



HAL
open science

Perturbation approaches of a planar crack in linear elastic fracture mechanics: A review

Veronique Lazarus

► **To cite this version:**

Veronique Lazarus. Perturbation approaches of a planar crack in linear elastic fracture mechanics: A review. *Journal of the Mechanics and Physics of Solids*, 2011, 59 (2), pp.121-144. 10.1016/j.jmps.2010.12.006 . hal-01904324

HAL Id: hal-01904324

<https://hal.science/hal-01904324>

Submitted on 29 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Perturbation approaches of a planar crack in Linear Elastic Fracture Mechanics: a review

V. Lazarus

UPMC Univ Paris 6, Univ Paris-Sud, CNRS, UMR 7608, Lab FAST, Bat 502, Campus Univ, F-91405, Orsay, France.

Abstract

One current challenge of Linear Elastic Fracture Mechanics (LEFM) is to take into account the non-linearities induced by the crack front deformations. For this, a suitable approach is the crack front perturbation method initiated by Rice (1985). It allows to update the Stress Intensity Factors (SIF) when the crack front of a planar crack is perturbed in its plane. This approach and its later extensions to more complex cases are recalled in this review. Applications concerning the deformation of the crack front when it propagates quasistatically in an homogeneous or heterogeneous media have been considered in brittle fracture, fatigue or subcritical propagation. The crack shapes corresponding to uniform SIF have been derived: cracks with straight or circular fronts, but also when bifurcations exist, with wavy front. For an initial straight crack, it has been shown that, in homogeneous media, in the quasistatic case, perturbations of all lengthscales progressively disappear unless disordered fracture properties yields Family and Vicsek (1985) roughness of the crack front. Extension of those perturbation approaches to more realistic geometries and to coalescence of cracks is also envisaged.

Key words: LEFM, brittle fracture, fatigue, crack front deformations, perturbation approach

-
- 1 Consider a crack embedded in an elastic solid that is loaded quasistatically
 - 2 (fig. 1). (i) Under which conditions of loading, (ii) in which direction and
 - 3 (iii) along which distance will this crack propagate? Linear Elastic Fracture

Email address: veronique.lazarus@upmc.fr (V. Lazarus)

URL: <http://www.fast.u-psud.fr/> (V. Lazarus)

Preprint submitted to Journal of the Mechanics and Physics of Solids December 20, 2010

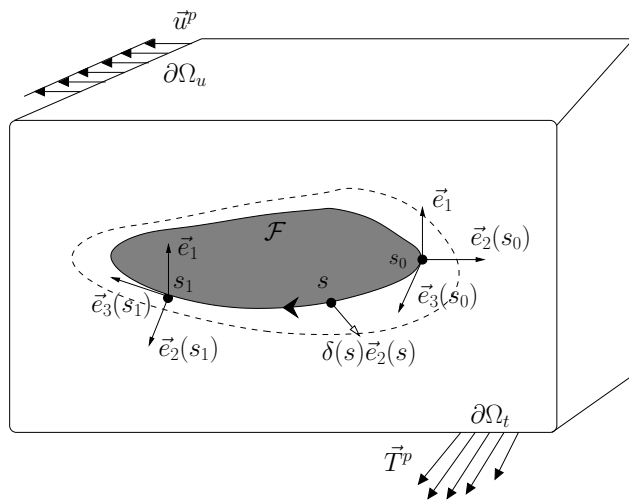


Figure 1: An example of three-dimensional LEFM problem: a three-dimensional planar crack (in grey) embedded in an arbitrary body loaded by external load \vec{T}^p , \vec{u}^p on $\partial\Omega_T$, $\partial\Omega_u$. The dashed line corresponds to the position of the initial front \mathcal{F} after an in-plane $\delta(s)\vec{e}_2(s)$ advance.

4 Mechanics (LEFM) aims at answering these questions. It is widely applied in
5 several fields, for instance: in engineering applications (Anderson, 1991) for
6 obvious safety reasons; in geological applications (Aydin and Pollard, 1988;
7 Atkinson, 1987) as earthquakes (Liu and Rice, 2005; Fisher et al., 1997; Grob
8 et al., 2009; Bonamy, 2009), basalt columns (Goehring et al., 2009) or in the
9 soft matter domain (Gauthier et al., 2010). The principal aim of this review,
10 in addition to a brief overview of the answers to the first two questions, is
11 linked to the third question and concerns the crack front deformation of a
12 three-dimensional planar crack during its propagation.

13
14 Nowadays, several different approaches are developed within LEFM. *Tra-*
15 *ditional* approach, in the continuity of the pioneer works of Griffith (1920)
16 and Irwin (1958), is based on the local mechanical and energetic fields near
17 the crack front and uses propagation criterions based on the Stress Inten-
18 sity Factors K_1 , K_2 , K_3 or the elastic energy release rate \mathcal{G} . This approach
19 aims at predicting the propagation conditions (loading, path) of a preexisting
20 crack, but is unable to deal with the crack initiation problem.

21 Recently new approaches have been developed that are able, in theory,

22 to deal with both the crack initiation and propagation paths: the *energetic*
23 *variational minimisation* approach to fracture and the *phase-field* method.
24 The first (i) has been shown to include the traditional approach (Francfort
25 and Marigo, 1998; Bourdin et al., 2008), (ii) has been approximated for nu-
26 merical purposes by non-local damage first-gradient model (Bourdin et al.,
27 2000) and (iii) has been applied to several brittle fracture problems as sta-
28 bility problems (Benallal and Marigo, 2007), the deterioration of the French
29 Panthéon (Lancioni and Royer-Carfagni, 2009), the propagation direction in
30 presence of mode 2 (Chambolle et al., 2009), crack patterns in shrinkage, dry-
31 ing or cooling, problems (Lazarus et al., 2009). The second, first developed
32 for solidification front (Caginalp and Fife, 1986; Collins and Levine, 1985)
33 was further extended to brittle fracture (Karma et al., 2001) and has shown
34 their efficiency to resolve problems of path determination in 2D (Henry and
35 Levine, 2004; Hakim and Karma, 2009; Corson et al., 2009) and 3D (Pons
36 and Karma, 2010).

37 But the traditional approach has still its place thanks to the maturity
38 acquired by its longer history: in particular, to deal with the crack front shape
39 deformations an efficient perturbation method of the crack front, pioneered
40 by Rice (1985), has been used. The method allows to update the stress
41 intensity factors (that are prerequisite to any crack propagation prediction)
42 for any small perturbation of the crack front without resolving the whole
43 elasticity problem. On the one hand, the initial method of Rice (1985) has
44 been applied to the propagation of cracks in disordered heterogeneous by the
45 statistical physics community. On the other, it has been extended and applied
46 to more and more complex problems by the fracture mechanics community.
47 It has recently gained a renewal of interest (Dalmas et al., 2009) with the
48 creation of new intelligent materials by deposition of a stack of nanometric
49 thin layers having specific functions (as for instance, optical, thermal, self-
50 cleaning) that may perturb locally the crack front.

51 A review concerning the roughening of the front in disordered heteroge-
52 neous materials has recently been done by Bonamy (2009) from a statistical
53 physics point of view. Here, the aim is to do a review of the crack front
54 perturbation approaches from a mechanical point of view. It is focused, as
55 in Bonamy (2009), mainly on the slow crack growth regime in which the
56 crack speed is negligible in comparison to Rayleigh speed. The high speed
57 regime is beyond the scope of this paper although at some places some key
58 references are given.

59 The outline of the paper is as follows. After a brief overview of the

60 traditional LEFM approach (section 1), the perturbation method is presented
 61 in the general case (section 2) and then for some model selected problems
 62 (section 3). Application to crack propagation in homogeneous media (section
 63 4), crack trapping by tougher obstacles (section 5), propagation in disordered
 64 media (section 6) are then developed.

65 1. Overview of the traditional LEFM approach

66 1.1. Definition of the SIFs

67 Consider a crack embedded in a linear elastic body. Suppose that the
 68 evolution is quasistatic¹. Let \mathcal{F} denote the crack front and s some curvilinear
 69 abscissa along it. The front is supposed to be smooth so that at each point s
 70 of \mathcal{F} , one can define a local basis of vectors $(\vec{e}_1(s), \vec{e}_2(s), \vec{e}_3(s))$ in the following
 71 way :

- 72 1. $\vec{e}_3(s)$ is tangent to \mathcal{F} and oriented in the same direction as the curvi-
 73 linear abscissa s ;
- 74 2. $\vec{e}_2(s)$ is in the crack plane, orthogonal to \mathcal{F} and oriented in the direction
 75 of propagation;
- 76 3. $\vec{e}_1(s) \equiv \vec{e}_1$ is orthogonal to the crack plane and oriented in such a way
 77 that the basis $(\vec{e}_2(s), \vec{e}_1, \vec{e}_3(s))$ is direct².

The SIFs $K_j(s)$, $j = 1, 2, 3$ at point s are then defined by the following
 formula, where Einstein's summation convention is employed :

$$\lim_{r \rightarrow 0} \sqrt{\frac{2\pi}{r}} \llbracket \vec{u}(s, r) \rrbracket \equiv 8\Lambda_{ij} K_j(s) \vec{e}_i(s). \quad (1)$$

In this expression $\llbracket \vec{u}(s, r) \rrbracket$ denotes the displacement discontinuity across the
 crack plane, oriented by the vector \vec{e}_1 , at the distance r behind the point s
 of \mathcal{F} , in the direction of the vector $-\vec{e}_2(s)$. Also, $(\Lambda_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 3} \equiv \mathbf{\Lambda}$ is the

¹In the dynamic case, one may refer e.g. to Freund (1972b, 1973, 1974), Kostrov (1975)
 for pioneer works and Ravi-Chandar (1998), Fineberg and Marder (1999), Freund (2000),
 Bouchbinder et al. (2010) for more recent publications.

²the order of the vectors may seem a little surprising but this definition of the local basis
 presents the advantage that the mode 1, (resp. 2 and 3) corresponds to the displacement
 jump along the vector with the same numbering, that is \vec{e}_1 (resp. \vec{e}_2 and \vec{e}_3)

diagonal matrix defined by

$$\mathbf{\Lambda} \equiv \frac{1}{E} \begin{pmatrix} 1 - \nu^2 & 0 & 0 \\ 0 & 1 - \nu^2 & 0 \\ 0 & 0 & 1 + \nu \end{pmatrix} \quad (2)$$

78 where E denotes Young's modulus and ν Poisson's ratio³.

79 *1.2. Crack advance versus loading criterions*

80 One shall distinguish several cases depending on the material and the
81 type of loading. The crack advance may occur dramatically above a certain
82 threshold (*brittle fracture*) but also below this threshold at a slower rate in
83 the cases of *subcritical* propagation due for instance to stress-corrosion or
84 *fatigue* propagation due to cyclic loading.

85 *1.2.1. Brittle fracture*

Concerning brittle fracture, Griffith (1920)'s criterion is extensively used: it states that the crack propagation occurs if the elastic energy released by the crack propagation \mathcal{G} is sufficient to counterbalance the fracture energy Γ necessary to create new surfaces:

$$\mathcal{G} < \Gamma \Rightarrow \text{no propagation}, \quad (3)$$

$$\mathcal{G} = \Gamma \Rightarrow \text{possible propagation}. \quad (4)$$

SIFs and energy release rate \mathcal{G} are linked by Irwin (1957)'s formula (Einstein summation convention is used):

$$\mathcal{G} = K_i \Lambda_{ij} K_j, \quad (5)$$

so that, in mode 1 ($K_2 = 0$, $K_3 = 0$), Griffith (1920)'s criterion is equivalent to Irwin (1958)'s criterion which states that the crack propagates if the Stress Intensity Factor K_1 at the crack tip exceeds the local toughness K_c :

$$K_1 < K_c \Rightarrow \text{no propagation}, \quad (6)$$

$$K_1 = K_c \Rightarrow \text{possible propagation}. \quad (7)$$

³A similar formula holds for an arbitrary anisotropic medium but the matrix $\mathbf{\Lambda}$ is then no longer diagonal (Broberg, 1999, §4.14), and also for the dynamic case, but then $\mathbf{\Lambda}$ depends on the crack velocity (Freund, 2000, §5.3).

Above the threshold, Griffith's criterion remains valid (Sharon and Fineberg, 1999) provided that the velocity dependence of Λ is taken into account (Freund, 2000) in equation (2). In particular, in mode 1 and in the slow growth regime ($v \ll C_R$), it gives for the crack velocity v :

$$v = \frac{2C_R}{K_c}(K_1 - K_c) \text{ for } K_1 \geq K_c \quad (8)$$

⁸⁶ C_R being the Rayleigh speed.

⁸⁷ *1.2.2. Subcritical or fatigue propagation*

Concerning the subcritical propagation below the brittle fracture threshold (K_c or Γ), Paris' law (Paris et al., 1961; Erdogan and Paris, 1963) with a threshold \mathcal{G}_0 or without ($\mathcal{G}_0 = 0$) is often used. It states that the crack velocity v goes as a power-law with the excess energy release rate \mathcal{G} :

$$v = C(\mathcal{G} - \mathcal{G}_0)^\beta, \quad (9)$$

In the case of mode 1 with $\mathcal{G}_0 = 0$, it is equivalent to (use (5) and (2)):

$$v = C'K_1^N \text{ with } N = 2\beta \text{ and } C' = C \left(\frac{1 - \nu^2}{E} \right)^\beta \quad (10)$$

⁸⁸ For fatigue, the same expressions (9) and (10) are valid if v is re-interpreted
⁸⁹ as advance during one cycle and \mathcal{G} as variation of the energy release rate dur-
⁹⁰ ing one cycle. An overview of the values of β or N and C for engineering
⁹¹ materials can be found in Fleck et al. (1994). It's physical background (Vieira
⁹² et al., 2008) and its validity field (Ciavarella et al., 2008) are still the subject
⁹³ of many research papers.

⁹⁴

⁹⁵ *1.3. Crack propagation direction criterions*

⁹⁶ In homogeneous isotropic elastic media, except in some special condi-
⁹⁷ tions, it is well known that whatever the external loading, the crack front
⁹⁸ bifurcates in order to reach a situation of pure tension loading as the crack
⁹⁹ propagates (Hull, 1993). Hence, planar crack propagation is generally stable
¹⁰⁰ under mode 1 loading and unstable under mode 2 or 3. A literature survey
¹⁰¹ of mixed mode crack growth can be found in Qian and Fatemi (1996). Under
¹⁰² mode (1+2) conditions, the crack kinks to make mode 2 vanish. The value

103 of the corresponding kink angle can be obtained, for instance, by the Prin-
104 ciple of Local Symmetry (PLS) of Goldstein and Salganik (1974) or by the
105 maximum tangential stress criterion (MTS; Erdogan and Sih (1963)). The
106 difference between these two criteria has been discussed by Amestoy and
107 Leblond (1992). In presence of mode 3, the problem becomes threedimen-
108 sional and seldom papers deal with the prediction of the propagation path
109 in this condition. Among them, Lazarus and Leblond (1998a), Lazarus et al.
110 (2001b), Lazarus et al. (2008) consider the particular case of 3 or 4 point
111 bending experiments and Cooke and Pollard (1996), Lin et al. (2010), Pons
112 and Karma (2010) the segmentation of the front. Theoretical (Karma et al.,
113 2011) and experimental (Lazarus et al., 2011a) papers on segmentation and
114 coarsening are also currently under progress.

115 In some particular situations, even in presence of mode 2 or 3, planar
116 crack propagation may be stable. It is the case for instance when the crack
117 is channelled along a planar surface of low fracture resistance, which can be
118 the case for instance along a geological fault or in composite materials. It
119 may also be the case in fatigue due to the presence of friction (Doquet and
120 Bertolino, 2008).

121 *1.4. Crack perturbation approaches*

122 To predict the propagation path applying crack advance and propagation
123 criteria described above, perturbation methods have been used. They are
124 based on the pioneer works of Bueckner (1970), Rice (1972, 1985). Their
125 method has been extended in both the static and dynamic case.

