}}\,. \tag{67}$$

The point of choosing such a set $\mathfrak{F}$ is that we can deduce from the fact that the quantity $\sup_{f\in\mathfrak{F}} |\mu_n(f) - \nu_{W+z_i}(f)|$ is small that the associated symmetric difference $\mathcal{L}^d((P_n \cap B(\lfloor z_i\rfloor, r))\Delta(z_i+W_p))$ is small. Indeed, we have

$$\begin{aligned}
&\mathcal{L}^d((P_n \cap B(\lfloor z_i\rfloor, r))\Delta(z_i+W_p)) \\
&\quad= \int_{(P_n\cap B(\lfloor z_i\rfloor,r))\setminus(z_i+W_p)} 1 d\mathcal{L}^d(x) + \int_{(z_i+W_p)\setminus P_n} 1 d\mathcal{L}^d(x) \\
&\quad\leq \int_{(P_n\cap B(\lfloor z_i\rfloor,r))\setminus(z_i+W_p)} g_i(x) d\mathcal{L}^d(x) - \int_{(z_i+W_p)\setminus P_n} g_i(x) d\mathcal{L}^d(x) \\
&\qquad + \mathcal{L}^d(W_p^\delta \setminus W_p^{-\delta}) \\
&\quad= |\nu_n(g_i\mathbb{1}_{B(\lfloor z_i\rfloor,r))}) - \nu_{W+z_i}(g_i\mathbb{1}_{B(\lfloor z_i\rfloor,r))})| + \mathcal{L}^d(W_p^\delta \setminus W_p^{-\delta}) \\
&\quad\leq \sup_{f\in\mathfrak{F}} |\mu_n(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)}) - \nu_n(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)})| \\
&\qquad + \sup_{f\in\mathfrak{F}} |\mu_n(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)}) - \nu_{W+z_i}(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)})| + \mathcal{L}^d(W_p^\delta \setminus W_p^{-\delta})\,.
\end{aligned} \tag{68}$$

So we choose δ small enough so that

$$\mathcal{L}^d(W_p^\delta \setminus W_p^{-\delta}) \leq \frac{\varepsilon_W}{4}\,. \tag{69}$$

Moreover, we have

$$\begin{aligned}\underline{\mathbb{P}}\Big[\exists G_n \in \mathcal{G}_n,\, |\partial^o G_n| \leq (1-w)\mathcal{I}_p(W_p)n^{d-1},\, \mu_n \in \mathcal{V}\big(\widetilde{\mathcal{W}}, \mathfrak{F}, \varepsilon_W/2\big)\Big] \\ \leq \sum_{i=1}^{M'} \mathbb{P}\left[\begin{array}{c}\exists G_n \in \mathcal{G}_n,\, |\partial^o G_n| \leq (1-w)\mathcal{I}_p(W_p)n^{d-1}, \\ \sup_{f\in\mathfrak{F}} |\mu_n(f) - \nu_{W+z_i}(f)| \leq \varepsilon_W/2\end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right]\,. \end{aligned} \tag{70}$$

Using inequalities (67), (68) and (69), we obtain

$$\begin{aligned}&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\ \begin{array}{c}|\partial^o G_n| \leq (1-w)\mathcal{I}_p(W_p)n^{d-1}, \\ \sup_{f\in\mathfrak{F}} |\mu_n(f) - \nu_{W+z_i}(f)| \leq \varepsilon_W/2\end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\ &\leq \underline{\mathbb{P}}\left[\exists G_n \in \mathcal{G}_n,\ \begin{array}{c}|\partial^o G_n| \leq (1-w)\mathcal{I}_p(W_p)n^{d-1}, \\ \sup_{f\in\mathfrak{F}} |\mu_n(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)}) - \nu_{W+z_i}(f\mathbb{1}_{B(\lfloor z_i\rfloor,r)})| \leq \varepsilon_W/2\end{array}\right] \\ &\leq \underline{\mathbb{P}}\left[\exists G_n \in \mathcal{G}_n,\ \begin{array}{c}|\partial^o G_n| \leq (1-w)\mathcal{I}_p(W_p)n^{d-1}, \\ \mathcal{L}^d((P_n \cap B(\lfloor z_i\rfloor, r))\Delta(z_i + W_p)) \leq \varepsilon_W\end{array}\right] + C_1\,\mathrm{e}^{-c_2 n^{1-3/2d}}\,. \end{aligned} \tag{71}$$

