Bridging the Hybrid High-Order and Virtual Element methods

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Abstract

We present a unifying viewpoint at Hybrid High-Order and Virtual Element methods on general polytopal meshes in dimension 2 or 3, both in terms of formulation and analysis. We focus on a model Poisson problem. To build our bridge, (i) we transcribe the (conforming) Virtual Element method into the Hybrid High-Order framework, and (ii) we prove H^m approximation properties for the local polynomial projector in terms of which the local Virtual Element discrete bilinear form is defined. This allows us to perform a unified analysis of Virtual Element/Hybrid High-Order methods, that differs from standard Virtual Element analyses by the fact that the approximation properties of the underlying virtual space are not explicitly used. As a complement to our unified analysis, we also study interpolation in local virtual spaces, shedding light on the differences between the conforming and nonconforming cases.

1 Introduction

The design of arbitrary-order Galerkin methods that support meshes with general polygonal/polyhedral cells has been attracting the attention of the community for more than 40 years now. In practice, the use of general meshes, when not an inherent constraint like e.g. in subsurface modelling, can bring major advantages. In particular, it increases the flexibility in meshing complex geometries (interfaces, cut cells...), and simplifies the refinement/coarsening procedures in adaptive simulations. Standard arbitrary-order polytopal discretisation approaches encompass the (polytopal) Finite Element (FE) method [48, 46], and the (polytopal) discontinuous Galerkin (dG) method [45, 42, 3, 32, 5, 16]. The construction of FE shape functions on arbitrarily-shaped cells that both (i) satisfy the desired conformity prescriptions, and (ii) for which closed-form expressions can be obtained (and numerically integrated), is highly challenging. However, when such shape functions are available, one can fully benefit from the fact that FE belong to the class of skeletal methods. We refer to Section 3 for a precise definition of skeletal methods, but basically they are those methods featuring bulk and skeletal degrees of freedom that are amenable to static condensation (bulk degrees of freedom can be locally eliminated in terms of the skeletal degrees of freedom, hence reducing the global linear system to a system posed in terms of the skeletal unknowns only). On the opposite side of the spectrum, the dG method, which is not (without further modification/hybridisation) a skeletal method, is based on completely nonconforming discrete spaces. One hence has the opportunity to consider simple polynomial local approximation spaces. However, the price to pay for such a flexibility is an increased number of (globally coupled) degrees of freedom, which makes of dG a computationally more expensive method than (statically condensed) FE on standard meshes. This is all the more true that the order of approximation increases. When considering meshes featuring cells with an important number of faces, things are not that clear anymore, and dG may definitely become a competitive computational approach. However, on general cells, the existence of Fortin operators for dG spaces is not clear. This may become limiting when it comes to robustly approximate tricky operators like the divergence (think, e.g., of a linear elasticity model in the quasi-incompressible limit) or curl operators.

More recently, a new paradigm has emerged. The idea is to define a finite element whose construction is generic with respect to the shape of the element. The underlying local approximation space (i) is spanned by functions that are (at least for some of them) implicitly defined (usually as the solutions to some PDEs posed in the cell), (ii) is built so that the desired conformity properties can be obtained at the global level, and (iii) is constructed so as to enjoy sufficient approximation properties (for instance, so as to contain the polynomial functions up to a given degree). The fact that one cannot obtain a closed-form

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expression for all shape functions is the reason why they are called virtual in that context. In practice, the numerical method is defined using computable (in terms of the degrees of freedom) projections of the virtual functions, and is stabilised through a subtle penalisation, that shall also be computable. The most salient example of such an approach is the (polynomial-based) Virtual Element (VE) method [7, 8], which has first been introduced under its conforming version (denoted c-VE). Another example is the Hybrid High-Order (HHO) method [35], first introduced/analysed for linear elasticity [33], and then for the Poisson problem [34]. Whereas the VE method is classically defined in terms of (virtual) functions, the HHO method is directly defined in terms of the degrees of freedom. For this reason, in the sequel, we will refer to the VE framework as "functional", whereas we will refer to the HHO one as "algebraic". In [24, Section 2.4], the HHO method has been proven equivalent (up to identical bulk polynomial degree, choice of stabilisation, and treatment of the right-hand side) to the nonconforming version of the VE method (denoted nc-VE) introduced in [43], and posteriorly analysed in [4]. In [24], the HHO method has also been bridged to the Hybridizable Discontinuous Galerkin (HDG) method [27, 23], in the sense that it is possible to recast the HHO method as a HDG method, with distinctive numerical flux trace. This work has shed light on the fact that the quite subtle choice of stabilisation advocated in HHO results in HDG formulation in a numerical flux trace that ensures superconvergence on general polytopal meshes. Note that efforts towards superconvergence for standard HDG methods (and the hybridised version of mixed methods) have also been undertaken (cf., e.g., [25], which gives a general theory of how to do so). All these methods belong to the class of skeletal methods (cf. Section 3). Finally, in [31], the nc-VE/HHO methods are proven to be Gradient Discretisation methods [37].

Let Ω be a bounded and connected open subset of $\mathbb{R}^d$, $d \in \{2,3\}$, whose boundary is assumed to be composed of a finite union of portions of affine hyperplanes. We focus on the following model Poisson problem: find $u \in H^1_0(\Omega)$ solution to

$$a(u,v) := \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = \int_\Omega f\, v =: l(v) \qquad \text{for all } v \in H^1_0(\Omega), \tag{1}$$

with source term $f \in L^2(\Omega)$. In this work, we complete the construction of the bridge between HHO and VE undertaken in [24, Section 2.4]. To do so, (i) we transcribe the c-VE method into the HHO algebraic framework, and (ii) we prove H^m approximation properties for the local polynomial projector in terms of which the local VE discrete bilinear form is defined. This allows us to perform a unified study of VE/HHO methods, that differs from standard VE analyses by the fact that the approximation properties of the underlying virtual space are not explicitly used. We build upon existing contributions, especially [30] on the analysis of schemes in fully discrete formulation, [17] on the unified analysis of c/nc-VE methods, and [44, 13, 19] (see also [10, 15, 18] for the treatment of faces with arbitrarily small measure, that our mesh assumptions will forbid in the present work) on the analysis of c-VE. We consider throughout this paper standard VE spaces (i.e., neither enhanced VE spaces [1], nor Serendipity VE spaces [9]), with bulk polynomial degree $k-1$ ($k \geqslant 1$), which simplifies the treatment of the lowest-order case without compromising the computational efficiency (since bulk unknowns are statically condensed). The aim of the present work is threefold. First, the transcription of the VE method into the HHO framework (and its analysis) is intended to contribute in a better understanding, by those more familiar with VE, of the HHO standpoint, and reciprocally. Second, we believe that the derivation, in the conforming case, of H^m approximation properties for the local polynomial projector in terms of which the local VE discrete bilinear form is defined may be of interest (up to its adaptation to a more general L^p setting) for the VE analysis of nonlinear problems. Finally, and as a complement to our unified analysis, we aim at clarifying what are the differences in terms of interpolation between the conforming and nonconforming cases.

The main results of this paper are contained in Sections 4 and 5.

In Section 4, we focus on interpolation in local virtual spaces. We study, in the conforming and nonconforming cases, the approximation properties of the local canonical interpolation operator (denoted $\mathcal{I}_T$). We derive L^2-norm and H^1-seminorm approximation results for $\mathcal{I}_T$ (cf. Theorems 4.13 and 4.19). In the conforming case, such results are not new. They can be found in [13]. Nevertheless, we include them for two reasons. First, we propose a new path to derive them. In particular, our analysis does not explicitly hinge on an inverse inequality for virtual functions as it is the case in [13]. Second, we make a comparison between the conforming and nonconforming cases. A crucial observation that is made clear in this paper is the following. The main difference between those two cases is that, in the nonconforming case, the local canonical interpolation operator $\mathcal{I}_T$ is the elliptic projector onto the local virtual space (cf. Eq. (33)). This is not true in the conforming case, which happens to complicate the analysis. In this latter case, in order to prove stability properties for $\mathcal{I}_T$ (which is actually the cornerstone of the analysis), one has to estimate a dual norm of the boundary normal flux of virtual functions (cf. Lemmas 4.7 and 4.12, and Remark 4.8).

For that, one has to make the following assumption.

Assumption 1.1. *For any function v on ∂T that is equal to the trace of a virtual function, there exists a lifting $\mathcal{L}_T v \in H^1(T)$ such that $\mathcal{L}_T v_{|\partial T} = v$, and satisfying the following scaled estimate:*

$$h_T^{-1}\|\mathcal{L}_T v\|_T + \|\boldsymbol{\nabla}\mathcal{L}_T v\|_T \leqslant c\left(h_T^{-1/2}\|v\|_{\partial T} + h_T^{1/2}\|\boldsymbol{\nabla}_t v\|_{\partial T}\right), \tag{2}$$

where $\boldsymbol{\nabla}_t$ denotes the tangential derivative, and $c > 0$ is a constant independent of h_T.

When T is star-shaped with respect to a ball of radius comparable to h_T, the result (2) does hold (cf. [13, Eq. (2.48) for $d = 2$ and Lemma 5.3 for $d = 3$]). When T is not star-shaped, which is a case our mesh assumptions will allow, we are not aware of a proof of (2). In the Section 4 of [13] dedicated to the relaxation of the star-shapedness assumption, this tricky aspect (especially in 3D) is eluded. In [19], the star-shapedness assumption is weakened, but the arguments are restricted to the 2D case, and to mesh assumptions that are a bit less general than those we consider here. We will hence keep (2) as an assumption of our analysis in the conforming case (cf. Remark 4.8). As opposed to the conforming case, in the nonconforming one, owing to the fact that $\mathcal{I}_T$ is the elliptic projector onto the local virtual space, an estimate such as (2) is not needed to prove stability for $\mathcal{I}_T$ (cf. Lemma 4.17 and Remark 4.18).

In Section 5, we perform our unified analysis of VE/HHO methods. Letting Π_T denote the elliptic projector onto the polynomial subspace $\mathbb{P}(T)$ of the local virtual space, the local polynomial projector $\mathcal{P}_T$ in terms of which the local VE discrete bilinear form is defined is actually equal to $\Pi_T \circ \mathcal{I}_T$. Splitting the error in broken H^1-seminorm (recall that the only computable/computed quantity is $\mathcal{P}_h u_h = \Pi_h u_h$) along

$$\|\boldsymbol{\nabla}_h(u - \Pi_h u_h)\| \leqslant \|\boldsymbol{\nabla}_h(u - \mathcal{P}_h u)\| + \|\boldsymbol{\nabla}_h\Pi_h(\mathcal{I}_h u - u_h)\|, \tag{3}$$

one is left with estimating the two terms in the right-hand side of (3). The first term is an approximation error, whereas the second is bounded by the consistency error of the scheme (discrete energy-norm error); cf. [30, Section 2.4.1]. In standard c-VE analyses (including [30, Theorem 19]), the first term in the right-hand side is split along

$$\|\boldsymbol{\nabla}_h(u - \mathcal{P}_h u)\| \leqslant \|\boldsymbol{\nabla}_h(u - \Pi_h u)\| + \|\boldsymbol{\nabla}_h\Pi_h(u - \mathcal{I}_h u)\| \leqslant \|\boldsymbol{\nabla}_h(u - \Pi_h u)\| + \|\boldsymbol{\nabla}_h(u - \mathcal{I}_h u)\|,$$

where one uses the stability property of Π_h. Similar splittings are deployed to take care of the consistency term. With such splittings, the approximation properties of the local virtual spaces invite themselves into the picture. To perform the analysis, one thus has to use the interpolation results of Section 4. In the nonconforming case, since $\mathcal{I}_T$ is the elliptic projector onto the local virtual space, one actually has $\mathcal{P}_T = \Pi_T$ (this has already been pointed out in [30, Remark 25]). The analysis hence inherently simplifies, as one can conclude by standard approximation results on Π_h. This explains why the approximation properties of the underlying virtual space do not appear explicitly in standard HHO error bounds. The virtual space is anyway not even introduced. In this article, we directly investigate the approximation properties of the polynomial projector $\mathcal{P}_T$ (especially in the conforming case), for which we prove H^m approximation properties (cf. Theorem 5.4). Interestingly, Assumption 1.1 is not needed to prove so (cf. Remark 5.5). Irrespectively of the conformity of the underlying global virtual space, we then split the error in broken H^1-seminorm along (3), and we perform the analysis only using the H^m approximation properties of $\mathcal{P}_h$. We end up with a factorised analysis inspired from that of HHO methods (cf. Theorem 5.12 and Corollary 5.14), that differs from standard VE ones by the fact that the approximation properties of the underlying virtual space are not explicitly used. We also perform, in the same vein, a factorised L^2-norm analysis (cf. Theorem 5.16 and Corollary 5.17).

The material is organized as follows. In Section 2, we introduce the notation, we detail our admissibility assumptions on mesh sequences, and we introduce a number of analysis tools that will be useful in the sequel. In Section 3, we undertake a general description of skeletal methods, such as the VE or HHO methods. In Section 4, we study interpolation in local virtual spaces. Finally, in Section 5, we provide our unified formulation/study of VE/HHO methods, as well as a description of the general workflow of the methods.

2 Notation, mesh assumptions, and basic analysis tools

We collect in this section all the conventions, tools, and results that will be useful in the sequel.

2.1 Notation

2.1.1 Geometry

For $l \in \{1, \dots, d\}$, we let $|\cdot|_l$ denote the l-dimensional Hausdorff measure. In what follows, the term polytope refers to polygons if $d = 2$, and to polyhedra if $d = 3$. The discretisation of the domain Ω is described in the following manner.

- $\mathcal{T}_h$ denotes a mesh of the domain Ω, i.e. a collection of disjoint open polytopes T (the cells) such that $\bigcup_{T\in\mathcal{T}_h} \overline{T} = \overline{\Omega}$. The parameter h is the meshsize, defined as $h := \max_{T\in\mathcal{T}_h} h_T$, where h_T stands for the diameter of the cell T.

- $\mathcal{F}_h$ denotes the collection of faces of the mesh $\mathcal{T}_h$. Since the cells of $\mathcal{T}_h$ are polytopes, their boundary is composed of a finite union of (closed) portions of affine hyperplanes, called facets. A closed subset F of $\overline{\Omega}$ with $|F|_{d-1} \neq 0$ is a face as soon as (i) F is equal to the intersection, for T_1, T_2 two cells of $\mathcal{T}_h$, of a facet of T_1 and a facet of T_2, or (ii) F is equal to the intersection, for T cell of $\mathcal{T}_h$, of a facet of T and a facet of Ω. In the first case, F is termed an interface, whereas in the second, F is termed a boundary face. Interfaces are collected in the set $\mathcal{F}_h^{\mathrm{i}}$, boundary faces in the set $\mathcal{F}_h^{\mathrm{b}}$, in such a way that $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\mathrm{b}}$. For a cell $T \in \mathcal{T}_h$, we let $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ be the collection of faces composing its boundary, and $\boldsymbol{n}_T$ be the unit normal vector to ∂T pointing outward T (that is defined almost everywhere on ∂T). For $F \in \mathcal{F}_T$, we also let $\boldsymbol{n}_{T,F} := \boldsymbol{n}_{T|F}$; remark that $\boldsymbol{n}_{T,F}$ is a constant vector on F since F is planar. Finally, for any face $F \in \mathcal{F}_h$, we denote h_F its diameter (and, for $T \in \mathcal{T}_h$, we let $h_{\partial T}$ be such that $h_{\partial T|F} := h_F$ for all $F \in \mathcal{F}_T$), and we let $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$ be the collection of cells sharing F (two cells for an interface, one for a boundary face).

- $\partial\mathcal{T}_h$ denotes the $(d-1)$-dimensional skeleton of the mesh $\mathcal{T}_h$, that is $\partial\mathcal{T}_h = \bigcup_{F\in\mathcal{F}_h} F$.

- When $d = 3$, $\mathcal{E}_h$ denotes the collection of edges e, $|e|_1 \neq 0$, of the mesh $\mathcal{T}_h$, defined from the collection $\mathcal{F}_h$ of faces. For a cell $T \in \mathcal{T}_h$ (respectively a face $F \in \mathcal{F}_h$), we let $\mathcal{E}_T := \{e \in \mathcal{E}_h \mid e \subset \partial T\}$ (respectively $\mathcal{E}_F := \{e \in \mathcal{E}_h \mid e \subset \partial F\}$, that will also be denoted $\mathcal{F}_F$ with a slight abuse in notation) be the collection of edges composing its boundary.

- $\mathcal{V}_h$ denotes the collection of vertices ν of the mesh $\mathcal{T}_h$. For a cell $T \in \mathcal{T}_h$ (respectively a face $F \in \mathcal{F}_h$), we let $\mathcal{V}_T := \{\nu \in \mathcal{V}_h \mid \nu \in \partial T\}$ (respectively $\mathcal{V}_F := \{\nu \in \mathcal{V}_h \mid \nu \in \partial F\}$) be the collection of its vertices. The position of any vertex $\nu \in \mathcal{V}_h$ is denoted $\boldsymbol{x}_\nu \in \Omega$.

When $d = 2$, faces are sometimes called edges in the literature. We will not use this vocable in this article. The term edge will always refer to a 1-manifold in dimension $d = 3$. We finally introduce, for $\nu \in \mathcal{V}_h$, the set $\mathcal{F}_\nu := \{F \in \mathcal{F}_h \mid \nu \in \mathcal{V}_F\}$ and, when $d = 3$, for $e \in \mathcal{E}_h$, the set $\mathcal{F}_e := \{F \in \mathcal{F}_h \mid e \in \mathcal{E}_F\}$.

2.1.2 Functions spaces

For $X \subset \overline{\Omega}$, and $m \geqslant 0$, we let $|\cdot|_{m,X}$ and $\|\cdot\|_{m,X}$ respectively denote the seminorm and norm on the Sobolev space $H^m(X;\mathbb{R}^l)$, $l \in \{1, d\}$, with the convention that $H^0(X;\mathbb{R}^l) \equiv L^2(X;\mathbb{R}^l)$ (hence, $|\cdot|_{0,X} \equiv \|\cdot\|_{0,X}$). We also define $\|\cdot\|_{\infty,X}$ as the norm on $L^\infty(X)$. We finally let $\langle\cdot,\cdot\rangle_{-m,X}$ be the duality pairing between $H^{-m}(X)$ and its topological dual.

For $q \in \mathbb{N}$ and $l \in \{1, \dots, d\}$, we let $\mathbb{P}_l^q$ be the vector space of l-variate polynomial functions of total degree less than or equal to q. We also let

$$\mathrm{N}_l^q := \dim(\mathbb{P}_l^q) = \binom{q+l}{q},$$

and we adopt the conventions $\mathbb{P}_l^{-1} = \{0\}$ and $\mathrm{N}_l^{-1} = 0$. For $T \in \mathcal{T}_h$, we define $\mathbb{P}_d^q(T)$ as the restriction of $\mathbb{P}_d^q$ to T. For $F \in \mathcal{F}_h$, we let $\mathbb{P}_{d-1}^q(F)$ be the restriction of $\mathbb{P}_d^q$ to F. When $d = 3$, for $e \in \mathcal{E}_h$, we let $\mathbb{P}_1^q(e)$ be the restriction of $\mathbb{P}_3^q$ to e. For $T \in \mathcal{T}_h$, we also define the broken space

$$\mathbb{P}_{d-1}^q(\mathcal{F}_T) := \left\{v \in L^2(\partial T) \mid v_{|F} \in \mathbb{P}_{d-1}^q(F)\ \forall F \in \mathcal{F}_T\right\},$$

and when $d = 3$, for $F \in \mathcal{F}_h$, we let $\mathbb{P}_1^q(\mathcal{E}_F) := \{v \in L^2(\partial F) \mid v_{|e} \in \mathbb{P}_1^q(e)\ \forall e \in \mathcal{E}_F\}$. We finally introduce, for any $T \in \mathcal{T}_h$, a set of basis functions for $\mathbb{P}_d^q(T)$, that we denote $\{\psi_{T,i}^q\}_{i\in\{1,\dots,\mathrm{N}_d^q\}}$, and for any $F \in \mathcal{F}_h$, a set of basis functions for $\mathbb{P}_{d-1}^q(F)$, that we denote $\{\psi_{F,j}^q\}_{j\in\{1,\dots,\mathrm{N}_{d-1}^q\}}$. When $d = 3$, we further introduce, for any $e \in \mathcal{E}_h$, a set of basis functions for $\mathbb{P}_1^q(e)$, denoted $\{\psi_{e,m}^q\}_{m\in\{1,\dots,\mathrm{N}_1^q\}}$.

Given a mesh $\mathcal{T}_h$ of Ω, we introduce the following notation for broken functions spaces on $\mathcal{T}_h$:

$$X(\mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_{h|T} \in X(T)\, \forall T \in \mathcal{T}_h\}.$$

We introduce on $H^1(\mathcal{T}_h)$ the so-called broken gradient operator $\boldsymbol{\nabla}_h : H^1(\mathcal{T}_h) \to L^2(\Omega)$ such that, for any $v_h \in H^1(\mathcal{T}_h)$ and $T \in \mathcal{T}_h$, $\boldsymbol{\nabla}_h v_{h|T} := \boldsymbol{\nabla}(v_{h|T})$. For any interface $F \in \mathcal{F}_h^{\mathrm{i}}$ with $\mathcal{T}_F = \{T_1, T_2\}$, we define the jump $[\![v_h]\!]_F$ along F of $v_h \in H^s(\mathcal{T}_h)$, $s > 1/2$, by $[\![v_h]\!]_F := (v_{h|T_1})_{|F} - (v_{h|T_2})_{|F}$, and we let $\boldsymbol{n}_F := \boldsymbol{n}_{T_1,F}$. For any $F \in \mathcal{F}_h^{\mathrm{b}}$ with $\mathcal{T}_F = \{T\}$, we let $[\![v_h]\!]_F := (v_{h|T})_{|F}$, and $\boldsymbol{n}_F := \boldsymbol{n}_{T,F}$. We finally introduce the operator $[\![\cdot]\!] : H^s(\mathcal{T}_h) \to L^2(\partial\mathcal{T}_h)$ such that, for any $v_h \in H^s(\mathcal{T}_h)$, $[\![v_h]\!]_{|F} := [\![v_h]\!]_F$ for all $F \in \mathcal{F}_h$. Assume that $v_h \in H^s(\mathcal{T}_h)$ is such that $v_{h|T} \in C^0(\overline{T})$ for all $T \in \mathcal{T}_h$. Then, the quantity $[\![v_h]\!]$ is piecewise continuous on the skeleton, with (potential) discontinuities at vertices when $d = 2$ and on edges/vertices when $d = 3$. Indeed, considering a vertex $\nu \in \mathcal{V}_h$, there are $\mathrm{card}(\mathcal{F}_\nu)$ potentially different values for $[\![v_h]\!]$ at $\boldsymbol{x}_\nu$, that are the $([\![v_h]\!]_F(\boldsymbol{x}_\nu))_{F\in\mathcal{F}_\nu}$. When $d = 3$, considering an edge $e \in \mathcal{E}_h$, there are, identically, $\mathrm{card}(\mathcal{F}_e)$ potentially different functions $[\![v_h]\!]$ on e, that are the $\big([\![v_h]\!]_{F|e}\big)_{F\in\mathcal{F}_e}$.

2.2 Mesh assumptions

We define the notion of admissible mesh family.