126 Nevertheless, in this review we focus further on the static case⁴. A first
127 set of papers considers the out-of-plane perturbation of the faces of a planar
128 crack: the first order variation of the SIF are given in Movchan et al. (1998).
129 It is applied by Obrezanova et al. (2002b) to generalise the Cotterell and
130 Rice (1980)'s 2D stability analysis of a crack to small out-of-plane deviation
131 of its path. We are currently extending this approach to 3D out of plane
132 instabilities in mixed mode 1+3 (Leblond et al., 2011). A second set of papers
133 gives the expressions of the SIFs along the front of an arbitrary kinked and
134 curved infinitesimal extension of some arbitrary crack: Leblond (1989) and
135 Amestoy and Leblond (1992) in 2D, and Leblond (1999) and Leblond et al.

⁴In the dynamic fracture case some key references are Rice et al. (1994), Willis and Movchan (1995), Willis and Movchan (1997), Woolfries and Willis (1999), Obrezanova et al. (2002a).

136 (1999) in 3D. These expressions have been applied for instance, to show that
 137 the PLS and the MTS yield very close but distinct kink angles (Amestoy
 138 and Leblond, 1992) or to the crack front rotation and segmentation in mixed
 139 mode 1+3 or 1+2+3 (Lazarus and Leblond, 1998a; Lazarus et al., 2001a,b).
 140 A third set of papers consider the same problem then the second in the
 141 particular case of a planar crack with a coplanar extension. They are the
 142 main object of this review and are developed below.

143 2. In-plane crack front perturbation approaches for an arbitrary 144 planar crack

145 Consider a plane crack of arbitrary shape embedded in some isotropic
 146 static elastic medium subjected to some arbitrary loading: given forces \vec{T}^p
 147 along $\partial\Omega_T$ and given displacements \vec{u}^p along $\partial\Omega_u$ (Figure 1). The aim of this
 148 section is to give the first order variation of the stress intensity factors due to
 149 small in-plane perturbation of the crack front. Such formulae have first been
 150 derived in several particular cases notably by Rice and coworkers, and then
 151 generalised to more arbitrary problems. Here the historical chronology is not
 152 respected: first, in the present section, the most general formulae are recalled
 153 by relying on the paper of Favier et al. (2006a). They are then particularised
 154 to some model problems in section 3.

155 2.1. Definitions and elementary properties of weight functions

156 *Definitions.* Let $k_{ij}(\mathcal{F}; s'; s, r)$ denote the i -th SIF at the point s' of the crack
 157 front \mathcal{F} resulting from application of a pair of opposite unit point forces equal
 158 to $\pm\vec{e}_j(s)$ on the upper (+) and lower (−) crack surfaces at a distance r
 159 behind the point s of the crack front the other loading being supposed to be
 160 zero ($\vec{T}^p = \vec{0}$ along $\partial\Omega_T$ and $\vec{u}^p = \vec{0}$ along $\partial\Omega_u$). These nine scalar functions
 161 are called the *crack face weight functions* (CFWFs).

The functions $k_{ij}(\mathcal{F}; s'; s, r)/\sqrt{r}$ are known to have a well-defined limit
 for $r \rightarrow 0$ (see for instance Leblond et al. (1999)). We then define the matrix
 $(W_{ij}(s', s))_{1 \leq i \leq 3, 1 \leq j \leq 3} \equiv \mathbf{W}(s', s)$ by the formula

$$W_{ij}(s', s) \equiv \pi \sqrt{\frac{\pi}{2}} D^2(s, s') \lim_{r \rightarrow 0} \frac{k_{ij}(\mathcal{F}; s'; s, r)}{\sqrt{r}} \quad (11)$$

162 where $D(s, s')$ denotes the cartesian distance between points s and s' . The
 163 functions $W_{ij}(s', s)$ in fact depend on the crack front shape, just like the

164 CFWFs, but the argument \mathcal{F} is omitted here for conciseness. They will be
 165 called the *fundamental kernels* (FKs) or more shortly the *kernels*.
 166 Although the SIFs depends on the loading intensity and position, the
 167 CFWFs, hence the FKs depend on it only through the definitions of $\partial\Omega_T$
 168 and $\partial\Omega_u$.

Properties. The CFWFs are positively homogeneous of degree $-3/2$; that is, if all distances are multiplied by some positive factor λ , the CFWFs are multiplied by $\lambda^{-3/2}$. The definition (11) of the functions $W_{ij}(s', s)$ then implies that they are positively homogeneous of degree 0:

$$\mathbf{W}(\lambda s', \lambda s) = \mathbf{W}(s', s) \quad \forall \lambda > 0 \quad (12)$$

Since tensile and shear problems are uncoupled for a planar crack in an infinite body, whatever the shape of the crack front, the components k_{12} , k_{13} , k_{21} and k_{31} of the CFWFs are zero, so that by equation (11),

$$W_{12}(s', s) \equiv W_{13}(s', s) \equiv W_{21}(s', s) \equiv W_{31}(s', s) \equiv 0. \quad (13)$$

Considering two problems, one with point forces equal to $\pm\vec{e}_i$ exerted on the crack faces at (s, r) and one with point forces equal to $\pm\vec{e}_j$ exerted on the crack faces at (s', r') , applying Betti's theorem, and using equations (1) and (11), one sees that the kernels obey the following ‘‘symmetry’’ property :

$$\Lambda_{im} W_{mj}(s, s') = \Lambda_{jm} W_{mi}(s', s). \quad (14)$$

Finally, Leblond et al. (1999) have shown that the limit of $\mathbf{W}(s, s')$ when $s' \rightarrow s$ is universal, i.e. that it does not depend on the geometry. It is linked to the behaviour of the weight-functions when the point of application of the loading tends toward the point of observation of the SIF which is a local property independent of the far geometry. The values of this limit are:

$$\left\{ \begin{array}{l} \lim_{s' \rightarrow s} W_{11}(s, s') = 1 \\ \lim_{s' \rightarrow s} W_{22}(s, s') = \frac{2 - 3\nu}{2 + \nu} \\ \lim_{s' \rightarrow s} W_{33}(s, s') = \frac{2 + \nu}{2 - \nu} \\ \lim_{s' \rightarrow s} W_{23}(s, s') = 0 \end{array} \right. \quad (15)$$

169 *2.2. First order variation of the stress intensity factors*

Let us now assume that the crack advances, under constant loading, by a small distance $\delta(s)$ within its plane in the direction perpendicular to its front (fig. 1). It has been shown in Favier et al. (2006a) that, for any loading, if $\delta(s_0) = 0$,

$$\delta\mathbf{K}(s_0) = \mathbf{N} \cdot \mathbf{K}(s_0)\delta'(s_0) + \frac{1}{2\pi} \text{PV} \int_{\mathcal{F}} \frac{\mathbf{W}(s_0, s)}{D^2(s_0, s)} \cdot \mathbf{K}(s)\delta(s)ds. \quad (16)$$

The condition $\delta(s_0) = 0$ ensures the existence of the Principal Value integral $\text{PV} \int$. The quantities $\mathbf{K}(s) \equiv (K_i(s))_{1 \leq i \leq 3}$ and $\delta\mathbf{K}(s) \equiv (\delta K_i(s))_{1 \leq i \leq 3}$ here are the column vectors of initial SIFs and variations of these SIFs, and $\mathbf{N} \equiv (N_{ij})_{1 \leq i \leq 3, 1 \leq j \leq 3}$ is the matrix defined by

$$\mathbf{N} \equiv \frac{2}{2 - \nu} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 - \nu & 0 \end{pmatrix}. \quad (17)$$

170 Equation (16) is identical to Leblond et al. (1999)'s general equation (30)
 171 (with the notation $\frac{1}{2\pi} \frac{\mathbf{W}(s_0, s)}{D^2(s_0, s)}$ instead of $\mathbf{Z}(s_0, s)$), in the special case of
 172 a planar crack with coplanar extension (and zero crack advance at the point
 173 s_0). It shall be noticed that the variation of the SIFs at a particular point
 174 s_0 depends in a non-local manner on the crack perturbation along all the
 175 front. It is due to long-range elastic interactions.

176 The restriction $\delta(s_0) = 0$ will now be removed by two methods:

177 1. Using a trick of Rice (1989), that consists of decomposing an arbitrary
 178 motion of the crack front defined by the normal advance $\delta(s)$ into two
 179 steps :

(a) A translatory motion of displacement vector $\delta(s_0)\vec{e}_2(s_0)$. This motion brings the point s_0 to its correct final position while leaving the crack front shape unchanged. The corresponding normal advance $\delta_*(s)$ is given, to first order in $\delta(s)$, by

$$\delta_*(s) = \delta(s_0) \vec{e}_2(s_0) \cdot \vec{e}_2(s). \quad (18)$$

180 The associated variation of $\mathbf{K}(s)$ will be denoted $\delta_*\mathbf{K}(s)$.

181 (b) A motion with normal advance given by $\delta(s) - \delta_*(s)$. This advance
 182 is zero at point s_0 so that the corresponding variation of $\mathbf{K}(s_0)$ is
 183 given by equation (16), with $\delta'(s_0) - \delta'_*(s_0) = \delta'(s_0)$ since $\delta'_*(s_0) =$
 184 0 by equation (18).

Adding up the contributions from these two motions, one gets the final expression of the variation of the SIFs under constant loading in the general case:

$$\begin{aligned} \delta\mathbf{K}(s_0) &= \delta_*\mathbf{K}(s_0) + \mathbf{N} \cdot \mathbf{K}(\mathbf{s}_0)\delta'(s_0) \\ &+ \frac{1}{2\pi} \text{PV} \int_{\mathcal{F}} \frac{\mathbf{W}(s_0, s)}{D^2(s_0, s)} \cdot \mathbf{K}(s) [\delta(s) - \delta_*(s)] ds. \end{aligned} \quad (19)$$

This expression allows to update the SIFs knowing the initial SIFs, FK and the displacement provided that the quantity $\delta_*\mathbf{K}$ can be calculated. The unknown quantity $\delta_*\mathbf{K}(s_0)$ is equal to zero if the translatory motion $\delta(s_0)\vec{e}_2(s_0)$ leaves the problem unchanged. It is for instance the case if the crack front is far from any boundary so that the media can be assumed to be infinite submitted to remote stress loading. Then, the first order formula simply becomes:

$$\begin{aligned} \delta\mathbf{K}(s_0) &= \mathbf{N} \cdot \mathbf{K}(\mathbf{s}_0)\delta'(s_0) \\ &+ \frac{1}{2\pi} \text{PV} \int_{\mathcal{F}} \frac{\mathbf{W}(s_0, s)}{D^2(s_0, s)} \cdot \mathbf{K}(s) [\delta(s) - \delta(s_0) \vec{e}_2(s_0) \cdot \vec{e}_2(s)] ds. \end{aligned} \quad (20)$$

2. Another possibility is to proceed as Leblond et al. (1999) and to decompose the normal advance $\delta(s)$ into a uniform advance $\delta(s_0)$ (denote $[\delta\mathbf{K}(s_0)]_{\delta(s)\equiv\delta(s_0)}$ the corresponding first order variation of the SIFs) and the advance $\delta(s) - \delta(s_0)$ for which the equation (16) can be used. The final expression then reads:

$$\begin{aligned} \delta\mathbf{K}(s_0) &= [\delta\mathbf{K}(s_0)]_{\delta(s)\equiv\delta(s_0)} + \mathbf{N} \cdot \mathbf{K}(\mathbf{s}_0)\delta'(s_0) \\ &+ \frac{1}{2\pi} \text{PV} \int_{\mathcal{F}} \frac{\mathbf{W}(s_0, s)}{D^2(s_0, s)} \cdot \mathbf{K}(s) [\delta(s) - \delta(s_0)] ds. \end{aligned} \quad (21)$$

185 This expression is useful if one can calculate $[\delta\mathbf{K}(s_0)]_{\delta(s)\equiv\delta(s_0)}$. It is the case
 186 for instance if the uniform advance $\delta(s) \equiv \delta(s_0)$ doesn't change the geometry
 187 of the problem as for a circular, straight half-plane or tunnel crack.

188

189 Formula (16) and its corollaries (19), (20), (21) have been derived for
 190 homogeneous isotropic elastic solids. For cracks at the interface between two
 191 elastic solids, such a formula exists in the sole case of a half-plane crack:
 192 the first order variation of the SIFs can be found in Lazarus and Leblond
 193 (1998b) and using an other formalism (Wiener-Hopf analysis) in Bercial-Velez
 194 et al. (2005), the connection between the two methods having been done by
 195 Piccolroaz et al. (2007).

196 *2.3. First order variation of the fundamental kernel*

To derive higher order variation of the SIFs, the first order variation of the fundamental kernel is necessary. It has been shown by Rice (1989) in mode 1 and Favier et al. (2006a) in modes 2+3 that:

$$\begin{aligned} \delta \mathbf{W}(s_0, s_1) &= \mathbf{N} \cdot \mathbf{W}(s_0, s_1) \delta'(s_0) - \mathbf{W}(s_0, s_1) \cdot \mathbf{N} \delta'(s_1) \\ &+ \frac{D^2(s_0, s_1)}{2\pi} \text{PV} \int_{\mathcal{F}} \frac{\mathbf{W}(s_0, s) \cdot \mathbf{W}(s, s_1)}{D^2(s_0, s) D^2(s_1, s)} \delta(s) ds, \end{aligned} \quad (22)$$

if $\delta(s_0) = \delta(s_1) = 0$. In order to get rid of these conditions, one must imagine a motion $\delta_{**}(s)$ such as $\delta_{**}(s_0) = \delta(s_0)$ and $\delta_{**}(s_1) = \delta(s_1)$. Denote $\delta_{**} \mathbf{W}(s_0, s_1)$ the corresponding variation of the kernel. Equation (22) then becomes:

$$\begin{aligned} \delta \mathbf{W}(s_0, s_1) &= \delta_{**} \mathbf{W}(s_0, s_1) + \mathbf{N} \cdot \mathbf{W}(s_0, s_1) [\delta'(s_0) - \delta'_{**}(s_0)] \\ &- \mathbf{W}(s_0, s_1) \cdot \mathbf{N} [\delta'(s_1) - \delta'_{**}(s_1)] \\ &+ \frac{D^2(s_0, s_1)}{2\pi} \text{PV} \int_{\mathcal{F}} \frac{\mathbf{W}(s_0, s) \cdot \mathbf{W}(s, s_1)}{D^2(s_0, s) D^2(s_1, s)} [\delta(s) - \delta_{**}(s)] ds. \end{aligned} \quad (23)$$

A difficulty is to be able to define $\delta_{**}(s)$ such as $\delta_{**} \mathbf{W}(s_0, s_1)$ can be calculated. This problem has not been solved at present in the general case. In the particular case of an infinite body subjected to uniform remote loading, one can always find a combination of a translatory motion, a rotation and a homothetical transformation bringing two distinct points s_0, s_1 from any initial positions to any final positions (This is obvious using a complex variable formalism and noting that such transformations are of the form $f(z) = az + b$ where a and b are arbitrary complex parameters). Such a combination leaves

the kernels unaffected so that $\delta_{**}W(s_0, s_1) = 0$. Equation (23) then yields:

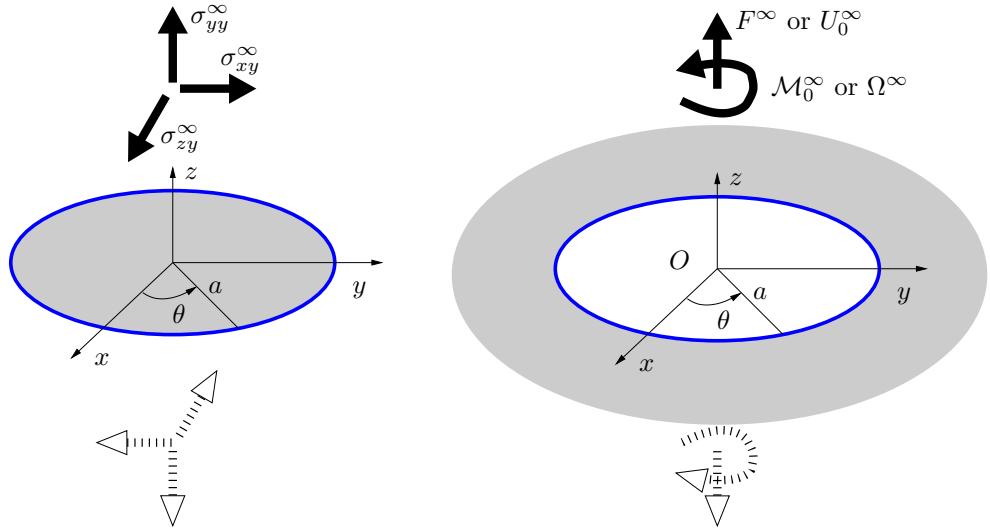
$$\begin{aligned} \delta \mathbf{W}(s_0, s_1) = & \mathbf{N} \cdot \mathbf{W}(\mathbf{s}_0, \mathbf{s}_1) [\delta'(s_0) - \delta'_{**}(s_0)] \\ & - \mathbf{W}(s_0, s_1) \cdot \mathbf{N} [\delta'(s_1) - \delta'_{**}(s_1)] \\ & + \frac{D^2(s_0, s_1)}{2\pi} \text{PV} \int_{\mathcal{F}} \frac{\mathbf{W}(s_0, s) \cdot \mathbf{W}(s, s_1)}{D^2(s_0, s)D^2(s_1, s)} [\delta(s) - \delta_{**}(s)] ds. \end{aligned} \quad (24)$$

197 Note that quantities $\delta'_{**}(s_0)$ and $\delta'_{**}(s_1)$ here are nonzero, unlike quantity
198 $\delta'_*(s_0)$ in equation (19).