Finally, we proceed as in inequality (62) in the proof of Theorem 5:

$$\begin{aligned}&\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\ \begin{array}{c}|\partial^o G_n| \leq (1-w)\mathcal{I}_p(W_p)n^{d-1}, \\ \mathcal{L}^d((P_n \cap B(\lfloor z_i\rfloor, r))\Delta(z_i + W_p)) \leq \varepsilon_W\end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\ &\qquad\qquad \leq \frac{1}{\theta_p}\sum_{k\in J} \mathbb{P}\left[G(z_i + x_k, \rho_k, n_{W_p}(x_k), w, \delta_2)\right]\,. \end{aligned} \tag{72}$$

It is possible to choose δ_2 depending on W, G and w (see again section 6 in [8]) such that there exist positive constants $C_{1,k}$ and $C_{2,k}$ depending on G, d, W, k and w so that for all $k \in J$,

$$\mathbb{P}[G(x_k, \rho_k, n_{W_p}(x_k), w, \delta_2)] \leq C_{1,k}\exp(-C_{2,k}n^{d-1})\,.$$

So combining inequalities (70), (71) and (72), we obtain

$$\begin{aligned}\underline{\mathbb{P}}\Big[\exists G_n \in \mathcal{G}_n,\ |\partial^o G_n| \leq (1-w)\mathcal{I}_p(W_p)n^{d-1},\ \mu_n \in \mathcal{V}\big(\widetilde{\mathcal{W}}, \mathfrak{F}, \varepsilon_W/2\big)\Big] \\ \leq M'\Big(C_1\,\mathrm{e}^{-c_2 n^{1-3/2d}} + \frac{1}{\theta_p}\sum_{k\in J} C_{1,k}\exp(-C_{2,k}n^{d-1})\Big)\,. \end{aligned} \tag{73}$$

Moreover, we have

$$\begin{aligned}&\underline{\mathbb{P}}\left[\exists G_n \in \mathcal{G}_n,\, \frac{|G_n|}{n^d} \geq (1+w)\theta_p\mathcal{L}^d(W_p),\, \mu_n \in \mathcal{V}\big(\widetilde{\mathcal{W}}, \mathfrak{F}, \varepsilon_W/2\big)\right] \\ &\qquad \leq \sum_{i=1}^{M'} \mathbb{P}\left[\begin{array}{c}\exists G_n \in \mathcal{G}_n,\, \frac{|G_n|}{n^d} \geq (1+w)\theta_p\mathcal{L}^d(W_p), \\ |\mu_n(1) - \nu_{W+z_i}(1)| \leq \varepsilon_W/2\end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \\ &\qquad \leq \sum_{i=1}^{M'} \mathbb{P}\left[\begin{array}{c}\exists G_n \in \mathcal{G}_n,\, \frac{|G_n|}{n^d} \geq (1+w)\theta_p\mathcal{L}^d(W_p), \\ \left|\frac{|G_n|}{n^d} - \theta_p\mathcal{L}^d(W_p)\right| \leq \varepsilon_W/2\end{array} \,\middle|\, 0 \in \mathcal{C}_\infty\right] \end{aligned} \tag{74}$$

where we recall that $\theta_p \mathcal{L}^d(W_p) = 1$, so up to choosing a smaller ε_W, we assume that $\varepsilon_W \leq 2w$ so that the probability in the sum is equal to 0. Finally, combining inequalities (73) and (74), we obtain

$$\mathbb{P}\left[n\varphi_n \geq \frac{1-w}{1+w} \frac{\mathcal{I}_p(W_p)}{\theta_p \mathcal{L}^d(W_p)} \,\Big|\, 0 \in \mathcal{C}_\infty \right] \leq \underline{\mathbb{P}}\left[\exists G_n \in \mathcal{G}_n,\, \mu_n \notin \mathcal{V}\big(\widetilde{\mathcal{W}}, \mathfrak{F}, \varepsilon_W/2\big) \right] + M'\Big(C_1 \exp(-C_2 n) + \frac{1}{\theta_p} \sum_{k \in J} C_{1,k} \exp(-C_{2,k} n^{d-1}) \Big) . \quad (75)$$

Thanks to Theorem 5, there exist positive constants C_1', C_2', depending on p, u, ε_W and d such that

$$\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\, \mu_n \notin \mathcal{V}\big(\widetilde{\mathcal{W}}, \mathfrak{F}, \varepsilon_W/2\big) \,\Big|\, 0 \in \mathcal{C}_\infty \right] \leq C_1' \exp(-C_2' n^{1-3/2d}) .$$