Definition 2.1. *The mesh family $(\mathcal{T}_h)_h$ is admissible if, for all h, $\mathcal{T}_h$ admits a matching simplicial submesh, denoted $\mathcal{S}_h$, and there exists $\gamma > 0$, called mesh regularity parameter, so that, for all h,*

(i) for all $S \in \mathcal{S}_h$ of diameter h_S and inradius r_S, $\gamma h_S \leqslant r_S$ (in other words, $\mathcal{S}_h$ is shape-regular);

(ii) for all $T \in \mathcal{T}_h$, and all $S \in \mathcal{S}_T := \{S \in \mathcal{S}_h \mid S \subseteq T\}$, $\gamma h_T \leqslant h_S$.

By matching simplicial submesh, we mean that $\mathcal{S}_h$ is a (hanging node free) simplicial mesh satisfying: for all $S \in \mathcal{S}_h$, there exists a unique $T \in \mathcal{T}_h$ such that $S \subseteq T$, and for all $Z \in \mathcal{Z}_h$, where $\mathcal{Z}_h$ collects the faces of $\mathcal{S}_h$, there exists at most one $F \in \mathcal{F}_h$ such that $Z \subseteq F$ (cf. [32, Definition 1.37]). Henceforth, we will use the symbol $\lesssim$ to indicate that an estimate is valid up to a multiplicative constant $c > 0$, with c only depending on the dimension d, the mesh regularity parameter γ, and, if need be, the underlying polynomial degree; in particular, the bound is uniform with respect to the meshsize.

Let us mention three important consequences of Definition 2.1: for all h, and all $T \in \mathcal{T}_h$,

(a) for all $S \in \mathcal{S}_T$, $\gamma h_T \leqslant h_S \leqslant h_T$, and $\mathrm{card}(\mathcal{S}_T) \lesssim 1$ (cf. [32, Lemma 1.40]);

(b) for all $F \in \mathcal{F}_T$, $\mathrm{card}(\mathcal{Z}_F) \lesssim 1$, where $\mathcal{Z}_F := \{Z \in \mathcal{Z}_h \mid Z \subseteq F\}$ (cf. [32, Lemma 1.41]);

(c) for all $F \in \mathcal{F}_T$, $\gamma^2 h_T \leqslant h_F \leqslant h_T$, and $\mathrm{card}(\mathcal{F}_T) \lesssim 1$ (cf. [32, Lemmas 1.42 and 1.41]).

From (a) and (b), one can picture the general outline to prove inverse and trace inequalities on arbitrarily-shaped (admissible) cells. One first considers the case of a simplex satisfying (i) of Definition 2.1, for which these inequalities are standard. Then, the passage to arbitrary geometries follows from the fact that any admissible cell is composed of an uniformly bounded number of simplices satisfying (i) (and, any admissible face is composed of an uniformly bounded number of subfaces belonging to simplices satisfying (i)), and whose diameters are comparable to the diameter of the cell under consideration (cf. Section 2.3.1). Note that the notion of admissible mesh we consider here allows for cells that are not necessarily star-shaped. The assumptions of Definition 2.1 are also sufficient to prove H^m approximation properties for standard polynomial projectors (cf. Section 2.3.3). As for point (c), it is instrumental in the analysis of numerical methods based on $\mathcal{T}_h$. When $d = 3$, it is an easy matter to show that, under the assumptions of Definition 2.1, one also has, for any $T \in \mathcal{T}_h$, $F \in \mathcal{F}_T$, and $e \in \mathcal{E}_F$, $\gamma^3 h_F \leqslant h_e \leqslant h_F$, and $\mathrm{card}(\mathcal{E}_F) \lesssim 1$.

2.3 Basic analysis tools

Henceforth, $\mathcal{T}_h$ denotes a member of an admissible mesh family in the sense of Definition 2.1.

2.3.1 Useful inequalities

On any $T \in \mathcal{T}_h$, the following inequalities hold:

- *inverse inequality:*

$$\forall v \in \mathbb{P}_d^q(T), \qquad |v|_{1,T} \lesssim h_T^{-1} \|v\|_{0,T}; \tag{4}$$

- *continuous trace inequality:* for any $F \in \mathcal{F}_T$,

$$\forall v \in H^1(T), \qquad \|v_{|F}\|_{0,F} \lesssim h_T^{-1/2}\|v\|_{0,T} + h_T^{1/2}|v|_{1,T}; \tag{5}$$

- *discrete trace inequality:* for any $F \in \mathcal{F}_T$,

$$\forall v \in \mathbb{P}_d^q(T), \qquad \|v_{|F}\|_{0,F} \lesssim h_T^{-1/2}\|v\|_{0,T}. \tag{6}$$

For the proofs of these different results, we refer to [32, Section 1.4.3] (note that therein, faces may even be nonplanar). We also state the classical Poincaré inequality:

$$\forall v \in H^1(T) \text{ such that } \int_T v = 0, \qquad \|v\|_{0,T} \leqslant c_{\mathrm{P}} h_T |v|_{1,T}. \tag{7}$$

If T is convex, $c_{\mathrm{P}} = \pi^{-1}$ is optimal, independently of the ambient dimension; cf. [6]. For some insight on the value of c_{P} on more general element shapes, we refer to [47]. In the forthcoming analysis, we will also need (i) the following nonstandard inverse and discrete trace inequalities, whose proofs can be found in [22, Lemma 4.4 (take $\mathbb{A}_\varepsilon = \mathbb{I}_d$)]: for all $v \in H^1(T)$ such that $\triangle v \in \mathbb{P}_d^q(T)$ for some $q \in \mathbb{N}$, there holds

$$\|\triangle v\|_{0,T} \lesssim h_T^{-1}|v|_{1,T}; \tag{8}$$

if, in addition, for some $F \in \mathcal{F}_T$, $\boldsymbol{\nabla} v_{|F} \cdot \boldsymbol{n}_{T,F} \in \mathbb{P}_{d-1}^q(F)$, then there also holds

$$\|\boldsymbol{\nabla} v_{|F} \cdot \boldsymbol{n}_{T,F}\|_{0,F} \lesssim h_T^{-1/2}|v|_{1,T}; \tag{9}$$

(ii) the following version of Sobolev inequality (cf., e.g., [14, Lemma 4.3.4]):

$$\forall v \in H^2(T), \qquad \|v\|_{\infty,T} \lesssim h_T^{-\frac{d}{2}}\|v\|_{0,T} + h_T^{\frac{(2-d)}{2}}|v|_{1,T} + h_T^{\frac{(4-d)}{2}}|v|_{2,T}; \tag{10}$$

and (iii) the following estimate on a dual norm of the boundary normal flux, whose proof is postponed until Appendix A.1, that is valid under Assumption 1.1: for any $v \in H^1(T)$ such that $\triangle v \in L^2(T)$,

$$\sup_{z \in \mathfrak{T}(\partial T)\setminus\{0\}} \frac{\langle \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T, z\rangle_{-\frac{1}{2},\partial T}}{h_T^{-1/2}\|z\|_{0,\partial T} + h_T^{1/2}|z|_{1,\partial T}} \lesssim |v|_{1,T} + h_T\|\triangle v\|_{0,T}, \tag{11}$$

where $\mathfrak{T}(\partial T) \subset H^1(\partial T)$ denotes the space of traces on ∂T of virtual functions (to be made precise in each situation) that is referred to in Assumption 1.1.

2.3.2 Finite element in the sense of Ciarlet

The following definition is directly inspired from [20, p. 94]. Let $l \in \{1, \dots, d\}$.

Definition 2.2. *A finite element consists in a triple $(X, V(X), \Sigma_X)$ where*

- *X is a bounded and connected Lipschitz subset of $\mathbb{R}^d$ such that $|X|_l \neq 0$;*
- *$V(X)$ is a finite-dimensional vector space of functions $v : X \to \mathbb{R}$;*
- *$\Sigma_X := \{\sigma_X^1, \dots, \sigma_X^{n_X}\}$, $n_X \in \mathbb{N}^\star$, is a collection of linear forms on $V(X)$ such that the mapping*

$$\underline{\Sigma}_X : V(X) \ni v \mapsto (\sigma_X^1(v), \dots, \sigma_X^{n_X}(v))^\intercal \in \mathbb{R}^{n_X}$$

is bijective (we then have $\dim(V(X)) = n_X$).

The operator $\underline{\Sigma}_X$ is the so-called (local) reduction operator, and $\underline{\Sigma}_X(v)$ is the so-called vector of (local) degrees of freedom (DoF). The bijectivity of $\underline{\Sigma}_X$ is in general referred to as unisolvence in the literature. The following proposition is a direct consequence of the unisolvence property.

Proposition 2.3. *Let $(X, V(X), \Sigma_X)$ be a finite element. There exists a basis $\{\varphi_{X,1}, \dots, \varphi_{X,n_X}\}$ (referred to as canonical) of $V(X)$ such that $\sigma_X^i(\varphi_{X,j}) = \delta_{ij}$ for all $i, j \in \{1, \dots, n_X\}$. The $(\varphi_{X,i})_{i\in\{1,\dots,n_X\}}$ are the so-called (local) shape functions.*

Let $r_X : \mathbb{R}^{n_X} \to V(X)$ be the operator such that, for $\underline{\mathrm{v}}_X := (\mathrm{v}_X^i)_{i\in\{1,\dots,n_X\}}^\intercal \in \mathbb{R}^{n_X}$, $r_X\underline{\mathrm{v}}_X := \sum_{i=1}^{n_X} \mathrm{v}_X^i \varphi_{X,i}$. One can easily remark that, for all $\underline{\mathrm{v}}_X \in \mathbb{R}^{n_X}$, $\underline{\Sigma}_X(r_X\underline{\mathrm{v}}_X) = \underline{\mathrm{v}}_X$, hence $r_X = \underline{\Sigma}_X^{-1}$ and, for any $v \in V(X)$,

$$v = r_X(\underline{\Sigma}_X(v)) = \sum_{i=1}^{n_X} \sigma_X^i(v)\,\varphi_{X,i}.$$

r_X is the so-called (local, canonical) reconstruction operator.

Assume that there exists a normed vector space $W(X)$ of functions $w : X \to \mathbb{R}$ such that (i) $V(X) \subset W(X)$, and such that (ii) every linear form σ_X^i of Σ_X can be extended as a linear form $\overline{\sigma}_X^i$ on $W(X)$. We denote by $\overline{\Sigma}_X$ this new collection of extended linear forms, and by $\underline{\overline{\Sigma}}_X$ the corresponding (local) reduction operator. We can then introduce the operator $\mathcal{I}_X : W(X) \to V(X)$ such that, for any $w \in W(X)$, $\mathcal{I}_X w := r_X(\underline{\overline{\Sigma}}_X(w)) = \sum_{i=1}^{n_X} \overline{\sigma}_X^i(w)\varphi_{X,i}$. With such a definition, there holds $\underline{\Sigma}_X(\mathcal{I}_X w) = \underline{\overline{\Sigma}}_X(w)$. The operator $\mathcal{I}_X$ is the so-called (local, canonical) interpolation operator (in a broad sense). Of course, for all $w \in V(X)$, $\mathcal{I}_X w = w$ since $\underline{\overline{\Sigma}}_{X|V(X)} = \underline{\Sigma}_X = r_X^{-1}$.

2.3.3 Standard polynomial projectors

For $T \in \mathcal{T}_h$, we define the L^2-orthogonal and elliptic projectors, respectively $\pi_T^q : L^2(T) \to \mathbb{P}_d^q(T)$ and $\Pi_T^q : H^1(T) \to \mathbb{P}_d^q(T)$, so that

$$\text{for all } v \in L^2(T), \qquad \int_T \pi_T^q v\, w = \int_T v\, w \qquad \forall w \in \mathbb{P}_d^q(T);$$

$$\text{for all } v \in H^1(T), \qquad \begin{cases} \displaystyle\int_T \boldsymbol{\nabla}\Pi_T^q v\cdot\boldsymbol{\nabla} w = \int_T \boldsymbol{\nabla} v\cdot\boldsymbol{\nabla} w \qquad \forall w \in \mathbb{P}_d^q(T), \\ \displaystyle\int_T \Pi_T^q v = \int_T v. \end{cases} \tag{12}$$

Remark that $\Pi_T^0 = \pi_{T|H^1(T)}^0$. For $l \in \{1,\dots,\mathrm{N}_d^q\}$, we let $\pi_T^{q,l}$ denote the linear form on $L^2(T)$ such that, for any $v \in L^2(T)$, $\pi_T^{q,l}(v) \in \mathbb{R}$ is the l^{th} coordinate of $\pi_T^q v$ on the basis $\{\psi_{T,i}^q\}_{i\in\{1,\dots,\mathrm{N}_d^q\}}$ of $\mathbb{P}_d^q(T)$.

Proposition 2.4 (Properties of π_T^q and Π_T^q)**.** *Let $T \in \mathcal{T}_h$ and $v \in L^2(T)$.*

- preservation of polynomials: *if $v \in \mathbb{P}_d^q(T)$, $\pi_T^q v = \Pi_T^q v = v$;*
- stability: $\|\pi_T^q v\|_{0,T} \leqslant \|v\|_{0,T}$ *and, if $v \in H^1(T)$,* $\|\Pi_T^q v\|_{0,T} \leqslant \|v\|_{0,T} + 2c_{\mathrm{P}}\, h_T |v|_{1,T}$;
- optimality: $\|v - \pi_T^q v\|_{0,T} = \min\limits_{z\in\mathbb{P}_d^q(T)} \|v - z\|_{0,T}$ *and, if $v \in H^1(T)$,* $|v - \Pi_T^q v|_{1,T} = \min\limits_{z\in\mathbb{P}_d^q(T)} |v - z|_{1,T}$;
- H^m approximation: *for $v \in H^s(T)$, $s \in \{1,\dots,q+1\}$, there holds, with $\varpi_T^q \in \{\pi_T^q; \Pi_T^q\}$,*

$$|v - \varpi_T^q v|_{m,T} \lesssim h_T^{s-m} |v|_{s,T} \qquad \text{for } m \in \{0,\dots,s\}, \tag{13}$$

and, for any $F \in \mathcal{F}_T$, and any $(\zeta_1,\dots,\zeta_d) \in \mathbb{N}^d$ such that $\sum_{i=1}^d \zeta_i = m$,

$$\left\|[\partial_{x_1}^{\zeta_1}\dots\partial_{x_d}^{\zeta_d}(v - \varpi_T^q v)]_{|F}\right\|_{0,F} \lesssim h_T^{s-m-1/2} |v|_{s,T} \qquad \text{for } m \in \{0,\dots,s-1\}. \tag{14}$$

The proof of the last result of Proposition 2.4 relies on (i) the ideas of [14, Chapter 4] and [38, Section 7] (cf. also [13, Section 4]), and on (ii) two important features of ϖ_T^q, that are the preservation of polynomials and its stability in the $L^2(T)$-norm. For a detailed proof, and an extension to more general Sobolev seminorms, we refer to [28] (L^2-orthogonal projector) and [29] (elliptic projector). We also introduce $\pi_h^q : L^2(\Omega) \to \mathbb{P}_d^q(\mathcal{T}_h)$ and $\Pi_h^q : H^1(\mathcal{T}_h) \to \mathbb{P}_d^q(\mathcal{T}_h)$ such that, for any $v \in L^2(\Omega)$ (resp., $v \in H^1(\mathcal{T}_h)$), $\pi_h^q v_{|T} := \pi_T^q(v_{|T})$ (resp., $\Pi_h^q v_{|T} := \Pi_T^q(v_{|T})$) for all $T \in \mathcal{T}_h$.

For $F \in \mathcal{F}_h$, we define $\pi_F^q : L^2(F) \to \mathbb{P}_{d-1}^q(F)$ so that, for all $v \in L^2(F)$,

$$\int_F \pi_F^q v\, w = \int_F v\, w \qquad \forall w \in \mathbb{P}_{d-1}^q(F).$$

For $l \in \{1,\dots,\mathrm{N}_{d-1}^q\}$, we let $\pi_F^{q,l}$ denote the linear form on $L^2(F)$ such that, for any $v \in L^2(F)$, $\pi_F^{q,l}(v) \in \mathbb{R}$ is the l^{th} coordinate of $\pi_F^q v$ on the basis $\{\psi_{F,j}^q\}_{j\in\{1,\dots,\mathrm{N}_{d-1}^q\}}$ of $\mathbb{P}_{d-1}^q(F)$. We also define, for $T \in \mathcal{T}_h$, $\pi_{\partial T}^q$:

$L^2(\partial T) \to \mathbb{P}^q_{d-1}(\mathcal{F}_T)$ such that, for any $v \in L^2(\partial T)$, $\pi^q_{\partial T} v_{|F} := \pi^q_F(v_{|F})$ for all $F \in \mathcal{F}_T$. When $d = 3$, for $e \in \mathcal{E}_h$, we define $\pi^q_e : L^2(e) \to \mathbb{P}^q_1(e)$ so that, for all $v \in L^2(e)$,

$$\int_e \pi^q_e v\, w = \int_e v\, w \qquad \forall w \in \mathbb{P}^q_1(e).$$

For $l \in \{1, \ldots, \mathrm{N}^q_1\}$, we let $\pi^{q,l}_e$ denote the linear form on $L^2(e)$ such that, for any $v \in L^2(e)$, $\pi^{q,l}_e(v) \in \mathbb{R}$ is the l^{th} coordinate of $\pi^q_e v$ on the basis $\{\psi^q_{e,m}\}_{m\in\{1,\ldots,\mathrm{N}^q_1\}}$ of $\mathbb{P}^q_1(e)$.

Remark 2.5 (Computation of $\pi^q_X v$)**.** *It can be easily seen that, for $X \in \{T, F, e\}$ and $v \in L^2(X)$, $\pi^0_X v = \frac{1}{|X|_l}\int_X v$ with $l \in \{d, d-1, 1\}$ respectively. To compute $\pi^q_X v$ for $q \geqslant 1$, it suffices to solve a symmetric positive-definite (SPD) system of size N^q_l. The procedure of choice is a Cholesky factorisation.*

Remark 2.6 (Computation of $\Pi^q_T v$, $q \geqslant 1$)**.** *One possibility (another one is to solve a problem posed on the quotient space $\mathbb{P}^q_d(T)/\mathbb{P}^0_d(T)$) to compute $\Pi^q_T v$ is to consider the following coercive problem: find $z \in \mathbb{P}^q_d(T)$ such that, for all $w \in \mathbb{P}^q_d(T)$,*

$$\int_T \boldsymbol{\nabla} z\cdot\boldsymbol{\nabla} w + \int_T \pi^0_T z\, \pi^0_T w = \int_T \boldsymbol{\nabla} v\cdot\boldsymbol{\nabla} w = -\int_T v\, \triangle w + \int_{\partial T} v_{|\partial T}\, \boldsymbol{\nabla} w_{|\partial T}\cdot \boldsymbol{n}_T.$$

The elliptic projection is obtained by $\Pi^q_T v = z + \pi^0_T v$. According to the expression above, to compute $\Pi^q_T v$, it suffices to know selected moments of $v \in H^1(T)$, namely $\pi^\mu_T v$ for $\mu = \max(q-2, 0)$ and $\pi^{q-1}_{\partial T}(v_{|\partial T})$. The computation of $\Pi^q_T v$ then requires to solve a SPD system of size N^q_d. The procedure of choice is again a Cholesky factorisation.

3 Skeletal methods

3.1 Generic structure

Galerkin methods on $\mathcal{T}_h$, among which c/nc-FE, c/nc-VE, or dG methods, seek for an approximation u_h of the solution $u \in H^1_0(\Omega)$ to Problem (1) in a broken space $V_{h,0} \subseteq V(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$, with $V(\mathcal{T}_h)$ such that, for all $T \in \mathcal{T}_h$, $V(T)$ is a finite-dimensional vector space of functions on which exists a collection Σ_T of linear forms so that $(T, V(T), \Sigma_T)$ is a finite element in the sense of Ciarlet. The linear forms collected in Σ_T are the local DoF of the method. For dG methods, $V_{h,0} = V(\mathcal{T}_h)$, whereas for c/nc-FE/VE, $V_{h,0}$ is a strict subspace of $V(\mathcal{T}_h)$ that embeds more or less stringent conformity prescriptions (with respect to $H^1_0(\Omega)$); for c-FE/VE for instance, $V_{h,0}$ is a subspace of $H^1_0(\Omega)$.

For those methods (like c/nc-FE/VE) based on a space $V_{h,0}$ embedding conformity prescriptions, locally to any $T \in \mathcal{T}_h$, the collection Σ_T of DoF can be split into (i) linear forms on $V(\partial T) := \{v_{|\partial T},\, v \in V(T)\}$ (collected in the set Σ^∂_T) and, (ii) if need be, those linear forms on $V(T)$ that cannot be written as linear forms on $V(\partial T)$ (collected in the set Σ°_T). The linear forms in Σ^∂_T are skeletal DoF, whereas those in Σ°_T are bulk DoF. We let $n^\partial_T := \mathrm{card}(\Sigma^\partial_T)$, $n^\circ_T := \mathrm{card}(\Sigma^\circ_T)$, and $n_T := n^\partial_T + n^\circ_T = \dim(V(T))$. From a global viewpoint, the collection of global DoF of the method splits into (i) linear forms on

$$V(\mathcal{F}_h) := \Big\{ v_h \in L^2(\partial\mathcal{T}_h) \mid v_{h|F} \in \bigoplus\nolimits_{T\in\mathcal{T}_F} V_T(F)\, \forall F \in \mathcal{F}_h \Big\}$$

with $V_T(F) := \{v_{|F},\, v \in V(T)\}$ (collected in the set Σ^∂_h) and, (ii) if need be, linear forms on $V(\mathcal{T}_h)$ that cannot be written as linear forms on $V(\mathcal{F}_h)$ (collected in the set Σ°_h). Whereas $\Sigma^\circ_h := \bigcup_{T\in\mathcal{T}_h} \check{\Sigma}^\circ_T$ where, for any $T \in \mathcal{T}_h$, $\check{\Sigma}^\circ_T$ is the collection of linear forms on $V(\mathcal{T}_h)$ given by $\check{\Sigma}^\circ_T := \big\{\sigma^\circ_T(\cdot_{|T}),\, \sigma^\circ_T \in \Sigma^\circ_T\big\}$, the global skeletal DoF in Σ^∂_h (that are linear forms on $V(\mathcal{F}_h)$) are intrinsically defined, and their local counterparts are obtained by localization. More precisely, for any $T \in \mathcal{T}_h$, letting

$$V_T(\mathcal{F}_h) := \big\{ v_h \in V(\mathcal{F}_h) \mid v_{h|\partial T} \in V(\partial T),\, v_{h|F} \equiv 0\, \forall F \in \mathcal{F}_h\backslash\mathcal{F}_T \big\} \text{ and } \hat{\Sigma}^\partial_T := \Big\{ \sigma^\partial_{|V_T(\mathcal{F}_h)} \not\equiv 0,\, \sigma^\partial \in \Sigma^\partial_h \Big\},$$

one has

$$\Sigma^\partial_T := \Big\{ c\,(\hat{\sigma}^\partial_T \circ \mathcal{Z}_h),\, \hat{\sigma}^\partial_T \in \hat{\Sigma}^\partial_T \Big\}, \tag{15}$$

where $\mathcal{Z}_h$ is the zero extension operator from $V(\partial T)$ to $V_T(\mathcal{F}_h)$, and c is a constant depending on the skeletal DoF under consideration and on T, whose default value is 1 (cf. Remark 3.1). Note that whereas $\mathrm{card}(\Sigma^\circ_h) = \sum_{T\in\mathcal{T}_h} n^\circ_T$, $\mathrm{card}(\Sigma^\partial_h) < \sum_{T\in\mathcal{T}_h} n^\partial_T$.