199 2.4. Some expressions of the fundamental kernel \mathbf{W}

200 To initiate the perturbation approach, the FKs must be known for the
201 unperturbed configurations. It is the case for some seldom geometries that
202 are depicted in figures 2, 3, 4. In those figures, the crack front are coloured
203 in blue and the faces in grey.

204 2.4.1. Circular cracks



(a) An *internal circular crack* in an infinite media loaded by remote tensile and shear stresses

(b) An *external circular crack* in an infinite media loaded by remote (i) tensile net centred force or displacement and (ii) momentum or rotation

Figure 2: Several problems involving a circular crack.

205 Two cases are considered, an internal and an external crack:

- For the internal circular crack such as $\partial\Omega_u = \emptyset$, for instance loaded by remote stresses (fig. 2(a)), the value of the non-zero components of the kernel \mathbf{W} are (Kassir and Sih, 1975; Tada et al., 1973; Bueckner, 1987; Gao and Rice, 1987b; Gao, 1988):

$$\left\{ \begin{array}{l} W_{11}(\theta_0, \theta_1) = 1 \\ W_{22}(\theta_0, \theta_1) = \frac{2 \cos(\theta_0 - \theta_1) - 3\nu}{2 - \nu} \\ W_{33}(\theta_0, \theta_1) = \frac{2(1 - \nu) \cos(\theta_0 - \theta_1) + 3\nu}{2 - \nu} \\ W_{23}(\theta_0, \theta_1) = \frac{1}{1 - \nu} W_{32}(\theta_1, \theta_0) = \frac{2 \sin(\theta_0 - \theta_1)}{2 - \nu}. \end{array} \right. \quad (25)$$

- 206 • For the external circular crack (fig. 2(b)), only the value in mode 1 is
 207 known (Stallybrass, 1981; Gao and Rice, 1987a; Rice, 1989) for several
 208 cases of remote boundary conditions:

- when remote points are clamped (given $U_0^\infty = 0, \Omega^\infty = 0$):

$$W_{11}(\theta_0, \theta_1) = 1 \quad (26)$$

- when remote points can not rotate but can move in the \vec{e}_1 direction (given $F^\infty = 0, \Omega^\infty = 0$):

$$W_{11}(\theta_0, \theta_1) = 1 + 4 \sin^2 \left(\frac{\theta_0 - \theta_1}{2} \right) \quad (27)$$

- when remote points can not move in the \vec{e}_1 direction, but can rotate (given $U_0^\infty = 0, \mathcal{M}_0^\infty = 0$):

$$W_{11}(\theta_0, \theta_1) = 1 + 24 \sin^2 \left(\frac{\theta_0 - \theta_1}{2} \right) \cos(\theta_0 - \theta_1) \quad (28)$$

- when remote points are constrained against any motion (given $F^\infty = 0, \mathcal{M}_0^\infty = 0$):

$$W_{11}(\theta_0, \theta_1) = 1 + 4 \sin^2 \left(\frac{\theta_0 - \theta_1}{2} \right) [1 + 6 \cos(\theta_0 - \theta_1)] \quad (29)$$

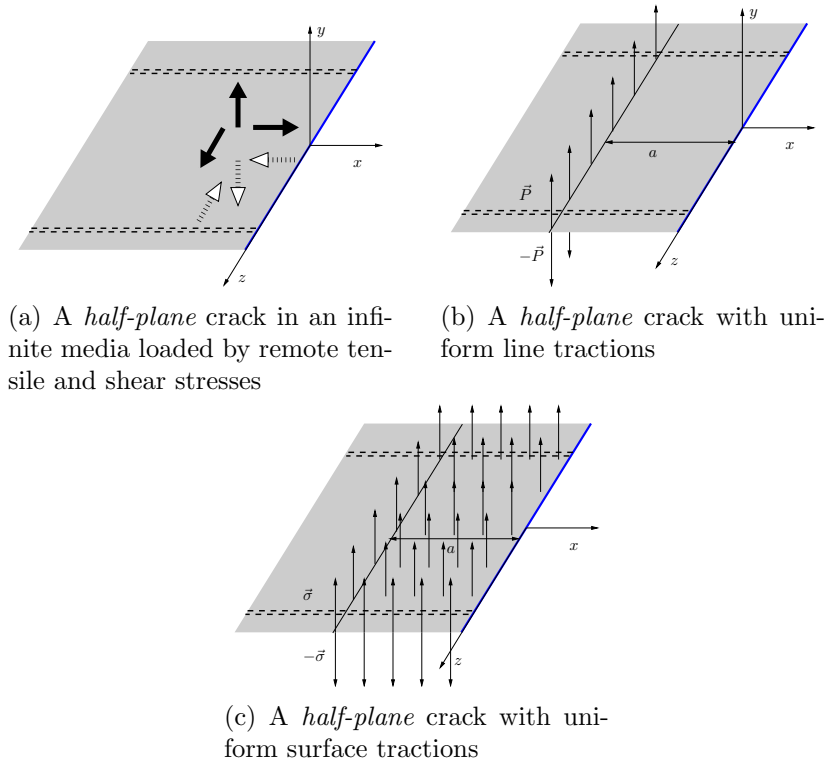


Figure 3: Several problems involving a half-plane crack

209 *2.4.2. Half-plane crack*

In the case of a half-plane crack with $\partial\Omega_u = \emptyset$, loaded for instance by remote stresses (fig. 3(a)) or line (fig. 3(b)) or surface traction (fig. 3(c)), the kernel is (Meade and Keer, 1984; Bueckner, 1987; Rice, 1985; Gao and Rice, 1986):

$$\begin{cases} W_{11}(z_1, z_0) = 1 \\ W_{22}(z_1, z_0) = \frac{2 - 3\nu}{2 - \nu} \\ W_{33}(z_1, z_0) = \frac{2 + \nu}{2 - \nu} \\ W_{23}(z_1, z_0) = 0 \end{cases} \quad (30)$$

210 *2.4.3. Tunnel-cracks*

211 The model of half-plane crack is widely used due to its simplicity, but it
 212 lacks a lengthscale. To introduce a lengthscale, Leblond and coauthors have

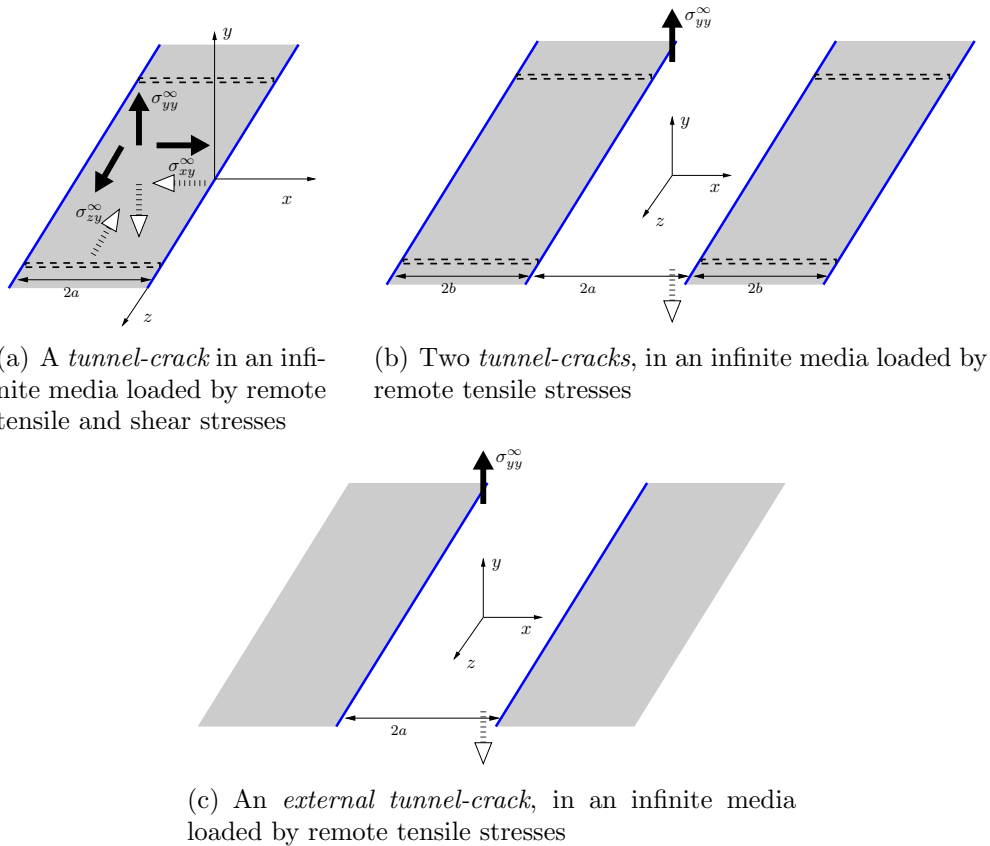


Figure 4: Several problems involving a tunnel-crack.

213 studied several cases involving a tunnel-crack (fig. 4) with $\partial\Omega_u = \emptyset$: Leblond
 214 et al. (1996) for the tensile tunnel-crack, Lazarus and Leblond (2002c) for
 215 the shear tunnel-crack (fig. 4(a)), Pindra et al. (2010b) for two coplanar
 216 tensile tunnel-cracks (fig. 4(b)), Legrand and Leblond (2010b) for an external
 217 tunnel-crack (fig. 4(c))⁵.

⁵External cracks give rise to traditional ambiguities on the external load, since they cannot withstand uniform tractions exerted at infinity. Here the situation considered unambiguously consists of two tunnel-cracks (fig. 4(b)) in the limiting case where $b \gg a$.

218 **3. Particular model case of tensile straight crack fronts**

219 The aim here is to introduce formulas that are useful further on for the
220 study of some crack propagation problems involving an initially straight crack
221 front (for sections 4.1 and 6 in particular).

222 *3.1. Unperturbed geometries and loading*

223 For simplicity, only mode 1 is considered and K_1 is re-noted K . To
224 study the propagation of a straight crack front, the most natural and simple
225 model is the half-plane crack loaded by remote stresses (fig. 3(a)). Even if a
226 certain number of results can be obtained with this model, it lacks crucially a
227 lengthscale. To fill this gap, this simple model has progressively be enriched.
228 Here, the following models are considered more specifically:

- 229 1. a half-plane crack loaded by remote stresses (fig. 3(a)), then $K(s) = K$
230 is a constant due to the lack of any lengthscale in this problem;
- 231 2. a half-plane crack with uniform line tractions P at a distance a of the
232 front (fig. 3(b)), then $K(z) = \sqrt{\frac{2}{\pi}}Pa^{-1/2}$;
- 233 3. a half-plane crack with uniform surface tractions σ in a band of width
234 a (fig. 3(c)), then $K(z) = 2\sqrt{\frac{2}{\pi}}\sigma a^{1/2}$;
- 235 4. a tunnel-crack loaded by remote stresses (fig. 4(a)) then $K(z) =$
236 $\sigma\sqrt{\pi}a^{1/2}$;

All these problems can be included in the more general framework for
which the initial SIF can be written under the form:

$$K(a) = ka^\alpha \tag{31}$$

237 where k depends on the loading level but is independent of a . The value of
238 α are 0 in the case 1, $-1/2$ in case 2, $1/2$ in cases 3 and 4. The sign of α
239 is of uppermost importance in the sequel. If $\alpha < 0$ the propagation is stable
240 under constant loading and if $\alpha > 0$ unstable⁶.

⁶This terminology makes an implicit reference to Irwin's propagation law (7) for which crack propagation occurs when the SIF reaches some critical value. For such a law and under constant loading, after the onset of crack propagation, the velocity of the crack goes immediately down to zero if the SIF decreases with distance, but continuously increases in the opposite case; hence the expressions "stable propagation", "unstable propagation".

241 3.2. Fourier transform of the first order variation of the SIF

Define the Fourier Transform $\widehat{\phi}(k)$ of some arbitrary function $\phi(z)$ by

$$\widehat{\phi}(k) \equiv \int_{-\infty}^{+\infty} \phi(z)e^{ikz} dz \quad \Leftrightarrow \quad \phi(z) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{\phi}(k)e^{-ikz} dk. \quad (32)$$

Using this definition and equation (21) applied to the geometries listed in section 3.1, Fourier components $\delta\widehat{K}(k)$ of the first order variation of the mode 1 SIF $\delta K(z)$ can be written under the following form:

$$\frac{\delta\widehat{K}(k, a)}{K(a)} = \left[\frac{\frac{dK(a)}{da}}{K(a)} - a^{-1}F(p) \right] \widehat{\delta}(k) = (\alpha - F(p)) \frac{\widehat{\delta}(k)}{a} \quad (33)$$

242 Here, k is the wavenumber and $p = ka$ the dimensionless one. In the case
 243 of the tunnel-crack geometry, we have supposed that the perturbations are
 244 the same for all the fronts for simplicity, so that if we denote $\delta_n(z)$ the
 245 perturbation of \mathcal{F}_n , it exists a function $\delta(z)$ such as $\delta_n(z) = \delta(z)$ whatever
 246 $n = 1, N$.

$F(p)$ can be derived from the expressions of the fundamental kernels listed in section 2.4. For instance, for the half-plane crack it reads:

$$F(p) = \frac{p}{2} \quad (34)$$

247 and for all the geometries of §3.1, it can be verified that (i) $F(0) = 0$ and (ii)
 248 $F(p)$ increases monotonically to finally behaves as $p/2$ for $p \rightarrow \infty$. This last
 249 behaviour is closely linked to the universal behaviour of $W_{11}(s, s')$ for $s' \rightarrow s$
 250 (eq. 15).

251 The general formulas for the tunnel-crack, without the symmetry hypoth-
 252 esis $\delta_n = \delta$ of the crack advance, can be found in Favier et al. (2006b). A
 253 formula similar to (33) can be find in Gao and Rice (1987b) (resp. Gao
 254 and Rice (1987a)) for an internal (resp. external) circular crack, in Pindra
 255 et al. (2010b) for two tunnel-cracks and in Legrand and Leblond (2010a) for
 256 an external tunnel-crack. For shear loading, see Gao and Rice (1986) for
 257 the half-plane crack, Pindra et al. (2008) for the interfacial half-plane crack,
 258 Pindra et al. (2010a) for the tunnel-crack.

259 **4. Crack propagation in an homogeneous media**

260 The aim of this section is to study the crack front shape changes arising
 261 from the quasistatic propagation in an homogeneous media. First the
 262 problem of crack shape bifurcation and stability (section 4.1) is studied ana-
 263 lytically by linear approaches, then large scale deformations (section 4.2) are
 264 presented using incremental non linear numerical simulations.