By choosing w small enough, we obtain

$$\mathbb{P}\left[n\varphi_n \geq (1-\varepsilon) \frac{\mathcal{I}_p(W_p)}{\theta_p \mathcal{L}^d(W_p)} \,\Big|\, 0 \in \mathcal{C}_\infty \right] \leq C_1' \exp(-C_2' n^{1-3/2d}) + M'\Big(C_1 \exp(-C_2 n) + \sum_{k \in J} C_{1,k} \exp(-C_{2,k} n^{d-1}) \Big) .$$

As M' is polynomial in n, the result follows.

6.3 Proof of Theorem 2

Let $\varepsilon > 0$. As in the proof of Theorem 4, there exists an integer M' that is polynomial in n and $z_1, \ldots, z_{M'}$ points of $B(0, n^{d-1})$ such that for any finite set $\mathfrak{F}$ of continuous functions of infinite norm at most 1, if $\mu_n \in \mathcal{V}(\mathcal{W}, \mathfrak{F}, \varepsilon)$ then $\mu_n \in \mathcal{V}(\widetilde{\mathcal{W}}, \mathfrak{F}, 2\varepsilon)$ where $\widetilde{\mathcal{W}} = \{ \nu_{z_i + W_p}, i = 1, \ldots, M' \}$. Let $\delta > 0$ we will choose later. Let us define f and g as

$$f(x) = \min(d_2(x, \mathbb{R}^d \setminus W_p^\delta)/\delta, 1), \text{ for } x \in \mathbb{R}^d$$

and

$$g(x) = \min(d_2(x, W_p)/\delta, 1), \text{ for } x \in \mathbb{R}^d .$$

The functions f and g are uniformly continuous and satisfy $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$. For each $i \in \{1, \ldots, M'\}$, we define f_i by $f_i(x) = f(x - z_i)$ and g_i by $g_i(x) = f(x - z_i)$ for $x \in \mathbb{R}^d$. We define

$$\mathfrak{F} = \{ f_i,\, 1 \leq i \leq M' \} \cup \{ g_i,\, 1 \leq i \leq M' \} .$$

Let $G_n \in \mathcal{G}_n$. Let $i \in \{1, \ldots, M'\}$. We have

$$|G_n \Delta ((n(W_p + z_i)) \cap \mathcal{C}_\infty)| = |G_n \setminus n(W_p + z_i)| + |(n(W_p + z_i) \cap \mathcal{C}_\infty) \setminus G_n| . \quad (76)$$

Using a renormalization argument as in the proof of Theorem 3, there exist positive constants C_1 and C_2 depending on p, ε and d such that for all $i \in \{1, \ldots, M'\}$,

$$\mathbb{P}\left[\left| \frac{|(n(W + z_i)) \cap \mathcal{C}_\infty|}{n^d} - \theta_p \mathcal{L}^d(W_p) \right| \geq \varepsilon \,\Big|\, 0 \in \mathcal{C}_\infty \right] \leq C_1 \exp(-C_2 n) .$$

As $G_n \cap (n(W_p + z_i)) \subset (n(W_p + z_i)) \cap \mathcal{C}_\infty$, we have with probability at least $1 - C_1 \exp(-C_2 n)$,

$$\begin{aligned}
&|((n(W_p + z_i)) \cap \mathcal{C}_\infty) \setminus G_n| \\
&\quad = |(n(W_p + z_i)) \cap \mathcal{C}_\infty| - |G_n \cap (n(W_p + z_i))| \\
&\quad \leq \theta_p \mathcal{L}^d(W_p) n^d + n^d \varepsilon - n^d \mu_n(f_i) + |n((W_p^\delta + z_i) \setminus (W_p + z_i)) \cap \mathbb{Z}^d| \,.
\end{aligned}$$

We can find a constant $c(\delta)$ depending only on δ, p and d, such that $c(\delta)$ goes to 0 when δ goes to 0 and for all $z \in \mathbb{R}^d$

$$|n((W_p^\delta + z) \setminus (W_p + z)) \cap \mathbb{Z}^d| \leq c(\delta) n^d \,,$$

so that,

$$\begin{aligned}
|((n(W_p + z_i)) \cap \mathcal{C}_\infty) \setminus G_n| &\leq n^d |\nu_{W_p + z_i}(f_i) - \mu_n(f_i)| + (\varepsilon + c(\delta)) n^d \\
&\leq n^d \sup_{h \in \mathfrak{F}} |\nu_{W_p + z_i}(h) - \mu_n(h)| + (\varepsilon + c(\delta)) n^d \,. \quad (77)
\end{aligned}$$