Remark 3.1 (Role of the constant c in (15)). *Assume that $V(T) \subset C^0(\overline{T})$ for all $T \in \mathcal{T}_h$, which is relevant in practice. Then, functions in $V(\mathcal{F}_h)$ are piecewise continuous on the skeleton, with (potential) discontinuities at vertices when $d = 2$ and on edges/vertices when $d = 3$. Yet, in the conforming case, in order to prescribe conformity, one has to define DoF at vertices when $d = 2$ and on edges/vertices when $d = 3$. Let us then describe how to do so. For a vertex $\nu \in \mathcal{V}_h$, the (global) skeletal DoF $\sigma^{\partial,\nu} \in \Sigma_h^\partial$ associated to a pointwise evaluation at $\boldsymbol{x}_\nu$ is defined, for any $v_h \in V(\mathcal{F}_h)$, by $\sigma^{\partial,\nu}(v_h) := \frac{1}{\mathrm{card}(\mathcal{F}_\nu)} \sum_{F\in\mathcal{F}_\nu} v_{h|F}(\boldsymbol{x}_\nu)$. Such a formula degenerates towards $v_h(\boldsymbol{x}_\nu)$ whenever v_h is continuous at $\boldsymbol{x}_\nu$. Besides, letting $c_T^\nu = \frac{\mathrm{card}(\mathcal{F}_\nu)}{\mathrm{card}(\mathcal{F}_\nu\cap\mathcal{F}_T)}$, one obtains as expected that the restriction of $\sigma^{\partial,\nu}$ to a cell $T \in \mathcal{T}_h$ such that $\nu \in \mathcal{V}_T$ is given, for any $v \in V(\partial T)$, by $\sigma_T^{\partial,\nu}(v) = v(\boldsymbol{x}_\nu)$. Edge DoF when $d = 3$ can be handled in a similar way. For face DoF, one can always take $c = 1$.*

For any $F \in \mathcal{F}_h$ and $v_h \in V(\mathcal{F}_h)$, we let $\mathcal{R}_F(v_h) \in V(\mathcal{F}_h)$ be such that $\mathcal{R}_F(v_h)_{|F} = v_{h|F}$ and $\mathcal{R}_F(v_h)_{|F'} \equiv 0$ for all $F' \in \mathcal{F}_h\backslash F$. The approximation space, that also takes into account the boundary conditions, is

$$V_{h,0} := \left\{ v_h \in V(\mathcal{T}_h) \mid (\sigma^\partial \circ \mathcal{R}_F)(\llbracket v_h \rrbracket) = 0 \, \forall F \in \mathcal{F}_h, \, \forall \sigma^\partial \in \Sigma_h^\partial \right\}, \tag{16}$$

with dimension $n_{h,0} := \dim(V_{h,0}) < \dim(V(\mathcal{T}_h)) =: n_h$ (remark that $n_h = \sum_{T\in\mathcal{T}_h} n_T$). The conditions on the jumps of discrete functions enforce the conformity prescriptions. The discrete problem then reads as follows: find $u_h \in V_{h,0}$ such that

$$a_h(u_h, v_h) := \sum_{T\in\mathcal{T}_h} a_T(u_{h|T}, v_{h|T}) = \sum_{T\in\mathcal{T}_h} l_T(v_{h|T}) =: l_h(v_h) \qquad \text{for all } v_h \in V_{h,0}, \tag{17}$$

where the (bi)linear forms $a_h : V(\mathcal{T}_h) \times V(\mathcal{T}_h) \to \mathbb{R}$ and $l_h : V(\mathcal{T}_h) \to \mathbb{R}$ are the sums of local contributions expressed by the local forms $a_T : V(T) \times V(T) \to \mathbb{R}$ and $l_T : V(T) \to \mathbb{R}$. This special structure of the discrete problem ensures that the potential bulk DoF of the method are not coupled between adjacent cells and can be eliminated locally in each cell $T \in \mathcal{T}_h$ in terms of the local skeletal DoF. Algebraically, the elimination consists in computing the Schur complement of the bulk-bulk block of the global linear system, which is quite inexpensive as this block is itself block-diagonal. After elimination, the global linear system to solve is expressed in terms of the skeletal DoF only. This explains why methods that are based on a discrete space like (16) and on a variational formulation like (17) are referred to as *skeletal*.

Without describing too much the space $V_{h,0}$, one can prove interesting structural properties on it. Let us denote, for any $F \in \mathcal{F}_h$, by $\check{\pi}_F^0$ the linear form on $V(\mathcal{F}_h)$ so that $\check{\pi}_F^0 := \pi_F^0(\cdot_{|F})$, with the convention (henceforth adopted) that $\mathbb{P}_l^0$ is identified to $\mathbb{R}$.

Lemma 3.2. *If $\{\check{\pi}_F^0\}_{F\in\mathcal{F}_h} \subseteq \Sigma_h^\partial$, then $\|\boldsymbol{\nabla}_h \cdot\|_{0,\Omega}$ defines a norm on $V_{h,0}$, and the following discrete Poincaré inequality holds:*

$$\forall v_h \in V_{h,0}, \qquad \|v_h\|_{0,\Omega} \lesssim \|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}. \tag{18}$$

Lemma 3.2, whose proof (which is quite classical but recalled for completeness) is postponed until Appendix A.2, gives sufficient conformity conditions for $V_{h,0}$ (that are reminiscent of lowest-order nc-FE) so as to ensure that a discrete Poincaré inequality holds on it.

Remark 3.3. *Obviously, the result of Lemma 3.2 remains valid under more stringent conformity prescriptions (like it is for instance the case for c-FE).*

3.2 Equivalent algebraic viewpoint

Since, for all $T \in \mathcal{T}_h$, $(T, V(T), \Sigma_T)$ is a finite element in the sense of Ciarlet, any function $v_h \in V(\mathcal{T}_h)$ can be equivalently written as a vector $\mathbb{R}^{n_h} \ni \underline{v}_h := (\underline{v}_T^\intercal)_{T\in\mathcal{T}_h}^\intercal$ with $\mathbb{R}^{n_T} \ni \underline{v}_T := \left(\underline{v}_T^{\partial\intercal}, \underline{v}_T^{\circ\intercal}\right)^\intercal$, $\underline{v}_T^\partial \in \mathbb{R}^{n_T^\partial}$ and $\underline{v}_T^\circ \in \mathbb{R}^{n_T^\circ}$. The vector $\underline{v}_T$, that satisfies $\underline{v}_T = \underline{\Sigma}_T(v_{h|T})$, is the restriction of $\underline{v}_h$ to T. For $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, we let $\underline{v}_{T,F}^\partial \in \mathbb{R}^{n_{T,F}^\partial}$ be the restriction of $\underline{v}_T^\partial$ to the face F (with possible overlaps between faces, i.e. $\sum_{F\in\mathcal{F}_T} n_{T,F}^\partial \geqslant n_T^\partial$). Since the skeletal DoF are intrinsically defined at the global level, we necessarily have that, for all $F \in \mathcal{F}_h^{\mathrm{i}}$ such that $\mathcal{T}_F = \{T_1, T_2\}$, the vectors $\underline{v}_{T_1,F}^\partial$ and $\underline{v}_{T_2,F}^\partial$ are local reductions obtained through the same collection of global linear forms. In particular, they have the same size, and we can let $[\underline{v}_h]_F := \underline{v}_{T_1,F}^\partial - \underline{v}_{T_2,F}^\partial$. For $F \in \mathcal{F}_h^{\mathrm{b}}$ such that $\mathcal{T}_F = \{T\}$, we let $[\underline{v}_h]_F := \underline{v}_{T,F}^\partial$. We introduce

$$\mathbb{V}_{h,0} := \left\{ \underline{v}_h \in \mathbb{R}^{n_h} \mid [\underline{v}_h]_F = \underline{0} \, \forall F \in \mathcal{F}_h \right\}. \tag{19}$$

There holds $\dim(\mathbb{V}_{h,0}) = n_{h,0}$. For vectors $\underline{\mathrm{v}}_h \in \mathbb{V}_{h,0}$, for every $F \in \mathcal{F}_h^{\mathrm{i}}$ we have $\underline{\mathrm{v}}_{T_1,F}^{\partial} = \underline{\mathrm{v}}_{T_2,F}^{\partial}$, and for every $F \in \mathcal{F}_h^{\mathrm{b}}$ we have $\underline{\mathrm{v}}_{T,F}^{\partial} = \underline{0}$. Problem (17) can be equivalently rewritten: find $\underline{\mathrm{u}}_h \in \mathbb{V}_{h,0}$ such that

$$\mathrm{a}_h(\underline{\mathrm{u}}_h, \underline{\mathrm{v}}_h) := \sum_{T\in\mathcal{T}_h} \mathrm{a}_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) = \sum_{T\in\mathcal{T}_h} \mathrm{l}_T(\underline{\mathrm{v}}_T) =: \mathrm{l}_h(\underline{\mathrm{v}}_h) \qquad \text{for all } \underline{\mathrm{v}}_h \in \mathbb{V}_{h,0}, \tag{20}$$

where the (bi)linear forms $\mathrm{a}_h : \mathbb{R}^{n_h} \times \mathbb{R}^{n_h} \to \mathbb{R}$ and $\mathrm{l}_h : \mathbb{R}^{n_h} \to \mathbb{R}$ are expressed in terms of the local forms $\mathrm{a}_T : \mathbb{R}^{n_T} \times \mathbb{R}^{n_T} \to \mathbb{R}$ and $\mathrm{l}_T : \mathbb{R}^{n_T} \to \mathbb{R}$ such that $\mathrm{a}_T := a_T(r_T\cdot, r_T\cdot)$ and $\mathrm{l}_T := l_T(r_T\cdot)$, where r_T is the (local, canonical) resconstruction operator of the finite element $(T, V(T), \Sigma_T)$. For all $T \in \mathcal{T}_h$, there holds $\underline{\mathrm{u}}_T = \underline{\Sigma}_T(u_{h|T})$, where $\underline{\mathrm{u}}_T \in \mathbb{R}^{n_T}$ is the restriction to T of $\underline{\mathrm{u}}_h \in \mathbb{V}_{h,0}$ solution to Problem (20), and $u_h \in V_{h,0}$ is the solution to Problem (17). Problem (17) and Problem (20) are strictly equivalent. They only differ by the fact that the former is written in a functional framework, whereas the latter is written in an (equivalent) algebraic framework, i.e. in terms of the discrete unknowns. In the rest of this section, we will stick to the functional notation used for Problem (17), but in Sections 4 and 5 we will adopt the algebraic notation used for Problem (20). In algebraic notation, roman fonts are used and vectorial quantities are underlined.

3.3 Examples of skeletal methods

The most famous examples of skeletal methods are surely given by the c/nc-FE methods. In that case, one considers, for all $T \in \mathcal{T}_h$,

$$a_T(w, v) := \int_T \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} v \text{ and } l_T(v) := \int_T f_{|T}\, v \qquad \text{for all } w, v \in V(T),$$

in such a way that the discrete problem writes: find $u_h \in V_{h,0}$ such that

$$a_h(u_h, v_h) = \int_\Omega \boldsymbol{\nabla}_h u_h \cdot \boldsymbol{\nabla}_h v_h = \int_\Omega f\, v_h = l_h(v_h) \qquad \text{for all } v_h \in V_{h,0}. \tag{21}$$

In the conforming case, since $V_{h,0} \subset H_0^1(\Omega)$, one can drop some h and directly write a, l, and $\boldsymbol{\nabla}$. For FE methods, it is assumed that closed-form expressions are available for the local shape functions. In that case, since $a_h(\cdot,\cdot) = \|\boldsymbol{\nabla}_h \cdot\|_{0,\Omega}^2$, Lemma 3.2 gives a sufficient condition so as to ensure well-posedness of the discrete problem in the sense of Hadamard (from (18) directly results the stability estimate $\|\boldsymbol{\nabla}_h u_h\|_{0,\Omega} \lesssim \|f\|_{0,\Omega}$ on the solution u_h to Problem (21)). Under that same (sufficient) condition, as soon as the space $V_{h,0}$ guarantees as well some approximability in a dense subspace of $H_0^1(\Omega)$, one can classically infer strong convergence in $H_0^1(\Omega)$ of u_h to the solution u to Problem (1) [37]. The result of Lemma 3.2, that does not assume that discrete functions in $V_{h,0}$ are piecewise polynomial, also applies to the mixed-order multiscale HHO method of [22, Section 5.1], for which the local (oscillatory) basis functions are considered to be all explicitly known on each coarse cell (in practice, they are in fact approximated using a fine submesh of the coarse cell).

Other examples of skeletal methods are given by the hybridised version of mixed FE methods, that can be recast under the form (17) after local elimination of the flux variable [40, 2, 26].

Other, more recent examples of skeletal methods are the c/nc-VE methods, as well as the HDG and HHO methods (cf. [11]). The specificity of VE methods with respect to FE methods is to consider virtual local spaces $V(T)$, i.e. local spaces which are spanned by functions that are (i) in general, not all computable, and (ii) by definition, never all computed. The local virtual functions are usually implicitly defined as the solutions to some PDEs posed in the cell. The VE methods are defined using computable (in terms of the DoF) projections of the virtual functions, and are stabilized through computable penalisations. Polynomial-based VE methods hinge on local virtual spaces (i) spanned by the solutions to PDEs featuring polynomial data, and (ii) that contain the space $\mathbb{P}_d^k$ for some $k \geqslant 1$. They are the VE methods that are classically encountered in the literature. As such, we will henceforth refer to them simply as VE methods, and we will exclusively focus on them in the sequel. An example of (nonconforming) method that hinges on a different kind of virtual space is given by the equal-order multiscale HHO method of [22, Section 5.2], for which local virtual functions do solve (oscillatory) PDEs with polynomial data, but the local virtual space does not contain polynomials in general.

4 Interpolation in local virtual spaces

In this section and in the following, we will make an extensive use of the notation introduced in Sections 2 and 3. We let $k \geqslant 1$ be a given integer, that will stand for the order of the method. Doing so, we adopt

the classical VE notation. HHO methods of the same order are classically defined using $k' = k-1 \geqslant 0$. Let $T \in \mathcal{T}_h$ be a given cell.

4.1 Conforming case

For any $\nu \in \mathcal{V}_T$ (respectively $\nu \in \mathcal{V}_F$, for $F \in \mathcal{F}_T$), we let ι_T^ν (respectively ι_F^ν) denote the linear form on $C^0(\partial T)$ (respectively $C^0(\partial F)$) so that, for all $v \in C^0(\partial T)$ (respectively $v \in C^0(\partial F)$), $\iota_T^\nu(v) = v(\boldsymbol{x}_\nu)$ (respectively $\iota_F^\nu(v) = v(\boldsymbol{x}_\nu)$).

4.1.1 Preliminary results

Let $d = 2$, and consider a face $F \in \mathcal{F}_T$ such that $F := [\boldsymbol{x}_{\nu_1}, \boldsymbol{x}_{\nu_2}]$. Then, we have the following classical result.

Proposition 4.1. *The triple $(F, \mathbb{P}_1^k(F), \Sigma_{1,F}^k)$, where the collection $\Sigma_{1,F}^k$ splits into $\Sigma_{1,F}^{k,\partial} := \{\iota_F^\nu\}_{\nu\in\mathcal{V}_F}$ and $\Sigma_{1,F}^{k,\mathrm{o}} := \{\pi_F^{k-2,j}\}_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}$, is a finite element in the sense of Ciarlet.*

To alleviate the notation, we have kept in the collection $\Sigma_{1,F}^{k,\mathrm{o}}$ the symbol $\pi_F^{k-2,j}$ to actually denote $\pi_{F|\mathbb{P}_1^k(F)}^{k-2,j}$. This slight abuse in notation will henceforth be adopted. Letting $n_{1,F}^k := \dim(\mathbb{P}_1^k(F))$, we introduce the operator $r_{1,F}^k : \mathbb{R}^{n_{1,F}^k} \to \mathbb{P}_1^k(F)$ such that, for any

$$\underline{\mathrm{v}}_F := \big((\mathrm{v}_F^{\partial,\nu})_{\nu\in\mathcal{V}_F}, (\mathrm{v}_F^{\mathrm{o},j})_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}\big)^\mathsf{T} \in \mathbb{R}^{n_{1,F}^k},$$

$r_{1,F}^k \underline{\mathrm{v}}_F \in \mathbb{P}_1^k(F)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_F (r_{1,F}^k \underline{\mathrm{v}}_F)' w' = -\int_F \mathrm{v}_F^{\mathrm{o}}\, w'' + [\mathrm{v}_F^{\partial,\nu_2} w'(\boldsymbol{x}_{\nu_2}) - \mathrm{v}_F^{\partial,\nu_1} w'(\boldsymbol{x}_{\nu_1})] \qquad \forall w \in \mathbb{P}_1^k(F), \\ r_{1,F}^k \underline{\mathrm{v}}_F(\boldsymbol{x}_{\nu_1}) = \mathrm{v}_F^{\partial,\nu_1}, \end{cases} \tag{22}$$

where we have introduced the notation $\mathrm{v}_F^{\mathrm{o}} := \sum_{j=1}^{\mathrm{N}_1^{k-2}} \mathrm{v}_F^{\mathrm{o},j} \psi_{F,j}^{k-2} \in \mathbb{P}_1^{k-2}(F)$. Remark that we also have $r_{1,F}^k \underline{\mathrm{v}}_F(\boldsymbol{x}_{\nu_2}) = \mathrm{v}_F^{\partial,\nu_2}$ (it suffices to test (22) against any $w \in \mathbb{P}_1^1(F)$).

Proposition 4.2. *The operator $r_{1,F}^k$ defined by (22) coincides with the (canonical) reconstruction operator $(\underline{\Sigma}_{1,F}^k)^{-1}$.*

Proof. Let $\mathbb{R}^{n_{1,F}^k} \ni \underline{\mathrm{v}}_F := \underline{\Sigma}_{1,F}^k(v)$ for some $v \in \mathbb{P}_1^k(F)$. Plugging $\underline{\mathrm{v}}_F$ into (22), we obtain by integration by parts (remark that $\mathrm{v}_F^{\mathrm{o}} = \pi_F^{k-2} v$)

$$\int_F (r_{1,F}^k \underline{\mathrm{v}}_F)' w' = \int_F v'\, w' \qquad \forall w \in \mathbb{P}_1^k(F),$$

which, combined to the condition $r_{1,F}^k \underline{\mathrm{v}}_F(\boldsymbol{x}_{\nu_1}) = \mathrm{v}_F^{\partial,\nu_1} = v(\boldsymbol{x}_{\nu_1})$, yields that $r_{1,F}^k \underline{\mathrm{v}}_F = r_{1,F}^k(\underline{\Sigma}_{1,F}^k(v)) = v$. This is true for any $v \in \mathbb{P}_1^k(F)$, hence $r_{1,F}^k = (\underline{\Sigma}_{1,F}^k)^{-1}$. □

The collection $\Sigma_{1,F}^k$ of linear forms on $\mathbb{P}_1^k(F)$ can patently be extended to a collection $\overline{\Sigma}_{1,F}^k$ of linear forms on $C^0(F)$. We can hence define the interpolation operator $\mathcal{I}_{1,F}^k : C^0(F) \to \mathbb{P}_1^k(F)$ such that $\mathcal{I}_{1,F}^k := r_{1,F}^k \circ \overline{\Sigma}_{1,F}^k$.

Lemma 4.3 (Stability of $\mathcal{I}_{1,F}^k$)**.** *For all $v \in C^0(F)$, there holds $\|\mathcal{I}_{1,F}^k v\|_{\infty,F} \lesssim \|v\|_{\infty,F}$.*

Proof. To prove the result, we test (22) with $\underline{\mathrm{v}}_F := \overline{\Sigma}_{1,F}^k(v)$. We get

$$\int_F (\mathcal{I}_{1,F}^k v)'\, w' = -\int_F v\, w'' + \big[v(\boldsymbol{x}_{\nu_2}) w'(\boldsymbol{x}_{\nu_2}) - v(\boldsymbol{x}_{\nu_1}) w'(\boldsymbol{x}_{\nu_1})\big] \qquad \forall w \in \mathbb{P}_1^k(F),$$

where we have used that $w'' \in \mathbb{P}_1^{k-2}(F)$. By Cauchy–Schwarz inequality, an inverse inequality on the 1-simplex F (cf., e.g., [39, Lemma 1.138]) and a reverse Lebesgue embedding (cf., e.g., [28, Lemma 5.1]) for $w' \in \mathbb{P}_1^{k-1}(F)$, we infer

$$\left|\int_F (\mathcal{I}_{1,F}^k v)'\, w'\right| \lesssim h_F^{-1/2} \|v\|_{\infty,F} |w|_{1,F} \qquad \forall w \in \mathbb{P}_1^k(F).$$

Taking $w = \mathcal{I}_{1,F}^k v \in \mathbb{P}_1^k(F)$ yields

$$\left|\mathcal{I}_{1,F}^k v\right|_{1,F} \lesssim h_F^{-1/2} \|v\|_{\infty,F}. \tag{23}$$

Now, using the fact that $\mathcal{I}_{1,F}^k v(\boldsymbol{x}_{\nu_1}) = v(\boldsymbol{x}_{\nu_1})$ and writing

$$\mathcal{I}_{1,F}^k v(\boldsymbol{x}) = v(\boldsymbol{x}_{\nu_1}) + \int_{\boldsymbol{x}_{\nu_1}}^{\boldsymbol{x}} (\mathcal{I}_{1,F}^k v)',$$

we finally obtain, by Cauchy–Schwarz inequality and (23), that

$$\left\|\mathcal{I}_{1,F}^k v\right\|_{\infty,F} \leqslant \|v\|_{\infty,F} + h_F^{1/2} \left|\mathcal{I}_{1,F}^k v\right|_{1,F} \lesssim \|v\|_{\infty,F}.$$

□

Remark 4.4. *One can equivalently choose, to define the finite element $(F, \mathbb{P}_1^k(F), \Sigma_{1,F}^k)$, to consider $\Sigma_{1,F}^{k,\mathrm{o}} := \{\iota_F^\nu\}_{\nu\in\mathcal{V}_F^\mathrm{o}}$, where $\mathcal{V}_F^\mathrm{o}$ is a set of $k-1$ internal points of F chosen such that the matrix defined by $\mathbb{M} := \left(\psi_{F,j}^k(\boldsymbol{x}_\nu)\right)_{\nu\in\mathcal{V}_F\cup\mathcal{V}_F^\mathrm{o},\, j\in\{1,\dots,\mathrm{N}_1^k\}}$ is invertible. The operator $r_{1,F}^k$ is then given by $r_{1,F}^k \underline{\mathrm{v}}_F := \sum_{j=1}^{\mathrm{N}_1^k} \left(\mathbb{M}^{-1}\underline{\mathrm{v}}_F\right)_j \psi_{F,j}^k$, and one can easily prove L^∞-stability for $\mathcal{I}_{1,F}^k$.*

4.1.2 The case $d = 2$

If $d = 2$, we let our conforming local virtual space on T be defined by

$$V_2^k(T) := \left\{ v \in H^1(T) \mid \triangle v \in \mathbb{P}_2^{k-1}(T),\, v_{|\partial T} \in \mathbb{P}_1^{k,c}(\mathcal{F}_T) \right\}, \tag{24}$$

where $\mathbb{P}_1^{k,c}(\mathcal{F}_T) := \mathbb{P}_1^k(\mathcal{F}_T) \cap C^0(\partial T)$. One can show (cf., e.g., [13, Remark 2.3]) that functions in $V_2^k(T)$ belong to $C^0(\overline{T})$. Besides, $\mathbb{P}_2^k(T) \subset V_2^k(T)$. The following result is standard.