265 *4.1. Crack front shape linear bifurcation and stability analysis*

266 First order perturbation approaches are extensively used in linear bifur-
 267 cation and stability analysis in various problems (Drazin, 1992; Bazant and
 268 Cedolin, 2003; Nguyen, 2000). Here the problem of configurational bifurca-
 269 tion and stability of a straight crack front is considered.

270 *4.1.1. Bifurcation*

271 Consider one of the model problems of section 3 and suppose that $K(z) =$
 272 K_c for all z . The configurational bifurcation problem aims at answering
 273 the following unicity question: can one find any configuration satisfying the
 274 condition that the SIF be equal to a constant along the crack front, other
 275 than the initial straight one?

The linear bifurcation problem amounts to search for a crack front per-
 turbation $\delta(s) \neq 0$ such as the first order variation $\delta K(s)$ of the SIF is zero.
 By equation (33), this reads:

$$[\alpha - F(p)]\widehat{\delta}(k) = 0, \quad (35)$$

so that non-zero solution exists if $[\alpha - F(p)] = 0 \forall p$. Since $F(p) \geq 0$, it
 exists only if $\alpha \geq 0$ that is if the propagation is unstable under constant
 loading. The bifurcation corresponds to a sinusoidal perturbation of critical
 wavelength λ_c solution of (α has been introduced in 31):

$$\lambda_c = \lambda_c^* a, \text{ where } \lambda_c^* = \frac{2\pi}{F^{-1}(\alpha)} \quad (36)$$

For the half-plane crack, it corresponds to Rice (1985)'s result:

$$\lambda_c = \frac{\pi K(a)}{\frac{dK(a)}{da}}, \quad (37)$$

276 which gives $\lambda_c = 2\pi a \sim 6.283 a$ in the case of surface tractions (fig. 3(c)). For
 277 the single tunnel-crack under remote loading, $\lambda_c = 6.793 a$ (Leblond et al.,
 278 1996) and two interacting tunnel cracks $\lambda_c = 18.426 a$ when $a \ll (b + a)$ (fig.
 279 4(b)).

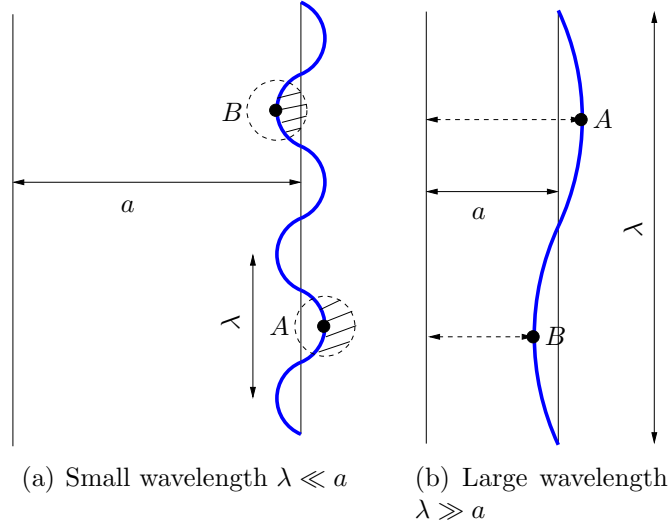


Figure 5: Sinusoidal perturbation of the crack front

280 The existence of a bifurcation if $\frac{dK(a)}{da} > 0$ and the nonexistence if $\frac{dK(a)}{da} <$
 281 0 has still been noticed by Nguyen (2000) in the case of thin films. It may
 282 be rationalised as follows:

- 283 1. Consider first a perturbation of the crack fronts of small wavelength,
 284 $\lambda \ll a$ (Figure 5(a)). The crack advance is maximum at point A and
 285 minimum at point B . Draw small circles centred at these points. That
 286 part of the interior of the circle occupied by the unbroken ligament
 287 (hatched in Figure 5(a)) is larger at point A than at point B , so that
 288 the opening of the crack is more hindered near the first point than near
 289 the second. Thus the stress intensity factors $K(A)$, $K(B)$ at points A
 290 and B obey the inequality $K(A) < K(B)$.
- 291 2. Consider now a perturbation of large wavelength, $\lambda \gg a$, and again
 292 points A and B where the crack advance is respectively maximum and
 293 minimum (Figure 5(b)). The stress intensity factors at points A and
 294 B are almost the same as for ligaments of uniform width equal to

295 the local width at these points (indicated by dashed double arrows in
 296 Figure 5(b)). It follows that $K(A) > K(B)$ if $\frac{dK(a)}{da} > 0$ and that
 297 $K(A) < K(B)$ if $\frac{dK(a)}{da} < 0$.

298 This implies:

- 299 • In the case $\frac{dK(a)}{da} > 0$, the difference $K(A) - K(B)$ is negative for small
 300 λ and positive for large λ , and obviously varies continuously with this
 301 parameter. Hence some special value λ_c such that $K(A) - K(B) = 0$
 302 must necessarily exist.
- 303 • In the case $\frac{dK(a)}{da} < 0$, the difference $K(A) - K(B)$ is negative for all
 304 λ , so that no bifurcation is possible.

305 In the case of shear loading, the results are more complex and can be
 306 found in Gao and Rice (1987b) for the internal circular crack, in Gao and
 307 Rice (1986) for the half-plane crack and in Lazarus and Leblond (2002b) for
 308 the tunnel-crack. For thin films, crack front bifurcation has also be stud-
 309 ied by Nguyen (2000); Adda-Bedia and Mahadevan (2006) and observed in
 310 experiments (Ghatak and Chaudhury, 2003).

311 4.1.2. Stability

312 The question here is as follows: if the crack front is slightly perturbed
 313 within the crack plane, will the perturbation increase (instability) or decay
 314 (stability) in time? Equivalently, will the crack front depart more and more
 315 from its initial configuration or tend to keep it? We restrict our attention
 316 here to the cases listed in section 3 for which the SIF $\mathbf{K}(z)$ in the initial
 317 configuration are uniform independent of z .

318
 319 When the extrema of $\delta(z)$ and $\delta G(z)$ coincide, stability or instability pre-
 320 vails according to whether the maxima of $\delta G(z)$ correspond to the minima or
 321 maxima of $\delta(z)$ (Rice, 1985; Gao and Rice, 1986, 1987b; Gao, 1988; Leblond
 322 et al., 1996; Lazarus and Leblond, 1998b; Legrand and Leblond, 2010a).
 323 Hence the answer to the question can simply be derived from the above bi-
 324 furcation discussion: sinusoidal perturbations are stable if $\alpha - F(p) < 0$ that
 325 is for wavelength smaller than the bifurcation wavelength λ_c (eq. 36) and
 326 unstable for $\lambda > \lambda_c$. In the case of non-existence of a bifurcation (stable
 327 propagation $\alpha < 0$), stability is thus achieved whatever the wavelength. In
 328 the case $\alpha > 0$, the critical wavelength is proportional to a , thus continuously

329 increases during propagation, stability ultimately prevails for all wavelengths.

330

331 But when the extrema of $\delta(z)$ and $\delta G(z)$ do not coincide, as is for in-
 332 stance the case of the tunnel-crack under shear loading (Lazarus and Leblond,
 333 2002b), it appears quite desirable to then discuss the stability issue in full gen-
 334 erality, without enforcing such a coincidence (Lazarus and Leblond, 2002a).
 335 It is then necessary to introduce a time dependent advance law.

Let us use here the Paris law (10). Its leading term reads:

$$\frac{da(t)}{dt} = CK^N \quad (38)$$

where $a(t)$ is the mean position of the crack front at instant t . Considering henceforward all perturbations as functions of the mean position a of the crack instead of time t , one gets the first order advance equation:

$$\frac{\partial \delta(k, a)}{\partial a} = N \frac{\delta K(k, a)}{K}, \quad (39)$$

which yields (Favier et al., 2006b) after use of FT (33) and integration (a_0 denotes the initial value of a):

$$\frac{\widehat{\delta}(k, a)}{\widehat{\delta}(k, a_0)} = \exp \left[N \int_{ka_0}^{ka} (\alpha - F(p)) \frac{dp}{p} \right] \quad (40)$$

or using the property $F(0) = 0$:

$$\frac{\widehat{\delta}(k, a)}{\widehat{\delta}(k, a_0)} = \left(\frac{a}{a_0} \right)^{N\alpha} \left(\frac{\psi(ka)}{\psi(ka_0)} \right)^{N\alpha}, \quad (41)$$

where $\psi(p)$ is defined by:

$$\psi(p) = \exp \left[- \int_0^p \frac{F(q)}{q} dq \right] \quad (42)$$

336 For the half-plane crack, its value is $\psi(p) = \exp(-\frac{p}{2})$, for the tunnel-crack it
 337 can be found in Favier et al. (2006b). For the sequel, it is useful to note that
 338 whatever the geometry (half-plane or tunnel), this function $\psi(p)$ decreases
 339 from 1 to 0 when p varies from 0 to $+\infty$.

340 From equation (40), it is clear that:

- 341 • If $\frac{dK(a)}{da} < 0$, then $\alpha - F(p) < 0$ so that all Fourier components of any
 342 wavelength decrease with crack growth a .
- 343 • If $\frac{dK(a)}{da} > 0$, for any given k , $\left|\widehat{\delta}(k, a)\right|^2$ increases as long as a remains
 344 smaller than $\frac{2\pi}{k\lambda_c^*}$ and decreases afterwards.

For $t \rightarrow \infty$, one can show that:

$$\begin{cases} \text{For } k = 0, & \left| \frac{\widehat{\delta}(0, a)}{\widehat{\delta}(k, a_0)} \right|^2 = \left(\frac{a}{a_0} \right)^{2N\alpha} \\ \text{For } k \neq 0, & \left| \frac{\widehat{\delta}(k, a)}{\widehat{\delta}(k, a_0)} \right|^2 \sim \left(\frac{a}{a_0} \right)^{2N\alpha} \exp(-N |k| a) \rightarrow 0 \end{cases} \quad (43)$$

345 so that:

- 346 • If $\frac{dK(a)}{da} < 0$, any initial perturbation disappears.
- 347 • If $\frac{dK(a)}{da} > 0$, the moduli of all Fourier components ultimately decay,
 348 except for $k = 0$ which continuously increases. This phenomenon is
 349 due to the fact that for all values of λ except $+\infty$, λ always ultimately
 350 becomes smaller than $\lambda_c(a)$ since the former wavelength is fixed whereas
 351 the latter increases in proportion with a .

352 Thus, one can conclude that whatever the small perturbation of crack
 353 front, the initial configuration is finally retrieved⁷. In the case of stable crack
 354 propagation $\frac{dK(a)}{da} < 0$, the stability prevails at all lengthscales from the
 355 beginning. In the case of unstable crack propagation $\frac{dK(a)}{da} > 0$, instability
 356 first prevails for all lengthscales such as $\lambda > \lambda_c$, but since λ_c is a growing
 357 function of the crack advance a , all wavelengths finally becomes stable so
 358 that the perturbation finally disappears. This is true for all the problems
 359 listed in section 3 provided that the first order study stays valid. To extend
 360 them to large perturbations, higher order terms must be taken into account.
 361 This is the subject of next section.

⁷the wavelength $k = 0$ corresponds indeed to a infinite wavelength that is an almost straight crack front.

362 *4.2. Largescale propagation simulations*

363 In the previous sections, the perturbation approach was applied to small
 364 perturbations of the crack front. Following an original idea of Rice (1989),
 365 Bower and Ortiz (1990) first extended the method to the study of arbitrary
 366 large propagation of a tensile crack leading the way to the numerical res-
 367 olution of some complex three dimensional crack problems. It consists in
 368 applying numerically the perturbation approach described in section 2, to a
 369 succession of small perturbations arising in arbitrary large ones. The media
 370 is assumed to be infinite loaded by remote stresses so that the SIF can be
 371 updated using formula (20) and the FKs using formula (24). The crack front
 372 shape at each instant is obtained by the inversion of heavily implicit systems
 373 of equations resulting from the direct application of Irwin’s criterion (7).
 374 The method was then extended and simplified by Lazarus (2003); notably
 375 a unified Paris’ type law (10) formulation for fatigue and brittle fracture
 376 ($N \rightarrow +\infty$) is proposed that gives the advance of the crack front in explicit
 377 form once the SIF is known. Extension to shear loading is performed in
 378 Favier et al. (2006a).

379
 Concerning propagation in an homogeneous media, Lazarus (2003) stud-
 ied the asymptotic behaviour of the SIF near an angular point of the front
 and retrieved the theoretical results of Leblond and Leguillon (1999) about
 the SIF singularity around a corner point of the front, the fatigue and brittle
 propagation paths of some special crack shapes (elliptical, rectangular, heart
 shaped ones) (fig. 6) loaded by remote tensile stresses. It appears that in
 all the cases studied, the crack becomes and stays circular after a certain
 time emphasizing that among all the configurations studied only the circular
 crack shape is stationnary. In the case of shear loading (Favier et al., 2006a),
 it appears (fig. 7) that the stationnary shape is nearly elliptic, the ratio of
 the axes being well approximated by:

$$\frac{a}{b} = (1 - \nu)^{\frac{\beta}{\beta+1}}, \quad (44)$$

380 β is the Paris law exponent in mixed mode loading (9), b corresponds to
 381 the axis in the direction of the shear loading. Whether all embedded plane
 382 cracks tend toward a configuration with uniform value of $\mathcal{G}(s)$ is a general
 383 result, has however, to my best knowledge, not been demonstrated, even if
 384 one guess that energy minimisation is the physical ground.

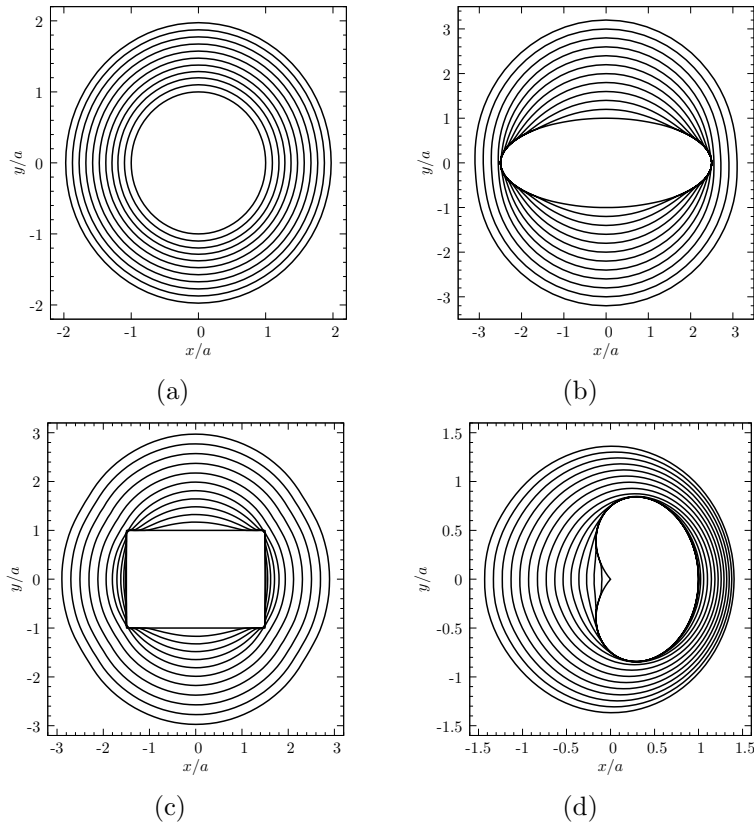


Figure 6: Successive crack fronts of pure tensile mode cracks in brittle fracture (eq. 10 with $N = 50$). Similar figures for fatigue can be found in Lazarus (2003) and show the same circular stationary crack shape.