Moreover, noticing that $\nu_{W_p + z_i}(g_i) = 0$, we obtain

$$\begin{aligned}
|G_n \setminus n(W + z_i)| &\leq n^d \mu_n(g_i) + |n((W_p^\delta + z_i) \setminus (W_p + z_i)) \cap \mathbb{Z}^d| \\
&\leq n^d |\mu_n(g_i) - \nu_{W_p + z_i}(g_i)| + n^d c(\delta) \\
&\leq n^d \sup_{h \in \mathfrak{F}} |\nu_{W_p + z_i}(h) - \mu_n(h)| + n^d c(\delta) \,. \quad (78)
\end{aligned}$$

Combining inequalities (76), (77) and (78), with high probability, we have

$$\begin{aligned}
&\inf_{z \in \mathbb{R}^d} \frac{1}{n^d} |G_n \Delta ((n(W_p + z)) \cap \mathcal{C}_\infty)| \\
&\quad \leq \min_{1 \leq i \leq M'} \frac{1}{n^d} |G_n \Delta ((n(W_p + z_i)) \cap \mathcal{C}_\infty)| \\
&\quad \leq \min_{\nu \in \widetilde{\mathcal{W}}} \left\{ \sup_{h \in \mathfrak{F}} |\nu(h) - \mu_n(h)| + \sup_{h \in \mathfrak{F}} |\nu(h) - \mu_n(h)| \right\} + \varepsilon + 2c(\delta) \\
&\quad \leq 2 \min_{\nu \in \widetilde{\mathcal{W}}} \sup_{h \in \mathfrak{F}} |\nu(h) - \mu_n(h)| + \varepsilon + 2c(\delta) \,.
\end{aligned}$$

Let us define for any $\iota > 0$,

$$u_g(\iota) = \min \left(\sup \left\{ \delta > 0, \, \forall x, y \in \mathbb{R}^d, \, \|x - y\|_2 \leq \delta \implies |g(x) - g(y)| \leq \iota \right\}, 1 \right) \,,$$

$$u_f(\iota) = \min \left(\sup \left\{ \delta > 0, \, \forall x, y \in \mathbb{R}^d, \, \|x - y\|_2 \leq \delta \implies |f(x) - f(y)| \leq \iota \right\}, 1 \right)$$

and $u = \min(u_f, u_g)$. This function is positive because the function f and g are uniformly continuous. It is easy to check that $\mathfrak{F}$ satisfies the condition required in Theorem 5 associated with the function u. Thus, there exist positive constants c_1 and c_2 depending on p, u, ε and d such that

$$\mathbb{P} \left[\exists G_n \in \mathcal{G}_n, \, \inf_{\nu \in \mathcal{W}} \sup_{h \in \mathfrak{F}} |\nu(h) - \mu_n(h)| \geq \varepsilon \,\middle|\, 0 \in \mathcal{C}_\infty \right] \leq c_1 \, \mathrm{e}^{-c_2 n^{1 - 3/2d}}$$

and so

$$\mathbb{P} \left[\exists G_n \in \mathcal{G}_n, \, \min_{\nu \in \widetilde{\mathcal{W}}} \sup_{h \in \mathfrak{F}} |\nu(h) - \mu_n(h)| \geq 2\varepsilon \,\middle|\, 0 \in \mathcal{C}_\infty \right] \leq c_1 \, \mathrm{e}^{-c_2 n^{1 - 3/2d}} \,.$$

We now choose δ small enough such that $c(\delta) \leq \varepsilon$ so that

$$\mathbb{P}\left[\exists G_n \in \mathcal{G}_n,\ \inf_{z\in\mathbb{R}^d} \frac{1}{n^d}|G_n \Delta((n(W_p+z)) \cap \mathcal{C}_\infty)| \geq 7\varepsilon \,\middle|\, 0 \in \mathcal{C}_\infty\right]$$
$$\leq c_1\, \mathrm{e}^{-c_2 n^{1-3/2d}} + M' C_1 \exp(-C_2 n)\,.$$

As M' is polynomial in n, this yields the result.

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