Proposition 4.5. *The triple $(T, V_2^k(T), \Sigma_{2,T}^k)$, where $V_2^k(T)$ is given by (24) and the collection $\Sigma_{2,T}^k$ splits into*

$$\Sigma_{2,T}^{k,\partial} := \{\iota_T^\nu\}_{\nu\in\mathcal{V}_T} \cup \left\{\pi_F^{k-2,j}(\cdot_{|F})\right\}_{j\in\{1,\dots,\mathrm{N}_1^{k-2}\}}^{F\in\mathcal{F}_T} \quad \text{and} \quad \Sigma_{2,T}^{k,\mathrm{o}} := \left\{\pi_T^{k-1,i}\right\}_{i\in\{1,\dots,\mathrm{N}_2^{k-1}\}},$$

is a finite element in the sense of Ciarlet.

Letting $n_{2,T}^k := \dim(V_2^k(T))$, for any

$$\underline{\mathrm{v}}_T := \left((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\dots,\mathrm{N}_1^{k-2}\}}^{F\in\mathcal{F}_T}, (\mathrm{v}_T^{\mathrm{o},i})_{i\in\{1,\dots,\mathrm{N}_2^{k-1}\}}\right)^\intercal \in \mathbb{R}^{n_{2,T}^k},$$

we have $\underline{\mathrm{v}}_T^\partial = \left((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\dots,\mathrm{N}_1^{k-2}\}}^{F\in\mathcal{F}_T}\right)^\intercal \in \mathbb{R}^{n_{2,T}^{k,\partial}}$, $\underline{\mathrm{v}}_{T,F}^\partial = \left((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_F}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\dots,\mathrm{N}_1^{k-2}\}}\right)^\intercal \in \mathbb{R}^{n_{1,F}^k}$ for any $F \in \mathcal{F}_T$, and we let $\mathrm{v}_T^\mathrm{o} := \sum_{i=1}^{\mathrm{N}_2^{k-1}} \mathrm{v}_T^{\mathrm{o},i} \psi_{T,i}^{k-1} \in \mathbb{P}_2^{k-1}(T)$. Defining $r_{1,\partial T}^k : \mathbb{R}^{n_{2,T}^{k,\partial}} \to \mathbb{P}_1^{k,c}(\mathcal{F}_T)$ so that, for any $\underline{\mathrm{v}}_T^\partial \in \mathbb{R}^{n_{2,T}^{k,\partial}}$, $r_{1,\partial T}^k \underline{\mathrm{v}}_{T|F}^\partial := r_{1,F}^k \underline{\mathrm{v}}_{T,F}^\partial$ for all $F \in \mathcal{F}_T$ where $r_{1,F}^k$ is defined by (22), we are now in position to introduce the operator $r_{2,T}^k : \mathbb{R}^{n_{2,T}^k} \to V_2^k(T)$ such that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{2,T}^k}$, $r_{2,T}^k \underline{\mathrm{v}}_T \in V_2^k(T)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla} r_{2,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^\mathrm{o} \triangle w + \left\langle \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T, r_{1,\partial T}^k \underline{\mathrm{v}}_T^\partial \right\rangle_{-\frac{1}{2},\partial T} & \forall w \in V_2^k(T), \\ \displaystyle\int_T r_{2,T}^k \underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^\mathrm{o}. \end{cases} \tag{25}$$

Proposition 4.6. *The operator $r_{2,T}^k$ defined by (25) coincides with the (canonical) reconstruction operator $(\underline{\Sigma}_{2,T}^k)^{-1}$.*

Proof. Let $\mathbb{R}^{n_{2,T}^k} \ni \underline{\mathrm{v}}_T := \underline{\Sigma}_{2,T}^k(v)$ for some $v \in V_2^k(T)$. Plugging $\underline{\mathrm{v}}_T$ into (25), we infer by integration by parts (remark that $\mathrm{v}_T^\mathrm{o} = \pi_T^{k-1} v$)

$$\int_T \boldsymbol{\nabla} r_{2,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w = -\int_T \pi_T^{k-1} v \triangle w + \left\langle \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T, r_{1,\partial T}^k \left(\underline{\Sigma}_{2,T}^{k,\partial}(v)\right) \right\rangle_{-\frac{1}{2},\partial T} = \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} w \quad \forall w \in V_2^k(T),$$

where we have used that $\triangle w \in \mathbb{P}_2^{k-1}(T)$ and that, for all $F \in \mathcal{F}_T$, $r_{1,\partial T}^k \left(\underline{\Sigma}_{2,T}^{k,\partial}(v)\right)_{|F} = r_{1,F}^k \left(\underline{\Sigma}_{2,T,F}^{k,\partial}(v)\right) = r_{1,F}^k \left(\underline{\Sigma}_{1,F}^k(v_{|F})\right) = v_{|F}$ (cf. Proposition 4.2). Since $\int_T r_{2,T}^k \underline{\mathrm{v}}_T = \int_T \pi_T^{k-1} v = \int_T v$, we finally infer that $r_{2,T}^k \underline{\mathrm{v}}_T = r_{2,T}^k \left(\underline{\Sigma}_{2,T}^k(v)\right) = v$. This is true for any $v \in V_2^k(T)$, hence $r_{2,T}^k = (\underline{\Sigma}_{2,T}^k)^{-1}$. □

Let us introduce the operator $\mathcal{I}^k_{1,\partial T} : C^0(\partial T) \to \mathbb{P}^{k,c}_1(\mathcal{F}_T)$ so that, for any $v \in C^0(\partial T)$, $\mathcal{I}^k_{1,\partial T} v_{|F} := \mathcal{I}^k_{1,F}(v_{|F})$ for all $F \in \mathcal{F}_T$. It is clear that the collection $\Sigma^k_{2,T}$ of linear forms on $V^k_2(T)$ can be extended to a collection $\overline{\Sigma}^k_{2,T}$ of linear forms on

$$H^{1,c}(T) := H^1(T) \cap C^0(\overline{T}).$$

We can hence define the interpolation operator $\mathcal{I}^k_{2,T} : H^{1,c}(T) \to V^k_2(T)$ such that $\mathcal{I}^k_{2,T} := r^k_{2,T} \circ \overline{\Sigma}^k_{2,T}$. Besides, we remark that, for all $v \in H^{1,c}(T)$, $r^k_{1,\partial T}\big(\overline{\Sigma}^{k,\partial}_{2,T}(v)\big) = \mathcal{I}^k_{1,\partial T}(v_{|\partial T})$. We have the following stability result for $\mathcal{I}^k_{2,T}$.

Lemma 4.7 (Stability of $\mathcal{I}^k_{2,T}$)**.** *Under Assumption 1.1, for all $v \in H^2(T)$, there holds*

- $\big|\mathcal{I}^k_{2,T} v\big|_{1,T} \lesssim |v|_{1,T} + h_T |v|_{2,T}$;
- $\big\|\mathcal{I}^k_{2,T} v\big\|_{0,T} \lesssim \|v\|_{0,T} + h_T |v|_{1,T} + h^2_T |v|_{2,T}$.

Proof. First, remark that if $v \in H^2(T)$, $v \in C^0(\overline{T})$ and hence $v \in H^{1,c}(T)$; consequently, $\mathcal{I}^k_{2,T} v$ has a sense. Let us test (25) with $\underline{\mathrm{v}}_T := \overline{\Sigma}^k_{2,T}(v)$. There holds

$$\int_T \boldsymbol{\nabla} \mathcal{I}^k_{2,T} v \cdot \boldsymbol{\nabla} w = -\int_T v \,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T, \mathcal{I}^k_{1,\partial T}(v_{|\partial T}) \big\rangle_{-\frac{1}{2},\partial T} \qquad \forall w \in V^k_2(T),$$

where we have used that $\triangle w \in \mathbb{P}^{k-1}_2(T)$ and that $r^k_{1,\partial T}\big(\overline{\Sigma}^{k,\partial}_{2,T}(v)\big) = \mathcal{I}^k_{1,\partial T}(v_{|\partial T})$. We equivalently rewrite the equality above as

$$\int_T \boldsymbol{\nabla} \mathcal{I}^k_{2,T} v \cdot \boldsymbol{\nabla} w = -\int_T (v - \pi^0_T v) \,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T, \mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v) \big\rangle_{-\frac{1}{2},\partial T} \qquad \forall w \in V^k_2(T)$$

and, by Cauchy–Schwarz inequality we infer, for any $w \in V^k_2(T)$,

$$\begin{aligned}\left|\int_T \boldsymbol{\nabla} \mathcal{I}^k_{2,T} v \cdot \boldsymbol{\nabla} w\right| \leqslant \|v - \pi^0_T v\|_{0,T} \|\triangle w\|_{0,T} + \Big(h_T^{-\frac{1}{2}} \big\|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big\|_{0,\partial T} \\ + h_T^{\frac{1}{2}} \big|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big|_{1,\partial T} \Big) \sup_{z \in \mathbb{P}^{k,c}_1(\mathcal{F}_T)\setminus\{0\}} \frac{\big\langle \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T, z \big\rangle_{-\frac{1}{2},\partial T}}{h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}} =: \mathfrak{T}_1 + \mathfrak{T}_2.\end{aligned}$$

To estimate $\mathfrak{T}_1$, we apply (8) to $w \in V^k_2(T)$, and the Poincaré inequality (7) to $(v - \pi^0_T v)$. We get

$$\mathfrak{T}_1 \lesssim |v|_{1,T} |w|_{1,T}.$$

To estimate the second factor in $\mathfrak{T}_2$, we make use of the estimate (11) with $\mathfrak{T}(\partial T) = \mathbb{P}^{k,c}_1(\mathcal{F}_T) \subset H^1(\partial T)$. This yields, using again (8),

$$\sup_{z \in \mathbb{P}^{k,c}_1(\mathcal{F}_T)\setminus\{0\}} \frac{\big\langle \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T, z \big\rangle_{-\frac{1}{2},\partial T}}{h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}} \lesssim |w|_{1,T}.$$

To estimate the first factor in $\mathfrak{T}_2$, we first use an inverse inequality (cf., e.g., [39, Lemma 1.138]) for $\mathcal{I}^k_{1,F}\big(v_{|F} - \pi^0_T v\big) \in \mathbb{P}^k_1(F)$ on the 1-simplex F for all $F \in \mathcal{F}_T$, to infer

$$\begin{aligned} h_T^{-\frac{1}{2}} \big\|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big\|_{0,\partial T} + h_T^{\frac{1}{2}} \big|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big|_{1,\partial T} \\ \lesssim h_T^{-\frac{1}{2}} \big\|\mathcal{I}^k_{1,\partial T}\big(v_{|\partial T} - \pi^0_T v\big)\big\|_{0,\partial T} \lesssim \big\|\mathcal{I}^k_{1,\partial T}\big(v_{|\partial T} - \pi^0_T v\big)\big\|_{\infty,\partial T}.\end{aligned}$$

Then, applying Lemma 4.3, and since $v \in C^0(\overline{T})$, there holds

$$h_T^{-\frac{1}{2}} \big\|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big\|_{0,\partial T} + h_T^{\frac{1}{2}} \big|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big|_{1,\partial T} \lesssim \big\|v_{|\partial T} - \pi^0_T v\big\|_{\infty,\partial T} \lesssim \big\|v - \pi^0_T v\big\|_{\infty,T},$$

which, by application of the Sobolev inequality (10), and of the Poincaré inequality (7), yields

$$h_T^{-\frac{1}{2}} \big\|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big\|_{0,\partial T} + h_T^{\frac{1}{2}} \big|\mathcal{I}^k_{1,\partial T}(v_{|\partial T} - \pi^0_T v)\big|_{1,\partial T} \lesssim |v|_{1,T} + h_T |v|_{2,T}.$$

Finally, we get

$$\mathfrak{T}_2 \lesssim \big(|v|_{1,T} + h_T|v|_{2,T}\big)|w|_{1,T}.$$

In conclusion, there holds, for any $w \in V_2^k(T)$,

$$\left|\int_T \boldsymbol{\nabla}\mathcal{I}_{2,T}^k v \cdot \boldsymbol{\nabla} w\right| \lesssim \big(|v|_{1,T} + h_T|v|_{2,T}\big)|w|_{1,T}.$$

Taking $w = \mathcal{I}_{2,T}^k v \in V_2^k(T)$ provides the expected estimate in the $H^1(T)$-seminorm. To obtain the estimate in the $L^2(T)$-norm, it suffices to remark that $\int_T \mathcal{I}_{2,T}^k v = \int_T \pi_T^{k-1} v = \int_T v$. Hence, by the triangle inequality and the Poincaré inequality (7), we infer

$$\big\|\mathcal{I}_{2,T}^k v\big\|_{0,T} \leqslant \|v\|_{0,T} + \big\|v - \mathcal{I}_{2,T}^k v\big\|_{0,T} \lesssim \|v\|_{0,T} + h_T\big(|v|_{1,T} + \big|\mathcal{I}_{2,T}^k v\big|_{1,T}\big).$$

The conclusion then follows from the estimate in the $H^1(T)$-seminorm. □

Remark 4.8. *We note that the only moment in the proof of Lemma 4.7 where Assumption 1.1 is needed is when it comes to use (11). The result (11) provides an estimate on a dual norm of the boundary normal flux of virtual functions, which is actually what we need. Indeed, owing to the lack of a priori regularity of virtual functions ($H^{3/2-\varepsilon}(T)$ for any $\varepsilon > 0$; cf. [13, Remark 2.3]), one cannot expect to control a L^2-norm of their boundary normal flux. The proof of (11) requires the result (2) to hold. This latter states the existence of a lifting operator with optimal scaling for traces of virtual functions (against which we test). Since we are not aware of a proof of (2) that is valid under our mesh assumptions (we recall that our cells may not be star-shaped), we keep (2) as an assumption.*

With a view towards the case $d = 3$, we state a sharper stability estimate for the interpolation operator.

Lemma 4.9 (Sharper stability estimate for $\mathcal{I}_{2,T}^k$)**.** *Under Assumption 1.1, for all $v \in H^{3/2}(T)$, there holds*

$$\big\|\mathcal{I}_{2,T}^k v\big\|_{0,T} + h_T\big|\mathcal{I}_{2,T}^k v\big|_{1,T} \lesssim \|v\|_{0,T} + h_T|v|_{1,T} + h_T^{3/2}|v|_{\frac{3}{2},T}.$$

Proof. The proof exactly follows the one of Lemma 4.7, with a slight variation when it comes to apply the Sobolev inequality (10). We make use of the following sharper estimate (cf. e.g., [15, Eq. (2.4)]): for all $z \in H^{3/2}(T)$,

$$\|z\|_{\infty,T} \lesssim h_T^{-1}\|z\|_{0,T} + |z|_{1,T} + h_T^{1/2}|z|_{\frac{3}{2},T}.$$

We thus obtain

$$\mathfrak{T}_2 \lesssim \big(|v|_{1,T} + h_T^{1/2}|v|_{\frac{3}{2},T}\big)|w|_{1,T},$$

which finally yields the desired estimate. □

4.1.3 The case $d = 3$

If $d = 3$, we let our conforming local virtual space on T be defined by

$$V_3^k(T) := \Big\{v \in H^1(T) \mid \triangle v \in \mathbb{P}_3^{k-1}(T),\, v_{|\partial T} \in V_2^{k,c}(\mathcal{F}_T)\Big\}, \tag{26}$$

where $V_2^{k,c}(\mathcal{F}_T) := V_2^k(\mathcal{F}_T) \cap C^0(\partial T)$ with broken space

$$V_2^k(\mathcal{F}_T) := \{v \in L^2(\partial T) \mid v_{|F} \in V_2^k(F)\, \forall F \in \mathcal{F}_T\},$$

where $V_2^k(F)$ is given by (24) with $T \leftarrow F$ (recall that $\mathcal{F}_F = \mathcal{E}_F$). One can show (cf. e.g., [15, Remark 5.1]) that functions in $V_3^k(T)$ belong to $C^0(\overline{T})$. Besides, $\mathbb{P}_3^k(T) \subset V_3^k(T)$. The following result is standard.

Proposition 4.10. *The triple $(T, V_3^k(T), \Sigma_{3,T}^k)$, where $V_3^k(T)$ is given by (26) and the collection $\Sigma_{3,T}^k$ splits into*

- $\Sigma_{3,T}^{k,\partial} := \{\iota_T^\nu\}_{\nu\in\mathcal{V}_T} \cup \big\{\pi_e^{k-2,m}(\cdot_{|e})\big\}_{m\in\{1,\dots,\mathrm{N}_1^{k-2}\}}^{e\in\mathcal{E}_T} \cup \big\{\pi_F^{k-1,j}(\cdot_{|F})\big\}_{j\in\{1,\dots,\mathrm{N}_2^{k-1}\}}^{F\in\mathcal{F}_T}$ *and*
- $\Sigma_{3,T}^{k,\mathrm{o}} := \big\{\pi_T^{k-1,i}\big\}_{i\in\{1,\dots,\mathrm{N}_3^{k-1}\}}$,

is a finite element in the sense of Ciarlet.

Letting $n_{3,T}^k := \dim(V_3^k(T))$, for any

$$\underline{\mathrm{v}}_T := \big((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_T}, (\mathrm{v}_{T,e}^{\partial,m})_{m\in\{1,\dots,\mathrm{N}_1^{k-2}\}}^{e\in\mathcal{E}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\dots,\mathrm{N}_2^{k-1}\}}^{F\in\mathcal{F}_T}, (\mathrm{v}_T^{\mathrm{o},i})_{i\in\{1,\dots,\mathrm{N}_3^{k-1}\}}\big)^\intercal \in \mathbb{R}^{n_{3,T}^k},$$

we have $\underline{\mathrm{v}}_T^\partial = \big((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_T}, (\mathrm{v}_{T,e}^{\partial,m})_{m\in\{1,\dots,\mathrm{N}_1^{k-2}\}}^{e\in\mathcal{E}_T}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\dots,\mathrm{N}_2^{k-1}\}}^{F\in\mathcal{F}_T}\big)^\intercal \in \mathbb{R}^{n_{3,T}^{k,\partial}}$,

$$\underline{\mathrm{v}}_{T,F}^\partial = \big((\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_F}, (\mathrm{v}_{T,e}^{\partial,m})_{m\in\{1,\dots,\mathrm{N}_1^{k-2}\}}^{e\in\mathcal{E}_F}, (\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\dots,\mathrm{N}_2^{k-1}\}}\big)^\intercal \in \mathbb{R}^{n_{2,F}^k} \text{ for any } F\in\mathcal{F}_T,$$

and we let $\mathrm{v}_T^{\mathrm{o}} := \sum_{i=1}^{\mathrm{N}_3^{k-1}} \mathrm{v}_T^{\mathrm{o},i}\psi_{T,i}^{k-1} \in \mathbb{P}_3^{k-1}(T)$ and, for further use, $\mathrm{v}_T^\partial \in \mathbb{P}_2^{k-1}(\mathcal{F}_T)$ so that $\mathrm{v}_{T|F}^\partial := \sum_{j=1}^{\mathrm{N}_2^{k-1}} \mathrm{v}_{T,F}^{\partial,j}\psi_{F,j}^{k-1} \in \mathbb{P}_2^{k-1}(F)$ for all $F\in\mathcal{F}_T$. Defining $r_{2,\partial T}^k : \mathbb{R}^{n_{3,T}^{k,\partial}} \to V_2^{k,c}(\mathcal{F}_T)$ so that, for any $\underline{\mathrm{v}}_T^\partial \in \mathbb{R}^{n_{3,T}^{k,\partial}}$, $r_{2,\partial T}^k\underline{\mathrm{v}}_{T|F}^\partial := r_{2,F}^k\underline{\mathrm{v}}_{T,F}^\partial$ for all $F\in\mathcal{F}_T$ with $r_{2,F}^k$ given by (25) with $T\leftarrow F$, we can now introduce the operator $r_{3,T}^k : \mathbb{R}^{n_{3,T}^k} \to V_3^k(T)$ such that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{3,T}^k}$, $r_{3,T}^k\underline{\mathrm{v}}_T \in V_3^k(T)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla} r_{3,T}^k\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^{\mathrm{o}}\,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, r_{2,\partial T}^k\underline{\mathrm{v}}_T^\partial\big\rangle_{-\frac12,\partial T} & \forall w\in V_3^k(T),\\ \displaystyle\int_T r_{3,T}^k\underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^{\mathrm{o}}. \end{cases} \tag{27}$$

Proposition 4.11. *The operator $r_{3,T}^k$ defined by (27) coincides with the (canonical) reconstruction operator $(\underline{\Sigma}_{3,T}^k)^{-1}$.*

Proof. Let $\mathbb{R}^{n_{3,T}^k} \ni \underline{\mathrm{v}}_T := \underline{\Sigma}_{3,T}^k(v)$ for some $v\in V_3^k(T)$. Plugging $\underline{\mathrm{v}}_T$ into (27), we infer by integration by parts (remark that $\mathrm{v}_T^{\mathrm{o}} = \pi_T^{k-1}v$)

$$\int_T \boldsymbol{\nabla} r_{3,T}^k\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla} w = -\int_T \pi_T^{k-1}v\,\triangle w + \Big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, r_{2,\partial T}^k\big(\underline{\Sigma}_{3,T}^{k,\partial}(v)\big)\Big\rangle_{-\frac12,\partial T} = \int_T \boldsymbol{\nabla} v\cdot\boldsymbol{\nabla} w \quad \forall w\in V_3^k(T),$$

where we have used that $\triangle w\in\mathbb{P}_3^{k-1}(T)$ and that, for all $F\in\mathcal{F}_T$, $r_{2,\partial T}^k\big(\underline{\Sigma}_{3,T}^{k,\partial}(v)\big)_{|F} = r_{2,F}^k\big(\underline{\Sigma}_{3,T,F}^{k,\partial}(v)\big) = r_{2,F}^k\big(\underline{\Sigma}_{2,F}^k(v_{|F})\big) = v_{|F}$ (cf. Proposition 4.6 with $T\leftarrow F$). Since $\int_T r_{3,T}^k\underline{\mathrm{v}}_T = \int_T \pi_T^{k-1}v = \int_T v$, we finally infer that $r_{3,T}^k\underline{\mathrm{v}}_T = r_{3,T}^k\big(\underline{\Sigma}_{3,T}^k(v)\big) = v$. This is valid for any $v\in V_3^k(T)$, hence $r_{3,T}^k = (\underline{\Sigma}_{3,T}^k)^{-1}$. □

Let us introduce the operator $\mathcal{I}_{2,\partial T}^k : C^0(\partial T) \to V_2^{k,c}(\mathcal{F}_T)$ so that, for any $v\in C^0(\partial T)$, $\mathcal{I}_{2,\partial T}^k v_{|F} := \mathcal{I}_{2,F}^k(v_{|F})$ for all $F\in\mathcal{F}_T$ with $\mathcal{I}_{2,F}^k$ being $\mathcal{I}_{2,T}^k$ defined in Section 4.1.2 with $T\leftarrow F$. The collection $\Sigma_{3,T}^k$ of linear forms on $V_3^k(T)$ can clearly be extended to a collection $\overline{\Sigma}_{3,T}^k$ of linear forms on $H^{1,c}(T)$. Thus, we can define the interpolation operator $\mathcal{I}_{3,T}^k : H^{1,c}(T) \to V_3^k(T)$ such that $\mathcal{I}_{3,T}^k := r_{3,T}^k \circ \overline{\Sigma}_{3,T}^k$. We remark that, for all $v\in H^{1,c}(T)$, $r_{2,\partial T}^k\big(\overline{\Sigma}_{3,T}^{k,\partial}(v)\big) = \mathcal{I}_{2,\partial T}^k(v_{|\partial T})$. We have the following stability result for $\mathcal{I}_{3,T}^k$.