385 5. Crack trapping by tougher obstacles

386 In previous section, all the material constants (elasticity coefficients, frac-
 387 ture toughness) were supposed to be homogeneous throughout the media. In
 388 the sequel, we are still considering the problem of a crack propagating in
 389 the slow velocity regime as in section 4 but with the toughness becoming
 390 heterogeneous. The elasticity coefficients are supposed to remain constant
 391 so that the perturbation approach of section 2 remains valid. If the tough-
 392 ness is heterogeneous, the crack advance changes from point to point and the
 393 crack front shape changes during propagation even if the SIFs were initially
 394 uniform. In this section, the propagation of the front through well defined

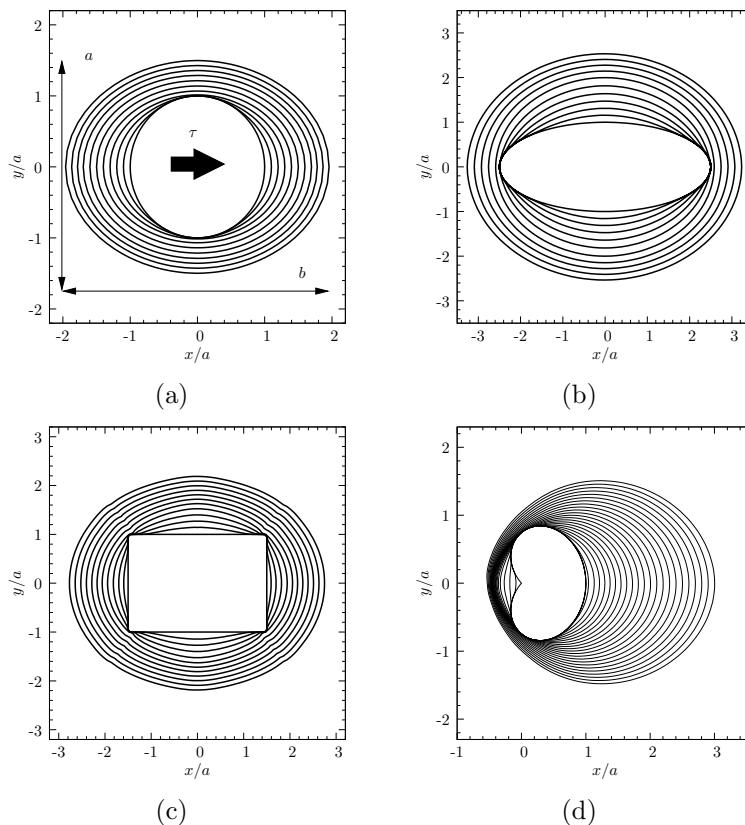


Figure 7: Successive crack fronts of pure shear mode cracks in brittle fracture (eq. 9 with $\beta = 25$, $\mathcal{G}_0 = 0$). Shear is along the x -axis. Similar figures for fatigue can be found in Favier et al. (2006a) and show also a quasielliptic stationary crack shape.

395 tougher obstacles is studied. In section 6, the toughness is supposed to be
 396 disordered so that a statistical approach is necessary.

397 Tougher inclusions may prevent or hinder the final breaking of a solid by
 398 two mechanisms: crack bridging and crack front trapping. The mechanism
 399 of toughening referred to as bridging occurs when unbroken inclusions lag
 400 behind a main crack front hindering its opening by pinning or friction; and
 401 as trapping when the crack front is deformed when it penetrates into the
 402 tougher zone or bows out it (Lange, 1970).

Whereas some aspects of crack bridging can be studied by 2D elasticity
 problems (Budiansky et al., 1988), crack trapping induces crack front shape
 deformations that makes the elasticity problem fully 3D. To understand the

mechanism, let us consider a tunnel-crack loaded by remote tensile loading σ (fig. 4(a)). The SIF for the straight-crack front then reads:

$$K = \sigma\sqrt{\pi a} \quad (45)$$

403 In the absence of obstacles, the propagation is unstable under constant load-
 404 ing so that the breakdown of the solid occurs as soon as the threshold is
 405 reached, unless the loading is decreased to ensure that $K(s) \leq K_c$ at each
 406 instant: $\sigma = K_c/\sqrt{\pi a}$. In the presence of tougher ($K_c^p > K_c^m$, K_c^p , K_c^m
 407 being resp. the matrix and particles toughness), the crack propagation of a
 408 tunnel-crack of width $2a_0$ starts when $\sigma = \sigma_0$, where $\sigma_0 = K_c^m/\sqrt{\pi a_0}$. Then,

- 409 • either the spacing between the obstacles is large enough, so the SIF
 410 in the matrix still increases at constant loading. The final breakdown
 411 then occurs for $\sigma = \sigma_0$.
- 412 • either the spacing is small enough, so the SIF in the matrix decreases
 413 at constant loading. Then a transient period of stable propagation at
 414 constant loading exists, so that the unstable propagation loading, that
 415 is the breakdown one, is increased.

416 In this last case, the breakdown may appear in two different situations:

1. The crack front has penetrated under a stable manner into the obstacles until reaching a configuration such as $K(s) = K_c(s)$ for all s at unstable breakdown instant. The corresponding loading level σ_c can then be determined by the following relation derived by Rice (1988); Gao and Rice (1989):

$$\frac{\sigma_c^2}{\sigma_0^2} = \frac{\langle K_c^2 \rangle}{(K_c^m)^2} \quad (46)$$

417 This situation is called "regular" by Gao and Rice (1989) and "weak
 418 pinning" by Roux et al. (2003). It appears for not too large toughness
 419 differences (to ensure not too large crack front deformations) and par-
 420 ticles that are large enough in the direction of propagation (to have
 421 the time to reach the equilibrium position). First order simulations
 422 of this regime have been performed by Gao and Rice (1989) using the
 423 perturbation approach described in section 2.2. The results have been
 424 compared to simulations of the same problem performed using a Bound-
 425 ary Element Method showing the accuracy and the limits of the first

426 order approach. They have also been compared to experiments by
427 Dalmas et al. (2009): good agreement between the theoretical and ex-
428 perimental crack shapes have been shown. The work of Gao and Rice
429 (1989) realized in mode 1 is extended to modes 2 and 3 in Gao et al.
430 (1991). Numerical large scale propagation simulations of this case have
431 also been performed by Bower and Ortiz (1990, 1991, 1993) for the
432 half-plane crack and Lazarus (2003) for the circular one.

433 2. The unstable breakdown occurs before $K(s) = K_c(s)$ is reached for all
434 s , so that only a part of the front propagates at the breakdown instant,
435 $K(s)$ being lower than $K_c(s)$ on the other part. In this situation,
436 called "irregular" by Gao and Rice (1989), "unstable" by Bower and
437 Ortiz (1991), and "strong pinning" by Roux et al. (2003) the value of
438 σ_c can only be determined numerically in each particular case. It has
439 been done by Bower and Ortiz (1991), using the incremental method
440 described in section 4.2, for a half plane crack propagating through an
441 array of particles. Their results concerning the bow out of the crack
442 front segments beyond an unbroken particle when the toughness of the
443 particles is high enough to prevent the penetration of the front in the
444 particle are compared to experiments by Mower and Argon (1995) and
445 show good agreement in general.

446 6. Crack propagation in a disordered media

447 In the last two decades, a number of works have been devoted to the
448 study of the evolution in time of the shape of the front of planar cracks
449 propagating in mode 1 in an elastic solid with randomly variable fracture
450 toughness. These works can be roughly divided into three groups. The
451 first group includes the works of Perrin and Rice (1994), Ramanathan and
452 Fisher (1997) and Morrissey and Rice (1998, 2000). They were devoted to
453 the theoretical study of some statistical features of the geometry of the front
454 of a tensile half-plane crack propagating dynamically. The second group of
455 papers consists of some experimental studies of the evolution in time of the
456 deformation of the front propagating quasistatically; see e.g. Schmittbuhl
457 and Måløy (1997), Delaplace et al. (1999), Schmittbuhl and Vilotte (1999),
458 Schmittbuhl et al. (2003a). The third group studied statistical properties
459 of the shape of crack fronts for a straight crack (half-plane or tunnel-crack)
460 propagating quasistatically: on the one hand, Schmittbuhl et al. (1995) and
461 Katzav and Adda-Bedia (2006) focusing notably on self-affine properties of

462 the crack front shape and on the other hand Favier et al. (2006b), Pindra
 463 et al. (2008), Pindra et al. (2010a), Legrand and Leblond (2010a) focusing
 464 essentially on the time evolution of the statistical properties (correlation
 465 functions and power spectra).

466 The main results of this last set of somewhat complex papers, concerning
 467 quasistatic propagation in an heterogeneous media, in particular of Favier
 468 et al. (2006b) and Pindra et al. (2008), are recalled in this section under a
 469 new simplified form in the spirit to be comprehensible by a broader audience.
 470 The model problems listed in section 3 are considered, in the case of brittle
 471 fracture (§6.1, §6.4), and in the case of subcritical or fatigue growth (§6.2).
 472 In §6.3 a synthetic table is presented showing the main analytical results.
 473 Finally the major factors having an influence on the crack front fluctuations
 474 are recalled (§6.5).

475 *6.1. Brittle fracture: case $K(x, z) = K_c(x, z) \forall(x, z)$ (weak pinning)*

476 Let us consider a half-plane or tunnel crack. We suppose that the SIF
 477 for the straight configuration is given by equation (31). The aim of this
 478 paragraph is to describe the crack front shape corresponding to $K(x, z) =$
 479 $K_c(x, z)$ for all points of the front when $K_c(x, z)$ is varying randomly.

480 *6.1.1. Fourier transforms of the crack front fluctuations versus toughness*
 481 *fluctuations*

If the toughness $K_c(x, z)$ is uniformly equal to a constant K_c , the crack
 front remains straight during propagation (unless bifurcations occur, see
 §4.1). Then the loading $k(t)$ corresponding to position $a(t)$ of the front
 at time t verifies:

$$k(t) = K_c a(t)^{-\alpha} \quad (47)$$

482 The parameter t , called "time" for convenience, appearing in this equation
 483 is not a physical time but a purely kinematical time, enabling us to locate
 484 the equilibrium position corresponding to a given loading $k(t)$. Remember
 485 that the propagation is stable (resp. unstable) if $\alpha < 0$ (resp. $\alpha > 0$), in the
 486 sense that the loading has to be increased (resp. decreased) for the crack to
 487 advance (a increases).
 488

Now introduce some small quenched (independent of time at a give ma-
 terial point) fluctuations of the toughness :

$$K_c(z, x) = \overline{K}_c(1 + \kappa(z, x)), \quad |\kappa| \ll 1 \quad (48)$$

It produces small fluctuations $\delta(z, a(t))$ and $\delta K(z, a(t))$ of the crack front position $a(z, t)$ and of the SIF $K(z, t)$ around its mean values $a(t)$ and $K(a(t))$ so that :

$$\begin{cases} a(z, t) & \equiv a(t) + \delta(z, a(t)), & |\delta(z, a(t))| & \ll a(t) \\ K(z, a(t)) & \equiv K(a(t)) + \delta K(z, a(t)), & |\delta K(z, a(t))| & \ll K(a(t)) \end{cases} \quad (49)$$

Expanding Irwin's criterion (7) to first order, identifying terms of order 0 and 1 and replacing the kinematical time t by the mean crack position a , one gets :

$$\begin{cases} K(a) = \overline{K}_c \\ \frac{\delta K(z, a)}{K(a)} = \kappa(z, a) \end{cases} \quad (50)$$

489 It shall be noticed that in this first order expansion, the quenched fluctuations
 490 are transferred on the mean crack position so that this approach does not
 491 permit to distinguish between annealed (time dependent fluctuations at a
 492 given position) and quenched noise.

Now, taking the Fourier transform of the equation (50.2) and using equation (33) giving the first order variation of the SIF as a function of the crack perturbation, one gets:

$$\hat{\delta}(k, a) = \frac{a\hat{\kappa}(k, a)}{\alpha - F(ka)} \quad (51)$$

Unfortunately, if $\alpha > 0$, the expression (51) is meaningless because the FT is divergent for p such as $\alpha - F(p) = 0$. This is linked to the existence of bifurcations (see section 4.1). We shall therefore consider the sole case of stable 2D crack propagation ($\alpha < 0$) henceforward. Eq. (51) then takes the form:

$$\hat{\delta}(k, a) = -\frac{a\hat{\kappa}(k, a)}{|\alpha| + F(p)} \quad (52)$$

493 This equation allows to obtain the first order crack front fluctuations δ for
 494 any given small toughness fluctuation κ . Notice that it is entirely determined
 495 by the instantaneous distribution of the toughness. It may be used to study
 496 the shape deformations during trapping by tougher obstacles. In the sequel,
 497 however we suppose the material to be disordered so that only statistical
 498 properties of κ are known and statistical study becomes necessary.

499 *6.1.2. Power spectrum of the crack front fluctuations versus toughness fluctu-*
500 *tations*

From equation (52), one gets for the power spectrum $\widehat{\mathcal{A}}(k, a)$ of the fluctuation $\delta(z)$ of the crack front:

$$\widehat{\mathcal{A}}(k, a) = a^2 \frac{\widehat{\mathcal{K}}(k)}{(|\alpha| + F(p))^2} \quad (53)$$

501 where $\widehat{\mathcal{K}}(k)$ is the power spectrum associated with the toughness fluctuations
502 κ supposed statistical homogeneous so that $\widehat{\mathcal{K}}(k)$ is independent of a . This
503 expression is quiet general. Some properties of it, in the particular case of
504 uncorrelated fluctuations are given in the sequel.

505

In the case of white noise $\widehat{\mathcal{K}}(k, a) = \widehat{\mathcal{K}}_0$, equation (53) gives $\frac{\widehat{\mathcal{A}}(k, a)}{\widehat{\mathcal{K}}_0 a^2}$ under an analytical form:

$$\frac{\widehat{\mathcal{A}}(k, a)}{\widehat{\mathcal{K}}_0 a^2} = \frac{1}{(|\alpha| + F(p))^2} \quad (54)$$

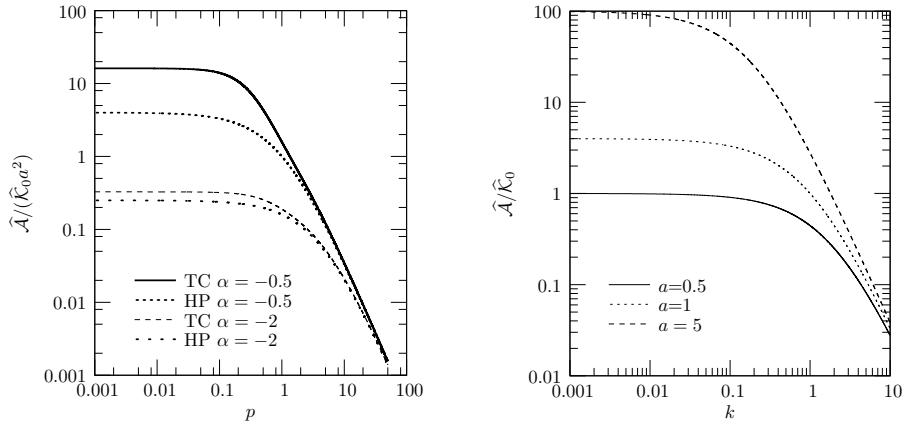
506 It is plotted as a function of $p = ka$ in figure 8(a) for several values of α
507 and a tunnel or a half-plane crack. In all these cases, one can notice the
508 presence of two regimes, with a transition between them depending on the
509 crack geometry and on the loading: one universal regime (independent of
510 the geometry) for $p = ka \gg 1$ that is small wavelengths $\lambda \ll a$ where $\frac{\widehat{\mathcal{A}}(k, a)}{\widehat{\mathcal{K}}_0 a^2}$
511 decreases with ka and a second (geometry dependent one) for $p = ka \ll 1$
512 that is large wavelengths $\lambda \gg a$ corresponding to a saturation. The existence
513 of this second regime is closely linked to the finite size of the system and can
514 not be obtained by the model of a half-plane crack loaded by remote tensile
515 stresses that, we recall, lacks any lengthscale.