Lemma 4.12 (Stability of $\mathcal{I}_{3,T}^k$)**.** *Assume that the assumptions of Lemma 4.9 where $T\leftarrow F$ are met for all $F\in\mathcal{F}_T$. Then, under Assumption 1.1, for all $v\in H^2(T)$, one has*

- $\big|\mathcal{I}_{3,T}^k v\big|_{1,T} \lesssim |v|_{1,T} + h_T|v|_{2,T}$;
- $\big\|\mathcal{I}_{3,T}^k v\big\|_{0,T} \lesssim \|v\|_{0,T} + h_T|v|_{1,T} + h_T^2|v|_{2,T}$.

Proof. First, remark that if $v\in H^2(T)$, $v\in C^0(\overline{T})$ also in that case and hence $v\in H^{1,c}(T)$; consequently, $\mathcal{I}_{3,T}^k v$ has a sense. Let us test (27) with $\underline{\mathrm{v}}_T := \overline{\Sigma}_{3,T}^k(v)$. There holds

$$\int_T \boldsymbol{\nabla}\mathcal{I}_{3,T}^k v\cdot\boldsymbol{\nabla} w = -\int_T v\,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, \mathcal{I}_{2,\partial T}^k(v_{|\partial T})\big\rangle_{-\frac12,\partial T} \qquad \forall w\in V_3^k(T).$$

We equivalently rewrite the equality above as

$$\int_T \boldsymbol{\nabla}\mathcal{I}_{3,T}^k v\cdot\boldsymbol{\nabla} w = -\int_T (v-\pi_T^0 v)\,\triangle w + \big\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, \mathcal{I}_{2,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\big\rangle_{-\frac12,\partial T} \qquad \forall w\in V_3^k(T),$$

and, by Cauchy–Schwarz inequality, we infer, for any $w \in V_3^k(T)$,

$$\left|\int_T \boldsymbol{\nabla}\mathcal{I}_{3,T}^k v\cdot\boldsymbol{\nabla} w\right| \leqslant \|v-\pi_T^0 v\|_{0,T}\|\triangle w\|_{0,T} + \Big(h_T^{-\frac12}\|\mathcal{I}_{2,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\|_{0,\partial T} + h_T^{\frac12}|\mathcal{I}_{2,\partial T}^k(v_{|\partial T}-\pi_T^0 v)|_{1,\partial T}\Big) \sup_{z\in V_2^{k,c}(\mathcal{F}_T)\setminus\{0\}} \frac{\langle \boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T, z\rangle_{-\frac12,\partial T}}{h_T^{-1/2}\|z\|_{0,\partial T}+h_T^{1/2}|z|_{1,\partial T}} =: \mathfrak{T}_1+\mathfrak{T}_2.$$

The term $\mathfrak{T}_1$ and the second factor in $\mathfrak{T}_2$ can be handled as in the proof of Lemma 4.7 (here, $\mathfrak{T}(\partial T) = V_2^{k,c}(\mathcal{F}_T) \subset H^1(\partial T)$). To estimate the first factor in $\mathfrak{T}_2$, we remark that for all $F\in\mathcal{F}_T$, $\big(v_{|F}-\pi_T^0 v\big)\in H^{3/2}(F)$. We can hence apply Lemma 4.9 where $T\leftarrow F$ to infer

$$h_F^{-1/2}\|\mathcal{I}_{2,F}^k(v_{|F}-\pi_T^0 v)\|_{0,F} + h_F^{1/2}|\mathcal{I}_{2,F}^k(v_{|F}-\pi_T^0 v)|_{1,F} \lesssim h_F^{-1/2}\|v_{|F}-\pi_T^0 v\|_{0,F} + h_F^{1/2}|v_{|F}|_{1,F} + h_F|v_{|F}|_{\frac32,F}.$$

By (a sharper version of) the continuous trace inequality (5), and the Poincaré inequality (7), we obtain, summing over $F\in\mathcal{F}_T$,

$$h_T^{-\frac12}\|\mathcal{I}_{2,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\|_{0,\partial T} + h_T^{1/2}|\mathcal{I}_{2,\partial T}^k(v_{|\partial T}-\pi_T^0 v)|_{1,\partial T} \lesssim |v|_{1,T}+h_T|v|_{2,T}.$$

Hence, there holds, for any $w\in V_3^k(T)$,

$$\left|\int_T \boldsymbol{\nabla}\mathcal{I}_{3,T}^k v\cdot\boldsymbol{\nabla} w\right| \lesssim \big(|v|_{1,T}+h_T|v|_{2,T}\big)|w|_{1,T}.$$

Taking $w=\mathcal{I}_{3,T}^k v\in V_3^k(T)$ provides the expected estimate in the $H^1(T)$-seminorm. To obtain the estimate in the $L^2(T)$-norm, we follow the same reasoning as in the proof of Lemma 4.7. □

4.1.4 Approximation properties

Theorem 4.13 (Approximation properties for $\mathcal{I}_{d,T}^k$, conforming case)**.** *Assume that the assumptions of Lemma 4.7 (when $d=2$) or 4.12 (when $d=3$) are met. Let $v\in H^s(T)$, for $s\in\{2,\dots,k+1\}$. Then, there holds*

$$\|v-\mathcal{I}_{d,T}^k v\|_{0,T} + h_T|v-\mathcal{I}_{d,T}^k v|_{1,T} \lesssim h_T^s|v|_{s,T}, \tag{28}$$

and, for any $F\in\mathcal{F}_T$,

$$\big\|\big(v-\mathcal{I}_{d,T}^k v\big)_{|F}\big\|_{0,F} \lesssim h_T^{s-1/2}|v|_{s,T}. \tag{29}$$

Proof. We follow the ideas of [14, Chapter 4] and [38, Section 7] (cf. also [13, Section 4]). Under our mesh assumptions, T is indeed a finite union of star-shaped subcells. We proceed by density of $C^\infty(\overline{T})$ in $H^s(T)$. The function $v\in C^\infty(\overline{T})$ admits the following Sobolev representation:

$$v = Q_T^s v + R_T^s v,$$

where $Q_T^s v\in\mathbb{P}_d^{s-1}(T)\subseteq\mathbb{P}_d^k(T)$ is an averaged Taylor polynomial, and the remainder $R_T^s v$ satisfies, for $r\in\{0,\dots,s\}$, the Bramble–Hilbert lemma:

$$|R_T^s v|_{r,T} \lesssim h_T^{s-r}|v|_{s,T}. \tag{30}$$

One can easily see that, since $Q_T^s v\in\mathbb{P}_d^k(T)$ and $\mathcal{I}_{d,T}^k v = v$ for all $v\in\mathbb{P}_d^k(T)\subset V_d^k(T)$, there holds $v-\mathcal{I}_{d,T}^k v = R_T^s v - \mathcal{I}_{d,T}^k(R_T^s v)$. Thus, for $m\in\{0,1\}$,

$$|v-\mathcal{I}_{d,T}^k v|_{m,T} \leqslant |R_T^s v|_{m,T} + |\mathcal{I}_{d,T}^k(R_T^s v)|_{m,T}.$$

By the stability result of Lemma 4.7 or 4.12, we infer

$$|v-\mathcal{I}_{d,T}^k v|_{m,T} \lesssim |R_T^s v|_{m,T} + \sum_{\alpha=m}^{2} h_T^{\alpha-m}|R_T^s v|_{\alpha,T}.$$

Applying the Bramble–Hilbert lemma (30) for $r \in \{m, \dots, 2\}$, we finally obtain (28). For $F \in \mathcal{F}_T$ now, the continuous trace inequality (5) yields

$$\left\| \left(v - \mathcal{I}_{d,T}^k v\right)_{|F} \right\|_{0,F} \lesssim h_T^{-1/2} \left\| v - \mathcal{I}_{d,T}^k v \right\|_{0,T} + h_T^{1/2} \left| v - \mathcal{I}_{d,T}^k v \right|_{1,T}.$$

The conclusion then follows from (28). □

Remark 4.14. *Note that general H^m approximation properties for $\mathcal{I}_{d,T}^k$ cannot be obtained easily because of the fact that we do not know if inverse inequalities hold for the derivatives of virtual functions (as opposed to polynomials, derivatives of virtual functions may not be virtual functions). This remark remains valid in the nonconforming case.*

4.2 Nonconforming case

We let our nonconforming local virtual space on T in dimension d be defined by

$$V_d^k(T) := \left\{ v \in H^1(T) \mid \triangle v \in \mathbb{P}_d^{k-1}(T),\ \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T) \right\}. \tag{31}$$

As opposed to the conforming case, the definition of the nonconforming local virtual space does not depend on the ambient dimension. We have $\mathbb{P}_d^k(T) \subset V_d^k(T)$. The following result is standard.

Proposition 4.15. *The triple $(T, V_d^k(T), \Sigma_{d,T}^k)$, where $V_d^k(T)$ is given by (31) and the collection $\Sigma_{d,T}^k$ splits into*

$$\Sigma_{d,T}^{k,\partial} := \left\{ \pi_F^{k-1,j}(\cdot_{|F}) \right\}_{j \in \{1,\dots,\mathrm{N}_{d-1}^{k-1}\}}^{F \in \mathcal{F}_T} \quad \text{and} \quad \Sigma_{d,T}^{k,\mathrm{o}} := \left\{ \pi_T^{k-1,i} \right\}_{i \in \{1,\dots,\mathrm{N}_d^{k-1}\}},$$

is a finite element in the sense of Ciarlet.

Letting $n_{d,T}^k := \dim(V_d^k(T))$, for any

$$\underline{\mathrm{v}}_T := \left((\mathrm{v}_{T,F}^{\partial,j})_{j \in \{1,\dots,\mathrm{N}_{d-1}^{k-1}\}}^{F \in \mathcal{F}_T}, (\mathrm{v}_T^{\mathrm{o},i})_{i \in \{1,\dots,\mathrm{N}_d^{k-1}\}} \right)^{\mathsf{T}} \in \mathbb{R}^{n_{d,T}^k},$$

we let $\mathrm{v}_T^{\mathrm{o}} := \sum_{i=1}^{\mathrm{N}_d^{k-1}} \mathrm{v}_T^{\mathrm{o},i} \psi_{T,i}^{k-1} \in \mathbb{P}_d^{k-1}(T)$ and $\mathrm{v}_T^{\partial} \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$ so that $\mathrm{v}_{T|F}^{\partial} := \sum_{j=1}^{\mathrm{N}_{d-1}^{k-1}} \mathrm{v}_{T,F}^{\partial,j} \psi_{F,j}^{k-1} \in \mathbb{P}_{d-1}^{k-1}(F)$ for all $F \in \mathcal{F}_T$. We can now introduce the operator $r_{d,T}^k : \mathbb{R}^{n_{d,T}^k} \to V_d^k(T)$ such that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{d,T}^k}$, $r_{d,T}^k \underline{\mathrm{v}}_T \in V_d^k(T)$ solves the well-posed problem

$$\begin{cases} \displaystyle\int_T \boldsymbol{\nabla} r_{d,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^{\mathrm{o}} \triangle w + \int_{\partial T} \mathrm{v}_T^{\partial} \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T & \forall w \in V_d^k(T), \\ \displaystyle\int_T r_{d,T}^k \underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^{\mathrm{o}}. \end{cases} \tag{32}$$

Proposition 4.16. *The operator $r_{d,T}^k$ defined by (32) coincides with the (canonical) reconstruction operator $(\underline{\Sigma}_{d,T}^k)^{-1}$.*

Proof. Let $\mathbb{R}^{n_{d,T}^k} \ni \underline{\mathrm{v}}_T := \underline{\Sigma}_{d,T}^k(v)$ for some $v \in V_d^k(T)$. Plugging $\underline{\mathrm{v}}_T$ into (32), we infer by integration by parts (remark that $\mathrm{v}_T^{\mathrm{o}} = \pi_T^{k-1} v$ and $\mathrm{v}_T^{\partial} = \pi_{\partial T}^{k-1}(v_{|\partial T})$)

$$\int_T \boldsymbol{\nabla} r_{d,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} w = -\int_T \pi_T^{k-1} v \triangle w + \int_{\partial T} \pi_{\partial T}^{k-1}(v_{|\partial T}) \boldsymbol{\nabla} w_{|\partial T} \cdot \boldsymbol{n}_T = \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} w \quad \forall w \in V_d^k(T),$$

where we have used that $\triangle w \in \mathbb{P}_d^{k-1}(T)$ and that, for all $F \in \mathcal{F}_T$, $\boldsymbol{\nabla} w_{|F} \cdot \boldsymbol{n}_{T,F} \in \mathbb{P}_{d-1}^{k-1}(F)$. Since $\int_T r_{d,T}^k \underline{\mathrm{v}}_T = \int_T \pi_T^{k-1} v = \int_T v$, we finally deduce that $r_{d,T}^k \underline{\mathrm{v}}_T = r_{d,T}^k\big(\underline{\Sigma}_{d,T}^k(v)\big) = v$. This is valid for any $v \in V_d^k(T)$, hence $r_{d,T}^k = (\underline{\Sigma}_{d,T}^k)^{-1}$. □

The collection $\Sigma_{d,T}^k$ of linear forms on $V_d^k(T)$ can be patently extended to a collection $\overline{\Sigma}_{d,T}^k$ of linear forms on $H^1(T)$. Thus, we can define the interpolation operator $\mathcal{I}_{d,T}^k : H^1(T) \to V_d^k(T)$ such that $\mathcal{I}_{d,T}^k := r_{d,T}^k \circ \overline{\Sigma}_{d,T}^k$. We have the following stability result for $\mathcal{I}_{d,T}^k$.

Lemma 4.17 (Stability of $\mathcal{I}_{d,T}^k$)**.** *For all $v \in H^1(T)$, there holds*

- $\left|\mathcal{I}_{d,T}^k v\right|_{1,T} \leqslant |v|_{1,T}$;
- $\left\|\mathcal{I}_{d,T}^k v\right\|_{0,T} \leqslant \|v\|_{0,T} + 2c_{\mathrm{P}}\, h_T |v|_{1,T}$.

Proof. We test (32) with $\underline{v}_T := \overline{\Sigma}_{d,T}^k(v)$, for some $v \in H^1(T)$. By integration by parts, there holds

$$\int_T \boldsymbol{\nabla}\mathcal{I}_{d,T}^k v \cdot \boldsymbol{\nabla} w = \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} w \qquad \text{for all } w \in V_d^k(T). \tag{33}$$

Testing with $w = \mathcal{I}_{d,T}^k v \in V_d^k(T)$, we immediately obtain the estimate in the $H^1(T)$-seminorm. Then, since $\int_T \mathcal{I}_{d,T}^k v = \int_T \pi_T^{k-1} v = \int_T v$, by the triangle and Poincaré (7) inequalities, there holds

$$\left\|\mathcal{I}_{d,T}^k v\right\|_{0,T} \leqslant \|v\|_{0,T} + \left\|v - \mathcal{I}_{d,T}^k v\right\|_{0,T} \leqslant \|v\|_{0,T} + c_{\mathrm{P}}\, h_T \big(|v|_{1,T} + \left|\mathcal{I}_{d,T}^k v\right|_{1,T}\big),$$

which enables to conclude. □

The equation (33) shows that, in the nonconforming case, the local (canonical) interpolator $\mathcal{I}_{d,T}^k$ is actually the elliptic projector onto the local virtual space. This is not true in the conforming case.

Remark 4.18. *In the proof of Lemma 4.17, and as opposed to the proofs of Lemmas 4.7 and 4.12, no estimate on a dual norm of the boundary normal flux is needed for virtual functions. As a consequence, one does not need Assumption 1.1. The difference results from the fact that, in the nonconforming case, virtual functions have L^2 boundary normal flux. In that case, one can actually prove an approximation result on the boundary normal flux (see (35)).*

Theorem 4.19 (Approximation properties for $\mathcal{I}_{d,T}^k$, nonconforming case)**.** *Let $v \in H^s(T)$, for $s \in \{1, \ldots, k+1\}$. Then, the following holds:*

$$\left\|v - \mathcal{I}_{d,T}^k v\right\|_{0,T} + h_T \left|v - \mathcal{I}_{d,T}^k v\right|_{1,T} \lesssim h_T^s |v|_{s,T}, \tag{34}$$

and, for any $F \in \mathcal{F}_T$,

$$\left\|\big(v - \mathcal{I}_{d,T}^k v\big)_{|F}\right\|_{0,F} + \delta_{s>1} h_T \left\|\boldsymbol{\nabla}\big(v - \mathcal{I}_{d,T}^k v\big)_{|F} \cdot \boldsymbol{n}_{T,F}\right\|_{0,F} \lesssim h_T^{s-1/2} |v|_{s,T}. \tag{35}$$

Proof. From (33), the fact that $\int_T \mathcal{I}_{d,T}^k v = \int_T v$ combined with Poincaré inequality (7), and the fact that $\mathbb{P}_d^k(T) \subset V_d^k(T)$, we infer

$$\left\|v - \mathcal{I}_{d,T}^k v\right\|_{0,T} + h_T \left|v - \mathcal{I}_{d,T}^k v\right|_{1,T} \leqslant (c_{\mathrm{P}} + 1)\, h_T \min_{w \in V_d^k(T)} |v - w|_{1,T} \leqslant (c_{\mathrm{P}} + 1)\, h_T \min_{w \in \mathbb{P}_d^k(T)} |v - w|_{1,T}.$$

The derivation of (34) is then straightforward using the approximation properties of standard polynomial projectors (cf. Proposition 2.4). Concerning (35), the first trace estimate can be easily proven as in Theorem 4.13, whereas the proof of the second is based on (9). We write, when $s > 1$,

$$\left\|\boldsymbol{\nabla}\big(v - \mathcal{I}_{d,T}^k v\big)_{|F} \cdot \boldsymbol{n}_{T,F}\right\|_{0,F} \leqslant \left\|\boldsymbol{\nabla}\big(v - \pi_T^k v\big)_{|F} \cdot \boldsymbol{n}_{T,F}\right\|_{0,F} + \left\|\boldsymbol{\nabla}\big(\pi_T^k v - \mathcal{I}_{d,T}^k v\big)_{|F} \cdot \boldsymbol{n}_{T,F}\right\|_{0,F}.$$

Since $\mathbb{P}_d^k(T) \subset V_d^k(T)$, one has $(\pi_T^k v - \mathcal{I}_{d,T}^k v) \in V_d^k(T)$ and can apply (9) to infer

$$\left\|\boldsymbol{\nabla}\big(\pi_T^k v - \mathcal{I}_{d,T}^k v\big)_{|F} \cdot \boldsymbol{n}_{T,F}\right\|_{0,F} \lesssim h_T^{-1/2} \left|\pi_T^k v - \mathcal{I}_{d,T}^k v\right|_{1,T}.$$

Adding/subtracting v, and using the result in the $H^1(T)$-seminorm and the approximation results on the L^2-orthogonal projector of Proposition 2.4, yields the conclusion. □

5 Bridging the Hybrid High-Order and Virtual Element methods

In this section, we formulate and analyse the VE/HHO methods within a unified algebraic framework inspired from HHO methods.

5.1 Local polynomial projector

Let $T \in \mathcal{T}_h$ be a given cell. In this section, we introduce the local polynomial projector $p_{d,T}^k$ in terms of which the local VE/HHO discrete bilinear form is defined.

Let l be an integer such that $l = k$ in the conforming case with $d = 2$, and $l = k-1$ in the conforming case with $d = 3$ and in the nonconforming case. Recall that $n_{d,T}^k$ denotes the dimension of $V_d^k(T)$. For any $\underline{\mathrm{v}}_T^\partial \in \mathbb{R}^{n_{d,T}^{k,\partial}}$, we define $\mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial \in \mathbb{P}_{d-1}^l(\mathcal{F}_T)$ such that

- $\mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial := r_{1,\partial T}^k \underline{\mathrm{v}}_T^\partial \in \mathbb{P}_1^{k,c}(\mathcal{F}_T)$ in the conforming case with $d = 2$;
- $\mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial := \mathrm{v}_T^\partial \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$ in the conforming case with $d = 3$ and in the nonconforming case.

We introduce the operator $p_{d,T}^k : \mathbb{R}^{n_{d,T}^k} \to \mathbb{P}_d^k(T)$ such that, for any $\underline{\mathrm{v}}_T \in \mathbb{R}^{n_{d,T}^k}$, $p_{d,T}^k\underline{\mathrm{v}}_T \in \mathbb{P}_d^k(T)$ solves the well-posed problem

$$\left\{\begin{aligned} &\int_T \boldsymbol{\nabla} p_{d,T}^k\underline{\mathrm{v}}_T\cdot\boldsymbol{\nabla} w = -\int_T \mathrm{v}_T^{\mathrm{o}}\,\triangle w + \int_{\partial T} \mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial\,\boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T \qquad \forall w \in \mathbb{P}_d^k(T),\\ &\int_T p_{d,T}^k\underline{\mathrm{v}}_T = \int_T \mathrm{v}_T^{\mathrm{o}}. \end{aligned}\right. \tag{36}$$

We note that, in each case, the definition of $p_{d,T}^k$ is the same as the definition of $r_{d,T}^k$ (see Section 4), up to the fact of testing against a strict subspace of $V_d^k(T)$, that is $\mathbb{P}_d^k(T)$ (in the conforming case with $d = 3$, remark that v_T^∂ is equal to $\pi_{\partial T}^{k-1}(r_{2,\partial T}^k\underline{\mathrm{v}}_T^\partial)$). It is an easy matter to see that actually $p_{d,T}^k = \Pi_T^k \circ r_{d,T}^k$, with Π_T^k defined by (12).