One can notice that such a behaviour (see Barabási and Stanley (1995)) corresponds to a Family and Vicsek (1985) scaling defined by $\widehat{\mathcal{A}}(k, a) = a^{\frac{1+2\zeta}{\tau}} G(ka^{1/\tau})$ where $G(x)$ is constant for $x \ll 1$ and $G(x) \sim x^{-1-2\zeta}$ for $x \gg 1$. Comparison with (54) gives indeed:

$$G(x) = \widehat{\mathcal{K}}_0 (|\alpha| + F(x))^{-2} \quad (55)$$

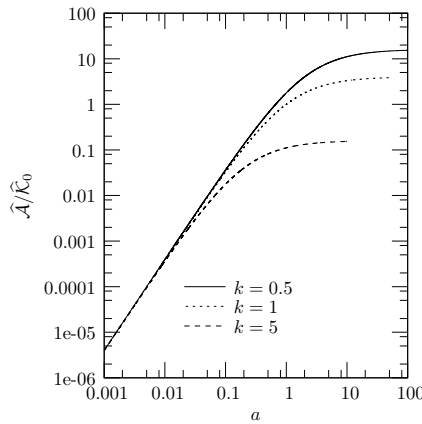
$\zeta = 0.5$ (roughness exponent) and $\tau = 1$ (dynamic exponent).

516 To better understand the Family and Vicsek (1985) scaling, let us study
517 the dependence over k and a of $\frac{\widehat{\mathcal{A}}(k, a)}{\widehat{\mathcal{K}}_0}$ and consider the particular case of a



(a) Master curves for different geometries (TC=tunnel-crack, HP=half-plane crack) and values of α .

(b) Dependence toward k for several given values of a (half-plane crack with $\alpha = -1/2$).



(c) Dependence toward a for several given values of k (half-plane crack with $\alpha = -1/2$).

Figure 8: Power spectrum of the crack front fluctuations for white noise toughness fluctuations.

518 half-plane crack with $\alpha = -1/2$. One can notice on figure 8(b), that for a
 519 given value of the mean position a :

- 520 • The large wavelength components are more developed than the small
 521 ones and are constant above a certain threshold.

522 • When a increases, the large wavelength components increases, but the
523 small ones are steadystate and more and more components are in this
524 last steadystate regime.

525 One can notice on figure 8(c), that for a given value of k , that is λ :

- 526 • The components increases with a until reaching a saturation.
- 527 • The increasing regime rate is similar whatever the wavelength, but
528 when k decreases, that is λ increases, the increasing regime lasts longer
529 so that the final amplitude increases with wavelength.

One can also derive the asymptotic expressions of the power spectrum $\hat{\mathcal{A}}(k, a)$:

$$\hat{\mathcal{A}}(k) = \frac{4\hat{\mathcal{K}}(k)}{k^2} \text{ for } k \neq 0 \quad \text{and} \quad \hat{\mathcal{A}}(0) = \frac{\hat{\mathcal{K}}(0)a^2}{\alpha^2} \text{ for } k = 0 \quad (56)$$

530 One shall notice that the convergence is not uniform so that the asymptotic
531 behaviour of the autocorrelation function can not be obtained by simply
532 inversion of the asymptotic behaviour of its Fourier transform. For the results
533 concerning the autocorrelation or related function as the square fluctuations,
534 the reader is invited to refer at Pindra et al. (2008) or to the table 1. One
535 shall however notice that relation (53) is the cornerstone for such a derivation.
536 It shall also allow to perform numerical simulations of the evolution of the
537 power spectrum or the functions related to the autocorrelation by inverse
538 Fourier Transform if the toughness fluctuation power spectrum is given. Such
539 developments is left for further work.

540 Similar results for an interfacial half-plane crack have been derived by
541 Pindra et al. (2008): the mismatch of elastic properties between the materials
542 introduces oscillations in the longtime behaviour but no significant difference
543 in the mean behaviour. The case of a shear tunnel-crack has been considered
544 by Pindra et al. (2010a): the results are rather similar to those previously
545 obtained for mode 1; one novelty, however, is that, the fronts no longer tend
546 to become symmetrical in time as in mode 1 (Favier et al., 2006b), so that
547 correlations between crack front fluctuations at two points are higher for
548 points located on the same front than for points located on distinct ones.

549 *6.2. Subcritical or fatigue propagation*

Let us suppose now that the crack advance is given by Paris' law (10). The inhomogeneity of the material is modelled by assuming the Paris constant to slightly fluctuated around its mean value; the Paris exponent N being considered as uniform for simplicity:

$$C(z, x) = C(1 + \delta c(z, x)), \quad |\delta c(z, x) \ll 1| \quad (57)$$

550 It produces small fluctuations $\delta(z, a(t))$ and $\delta K(z, a(t))$ of the crack position
551 and the SIF around its mean position a (eq. 49).

552 *6.2.1. Evolution of the perturbation of the crack front*

553 Expanding first the propagation law to first order in $\delta(z, t)$, $\delta K(z, t)$ and
554 identifying terms of order 0 and 1, one gets

$$\begin{cases} \frac{da}{dt}(t) &= C [K(t)]^N \\ \frac{\partial \delta}{\partial t}(z, t) &= C [K(t)]^N \left[N \frac{\delta K(z, a(t))}{K(t)} + \delta c(z, a(t)) \right]. \end{cases}$$

555 Eliminating dt between these expressions and considering henceforward
556 all perturbations as functions of the mean position $a(t) \equiv a$ of the crack
557 instead of time, one gets

$$\frac{\partial \delta}{\partial a}(z, a) = N \frac{\delta K(z, a)}{K(a)} + \delta c(z, a).$$

Upon use of the Fourier decompositions of $\delta(z, a)$, $\delta K(z, a)$, $\delta c(z, a)$ and equation (33), this finally yields the evolution equation of the Fourier transform $\widehat{\delta}(k, a)$ of the perturbation of the crack front:

$$\frac{\partial \widehat{\delta}}{\partial a}(k, a) = \frac{N}{a} [\alpha - F(ka)] \widehat{\delta}(k, a) + \widehat{\delta c}(k, a). \quad (58)$$

Assuming the crack to be initially straight and integrating the linear, inhomogeneous, first-order differential equation (58) by standard methods, one gets

$$\widehat{\delta}(k, a) = \int_{a_0}^a \left(\frac{a}{a'} \right)^{N\alpha} \left[\frac{\psi(ka)}{\psi(ka')} \right]^N \widehat{\delta c}(k, a') da' \quad (59)$$

558 where a_0 denotes the initial value of a and ψ the function defined by equation
559 (42).

560 Notice that contrary to the brittle case (eq. 52), this equation is valid
 561 whatever the sign of α and that an effect of memory of previous configurations
 562 of the crack front is present here. This equation allows to obtain the crack
 563 front fluctuations δ for any given Paris' constant fluctuation δc . This shall
 564 be done numerically and is leaved for a further work. In the sequel, however
 565 only statistical properties of δc are known, so that statistical study becomes
 566 necessary.

567 *6.2.2. General formula for the power spectrum of the perturbation of the crack*
 568 *front*

By the expression (59) of $\widehat{\delta}(k, a)$, one gets for the power spectrum of the crack front fluctuations:

$$\widehat{\mathcal{A}}(k, a) = \int_{a_0}^a \int_{a_0}^a \left(\frac{a^2}{a_1 a_2} \right)^{N\alpha} \left(\frac{[\psi(ka)]^2}{\psi(ka_1)\psi(ka_2)} \right)^N \widehat{\mathcal{C}}(k, a_2 - a_1) da_1 da_2, \quad (60)$$

569 where $\widehat{\mathcal{C}}(k, a_2 - a_1)$ is the power spectrum of the Paris' constant fluctuations
 570 δc .

Due to the memory effect, this equation is more complex than the equivalent (53) one in brittle fracture. Its properties for any value of a has not at present been studied. Its asymptotic behaviour for $a \rightarrow \infty$ has however be obtained by Favier et al. (2006b) and Pindra et al. (2008). For $k = 0$, one gets:

$$\widehat{\mathcal{A}}(0, a) \sim \begin{cases} \frac{\widetilde{\mathcal{C}}(0, 0)}{2N\alpha - 1} a_0 \left(\frac{a}{a_0} \right)^{2N\alpha} & \text{if } \alpha > \frac{1}{2N}, \\ \frac{\widetilde{\mathcal{C}}(0, 0)}{1 - 2N\alpha} a & \text{if } \alpha < \frac{1}{2N}, \end{cases} \quad (61)$$

and for $k \neq 0$:

$$\widehat{\mathcal{A}}(k, a) \sim \frac{\widetilde{\mathcal{C}}(k, 0)}{N|k|} \quad (62)$$

571 where $\widetilde{\mathcal{C}}(k, k')$ is the double z, x -Fourier transform of the function \mathcal{C} .

572 One can notice, like in brittle fracture that the large wavelengths are preferentially selected by the system and that ultimately only the zero wavenumber $k = 0$ component still evolves with a , the other components being in a steadystate (independent of a) rough regime. The first order study is not sufficient to determine the roughness exponent in this case (it gives indeed $\zeta = 0$). This is probably linked to the memory effect that delay the development of this regime. A second order study is then necessary. It has
 578

579 been performed by Katzav and Adda-Bedia (2006) who obtains a roughness
580 exponent of $\zeta = 0.5$.

581 *6.3. Synthesis of the theoretical analytical results*

582 The previous results derived for the half-plane or tunnel crack, with ad-
583 ditional ones derived from Favier et al. (2006b) and Pindra et al. (2008) are
584 summarized in table 1.

Table 1: Main analytical results concerning the propagation in a disordered media

	fatigue $\alpha > \frac{1}{2N}$	fatigue $\alpha < \frac{1}{2N}$	brittle $\alpha < 0$
For any a :			
$\hat{\delta}$	$\int_{a_0}^a \left(\frac{\psi(ka)}{\psi(ka')} \right)^N \left(\frac{a}{a'} \right)^{N\alpha} \widehat{\delta}c(k, a') da'$	idem	$-\frac{a\widehat{\mathcal{K}}(k, a)}{ \alpha + F(ka)}$
$\widehat{\mathcal{A}}$	$\int_{a_0}^a \int_{a_0}^a \left(\frac{a^2}{a_1 a_2} \right)^{N\alpha} \left(\frac{[\psi(ka)]^2}{\psi(ka_1)\psi(ka_2)} \right)^N \widehat{\mathcal{C}}(k, a_2 - a_1) da_1 da_2$	idem	$a^2 \frac{\widehat{\mathcal{K}}(k)}{(\alpha + F(ka))^2}$
For $a \rightarrow \infty$:			
$\widehat{\mathcal{A}}(k, a)$	$\frac{\widetilde{\mathcal{C}}(k, 0)}{N k }$	$\frac{\widetilde{\mathcal{C}}(k, 0)}{N k }$	$\frac{4\widehat{\mathcal{K}}(k)}{k^2}$
$\widehat{\mathcal{A}}(0, a)$	$\frac{\widetilde{\mathcal{C}}(0, 0)}{2N\alpha - 1} a_0 \left(\frac{a}{a_0} \right)^{2N\alpha}$	$\frac{\widetilde{\mathcal{C}}(0, 0)}{1 - 2N\alpha} a$	$\widehat{\mathcal{K}}(0) \frac{a^2}{\alpha^2}$
$\mathcal{A}(z)$	$\frac{\widetilde{\mathcal{C}}(0, 0)}{\pi(2N\alpha - 1)} \left(\frac{a}{a_0} \right)^{2N\alpha - 1} \int_{-\infty}^{+\infty} \psi(p)^{2N} dp$	$\frac{\widetilde{\mathcal{C}}(0, 0)}{\pi N} \ln a$	$\frac{2a\widehat{\mathcal{K}}(0)}{\pi \alpha }$

585 We recall that in this table $F(p)$ is the function introduced in §3.2. This
 586 function is such as $F(0) = 0$ and increases monotonically to finally behaves
 587 as $p/2$ for $p \rightarrow \infty$. The function ψ is defined by eq. (42) and decreases from
 588 1 to 0 when p varies from 0 to $+\infty$.

589 One notices that in all the cases the system preferentially "selects" per-
 590 turbations of the crack front with small wavenumbers k , that is, large wave-
 591 lengths $\lambda = 2\pi/|k|$. Physically it is linked to the process explained in §4.1.
 592 One can also easily discuss, using the table 1, the differences between brittle
 593 fracture and fatigue, the role of the loading type (sign of α) and of the crack
 594 geometry (function $F(p)$ and ψ):

- 595 • Concerning the crack advance law, one can notice by comparison of
 596 columns 3 and 4, that the relations are less complex in brittle fracture
 597 than in fatigue since there is no time dependence of the response in the
 598 first case contrary to the second. One can also notice that the disorder
 599 grows faster in brittle fracture than in fatigue, the development being
 600 slowed down by a memory effect. And finally, for $\alpha > 0$, the treatment
 601 is possible only in fatigue, since in brittle fracture the appearance of
 602 bifurcation renders the problem ill-posed.
- 603 • Thus, the dependence upon the sign of α can be considered only in
 604 fatigue. Comparison of columns 2 and 3 shows that the disorder grows
 605 faster for $\alpha > \frac{1}{2N}$ than for $\alpha < \frac{1}{2N}$. It is obvious since unstable wave-
 606 lengths exist for $\alpha > 0$ and not for $\alpha < 0$. The selection of the large
 607 wavelengths remains however since the large ones grow faster than the
 608 small ones.
- 609 • Concerning the crack geometry, one can notice that the asymptotic
 610 behaviour for $a \rightarrow \infty$ is independent of F , that is the same for the
 611 half-plane and tunnel cracks. Moreover, we have seen in section 6.1.2,
 612 that in brittle fracture, the power spectrum satisfies a Family-Vicsek
 613 scaling in both cases and the geometry introduces a difference only
 614 for the transition toward the large wavelength saturation regime (fig.
 615 8(a)).

616 6.4. Brittle fracture: case $\exists x, z$ such as $K(x, z) < K_c(x, z)$ (strong pinning)

617 In the case of strong pinning, when points of the crack front such as
 618 $K(x, z) < K_c(x, z)$ exist, an analytical approach is no more possible. Using
 619 a numerical approach based on the perturbation approach for the half-plane

620 crack of Rice (1985), Schmittbuhl et al. (1995); Schmittbuhl and Vilotte
 621 (1999); Rosso and Krauth (2002) have obtained a different value of the
 622 roughness exponent: $\zeta \sim 0.35 - 0.4$. In these simulations, the toughness
 623 fluctuations were supposed to be quenched.

624 The reason for the difference between the analytical value $\zeta \sim 0.5$ ob-
 625 tained for weak pinning and this last result shall be clarified. It shall be due
 626 to the difference weak/strong pinning, to the difference quenched/annealed
 627 still noticed in other problems (Kardar, 1998) but also to numerical biases.
 628 The analytical results may for this last point serve as validation for the nu-
 629 merical procedures.

630 *6.5. Synthesis of the major results*

631 Now, let us list the factors that have an influence on the crack fluctuations
 632 resulting from toughness ones. Among them, we have seen that some of
 633 them have a quantitative influence on the statistical properties of the crack
 634 fluctuations, but a minor influence on the qualitative type of behaviour:

- 635 • Whether the crack propagates in an homogeneous media or along an
 636 interface (Pindra et al., 2008);
- 637 • Whether the loading is in tensile or shear mode (Pindra et al., 2010a);
- 638 • Whether the FK is the one of a half-plane or a tunnel-crack.

639 Some others have a major influence, namely:

- 640 • The advance law: Paris' law gives nonlocal time-dependent laws for
 641 the evolution of the crack fluctuation in contrary to Irwin's criterion.
 642 Moreover, strong pinning (see §5 for the definition) seems to lead to
 643 different value of the roughness exponent: $\zeta \sim 0.35 - 0.4$ (Schmittbuhl
 644 et al., 1995; Rosso and Krauth, 2002) then weak pinning $\zeta \sim 0.5$.
- 645 • The loading: we have seen previously the dependance of the result with
 646 the sign of $\frac{dK(a)}{da}$.
- 647 • Uncorrelated/correlated fluctuations: equation (53) shows the depen-
 648 dence of the result with the toughness fluctuations (see also Schmit-
 649 tbuhl and Vilotte (1999)).
- 650 • Annealed/quenched noise: has been shown to give different results in
 651 other problems (Kardar, 1998).