Remark 5.1. *As opposed to $r_{d,T}^k$ whose computation necessitates the knowledge of a basis for $V_d^k(T)$ (and even for $V_{d-1}^k(\mathcal{F}_T)$ in the conforming case with $d = 3$), the operator $p_{d,T}^k$ is entirely computable. It depends on the DoF through $\mathrm{v}_T^{\mathrm{o}} \in \mathbb{P}_d^{k-1}(T)$ and $\mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial \in \mathbb{P}_{d-1}^l(\mathcal{F}_T)$. In the conforming case with $d = 2$, $\mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial \in \mathbb{P}_1^{k,c}(\mathcal{F}_T)$ depends on $(\mathrm{v}_T^{\partial,\nu})_{\nu\in\mathcal{V}_T}$ and $(\mathrm{v}_{T,F}^{\partial,j})_{j\in\{1,\ldots,\mathrm{N}_1^{k-2}\}}^{F\in\mathcal{F}_T}$ through (22). In the conforming case with $d = 3$ and in the nonconforming case, $\mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T) \ni \mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial = \mathrm{v}_T^\partial$.*

Let us set $W_d^k(T) := H^{1,c}(T)$ in the conforming case and $W_d^k(T) := H^1(T)$ in the nonconforming case. We can now define the operator $\mathcal{P}_{d,T}^k : W_d^k(T) \to \mathbb{P}_d^k(T)$ such that $\mathcal{P}_{d,T}^k := p_{d,T}^k \circ \underline{\Sigma}_{d,T}^k$. We have $\mathcal{P}_{d,T}^k = \Pi_T^k \circ \mathcal{I}_{d,T}^k$. Since $\mathbb{P}_d^k(T) \subset W_d^k(T)$ and $\mathcal{P}_{d,T}^k v = v$ for all $v \in \mathbb{P}_d^k(T)$, the operator $\mathcal{P}_d^k(T)$ is a (polynomial) projector. It satisfies the following properties:

- $\mathcal{P}_{d,T}^k v = \Pi_T^k v$ for all $v \in V_d^k(T)$;
- in the nonconforming case, since $\mathcal{I}_{d,T}^k$ is the elliptic projector onto $V_d^k(T)$ and $\mathbb{P}_d^k(T) \subset V_d^k(T)$, $\mathcal{P}_{d,T}^k = \Pi_T^k$ (this has already been pointed out in [30, Remark 25]).

Remark 5.2. *With the specific choice of virtual space we have made in the conforming case with $d = 3$, we actually also have in that case $\mathcal{P}_{3,T}^k = \Pi_{T|H^{1,c}(T)}^k$. Yet, in that case, $\mathcal{I}_{3,T}^k$ is not the elliptic projector onto the local virtual space.*

Let $\underline{s}$ be an integer such that $\underline{s} = 2$ in the conforming case and $\underline{s} = 1$ in the nonconforming case.

Lemma 5.3 (Stability of $\mathcal{P}_{d,T}^k$)**.** *For all $v \in H^{\underline{s}}(T)$, there holds*

$$\left\|\mathcal{P}_{d,T}^k v\right\|_{0,T} \lesssim \|v\|_{0,T} + h_T|v|_{1,T} + \delta_{c,d=2}\,h_T^2|v|_{2,T}. \tag{37}$$

Proof. In the nonconforming case and in the conforming case with $d = 3$ (cf. Remark 5.2), the result is a direct consequence of Proposition 2.4. In the conforming case with $d = 2$, $\mathcal{P}_{2,T}^k v$ satisfies, for all $w \in \mathbb{P}_2^k(T)$,

$$\begin{aligned}\int_T \boldsymbol{\nabla}\mathcal{P}_{2,T}^k v\cdot\boldsymbol{\nabla} w &= -\int_T \pi_T^{k-1} v\triangle w + \int_{\partial T}\mathcal{I}_{1,\partial T}^k(v_{|\partial T})\boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T\\ &= -\int_T v\triangle w + \int_{\partial T}\mathcal{I}_{1,\partial T}^k(v_{|\partial T})\boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T,\end{aligned}$$

where we have used that $\triangle w \in \mathbb{P}_2^{k-2}(T)$. We equivalently rewrite the equality above as

$$\int_T \boldsymbol{\nabla}\mathcal{P}_{2,T}^k v\cdot\boldsymbol{\nabla} w = -\int_T (v-\pi_T^0 v)\triangle w + \int_{\partial T} \mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T,$$

and, by Cauchy–Schwarz inequality we infer, for any $w \in \mathbb{P}_2^k(T)$,

$$\left|\int_T \boldsymbol{\nabla}\mathcal{P}_{2,T}^k v\cdot\boldsymbol{\nabla} w\right| \leqslant \left\|v-\pi_T^0 v\right\|_{0,T}\|\triangle w\|_{0,T} + \left\|\mathcal{I}_{1,\partial T}^k(v_{|\partial T}-\pi_T^0 v)\right\|_{0,\partial T}\left\|\boldsymbol{\nabla} w_{|\partial T}\cdot\boldsymbol{n}_T\right\|_{0,\partial T}.$$

By the Poincaré inequality (7) and the inverse inequality (4) for the first term of the right-hand side, and by the stability result of Lemma 4.3 and the discrete trace inequality (6) for the second, we infer the estimate

$$\left|\int_T \boldsymbol{\nabla}\mathcal{P}_{2,T}^k v\cdot\boldsymbol{\nabla} w\right| \lesssim |v|_{1,T}|w|_{1,T} + \left\|v_{|\partial T}-\pi_T^0 v\right\|_{\infty,\partial T}|w|_{1,T}.$$

The Sobolev inequality (10) combined to the Poincaré inequality (7) then yields

$$\left|\int_T \boldsymbol{\nabla}\mathcal{P}_{2,T}^k v\cdot\boldsymbol{\nabla} w\right| \lesssim \left(|v|_{1,T}+h_T|v|_{2,T}\right)|w|_{1,T}.$$

Taking $w = \mathcal{P}_{2,T}^k v \in \mathbb{P}_2^k(T)$, we infer

$$\left|\mathcal{P}_{2,T}^k v\right|_{1,T} \lesssim |v|_{1,T} + h_T|v|_{2,T}.$$

The triangle inequality, the Poincaré inequality (7), and the fact that $\int_T \mathcal{P}_{2,T}^k v = \int_T v$ finally proves the result in the $L^2(T)$-norm. □

Theorem 5.4 (H^m approximation properties for $\mathcal{P}_{d,T}^k$)**.** *Let $v \in H^s(T)$, for $s \in \{\underline{s},\dots,k+1\}$. Then, the following holds:*

$$\left|v-\mathcal{P}_{d,T}^k v\right|_{m,T} \lesssim h_T^{s-m}|v|_{s,T} \qquad \text{for } m \in \{0,\dots,s\}, \tag{38}$$

and, for any $F \in \mathcal{F}_T$, and any $(\zeta_1,\dots,\zeta_d) \in \mathbb{N}^d$ such that $\sum_{i=1}^d \zeta_i = m$,

$$\left\|\left[\partial_{x_1}^{\zeta_1}\dots\partial_{x_d}^{\zeta_d}\left(v-\mathcal{P}_{d,T}^k v\right)\right]_{|F}\right\|_{0,F} \lesssim h_T^{s-m-1/2}|v|_{s,T} \qquad \text{for } m \in \{0,\dots,s-1\}. \tag{39}$$

Proof. In the nonconforming case and in the conforming case with $d=3$ (cf. Remark 5.2), the result directly follows from (13) and (14) of Proposition 2.4. To prove the result in the conforming case with $d=2$, we follow the ideas of [14, Chapter 4] and [38, Section 7] (cf. also [13, Section 4]). Under our mesh assumptions, T is indeed a finite union of star-shaped subcells. We proceed by density of $C^\infty(\overline{T})$ in $H^s(T)$. The function $v \in C^\infty(\overline{T})$ admits the following Sobolev representation:

$$v = Q_T^s v + R_T^s v,$$

where $Q_T^s v \in \mathbb{P}_2^{s-1}(T) \subseteq \mathbb{P}_2^k(T)$ is an averaged Taylor polynomial, and the remainder $R_T^s v$ satisfies, for $r \in \{0,\dots,s\}$, the Bramble–Hilbert lemma (30). One can easily see that, since $Q_T^s v \in \mathbb{P}_2^k(T)$ and $\mathcal{P}_{2,T}^k v = v$ for all $v \in \mathbb{P}_2^k(T)$, there holds $v - \mathcal{P}_{2,T}^k v = R_T^s v - \mathcal{P}_{2,T}^k(R_T^s v)$. Thus,

$$\left|v-\mathcal{P}_{2,T}^k v\right|_{m,T} \leqslant |R_T^s v|_{m,T} + \left|\mathcal{P}_{2,T}^k(R_T^s v)\right|_{m,T}.$$

Applying m times the inverse inequality (4) to $\mathcal{P}_{2,T}^k(R_T^s v) \in \mathbb{P}_2^k(T)$, and then the stability result (37) for $\mathcal{P}_{2,T}^k$ in the $L^2(T)$-norm, we infer

$$\left|v-\mathcal{P}_{2,T}^k v\right|_{m,T} \lesssim |R_T^s v|_{m,T} + h_T^{-m}\left\|\mathcal{P}_{2,T}^k(R_T^s v)\right\|_{0,T} \lesssim |R_T^s v|_{m,T} + \sum_{\alpha=0}^{2} h_T^{\alpha-m}|R_T^s v|_{\alpha,T}.$$

By the Bramble–Hilbert lemma (30) for $r=m$ and $r=\alpha \in \{0,\dots,2\}$, we finally obtain (38). For $F \in \mathcal{F}_T$ now, the continuous trace inequality (5) yields

$$\left\|\left[\partial_{x_1}^{\zeta_1}\dots\partial_{x_d}^{\zeta_d}\left(v-\mathcal{P}_{2,T}^k v\right)\right]_{|F}\right\|_{0,F} \lesssim h_T^{-1/2}\left|v-\mathcal{P}_{2,T}^k v\right|_{m,T} + h_T^{1/2}\left|v-\mathcal{P}_{2,T}^k v\right|_{m+1,T}.$$

The conclusion then follows from (38) (remark that, for $m \in \{0,\dots,s-1\}$, $m+1 \in \{1,\dots,s\}$). □

Remark 5.5. *Note that the proofs of Lemma 5.3 and Theorem 5.4 do not rely, even in the conforming case and as opposed to those of Section 4.1, on the use of the result* (11)*. We hence do not need Assumption 1.1.*

5.2 Discrete problem

Let us define, for any $T \in \mathcal{T}_h$, the following seminorm on $\mathbb{R}^{n^k_{d,T}}$: for all $\underline{\mathrm{v}}_T \in \mathbb{R}^{n^k_{d,T}}$,

- $|\underline{\mathrm{v}}_T|_T^2 := \|\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}}\|_{0,T}^2 + \left\|h_{\partial T}^{-1/2}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^{\partial}\big)\right\|_{0,\partial T}^2$ in the conforming case with $d = 2$ and in the non-conforming case;
- $|\underline{\mathrm{v}}_T|_T^2 := \|\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}}\|_{0,T}^2 + \left\|h_{\partial T}^{-1/2}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{v}_T^{\partial}\big)\right\|_{0,\partial T}^2 + \sum_{F\in\mathcal{F}_T}\left\|\mathrm{v}_{T|\partial F}^{\mathrm{o}} - r^k_{1,\partial F}\underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F}^2$ in the conforming case with $d = 3$, where $r^k_{1,\partial F}$ is $r^k_{1,\partial T}$ defined in Section 4.1.2 where $T \leftarrow F$.

Letting $|\underline{\mathrm{v}}_h|_h^2 := \sum_{T\in\mathcal{T}_h}|\underline{\mathrm{v}}_T|_T^2$ for any $\underline{\mathrm{v}}_h \in \mathbb{R}^{n^k_{d,h}}$, one can easily see that $|\cdot|_h$ defines a norm on $\mathbb{V}^k_{d,h,0}$ defined by (19).

Let us write down the discrete problem. We consider Problem (20), where the local linear form is given by

$$\mathrm{l}_T(\underline{\mathrm{v}}_T) := \int_T f_{|T}\mathrm{v}_T^{\mathrm{o}},$$

and the (symmetric) local bilinear form, based on the local polynomial projector of Section 5.1, by

$$\mathrm{a}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) := \int_T \boldsymbol{\nabla}p^k_{d,T}\underline{\mathrm{w}}_T\cdot\boldsymbol{\nabla}p^k_{d,T}\underline{\mathrm{v}}_T + \mathrm{s}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T),$$

with (symmetric positive semidefinite) stabilisation

- $\mathrm{s}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) := \int_{\partial T} h_{\partial T}^{-1}\pi^l_{\partial T}\big(\delta^k_{d,T}\underline{\mathrm{w}}_{T|\partial T} - \mathrm{t}_{\partial T}\underline{\mathrm{w}}_T^{\partial}\big)\pi^l_{\partial T}\big(\delta^k_{d,T}\underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^{\partial}\big)$ in the conforming case with $d = 2$ and in the nonconforming case,
- $\mathrm{s}_T(\underline{\mathrm{w}}_T, \underline{\mathrm{v}}_T) := \int_{\partial T} h_{\partial T}^{-1}\pi^{k-1}_{\partial T}\big(\delta^k_{3,T}\underline{\mathrm{w}}_{T|\partial T} - \mathrm{w}_T^{\partial}\big)\pi^{k-1}_{\partial T}\big(\delta^k_{3,T}\underline{\mathrm{v}}_{T|\partial T} - \mathrm{v}_T^{\partial}\big)$
  $+ \sum_{F\in\mathcal{F}_T}\int_{\partial F}\big(\delta^k_{3,T}\underline{\mathrm{w}}_{T|\partial F} - r^k_{1,\partial F}\underline{\mathrm{w}}_{T,F}^{\partial,\partial}\big)\big(\delta^k_{3,T}\underline{\mathrm{v}}_{T|\partial F} - r^k_{1,\partial F}\underline{\mathrm{v}}_{T,F}^{\partial,\partial}\big)$ in the conforming case with $d = 3$,

where $\mathbb{P}^k_d(T) \ni \delta^k_{d,T}\underline{\mathrm{v}}_T := p^k_{d,T}\underline{\mathrm{v}}_T + \pi_T^{k-1}\big(\mathrm{v}_T^{\mathrm{o}} - p^k_{d,T}\underline{\mathrm{v}}_T\big)$. Note that the stabilisation bilinear form is entirely computable in terms of the DoF. Other choices of stabilisation are possible (cf., e.g., [10]). Basically, any stabilisation satisfying the assumptions of [36, Assumption 4.1] is admissible. Here, we build upon the standard HHO choice of stabilisation (see [33, 34]). To the best of our knowledge, in the conforming case, the stabilisation we propose is new (it is a close variant of existing ones).

Remark 5.6. *As expected, one can verify that our stabilisation bilinear form vanishes when one of its arguments is the reduction of a polynomial in $\mathbb{P}^k_d(T)$. Furthermore, if all the occurrences of $p^k_{d,T}$ in the expression of a_T were replaced by $r^k_{d,T}$ (assuming that it is computable), then the stabilisation bilinear form would be identically zero, and one would recover a standard FE method (up to the treatment of the right-hand side) on the space $\mathbb{V}^k_{d,h,0}$.*

Remark 5.7 (Equivalent functional viewpoint)**.** *Letting $w, v \in V^k_d(T)$ such that $w := r^k_{d,T}\underline{\mathrm{w}}_T$ and $v := r^k_{d,T}\underline{\mathrm{v}}_T$, and since $p^k_{d,T} = \Pi^k_T \circ r^k_{d,T}$, one can equivalently consider Problem (17), with local (bi)linear forms*

$$l_T(v) = \int_T f_{|T}\pi_T^{k-1}v, \qquad a_T(w,v) = \int_T \boldsymbol{\nabla}\Pi^k_T w\cdot\boldsymbol{\nabla}\Pi^k_T v + s_T(w,v),$$

with stabilisation

$$s_T(w,v) = \int_{\partial T} h_{\partial T}^{-1}\pi^l_{\partial T}\big([\Delta^k_{d,T}w - w]_{|\partial T}\big)\pi^l_{\partial T}\big([\Delta^k_{d,T}v - v]_{|\partial T}\big)$$

in the conforming case with $d = 2$ and in the nonconforming case, and stabilisation supplemented with the term

$$\sum_{F\in\mathcal{F}_T}\int_{\partial F}\big(\Delta^k_{3,T}w - w\big)_{|\partial F}\big(\Delta^k_{3,T}v - v\big)_{|\partial F}$$

in the conforming case with $d = 3$, where $\Delta^k_{d,T}v := \Pi^k_T v + \pi_T^{k-1}\big(v - \Pi^k_T v\big)$.

We now prove well-posedness for Problem (20).

Lemma 5.8 (Local coercivity and boundedness)**.** *For all $T \in \mathcal{T}_h$, and all $\underline{\mathrm{v}}_T \in \mathbb{R}^{n^k_{d,T}}$, the following holds:*

$$|\underline{\mathrm{v}}_T|_T^2 \lesssim \mathrm{a}_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T) \lesssim |\underline{\mathrm{v}}_T|_T^2.$$

Proof. Let us show local coercivity. Testing (36) with $w = \mathrm{v}_T^{\mathrm{o}} \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$, we infer

$$-\int_T \mathrm{v}_T^{\mathrm{o}} \triangle \mathrm{v}_T^{\mathrm{o}} = \int_T \boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} \mathrm{v}_T^{\mathrm{o}} - \int_{\partial T} \mathrm{t}_{\partial T} \underline{\mathrm{v}}_T^{\partial} \boldsymbol{\nabla} \mathrm{v}_{T|\partial T}^{\mathrm{o}} \cdot \boldsymbol{n}_T.$$

Integrating by parts the left-hand side, there holds

$$\|\boldsymbol{\nabla} \mathrm{v}_T^{\mathrm{o}}\|_{0,T}^2 = \int_T \boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T \cdot \boldsymbol{\nabla} \mathrm{v}_T^{\mathrm{o}} + \int_{\partial T} \big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T} \underline{\mathrm{v}}_T^{\partial}\big) \boldsymbol{\nabla} \mathrm{v}_{T|\partial T}^{\mathrm{o}} \cdot \boldsymbol{n}_T,$$

which yields, by Cauchy–Schwarz inequality, and the discrete trace inequality (6),

$$\|\boldsymbol{\nabla} \mathrm{v}_T^{\mathrm{o}}\|_{0,T} \lesssim \|\boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T\|_{0,T} + \Big\| h_{\partial T}^{-1/2} \big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T} \underline{\mathrm{v}}_T^{\partial}\big) \Big\|_{0,\partial T}. \tag{40}$$

Now, adding/subtracting $\pi_{\partial T}^l \big(\delta_{d,T}^k \underline{\mathrm{v}}_{T|\partial T}\big)$, by stability of π_F^l in the $L^2(F)$-norm for all $F \in \mathcal{F}_T$, we infer

$$\begin{aligned}\Big\| h_{\partial T}^{-1/2} \big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T} \underline{\mathrm{v}}_T^{\partial}\big) \Big\|_{0,\partial T} \leqslant{}& \Big\| h_{\partial T}^{-1/2} \big(p_{d,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{d,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial T} \Big\|_{0,\partial T} \\ &+ \Big\| h_{\partial T}^{-1/2} \pi_{\partial T}^l \big(\delta_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T} \underline{\mathrm{v}}_T^{\partial}\big) \Big\|_{0,\partial T}.\end{aligned}$$

By the discrete trace inequality (6), and an application of (13) with $s = 1$ and $m = 0$, we infer

$$\Big\| h_{\partial T}^{-1/2} \big(\mathrm{v}_{T|\partial T}^{\mathrm{o}} - \mathrm{t}_{\partial T} \underline{\mathrm{v}}_T^{\partial}\big) \Big\|_{0,\partial T} \lesssim \|\boldsymbol{\nabla} p_{d,T}^k \underline{\mathrm{v}}_T\|_{0,T} + \Big\| h_{\partial T}^{-1/2} \pi_{\partial T}^l \big(\delta_{d,T}^k \underline{\mathrm{v}}_{T|\partial T} - \mathrm{t}_{\partial T} \underline{\mathrm{v}}_T^{\partial}\big) \Big\|_{0,\partial T}. \tag{41}$$

In the conforming case with $d = 2$ and in the nonconforming case, we infer local coercivity as a direct consequence of (40) and (41). In the conforming case with $d = 3$, one has to estimate two terms in the stabilisation part of the seminorm. The first one is handled as above ($l = k-1$). It then remains to estimate the term $\big\| \mathrm{v}_{T|\partial F}^{\mathrm{o}} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial} \big\|_{0,\partial F}$ for all $F \in \mathcal{F}_T$. Adding/subtracting $\delta_{3,T}^k \underline{\mathrm{v}}_{T|\partial F}$, we have for any $F \in \mathcal{F}_T$,

$$\Big\| \mathrm{v}_{T|\partial F}^{\mathrm{o}} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial} \Big\|_{0,\partial F} \leqslant \Big\| \big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial F} \Big\|_{0,\partial F} + \Big\| \delta_{3,T}^k \underline{\mathrm{v}}_{T|\partial F} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial} \Big\|_{0,\partial F}.$$

Since $\big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|F} \in \mathbb{P}_2^k(F)$, we can apply a first time the discrete trace inequality (6) (with $T \leftarrow F$) to obtain

$$\Big\| \big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial F} \Big\|_{0,\partial F} \lesssim h_F^{-1/2} \Big\| \big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|F} \Big\|_{0,F},$$

and a second time (as it is) to obtain

$$\Big\| \big(p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\big)_{|\partial F} \Big\|_{0,\partial F} \lesssim h_T^{-1} \|p_{3,T}^k \underline{\mathrm{v}}_T - \pi_T^{k-1}(p_{3,T}^k \underline{\mathrm{v}}_T)\|_{0,T}.$$

An application of (13) with $s = 1$ and $m = 0$ finally yields

$$\Big\| \mathrm{v}_{T|\partial F}^{\mathrm{o}} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial} \Big\|_{0,\partial F} \lesssim \|\boldsymbol{\nabla} p_{3,T}^k \underline{\mathrm{v}}_T\|_{0,T} + \Big\| \delta_{3,T}^k \underline{\mathrm{v}}_{T|\partial F} - r_{1,\partial F}^k \underline{\mathrm{v}}_{T,F}^{\partial,\partial} \Big\|_{0,\partial F},$$

which concludes the proof of local coercivity in the conforming case with $d = 3$. We omit the proof of local boundedness, which relies on the same kind of arguments. □

Well-posedness follows as an immediate consequence of Lemma 5.8, and of the fact that $|\cdot|_h$ defines a norm on $\mathbb{V}_{d,h,0}^k$.

Corollary 5.9 (Well-posedness). *For all $\underline{\mathrm{v}}_h \in \mathbb{R}^{n_{d,h}^k}$, there holds*

$$|\underline{\mathrm{v}}_h|_h^2 \lesssim \mathrm{a}_h(\underline{\mathrm{v}}_h, \underline{\mathrm{v}}_h).$$

As a consequence, Problem (20) is well-posed.

Before proceeding with the convergence analysis, let us investigate the consistency of our stabilisation.

Lemma 5.10 (Consistency of the stabilisation). *Let $v \in H^s(T)$, for $s \in \{\underline{s}, \dots, k+1\}$. Then, letting $\underline{\mathrm{v}}_T = \overline{\Sigma}_{d,T}^k(v)$, the following holds:*

$$\mathrm{s}_T(\underline{\mathrm{v}}_T, \underline{\mathrm{v}}_T)^{1/2} \lesssim h_T^{s-1}|v|_{s,T}. \tag{42}$$

Proof. In the conforming case with $d = 3$ and in the nonconforming case, we have to estimate the quantity $\left\|h_{\partial T}^{-1/2}\pi_{\partial T}^{k-1}\big(\delta_{d,T}^k\underline{\mathrm{v}}_{T|\partial T} - \mathrm{v}_T^\partial\big)\right\|_{0,\partial T}$. We have $\mathrm{v}_T^\partial = \pi_{\partial T}^{k-1}\big(v_{|\partial T}\big)$ and $\mathrm{v}_T^{\mathrm{o}} = \pi_T^{k-1}v$, hence, by stability of $\pi_{\partial T}^{k-1}$, there holds

$$\begin{aligned}\left\|h_{\partial T}^{-1/2}\pi_{\partial T}^{k-1}\big(\delta_{d,T}^k\underline{\mathrm{v}}_{T|\partial T} - \mathrm{v}_T^\partial\big)\right\|_{0,\partial T} \leqslant{}& \left\|h_{\partial T}^{-1/2}\big(\mathcal{P}_{d,T}^k v - v\big)_{|\partial T}\right\|_{0,\partial T}\\ &+ \left\|h_{\partial T}^{-1/2}\pi_T^{k-1}\big(v - \mathcal{P}_{d,T}^k v\big)_{|\partial T}\right\|_{0,\partial T}.\end{aligned}$$

To estimate the first term in the right-hand side, we directly apply the result (39) for $m = 0$. To estimate the second term, we successively use the discrete trace inequality (6), the stability of π_T^{k-1} in the $L^2(T)$-norm, and the result (38) for $m = 0$. The conclusion easily follows. In the conforming case with $d = 2$, we have to estimate the quantity

$$\left\|h_{\partial T}^{-1/2}\pi_{\partial T}^k\big(\delta_{2,T}^k\underline{\mathrm{v}}_{T|\partial T} - r_{1,\partial T}^k\underline{\mathrm{v}}_T^\partial\big)\right\|_{0,\partial T} = \left\|h_{\partial T}^{-1/2}\big(\delta_{2,T}^k\underline{\mathrm{v}}_{T|\partial T} - r_{1,\partial T}^k\underline{\mathrm{v}}_T^\partial\big)\right\|_{0,\partial T}.$$

Since $r_{1,\partial T}^k\underline{\mathrm{v}}_T^\partial = \mathcal{I}_{1,\partial T}^k(v_{|\partial T})$, there holds

$$\begin{aligned}\left\|h_{\partial T}^{-1/2}\pi_{\partial T}^k\big(\delta_{2,T}^k\underline{\mathrm{v}}_{T|\partial T} - r_{1,\partial T}^k\underline{\mathrm{v}}_T^\partial\big)\right\|_{0,\partial T} \leqslant{}& \left\|h_{\partial T}^{-1/2}\big(\mathcal{P}_{2,T}^k v_{|\partial T} - \mathcal{I}_{1,\partial T}^k(v_{|\partial T})\big)\right\|_{0,\partial T}\\ &+ \left\|h_{\partial T}^{-1/2}\pi_T^{k-1}\big(v - \mathcal{P}_{2,T}^k v\big)_{|\partial T}\right\|_{0,\partial T}.\end{aligned}$$

The second term in the right-hand side can be estimated as previously. The first one is handled remarking that $\mathcal{P}_{2,T}^k v_{|\partial T} - \mathcal{I}_{1,\partial T}^k(v_{|\partial T}) = \mathcal{I}_{1,\partial T}^k\big([\mathcal{P}_{2,T}^k v - v]_{|\partial T}\big)$. Applying Lemma 4.3, we thus have

$$\left\|h_{\partial T}^{-1/2}\big(\mathcal{P}_{2,T}^k v_{|\partial T} - \mathcal{I}_{1,\partial T}^k(v_{|\partial T})\big)\right\|_{0,\partial T} \lesssim \left\|(v - \mathcal{P}_{2,T}^k v)_{|\partial T}\right\|_{\infty,\partial T} \lesssim \left\|v - \mathcal{P}_{2,T}^k v\right\|_{\infty,T}.$$

By the Sobolev inequality (10) for $d = 2$, combined to (38) with $m = \alpha \in \{0, \dots, 2\}$, we finally infer

$$\left\|h_{\partial T}^{-1/2}\big(\mathcal{P}_{2,T}^k v_{|\partial T} - \mathcal{I}_{1,\partial T}^k(v_{|\partial T})\big)\right\|_{0,\partial T} \lesssim \sum_{\alpha=0}^{2} h_T^{\alpha-1}\left|v - \mathcal{P}_{2,T}^k v\right|_{\alpha,T} \lesssim h_T^{s-1}|v|_{s,T}.$$

We are left with estimating, in the conforming case with $d = 3$, $\left\|\delta_{3,T}^k\underline{\mathrm{v}}_{T|\partial F} - r_{1,\partial F}^k\underline{\mathrm{v}}_{T,F}^{\partial,\partial}\right\|_{0,\partial F}$ for all $F \in \mathcal{F}_T$. We follow the same path as previously, and end up having to estimate $\left\|\mathcal{I}_{1,\partial F}^k\big([\mathcal{P}_{3,T}^k v - v]_{|\partial F}\big)\right\|_{0,\partial F}$. By Lemma 4.3 and (10) for $d = 3$, we infer

$$\left\|\mathcal{I}_{1,\partial F}^k\big([\mathcal{P}_{3,T}^k v - v]_{|\partial F}\big)\right\|_{0,\partial F} \lesssim h_F^{1/2}\left(h_T^{-\frac{3}{2}}\left\|v - \mathcal{P}_{3,T}^k v\right\|_{0,T} + h_T^{-\frac{1}{2}}\left|v - \mathcal{P}_{3,T}^k v\right|_{1,T} + h_T^{\frac{1}{2}}\left|v - \mathcal{P}_{3,T}^k v\right|_{2,T}\right).$$

The conclusion then follows from (38) with $m \in \{0, \dots, 2\}$. □

5.3 Convergence analysis

Let us begin this section with the following remark.

Remark 5.11 (Regularity of the solution to (1)). *Since the boundary of the domain Ω is assumed to be composed of a finite union of portions of affine hyperplanes, one has (see, e.g., [41, Theorem 4.4.3.7]) the following elliptic regularity result: there is $\varepsilon \in (0, 1/2]$ so that $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)$ and*

$$\|u\|_{\frac{3}{2}+\varepsilon,\Omega} \lesssim \|f\|_{0,\Omega}.$$

If Ω is in addition convex, one can actually prove full elliptic regularity, i.e. $u \in H^2(\Omega)$ and

$$\|u\|_{2,\Omega} \lesssim \|f\|_{0,\Omega}. \tag{43}$$

In any case, and since $\mathrm{div}(\boldsymbol{\nabla} u) = -f \in L^2(\Omega)$, there holds: for all $F \in \mathcal{F}_h^{\mathrm{i}}$, $[\![\boldsymbol{\nabla} u]\!]_F \cdot \boldsymbol{n}_F = 0$ a.e. on F.

Let $|\cdot|_{\mathrm{e},h}^2 := \mathrm{a}_h(\cdot,\cdot)$. According to Corollary 5.9, $|\cdot|_{\mathrm{e},h}$ defines a norm on $\mathbb{V}_{d,h,0}^k$. For $\boldsymbol{\beta} := (\beta_T)_{T\in\mathcal{T}_h}^\intercal \in \{2,\dots,k+1\}^{\mathrm{card}(\mathcal{T}_h)}$, we define

$$H^{\boldsymbol{\beta}}(\mathcal{T}_h) := \left\{ v_h \in L^2(\Omega) \mid v_{h|T} \in H^{\beta_T}(T)\ \forall T\in\mathcal{T}_h \right\}.$$

Theorem 5.12 (Discrete energy-norm error estimate)**.** *Assume that the solution $u\in H_0^1(\Omega)$ to Problem (1) further belongs to $H^{\boldsymbol{\beta}}(\mathcal{T}_h)$. Then, the following estimate holds:*

$$\left|\underline{\Sigma}_{d,h}^k(u) - \underline{\mathrm{u}}_h\right|_{\mathrm{e},h} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\right)^{1/2}, \tag{44}$$

where $\mathbb{R}^{n_{d,h}^k} \ni \underline{\Sigma}_{d,h}^k(u) := \big(\underline{\Sigma}_{d,T}^k(u_{|T})^\intercal\big)_{T\in\mathcal{T}_h}^\intercal$, and $\underline{\mathrm{u}}_h \in \mathbb{V}_{d,h,0}^k$ is the unique solution to Problem (20).

Proof. Since $u\in H^{\frac32+\varepsilon}(\Omega)\cap H_0^1(\Omega)$ (cf. Remark 5.11), $u\in C^0(\overline{\Omega})$ and $u_{|\partial\Omega}\equiv 0$, hence $\underline{\Sigma}_{d,h}^k(u)\in\mathbb{V}_{d,h,0}^k$, and so does the difference $\big(\underline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h\big)$. We can then write

$$\left|\underline{\Sigma}_{d,h}^k(u) - \underline{\mathrm{u}}_h\right|_{\mathrm{e},h} = \max_{\underline{\mathrm{v}}_h\in\mathbb{V}_{d,h,0}^k,\,|\underline{\mathrm{v}}_h|_{\mathrm{e},h}=1} \mathrm{a}_h\big(\underline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h,\underline{\mathrm{v}}_h\big). \tag{45}$$

Since $\underline{\mathrm{u}}_h$ solves Problem (20), we have to estimate, for any $\underline{\mathrm{v}}_h\in\mathbb{V}_{d,h,0}^k$ such that $|\underline{\mathrm{v}}_h|_{\mathrm{e},h}=1$,

$$\mathcal{C}_h(\underline{\mathrm{v}}_h) := \sum_{T\in\mathcal{T}_h}\mathcal{C}_T(\underline{\mathrm{v}}_T),\quad \text{with}\quad \mathcal{C}_T(\underline{\mathrm{v}}_T) := \mathrm{a}_T\big(\underline{\Sigma}_{d,T}^k(u_{|T}),\underline{\mathrm{v}}_T\big) - \int_T f_{|T}\mathrm{v}_T^{\mathrm{o}}.$$

Using the strong form of Problem (1), and integrating by parts, we infer, for any $T\in\mathcal{T}_h$,

$$\mathcal{C}_T(\underline{\mathrm{v}}_T) = \int_T \boldsymbol{\nabla}\mathcal{P}_{d,T}^k(u_{|T})\cdot\boldsymbol{\nabla}p_{d,T}^k\underline{\mathrm{v}}_T - \int_T\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}} + \int_{\partial T}\mathrm{v}_{T|\partial T}^{\mathrm{o}}\boldsymbol{\nabla}u_{|\partial T}\cdot\boldsymbol{n}_T + \mathrm{s}_T\big(\underline{\Sigma}_{d,T}^k(u_{|T}),\underline{\mathrm{v}}_T\big).$$

Using the definition (36) of $p_{d,T}^k\underline{\mathrm{v}}_T$ (where the first term in the right-hand side is integrated by parts), and since $\mathcal{P}_{d,T}^k(u_{|T})\in\mathbb{P}_d^k(T)$, we then have

$$\begin{aligned}\mathcal{C}_T(\underline{\mathrm{v}}_T) = \int_T\boldsymbol{\nabla}\big(\mathcal{P}_{d,T}^k(u_{|T})-u\big)\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}} - \int_{\partial T}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}}-\mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial\big)\boldsymbol{\nabla}\mathcal{P}_{d,T}^k(u_{|T})_{|\partial T}\cdot\boldsymbol{n}_T\\ + \int_{\partial T}\mathrm{v}_{T|\partial T}^{\mathrm{o}}\boldsymbol{\nabla}u_{|\partial T}\cdot\boldsymbol{n}_T + \mathrm{s}_T\big(\underline{\Sigma}_{d,T}^k(u_{|T}),\underline{\mathrm{v}}_T\big).\end{aligned}$$

Summing over $T\in\mathcal{T}_h$, and invoking the continuity of the boundary normal flux of the exact solution along interfaces (cf. Remark 5.11), combined to the fact that $\underline{\mathrm{v}}_h\in\mathbb{V}_{d,h,0}^k$, we infer

$$\begin{aligned}\mathcal{C}_h(\underline{\mathrm{v}}_h) = \sum_{T\in\mathcal{T}_h}\Bigg(\int_T\boldsymbol{\nabla}\big(\mathcal{P}_{d,T}^k(u_{|T})-u\big)\cdot\boldsymbol{\nabla}\mathrm{v}_T^{\mathrm{o}}\\ + \int_{\partial T}\big(\mathrm{v}_{T|\partial T}^{\mathrm{o}}-\mathrm{t}_{\partial T}\underline{\mathrm{v}}_T^\partial\big)\boldsymbol{\nabla}\big(u-\mathcal{P}_{d,T}^k(u_{|T})\big)_{|\partial T}\cdot\boldsymbol{n}_T + \mathrm{s}_T\big(\underline{\Sigma}_{d,T}^k(u_{|T}),\underline{\mathrm{v}}_T\big)\Bigg).\end{aligned} \tag{46}$$

Applying Cauchy–Schwarz inequality, and the approximation results (38)–(39) with $s\in\{\beta_T\}_{T\in\mathcal{T}_h}$ and $m=1$, we obtain

$$\mathcal{C}_h(\underline{\mathrm{v}}_h) \lesssim \left(\sum_{T\in\mathcal{T}_h}\left(h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2 + \mathrm{s}_T\big(\underline{\Sigma}_{d,T}^k(u_{|T}),\underline{\Sigma}_{d,T}^k(u_{|T})\big)\right)\right)^{1/2}\left(|\underline{\mathrm{v}}_h|_h + |\underline{\mathrm{v}}_h|_{\mathrm{e},h}\right).$$

The conclusion then follows from Lemma 5.10 with $s\in\{\beta_T\}_{T\in\mathcal{T}_h}$, Corollary 5.9, and the fact that $|\underline{\mathrm{v}}_h|_{\mathrm{e},h}=1$ by assumption. □

Remark 5.13. *In the nonconforming case, to prove that $\underline{\Sigma}_{d,h}^k(u)$ belongs to $\mathbb{V}_{d,h,0}^k$, it is sufficient to use that $u\in H_0^1(\Omega)$. Furthermore, the first term in the right-hand side of (46) is identically zero. Indeed, in that case, for all $T\in\mathcal{T}_h$, $\mathcal{P}_{d,T}^k=\Pi_T^k$ (recall that $\mathrm{v}_T^{\mathrm{o}}\in\mathbb{P}_d^{k-1}(T)\subset\mathbb{P}_d^k(T)$). Such a property is not true in the conforming case.*

Corollary 5.14 ($H^1(\mathcal{T}_h)$-seminorm error estimate)**.** *Under the regularity assumption of Theorem 5.12, the following estimate holds:*

$$\left\|\boldsymbol{\nabla}_h\left(u-p_{d,h}^k\underline{\mathrm{u}}_h\right)\right\|_{0,\Omega}\lesssim\left(\sum_{T\in\mathcal{T}_h}h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\right)^{1/2},\tag{47}$$

where $p_{d,h}^k:\mathbb{R}^{n_{d,h}^k}\to\mathbb{P}_d^k(\mathcal{T}_h)$ is such that, for all $T\in\mathcal{T}_h$, $p_{d,h}^k\underline{\mathrm{v}}_{h|T}:=p_{d,T}^k\underline{\mathrm{v}}_T$.

Proof. Letting $\mathcal{P}_{d,h}^k:=p_{d,h}^k\circ\overline{\Sigma}_{d,h}^k$, by a simple triangle inequality, we infer

$$\left\|\boldsymbol{\nabla}_h\left(u-p_{d,h}^k\underline{\mathrm{u}}_h\right)\right\|_{0,\Omega}\le\left\|\boldsymbol{\nabla}_h\left(u-\mathcal{P}_{d,h}^ku\right)\right\|_{0,\Omega}+\left\|\boldsymbol{\nabla}_hp_{d,h}^k\left(\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h\right)\right\|_{0,\Omega}.$$

Using the definition of $|\cdot|_{\mathrm{e},h}$, we obtain

$$\left\|\boldsymbol{\nabla}_h\left(u-p_{d,h}^k\underline{\mathrm{u}}_h\right)\right\|_{0,\Omega}\le\left\|\boldsymbol{\nabla}_h\left(u-\mathcal{P}_{d,h}^ku\right)\right\|_{0,\Omega}+\left|\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h\right|_{\mathrm{e},h}.\tag{48}$$

The conclusion follows from the approximation result (38) with $s\in\{\beta_T\}_{T\in\mathcal{T}_h}$ and $m=1$, and from the discrete energy-norm error estimate (44). □

The proof of Theorem 5.12–Corollary 5.14 is inspired from the one for HHO methods, but is here valid whatever the conformity of the underlying global virtual space. The derivation of our error estimate in $H^1(\mathcal{T}_h)$-seminorm is based on a splitting (48) of the error that follows [30, Section 2.4.1]. Basically, the error splits into an approximation error, and the discrete energy-norm error, that is nothing but the consistency error of the scheme (see (45)). All the analysis can be performed only resorting to the H^m approximation properties of the polynomial projector $\mathcal{P}_{d,h}^k$. In that respect, in the conforming case, it constitutes an alternative to standard VE analyses, where the approximation properties of the local virtual spaces are usually explicitly used (hence under Assumption 1.1). The same remark also holds for the $L^2(\Omega)$-norm analysis below.

Remark 5.15. *Assume that we want to derive a $H^1(\mathcal{T}_h)$-seminorm error estimate on the difference $(u-u_h)$, where $u_h\in V_{d,h,0}^k$ is such that $u_{h|T}:=r_{d,T}^k\underline{\mathrm{u}}_T$ for all $T\in\mathcal{T}_h$. In view of Lemma 3.2 and Remark 3.3, it is clear that, for $\underline{\mathrm{v}}_h\in\mathbb{R}^{n_{d,h}^k}$, the quantity $\left(\sum_{T\in\mathcal{T}_h}\left\|\boldsymbol{\nabla}r_{d,T}^k\underline{\mathrm{v}}_T\right\|_{0,T}^2\right)^{1/2}$, which can be equivalently written $\|\boldsymbol{\nabla}_hv_h\|_{0,\Omega}$ for $v_h\in V_d^k(\mathcal{T}_h)$ such that $v_{h|T}:=r_{d,T}^k\underline{\mathrm{v}}_T$ for all $T\in\mathcal{T}_h$, defines a norm on $\mathbb{V}_{d,h,0}^k$. Furthermore, one can easily prove (we omit the proof for brevity) that, for any $\underline{\mathrm{v}}_h\in\mathbb{R}^{n_{d,h}^k}$,*

$$\left(\sum_{T\in\mathcal{T}_h}\left\|\boldsymbol{\nabla}r_{d,T}^k\underline{\mathrm{v}}_T\right\|_{0,T}^2\right)^{1/2}\lesssim|\underline{\mathrm{v}}_h|_h.\tag{49}$$

To prove (49) in the conforming case, one has to use on all $T\in\mathcal{T}_h$ the same arguments as in the proof of Lemma 4.7 (when $d=2$) or 4.12 (when $d=3$). We will hence assume, in the conforming case, that the assumptions of the corresponding lemma are met for all $T\in\mathcal{T}_h$. Let us apply (49) to $\underline{\mathrm{v}}_h=\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h$. We get, by Corollary 5.9,

$$\left\|\boldsymbol{\nabla}_h\left(\mathcal{I}_{d,h}^ku-u_h\right)\right\|_{0,\Omega}\lesssim\left|\overline{\Sigma}_{d,h}^k(u)-\underline{\mathrm{u}}_h\right|_{\mathrm{e},h},\tag{50}$$

where $\mathcal{I}_{d,h}^ku\in V_{d,h,0}^k$ is such that $\mathcal{I}_{d,h}^ku_{|T}:=\mathcal{I}_{d,T}^k(u_{|T})=r_{d,T}^k\left(\overline{\Sigma}_{d,T}^k(u_{|T})\right)$ for all $T\in\mathcal{T}_h$. Since, by the triangle inequality,

$$\left\|\boldsymbol{\nabla}_h\left(u-u_h\right)\right\|_{0,\Omega}\le\left\|\boldsymbol{\nabla}_h\left(u-\mathcal{I}_{d,h}^ku\right)\right\|_{0,\Omega}+\left\|\boldsymbol{\nabla}_h\left(\mathcal{I}_{d,h}^ku-u_h\right)\right\|_{0,\Omega},$$

we can prove, using (28)–(34) with $s\in\{\beta_T\}_{T\in\mathcal{T}_h}$ and $m=1$ (same assumptions as above in the conforming case) for the first term in the right-hand side, and (50) combined to (44) for the second, that under the same regularity assumption as in Theorem 5.12, there holds

$$\left\|\boldsymbol{\nabla}_h\left(u-u_h\right)\right\|_{0,\Omega}\lesssim\left(\sum_{T\in\mathcal{T}_h}h_T^{2(\beta_T-1)}|u|_{\beta_T,T}^2\right)^{1/2}.\tag{51}$$

Obviously, the discrete solution u_h is not computable/computed in practice, only the polynomial projection $p_{d,h}^k\underline{\mathrm{u}}_h=\Pi_h^ku_h$ is.

Let us now derive an estimate on the error between $p_{d,h}^k \underline{\mathrm{u}}_h$ and u in the $L^2(\Omega)$-norm. To prove our result, we reuse some ideas from the proof of [30, Theorem 22]. We assume that $k \geqslant 2$. The lowest-order case would require a specific treatment, and additional regularity on the source term. We omit its study here for brevity.

Theorem 5.16 (Supercloseness of bulk unknowns)**.** *Let $k \geqslant 2$. Assume that the solution $u \in H_0^1(\Omega)$ to Problem (1) further belongs to $H^\beta(\mathcal{T}_h)$. Assume full elliptic regularity for Problem (1) (cf. Remark 5.11). Then, there holds*

$$\left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega} \lesssim h \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T,T}^2\right)^{1/2}, \tag{52}$$

where $\mathrm{u}_h^{\mathrm{o}} \in \mathbb{P}_d^{k-1}(\mathcal{T}_h)$ is such that $\mathrm{u}_{h|T}^{\mathrm{o}} := \mathrm{u}_T^{\mathrm{o}}$ for all $T \in \mathcal{T}_h$.