652 The model is however valid only:

- 653 • In the quasistatic case. When dynamic effects become important, one
654 may refer for instance to Ramanathan and Fisher (1997) or Morrissey
655 and Rice (1998).
- 656 • Outside the Process Zone. Inside percolation damage models have been
657 used (Hansen and Schmittbuhl, 2003).
- 658 • For small fluctuation variations. No full description of large variations
659 has until now been developed.
- 660 • Far from any boundary. Interactions with boundaries have been show
661 to be important, but have seldom be studied (Gao et al. (1991), Pindra
662 et al. (2010b), Legrand and Leblond (2010b)).

663 Due to the difficulty of observation of the crack front during the prop-
664 agation, few different experiments exist at present, that allow to study the
665 crack front deformations. Among them Dagquier et al. (1995) used ink injec-
666 tions to follow the crack front in brittle fracture and fatigue. Schmittbuhl
667 and Måløy (1997) studied the in-plane propagation through a transparent
668 plexiglas block, the toughness fluctuations being obtained by sand blasting
669 the surface of two plexiglas plates before to weld them together. Despite the
670 numerous papers written on the subject (Schmittbuhl and Måløy, 1997; De-
671 laplace et al., 1999; Maloy and Schmittbuhl, 2001; Schmittbuhl et al., 2003b;
672 Hansen and Schmittbuhl, 2003; Schmittbuhl et al., 2003a; Santucci et al.,
673 2009), the debate seems not closed. All the previous points shall be verified
674 to clarify the situation.

675 7. Conclusion

676 Crack front small perturbation approach initiated by Rice (1985) and
677 later extended to more complex cases has been recalled. This approach al-
678 lows to update the stress intensity factors when the crack front is slightly
679 perturbed in its plane. Applications concerning the deformation of the crack
680 front when it propagates quasistatically in an homogeneous or heterogeneous
681 media have been considered in brittle fracture or fatigue/subcritical propa-
682 gation. Only the case of one crack propagating in an infinite plane without
683 interaction with a boundary has been considered. The stable shapes corre-
684 sponding to uniform SIF have been derived: straight or circular, but also

685 when bifurcations exists, wavy crack fronts. For a straight crack, it has been
686 shown that perturbation of all lengthscales progressively disappears unless
687 disordered fracture properties yields Family and Vicsek (1985) roughness of
688 the crack front.

689 This approach has recently be extended to interaction between several
690 cracks: the FK for two tunnel-cracks has been derived (Pindra et al. (2010b),
691 Legrand and Leblond (2010b)) and the disorder during their coalescence
692 studied (Legrand and Leblond, 2010a). Interestingly, it has been shown
693 that, stability first prevails for all lengthscales such as $\lambda < \lambda_c$, but since λ_c is
694 this time a decreasing function of the crack advance, all wavelengths finally
695 become unstable so that the perturbation does not vanish. This underlines
696 that stability results depend on the interaction of the crack with other cracks
697 and more generally also, with obstacles or solid boundaries. Such a conclusion
698 has still been obtained by Gao et al. (1991) who studied the stability issue
699 when the front approaches a stress free plate boundary: when the crack is far
700 enough from the boundary, the critical wavelength λ_c increases with crack
701 growth (the model of infinite solid is then valid) and when it approaches the
702 boundary, λ_c decreases with crack growth.

703 Now, comparison of the theoretical results recalled in this paper with
704 experiments are seldom and have shown more or less success, in particular if
705 one looks for quantitative agreement. Thus, to make them useful in particular
706 for the engineering sciences, comparison with experiments have to be done
707 in a deeper and more extensive way.

708 It is planed in the next years during the ANR Programme SYSCOMM
709 (ANR-09-SYSC-006 Mechanics and Statistical Physics of Rupture in Brittle
710 Heterogeneous Materials) that supported this work.

711 **8. Acknowledgements**

712 The support of the ANR Programme SYSCOMM (ANR-09-SYSC-006)
713 is gratefully acknowledged.

714 I would also like to greatly thank E. Bouchaud, J.R. Rice, L. Vanel, J.B.
715 Leblond, M. Adda-Bedia, J.J. Marigo, S. Roux, J. Schmittbuhl who kindly
716 accepted to participate to the examiners board of my "Habilitation à diriger
717 des Recherches" exam and who encouraged me to publish this work. A
718 particular thank goes to D. Bonamy for its careful review.

719 **References**

- 720 Adda-Bedia, M., Mahadevan, L., 2006. Crack-front instability in a confined
721 elastic film. *Proceedings of the royal society A-Mathematical physical and*
722 *engineering sciences* 462 (2075), 3233 – 3251.
- 723 Amestoy, M., Leblond, J.-B., 1992. Crack Paths in Plane Situations - II. De-
724 tailed Form of the Expansion of the Stress Intensity Factors. *International*
725 *Journal of Solids and Structures* 29, 465–501.
- 726 Anderson, T. L., 1991. *Fracture Mechanics: Fundamentals and Applications*.
727 CRC Press, Inc., Boca Raton, USA.
- 728 Atkinson, B. K., 1987. *Fracture mechanics of rock*. Academic Press geology
729 series.
- 730 Aydin, A., Pollard, D. D., 1988. Progress in understanding jointing over the
731 past century. *Geological Society of America Bulletin* 100, 1181–1204.
- 732 Barabási, A.-L., Stanley, H. E., 1995. *Fractal concepts in surface growth*.
- 733 Bazant, Z. P., Cedolin, L., 2003. *Stability of Structures: Elastic, Inelastic,*
734 *Fracture, and Damage Theories*, new ed edition (february 14, 2003) Edi-
735 tion. Dover Publications.
- 736 Benallal, A., Marigo, J.-J., 2007. Bifurcation and stability issues in gradient
737 theories with softening. *Modelling and Simulation in Materials Science and*
738 *Engineering* 15, S283–S295.
- 739 Bercial-Velez, J., Antipov, Y., Movchan, A., May 2005. High-order asymp-
740 totics and perturbation problems for 3d interfacial cracks. *Journal of the*
741 *Mechanics and Physics of Solids* 53 (5), 1128–1162.
- 742 Bonamy, D., 2009. Intermittency and roughening in the failure of brittle
743 heterogeneous materials. *Journal of Physics D: Applied Physics* 42 (21),
744 214014 (21pp).
- 745 Bouchbinder, E., Livne, A., Fineberg, J., 03 2010. Weakly nonlinear frac-
746 ture mechanics: experiments and theory. *International Journal of Fracture*
747 162 (1), 3–20.

- 748 Bourdin, B., Francfort, G., Marigo, J.-J., 2008. The variational approach to
749 fracture. *Journal of elasticity* 91 (1), 5 – 148.
- 750 Bourdin, B., Francfort, G. A., Marigo, J. J., 2000. Numerical experiments in
751 revisited brittle fracture. *Journal of the Mechanics and Physics of Solids*
752 48 (4), 797–826.
- 753 Bower, A. F., Ortiz, M., 1990. Solution of three-dimensional crack problems
754 by a finite perturbation method. *Journal of the Mechanics and Physics of*
755 *Solids* 38 (4), 443–480.
- 756 Bower, A. F., Ortiz, M., 1991. A three-dimensional analysis of crack trapping
757 and bridging by tough particles. *Journal of the Mechanics and Physics of*
758 *Solids* 39 (6), 815–858.
- 759 Bower, A. F., Ortiz, M., 1993. An analysis of crack trapping by residual
760 stresses in brittle solids. *Transactions of the ASME. Journal of Applied*
761 *Mechanics* 60 (1), 175–82.
- 762 Broberg, K. B., 1999. *Cracks and fracture*. Academic Press.
- 763 Budiansky, B., Amazigo, J. C., Evans, A. G., 1988. Small-scale crack bridging
764 and the fracture toughness of particulate-reinforced ceramics. *Journal of*
765 *the Mechanics and Physics of Solids* 36 (2), 167 – 187.
- 766 Bueckner, H. F., 1970. A novel principle for the computation of stress inten-
767 sity factors. *Zeitschrift fur Angewandte Mathematik und Mechanik* 50 (9),
768 529–546.
- 769 Bueckner, H. F., 1987. Weight functions and fundamental fields for the
770 penny-shaped and the half-plane crack in three-space. *International Jour-*
771 *nal of Solids and Structures* 23 (1), 57–93.
- 772 Caginalp, G., Fife, P., 1986. Phase field methods for interfacial boundaries.
773 *Physical Review B* 33, 7792–7794.
- 774 Chambolle, A., Francfort, G., Marigo, J.-J., 2009. When and how do cracks
775 propagate? *Journal of the Mechanics and Physics of Solids* 57 (9), 1614 –
776 1622.

- 777 Ciavarella, M., Paggi, M., Carpinteri, A., 2008. One, no one, and one hundred
778 thousand crack propagation laws: A generalized barenblatt and botvina
779 dimensional analysis approach to fatigue crack growth. *Journal of the Me-*
780 *chanics and Physics of Solids* 56 (12), 3416 – 3432.
- 781 Collins, J., Levine, H., 1985. Diffuse interface model of diffusion-limited
782 crystal-growth. *Physical review B* 31 (9), 6119–6122.
- 783 Cooke, M. L., Pollard, D. D., 1996. Fracture propagation paths under mixed
784 mode loading within rectangular blocks of polymethyl methacrylate. *Jour-*
785 *nal of Geophysical Research* 101 (B2), 3387–3400.
- 786 Corson, F., Adda-Bedia, M., Henry, H., Katzav, E., 2009. Thermal fracture
787 as a framework for quasi-static crack propagation. *International Journal*
788 *of Fracture* 158 (1), 1–14.
- 789 Cotterell, B., Rice, J. R., 1980. Slightly curved or kinked cracks. *International*
790 *Journal of Fracture* 16 (2), 155–169.
- 791 Daguier, P., Bouchaud, E., Lapasset, G., 1995. Roughness of a crack front
792 pinned by microstructural obstacles. *EPL (Europhysics Letters)* 31 (7),
793 367–372.
- 794 Dalmas, D., Barthel, E., Vandembroucq, D., 2009. Crack front pinning by
795 design in planar heterogeneous interfaces. *Journal of the Mechanics and*
796 *Physics of Solids* 57 (3), 446 – 457.
- 797 Delaplace, A., Schmittbuhl, J., Måløy, K. J., 1999. High resolution descrip-
798 tion of a crack front in a heterogeneous plexiglas block. *Physical Review E*
799 *(Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics)*
800 60 (2), 1337–43.
- 801 Doquet, V., Bertolino, G., 2008. Local approach to fatigue cracks bifurcation.
802 *International journal of fatigue* 30 (5), 942–950.
- 803 Drazin, P. G., 1992. *Nonlinear systems*. Press Syndicate of the University of
804 Cambridge, Cambridge.
- 805 Erdogan, G., Paris, P., 1963. A critical analysis of crack propagation laws.
806 *ASME J. Basic Engng Trans.* 85, 528–534.

- 807 Erdogan, G., Sih, G. C., 1963. On the crack extension in plates under plane
808 loading and transverse shear. *ASME J. Basic Engng* 85, 519–527.
- 809 Family, F., Vicsek, T., 1985. Scaling of the active zone in the eden process on
810 percolation networks and the ballistic deposition model. *Journal of Physics*
811 *A: Mathematical and General* 18 (2), L75.
- 812 Favier, E., Lazarus, V., Leblond, J.-B., 2006a. Coplanar propagation paths
813 of 3D cracks in infinite bodies loaded in shear. *International Journal of*
814 *Solids and Structures* 43 (7-8), 2091–2109.
- 815 Favier, E., Lazarus, V., Leblond, J.-B., 2006b. Statistics of the deformation
816 of the front of a tunnel-crack propagating in some inhomogeneous medium.
817 *Journal of the Mechanics and Physics of Solids* 54 (7), 1449–1478.
- 818 Fineberg, J., Marder, M., 1999. Instability in dynamic fracture. *Physics*
819 *Reports-Rev. Sec. Phys. Lett.* 313 (1-2), 2–108.
- 820 Fisher, D. S., Dahmen, K., Ramanathan, S., Ben-Zion, Y., 1997. Statistics
821 of earthquakes in simple models of heterogeneous faults. *Physical Review*
822 *Letters* 78 (25), 4885–8.
- 823 Fleck, N., Kang, K., Ashby, M., 1994. Overview no. 112: The cyclic properties
824 of engineering materials. *Acta Metallurgica et Materialia* 42 (2), 365 – 381.
- 825 Francfort, G. A., Marigo, J. J., 1998. Revisiting brittle fracture as an energy
826 minimization problem. *Journal of the Mechanics and Physics of Solids* 46,
827 1319–1342.
- 828 Freund, L., 1972a. Crack propagation in an elastic solid subjected to general
829 loading–i. constant rate of extension. *Journal of the Mechanics and Physics*
830 *of Solids* 20 (3), 129 – 140.
- 831 Freund, L., 1972b. Crack propagation in an elastic solid subjected to general
832 loading–ii. non-uniform rate of extension. *Journal of the Mechanics and*
833 *Physics of Solids* 20 (3), 141 – 152.
- 834 Freund, L., 1973. Crack propagation in an elastic solid subjected to general
835 loading–iii. stress wave loading. *Journal of the Mechanics and Physics of*
836 *Solids* 21 (2), 47 – 61.

- 837 Freund, L. B., 1974. Crack propagation in an elastic solid subjected to general
838 loading– iv. obliquely incident stress pulse. *Journal of the Mechanics and*
839 *Physics of Solids* 22 (3), 137 – 146.
- 840 Freund, L. B., 2000. Substrate curvature due to thin film mismatch strain in
841 the nonlinear deformation range. *Journal of the Mechanics and Physics of*
842 *Solids* 48 (6-7), 1159–1174.
- 843 Gao, H., 1988. Nearly circular shear mode cracks. *International Journal of*
844 *Solids and Structures* 24 (2), 177–193.
- 845 Gao, H., Rice, J. R., 1986. Shear stress intensity factors for planar crack
846 with slightly curved front. *ASME Journal of Applied Mechanics* 53 (4),
847 774–778.
- 848 Gao, H., Rice, J. R., 1987a. Nearly circular connections of elastic half spaces.
849 *ASME Journal of Applied Mechanics* 54 (4), 627–634.
- 850 Gao, H., Rice, J. R., 1987b. Somewhat circular tensile cracks. *International*
851 *Journal of Fracture* 33 (3), 155–174.
- 852 Gao, H., Rice, J. R., 1989. A first-order perturbation analysis of crack trap-
853 ping by arrays of obstacles. *Transactions of the ASME* 56, 828–836.
- 854 Gao, H., Rice, J. R., Lee, J., 1991. Penetration of a quasi-statically slipping
855 crack into a seismogenic zone of heterogeneous fracture resistance. *Journal*
856 *of Geophysical Research* 96 (B13), 21535–21548.
- 857 Gauthier, G., Lazarus, V., Pauchard, L., 2010. Shrinkage star-shaped cracks:
858 Explaining the transition from 90 degrees to 120 degrees. *EPL* 89, 26002.
- 859 Ghatak, A., Chaudhury, M. K., 2003. Adhesion-induced instability patterns
860 in thin confined elastic film. *Langmuir* 19 (7), 2621–2631.
- 861 Goehring, L., Mahadevan, L., Morris, S. W., 2009. Nonequilibrium scale
862 selection mechanism for columnar jointing. *Proceedings of the National*
863 *Academy of Sciences* 106 (2), 387–392.
- 864 Goldstein, R. V., Salganik, R. L., 1974. Brittle fracture of solids with arbi-
865 trary cracks. *International Journal of Fracture* 10, 507–523.