Proof. We follow the standard Aubin–Nitsche argument. Setting $g := (\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}) \in L^2(\Omega)$, we let $z \in H_0^1(\Omega)$ be the unique solution in $H^1(\Omega)$ to $-\triangle z = g$ in Ω, with $z = 0$ on $\partial\Omega$. By (full) elliptic regularity, $z \in H^2(\Omega)$ and, in view of (43), there holds

$$\|z\|_{2,\Omega} \lesssim \left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega}. \tag{53}$$

Writing

$$\left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega}^2 = -\sum_{T\in\mathcal{T}_h} \int_T \left(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}\right) \triangle z,$$

and integrating by parts, we infer, since $z \in H^2(\Omega)$ (hence $[\![\boldsymbol{\nabla} z]\!]_F \cdot \boldsymbol{n}_F = 0$ almost everywhere on interfaces $F \in \mathcal{F}_h^{\mathrm{i}}$), and $u \in H_0^1(\Omega) \cap H^2(\Omega)$ as well as $\underline{\mathrm{u}}_h \in \mathbb{V}_{d,h,0}^k$,

$$\begin{aligned}\left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega}^2 = \sum_{T\in\mathcal{T}_h} \Bigg(&\int_T \boldsymbol{\nabla}\left(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}\right)\cdot\boldsymbol{\nabla} z \\ &- \int_{\partial T} \left[\left(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}\right)_{|\partial T} - \left(\mathrm{t}_{\partial T}\left(\underline{\Sigma}_{d,T}^{k,\partial}(u_{|T})\right) - \mathrm{t}_{\partial T}\underline{\mathrm{u}}_T^{\partial}\right)\right] \boldsymbol{\nabla} z_{|\partial T}\cdot\boldsymbol{n}_T\Bigg).\end{aligned}$$

Adding/subtracting the term $\displaystyle\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\nabla} p_{d,T}^k\left(\overline{\Sigma}_{d,T}^k(u_{|T}) - \underline{\mathrm{u}}_T\right)\cdot\boldsymbol{\nabla}\mathcal{P}_{d,T}^k(z_{|T})$, and using the definition (36) (where the first term in the right-hand side is integrated by parts), we infer that

$$\begin{aligned}\left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega}^2 = \mathfrak{T}_1 + \Bigg(&\int_\Omega \boldsymbol{\nabla}_h\mathcal{P}_{d,h}^k u\cdot\boldsymbol{\nabla}_h\mathcal{P}_{d,h}^k z - \mathrm{a}_h\left(\underline{\mathrm{u}}_h, \overline{\Sigma}_{d,h}^k(z)\right)\Bigg) \\ &+ \sum_{T\in\mathcal{T}_h} \mathrm{s}_T\left(\underline{\mathrm{u}}_T, \overline{\Sigma}_{d,T}^k(z_{|T})\right) =: \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3,\end{aligned} \tag{54}$$

where

$$\begin{aligned}\mathfrak{T}_1 := \sum_{T\in\mathcal{T}_h} \Bigg(&\int_T \boldsymbol{\nabla}\left(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}\right)\cdot\boldsymbol{\nabla}\left(z - \mathcal{P}_{d,T}^k(z_{|T})\right) \\ &- \int_{\partial T} \left[\left(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}\right)_{|\partial T} - \left(\mathrm{t}_{\partial T}\left(\underline{\Sigma}_{d,T}^{k,\partial}(u_{|T})\right) - \mathrm{t}_{\partial T}\underline{\mathrm{u}}_T^{\partial}\right)\right] \boldsymbol{\nabla}\left(z - \mathcal{P}_{d,T}^k(z_{|T})\right)_{|\partial T}\cdot\boldsymbol{n}_T\Bigg).\end{aligned}$$

We now have to estimate the different terms in the right-hand side of (54). To estimate $\mathfrak{T}_1$, we apply successively the Cauchy–Schwarz inequality, (38)–(39) with $s = 2$ and $m = 1$, and Corollary 5.9. We get

$$\mathfrak{T}_1 \lesssim \left|\overline{\Sigma}_{d,h}^k(u) - \underline{\mathrm{u}}_h\right|_{\mathrm{e},h} h\,|z|_{2,\Omega},$$

which, in turn, by (44) and (53), gives

$$\mathfrak{T}_1 \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T,T}^2\right)^{1/2} h\left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega}. \tag{55}$$

To estimate $\mathfrak{T}_3$, we use the fact that

$$\sum_{T\in\mathcal{T}_h} \mathrm{s}_T\big(\underline{\mathrm{u}}_T, \overline{\Sigma}_{d,T}^k(z_{|T})\big) = \sum_{T\in\mathcal{T}_h} \mathrm{s}_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}), \overline{\Sigma}_{d,T}^k(z_{|T})\big) - \sum_{T\in\mathcal{T}_h} \mathrm{s}_T\big(\overline{\Sigma}_{d,T}^k(u_{|T}) - \underline{\mathrm{u}}_T, \overline{\Sigma}_{d,T}^k(z_{|T})\big)$$

to infer, using successively the Cauchy–Schwarz inequality, (42) with $s \in \{\beta_T\}_{T\in\mathcal{T}_h}$ for $u_{|T}$, (44), (42) with $s = 2$ for $z_{|T}$, and (53),

$$\mathfrak{T}_3 \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T,T}^2\right)^{1/2} h \left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega}. \tag{56}$$

We are left with estimating $\mathfrak{T}_2$. Recall that $\mathrm{a}_h\big(\underline{\mathrm{u}}_h, \overline{\Sigma}_{d,h}^k(z)\big) = \int_\Omega f \pi_h^{k-1} z$ since $\underline{\mathrm{u}}_h \in \mathbb{V}_{d,h,0}^k$ solves Problem (20). Adding/subtracting to $\mathfrak{T}_2$ the term $\displaystyle\int_\Omega \boldsymbol{\nabla}_h \Pi_h^k u \cdot \boldsymbol{\nabla}_h \mathcal{P}_{d,h}^k z$, there holds

$$\mathfrak{T}_2 = \left(\int_\Omega \boldsymbol{\nabla}_h \Pi_h^k u \cdot \boldsymbol{\nabla}_h \mathcal{I}_{d,h}^k z + \int_\Omega \triangle u\, \pi_h^{k-1} z\right) + \left(\int_\Omega \boldsymbol{\nabla}_h\big(\mathcal{I}_{d,h}^k u - u\big) \cdot \boldsymbol{\nabla}_h \mathcal{P}_{d,h}^k z\right) =: \mathfrak{T}_{2,1} + \mathfrak{T}_{2,2}.$$

Let us first estimate $\mathfrak{T}_{2,1}$. An integration by parts yields

$$\begin{aligned}\mathfrak{T}_{2,1} = \sum_{T\in\mathcal{T}_h} \left(-\int_T \triangle \Pi_T^k(u_{|T})\, \pi_T^{k-1}(z_{|T}) + \int_{\partial T} \pi_{\partial T}^{k-1}\left(\mathcal{I}_{d,T}^k(z_{|T})_{|\partial T}\right) \boldsymbol{\nabla}\Pi_T^k(u_{|T})_{|\partial T} \cdot \boldsymbol{n}_T\right) \\ + \sum_{T\in\mathcal{T}_h} \left(-\int_T \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \pi_T^{k-1}(z_{|T}) + \int_{\partial T} \pi_T^{k-1}(z_{|T})_{|\partial T} \boldsymbol{\nabla} u_{|\partial T} \cdot \boldsymbol{n}_T\right),\end{aligned}$$

where we have used that $-\triangle \Pi_T^k(u_{|T}) \in \mathbb{P}_d^{k-2}(T) \subset \mathbb{P}_d^{k-1}(T)$ and $\pi_T^{k-1}\left(\mathcal{I}_{d,T}^k(z_{|T})\right) = \pi_T^{k-1}(z_{|T})$, and that $\boldsymbol{\nabla}\Pi_T^k(u_{|T})_{|\partial T} \cdot \boldsymbol{n}_T \in \mathbb{P}_{d-1}^{k-1}(\mathcal{F}_T)$. A new integration by parts of the very first term then yields

$$\begin{aligned}\mathfrak{T}_{2,1} = \sum_{T\in\mathcal{T}_h} \left(\int_T \boldsymbol{\nabla}\Pi_T^k(u_{|T}) \cdot \boldsymbol{\nabla} \pi_T^{k-1}(z_{|T}) - \int_{\partial T} \left[\pi_T^{k-1}(z_{|T})_{|\partial T} - \pi_{\partial T}^{k-1}\left(\mathcal{I}_{d,T}^k(z_{|T})_{|\partial T}\right)\right] \boldsymbol{\nabla}\Pi_T^k(u_{|T})_{|\partial T} \cdot \boldsymbol{n}_T\right) \\ + \sum_{T\in\mathcal{T}_h} \left(-\int_T \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \pi_T^{k-1}(z_{|T}) + \int_{\partial T} \pi_T^{k-1}(z_{|T})_{|\partial T} \boldsymbol{\nabla} u_{|\partial T} \cdot \boldsymbol{n}_T\right),\end{aligned}$$

which finally provides, by orthogonality of Π_T^k combined to the fact that $\pi_T^{k-1}(z_{|T}) \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$, and since $u \in H^2(\Omega)$ and $z \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$\mathfrak{T}_{2,1} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \pi_{\partial T}^{k-1}\left[\pi_T^{k-1}(z_{|T})_{|\partial T} - \mathcal{I}_{d,T}^k(z_{|T})_{|\partial T}\right] \boldsymbol{\nabla}\big(u - \Pi_T^k(u_{|T})\big)_{|\partial T} \cdot \boldsymbol{n}_T.$$

By the same kind of arguments, we also obtain

$$\mathfrak{T}_{2,2} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \pi_{\partial T}^{k-1}\left[u_{|\partial T} - \mathcal{I}_{d,T}^k(u_{|T})_{|\partial T}\right] \boldsymbol{\nabla}\left(z - \mathcal{P}_{d,T}^k(z_{|T})\right)_{|\partial T} \cdot \boldsymbol{n}_T.$$

In the conforming case with $d = 3$ (cf. Remark 5.2) and in the nonconforming case, one has

$$\pi_{\partial T}^{k-1}\left(\mathcal{I}_{d,T}^k(v_{|T})_{|\partial T}\right) = \pi_{\partial T}^{k-1}(v_{|\partial T}),$$

hence $\mathfrak{T}_{2,2} = 0$ and $\mathfrak{T}_{2,1}$ can be estimated using successively (i) Cauchy–Schwarz inequality, (ii) (14) with $s \in \{\beta_T\}_{T\in\mathcal{T}_h}$ and $m = 1$, (iii) the stability of $\pi_{\partial T}^{k-1}$ in the $L^2(\partial T)$-norm, (iv) (14) with $s = 2$ and $m = 0$ (recall that $k \geqslant 2$), and (v) (53). One obtains

$$\mathfrak{T}_2 \lesssim h \left\|\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right\|_{0,\Omega} \left(\sum_{T\in\mathcal{T}_h} h_T^{2(\beta_T-1)} |u|_{\beta_T,T}^2\right)^{1/2}. \tag{57}$$

It remains to treat the conforming case with $d = 2$. In that case, $\mathcal{I}_{2,T}^k(v_{|T})_{|\partial T} = \mathcal{I}_{1,\partial T}^k(v_{|\partial T})$. The term $\mathfrak{T}_{2,1}$ can be estimated as previously remarking that

$$\pi_T^{k-1}(z_{|T})_{|\partial T} - \mathcal{I}_{1,\partial T}^k(z_{|\partial T}) = \mathcal{I}_{1,\partial T}^k\left(\pi_T^{k-1}(z_{|T})_{|\partial T} - z_{|\partial T}\right).$$

By the stability result of Lemma 4.3, and the Sobolev inequality (10) combined to (13) with $s = 2$ and $m \in \{0, \ldots, 2\}$ (recall that $k \geqslant 2$), we infer

$$\mathfrak{T}_{2,1} \lesssim h \left\| \pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}} \right\|_{0,\Omega} \left(\sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T - 1)} |u|_{\beta_T, T}^2 \right)^{1/2}.$$

The term $\mathfrak{T}_{2,2}$ can be estimated remarking that

$$u_{|\partial T} - \mathcal{I}_{1,\partial T}^k(u_{|\partial T}) = \left(u - \mathcal{P}_{2,T}^k(u_{|T})\right)_{|\partial T} - \mathcal{I}_{1,\partial T}^k \left(\left[u - \mathcal{P}_{2,T}^k(u_{|T}) \right]_{|\partial T} \right).$$

The conclusion then easily follows from the stability of $\pi_{\partial T}^{k-1}$ in the $L^2(\partial T)$-norm, from Lemma 4.3 and the Sobolev inequality (10), from the approximation results (38)–(39), and from (53). We infer the same kind of estimate as for $\mathfrak{T}_{2,1}$, and finally we infer (57), also in that case. Collecting (54), (55), (57), and (56), we conclude the proof. □

Corollary 5.17 ($L^2(\Omega)$-norm error estimate)**.** *Under the assumptions of Theorem 5.16, the following estimate holds:*

$$\left\| u - p_{d,h}^k \underline{\mathrm{u}}_h \right\|_{0,\Omega} \lesssim h \left(\sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T - 1)} |u|_{\beta_T, T}^2 \right)^{1/2}. \tag{58}$$

Proof. By the triangle inequality, and by stability of π_h^0 in the $L^2(\Omega)$-norm, we infer

$$\left\| u - p_{d,h}^k \underline{\mathrm{u}}_h \right\|_{0,\Omega} \leqslant \left\| \pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}} \right\|_{0,\Omega} + \left\| \left(u - p_{d,h}^k \underline{\mathrm{u}}_h\right) - \pi_h^0 \left(\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right) \right\|_{0,\Omega}.$$

We then remark that $\int_T \left(u - p_{d,T}^k \underline{\mathrm{u}}_T\right) = \int_T \pi_T^0 \left(\pi_T^{k-1}(u_{|T}) - \mathrm{u}_T^{\mathrm{o}}\right)$ for all $T \in \mathcal{T}_h$. Hence, by the Poincaré inequality (7), there holds

$$\left\| u - p_{d,h}^k \underline{\mathrm{u}}_h \right\|_{0,\Omega} \lesssim \left\| \pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}} \right\|_{0,\Omega} + h \left\| \boldsymbol{\nabla}_h \left(u - p_{d,h}^k \underline{\mathrm{u}}_h\right) \right\|_{0,\Omega}.$$

The conclusion then follows from (52) and (47). □

Remark 5.18. *Assume that we want to derive a $L^2(\Omega)$-norm error estimate on $(u - u_h)$. As in Remark 5.15, we assume, in the conforming case, that the assumptions of Lemma 4.7 (when $d = 2$) or 4.12 (when $d = 3$) are met for all $T \in \mathcal{T}_h$. Noticing that $\pi_h^0(u - u_h) = \pi_h^0\left(\pi_h^{k-1}(u - u_h)\right) = \pi_h^0\left(\pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}}\right)$, we can write, by stability of π_h^0 in the $L^2(\Omega)$-norm,*

$$\|u - u_h\|_{0,\Omega} \leqslant \left\| \pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}} \right\|_{0,\Omega} + \left\| (u - u_h) - \pi_h^0 (u - u_h) \right\|_{0,\Omega}.$$

Then, the Poincaré inequality (7) yields

$$\|u - u_h\|_{0,\Omega} \lesssim \left\| \pi_h^{k-1} u - \mathrm{u}_h^{\mathrm{o}} \right\|_{0,\Omega} + h \left\| \boldsymbol{\nabla}_h \left(u - u_h\right) \right\|_{0,\Omega}.$$

From (52) and (51), we finally infer, under the assumptions of Theorem 5.16, that

$$\|u - u_h\|_{0,\Omega} \lesssim h \left(\sum_{T \in \mathcal{T}_h} h_T^{2(\beta_T - 1)} |u|_{\beta_T, T}^2 \right)^{1/2}.$$

5.4 General workflow

As for any Galerkin method, the general workflow for solving Problem (20) with the VE/HHO methods splits into (i) an offline stage, that is independent of the source term and of the boundary conditions, and which aims at performing the assembly of the general problem matrix, and (ii) an online stage, that consists, for given source term and boundary conditions, in solving the resulting global system. A change in the data only affects the online stage (cf. Remark 5.19). The precomputations that are performed in the offline stage are all local; hence, this stage can naturally benefit from parallel architectures.

Let us describe, in details, these two stages for Problem (20), beginning with the offline stage.

1. In the conforming case with $d = 2$ (respectively, $d = 3$), one first computes the operator $r_{1,F}^k$ defined by (22) for all $F \in \mathcal{F}_h$ (respectively, $r_{1,e}^k$ for all $e \in \mathcal{E}_h$). This requires to solve a SPD system of size $k+1$ (cf. Remark 2.6), for $k+1$ right-hand sides. In both the conforming and nonconforming cases now, locally to any $T \in \mathcal{T}_h$, one computes the operator $p_{d,T}^k$ defined by (36). Its computation requires to solve a SPD system of size N_d^k (cf. also Remark 2.6), for a number of right-hand sides that is (i) $\mathrm{N}_2^{k-1} + \mathrm{card}(\mathcal{F}_T) \times (k-1) + \mathrm{card}(\mathcal{V}_T)$ in the conforming case with $d = 2$, (ii) $\mathrm{N}_3^{k-1} + \mathrm{card}(\mathcal{F}_T) \times \mathrm{N}_2^{k-1}$ in the conforming case with $d = 3$, and (iii) $\mathrm{N}_d^{k-1} + \mathrm{card}(\mathcal{F}_T) \times \mathrm{N}_{d-1}^{k-1}$ in the nonconforming case (cf. Remark 5.1). Once the operator $p_{d,T}^k$ has been computed, one computes, still locally to any $T \in \mathcal{T}_h$, the bilinear form, that writes in terms of the DoF and of the different already computed quantities.

2. As common to any skeletal method (cf. Section 3), bulk DoF are locally eliminated by static condensation in terms of the local skeletal DoF. Locally to each $T \in \mathcal{T}_h$, one has to solve a SPD system of size N_d^{k-1} for a number of right-hand sides that is the number of local skeletal DoF.

Let us now describe the online stage, for a given source term $f \in L^2(\Omega)$.

3. One computes the right-hand side, which requires, locally to any $T \in \mathcal{T}_h$, to integrate $f_{|T}$ against polynomials in $\mathbb{P}_d^{k-1}(T)$, and to perform its static condensation. Then, one eliminates the boundary (Dirichlet) DoF from the global system.

4. One solves the resulting SPD global system, that is of size

$$\mathrm{card}(\mathcal{F}_h^{\mathrm{i}}) \times (k-1) + \mathrm{card}(\mathcal{V}_h^{\mathrm{i}})$$

in the conforming case with $d = 2$,

$$\mathrm{card}(\mathcal{F}_h^{\mathrm{i}}) \times \mathrm{N}_2^{k-1} + \mathrm{card}(\mathcal{E}_h^{\mathrm{i}}) \times (k-1) + \mathrm{card}(\mathcal{V}_h^{\mathrm{i}})$$

in the conforming case with $d = 3$ (cf. Remark 5.21), and $\mathrm{card}(\mathcal{F}_h^{\mathrm{i}}) \times \mathrm{N}_{d-1}^{k-1}$ in the nonconforming case.

For an example of implementation of the method using generic programming, we refer to [21].

Remark 5.19. *Except point* 3*, the description above of the general workflow of the method applies verbatim to the case of a Problem* (1) *featuring nonhomogeneous mixed Dirichlet–Neumann boundary conditions.*

Remark 5.20. *In a multi-query context in which the datum is the diffusion coefficient, part of the offline stage becomes online (in particular, the static condensation part). Depending on how are defined the polynomial projectors (if their definition includes or not the diffusion coefficient), the part of the offline stage becoming online may be more or less important. In that case, one would better consider reduced basis like techniques.*

Remark 5.21. *In the conforming case with $d = 3$, one can reduce the size of the global system by using enhanced virtual spaces [1] or Serendipity spaces [9] on the faces $F \in \mathcal{F}_T$ of the cells $T \in \mathcal{T}_h$, instead of $V_2^k(F)$. Typically, with enhancement, one reduces the number of face DoF to $\mathrm{card}(\mathcal{F}_h^{\mathrm{i}}) \times \mathrm{N}_2^{k-2}$ instead of $\mathrm{card}(\mathcal{F}_h^{\mathrm{i}}) \times \mathrm{N}_2^{k-1}$. The inclusion of enhanced virtual spaces into our general framework will be the topic of a forthcoming work.*

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A Proofs of (11) and Lemma 3.2

A.1 Proof of (11)

By Assumption 1.1, for any $z \in \mathfrak{T}(\partial T) \subset H^1(\partial T)$, there exists $\overline{z} \in H^1(T)$ such that $\overline{z}_{|\partial T} = z$ and

$$h_T^{-1} \|\overline{z}\|_{0,T} + |\overline{z}|_{1,T} \lesssim h_T^{-1/2} \|z\|_{0,\partial T} + h_T^{1/2} |z|_{1,\partial T}. \tag{59}$$

Now, since $v \in H^1(T)$ is such that $\triangle v \in L^2(T)$, by the divergence formula there holds

$$\langle \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T, z \rangle_{-\frac{1}{2},\partial T} = \int_T \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} \overline{z} + \int_T \triangle v\, \overline{z},$$

and hence, by Cauchy–Schwarz inequality,

$$\langle \boldsymbol{\nabla} v_{|\partial T} \cdot \boldsymbol{n}_T, z \rangle_{-\frac{1}{2},\partial T} \leqslant |v|_{1,T} |\overline{z}|_{1,T} + h_T \|\triangle v\|_{0,T} h_T^{-1} \|\overline{z}\|_{0,T}.$$

The estimate (59) enables to conclude.

A.2 Proof of Lemma 3.2

Assume that, for $v_h \in V_{h,0}$, $\|\boldsymbol{\nabla}_h v_h\|_{0,\Omega} = 0$. Then, for all $T \in \mathcal{T}_h$, $\boldsymbol{\nabla}(v_{h|T}) = \boldsymbol{0}$ and there is $c_T \in \mathbb{R}$ such that $v_{h|T} = c_T$. Since $\{\check{\pi}_F^0\}_{F\in\mathcal{F}_h} \subseteq \Sigma_h^\partial$, then for all $F \in \mathcal{F}_h^{\mathrm{i}}$, $\int_F [\![v_h]\!]_F = 0$ and there exists $c \in \mathbb{R}$ such that $c_T = c$ for all $T \in \mathcal{T}_h$. The fact that $\int_F v_{h|F} = 0$ for some (here, all) $F \in \mathcal{F}_h^{\mathrm{b}}$ finally yields that $v_h \equiv 0$ on Ω. To prove (18), we start from the following discrete Poincaré inequality on $H^1(\mathcal{T}_h)$ (cf., e.g., [12]):

$$\forall v_h \in H^1(\mathcal{T}_h), \qquad \|v_h\|_{0,\Omega}^2 \lesssim \|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}^2 + \sum_{F\in\mathcal{F}_h} h_F^{-1} \|[\![v_h]\!]_F\|_{0,F}^2,$$

and we show that $\sum_{F\in\mathcal{F}_h} h_F^{-1} \|[\![v_h]\!]_F\|_{0,F}^2 \lesssim \|\boldsymbol{\nabla}_h v_h\|_{0,\Omega}^2$ for all $v_h \in V_{h,0}$. To prove so, since for $v_h \in V_{h,0}$, $\int_F [\![v_h]\!]_F = 0$ for all $F \in \mathcal{F}_h$, there holds

$$\|[\![v_h]\!]_F\|_{0,F}^2 = \int_F [\![v_h - \pi_h^0 v_h]\!]_F [\![v_h]\!]_F \leqslant \|[\![v_h - \pi_h^0 v_h]\!]_F\|_{0,F} \|[\![v_h]\!]_F\|_{0,F},$$

and we can use the continuous trace inequality (5) and the Poincaré inequality (7) to infer

$$\begin{aligned} h_F^{-\frac{1}{2}} \|[\![v_h]\!]_F\|_{0,F} &\lesssim h_F^{-\frac{1}{2}} \sum_{T\in\mathcal{T}_F} \left(h_T^{-\frac{1}{2}} \|v_{h|T} - \pi_T^0(v_{h|T})\|_{0,T} + h_T^{\frac{1}{2}} \|\boldsymbol{\nabla}(v_{h|T})\|_{0,T} \right) \\ &\lesssim h_F^{-\frac{1}{2}} \sum_{T\in\mathcal{T}_F} h_T^{\frac{1}{2}} \|\boldsymbol{\nabla}(v_{h|T})\|_{0,T}. \end{aligned}$$

Finally, since h_F is comparable to h_T for $T \in \mathcal{T}_F$, and $\mathrm{card}(\mathcal{F}_T) \lesssim 1$ for all $T \in \mathcal{T}_h$ (cf. Section 2.2), the conclusion follows.

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