- 866 Griffith, A. A., 1920. The phenomena of rupture and flow in solids. Philo-
867 sophical Transactions of the Royal Society of London 221, 163–198.
- 868 Grob, M., Schmittbuhl, J., Toussaint, R., Rivera, L., Santucci, S., Maloy,
869 K. J., JUL 2009. Quake Catalogs from an Optical Monitoring of an Inter-
870 facial Crack Propagation. *Pure and applied geophysics* 166 (5-7), 777–799.
- 871 Hakim, V., Karma, A., 2009. Laws of crack motion and phase-field models
872 of fracture. *Journal of the Mechanics and Physics of Solids* 57 (2), 342 –
873 368.
- 874 Hansen, A., Schmittbuhl, J., 2003. Origin of the universal roughness expo-
875 nent of brittle fracture surfaces: stress-weighted percolation in the damage
876 zone. *Physical Review Letters* 90 (4), 045504/1–4.
- 877 Henry, H., Levine, H., 2004. Dynamic instabilities of fracture under biaxial
878 strain using a phase field model. *Physical Review Letters* 93 (10), 105504.
- 879 Hull, D., 1993. Tilting cracks: the evolution of fracture surface topology in
880 brittle solids. *International Journal of Fracture* 62 (2), 119–138.
- 881 Irwin, G. R., 1957. Analysis of stresses and strains near the end of a crack
882 traversing a plate. *Journal of Applied Mechanics* 24, 361–364.
- 883 Irwin, G. R., 1958. *Fracture*. *Hand. der Physik*. Vol. IV. Springer, Berlin.
- 884 Kardar, M., 1998. Nonequilibrium dynamics of interfaces and lines. *Physics*
885 *Reports-Rev. Sec. Phys. Lett.* 301 (1-3), 85–112.
- 886 Karma, A., Kessler, D. A., Levine, H., 2001. Phase-field model of mode III
887 dynamic fracture. *Physical Review Letters* 87, 045501.
- 888 Karma, A., Leblond, J. B., Pons, A. J., Lazarus, V., 2011. Instability of
889 crack propagation with superposition of antiplane shear and tension (In
890 preparation).
- 891 Kassir, M. K., Sih, G., 1975. *Three Dimensional Crack Problems*. Nordhoff
892 International Publishing, Leyden, The Netherlands.
- 893 Katzav, E., Adda-Bedia, M., 2006. Roughness of tensile crack fronts in het-
894 erogenous materials. *Europhysics Letters* 76 (3), 450–456.

- 895 Kostrov, B., 1975. On the crack propagation with variable velocity. Interna-
896 tional Journal of Fracture 11, 47–56.
- 897 Lancioni, G., Royer-Carfagni, G., APR 2009. The Variational Approach to
898 Fracture Mechanics. A Practical Application to the French Panth, on in
899 Paris. Journal of elasticity 95 (1-2), 1–30.
- 900 Lange, F. F., 1970. The interaction of a crack front with a second-phase
901 dispersion. Philosophical Magazine 22 (179).
- 902 Lazarus, V., 2003. Brittle fracture and fatigue propagation paths of 3D plane
903 cracks under uniform remote tensile loading. International Journal of Frac-
904 ture 122 (1-2), 23–46.
- 905 Lazarus, V., Buchholz, F.-G., Fulland, M., Wiebesiek, J., 2008. Comparison
906 of predictions by mode II or mode III criteria on crack front twisting in
907 three or four point bending experiments. International Journal of Fracture
908 153, 141–151.
- 909 Lazarus, V., Gauthier, G., Pauchard, L., Maurini, C., Valdivia, C., July
910 2009. Basalt columns and crack formation during directional drying of
911 colloidal suspensions in capillary tubes. In: 12th International Conference
912 on Fracture (ICF12). Ottawa (Canada).
- 913 Lazarus, V., Leblond, J., Karma, A., 2011a. Segmentation and coarsening in
914 presence of mode 3: experimental study (In preparation).
- 915 Lazarus, V., Leblond, J.-B., 1998a. Crack paths under mixed mode (I+III)
916 or (I+II+III) loadings. Comptes Rendus de l'Académie des Sciences, Série
917 II (Mécanique, Physique, Astronomie) 326 (3), 171–177.
- 918 Lazarus, V., Leblond, J.-B., 1998b. Three-dimensional crack-face weight
919 functions for the semi-infinite interface crack. I. Variation of the stress
920 intensity factors due to some small perturbation of the crack front. Jour-
921 nal of the Mechanics and Physics of Solids 46 (3), 489–511.
- 922 Lazarus, V., Leblond, J.-B., 2002a. Crack front stability for a tunnel-crack
923 propagating along its plane in mode 2+3. Comptes Rendus de l'Académie
924 des Sciences Paris, Serie II (Mecanique, Physique, Astronomie) 330 (6),
925 437–443.

- 926 Lazarus, V., Leblond, J.-B., 2002b. In-plane perturbation of the tunnel-crack
927 under shear loading. I: Bifurcation and stability of the straight configura-
928 tion of the front. *International Journal of Solids and Structures* 39 (17),
929 4421–4436.
- 930 Lazarus, V., Leblond, J.-B., 2002c. In-plane perturbation of the tunnel-crack
931 under shear loading. II: determination of the fundamental kernel. *Internation-
932 al Journal of Solids and Structures* 39 (17), 4437–4455.
- 933 Lazarus, V., Leblond, J.-B., Mouchrif, S.-E., 2001a. Crack front rotation and
934 segmentation in mixed mode I+III or I+II+III - part I: Calculation of
935 Stress Intensity Factor. *Journal of the Mechanics and Physics of Solids*
936 49 (7), 1399–1420.
- 937 Lazarus, V., Leblond, J.-B., Mouchrif, S.-E., 2001b. Crack front rotation
938 and segmentation in mixed mode I+III or I+II+III - part II: Comparison
939 with experiments. *Journal of the Mechanics and Physics of Solids* 49 (7),
940 1421–1443.
- 941 Lazarus, V., Maurini, C., Gauthier, G., Bourdin, B., 2011b. Crack patterns
942 in directional drying: experiments and results of numerical energy mini-
943 mization (In preparation).
- 944 Leblond, J., Karma, A., Lazarus, V., 2011. Theoretical analysis of crack
945 front instability in mode I+III. *Journal of the mechanics and physics of
946 solides* (Submitted).
- 947 Leblond, J.-B., 1989. Crack Paths in Plane Situations - I. General Form of the
948 Expansion of the Stress Intensity Factors. *International Journal of Solids
949 and Structures* 25, 1311–1325.
- 950 Leblond, J.-B., 1999. Crack paths in three-dimensional elastic solids. i: two-
951 term expansion of the stress intensity factors—application to crack path
952 stability in hydraulic fracturing. *International Journal of Solids and Struc-
953 tures* 36 (1), 79 – 103.
- 954 Leblond, J.-B., Lazarus, V., Mouchrif, S.-E., 1999. Crack paths in three-
955 dimensional elastic solids. II. Three-term expansion of the stress intensity
956 factors - Applications and perspectives. *International Journal of Solids and
957 Structures* 36 (1), 105–142.

- 958 Leblond, J.-B., Leguillon, D., 1999. Asymptotic behavior of stress intensity
959 factors near an angular point of a crack front. *European Journal of Me-*
960 *chanics, A/Solids* 18 (1), 135–145.
- 961 Leblond, J.-B., Mouchrif, S.-E., Perrin, G., 1996. The tensile tunnel-crack
962 with a slightly wavy front. *International Journal of Solids and Structures*
963 33 (14), 1995–2022.
- 964 Legrand, L., Leblond, J. B., 2010a. Evolution of the shape of the fronts of
965 a pair of semi-infinite cracks during their coplanar coalescence. *ZAMM*
966 90 (10-11), 821–836.
- 967 Legrand, L., Leblond, J.-B., 2010b. In-plane perturbation of a system of two
968 coplanar slit-cracks - II: Case of a large distance between the outer crack
969 fronts. *International Journal of Solids and Structures* 47, 3504–3512.
- 970 Lin, B., Mear, M., Ravi-Chandar, K., 2010. Criterion for initiation of cracks
971 under mixed-mode I + III loading. *International Journal of Fracture* 165,
972 175–188.
- 973 Liu, Y., Rice, J., 2005. Aseismic slip transients emerge spontaneously in
974 three-dimensional rate and state modeling of subduction earthquake se-
975 quences. *Journal of geophysical research-Solid earth* 110 (B8).
- 976 Maloy, K., Schmittbuhl, J., 2001. Dynamical event during slow crack propa-
977 gation. *Physical Review Letters* 87 (10), 105502/1–4.
- 978 Meade, K. P., Keer, L. M., 1984. On the problem of a pair of point forces
979 applied to the faces of a semi-infinite plane crack. *Journal of Elasticity*
980 14 (1), 3–14.
- 981 Morrissey, J., Rice, J., 1998. Crack front waves. *Journal of the mechanics*
982 *and physics of solids* 46 (3), 467 – 487.
- 983 Morrissey, J., Rice, J., 2000. Perturbative simulations of crack front waves.
984 *Journal of the mechanics and physics of solids* 48 (6-7), 1229 – 1251.
- 985 Movchan, A. B., Gao, H., Willis, J. R., 1998. On perturbations of plane
986 cracks. *International Journal of Solids and Structures* 35 (26-27), 3419–
987 3453.

- 988 Mower, T., Argon, A., 1995. Experimental investigations of crack trapping
989 in brittle heterogeneous solids. *Mechanics of Materials* 19 (4), 343–64.
- 990 Nguyen, Q.-S., 2000. *Stability and nonlinear solid mechanics*. John Wiley.
- 991 Obrezanova, O., Movchan, A., Willis, J., 2002a. Dynamic stability of a prop-
992 agating crack. *Journal of the mechanics and physics of solids* 50 (12), 2637–
993 2668.
- 994 Obrezanova, O., Movchan, A., Willis, J., 2002b. Stability of an advancing
995 crack to small perturbation of its path. *Journal of the Mechanics and*
996 *Physics of Solids* 50 (1), 57–80.
- 997 Paris, P., Gomez, M., Anderson, W., 1961. A rational analytic theory of
998 fatigue. *Trends Eng.* 13, 94.
- 999 Perrin, G., Rice, J. R., 1994. Disorder of a dynamic planar crack front
1000 in a model elastic medium of randomly variable toughness. *Journal of the*
1001 *Mechanics and Physics of Solids* 42 (6), 1047–1064.
- 1002 Piccolroaz, A., Mishuris, G., Movchan, A. B., 2007. Evaluation of the
1003 Lazarus-Leblond constants in the asymptotic model of the interfacial wavy
1004 crack. *Journal of the Mechanics and Physics of Solids* 55 (8), 1575–1600.
- 1005 Pindra, N., Lazarus, V., Leblond, J.-B., 2008. The deformation of the front of
1006 a 3d interface crack propagating quasistatically in a medium with random
1007 fracture properties. *Journal of the Mechanics and Physics of Solids* 56 (4),
1008 1269–1295.
- 1009 Pindra, N., Lazarus, V., Leblond, J.-B., 2010a. Geometrical disorder of
1010 the fronts of a tunnel-crack propagating in shear in some heterogeneous
1011 medium. *Journal of the Mechanics and Physics of Solids* 58, 281–299.
- 1012 Pindra, N., Lazarus, V., Leblond, J.-B., 2010b. In-plane perturbation of a
1013 system of two coplanar slit-cracks - I: Case of arbitrarily spaced crack
1014 fronts. *International Journal of Solids and Structures* 47, 3489–3503.
- 1015 Pons, A. J., Karma, A., 2010. Helical crack-front instability in mixed-mode
1016 fracture. *Nature* 464, 85–89.
- 1017 Qian, J., Fatemi, A., 1996. Mixed mode fatigue crack growth: a literature
1018 survey. *Engineering Fracture Mechanics* 55 (6), 969.

- 1019 Ramanathan, S., Fisher, D. S., 1997. Dynamics and instabilities of planar
1020 tensile cracks in heterogeneous media. *Physical Review Letters* 79 (5),
1021 877–80.
- 1022 Ravi-Chandar, K., 1998. Dynamic fracture of nominally brittle materials.
1023 *International journal of fracture* 90 (1-2), 83–102.
- 1024 Rice, J., Ben-Zion, Y., Kim, K., 1994. 3-Dimensional perturbation solution
1025 for a dynamic planar crack moving unsteadily in a model elastic solid.
1026 *Journal of the mechanics and physics of solids* 42 (5), 813–843.
- 1027 Rice, J. R., 1972. Some remarks on elastic crack-tip stress fields. *International*
1028 *Journal of Solids and Structures* 8 (6), 751–758.
- 1029 Rice, J. R., 1985. First-order variation in elastic fields due to variation in lo-
1030 cation of a planar crack front. *ASME Journal of Applied Mechanics* 52 (3),
1031 571–579.
- 1032 Rice, J. R., 1988. Crack fronts trapped by arrays of obstacles: Solutions
1033 based on linear perturbation theory. In: Rosakis, A. J., Ravi-Chandar, K.,
1034 Rajapakse, Y. (Eds.), *Analytical, Numerical and Experimental Aspects of*
1035 *Three Dimensional Fracture Processes*. Vol. 91. ASME Applied Mechanics
1036 Division, American Society of Mechanical Engineers, New York, pp. 175–
1037 184.
- 1038 Rice, J. R., 1989. Weight function theory for three-dimensional elastic crack
1039 analysis. In: Wei, R. P., Gangloff, R. P. (Eds.), *Fracture Mechanics :
1040 Perspectives and Directions (Twentieth Symposium)*. American Society
1041 for Testing and Materials STP 1020, Philadelphia, USA, pp. 29–57.
- 1042 Rosso, A., Krauth, W., 2002. Roughness at the depinning threshold for a
1043 long-range elastic string. *Physical review E* 65 (2), 025101.
- 1044 Roux, S., Vandembroucq, D., Hild, F., 2003. Effective toughness of heteroge-
1045 neous brittle materials. *European Journal of Mechanics - A/Solids* 22 (5),
1046 743–749.
- 1047 Santucci, S., Grob, M., R.Toussaint, Schmittbuhl, J., Hansen, A., Måløy,
1048 K. J., July 2009. Crackling dynamics during the failure of heterogeneous
1049 material: Optical and acoustic tracking of slow interfacial crack growth.
1050 In: *12th International Conference on Fracture (ICF12)*. Ottawa (Canada).

- 1051 Schmittbuhl, J., Delaplace, A., Måløy, K. J., Perfettini, H., Vilotte, J. P.,
1052 2003a. Slow crack propagation and slip correlations. Pure and applied geo-
1053 physics 160 (5-6), 961–976.
- 1054 Schmittbuhl, J., Hansen, A., Batrouni, G., 2003b. Roughness of interfacial
1055 crack fronts: stress-weighted percolation in the damage zone. Physical Re-
1056 view Letters 90 (4), 045505/1–4.
- 1057 Schmittbuhl, J., Måløy, K. J., 1997. Direct observation of a self-affine crack
1058 propagation. Physical Review Letters 78 (20), 3888–91.
- 1059 Schmittbuhl, J., Roux, S., Vilotte, J. P., Maloy, K.-J., 1995. Interfacial crack
1060 pinning: effect of nonlocal interactions. Physical Review Letters 74 (10),
1061 1787–1790.
- 1062 Schmittbuhl, J., Vilotte, J.-P., 1999. Interfacial crack front wandering: influ-
1063 ence of quenched noise correlations. Physica A 270 (1-2), 42–56.
- 1064 Sharon, E., Fineberg, J., JAN 28 1999. Confirming the continuum theory of
1065 dynamic brittle fracture for fast cracks. NATURE 397 (6717), 333–335.
- 1066 Stallybrass, M., 1981. On the concentrated loading of certain elastic half-
1067 space problems and related external crack problems. A new approach. In-
1068 ternational Journal of Engineering Science 19 (8), 1123–1144.
- 1069 Tada, H., Paris, P. C., Irwin, G. R., 1973. The Stress Analysis of Cracks
1070 Handbook. Del Research Corporation, Hellertown, USA.
- 1071 Vieira, A. P., Jose S. Andrade, J., Herrmann, H. J., 2008. Subcritical crack
1072 growth: The microscopic origin of Paris’ law. Physical Review Letters
1073 100 (19), 195503, origine physic de la loi de Paris.
- 1074 Willis, J., Movchan, A., 1997. Three-dimensional dynamic perturbation of a
1075 propagating crack. Journal of the mechanics and physics of solids 45 (4),
1076 591–610.
- 1077 Willis, J. R., Movchan, A. B., 1995. Dynamic weight-functions for a moving
1078 crack. 1. mode-I loading. J. Mech. Phys. Solids 43 (3), 319–341.
- 1079 Woolfries, S., Willis, J. R., 1999. Perturbation of a dynamic planar crack
1080 moving in a model elastic solid. Journal of the Mechanics and Physics of
1081 Solids 47 (8), 1633–1661.